Discrete gauging in Coulomb branches of three dimensional $\mathcal{N} = 4$ supersymmetric gauge theories

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ABSTRACT: This paper tests a conjecture on discrete non-Abelian gauging of $3d \mathcal{N} = 4$ supersymmetric quiver gauge theories. Given a parent quiver with a bouquet of n nodes of rank 1, invariant under a discrete S_n global symmetry, one can construct a daughter quiver where the bouquet is substituted by a single adjoint n node. Based on the main conjecture in this paper, the daughter quiver corresponds to a theory where the S_n discrete global symmetry is gauged and the new Coulomb branch is a non-Abelian orbifold of the parent Coulomb branch. We demonstrate and test the conjecture for three simply laced families of bouquet quivers and a non-simply laced bouquet quiver with C_2 factor in the global symmetry.

KEYWORDS: Discrete Symmetries, Field Theories in Lower Dimensions, Global Symmetries, Supersymmetric Gauge Theory

ARXIV EPRINT: [1807.03221](https://arxiv.org/abs/1807.03221)

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Figure 1. $Q_{\{1^n\}}$ quiver.

Figure 2. $Q_{\{n\}}$ quiver.

1 Introduction

3d $\mathcal{N} = 4$ quiver gauge theory typically has a Coulomb branch of the moduli space that is a hyperkähler singularity [1–4]. The graph theoretical nature of such quivers opens a large field for the study of orbifolding and other actions of discrete groups [5]. This paper is devoted to a particular action on quivers which has both, a gauge theoretic, as well as a geometric interpretation.

We consider a large class of quivers which have a set of $U(1)$ nodes attached to a common pivot node. Apart from this *complete bouquet*¹ of nodes, the rest of the quiver is arbitrary as the statement to be made on the quiver is a purely local one. The action on the quiver can be summarized by taking the set of n $U(1)$ nodes and replacing them with an adjoint n node. The construction is formulated by Conjecture 1, which is a more general version of Conjecture 1 in [6].

Conjecture 1 (Discrete Gauging) *Given a 3d $\mathcal{N} = 4$ quiver $Q_{\{1^n\}}$ ² with n nodes of rank 1 attached to another node of rank k , (gauge node or global node³) (figure 1), one can construct a new 3d $\mathcal{N} = 4$ quiver $Q_{\{n\}}$ with an adjoint n node attached to k (figure 2). Then, the following relation between the Coulomb branches of these quivers holds*

$$\mathcal{C}(Q_{\{n\}}) = \mathcal{C}(Q_{\{1^n\}}) / S_n, \quad (1.1)$$

where S_n is the discrete symmetry group of permutations of n elements.

¹*Complete* means that all the nodes of the bouquet are of rank 1. Generically, bouquet can consist of nodes of any ranks, not necessarily the same within the bouquet.

²The partition notation for bouquet quivers is explained later in this section.

³The special case when the pivot node is a global flavor node is discussed in section 4.1 in [1].

The quiver in figure 1 has a natural S_n symmetry which permutes the $U(1)$ gauge nodes, and the corresponding Coulomb branch inherits this symmetry as a discrete global symmetry. A natural step in a geometric construction of moduli spaces is to gauge a subgroup of the discrete global symmetry, resulting in a new moduli space. From gauge theoretical perspective, one constructs a new theory, given by the quiver in figure 2, such that the Coulomb branches satisfy equation (1.1) of Conjecture 1.

Discrete gauging has been ascribed physical interpretation for a particular class of 6d $\mathcal{N} = (1, 0)$ supersymmetric theories that describe low energy physics of a set of n M5 branes on a $\mathbb{C}^2/\mathbb{Z}_k$ singularity in [6]. The Higgs branch of such theories at infinite coupling can be expressed as a Coulomb branch of a 3d $\mathcal{N} = 4$ quiver gauge theory. System of n separated M5 branes on $\mathbb{C}^2/\mathbb{Z}_k$ singularity, has a discrete S_n global symmetry on the moduli space. This arises from the manifest permutation symmetry of the corresponding M5 branes (i.e. the positions of the separated M5 branes). By making some of the M5 branes coincident a subgroup of the discrete global symmetry $H_\lambda \subseteq S_n$ is gauged. H_λ corresponds to a partition λ that describes subsets of M5 branes that are coincident. For every partition, different H_λ is gauged, producing a theory with a Coulomb branch that is an non-Abelian orbifold of the parent Coulomb branch ($\mathcal{C}_\lambda$, $\lambda = 1^n$), which corresponds to n separated M5 branes on a $\mathbb{C}^2/\mathbb{Z}_k$ singularity.

In this paper, Conjecture 1 applies to 3d $\mathcal{N} = 4$ quivers that need not necessarily describe low energy dynamics of systems of M5 branes on ALE singularities. In the present work, Conjecture 1 describes a phenomenon purely in 3d without any reference to other dimensions.

In order to test and provide evidence of Conjecture 1, we study a class of quivers which contain a sub-quiver such as depicted in figure 1, consisting of a *bouquet* of n rank 1 gauge nodes that stems from a rank k gauge node. Appendix A contains an example of a simple construction that can be employed to obtain bouquet quivers starting from a generic unitary quiver with flavors. In this paper, we perform the discrete gauging construction prescribed in Conjecture 1 for quivers with unitary gauge nodes, however, it should be emphasized that analogous construction can be formally defined for much broader class of quivers.⁴

To establish the notation, consider the bouquet quiver in figure 3, and let us make the following remarks. Firstly, in a computation of the Coulomb branch, a center of mass $U(1)$ always decouples. All the Coulomb branches in this paper are computed by decoupling the $U(1)$ on the central node since for simply laced quivers, the Coulomb branch does not change if one decouples the $U(1)$ on a different gauge node of the quiver.⁵ Secondly, various arrangements and ranks of the bouquet nodes of a given quiver are in one-to-one correspondence with partitions of n . In particular, in figure 3 and figure 4 the upper nodes are arranged in the form of $\mathcal{P}_{[1^n]}(n)$ and $\mathcal{P}_{[2, 1^{n-2}]}(n)$, respectively. Throughout this paper

⁴Strictly speaking, one only requires a presence of a bouquet without any additional requirements on the node from which the bouquet stems. In particular, the pivot node can be an ortho-symplectic (i.e. O , SO or Sp) gauge node.

⁵A non-simply laced quiver has, in general, K different Coulomb branches, where $K = \#_s + 1$. $\#_s$ is the number of short nodes modulo outer automorphisms of the quiver.

Figure 3. A series quiver with $\mathcal{P}_{[1^n]}(n)$ bouquet.

Figure 4. A-type quiver with $\mathcal{P}_{[2,1^{n-2}]}$ bouquet.

we use partition notation to describe the arrangements and ranks of the bouquet nodes.

In order to perform discrete gauging, let us gauge H , a subgroup of the discrete symmetry S_n , that acts on the bouquet of nodes in figure 3. For $H = \mathbb{Z}_2$, according to Conjecture 1, one obtains the daughter quiver depicted in figure 4.

The bouquet of the new quiver in figure 4 consists of $n - 2$ copies of $U(1)$ nodes and a single $U(2)$ node with an adjoint loop. The adjoint loop adds extra hypermultiplet contributions to the conformal dimension Δ ([7, 8], (2.4) in [1]) of BPS operators⁶ that live in that particular node. The addition of extra hypermultiplets is straightforwardly adjusted for, and implemented, in the *monopole formula*, (2.7) in [1], used for the computation of the Coulomb branch. Examples of quivers with adjoint nodes recently appeared in [10, 11].

It is not yet espied which families of quivers with bouquets are the most interesting for their physical or mathematical properties. From the possible landscape of bouquet quivers the following three families are studied in the present paper:

- Star-shaped quivers with a central 2 node and a bouquet of 1 nodes
- Quivers consisting of a chain of n_2 rank 2 nodes with two bouquets:
 - The first bouquet with n_1 rank 1 nodes is attached to the leftmost chain node
 - The second bouquet with two rank 1 nodes is attached to the rightmost chain node
- A-type quivers with outer $\mathbb{Z}_2$ automorphism symmetry and a bouquet that stems from the central node.⁷

⁶For detailed discussion of BPS operators, see f.i. [9].

⁷Compare with (2.12) in [6].

Figure 5. A series quiver with $\mathcal{P}_{[n]}(n)$ bouquet.

Figure 3 and 4 show examples of quivers which belong to the third family. We can parametrize this family by n and k . For $k = 2$ one recovers the first family of quivers. Considering the quivers belonging to the second family and setting $n_2 = 1$ (i.e. if the “chain” contains only a single 2 node), one also recovers the first family.

Starting with figure 3, one can draw quivers for all partitions of n . For each $U(r)$ node with $r > 1$ in the bouquet, one remembers to add an adjoint loop. By gauging the entire global S_n symmetry of the theory in figure 3, and using Conjecture 1, one obtains the quiver in figure 5, corresponding to the last partition $\mathcal{P}_{[n]}(n)$. It is natural to study the relations between Coulomb branches corresponding to the various partitions $\mathcal{P}(n)$. Let $\mathcal{P}_{[\lambda]}(n)$ be a daughter theory constructed from the parent complete bouquet quiver $\mathcal{P}_{[1^n]}(n)$. Conjecture 1 implies that the Coulomb branches satisfy

$$\mathcal{C}_{[\lambda]} = \mathcal{C}_{[1^n]}/\Gamma, \quad (1.2)$$

where Γ is a discrete symmetry group that corresponds to the difference of the global permutation symmetry between the complete $\mathcal{P}_{[1^n]}(n)$ bouquet and the $\mathcal{P}_{[\lambda]}(n)$ bouquet, respectively. Equation (1.2) has the following implication on the volumes of the two Coulomb branches. One can expand the unrefined Hilbert series around the $t = 1$ pole

$$\text{HS}(t) \big|_{t \rightarrow 1} \sim \frac{R}{(1-t)^d}, \quad (1.3)$$

where d is the complex dimension of the Coulomb branch and R denotes the value of the residue at the pole. Then, since equations (1.4) and (1.5)

$$\text{vol}(\mathcal{C}_{[\lambda]}) = R_\lambda \quad (1.4)$$

$$\text{vol}(\mathcal{C}_{[1^n]}) = R_{1^n} \quad (1.5)$$

define the volumes of the Coulomb branches, equation (1.6) is satisfied.

$$\frac{\text{vol}(\mathcal{C}_{[1^n]})}{\text{vol}(\mathcal{C}_{[\lambda]})} = \frac{R_{1^n}}{R_\lambda} = \text{ord}(\Gamma). \quad (1.6)$$

Note, that $\text{ord}(\Gamma)$ denotes the order of the discrete group Γ . In this note, the discrete gauging construction of Conjecture 1 is applied to all three aforementioned families of quivers. As a result, for each gauged subgroup $H_\lambda \subseteq S_n$ of the discrete global symmetry, one can study the obtained Coulomb branches and perform a collection of non-trivial tests

verifying that the daughter Coulomb branches are non-Abelian orbifolds of the parent Coulomb branch. The same construction is done for a particular representative of non-simply laced quivers with C_2 factor in the global symmetry. One of the motivations for including non-simply laced quivers is merely to emphasize that non-simply laced theories are equally important to study as the simply laced gauge theories. The comparison of the Coulomb branch volumes of the unrefined Hilbert Series is used as a necessary non-trivial test of equation (1.1). Direct comparison of the refined Hilbert Series can be used for an exact verification of Conjecture 1. For the latter, one needs to study how the refined Hilbert series of theory $\mathcal{P}_{[1^n]}(n)$ maps to that of the $\mathcal{P}_{[\lambda]}(n)$ theory. In particular, one can use the character maps between the corresponding character expansions of the Hilbert series.

1.1 The layout of the paper

In section 2 we set the stage by performing an analysis for the *first family* of bouquet quiver theories. The analysis has the following structure.

The analysis. Assuming that Conjecture 1 holds, for each partition (corresponding to different gauging of the discrete global symmetry), we display the corresponding quiver alongside with its imbalance and quaternionic dimension. The imbalance of the unbalanced node as well as the quaternionic dimension of the Coulomb branch are included in the captions of figures. The simple root fugacities used in the monopole formula computation of the Hilbert series are shown inside the quiver nodes or in a separate figure. The rest of the analysis aims to provide evidence for Conjecture 1.

First, we state the anticipation of the global symmetry (G_{global}) on the Coulomb branch based on a conjectured claim about the G_{global} of unbalanced quivers. We present the claim for *minimally unbalanced* quivers in the beginning of section 2. Section 4 contains an extended version of the claim for quivers with more than one unbalanced node. The analysis then further proceeds by the following steps:

- After a computation of the Hilbert series using simple root fugacities the unrefined Hilbert series (HS), obtained by setting all root fugacities to unity, is computed. The result is reported together with the corresponding expansion of the unrefined HS.
- The Plethystic Logarithm (PL) of the unrefined HS is taken.⁸ The t^2 coefficient is compared with the dimension of the adjoint representation of the expected G_{global} . This provides a necessary confirmation that the anticipated G_{global} is correct. In cases of Coulomb branches which have a free sector (this happens for quivers containing a node with a negative imbalance) the global symmetry has two parts:
 - Firstly, the freely generated part of the Coulomb branch is determined.
 - Secondly, this free sector is multiplied out so that the non-trivial part of the Coulomb branch can be further analyzed.

⁸For the definition of Plethystic Logarithm, see [12] or (4.2) in [13].

- The fugacity map, that turns the simple root fugacities into the appropriate fugacities of the G_{global} is given. One then shows that the t^2 coefficient of the refined HS is the character of the adjoint representation of G_{global} . This serves as a direct verification of the G_{global} of the theory.⁹ (In case of a theory which has a free sector, it appears in the form of a character coefficient in front of t in the refined Hilbert series. Before proceeding further with the refined analysis, the free sector is factored out by multiplying the refined HS with an inverse of the Plethystic Exponential¹⁰ (PE) of the character appearing in front of the t term.)
- Next, the Plethystic Logarithm (PL) of the refined HS is taken. The refined PL encodes the information about the number, degree and representation behavior of generators and relations that define the Coulomb branch as an affine algebraic variety.
- The representation content of the chiral ring can be described using a simple polynomial. This compact form is given by the Highest Weight Generating function (HWG), [14]. The HWGs for the first two families of quivers have simple forms and are therefore included. All HWGs are given in the form of Plethystic Exponential (PE).
- Finally, the Coulomb branch is identified.

After the exhaustion of all partitions for a given parent quiver (i.e. when all quivers obtainable by discrete gauging on the parent quiver are exhausted), the volumes of the Coulomb branches are compared. The corresponding ratios are summarized in tables at the end of each subsection. This serves as a non-trivial check that the Coulomb branches of the constructed daughter quivers are orbifolds of the parent Coulomb branch. Section 3 contains a natural generalization of the results of section 2 to theories with $SU(2)^{n_1} \times D_{n_2+1}$ global symmetry and an identification of a general pattern of HWG for the *second family* of quivers. In section 4 the *third family* of theories is studied. In particular, we examine the $k = n = 3$ representative of A-type bouquet quivers invariant under an additional $\mathbb{Z}_2$ outer automorphism symmetry. Section 5 discusses discrete gauging on Coulomb branches of non-simply laced theories. A particular example of a bouquet quiver with a factor of C_2 in the global symmetry is studied. We conclude and discuss possible directions of further investigations in section 6, where we also propose a broader generalization of the pattern of HWG formula (3.28) found at the end of section 3.

2 First family: quivers with central 2 node and a bouquet of 1 nodes

In order to begin, consider the theory in figure 3 and set $k = 2$. For $k = 2$ the discrete global symmetry of the bouquet enhances from S_n to S_{n+2} (i.e. the bouquet of n rank 1 nodes enhances to a bouquet of $n + 2$ nodes). One obtains the simplest quivers for the study of discrete gauging and orbifold actions.

⁹In case of quivers with large character coefficients, this step is by-passed by showing directly the refined PL in the next step.

¹⁰For details of Plethystic Exponential see [12] or (4.2) in [13].

Figure 6. $\mathcal{P}_{[1^3]}(3)$ quiver with $SU(2)^3 \subset Sp(4)$ global symmetry, $b = -1$, $\dim^{\mathbb{H}} \mathcal{M}_C = 4$.

