All (4,0): Sigma models with (4,0) off-shell supersymmetry

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ABSTRACT: Off-shell (4, 0) supermultiplets in 2-dimensions are formulated. These are used to construct sigma models whose target spaces are vector bundles over manifolds that are hyperkähler with torsion. The off-shell supersymmetry implies that the complex structures are simultaneously integrable and allows us to write actions using extended superspace and projective superspace, giving an explicit construction of the target space geometries.

KEYWORDS: Extended Supersymmetry, Superspaces

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1 Introduction

The (1,0) supersymmetric sigma model [1] has a target space $\mathcal{M}$ with a metric g and closed 3-form $H = dB$, together with a vector bundle over $\mathcal{M}$ with connection $A_i{}^B{}_C(X)$ and fibre metric $G_{AB}(X)$, where $A, B = 1, \dots, m$ are indices for the m -dimensional fibres. The action can be written in (1,0) superspace with coordinates $x^\mu = (x^+, x^-)$ and θ^- . The action is given in terms of scalar superfields $X^i(x, \theta)$ with $i = 1, \dots, D$ (where D is the real dimension of the target space) and m real fermionic spinor superfields $\Psi^A(x, \theta)$ with $A = 1, \dots, m$. The action is

$$S = \frac{1}{2} \int d^2x d\theta \left(\partial_- X^i (g + B)_{ij} D_+ X^j + G_{AB} \Psi_-^A \nabla_+ \Psi_-^B \right) \quad (1.1)$$

where

$$\nabla_+ \Psi^B = D_+ \Psi^B + D_+ X^i A_i{}^B{}_C \Psi^C \quad (1.2)$$

Only the antisymmetric part of the connection appears in the action so we can take $A_{iAB} \equiv G_{AB} A_i{}^B{}_C$ to be antisymmetric, $A_{iAB} = -A_{iBA}$, and the structure group G of the bundle to be in $O(m)$. The modified connection

$$\hat{A}_i{}^B{}_C = A_i{}^B{}_C + \frac{1}{2} G^{BA} G_{AC,i} \quad (1.3)$$

is a metric connection, $\hat{\nabla}_i G_{AB} = 0$ [1]. As replacing A with $\hat{A}$ in the action (1.1) doesn't change the action (the difference between the two actions vanishes), we can without loss of generality take A to be the metric connection [2], and we drop the hat from A in what follows.

The classical (1,0) sigma model will in fact be $(p, 0)$ supersymmetric provided that $\mathcal{M}$ admits $p - 1$ complex structures $J^{(A)}$ such that the geometry satisfies the conditions [3]

$$J^{(A)t} g J^{(A)} = g, \quad (J^{(A)})^2 = -\mathbb{1}, \quad \nabla^{(+)} J^{(A)} = 0, \quad (1.4)$$

where

$$\nabla^{(+)} := \left(\nabla^{(0)} + \frac{1}{2} g^{-1} H \right) \quad (1.5)$$

is the connection with torsion $g^{il}H_{ljk}$ added to the Levi-Civita connection $\nabla^{(0)}$. In addition it is required that the vector bundle is holomorphic with respect to each of the complex structures, i.e the field strength $F = dA + \frac{1}{2}[A, A]$ is a (1,1) form with respect to each of the complex structures [1–3]. The supersymmetry transformation of the bosonic superfields is

$$\delta X^i = \epsilon_A^+ (J_{(A)})^i_j D_+ X^j, \quad (1.6)$$

and that of the fermi superfields is

$$\delta \Psi^B = -(\delta X^i) A_i^B{}_C \Psi^C. \quad (1.7)$$

If there is a bundle tensor $I^A{}_B(X)$ that satisfies

$$G_{CA} I^A{}_B = G_{BA} I^A{}_C \quad (1.8)$$

and is covariantly constant

$$\nabla_i I^A{}_B \equiv \partial_i I^A{}_B + A_i^A{}_C I^C{}_B - I^A{}_C A_i^C{}_B = 0 \quad (1.9)$$