2.1 Case: $k = 2$, $n = 1$

Further, let's consider the $n = 1$ case. The trivial S_1 symmetry enhances to S_3 discrete global symmetry, which becomes the group of outer automorphisms of the quiver permuting the bouquet nodes. Correspondingly, the theory is denoted by $\mathcal{P}_{[1^3]}(3)$. The quiver forms the finite D_4 Dynkin diagram depicted in figure 6, which is the only Dynkin diagram with the *triality* property.

For a simply-laced quiver the balance of the i -th node is defined as [15]:

$$b_{ADE}(i) = \sum_{j \in \text{adjacent nodes}} N_j - 2N_i, \quad (2.1)$$

where N denotes the rank. Quivers with a single unbalanced node (i.e. single node with balance $b \neq 0$) are termed *minimally unbalanced*. Throughout this paper the unbalanced nodes are conveniently drawn red. The red node of the minimally unbalanced quiver in figure 6 has balance $b = 3 \times 1 - 2 \times 2 = -1$. Negative balance indicates that the theory has a free sector, which implies that either part of, or the entire Coulomb branch, is *freely generated*.¹¹ The identification of the global symmetry of a minimally unbalanced quiver is based on the following important claim:¹²

Given a minimally unbalanced quiver $\mathcal{Q}$, the global symmetry on the Coulomb branch is: $G_{\text{global}} = \prod_i G_i$, where G_i are groups corresponding to the Dynkin diagrams that are formed by the subsets of balanced nodes of $\mathcal{Q}$.

Since the balanced sub-quivers in figure 6 correspond to three A_1 Dynkin diagrams, the global symmetry is expected to be $SU(2) \times SU(2) \times SU(2)$. Moreover, each of the three bouquet nodes that connects to the unbalanced node contributes with a fundamental representation of $SU(2)$. As a consequence, there are 8 monopole operators transforming under the three-fundamental representation of $SU(2)^3$, denoted by Dynkin labels $[1; 1; 1]$. These monopole operators carry spin 1/2 charge under $SU(2)_R$ R-symmetry. As 8 is also the complex dimension of the Coulomb branch, we learn that the whole Coulomb branch is free and it is a copy of $\mathbb{H}^4$ with a global symmetry $Sp(4)$. Hence, for the global symmetry we can write:

$$G_{\text{global}} = SU(2) \times SU(2) \times SU(2) \subset Sp(4), \quad (2.2)$$

¹¹See observation 3.1 in [16].

¹²Extended version of this claim, applicable for quivers with two or more unbalanced nodes, is formulated in section 4.

Figure 7. $\mathcal{P}_{[1^3]}(3)$ quiver with fugacities.

where the explicit embedding is given in equation (2.3)

$$[1; 1; 1]_{\text{SU}(2) \times \text{SU}(2) \times \text{SU}(2)} \hookrightarrow [1, 0, 0, 0]_{\text{Sp}(4)} \quad (2.3)$$

In order to find the global symmetry explicitly, one computes the Hilbert Series, utilizing the monopole formula [1]. One first starts with the assignment of simple root fugacities given in figure 7. As is outlined in the introduction, one then proceeds by computing the unrefined Hilbert series (HS), which is obtained by setting all simple root fugacities to unity: $z_i = 1, i = 0, 1, 2, 3$. The unrefined Hilbert series is given by equation (2.4).

$$\text{HS}(t) = \frac{1}{(1-t)^8} \quad (2.4)$$

The expansion of the unrefined Hilbert series reads

$$\text{HS}(t) = 1 + 8t + 36t^2 + 120t^3 + 330t^4 + 792t^5 + O(t^6). \quad (2.5)$$

Taking the Plethystic Logarithm (PL) of the unrefined Hilbert series one finds

$$\text{PL} = 8t. \quad (2.6)$$

The following can be immediately observed:

- The absence of any negative contributions (absence of relations) signifies that the entire Coulomb branch is freely generated.
- The t coefficient corresponds to dimension of some representation of the G_{global} .

Based on the previous claim, the t coefficient corresponds to the dimension of the three-fundamental representation (rep) of $\text{SU}(2)^3$

$$\dim[1; 1; 1] = 2 \times 2 \times 2 = 8, \quad (2.7)$$

where $[a_1; a_2; a_3]$ denote the Dynkin labels of the three-representation of $\text{SU}(2) \times \text{SU}(2) \times \text{SU}(2) \subset \text{Sp}(4)$. Next step is to employ the fugacity map. The simple root fugacities $z_i, i = 1, 2, 3$ are mapped to the $\text{SU}(2)$ fundamental weight fugacities $x_i, i = 1, 2, 3$ according to prescription (2.8)

$$z_i \rightarrow x_i^2, \quad i = 1, 2, 3 \quad (2.8)$$

$$z_0 \rightarrow (z_1 z_2 z_3)^{-\frac{1}{2}}, \quad (2.9)$$

and the simple root fugacity of the unbalanced node, z_0 , is eliminated according to substitution (2.9). In case of a quiver with only gauge nodes, the mapping that eliminates the fugacity of an unbalanced node is canonically derived from the gauge fixing condition in the following way. Consider a minimally unbalanced quiver with simple root fugacities $z_i, i = 1, \dots, N$ and the corresponding node ranks $r_i, i = 1, \dots, N$. Without the loss of generality, let z_N be the fugacity of the unbalanced node and r_N its rank, respectively. Then, the elimination of z_N is derived from the constraint:

$$\prod_i^N z_i^{r_i} = 1 \implies z_N = \left(\prod_i^{N-1} z_i^{r_i} \right)^{-\frac{1}{r_N}}. \quad (2.10)$$

After the mapping, given by (2.8) and (2.9), the expansion of the refined HS is computed as

$$\begin{aligned} \text{HS}(x_i, t) = 1 + & \left(\frac{1}{x_1 x_2 x_3} + \frac{x_1}{x_2 x_3} + \frac{x_2}{x_1 x_3} + \frac{x_1 x_2}{x_3} + \frac{x_3}{x_1 x_2} + \frac{x_1 x_3}{x_2} + \frac{x_2 x_3}{x_1} + x_1 x_2 x_3 \right) t \\ & + O(t^2). \end{aligned} \quad (2.11)$$

Rewriting the t coefficient as

$$\left(x_1 + \frac{1}{x_1} \right) \left(x_2 + \frac{1}{x_2} \right) \left(x_3 + \frac{1}{x_3} \right) \quad (2.12)$$

one directly identifies the character of the three-fundamental representation of $\text{SU}(2)^3$, which verifies the expectation of the global symmetry. The expression of the refined HS is compactly written in equation (2.13)

$$\text{HS}(x_i, t) = \text{PE}[[1; 1; 1]t] = \prod_{\epsilon_i = \pm 1} \frac{1}{1 - x_1^{\epsilon_1} x_2^{\epsilon_2} x_3^{\epsilon_3} t}, \quad (2.13)$$

where $x_i, i = 1, 2, 3$ are the fugacities of the fundamental weights of $\text{SU}(2)^3$ and ϵ_i runs over the two weights of the fundamental representation of $\text{SU}(2)$. Equation (2.13) coincides with (4.1) in [13]. In general, as an (affine) algebraic variety, the Coulomb branch is specified by:

- Number and degree of generators
- Representation under which generators transform (to all relevant orders of t)
- Representations under which relations transform (to all relevant orders of t)

All this information is succinctly encoded in the Plethystic Logarithm (PL) of the refined Hilbert series. Taking the PL of the refined Hilbert series in equation (2.11) or (2.13) one obtains equation (2.14),

$$\text{PL} = [1; 1; 1]_8 t \quad (2.14)$$

where the subscript denotes the total complex dimension of the representation $\dim^{\mathbb{C}}[1; 1; 1] = 2 \times 2 \times 2 = 8$. The subscript notation of the refined PL is conveniently used

Figure 8. $\mathcal{P}_{[2,1]}(3)$ quiver with $SU(2)^2 \cong SO(4)$ global symmetry, $b = -1$, $\dim^{\mathbb{H}} \mathcal{M}_C = 4$.

throughout the paper to denote the dimensions of the corresponding representations. The Coulomb branch is a freely generated space of quaternionic dimension 4:

$$\mathcal{C}_{[1^3]} = \mathbb{H}^4. \quad (2.15)$$

The representational content of the chiral ring is neatly encoded by the highest weight generating function (HWG). The HWG for the theory in figure 6 is given by equation (2.16), which agrees with (4.3) in [13],

$$\text{HWG} = \text{PE} \left[\mu_1 \mu_2 \mu_3 t + \sum_{i=1}^3 \mu_i^2 t^2 + \mu_1 \mu_2 \mu_3 t^3 + t^4 - \mu_1^2 \mu_2^2 \mu_3^2 t^6 \right] \quad (2.16)$$

and where μ_i , $i = 1, 2, 3$ are the highest weight fugacities of the three $SU(2)$ representations. In terms of $Sp(4)$ representations, the HWG takes the simple form

$$\text{HWG} = \text{PE} [\mu_1 t], \quad (2.17)$$

where now μ_1 denotes a highest weight fugacity for $Sp(4)$.

2.1.1 Gauging $H_{\lambda} = \mathbb{Z}_2$

Next, we would like to construct a new theory with a Coulomb branch that is an orbifold of the Coulomb branch of the previously analyzed $\mathcal{P}_{[1^3]}(3)$ theory. Assuming Conjecture 1, let us gauge a subgroup $\mathbb{Z}_2 \subset S_3$ of the discrete global symmetry of the $\mathcal{P}_{[1^3]}(3)$ theory, which acts on the bouquet by permuting its three nodes. Following Conjecture 1, the bouquet of the constructed quiver consists of a single rank 1 node and a single adjoint 2 node. Accordingly, let's denote the newly constructed theory by $\mathcal{P}_{[2,1]}(3)$. The quiver and the explicit assignment of the simple root fugacities are depicted in figure 8. Note that the adjoint 2 node connected to a rank 2 node is balanced. More generally, any adjoint node, with rank N , connected to a rank 2 node is balanced because the extra hypermultiplet contributions coming from the adjoint loop exactly cancel the contributions from the vector multiplet. Since there are two balanced A_1 sub-quivers, the expected global symmetry is $SU(2) \times SU(2) \simeq SO(4)$. After the computation of the Hilbert series using the simple root fugacities, set $\forall i, z_i = 1$ to obtain the unrefined HS in equation (2.18).

$$\text{HS}(t) = \frac{1 + t^2}{(1 - t)^6 (1 - t^2)^2} \quad (2.18)$$

Equation (2.18) has the expansion of the form

$$\text{HS}(t) = 1 + 6t + 24t^2 + 74t^3 + 194t^4 + O(t^5). \quad (2.19)$$

Taking the PL of the unrefined HS one obtains equation (2.20)

$$\text{PL} = 6t + 3t^2 - t^4. \quad (2.20)$$

The term by term analysis of equation (2.20) implies the following:

- t : there is a freely generated part of the Coulomb branch with quaternionic dimension 3. Thus, the free part of the Coulomb branch is [10]: $\mathcal{C}_{f.g.} = \mathbb{H}^3$, which is generated by the fundamental representation of $\text{Sp}(3)$, denoted by $[1, 0, 0]$. Hence, the global symmetry has two constituent parts:

$$G_{\text{global}} = G_{\text{global,free}} \times G_{\text{global, non-trivial}}, \quad (2.21)$$

such that $G_{\text{global,free}} = \text{Sp}(3)$.

- t^2 : there is a non-trivial part of the Coulomb branch generated by a 3 dimensional representation of $G_{\text{global, non-trivial}}$. In order to analyze the non-trivial part of the Coulomb branch, the free part needs to be multiplied out.
- t^4 : there is a relation at this order that transforms as a singlet under G_{global} .

Utilize the fugacity map

$$z_1 \rightarrow x_1^2, \quad (2.22)$$

$$z_3 \rightarrow x_2^2, \quad (2.23)$$

$$z_2 \rightarrow (z_1^2 z_3)^{-\frac{1}{2}}, \quad (2.24)$$

where the simple root fugacities z_1, z_3 map to x_1 and x_2 , the fundamental weight fugacities of the two $\text{SU}(2)$ following prescriptions (2.22) and (2.23), respectively. Note, that the latter $\text{SU}(2)$ corresponds to the rank 1 bouquet node and the former $\text{SU}(2)$ corresponds to the adjoint 2 node. The z_2 root fugacity of the unbalanced node is eliminated according to prescription (2.24), which again, follows from constraint (2.10). The expansion of the refined HS takes the form

$$\begin{aligned} \text{HS}(x_1, x_2, t) = & 1 + \left(x_1^2 + 1 + \frac{1}{x_1^2}\right) \left(x_2 + \frac{1}{x_2}\right) t + \left(x_2^2 + 1 + \frac{1}{x_2^2}\right) t^2 \\ & + \left(x_2^4 + x_2^2 + 1 + \frac{1}{x_2^2} + \frac{1}{x_2^4}\right) t^4 + O(t^6) \end{aligned} \quad (2.25)$$

Comparing equations (2.25) and (2.20) one infers that the t coefficient in equation (2.20) can be regarded as the dimension of the $[2; 1]$ two-representation of $\text{SU}(2) \times \text{SU}(2)$ (or the dimension of the $[1, 0, 0]$ fundamental representation of $\text{Sp}(3)$). In this case one finds the embedding:¹³

$$[2; 1]_{\text{SO}(3) \times \text{Sp}(1)} \hookrightarrow [1, 0, 0]_{\text{Sp}(3)}. \quad (2.26)$$

¹³We write $\text{SO}(3)$ instead of $\text{SU}(2)$ given the universal double covering of the A_1 algebra and since the $[2]$ representation is real.

Observe in equation (2.25) that the $SU(2)$ symmetry of the adjoint 2 node appears in the symmetry of the free sector at order t but at higher orders of t only the $SU(2)$ that corresponds to the balanced 1 node plays a role. Let us now multiply out the free sector of the theory and continue with the analysis of the non-trivial part of the Hilbert Series. One takes the PE of the character in front of the t coefficient and multiplies the whole HS by the inverse of this PE. The obtained refined HS now describes the non-trivial part of the Coulomb branch. The corresponding refined PL can be written as:

$$PL = [2; 0]_3 t^2 - [0; 0]_1 t^4. \quad (2.27)$$

The t^2 coefficient can be regarded as the 3 dimensional representation of $SU(2) \times SU(2)$ with Dynkin labels $[2; 0]$. The relation at t^4 transforms as a singlet under the two-representation of $SU(2) \times SU(2)$, denoted by $[0; 0]$. Note that the simple PL in equation (2.27) describes a $\mathbb{C}^2/\mathbb{Z}_2$ singularity, also termed A_1 singularity in the literature. In summary, the Coulomb branch of the theory:

$$\mathcal{C}_{[2,1]} = \mathbb{H}^3 \times \mathbb{C}^2/\mathbb{Z}_2. \quad (2.28)$$

has two parts; a free sector in the form of $\mathbb{H}^3$ and a non-trivial part in the form of an A_1 singularity [17]. The representation content of the chiral ring on the non-trivial part of the Coulomb branch is described by the HWG in equation (2.29),

$$HWG = PE[\mu^2 t^2] \quad (2.29)$$

where μ denotes the highest weight fugacity of $SU(2)$.

2.1.2 Gauging $H_\lambda = S_3$

We saw that the discrete gauging of $H_\lambda = \mathbb{Z}_2 \subset S_3$ for the parent $\mathcal{P}_{[1^3]}(3)$ quiver produced the $\mathcal{P}_{[2,1]}(3)$ quiver theory with a Coulomb branch in the form of a discrete quotient:

$$\mathcal{C}_{[2,1]} = \mathcal{C}_{[1^3]}/\mathbb{Z}_2. \quad (2.30)$$

Given the success of this construction, let us consider the $\mathcal{P}_{[1^3]}(3)$ theory where the entire discrete S_3 global symmetry is gauged. According to Conjecture 1, this theory is described by the quiver, depicted with explicit assignment of the simple root fugacities in figure 9. The quiver consists of a single adjoint 3 node attached to the central 2 node. Accordingly, the theory is denoted by $\mathcal{P}_{[3]}(3)$. The balanced part of the quiver forms the A_1 Dynkin diagram, therefore, the anticipated global symmetry is $SU(2)$. The unrefined HS, obtained by setting all the simple root fugacities z_i to unity, takes the form given in equation (2.31)

$$HS(t) = \frac{1 + t^2 + 2t^3 + t^4 + t^6}{(1 - t)^4(1 - t^2)^2(1 - t^3)^2}. \quad (2.31)$$

The expansion of equation (2.31) yields

$$HS(t) = 1 + 4t + 13t^2 + 36t^3 + 87t^4 + 190t^5 + 386t^6 + O(t^7). \quad (2.32)$$

The PL of the unrefined expression (2.31) reads

$$PL = 4t + 3t^2 + 4t^3 - 2t^5 - 2t^6 + O(t^7). \quad (2.33)$$

Let us analyze the first two terms in the last expression:

Figure 9. $\mathcal{P}_{[3]}(3)$ quiver with $SU(2) \subset Sp(2)$ global symmetry, $b = -1$, $\dim \mathcal{M}_C^{\mathbb{H}} = 4$.

- t : there is a free sector corresponding to $\mathbb{H}^2$ generated by $G_{\text{global, free}} = Sp(2)$. In particular, it is generated by the fundamental representation of $Sp(2)$, denoted by $[1, 0]$.
- t^2 : the coefficient matches the dimension of the adjoint representation of $SU(2)$. Hence, the expected non-trivial global symmetry is: $G_{\text{global, non-trivial}} = SU(2)$.

The refined expression for Hilbert series is obtained using the fugacity map

$$z_1 \rightarrow x^2, \quad (2.34)$$

$$z_2 \rightarrow (z_1^3)^{-\frac{1}{2}}, \quad (2.35)$$

where the simple root fugacity z_1 maps to the $SU(2)$ fundamental weight fugacity x , and the root fugacity of the unbalanced node, z_2 , is eliminated according to the gauge fixing condition (2.35). The computation of the refined HS yields equation (2.36).

$$\begin{aligned} \text{HS}_{[3]}(t, x) = 1 + \left(\frac{1}{x^3} + \frac{1}{x} + x + x^3 \right) t + \left(\frac{1}{x^6} + \frac{1}{x^4} + \frac{3}{x^2} + 3x^6 + 3x^2 + x^4 + x^6 \right) t^2 \\ + O(t^3). \end{aligned} \quad (2.36)$$

We see that the free sector, corresponding to $\mathbb{H}^2$, is spanned by generators in the fundamental rep of $Sp(2)$, denoted by $[1, 0]$, or equivalently in the $[3]_4$ rep of $SU(2)$. The embedding found in this case can be written as:

$$[3]_{SU(2)} \hookleftarrow [1, 0]_{Sp(2)}. \quad (2.37)$$

Let us multiply out the free sector in an analogous manner than in the previous case. Again, one multiplies the refined HS by the inverse of the PE of the character in front of t in equation (2.36). The PL of the obtained refined HS for the non-trivial part of the Coulomb branch takes the form:

$$\begin{aligned} \text{PL} = \left(\frac{1}{x^2} + 1 + x^2 \right) t^2 + \left(\frac{1}{x^3} + \frac{1}{x} + x + x^3 \right) t^3 \\ - \left(\frac{1}{x} + x \right) t^5 - \left(\frac{1}{x^2} + 1 + x^2 \right) t^6 + O(t^7). \end{aligned} \quad (2.38)$$

The character appearing in front of t^2 is the character of the adjoint representation of the non-trivial global symmetry $SU(2)$. This verifies that

$$G_{\text{global, non-trivial}} = SU(2). \quad (2.39)$$

Now, the refined PL can be written as

$$\text{PL} = [2]_3 t^2 + [3]_4 t^3 - [1]_2 t^5 - [2]_3 t^6 + O(t^7). \quad (2.40)$$

where $[a]$ denotes the Dynkin labels of the representation of $\text{SU}(2)$. Comparing this expression with the refined HS:

$$\text{HS} = 1 + [2]t^2 + [3]t^3 + ([4] + [0])t^4 + ([5] + [3])t^5 + (2[6] + [2])t^6 + O(t^7) \quad (2.41)$$

one sees that at order t^5 there is one operator that must be set to zero, and at order t^6 there are two operators that satisfy one relation, hence must be proportional to each other. These observations are used below in the explicit construction of the Coulomb branch algebraic variety. The HWG for the non-trivially generated part of the Coulomb branch is given by equation (2.42)

$$\text{HWG} = \text{PE}[\mu^2 t^2 + \mu^3 t^3 + t^4 + \mu^3 t^5 - \mu^6 t^{10}] \quad (2.42)$$

where μ is the highest weight fugacity of $\text{SU}(2)$.

The $\mathcal{P}_{[3]}(3)$ theory has none of the S_3 global symmetry compared to the $\mathcal{P}_{[1^3]}(3)$ theory. Therefore, by Conjecture 1 it is implied that the $\mathcal{C}_{[3]}$ Coulomb branch is an S_3 orbifold of the parent $\mathcal{C}_{[1^3]}$ Coulomb branch. Indeed, in accord with equation (1.1), the computed Coulomb branch variety can be written as:

$$\mathcal{C}_{[3]} = \mathbb{H}^2 \times \mathbb{C}^4 / S_3. \quad (2.43)$$

This result is confirmed by explicit computations of the S_3 Molien invariant reproducing equation (2.31).

Let us analyze equation (2.40) and (2.41) in more detail. In (2.40) there are generators transforming under the adjoint $[2]$ rep of $\text{SU}(2)$ at t^2 , and additional generators at order t^3 transforming under the $[3]$ rep. Altogether we have 7 generators. There are relations at order t^5 and t^6 transforming under $[1]$ and $[2]$ reps, respectively. Explicitly, the generators at t^2 are:

$$M_{\alpha\beta}, \quad (2.44)$$

where $\alpha, \beta = 1, 2$, and they satisfy

$$M_{\alpha\beta} = M_{\beta\alpha}, \quad (2.45)$$

$$\deg(M) = 2, \quad (2.46)$$

where $\deg()$ denotes the degree of the generator which is associated with the power of t at which they appear. The generators at t^3 are:

$$N_{\alpha\beta\gamma}, \quad (2.47)$$

where $\alpha, \beta, \gamma = 1, 2$. These are also symmetric in all indices and with $\deg(N) = 3$. Now, remembering

$$[2] \otimes [3] = [5] \oplus [3] \oplus [1] \quad (2.48)$$

$$\text{Sym}^2[3]_{10} = [6]_7 \oplus [2]_3 \quad (2.49)$$

$$\text{Sym}^3[2]_{10} = [6]_7 \oplus [2]_3 \quad (2.50)$$

and observing that in expression (2.41) at order t^5 the $[1]$ is missing, one deduces that this must be a relation. Hence, the relation at order t^5 with degree 5 is:

$$\epsilon^{\beta\delta}\epsilon^{\alpha\gamma}M_{\alpha\beta}N_{\gamma\delta\epsilon}=0. \quad (2.51)$$

At t^6 the two operators of degree 6 satisfy equation (2.52).

$$M_{\alpha_1\alpha_2}M_{\alpha_3\alpha_4}M_{\alpha_5\alpha_6}\epsilon^{\alpha_2\alpha_3}\epsilon^{\alpha_4\alpha_5}=N_{\alpha_1\alpha_2\alpha_3}N_{\alpha_4\alpha_5\alpha_6}\epsilon^{\alpha_2\alpha_4}\epsilon^{\alpha_3\alpha_5} \quad (2.52)$$

Note that on the left hand side the operator transforms in the $[2]$ of (2.50) that is coming from the third symmetrization. The operator on the right hand side transforms in the $[2]$ in (2.49) that comes from the second symmetrization. Equations (2.51) and (2.52) produce $2+3=5$ equations that constrain the 7 generators. The Coulomb branch can be computed from this explicit analysis employing Maclauay2.¹⁴ The computation yields an unrefined HS of the form:

$$\text{HS}(t)=\frac{1-2t^5-3t^6+3t^8+2t^9-t^{14}}{(1-t^2)^3(1-t^3)^4} \quad (2.53)$$

which is precisely the unrefined HS obtained previously if one factors out the free sector in equation (2.31). Let us now use the comparison of the Coulomb branch volumes as a non-trivial test of Conjecture 1.

2.1.3 Comparison of the Coulomb branch volumes

Following the method outlined in section 1 for the $k=2$, $n=1$ theories one can compare the volumes of the Coulomb branches. Expanding the unrefined Hilbert series (2.5), (2.18) and (2.31) according to equation (1.3) and plugging into equation (1.6) one finds:

$$\frac{\text{vol}(\mathcal{C}_{[1^3]})}{\text{vol}(\mathcal{C}_{[2,1]})}=\frac{R_{[1^3]}}{R_{[2,1]}}=\frac{1}{\frac{1}{2}}=2=\text{ord}(\mathbb{Z}_2) \quad (2.54)$$

$$\frac{\text{vol}(\mathcal{C}_{[1^3]})}{\text{vol}(\mathcal{C}_{[3]})}=\frac{R_{[1^3]}}{R_{[3]}}=\frac{1}{\frac{1}{6}}=6=\text{ord}(S_3) \quad (2.55)$$

which are the expected ratios. Equations (2.54) and (2.55) provide a non-trivial test of Conjecture 1, namely, that the Coulomb branches of $\mathcal{P}_{[2,1]}(3)$ and $\mathcal{P}_{[3]}(3)$ are $\mathbb{Z}_2$ and S_3 orbifolds of the parent $\mathcal{P}_{[1^3]}(3)$ Coulomb branch, respectively. Note, that it follows that the Coulomb branch of $\mathcal{P}_{[3]}(3)$ is a $\mathbb{Z}_3$ quotient of the $\mathcal{P}_{[2,1]}(3)$ Coulomb branch. This can be tested explicitly employing the ideas of stepwise projection [18]. Table 1 summarizes the ratios of volumes between the Coulomb branches of $k=2$, $n=1$ theories. The relations between the Coulomb branches of $k=2$, $n=1$ theories are schematically depicted by the commutative diagram in figure 10, where the arrows denote quotients.

2.2 Case: $k=2$, $n=2$

Let us now turn to the case $k=n=2$. The S_2 discrete global symmetry of the bouquet enhances to S_4 . The five partitions of 4 are $\mathcal{P}(4)=\{[1^4],[2,1^2],[2^2,1],[3,1],[4]\}$. The first theory, corresponding to $\mathcal{P}_{[1^4]}(4)$ is shown in figure 11 together with the corresponding assignment of the simple root fugacities. The quiver is fully balanced and forms the $\hat{D}_4$

¹⁴Maclauay2 program for computation of algebraic varieties is available at:

<https://faculty.math.illinois.edu/Macaulay2/>

Volume Ratios of $k = 2, n = 1$ theories			
Partition	$[1^3]$	$[2, 1]$	$[3]$
$[1^3]$	1	2	6
$[2, 1]$		1	3
$[3]$			1

Table 1. Ratios of Coulomb branch volumes for $k = 2, n = 1$ theories.

Figure 10. Commutative diagram of Coulomb branch orbifolding for $k = 2, n = 1$ theories.

Figure 11. $\mathcal{P}_{[1^4]}(4)$ quiver with D_4 global symmetry, $b = 0$, $\dim \mathcal{M}_C^{\mathbb{H}} = 5$.

Dynkin diagram (i.e. the affine Dynkin diagram of D_4). After the decoupling of the center of mass $U(1)$, one expects to find the enhanced $D_4 \equiv SO(8) \supset SU(2)^4$ global symmetry. The $SU(2)^4$ is the maximal subgroup of $SO(8)$ that has the natural S_4 symmetry which plays a role in the following. The computation of the unrefined HS yields

$$\text{HS}_{[1^4]}(t) = \frac{(1 + t^2)(1 + 17t^2 + 48t^4 + 17t^6 + t^8)}{(1 - t^2)^{10}} \quad (2.56)$$

Equation (2.56) is consistent with previous results in [19] and in table 11 in [15]. Expanding equation (2.56), one finds

$$\text{HS}(t) = 1 + 28t^2 + 300t^4 + 1925t^6 + 8918t^8 + O(t^{10}). \quad (2.57)$$

The corresponding PL reads

$$\text{PL} = 28t^2 - 106t^4 + 833t^6 - 8400t^8 + O(t^{10}) \quad (2.58)$$

The t^2 coefficient agrees with the dimension of the adjoint representation of $SO(8)$:

$$\dim [0, 1, 0, 0]_{D_4} = 28. \quad (2.59)$$

There is a crucial difference between the PLs in the previous subsection and the PL in equation (2.58). The absence of t term in equation (2.58) implies that there is no free sector (i.e. no free hypers) in the theory. This follows from the absence of a node with negative imbalance in the quiver. The simple root fugacities, indicated in figure 11 are treated in the following manner. As previously, one of the fugacities is eliminated by the gauge fixing condition. Recall, that the elimination condition follows from constraint (2.10). In this case, one eliminates one of the bouquet fugacities such that the remaining fugacities form the shape of a D_4 Dynkin diagram. One declares the z_4 to be the null node (i.e. the affine node in the $\hat{D}_4$ Dynkin diagram) and the elimination of z_4 is thus based on prescription given by (2.60). Note that z_4 becomes the inverse of the adjoint weight fugacity. One uses the Cartan matrix of D_4 to map the remaining simple root fugacities z_i , $i = 1, 2, 3, 5$ to the fundamental weights of D_4 , such that the powers in the fugacity map are determined by the components of the Cartan matrix. The mapping is summarized by the following equations:

$$z_4 \rightarrow (z_1 z_2 z_3 z_5^2)^{-1} = y_2^{-1}, \quad (2.60)$$

$$z_1 \rightarrow y_1^2 y_2^{-1}, \quad z_2 \rightarrow y_2^2 (y_1 y_3 y_4)^{-1}, \quad (2.61)$$

$$z_3 \rightarrow y_3^2 y_2^{-1}, \quad z_5 \rightarrow y_4^2 y_2^{-1}. \quad (2.62)$$

Making use of this fugacity map the refined HS is computed. One finds that the t^2 coefficient is precisely the character of the adjoint representation of D_4 . which confirms that the global symmetry is $SO(8)$. For the purpose of brevity, we refrain from showing the character expansion of the refined Hilbert series and directly show the result of the computation of the refined PL:

$$\begin{aligned} \text{PL} = & [0, 1, 0, 0]_{28} t^2 - ([0, 0, 0, 0]_1 + [2, 0, 0, 0]_{35} + [0, 0, 2, 0]_{35} + [0, 0, 0, 2]_{35}) t^4 \\ & + ([2, 0, 0, 0]_{35} + [0, 1, 0, 0]_{28} + [0, 0, 2, 0]_{35} + 2[1, 0, 1, 1]_{350} + [0, 0, 0, 2]_{35}) t^6 \\ & + O(t^8), \end{aligned} \quad (2.63)$$

where $[d_1, d_2, d_3, d_4]_{\text{dim}}$ are the Dynkin labels for D_4 and the subscript denotes the dimension of the representation. Note that the relations at order t^4 are manifestly invariant under the triality of D_4 . The Coulomb branch is the reduced moduli space¹⁵ of one D_4 -instanton on $\mathbb{C}^2$ [19, 20]. Geometrically, the Coulomb branch is a simple algebraic variety which is a closure of minimal nilpotent orbit¹⁶ of D_4 :

$$\mathcal{C}_{[1^4]} = \overline{\min \mathcal{O}_{D_4}}. \quad (2.64)$$

All the information about the chiral ring is neatly encoded by the HWG in equation (2.65)

$$\text{HWG} = \text{PE}[\mu_2 t^2], \quad (2.65)$$

where μ_2 is the fugacity of the highest weight of D_4 .

¹⁵In the literature, reduced single instanton moduli spaces are also known under the abbreviation RSIMS.

¹⁶This space can be defined as a space of 8×8 matrices M that satisfy $M = -M^T$, $M^2 = 0$, $\text{rank}(M) \leq 2$.

Figure 12. $\mathcal{P}_{[2,1^2]}(4)$ quiver with B_3 global symmetry, $b = 0$, $\dim \mathcal{M}_C^{\mathbb{H}} = 5$.

Figure 13. Dynkin diagram of the affine B_3 algebra.