then the theory has a further fermionic symmetry [1] of the form

$$\delta \Psi_-^A = \zeta_- I^A{}_B \nabla_+ \Psi_-^B \quad (1.10)$$

where ζ_- is a spinorial parameter. The presence of such covariantly constant tensors reduces the structure group of the bundle. The non-trivial irreducible cases are the case in which the structure group is reduced to $U(m/2) \subset O(m)$ (m even) with one matrix I which is a complex structure on the fibres, and the case in which the structure group is reduced to $Sp(m/4) \subset O(m)$ ($m/4$ integral) with three complex structures on the fibres satisfying the quaternion algebra, giving three extra symmetries. However, the fermionic superfields vary into field equations under symmetries of the form (1.10), so that they are on-shell trivial symmetries that have no Noether charge or dynamical consequences, and commute with the usual supersymmetries on-shell. Nonetheless, such on-shell trivial transformations can be useful, as combining them with the supersymmetry transformations (1.7) can under certain conditions give a supersymmetry algebra that closes off-shell [2]. This will be useful here for constructing models with (4,0) supersymmetry off-shell. For each off-shell supersymmetry, there is a complex structure $J^i{}_j$ on the base and a complex structure $I^A{}_B$ on each fibre that combine to form a complex structure on the total space.

We will discuss a special class of (4,0) models in (2,0) superspace and in (4,0) projective superspace. (4,0) models have been discussed in (1,0) superspace, in harmonic superspace and in components in, e.g., [2, 4] and [3, 5]. In the quantum theory, anomaly cancellation requires modifying the geometry, with H modified by Chern-Simons terms so that $dH \propto \text{tr}(F^2 - R^2)$ [1]. The finiteness of (4,0) sigma models has been discussed in [6, 7].

2 (4,0) off-shell supermultiplets

In this section we formulate off-shell (4,0) supermultiplets that generalise the (4,1) supermultiplet introduced in [8] and the (4,4) supermultiplets of [9]. We use a (4,0) superspace with coordinates $x^+, x^-, \theta_a^+, \bar{\theta}^{+a}$ with θ_a^+ being complex. The index $a = 1, 2$ is an $SU(2)$ index, and the two right-handed complex spinorial covariant derivatives $\mathbb{D}_+^a$ satisfy

$$\{\mathbb{D}_{+a}, \bar{\mathbb{D}}_+^b\} = 2i\delta_a^b \partial_{++}, \quad a, b = 1, 2. \quad (2.1)$$

There is then a (4,0) multiplet, obtained by truncating the (4,1) multiplet (A.1), consisting of a pair of scalar (4,0) superfields ϕ, χ satisfying the constraints

$$\begin{aligned} \bar{\mathbb{D}}_+^1 \phi &= 0 = \mathbb{D}_{+2} \phi, & \bar{\mathbb{D}}_+^1 \chi &= 0 = \mathbb{D}_{+2} \chi, \\ \bar{\mathbb{D}}_+^2 \chi &= -i\bar{\mathbb{D}}_+^1 \bar{\phi}, & \bar{\mathbb{D}}_+^2 \phi &= i\bar{\mathbb{D}}_+^1 \bar{\chi}. \end{aligned} \quad (2.2)$$

In addition we introduce a fermi multiplet consisting of a pair of spinor superfields ψ_-, λ_- satisfying the following constraints

$$\begin{aligned} \bar{\mathbb{D}}_+^1 \psi_- &= 0 = \mathbb{D}_{+2} \psi_-, & \bar{\mathbb{D}}_+^1 \lambda_- &= 0 = \mathbb{D}_{+2} \lambda_-, \\ \bar{\mathbb{D}}_+^2 \lambda_- &= -i\bar{\mathbb{D}}_+^1 \bar{\psi}_-, & \bar{\mathbb{D}}_+^2 \psi_- &= i\bar{\mathbb{D}}_+^1 \bar{\lambda}_-. \end{aligned} \quad (2.3)$$