2.2.1 Gauging $H_\lambda = \mathbb{Z}_2$

Given the evidence for Conjecture 1 in the previous subsection for $k = 2$, $n = 1$ quivers, let us now use the $\mathcal{P}_{[1^4]}(4)$ quiver to construct theories for all other partitions of $\mathcal{P}(4)$. In order to construct the first theory, gauge $\mathbb{Z}_2 \subset S_4$, a subgroup of the discrete global symmetry of the parent $\mathcal{P}_{[1^4]}(4)$ quiver in figure 11. According to Conjecture 1, the bouquet of the constructed theory consists of an adjoint 2 node and two rank 1 nodes. We denote this theory by $\mathcal{P}_{[2,1^2]}(4)$. The corresponding quiver and the simple root fugacities are depicted in figure 12. Note that the adjoint node connected to rank 2 node is balanced. Since we are studying a quiver obtained by a $\mathbb{Z}_2$ quotient of a quiver with $\text{SO}(8)$ global symmetry and $\text{SO}(7)$ is a subgroup of $\text{SO}(8)$ that commutes with $\mathbb{Z}_2$, this provides the first indication for the expectation of the global symmetry. Another indication for the anticipated global symmetry comes from comparing the quiver in figure 12 with the affine B_3 Dynkin diagram, depicted in figure 13. When one eliminates one of the simply connected rank 1 nodes (using the gauge fixing condition), it is natural to expect:

$$G_{\text{global}} = B_3 \equiv \text{SO}(7). \quad (2.66)$$

Before we turn to the refined analysis, let us proceed by computing the HS using the simple root fugacities, and setting all to unity in order to obtain the expression of the unrefined HS:

$$\text{HS}_{[2,1^2]}(t) = \frac{(1+t^2)(1+10t^2+20t^4+10t^6+t^8)}{(1-t^2)^{10}}. \quad (2.67)$$

Indeed, note that equation (2.67) contains the HS of the next to minimal nilpotent orbit of B_3 , listed in table 10 in [15]. Expanding the unrefined HS, one obtains equation (2.68)

$$\text{HS}(t) = 1 + 21t^2 + 195t^4 + 1155t^6 + 5096t^8 + O(t^{10}), \quad (2.68)$$

which has the PL of the form

$$\text{PL} = 21t^2 - 36t^4 + 140t^6 - 784t^8 + O(t^{10}). \quad (2.69)$$

The t^2 coefficient in the last two expressions is the dimension of the adjoint representation of $B_3 \equiv \text{SO}(7)$:

$$\dim [0, 1, 0]_{B_3} = 21, \quad (2.70)$$

which agrees with the expected global symmetry. Next, perform the mapping:

$$z_4 \rightarrow (z_1 z_2^2 z_3^2)^{-1} = x_2^{-1}, \quad (2.71)$$

$$z_1 \rightarrow x_1^2 x_2^{-1}, \quad z_2 \rightarrow x_2^2 x_1^{-1} x_3^{-2}, \quad z_3 \rightarrow x_3^2 x_2^{-1}, \quad (2.72)$$

such that z_4 is eliminated by the gauge fixing condition (2.71). Typically, since z_4 is declared to be the null node, it maps to the inverse of the adjoint weight fugacity. The remaining fugacities are mapped to the fundamental weight fugacities of B_3 using the Cartan matrix. After the mapping, the refined HS is obtained. For brevity, we only show the t^2 coefficient of the expansion of the refined HS:

$$\begin{aligned} 3 + \frac{1}{x_1} + x_1 + \frac{1}{x_2} + \frac{x_1}{x_2} + \frac{x_1^2}{x_2} + x_2 + \frac{x_2}{x_1^2} + \frac{x_2}{x_1} + \frac{x_1}{x_3^2} + \frac{x_2}{x_3^2} \\ + \frac{x_2}{x_1 x_3^2} + \frac{x_1 x_2}{x_3^2} + \frac{x_2^2}{x_1 x_3^2} + \frac{x_3^2}{x_1} + \frac{x_1 x_3^2}{x_2^2} + \frac{x_3^2}{x_2} + \frac{x_3^2}{x_1 x_2} + \frac{x_1 x_3^2}{x_2} \end{aligned} \quad (2.73)$$

which coincides with the character of the adjoint representation of $\text{SO}(7)$. This confirms that the global symmetry is B_3 and allows us to write the refined PL in the form:

$$\text{PL} = [0, 1, 0]_{21} t^2 - ([0, 0, 0]_1 + [0, 0, 2]_{35}) t^4 + ([1, 1, 0]_{105} + [0, 0, 2]_{35}) t^6 + O(t^8), \quad (2.74)$$

where $[d_1, d_2, d_3]$ are the Dynkin labels of B_3 . Recall, that the subscripts denote the dimensions of the corresponding representations. As an algebraic variety, the Coulomb branch is a closure of next to minimal nilpotent orbit of $\mathfrak{so}(7)$ algebra:¹⁷

$$\mathcal{C}_{[2,1^2]} = \overline{\text{n. min } \mathcal{O}_{B_3}}. \quad (2.75)$$

The representation content of the chiral ring is summarized by the HWG in equation (2.76)

$$\text{HWG} = \text{PE}[\mu_2 t^2 + \mu_1^2 t^4], \quad (2.76)$$

where μ_i , $i = 1, 2, 3$ are the fugacities for the highest weights of B_3 . The computation of equation (2.76), which is done starting from the quiver in figure 12, provides an independent test that the Coulomb branch moduli space is given by equation (2.75) since it is consistent with results of table 10 in [15]. The refined analysis together with the fact that the algebraic variety is multiplicity-free determines the Coulomb branch uniquely. The $\mathbb{Z}_2$ quotient between HWG (2.65) and (2.76) maps the adjoint rep of D_4 into the adjoint and a vector rep of B_3 . Whereas the adjoint is invariant under this action, the vector transform non-trivially with a minus sign, and hence comes in form of the natural invariant $\mu_1^2 t^4$. Overall, the decomposition of $\text{SO}(8)$ into $\text{SO}(7)$ can be written as:

$$\mu_2 t^2 \rightarrow \mu_2 t^2 + \mu_1^2 t^4 \quad (2.77)$$

which is used in the analysis of the next case and the HWG derivation in appendix B.

¹⁷This space is defined as a space of 7×7 matrices M , satisfying: $M = -M^T$, $\text{Tr}(M^2) = 0$, $\text{rank}(M) \leq 2$. In the previous literature, this space is defined with the extra condition $M^3 = 0$ but equation (2.74) shows that this nilpotency condition is already implied by the rank and the trace conditions.

Figure 14. $\mathcal{P}_{[2^2]}(4)$ quiver with $A_3 \cong D_3$ global symmetry, $b = 0$, $\dim \mathcal{M}_C^{\mathbb{H}} = 5$.

2.2.2 Gauging $H_\lambda = \mathbb{Z}_2 \times \mathbb{Z}_2$

Let us now turn to the construction of the $\mathcal{P}_{[2^2]}(4)$ theory, which is obtained by gauging the subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset S_4$ of the original permutation symmetry of $\mathcal{P}_{[1^4]}(4)$. According to Conjecture 1, the desired quiver takes the form depicted, alongside with the assignment of the simple root fugacities, in figure 14. The quiver is fully balanced and contains a bouquet of two adjoint 2 nodes that stems from the central 2 node. The anticipated global symmetry on the Coulomb branch is $SU(4) \cong SO(6)$ since the balanced nodes form a Dynkin diagram of $A_3 \cong D_3$. Moreover, $SO(6)$ is also the subgroup which commutes with $\mathbb{Z}_2 \times \mathbb{Z}_2$ inside $SO(8)$. Compute the HS with the simple root fugacities, and set all the fugacities z_i , $i = 1, 2, 3$ to unity to find the unrefined HS in equation (2.78).

$$\text{HS}_{[2^2]}(t) = \frac{1 + 10t^2 + 55t^4 + 150t^6 + 288t^8 + 336t^{10} + 288t^{12} + 150t^{14} + 55t^{16} + 10t^{18} + t^{20}}{(1 - t^2)^{10}(1 + t^2)^5} \quad (2.78)$$

The expansion of the unrefined HS yields

$$\text{HS}(t) = 1 + 15t^2 + 125t^4 + 685t^6 + 2898t^8 + O(t^{10}). \quad (2.79)$$

Taking PL of equation (2.78) one obtains equation (2.80).

$$\text{PL} = 15t^2 + 5t^4 - 70t^6 + 273t^8 + O(t^{10}). \quad (2.80)$$

The t^2 coefficient in the last equation agrees with the expected global symmetry since the dimension of the adjoint representation of $SU(4)$ is

$$\dim [1, 0, 1]_{A_3} = 15. \quad (2.81)$$

In order to perform the refined analysis of the Coulomb branch in terms of the A_3 symmetry all three simple root fugacities are needed. This is a complication, however, since eliminating one of the three simple root fugacities by the gauge fixing condition will leave us with just two fugacities to work with. This indicates a presence of certain embedding of a lower rank symmetry inside the A_3 global symmetry. In order to do the analysis in terms of the A_3 , observe that the unrefined HS computed in equation (2.78) is also the unrefined HS for a Higgs branch quiver in figure 15, where gauge (round) and flavor (square) groups are denoted explicitly.¹⁸ Note that the O_1 nodes (or equivalently $\mathbb{Z}_2$) precisely realize the two $\mathbb{Z}_2$ actions on the Higgs branch of a $\text{Sp}(1)$ gauge theory with a D_4 flavor group and by

¹⁸The first computation of the Higgs branch quiver was done by Rudolph Kalveks.

Figure 15. Higgs branch quiver with $D_3 \equiv A_3$ global symmetry. Note that $O_1 \equiv \mathbb{Z}_2$ and C_1 denotes $\text{Sp}(2)$, $\dim \mathcal{M}_H^{\text{H}} = 5$.

gauging the two factors of $\mathbb{Z}_2$ in the global symmetry, one recovers the quiver depicted in figure 15, where the remaining global symmetry is $\text{SO}(6) \cong \text{SU}(4)$. Given the above motivations, let us bypass the problem constituted by the missing fugacities in figure 14 and use a computation of the Higgs branch of the quiver in figure 15 instead. After appropriate fugacity maps one can show that the refined Hilbert series are equal to each other. Let us start with the quiver in figure 14. After the computation of the HS using simple root fugacities $z_i, i = 1, 2, 3$ impose the gauge fixing condition:

$$z_3 \rightarrow (z_1^2 z_2^2)^{-1/2}. \quad (2.82)$$

eliminating the z_3 fugacity. Recall, that the gauge fixing follows from constraint (2.10). The obtained HS only contains z_1 and z_2 fugacities. The t^2 coefficient of the refined HS takes the form

$$3 + \frac{2}{z_1} + 2z_1 + \frac{1}{z_2} + \frac{1}{z_1^2 z_2} + \frac{2}{z_1 z_2} + z_2 + 2z_1 z_2 + z_1^2 z_2. \quad (2.83)$$

Written in terms of the simple roots, this is precisely the character of the adjoint representation of A_3 under the identification $z_3 \rightarrow z_1$!

On the other hand, the HWG for the Higgs branch quiver in figure 15 is given by equation (2.84)

$$\text{HWG} = \text{PE}[\mu_1 \mu_3 t^2 + (2\mu_2^2 + 1)t^4 + \mu_2^2 t^6 - \mu_2^4 t^{12}], \quad (2.84)$$

where $\mu_i, i = 1, 2, 3$ are the highest weight fugacities of A_3 . The derivation of HWG (2.84) is included in appendix B. One can turn this HWG into the refined Hilbert series which is expressed using the fundamental weight fugacities $x_i, i = 1, 2, 3$. Further, let's use the inverse of the Cartan matrix to map the x_i fugacities in the refined HS to the simple root fugacities $z_i, i = 1, 2, 3$. The desired fugacity map takes the form:

$$x_1 \rightarrow (z_1^3 z_2^2 z_3)^{\frac{1}{4}}, \quad x_2 \rightarrow (z_1 z_2^2 z_3)^{\frac{1}{2}}, \quad x_3 \rightarrow (z_1 z_2^2 z_3^3)^{\frac{1}{4}}. \quad (2.85)$$

At this stage, the refined HS for the Higgs branch quiver is expressed using all three simple root fugacities. As a final step make the same identification used to recover the correct character in front of t^2 coefficient in equation (2.83). Recall, the form of the identification:

$$z_3 \rightarrow z_1. \quad (2.86)$$

Finally, the two Hilbert series, obtained by working from both sides of the duality and using the fugacity maps prescribed above, are equal! This verifies that the global symmetry of the

Figure 16. Coulomb branch quiver with A_m global symmetry, $\dim \mathcal{M}_C^{\mathbb{H}} = 2m - 1$.

Figure 17. Higgs branch quiver with D_m global symmetry, $\dim \mathcal{M}_H^{\mathbb{H}} = 2m - 1$.

$\mathcal{P}_{[2^2]}$ theory in figure 14 is A_3 and the chiral ring is described by the HWG in equation (2.84). By the sequel, the PL of the refined HS can be written in the form:

$$\begin{aligned} \text{PL} = & [1, 0, 1]_{15} t^2 + ([0, 2, 0]_{20} - [1, 0, 1]_{15}) t^4 - ([0, 2, 0]_{20} - [2, 1, 0]_{45} - [0, 1, 2]_{45}) t^6 \\ & + ([2, 1, 0]_{45} + 2[1, 0, 1]_{15} + [1, 2, 1]_{175} + [0, 1, 2]_{45} - [0, 2, 0]_{20} - 2) t^8 \\ & + O(t^{10}), \end{aligned} \quad (2.87)$$

where $[d_1, d_2, d_3]$ are the Dynkin labels of the A_3 representations. Recall, that the subscripts denote the dimensions of the corresponding representations. Consider a quiver consisting of a chain of m rank 2 nodes such that the two boundary nodes are adjoint. The quiver is depicted in figure 16. A generalization of the last derivation, supported by computational evidence up to $m = 4$, implies that the following conjecture holds:

The Coulomb branch of quiver in figure 16 is equal to the Higgs branch of the quiver in figure 17.

One can consider a quiver for an $\text{Sp}(1)$ gauge theory with m flavors and realize the two $\mathbb{Z}_2$ actions. The same can be done for the Coulomb branch quiver in the form of the affine D_{m+2} Dynkin diagram such that the fork $\text{U}(1)$ nodes on both ends are substituted by adjoint 2 nodes (due to the two $\mathbb{Z}_2$ actions). The obtained Higgs and Coulomb branch quivers are precisely those in figures 16 and 17.

2.2.3 Gauging $H_\lambda = S_3$

The next theory is obtained by gauging an S_3 subgroup of the S_4 discrete global symmetry of the parent $\mathcal{P}_{[1^4]}(4)$ quiver. Conjecture 1 implies that the result of such discrete gauging produces the $\mathcal{P}_{[3,1]}(4)$ quiver, depicted in figure 18. The assignment of the simple root fugacities is also shown in figure 18. The anticipation of the global symmetry follows from the comparison of this quiver to an affine G_2 Dynkin diagram in a similar fashion as in the case of the $\mathcal{P}_{[2,1^2]}(4)$ quiver. Moreover, G_2 is also the a subgroup that commutes with S_3

Figure 18. $\mathcal{P}_{[3,1]}(4)$ quiver with G_2 global symmetry, $\dim \mathcal{M}_C^{\mathbb{H}} = 5$.

Figure 19. Affine Dynkin diagram of G_2 with Coxeter labels and simple root fugacities.

inside $SO(8)$. We proceed by computing the HS and unrefining by setting all simple root fugacities, $z_i, i = 1, 2, 3$, to 1. The unrefined HS is given by equation (2.88).

$$\text{HS}_{[3,1]}(t) = \frac{(1+t^2)(1+3t^2+6t^4+3t^6+t^8)}{(1-t^2)^{10}} \quad (2.88)$$

The expansion of the unrefined HS reads

$$\text{HS}_{[3,1]}(t) = 1 + 14t^2 + 104t^4 + 539t^6 + 2184t^8 + O(t^{10}). \quad (2.89)$$

Note, that (2.88) agrees with the result of the HS for the sub-regular nilpotent orbit of G_2 in table 3 in [21]. The unrefined PL takes the form

$$\text{PL} = 14t^2 - t^4 - 7t^6 + 7t^8 + O(t^9) \quad (2.90)$$

The t^2 coefficient of the PL is the dimension of the adjoint representation of G_2 :

$$\dim [1, 0]_{G_2} = 14 \quad (2.91)$$

In fact, the global symmetry for the quiver in figure 18 is argued to be G_2 in [22]. In order to confirm this expectation on a level of the refined HS, the following mappings need to be employed.

Mapping of $\mathcal{P}_{[3,1]}(4)$ simple root fugacities to the highest weight fugacities of G_2 : the procedure involves three steps. First step is to impose the usual gauge condition which eliminates the fugacity of the adjoint 3 node.¹⁹ The map is given by equation (2.92)

$$z_1 \rightarrow (z_2^2 z_3)^{-\frac{1}{3}}, \quad (2.92)$$

which leaves us with fugacities z_2 and z_3 . Second step is to map these fugacities to the simple root fugacities of G_2 (i.e. one needs to find a Dynkin map from A_2 to G_2). For this purpose, consider the affine Dynkin diagram of G_2 , depicted in figure 19. The Coxeter labels are indicated inside the nodes. In the Dynkin map, the r_0 fugacity of the affine node

¹⁹One could equally eliminate the rank 1 node and adjust for such change in the next mappings.

will play no role, hence we can express r_0 in terms of the other two:

$$r_0 = (r_1^2 r_2^3)^{-1} \quad (2.93)$$

Note that all fugacities are weighted by their Coxeter labels. Further, by comparing figures 18 and 19, one sees that, in the Dynkin map, z_2 should map to r_1 . Moreover, the z_3 fugacity maps to r_0 fugacity of the affine node as these are the corresponding rank 1 nodes. Hence, the desired Dynkin map is:

$$z_2 \rightarrow r_1 \quad (2.94)$$

$$z_3 \rightarrow r_0 = (r_1^2 r_2^3), \quad (2.95)$$

which concludes the second step. The last step is to map the G_2 simple root fugacities r_1, r_2 to the coordinates on the weight space of G_2 . Equations (2.96) and (2.97) contain the desired map.

$$r_1 \rightarrow y_1^2 y_3^{-3} \quad (2.96)$$

$$r_2 \rightarrow y_2^2 y_1^{-1}. \quad (2.97)$$

After these mappings, the t^2 coefficient of the refined HS is computed as

$$2 + \frac{1}{y_1} + y_1 + \frac{y_1}{y_3} + \frac{y_1^2}{y_3^2} + \frac{y_1}{y_2} + \frac{1}{y_2} + \frac{y_1}{y_2} + y_2 + \frac{y_2}{y_1} + \frac{y_2^2}{y_1} + \frac{y_2^3}{y_1^2} + \frac{y_2^3}{y_1}, \quad (2.98)$$

which is precisely the character of the 14 dimensional adjoint representation of G_2 . Thus, the expectation of global symmetry is verified and in agreement with arguments in [22]. Finally, one can write the refined PL in the form:

$$\text{PL} = [1, 0]_{14} t^2 - [0, 0]_1 t^4 - [0, 1]_7 t^6 + [0, 1]_7 t^8 + O(t^{10}). \quad (2.99)$$

There are two relations at: t^4 transforming as a singlet, and at t^6 transforming under the 7 dimensional $[0, 1]$ representation of G_2 , respectively. These relations can be summarized by the algebraic variety made out of 14 complex numbers in the adjoint representation of G_2 , M^a which satisfy the relations

$$M^a M^a = 0,$$

and

$$MMM|_{[0,1]} = 0.$$

The Coulomb branch of $\mathcal{P}_{[3,1]}(4)$ is the 10 dimensional sub-regular nilpotent orbit of G_2 [21]:

$$\mathcal{C}_{[3,1]} = \overline{\text{sub.reg.}\mathcal{O}_{G_2}}. \quad (2.100)$$

The formula for the HWG is given by equation (2.101), which is equation (3.37) in [23], where the authors used a different convention for the factor multiplying the conformal dimension in the monopole formula (i.e. all t powers are half of those herein).

$$\text{HWG} = \text{PE}[\mu_2 t^2 + \mu_1^2 t^4 + \mu_1^3 t^6 + \mu_2^2 t^8 + \mu_1^3 \mu_2 t^{10} - \mu_1^6 \mu_2^2 t^{20}]. \quad (2.101)$$

Let us refer to the quiver in figure 18 as the G_2 -tail for the following reason. Consider a construction defined by two steps:

Figure 20. $\mathcal{P}_{[4]}(4)$ quiver with A_2 global symmetry, $\dim \mathcal{M}_C^{\mathbb{H}} = 5$.

- Consider any quiver Q , with G_0 global symmetry and attach the G_2 -tail (which has a G_2 global symmetry) to this quiver via an adjoint node²⁰
- Multiply all the ranks of Q by 3

Then, the theory constructed by this procedure has a global symmetry G_{global} that satisfies equation (2.102).

$$G_{\text{global}} = G_0 \times G_2. \quad (2.102)$$

For detailed examples of this construction, see (5.24) and (5.25) and the consequent discussion in [10].

2.2.4 Gauging $H_\lambda = S_4$

In order to construct the last theory, gauge the entire discrete S_4 symmetry of the parent $\mathcal{P}_{[14]}(4)$ quiver. According to Conjecture 1, the desired $\mathcal{P}_{[4]}(4)$ quiver takes the form depicted in figure 20. The assignment of the simple root fugacities is also shown in figure 20. The balanced part of this quiver forms the A_2 Dynkin diagram therefore the expected global symmetry is $SU(3)$. It is also the subgroup that commutes with S_4 inside $SO(8)$. Moreover, as in the case of the $\mathcal{P}_{[22]}$ quiver, the number of fugacities after the gauge fixing is smaller than the rank of the expected global symmetry. Thus, one expects to find a certain embedding: $SU(2) \hookrightarrow SU(3)$. In fact, the set of all embedding of $\mathfrak{su}(2)$ inside a $\mathfrak{su}(n)$ algebra is in one-to-one correspondence with the set of all nilpotent orbits of $\mathfrak{su}(n)$. There is a bijection between nilpotent orbits and the partitions of $\mathcal{P}(n)$. As a first step, the HS is first computed using the simple root fugacities, and then unrefined by setting all fugacities to unity. The unrefined HS is given by equation (2.103).

$$\text{HS}_{[4]}(t) = \frac{1 + 3t^2 + 13t^4 + 25t^6 + 46t^8 + 48t^{10} + 46t^{12} + 25t^{14} + 13t^{16} + 3t^{18} + t^{20}}{(1 - t^2)^5(1 - t^4)^5}. \quad (2.103)$$

The expansion of the unrefined Hilbert series reads

$$\text{HS}_{[4]}(t) = 1 + 8t^2 + 48t^4 + 210t^6 + 771t^8 + O(t^{10}). \quad (2.104)$$

The expression of the unrefined PL takes the form

$$PL = 8t^2 + 12t^4 - 6t^6 - 21t^8 + O(t^{10}). \quad (2.105)$$

²⁰In case of A-series, one needs to attach the G_2 -tail via both of the adjoint nodes.

The t^2 coefficient agrees with the dimension of the adjoint representation of $SU(3)$:

$$\dim [1, 1]_{A_2} = 8 \quad (2.106)$$

Lets proceed by mapping the simple root fugacities $z_i, i = 1, 2$ according to (2.107) and (2.108)

$$z_2 \rightarrow (z_1^4)^{-1/2} \quad (2.107)$$

$$z_1 \rightarrow x^2, \quad (2.108)$$

where z_2 fugacity of the rank 2 node is eliminated by constraint (2.10), and x is the fugacity for the fundamental weight of $SU(2)$. Using this mapping, the expansion of the refined PL takes the form:

$$\begin{aligned} PL = & \left(2 + \frac{1}{x^4} + \frac{2}{x^2} + 2x^2 + x^4\right) t^2 + \left(4 + \frac{2}{x^4} + \frac{2}{x^2} + 2x^2 + 2x^4\right) t^4 \\ & - \left(2 + \frac{2}{x^2} + 2x^2\right) t^6 + \left(7 + \frac{3}{x^4} + \frac{4}{x^2} + 4x^2 + 3x^4\right) t^8 + O(t^9). \end{aligned} \quad (2.109)$$

The last expression can be written as:

$$PL = ([4]_5 + [2]_3)t^2 + (2[4]_5 + 2[0]_1)t^4 - (2[2]_3)t^6 - (3[4]_5 + [2]_3 + 3[0]_1)t^8 + O(t^9) \quad (2.110)$$

where $SU(2)$ representations are denoted by Dynkin label $[a]$. One can list representations at each order of t as follows:

- t^2 : generators transforming under $[4] + [2]$
- t^4 : generators transforming under $2[4] + 2[0]$
- t^6 : relations transforming under $[2]$
- t^8 : relations transforming under $3[4] + [2] + 3 \times [0]$

The obtained embedding of $SU(2)$ inside $SU(3)$ corresponds to the homomorphism embedding characterizing the maximal nilpotent orbit of $SU(3)$ (f.i. see the last row of second table in appendix B.1 in [15]):

$$[4]_5 \oplus [2]_3 \hookrightarrow [1, 1]_8 \quad (2.111)$$

This provides a verification of the expected global symmetry since the $\mathfrak{su}(2)$ embedding in case of the maximal nilpotent orbit of $\mathfrak{su}(3)$ is characterized by a map where the two fugacities of the A_2 , (y_1, y_2) map to $(x^2, 1)$, where x is the $SU(2)$ fugacity, and so in turn, the character of the adjoint representation of A_3 $[1, 1]$ becomes the character of $[4] \oplus [2]$ of A_1 , which is the t^2 coefficient in equation (2.109). In terms of the A_2 Dynkin labels, the refined PL can be written in the form:

$$PL = [1, 1]_8 t^2 + ([2, 0]_6 + [0, 2]_6) t^4 - ([1, 0]_3 + [0, 1]_3) t^6 - ([1, 1]_{15} + [2, 0]_6 + [0, 2]_6 + 1) t^8 + O(t^9). \quad (2.112)$$

The eight generators of the global symmetry transform under the adjoint representation of A_2 . At order t^4 there are generators transforming under the $[2, 0]$ and the conjugate $[0, 2]$ representation. The relation at order t^6 transforms under the fundamental and anti-fundamental representations denoted by $[1, 0]$ and $[0, 1]$, respectively. Finally, the relations at t^8 transform as $[1, 1] + [2, 0] + [0, 2] + 1$.

Ratios of Coulomb branch volumes for $n = 2, k = 2$ theories					
Partition	$[1^4]$	$[2, 1^2]$	$[2^2]$	$[3, 1]$	$[4]$
$[1^4]$	1	2	4	6	24
$[2, 1^2]$		1	2	3	12
$[2^2]$			1	$3/2$	6
$[3, 1]$				1	4
$[4]$					1

Table 2. Ratios of Coulomb branch volumes for $n = 2, k = 2$ theories.

2.2.5 Comparison of the Coulomb branch volumes

Consider the unrefined HS computed in this subsection for all five theories. Recall that these are: (2.56), (2.67), (2.78), (2.88) and (2.103). For each pair of theories, expand the unrefined Hilbert series according to equation (1.3) and plug into equation (1.6). The computed ratios of the Coulomb branch volumes of $k = n = 2$ theories are summarized in table 2. The parent Coulomb branch of $\mathcal{P}_{[1^4]}(4)$ is $\overline{\min \mathcal{O}_{D_4}}$. All the evidence for Conjecture 1 suggests that for the daughter Coulomb branches, there holds:

$$\mathcal{C} = \overline{\min \mathcal{O}_{D_4}} / \Gamma \quad (2.113)$$

where $\Gamma \subseteq S_4$ is a discrete group. In particular:

$$\Gamma = \begin{cases} S_2 \equiv \mathbb{Z}_2 & \text{for } [2, 1^2] \\ \mathbb{Z}_2 \times \mathbb{Z}_2 & \text{for } [2^2] \\ S_3 & \text{for } [3, 1] \\ S_4 & \text{for } [4] \end{cases} \quad (2.114)$$

Note that the obtained relations (2.115) and (2.116)

$$\overline{\min \mathcal{O}_{D_4}} / \mathbb{Z}_2 = \overline{\text{n.min } \mathcal{O}_{B_3}} \quad (2.115)$$

$$\overline{\min \mathcal{O}_{D_4}} / S_3 = \overline{\text{sub.reg } \mathcal{O}_{G_2}} \quad (2.116)$$

relate quotients of the closure of the minimal nilpotent orbit of $\mathfrak{so}(8)$ algebra to the (closures of the) next to minimal orbit of $\mathfrak{so}(7)$ and sub-regular nilpotent orbit of $\mathfrak{g}_2$, respectively. This is the classic result of Kostant and Brylinski [24].

3 Second family: bouquet quivers with $A_1^{n_1} \times D_{n_2+1}$ global symmetry

The analysis of the first family of bouquet quivers allows a generalization of the results of the HWG to a larger family of quivers. In the first part of this section we derive a general formula of the HWG for the *second family* of quivers with $A_1^{n_1} \times D_{n_2+1}$ global symmetry. In the second part of the section, we perform discrete gauging for a particular member of this family. The main focus in this section is the derivation of the general formula for the HWG and the analysis of the particular quivers is given in less detail. The simple root fugacities, indicated inside the quiver nodes in the figures, are denoted by z_i and w_i . The

Figure 21. $\mathcal{P}_{[1^{n_1}]}(n_1)$ quiver, $b = n_1 - 4$, $\dim \mathcal{M}_C^{\mathbb{H}} = n_1 + 1$.

fugacity maps throughout this section show the mappings of the simple root fugacities to the fundamental weight fugacities which are denoted by x_i and y_i . Finally, the highest weight fugacities used in the expressions of HWG are denoted by μ_i and ν_i .

3.1 Derivation of HWG

Consider the theory in figure 3 and set $k = 2$, $n = n_1$. The corresponding quiver is given in figure 21. The central node is balanced for a special case $n_1 = 4$, which is indicated by the radial color gradient of the node. The theory in figure 21 has a $SU(2)^{n_1}$ global symmetry which enhances²¹ to $SO(8)$ for $k_1 = 4$.

For $n_1 \neq 4$ all the bouquet nodes are balanced and the only unbalanced node is the central one, with balance $b = n_1 - 4$. Lets consider the $n_1 = 5$ case. The balanced subquivers form five A_1 Dynkin diagrams, therefore the expected global symmetry is $SU(2)^5$. Analogically to figure 11, the simple root fugacities are assigned such that z_0 is the simple root fugacity of the unbalanced node and $z_i, i = 1, \dots, 5$ are the simple root fugacities of the bouquet nodes. Computation of the unrefined HS yields

$$HS(t) = \frac{P_1(t)}{(1-t)^{12}(1+t)^4(1+t+t^2)^6} \quad (3.1)$$

where

$$P_1(t) = 1 - 2t + 12t^2 + 4t^3 + 21t^4 + 60t^5 + 54t^6 + 66t^7 + 120t^8 + \dots \text{palindrome} \dots + t^{16} \quad (3.2)$$

The expansion of the unrefined HS is given by equation (3.3).

$$HS(t) = 1 + 15t^2 + 32t^3 + 116t^4 + 352t^5 + 863t^6 + 2112t^7 + O(t^8) \quad (3.3)$$

The PL of the unrefined HS is computed as

$$PL = 15t^2 + 32t^3 - 4t^4 - 128t^5 - 285t^6 + 320t^7 + O(t^8). \quad (3.4)$$

The t^2 coefficient of the last expression is the dimension of the expected G_{global} :

$$5 \times \dim [2]_{A_1} = 15. \quad (3.5)$$

²¹See section 2.

Figure 22. $\{\mathcal{P}_{[1^3]}(3), \mathcal{P}_{[1^2]}(2)\}$ quiver with $SU(2)^3 \times D_3$, global symmetry, $b = 1$, $\dim \mathcal{M}_C^{\mathbb{H}} = 8$.