The (4,1) supermultiplet of [8] is very similar: there, the (4,1) superfields $\hat{\phi}, \hat{\chi}$ satisfied the constraint (2.1). Expanding the (4,1) superfields in the extra fermionic coordinate θ^- gives the (4,0) superfields given above. We obtain bosonic (4,0) superfields $\phi = \hat{\phi}|_{\theta^-=0}$ and $\chi = \hat{\chi}|_{\theta^-=0}$ satisfying (2.1) together with fermionic (4,0) superfields $\psi_- = D_- \hat{\phi}|_{\theta^-=0}$ and $\lambda_- = D_- \hat{\chi}|_{\theta^-=0}$ satisfying (2.3). The supersymmetry transformations that follow from the constraints above can be rewritten in (1,0) superspace; see the appendix for details.

(4,1) sigma models constructed with (4,1) superfields $\hat{\phi}^i, \hat{\chi}^i$ were formulated in [8], giving a $4d$ -dimensional geometry with coordinates $\hat{\phi}^i|_{\theta^-=0}$ and $\hat{\chi}^i|_{\theta^-=0}$, where $i = 1, \dots, d$. Then expanding into (4,0) superspace gives coordinate superfields ϕ^i, χ^i and fermionic superfields ψ_-^i, λ_-^i which correspond to sections of the tangent bundle of the target space (tensored with the negative chirality spinor bundle of the worldsheet). We will obtain more general models by allowing ψ_-, λ_- to be sections of an arbitrary vector bundle over the target space, instead of the tangent bundle.

3 (2,0) superspace formulation

The general (2,0) sigma model action can be written in (2,0) superspace as [10]

$$S = \int d^2x d^2\theta \left(k_\alpha \partial_- \varphi^\alpha + \bar{k}_{\bar{\alpha}} \partial_- \bar{\varphi}^{\bar{\alpha}} + e_{\mu\nu} \Lambda_-^\mu \Lambda_-^\nu + G_{\mu\bar{\nu}} \Lambda_-^\mu \bar{\Lambda}_-^{\bar{\nu}} + e_{\bar{\mu}\bar{\nu}} \bar{\Lambda}_-^{\bar{\mu}} \bar{\Lambda}_-^{\bar{\nu}} \right), \quad (3.1)$$

where $G_{\mu\bar{\mu}}$ is the fibre metric, and $e_{\bar{\mu}\bar{\nu}} = \overline{e_{\mu\nu}}$. Expanding in components or (1,0) superfields gives a Hermitian target space metric

$$g_{\alpha\bar{\beta}} = i(\partial_\alpha \bar{k}_{\bar{\beta}} - \partial_{\bar{\beta}} k_\alpha) \quad (3.2)$$

and a B -field which, in a gauge in which $B = B^{(2,0)} + B^{(0,2)}$, has

$$B_{\alpha\beta}^{(2,0)} = i(\partial_\alpha k_\beta - \partial_\beta k_\alpha), \quad (3.3)$$

and $B^{(0,2)}$ is the complex conjugate of this. The fields φ^α are $(2,0)$ chiral scalar superfields

$$\bar{\mathbb{D}}_+ \varphi^\alpha = 0, \quad (3.4)$$

and $\bar{\varphi}^{\bar{\alpha}}$ are their complex conjugates $\bar{\varphi}^{\bar{\alpha}} = (\varphi^\alpha)^*$. The fields Λ_-^μ are $(2,0)$ fermionic chiral spinor superfields

$$\bar{\mathbb{D}}_+ \Lambda_-^\mu = 0, \quad (3.5)$$

and $\bar{\Lambda}_-^{\bar{\mu}}$ are their complex conjugates.