Perform the mapping according to (3.6), (3.7) and (3.8)

$$z_i \rightarrow x_i^2, \quad i = 1, 2, 3 \quad (3.6)$$

$$z_4 \rightarrow y_1^2, \quad z_5 \rightarrow y_2^2 \quad (3.7)$$

$$z_0 \rightarrow (z_1 z_2 z_3 z_4 z_5)^{-\frac{1}{2}} = (x_1 x_2 x_3 y_1 y_2)^{-1}. \quad (3.8)$$

such that the unbalanced fugacity z_0 is eliminated²² and x_i and y_i are the fundamental weight fugacities of $SU(2)$. Note the splitting of fugacities of the bouquet nodes into x_1, x_2, x_3 and y_1, y_2 . The reason for such splitting will shortly become apparent. After the computation of the refined HS one makes use of the HWG to describe the chiral ring of the theory. The HWG takes the form given by equation (3.9) [13]:

$$\begin{aligned} \text{HWG} = \text{PE}[(\nu_1^2 + \nu_2^2 + \nu_3^2 + \mu_1^2 + \mu_2^2) t^2 + (\nu_1 \nu_2 \nu_3 \mu_1 \mu_2) (t^3 + t^5) \\ + (\mu_1^2 \mu_2^2) t^4 - (\mu_1^2 \mu_2^2) t^4 + t^4 - (\nu_1 \nu_2 \nu_3 \mu_1 \mu_2)^2 t^{10}] \end{aligned} \quad (3.9)$$

where μ_i , $i = 1, 2$ and ν_i , $i = 1, 2, 3$ are the fugacities for the highest weights of $SU(2)$. The t^2 terms in equation (3.9) are the usual contributions of the global symmetry for each of the balanced $SU(2)$ nodes. The imbalance of the central node, $b = 1$, produces the t^3 contribution in equation (3.9). Furthermore, since the 5 bouquet nodes are connected to the unbalanced node, the resulting operators transform in the multi-fundamental representation corresponding to all of bouquet nodes, denoted by $\nu_1 \nu_2 \mu_1 \mu_2 \mu_3$. The t^5 naturally comes from the tensor product of the adjoint and the multi-fundamental representation. It should also be emphasized that a zero in the form of $(\mu_1^2 \mu_2^2) t^4 - (\mu_1^2 \mu_2^2) t^4$ is added to expression 3.9 in anticipation of the $\mu_1^2 t^4$ term of 3.18. The t^4 singlet term shows up since the Casimir invariant of the five $SU(2)$ are all proportional to each other. Finally, there is a relation at t^{10} transforming under $[2; 2; 2; 2; 2]$ (i.e. the adjoint five-representation of $SU(2)^5$).

Now, let's study a quiver where we add a rank 2 node to obtain a chain of two 2 nodes. Furthermore, split the $\mathcal{P}_{[1^5]}(5)$ bouquet into $\mathcal{P}_{[1^3]}(3)$ bouquet attached to the first 2 node and a $\mathcal{P}_{[1^2]}(2)$ bouquet attached to the other 2 node. This splitting of nodes is signified by the splitting of the fundamental weight fugacities of the bouquet nodes into x_i and y_i in the previous case. This splitting carries over to the HWG such that the highest weight fugacities are split into μ_i and ν_i . The resulting quiver, which now corresponds to a pair of partitions $\{\mathcal{P}_{[1^3]}(3), \mathcal{P}_{[1^2]}(2)\}$, is depicted in figure 22. The balance of the unbalanced red node is $(3 * 1 + 2) - (2 * 2) = 1$. One expects $A_1^3 \times D_3 \equiv A_1^3 \times A_3$ global symmetry on

²²Recall, that the elimination follows from (2.10).

Figure 23. $\{\mathcal{P}_{[1^3]}(3), \mathcal{P}_{[1^2]}(2)\}$ quiver with simple root fugacities.

the Coulomb branch by simply looking at the balanced sub-quivers. The unrefined HS is computed as

$$\text{HS}(t) = \frac{P_2(t)}{(1-t)^{16}(1+t)^8(1+t+t^2)^8} \quad (3.10)$$

where

$$P_2(t) = 1 + 16t^2 + 40t^3 + 118t^4 + 336t^5 + 747t^6 + 1344t^7 + 2396t^8 + 3616t^9 + 4670t^{10} + 5568t^{11} + 6060t^{12} + \dots \text{palindrome} \dots + t^{24}. \quad (3.11)$$

The expansion of the unrefined HS yields

$$\text{HS}(t) = 1 + 24t^2 + 48t^3 + 282t^4 + 848t^5 + 2743t^6 + 7728t^7 + O(t^8), \quad (3.12)$$

and the corresponding unrefined PL takes the form

$$\text{PL} = 24t^2 + 48t^3 - 18t^4 - 304t^5 - 601t^6 + 1488t^7 + O(t^8). \quad (3.13)$$

The t^2 coefficient in the last equation can be identified with the dimension of the adjoint representations that form the global symmetry:

$$3 \times \dim [2]_{A_1} + \dim [0, 1, 1]_{D_3} = 24. \quad (3.14)$$

Given the simple root fugacity assignment in figure 23, perform a mapping according to equations (3.15) and (3.16)

$$z_1 \rightarrow x_1^2, \quad z_2 \rightarrow x_2^2, \quad z_3 \rightarrow x_3^2, \quad (3.15)$$

$$z_6 \rightarrow y_1^2 y_2^{-1} y_3^{-1}, \quad z_5 \rightarrow y_2^2 y_1^{-1}, \quad z_7 \rightarrow y_3^2 y_1^{-1} \quad (3.16)$$

$$z_4 \rightarrow (z_1 z_2 z_3 z_5^2 z_6 z_7)^{-\frac{1}{2}} = (x_1^{-2} x_2^{-2} x_3^{-2} y_1 y_2^{-3} y_3^{-1})^{\frac{1}{2}}. \quad (3.17)$$

where x_i , $i = 1, 2, 3$ are the $\text{SU}(2)$ fundamental weight fugacities, y_i , $i = 1, 2, 3$ are the D_3 fundamental weight fugacities, and z_4 , the fugacity of the red node is eliminated according to equation (3.17). The resulting HWG takes the form given by equation (3.18).

$$\text{HWG} = \text{PE}[(\nu_1^2 + \nu_2^2 + \nu_3^2 + \mu_2 \mu_3) t^2 + (\nu_1 \nu_2 \nu_3 \mu_1)(t^3 + t^5) + \mu_1^2 t^4 + t^4 - (\nu_1 \nu_2 \nu_3 \mu_1)^2 t^{10}] \quad (3.18)$$

Note the slight change in the structure of the terms appearing in equation (3.18) compared to equation (3.9). At order t^2 there is the adjoint $[0, 1, 1]$ rep of D_3 and the three-adjoint

Figure 24. $\{\mathcal{P}_{[1^3]}(3), \mathcal{P}_{[1^2]}(2)\}$ quiver with $SU(2)^3 \times D_4$ global symmetry, $b = 1$, $\dim \mathcal{M}_C^{\mathbb{H}} = 10$.

$[2; 2; 2]$ rep of $SU(2)^3$. Recall that $[d_1, d_2, d_3]$ and $[a_1; a_2; a_3]$ denote the Dynkin labels of D_3 and $SU(2) \times SU(2) \times SU(2)$, respectively. At t^3 and t^5 there are generators transforming under $[1; 1; 1; 1, 0, 0]$ representation of $SU(2)^3 \times D_3$. There are also generators transforming under $[0; 0; 0; 0, 0, 0]$ and $[0; 0; 0; 2, 0, 0]$ at t^4 . Finally, there is a relation at order t^{10} transforming under $[2; 2; 2; 2, 0, 0]$. Before the identification of a general pattern of HWG for this family of quivers is made possible, one more case needs to be considered. For this purpose, consider the quiver in figure 24, with the main chain consisting of three rank 2 nodes. The simple root fugacities are indicated inside the nodes in figure 24. The anticipated global symmetry, read off as the balanced sub-diagrams, is $A_1^3 \times D_4$. After the computation of the HS with the indicated simple root fugacities, the unrefined HS is obtained by setting all the fugacities to unity. The result is given in equation (3.19).

$$\text{HS}_{[1^3]}(t) = \frac{P_3(t)}{(-1+t)^{20}(1+t)^{12}(1+t+t^2)^{10}} \quad (3.19)$$

where

$$\begin{aligned} Q_1(t) = & 1 + 2t + 28t^2 + 108t^3 + 440t^4 + 1482t^5 + 4394t^6 + 11122t^7 + 25532t^8 \\ & + 52164t^9 + 95692t^{10} + 158586t^{11} + 239637t^{12} + 328584t^{13} + 410844t^{14} \\ & + 469872t^{15} + 491976t^{16} + \dots \text{palindrome} \dots + t^{32} \end{aligned} \quad (3.20)$$

The expansion of the unrefined HS has the form

$$\text{HS}(t) = 1 + 37t^2 + 64t^3 + 630t^4 + 1728t^5 + 7803t^6 + 22848t^7 + 75858t^8 + O(t^9). \quad (3.21)$$

Taking the PL of equation (3.19) one finds

$$\text{PL} = 37t^2 + 64t^3 - 73t^4 - 640t^5 - 715t^6 + 6208t^7 + 23614t^8 - O(t^9). \quad (3.22)$$

The t^2 coefficient in equation (3.22) matches the dimension of the adjoint representation of the global symmetry:

$$3 \times \dim [2]_{A_1} + \dim [0, 1, 0, 0]_{D_4} = 37. \quad (3.23)$$

Let us use the fugacity map

$$z_i \rightarrow x_i^2, \quad i = 1, 2, 3 \quad (3.24)$$

$$w_1 \rightarrow y_1^2 y_2^{-1}, \quad w_2 \rightarrow y_2^2 y_1^{-1} y_3^{-1} y_4^{-1}, \quad w_3 \rightarrow y_3^2 y_2^{-1}, \quad w_4 \rightarrow y_4^2 y_2^{-1} \quad (3.25)$$

$$z_0 \rightarrow (z_1 z_2 z_3 w_1^2 w_2^2 w_3 w_4)^{-\frac{1}{2}} = (x_1 x_2 x_3 y_1)^{-1}. \quad (3.26)$$

Figure 25. $\{\mathcal{P}_{[1^{n_1}]}(n_1), \mathcal{P}_{[1^2]}(2)\}$ quiver with $SU(2)^{n_1} \times D_{n_2+1}$ global symmetry, $b = n_1 + 2 - 4$, $\dim \mathcal{M}_C^{\mathbb{H}} = n_1 + 2n_2 + 1$.

Figure 26. $\{\mathcal{P}_{[1^{n_1}]}(n_1), \mathcal{P}_{[1^2]}(2)\}$ quiver with fugacity assignment.

whereupon the simple root fugacities z_i , $i = 1, 2, 3$, and w_j , $j = 1, 2, 3, 4$, map to the fundamental weight fugacities of the $SU(2)$ and D_4 , respectively, and z_0 , the fugacity of the red node, is substituted according to prescription (3.26). The HWG takes the form given by equation (3.27),

$$\text{HWG} = \text{PE}[(\nu_1^2 + \nu_2^2 + \nu_3^2 + \mu_2) t^2 + (\nu_1 \nu_2 \nu_3 \mu_1)(t^3 + t^5) + \mu_1^2 t^4 + t^4 - (\nu_1 \nu_2 \nu_3 \mu_1)^2 t^{10}]. \quad (3.27)$$

where, μ_2 is the adjoint weight fugacity of D_4 . Recall that the vector reps appear because the vector nodes of the balanced Dynkin sub-diagrams connect to the unbalanced node. Moreover, these appear at orders t^3 and t^5 since the imbalance is 1, same as in the previous case. This form of HWG is anticipated for all members of this family.

Increasing the number of nodes of the left bouquet results in a simple change of the form of the HWG. The number of 1 nodes in the left bouquet only changes the imbalance of the red node, therefore, for higher n_1 , the $(t^3 + t^5)$ terms will appear at higher orders of t [25]. For the $\{\mathcal{P}_{[1^4]}(4), \mathcal{P}_{[1^2]}(2)\}$ theory one expects the contribution to appear at $(t^4 + t^6)$. More generally, for a $\{\mathcal{P}_{[1^{n_1}]}(n_1), \mathcal{P}_{[1^2]}(2)\}$ theory, these contributions are expected at orders $(t^{n_1} + t^{n_1+2})$. We are in a position to write down a general expression of the HWG for a two parameter family of bouquet quivers of the form in figure 25. The simple root fugacity assignment is shown separately in figure 26 for clarity of presentation. Perform the mapping such that the simple root fugacities z_i , $i = 1, \dots, k_1$, map to the $SU(2)$ fundamental weight fugacities ν_i , $i = 1, \dots, n_1$ and, the root fugacities w_j , $j = 1, \dots, n_2 + 1$, map to D_{n_2+1} fundamental weight fugacities μ_j , $j = 1, \dots, n_2 + 1$. In full analogy to previous cases, this is achieved by deriving the fugacity map using the Cartan matrix, and the elimination of the z_0 fugacity of the red node follows from the gauge fixing condition (2.10). The HWG

Figure 27. $\{\mathcal{P}_{2,1}(3), n_2 = 3\}$ quiver with $SU(2)^2 \times D_4$ global symmetry, $b = 1$, $\dim \mathcal{M}_C^{\mathbb{H}} = 10$.

for the family of $\{\mathcal{P}_{[1^{n_1}]}(n_1), \mathcal{P}_{[1^2]}(2)\}$ quiver theories takes the form given by the general formula (3.28).

$$\text{HWG} = \text{PE} \left[\left(\sum_{i=1}^{n_1} \nu_i^2 + \mu_2 \right) t^2 + \left(\mu_1 \prod_{i=1}^{n_1} \nu_i \right) (t^{n_1} + t^{n_1+2}) + (\mu_1^2) t^4 + t^4 - \left(\mu_1 \prod_{i=1}^{n_1} \nu_i \right)^2 t^{10} \right] \quad (3.28)$$

Formula (3.28) contains the usual generators at order t^2 transforming under the adjoint representations corresponding to the D_{n_2+1} and $SU(2)$ nodes. In addition, there are generators in the vector representation of D_{n_2+1} since it is the vector Dynkin node that connects to the unbalanced node. For the same reason, the (n_1) -fundamental representation of the $SU(2)$ is present. Formula (3.28) is verified with an explicit computation of the HWG up to $n_1 = 4$ and $n_2 = 4$.

General analogue of formula (3.28) for bouquet quivers with ABCEFG factors in the global symmetry is conjectured in section 6.

3.2 Discrete gauging of $\{\mathcal{P}_{[1^3]}(3), n_2 = 3\}$ theory

3.2.1 Gauging $H_\lambda = \mathbb{Z}_2$

In this subsection, we use the unrefined description of the Coulomb branches to provide a non-trivial test of equation (1.1). Let us study the two parameter family of theories in figure 25. Note, that the quiver has discrete global symmetries S_{n_1} and S_2 corresponding to the left and the right bouquet, respectively. Lets focus on the first bouquet only, such that the rest of the quiver preserves a manifest D_{n_2+1} global symmetry. Such theories will be denoted by $\{\mathcal{P}_{[n_1]}(n_1), n_2\}$. The two parameters n_1 and n_2 correspond to the number of bouquet nodes and the number of rank 2 chain nodes, respectively. Let us study discrete gauging on a particular member of this family by setting $n_1 = n_2 = 3$. The considered theory is depicted in figure 24. The unrefined HS is given by equation (3.19) in the previous subsection.

Gauge a $\mathbb{Z}_2$ subgroup of the discrete global S_3 symmetry to construct a new theory. Following Conjecture 1 one obtain the quiver depicted in figure 27, which is accordingly denoted by $\{\mathcal{P}_{[2,1]}(3), 3\}$. The balanced subset of the quiver forms the $A_1^2 \times D_4$ Dynkin diagram. Hence, the anticipated global symmetry is $SU(2) \times SU(2) \times SO(8)$. The unrefined HS is given by equation (3.29)

$$\text{HS}_{[2,1]}(t) = \frac{Q_2(t)}{(-1+t)^{20}(1+t)^{14}(1+t^2)^2(1+t+t^2)^{10}}, \quad (3.29)$$

Figure 28. $\{\mathcal{P}_3(3), n_2 = 3\}$ quiver with $SU(2) \times D_4$ global symmetry, $b = 1$, $\dim \mathcal{M}_C^{\mathbb{H}} = 10$.

where

$$\begin{aligned}
 Q_2(t) = & 1 + 4t + 32t^2 + 146t^3 + 592t^4 + 2052t^5 + 6348t^6 + 17276t^7 + 42495t^8 \\
 & + 94722t^9 + 192829t^{10} + 359694t^{11} + 618737t^{12} + 983550t^{13} \\
 & + 1449871t^{14} + 1985584t^{15} + 2531833t^{16} + 3008328t^{17} + 3335694t^{18} \\
 & + 3452040t^{19} + \cdots \text{palindrome} \cdots + t^{38}
 \end{aligned} \tag{3.30}$$

From the unrefined PL given by equation (3.31),

$$\text{PL} = 34t^2 + 48t^3 - 66t^4 - 400t^5 - 129t^6 + 3744t^7 + 7875t^8 - 28352t^9 + O(t^{10}) \tag{3.31}$$

observe that the t^2 coefficient agrees with the expected global symmetry:

$$2 \times \dim[2]_{A_1} + \dim[0, 1, 0, 0]_{D_4} = 34. \tag{3.32}$$

3.2.2 Gauging $H_\lambda = S_3$

Finally, gauge $H = S_3$, the entire global symmetry of the $\{\mathcal{P}_{[1^3]}(3), n_2 = 3\}$ theory to obtain the $\{\mathcal{P}_{[3]}(3), n_2 = 3\}$ theory depicted in figure 28. The expected global symmetry is $A_1 \times D_4$. The unrefined HS is given by equation (3.33)

$$\text{HS}_{[3]}(t) = \frac{Q_3(t)}{(-1+t)^{20}(1+t)^{14}(1+t^2)^2(1+t+t^2)^{10}}, \tag{3.33}$$

where

$$\begin{aligned}
 Q_3(t) = & 1 + 4t + 29t^2 + 118t^3 + 436t^4 + 1342t^5 + 3754t^6 + 9232t^7 + 20764t^8 \\
 & + 42590t^9 + 80758t^{10} + 141402t^{11} + 230675t^{12} + 350568t^{13} + 498471t^{14} \\
 & + 663084t^{15} + 827454t^{16} + 968184t^{17} + 1064154t^{18} + 1097832t^{19} \\
 & + \cdots \text{palindrome} \cdots + t^{32}
 \end{aligned} \tag{3.34}$$

The unrefined PL takes the form given by equation (3.35),

$$\text{PL} = 31t^2 + 32t^3 - 65t^4 - 224t^5 + 249t^6 + 2192t^7 + 22t^8 - 21600t^9 + O(t^{10}), \tag{3.35}$$

which is in agreement with the expectation of the global symmetry since the t^2 coefficient equals

$$\dim[2]_{A_1} + \dim[0, 1, 0, 0]_{D_4} = 31. \tag{3.36}$$

3.2.3 Comparison of the Coulomb branch volumes

Expanding the unrefined Hilbert series (3.19), (3.29) and (3.33) according to equation (1.3) and plugging into (1.6) one finds:

$$\frac{\text{vol}(\mathcal{C}_{[1^3]})}{\text{vol}(\mathcal{C}_{[2,1]})} = \frac{R_{[1^3]}}{R_{[2,1]}} = \frac{\frac{56791}{3359232}}{\frac{56791}{6718464}} = 2 = \text{ord}(\mathbb{Z}_2) \quad (3.37)$$

$$\frac{\text{vol}(\mathcal{C}_{[1^3]})}{\text{vol}(\mathcal{C}_{[3]})} = \frac{R_{[1^3]}}{R_{[3]}} = \frac{\frac{56791}{3359232}}{\frac{56791}{20155392}} = 6 = \text{ord}(S_3) \quad (3.38)$$

which are the expected ratios. Obtained results (3.37) and (3.38) are in accord with Conjecture 1 and provide a non-trivial check that the Coulomb branches of $\{\mathcal{P}_{[2,1]}(3), n_2 = 2\}$ and $\{\mathcal{P}_{[3]}(3), n_2 = 2\}$ quivers are $\mathbb{Z}_2$ and S_3 orbifolds of the parent $\{\mathcal{P}_{[1^3]}(3), n_2 = 2\}$ Coulomb branch, respectively. Note, that it follows that the $\{\mathcal{P}_{[3]}(3), n_2 = 2\}$ Coulomb branch is a $\mathbb{Z}_3$ orbifold of the $\{\mathcal{P}_{[2,1]}(3), n_2 = 2\}$ Coulomb branch. Again, the explicit test involves the methods of the stepwise projection [18]. Let us now we turn to the third family of quivers.

4 Third family: A-type bouquet quivers with $U(1)^n \times A_2^2$ global symmetry

In this section, we consider unitary bouquet quivers with $U(1)^n \times A_{k-1}^2$ global symmetry, following the parametrization in figure 3. Let us set $k = n = 3$ to obtain the theory depicted in figure 29. As a gauge theory, the Coulomb branch quivers in this section correspond to the Higgs branches²³ of quivers describing a 6d $\mathcal{N} = (1, 0)$ low energy dynamics of a stack of three M5 branes on an A_2 singularity [6]. The arrangement of the bouquet nodes corresponds to three separated M5 branes, hence, we accordingly denote this theory by $\mathcal{P}_{[1^3]}(3)$. In the previous sections, only *fully balanced* or *minimally unbalanced* quivers were considered. Since in this section, one encounters quivers with more than one unbalanced node, the conjectured prescription for reading off the global symmetry from the quiver needs to be extended. For the global symmetry of a quiver with N , $N \geq 2$ unbalanced nodes, there holds

$$G_{\text{global}} = G_{\text{balanced}}^i \times U(1)^{N-1}, \quad (4.1)$$

where G_{balanced}^i is the symmetry group that corresponds to the Dynkin diagram formed by the i -th balanced subset of nodes. Moreover, there are $N - 1$ additional $U(1)$ factors such that the number of $U(1)$ Abelian factors in the global symmetry is one less than the number of unbalanced nodes. In the case of figure 29, one expects $4 - 1 = 3$ copies of such Abelian factors. Hence, the expected global symmetry is $U(1)^3 \times SU(3)^2$. There is an additional S_3 discrete global symmetry that permutes the bouquet nodes. The balance of all four unbalanced nodes is 1. Throughout this section, we refrain from showing explicit fugacity assignments and maps since the main objective is to test formula (1.1). The unrefined

²³Note, that herein, when we talk about the $U(1)$ factors in the global symmetry, we are referring to the 3d quivers only. It should be emphasized that in the 6d, the anomalous $U(1)$ factors are no longer part of the global symmetry. Nevertheless, they remain as part of the isometry of the moduli space.

Figure 29. $\mathcal{P}_{[13]}(3)$ quiver with $SU(3)^2 \times U(1)^3$ global symmetry, $b_i = 1$, $i = 1, 2, 3, 4$, $\dim \mathcal{M}_C^{\mathbb{H}} = 11$.

Hilbert Series for the $\mathcal{P}_{[13]}(3)$ quiver takes the form

$$HS_{[13]}(t) = \frac{P_1(t)}{(-1+t)^{22}(1+t)^{16}(1+t^2)^8(1+t+t^2)^{11}(1+t+t^2+t^3+t^4)^5}, \quad (4.2)$$

where

$$\begin{aligned} P_1(t) = & 1 + 10t + 66t^2 + 343t^3 + 1561t^4 + 6421t^5 + 24318t^6 + 85373t^7 + 279505t^8 \\ & + 856911t^9 + 2470009t^{10} + 6715986t^{11} + 17278135t^{12} + 42171723t^{13} \\ & + 97892626t^{14} + 216588291t^{15} + 457659547t^{16} + 925229636t^{17} \\ & + 1792503575t^{18} + 3332789141t^{19} + 5954799253t^{20} + 10236605469t^{21} \\ & + 16948970150t^{22} + 27055291005t^{23} + 41673945980t^{24} \\ & + 61990354851t^{25} + 89112653186t^{26} + 123875740431t^{27} \\ & + 166613606315t^{28} + 216934711187t^{29} + 273547259468t^{30} \\ & + 334183688804t^{31} + 395665660521t^{32} + 454128806740t^{33} \\ & + 505396609910t^{34} + 545458043162t^{35} + 570976321490t^{36} \\ & + 579740398924t^{37} + \dots \text{palindrome} \dots + t^{74}. \end{aligned} \quad (4.3)$$

Taking the PL of the unrefined HS yields

$$PL = 19t^2 + 24t^3 + 53t^4 + 36t^5 - 129t^6 - 588t^7 - 1347t^8 - O(t^9). \quad (4.4)$$

The t^2 coefficient agrees with the anticipated $G_{\text{global}} = U(1)^3 \times SU(3)^2$ since

$$3 \times \dim U(1) + 2 \times \dim [1, 1]_{A_2} = 19. \quad (4.5)$$

4.1 Gauging $H_\lambda = \mathbb{Z}_2$

Gauge a $\mathbb{Z}_2$ subgroup of the discrete S_3 symmetry such that, according to Conjecture 1, the obtained theory corresponding to $\mathcal{P}_{[2,1]}(3)$ is described by a quiver in figure 30. The two balanced sub-quivers form a $A_2 \times A_2$ global symmetry. Moreover, there are three unbalanced nodes which implies that there are additional two $U(1)$ factors in the global symmetry. Altogether, we have

$$G_{\text{global}} = U(1)^2 \times SU(3)^2. \quad (4.6)$$

The unrefined Hilbert Series is given by equation (4.7),

$$HS_{[2,1]}(t) = \frac{P_2(t)}{(-1+t)^{22}(1+t)^{16}(1+t^2)^8(1-t+t^2)(1+t+t^2)^{11}(1+t+t^2+t^3+t^4)^5}, \quad (4.7)$$

Figure 30. $\mathcal{P}_{[2,1]}(3)$ quiver with $SU(3)^2 \times U(1)^2$ global symmetry, $\dim \mathcal{M}_C^{\mathbb{H}} = 11$.

where

$$\begin{aligned}
 P_2(t) = & 1 + 9t + 56t^2 + 276t^3 + 1192t^4 + 4635t^5 + 16581t^6 + 55030t^7 + 170775t^8 \\
 & + 497861t^9 + 1369519t^{10} + 3566403t^{11} + 8819153t^{12} + 20761818t^{13} \\
 & + 46641268t^{14} + 100192056t^{15} + 206191600t^{16} + 407200034t^{17} \\
 & + 772867324t^{18} + 1411740354t^{19} + 2484834652t^{20} + 4219097138t^{21} \\
 & + 6917735891t^{22} + 10963035811t^{23} + 16806739624t^{24} + 24943050628t^{25} \\
 & + 35861261184t^{26} + 49977850045t^{27} + 67552995501t^{28} + 88601153016t^{29} \\
 & + 112810770236t^{30} + 139490143344t^{31} + 167556757817t^{32} + 195581752669t^{33} \\
 & + 221893836645t^{34} + 244734679875t^{35} + 262447986225t^{36} + 273674136136t^{37} \\
 & + 277519995798t^{38} + \dots \text{palindrome} \dots + t^{76}.
 \end{aligned} \tag{4.8}$$

The unrefined PL is given by equation (4.9).

$$\text{PL} = 18t^2 + 22t^3 + 36t^4 + 20t^5 - 71t^6 - 320t^7 - 615t^8 - O(t^9). \tag{4.9}$$

The t^2 coefficient agrees with the anticipation of the global symmetry since

$$2 \times \dim U(1) + 2 \times \dim [1, 1]_{A_2} = 18. \tag{4.10}$$

4.2 Gauging $H_\lambda = S_3$

Finally, gauge the entire S_3 of $\mathcal{P}_{[1^3]}(3)$ in figure 29. According to Conjecture 1 one obtains the $\mathcal{P}_{[3]}(3)$ quiver, depicted in figure 31. Since there are two unbalanced nodes, a single $U(1)$ factor is anticipated in the global symmetry. Hence, the expected global symmetry is:

$$G_{\text{global}} = U(1) \times SU(3) \times SU(3) \tag{4.11}$$

Balance of the central and adjoint 3 node is $b_3 = 1$ and $b_{\text{Adj}} = 1$, respectively. The lack of the S_3 symmetry of the bouquet is reflected in the form of the Hilbert Series (4.12),

$$\text{HS}_{[3]}(t) = \frac{P_3(t)}{(-1+t)^{22}(1+t)^{16}(1+t^2)^8(1-t+t^2)(1+t+t^2)^{11}(1+t+t^2+t^3+t^4)^5} \tag{4.12}$$

Figure 31. $\mathcal{P}_{[3]}(3)$ quiver with $SU(3)^2 \times U(1)$ global symmetry, $b_3 = 1$, $b_{Adj} = 1$ $\dim \mathcal{M}_C^{\text{H}} = 11$.

where

$$\begin{aligned}
 P_3(t) = & 1 + 9t + 55t^2 + 265t^3 + 1100t^4 + 4069t^5 + 13742t^6 + 42912t^7 + 125138t^8 \\
 & + 343023t^9 + 888619t^{10} + 2184322t^{11} + 5112353t^{12} + 11424591t^{13} + 24436388t^{14} \\
 & + 50131522t^{15} + 98823582t^{16} + 187490947t^{17} + 342838440t^{18} \\
 & + 604970597t^{19} + 1031345366t^{20} + 1700334084t^{21} + 2713413646t^{22} \\
 & + 4194680213t^{23} + 6286332847t^{24} + 9138877284t^{25} + 12895494665t^{26} \\
 & + 17670886241t^{27} + 23526392712t^{28} + 30444409900t^{29} + 38306534638t^{30} \\
 & + 46880165917t^{31} + 55818219780t^{32} + 64674799961t^{33} + 72937612669t^{34} \\
 & + 80074444293t^{35} + 85588479301t^{36} + 89074448896t^{37} + 90267198678t^{38} \\
 & + \cdots \text{palindrome} \cdots + t^{76}.
 \end{aligned} \tag{4.13}$$

Equation (4.14) contains the unrefined PL:

$$\text{PL} = 17t^2 + 20t^3 + 18t^4 + 2t^5 - 33t^6 - 122t^7 - 139t^8 + (t^9). \tag{4.14}$$

The t^2 coefficient matches the dimension of the expected global symmetry

$$\dim U(1) + 2 \times \dim [1, 1]_{A_2} = 17. \tag{4.15}$$

Let us now turn to the comparison of the Coulomb branch volumes.

4.3 Comparison of the Coulomb branch volumes

Expanding the unrefined Hilbert series (4.2), (4.7) and (4.12) according to equation (1.3) and plugging into (1.6) one finds the expected ratios:

$$\frac{\text{vol}(\mathcal{C}_{[1^3]})}{\text{vol}(\mathcal{C}_{[2,1]})} = \frac{R_{[1^3]}}{R_{[2,1]}} = \frac{\frac{689419303427}{773967052800000}}{\frac{689419303427}{1547934105600000}} = 2 = \text{ord}(\mathbb{Z}_2) \tag{4.16}$$

$$\frac{\text{vol}(\mathcal{C}_{[1^3]})}{\text{vol}(\mathcal{C}_{[3]})} = \frac{R_{[1^3]}}{R_{[3]}} = \frac{\frac{689419303427}{773967052800000}}{\frac{689419303427}{4643802316800000}} = 6 = \text{ord}(S_3) \tag{4.17}$$

Results (4.16) and (4.17) are in accord with Conjecture 1 and they provide a necessary non-trivial check that the Coulomb branches of $\mathcal{P}_{[2,1]}(3)$ and $\mathcal{P}_{[3]}(3)$ are $\mathbb{Z}_2$ and S_3 orbifolds of the parent $\mathcal{P}_{[1^3]}(3)$ Coulomb branch, respectively. Note, that the Coulomb branch of

Figure 32. Commutative diagram of Coulomb branch orbifold hierarchy for $n = 3, k = 3$ bouquet quivers.

Ratios of $k = 3, n = 3$ Coulomb branch volumes			
Partition	$[1^3]$	$[2, 1]$	$[3]$
$[1^3]$	1	2	6
$[2, 1]$		1	3
$[3]$			1

Table 3. Ratios of Coulomb branch volumes for $k = 3, n = 3$ family.

$\mathcal{P}_{[3]}(3)$ is a $\mathbb{Z}_3$ quotient of the Coulomb branch of $\mathcal{P}_{[2,1]}(3)$. The orbifold hierarchy for $k = 3, n = 3$ theories is symbolized by the commutative diagram in figure 32. Note, that this case is precisely analogical to the cases encountered earlier in this paper (i.e. the $k = 2, n = 1$ quivers of the first family). In figure 32, vertices denote the Coulomb branches of the three $k = 3, n = 3$ A-type bouquet quivers and arrows denote the quotients between the branches. The ratios of the Coulomb branch volumes are summarized in table 3. The ratios are in one-to-one correspondence with the ratios of the orders of the corresponding quotient groups. The same analysis was carried out for all members of this family up to $k = 4, n = 5$. Let us now test the discrete gauging construction of Conjecture 1 on the Coulomb branches of non-simply laced quivers.