Expanding the $(4,0)$ supermultiplets of the last section into $(2,0)$ superspace, using a similar procedure to that in [8], gives $(2,0)$ superfields. First, it gives chiral $(2,0)$ scalar superfields ϕ, χ that transform under the extra nonmanifest supersymmetries as

$$\bar{Q}_+ \phi = i\bar{\mathbb{D}}_+ \bar{\chi}, \quad \bar{Q}_+ \chi = -i\bar{\mathbb{D}}_+ \bar{\phi} \quad (3.6)$$

In addition it gives chiral fermionic $(2,0)$ spinor superfields ψ_-, λ_- transforming under the extra nonmanifest supersymmetries as

$$\bar{Q}_+ \psi_- = i\bar{\mathbb{D}}_+ \bar{\lambda}_-, \quad \bar{Q}_+ \lambda_- = -i\bar{\mathbb{D}}_+ \bar{\psi}_- \quad (3.7)$$

The action for d $(4,0)$ multiplets must take the form (3.1) when written in $(2,0)$ superspace, with $(2,0)$ chiral superfields $\varphi^\alpha = (\phi^i, \chi^i)$ with $i = 1, \dots, d$ and $\alpha = 1, \dots, 2d$ and fermionic chiral superfields $\Lambda_-^\mu = (\psi_-^a, \lambda_-^a)$ with $a = 1, \dots, n$ and $\mu = 1, \dots, 2n$ for some n . For the complex conjugate superfields, we use the notation $\bar{\varphi}^{\bar{\alpha}} = (\bar{\phi}^i, \bar{\chi}^i)$ and $\bar{\Lambda}_-^{\bar{\mu}} = (\bar{\psi}_-^a, \bar{\lambda}_-^a)$. The transformations can then be written as

$$\bar{\delta}\varphi = -\bar{\epsilon}^+ \hat{\sigma}_2 \bar{\mathbb{D}}_+ \bar{\varphi}, \quad \bar{\delta}\Lambda_- = -\bar{\epsilon}^+ \hat{\sigma}_2 \bar{\mathbb{D}}_+ \bar{\Lambda}_-, \quad (3.8)$$

where $\hat{\sigma}_2$ is $\hat{\sigma}_2 = \sigma_2 \otimes \mathbb{1}_{d \times d}$ when acting on the bosonic superfields and is $\hat{\sigma}_2 = \sigma_2 \otimes \mathbb{1}_{n \times n}$ when acting on the fermionic superfields, with σ_2 the usual Pauli matrix.

Expanding into $(1,0)$ superspace as outlined in the appendix gives three complex structures for the target space $\mathcal{M}$ and three complex structures for the fibres, and both sets take the form

$$\mathbb{I}^{(1)} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \otimes \mathbb{1}, \quad \mathbb{I}^{(2)} = \begin{pmatrix} 0 & i\hat{\sigma}_2 \\ i\hat{\sigma}_2 & 0 \end{pmatrix}, \quad \mathbb{I}^{(3)} = \begin{pmatrix} 0 & -\hat{\sigma}_2 \\ \hat{\sigma}_2 & 0 \end{pmatrix}. \quad (3.9)$$

where $\mathbb{1} = \mathbb{1}_{d \times d}$ when acting on the bosonic superfields and $\mathbb{1} = \mathbb{1}_{n \times n}$ when acting on the fermionic superfields. The complex structure $\mathbb{I}^{(3)}$ is the one corresponding to the supersymmetry transformations (3.8). The complex structures acting on the bosons and the complex structures acting on the fermions each satisfy the quaternion algebra

$$\mathbb{I}^{(I)} \mathbb{I}^{(J)} = -\delta^{IJ} + \epsilon^{IJK} \mathbb{I}^{(K)}. \quad (3.10)$$

For each label I , the complex structure on $\mathcal{M}$ and the complex structure on the fibres together represent a complex structure on the total space. Together the complete set defines a quaternionic structure on the total space.