5 Non-simply laced bouquet quivers

This section discusses discrete gauging on Coulomb branches of non-simply laced quiver theories. In a non-simply laced quiver, the non-simply laced edge points towards the short nodes. Accordingly, the side of the quiver that contains short nodes is called *short* and vice versa. Since the discrete gauging action is purely local, it does not distinguish between the *long* and the *short* side of the quiver and, hence, the main construction of this paper can be performed on Coulomb branches of non-simply laced theories in the same fashion as for the simply laced theories. Consider a simple complete bouquet quiver with a $SU(2)^3 \times C_2$ global symmetry. Following the previous notation we denote this theory by $\mathcal{P}_{[1^3]}$. The quiver is depicted in figure 33 (the simple root fugacities are shown inside the nodes for completeness). One first computes the refined HS using the simple root fugacities and sets

Figure 33. $\mathcal{P}_{[1^3]}(3)$ quiver with $SU(2)^3 \times C_2$ global symmetry, $b = 1$, $\dim \mathcal{M}_C^{\mathbb{H}} = 8$.

all to unity to obtain the unrefined Hilbert Series in equation (5.1),

$$HS_{[1^3]}(t) = \frac{P_4(t)}{(-1+t)^5(1+t)^3(-1+t^2)^{10}(1+t+t^2)^2(-1+t^3)^5(-1+t^4)^8}, \quad (5.1)$$

where

$$\begin{aligned} P_4(t) = & 1 + 6t^2 + 25t^3 + 48t^4 + 86t^5 + 174t^6 - 16t^7 - 479t^8 - 786t^9 - 1665t^{10} \\ & - 2343t^{11} - 426t^{12} + 4103t^{13} + 10658t^{14} + 16406t^{15} + 9016t^{16} - 13662t^{17} \\ & - 35689t^{18} - 50648t^{19} - 37611t^{20} + 15375t^{21} + 69626t^{22} + 106493t^{23} \\ & + 104738t^{24} + 22330t^{25} - 97700t^{26} - 180462t^{27} - 193479t^{28} - 82660t^{29} \\ & + 109804t^{30} + 228737t^{31} + \dots \text{palindrome} \dots + t^{63}. \end{aligned} \quad (5.2)$$

The PL of the unrefined HS is given by equation (5.3).

$$PL = 19t^2 + 32t^3 + 35t^4 - 64t^5 - 369t^6 - 832t^7 + O(t^8). \quad (5.3)$$

The t^2 coefficient of the unrefined PL is identified as

$$3 \times \dim [2]_{A_1} + \dim [2, 0]_{C_2} = 19, \quad (5.4)$$

which is the dimension of the adjoint representations of the expected global symmetries.

5.1 Gauging $H_\lambda = S_3$

Let us directly construct the last daughter theory, where the entire S_3 discrete global symmetry is gauged. This is achieved by gauging the entire S_3 on the Coulomb branch of the $\mathcal{P}_{[1^3]}(3)$ parent quiver. This amounts to the substitution of the original bouquet for a single adjoint 3 node. Conjecture 1 implies that the resulting quiver (with the root fugacities explicitly indicated inside the nodes for completeness) takes the form depicted in figure 34. The balanced part of the quiver corresponds to $A_1 \times C_2$ global symmetry. The computation of the unrefined Hilbert Series yields equation (5.5)

$$HS_{[3]}(t) = \frac{P_5(t)}{(-1+t)^5(1+t)^3(1-t^2)^{10}(1+t^2)^2(1+t+t^2)^2(-1+t^3)^5(1-t^4)^8(1+t^2+t^4)^2}, \quad (5.5)$$

Figure 34. $\mathcal{P}_{[3]}(3)$ quiver with $A_1 \times C_2$ global symmetry, $b = 1$, $\dim \mathcal{M}_C^{\mathbb{H}} = 8$.

where

$$\begin{aligned}
 P_5(t) = & 1 + 4t^2 + 9t^3 + 26t^4 + 52t^5 + 78t^6 + 34t^7 - 59t^8 - 194t^9 - 454t^{10} - 667t^{11} \\
 & - 911t^{12} - 918t^{13} - 48t^{14} + 1985t^{15} + 4650t^{16} + 7296t^{17} + 6956t^{18} \\
 & + 1882t^{19} - 6962t^{20} - 18740t^{21} - 25008t^{22} - 21570t^{23} - 6662t^{24} \\
 & + 17008t^{25} + 37396t^{26} + 47834t^{27} + 43231t^{28} + 24580t^{29} - 6046t^{30} \\
 & - 42257t^{31} - 77738t^{32} - 92718t^{33} - 69502t^{34} - 11234t^{35} + 68408t^{36} \\
 & + 120258t^{37} + \cdots \text{palindrome} \cdots + t^{75}.
 \end{aligned} \tag{5.6}$$

The unrefined PL takes the form:

$$\text{PL} = 13t^2 + 16t^3 + 26t^4 + 16t^5 - 49t^6 - 264t^7 + O(t^8). \tag{5.7}$$

Indeed, the t^2 coefficient agrees with the dimension of the adjoint representations of constituent groups of the global symmetry:

$$\dim [2]_{A_1} + \dim [2, 0]_{C_2} = 13. \tag{5.8}$$

5.2 Comparison of the Coulomb branch volumes

Let us compare the volumes of the two 16 dimensional Coulomb branches computed in this section. The two relevant Hilbert series are given by equations (5.1) and (5.5). Expand the HS according to equation (1.3) and plug into (1.6) to find:

$$\frac{\text{vol}(\mathcal{C}_{[1^3]})}{\text{vol}(\mathcal{C}_{[3]})} = \frac{\frac{3743}{186624}}{\frac{3743}{1119744}} = 6 \tag{5.9}$$

which matches the order of the quotient group S_3 . The ratio of the volumes of the Coulomb varieties in (5.9) provides a non-trivial check that the $\mathcal{C}_{[3]}$ Coulomb branch is a non-Abelian S_3 orbifold of the $\mathcal{C}_{[1^3]}$ Coulomb branch.

Table 4 lists ratios of the Coulomb branch volumes between a pair of theories of the same type²⁴ with bouquets invariant under a discrete G and H symmetry, respectively. Let us assume that Γ is the subgroup of G that is gauged on the Coulomb branch of the former quiver in order to construct the latter descending quiver. The $G/H = G/\Gamma$

²⁴By theories of the same type we understand theories given by quivers that differ only in the form of the bouquets.

Ratios of Quotients							
G/Γ	$\mathbb{I}$	$\mathbb{Z}_2$	$\mathbb{Z}_3$	S_3	S_4	S_5	S_6
$\mathbb{I}$	1	2	3	6	24	120	720
$\mathbb{Z}_2$		1	3/2	3	12	60	360
$\mathbb{Z}_3$			1	2	8	40	260
S_3				1	4	20	120
S_4					1	5	30
S_5						1	6
S_6							1

Table 4. Ratios of $t = 1$ Hilbert Series poles for $k = 3$, $n = 3$ family.

quotient corresponds to the amount of permutation symmetry that is lost by gauging a discrete subgroup of the global permutational symmetry of the parent quiver. Graph theoretically, it corresponds to the difference of permutation symmetry between the parent $\mathcal{P}_{[\lambda]}(n)$ bouquet and the descendant $\mathcal{P}_{[\chi]}(n)$ bouquet. The ratio of the Coulomb branch volumes is shown in the upper-diagonal part of table 4. The ratios below the diagonal are inverse values of those above. Higher order Abelian and non-Abelian discrete groups that naturally show up in orbifold actions on bouquet quivers, as well as products of two or more quotient groups of the form $H_1 \times H_2$, are not included in the table 4 for brevity. For example, the Coulomb branch of a quiver with a bouquet of two adjoint rank 2 nodes, denoted $\mathcal{C}_{[2^2]}$, is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold of the a parent Coulomb branch $\mathcal{C}_{[1^4]}$, corresponding to a quiver with a complete bouquet of four rank 1 nodes. We encounter such case in section 3. The Coulomb branches satisfy equation (5.10).

$$\frac{\text{vol}(\mathcal{C}_{[1^4]})}{\text{vol}(\mathcal{C}_{[2^2]})} = \text{ord}(\mathbb{Z}_2) \times \text{ord}(\mathbb{Z}_2) = 2 \times 2 = 4. \quad (5.10)$$

6 Conclusions and discussion

By the formulation of Conjecture 1 we introduce general construction for discrete gauging in Coulomb branches of $3d \mathcal{N} = 4$ quiver gauge theories. Although the construction is purely local and hence applicable to any quiver with a bouquet, it is demonstrated for three particular families of simply-laced bouquet quivers. The first and the third family are of particular interest since they serve as indispensable Coulomb branch tools for understanding Higgs branch phases of the $6d \mathcal{N} = (1, 0)$ world-volume theories of a stack of n M5 branes on the $\mathbb{C}^2/\mathbb{Z}_k$ singularity in M-theory. In section 5 the discrete gauging construction is extended to include a non-simply-laced quiver with a C_2 factor in the global symmetry. Central part of the work in this paper that concerns the first family of quivers aims to offer a detailed analysis in support of Conjecture 1. The remaining part of this paper uses the unrefined analysis of the quivers such that the ratios of the Coulomb branch volumes, defined in equation (1.6), are used as a non-trivial verification of equation (1.1). As a remark, a complementary perspective of discrete gauging and its manifestation as discrete quotients on Coulomb branches is presented in [26].

Figure 35. $\mathcal{P}_{[1^n]}(n)$ quiver with $SU(2)^n \times G$ global symmetry, where G is any Lie group.

Possibly, an analogue of the general formula (3.28) exists for other types of bouquet quivers. Consider a quiver that consists of two parts:

- A bouquet that stems from a rank 2 unbalanced node
- A second part, connected to the rank 2 unbalanced node, that is itself a balanced ABCDEFG Dynkin diagram.

Such quivers can be constructed by attaching a bouquet via the rank 2 unbalanced node to a minimally unbalanced quiver. Quivers constructed in this manner take the form schematically depicted in figure 35. The box on the right in figure 35 symbolizes the balanced part of a minimally unbalanced quiver. The classification of all minimally unbalanced quivers is developed in [25]. Recalling the general formula (3.28) in section 4, one can speculate that the HWG for quivers of the form in figure 35 would involve:

- Order t^2 : adjoint representations of the balanced sub-quivers (i.e. n copies of $SU(2)$ adjoint rep and a single adjoint rep corresponding to the balanced ABCDEFG part of the quiver). Let $\nu_i, i = 1, \dots, n$ denote the fugacities of the highest weights of the n $SU(2)$ representations.
- Order $t^n + t^{n+2}$: n -fundamental representation of $SU(2)$ combined with the representation that corresponds to the node of the ABCDEFG Dynkin diagram that is connected to the red node (i.e. the vector node in the case of D_n). Lets denote the highest weight fugacity for the representation of this ABCDEFG Dynkin node by the μ_{unbal} .
- Order t^4 : μ_{unbal}^2 contribution and the typical singlet contribution
- Order t^{10} : relation transforming under $(\nu_1 \dots \nu_n \mu_{\text{unbal}})^2$

Interesting feature of this conjecture lies in the possibility to take moduli space with any particular isometry on the Coulomb branch and use discrete gauging to obtain various non-Abelian orbifolds of the original space. This is a novel method for constructing non-Abelian orbifold geometrical spaces with certain isometry. On the level of direct computation, however, it is challenging to obtain the HWG and the explicit verification of this conjecture is left for future study.²⁵

²⁵The contribution appearing with the singlet at order t^4 is the most speculative part of the conjecture and requires verification.

Figure 36. Local part of a quiver with a n flavor node attached to a k gauge node.

Figure 37. Local part of $\mathcal{P}_{[1^n]}(n)$ complete bouquet quiver.

The investigation of the analogue of Conjecture 1 for ortho-symplectic quivers with bouquet nodes of type O/Sp is one possible future direction. Another possible direction for development is the study of the same phenomenon in the context that involves M5 branes on an different type of singularity (i.e. the D-type or E-type singularities). The analysis, however, is much more subtle due to the lack of intuition and complexity of the corresponding higher-dimensional physics.

Acknowledgments

A.Z. would like to express a special gratitude to Rudolph Kalveks for indispensable help with some of the computations. A.Z. would also like to thank Santiago Cabrera and Marcus Sperling for useful and enlightening discussions. A.H. would like to thank Ronen Plesser, Travis Maxfield, Gabi Zafrir, Santiago Cabrera, Rudolph Kalveks and Marcus Sperling for enlightening discussions. A.H. is supported by STFC Consolidated Grant ST/J0003533/1, and EPSRC Programme Grant EP/K034456/1.

A Construction of bouquet quivers

From a generic unitary $3d \mathcal{N} = 4$ quiver with a n flavor node attached to a k gauge node, one can obtain a *complete bouquet* quiver by simply gauging the whole global symmetry into n separate rank 1 nodes. As an example, consider the quiver in figure 36. Round and square nodes denote gauge and flavor groups, respectively. To obtain the complete bouquet quiver, gauge the flavor node into separate U(1) gauge nodes. The resulting quiver is shown in figure 37. The form of the bouquet arrangement is denoted by $\mathcal{P}_{[1^n]}(n)$. This notation accordingly signifies that there are n copies of rank 1 nodes.

The Coulomb brach of the quiver in figure 36, denoted by $\mathcal{C}_1$, and the Coulomb brach of the quiver in figure 37, denoted by $\mathcal{C}_2$ satisfy

$$\mathcal{C}_1 = \mathcal{C}_2 / \text{U}(1)^n \quad (\text{A.1})$$

where the $/$ denotes a hyperKahler quotient.

Figure 38. Sp(1) gauge theory with D_4 global symmetry.

B Derivation of the HWG for the $\mathcal{P}_{[2^2]}$ theory

In this appendix we derive the HWG in equation (2.84). One starts with the HWG for the Sp(1) gauge theory with D_4 flavor group, depicted in figure 38 which has the form previously given in (2.65):

$$\text{HWG} = \text{PE}[\mu_2 t^2]. \quad (\text{B.1})$$

Recall that under the action of the first $\mathbb{Z}_2$ the representation decomposition is given by (2.77). Hence, after the first $\mathbb{Z}_2$ action the HWG becomes

$$\text{HWG} = \text{PE}[\mu_2 t^2 + \mu_1^2 t^4], \quad (\text{B.2})$$

where μ_i are the fugacities for the highest weights of SO(7). In order to rewrite this HWG in terms of the SU(4) fugacities, remember that the representations decompose as:

$$\mu_2 \rightarrow \mu_1 \mu_3 + \mu_2 \quad (\text{B.3})$$

$$\mu_1^2 \rightarrow \mu_2^2 + \mu_2 + 1, \quad (\text{B.4})$$

where on the l.h.s. the μ_i are the highest weight fugacities of SO(7) and on the r.h.s. the μ_i are the highest weight fugacities of SU(4), respectively. Thus, in terms of SU(4), the HWG (B.2) can be written in the form:

$$\text{HWG} = \text{PE}[\mu_1 \mu_3 t^2 + \mu_2 t^2 + \mu_2^2 t^4 + \mu_2 t^4 + t^4 - \mu_2^2 t^8] \quad (\text{B.5})$$

Note, that $\mu_2^2 t^8$ is subtracted to account for the undesired product of $\mu_2^2 t^4$ and t^4 . The μ_2 transforms under the $\mathbb{Z}_2$ action with a minus sign therefore it must come in a form of a natural invariant $\mu_2^2 t^2$. Therefore, under the $\mathbb{Z}_2$ action, the HWG takes the form:

$$\text{HWG} = \text{PE}[\mu_1 \mu_3 t^2 + \mu_2^2 t^4 + \mu_2^2 t^4 + \mu_2^2 t^6 + t^4 + \mu_2^2 t^8 - \mu_2^2 t^8 - \mu_2^4 t^{12}] \quad (\text{B.6})$$

In summary, after the $\mathbb{Z}_2 \times \mathbb{Z}_2$ action the final HWG takes the form:

$$\text{HWG} = \text{PE}[\mu_1 \mu_3 t^2 + (2\mu_2^2 + 1)t^4 + \mu_2^2 t^6 - \mu_2^4 t^{12}]. \quad (\text{B.7})$$

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