The vector potential in (3.1) has components $k_\alpha = (k_{\phi^i}, k_{\chi^i})$ and the terms involving k will be invariant if k satisfies

$$\begin{aligned} g_{\phi^i \bar{\phi}^j} - g_{\chi^j \bar{\chi}^i} &= \bar{k}_{\bar{\phi}^i, \phi^j} - k_{\phi^j, \bar{\phi}^i} - \bar{k}_{\bar{\chi}^j, \chi^i} + k_{\chi^i, \bar{\chi}^j} = 0 \\ g_{\phi^i \bar{\chi}^j} &= \bar{k}_{\bar{\chi}^i, \phi^j} - k_{\phi^i, \bar{\chi}^j} = 0, \end{aligned} \quad (3.11)$$

which is the condition that the metric is hermitian with respect to $\mathbb{I}^{(3)}$, as well as

$$\begin{aligned} \frac{1}{2} \left(k_{\phi^j, \bar{\chi}^k} - \bar{k}_{\bar{\chi}^j, \phi^k} \right)_{,\bar{\beta}} - \bar{k}_{\bar{\beta}, \phi^j} \bar{\chi}^k &= 0 \\ \frac{1}{2} \left(\bar{k}_{\bar{\phi}^k, \phi^j} + k_{\chi^k, \bar{\chi}^j} + \bar{k}_{\bar{\chi}^j, \chi^k} + k_{\phi^j, \bar{\phi}^k} \right)_{,\bar{\beta}} - \bar{k}_{\bar{\beta}, \phi^j} \bar{\phi}^k - \bar{k}_{\bar{\beta}, \chi^k} \bar{\chi}^j &= 0 \\ \frac{1}{2} \left(k_{\chi^j, \bar{\phi}^k} - \bar{k}_{\bar{\phi}^j, \chi^k} \right)_{,\bar{\beta}} - \bar{k}_{\bar{\beta}, \chi^j} \bar{\phi}^k &= 0, \end{aligned} \quad (3.12)$$

which represents the covariant constancy of the complex structures. These conditions are the same as those derived for the $(4, 1)$ multiplet in [8].

Turning to the fermionic part of (3.1), the variation of the terms involving Λ gives

$$G_{\psi^a \bar{\lambda}^b} = 0, \quad G_{\psi^a \bar{\psi}^b} - G_{\lambda^b \bar{\lambda}^a} = 0 \quad (3.13)$$

expressing the hermiticity of the fibre metric, and

$$\begin{aligned} G_{\psi^a \bar{\nu}, \bar{\phi}^i} - 2e_{\bar{\nu} \bar{\lambda}^a, \chi^i} &= 0, & G_{\psi^a \bar{\nu}, \bar{\chi}^i} + 2e_{\bar{\nu} \bar{\lambda}^a, \phi^i} &= 0 \\ e_{\mu\nu, \chi^j} \bar{\phi}^i &= 0, & e_{\mu\nu, \phi^j} \bar{\phi}^i + e_{\mu\nu, \chi^i} \bar{\chi}^j &= 0, & e_{\mu\nu, \phi^j} \bar{\chi}^i &= 0, \end{aligned} \quad (3.14)$$

We use the notation that subscripts μ or $\bar{\mu}$ represent subscripts Λ^μ or $\bar{\Lambda}^\mu$.

When the vector bundle is the tangent bundle, the model has $(4, 1)$ supersymmetry and the complex structure $\mathbb{I}^{(3)}$ on the fibres is that resulting from the complex structure on the base. Then G and e are given in terms of the potential k by $e_{\alpha\beta} = 2k_{[\alpha, \beta]}$, $G_{\alpha\bar{\beta}} = k_{\alpha, \bar{\beta}} + \bar{k}_{\bar{\beta}, \alpha}$ and as a result (3.13) and (3.14) can be shown to agree with (3.11) and (3.12).

4 Projective superspace

Projective superspace was introduced to construct actions with manifest extended supersymmetry [11] and has been studied in various dimensions; see, e.g., [12–18]. A similar but distinct approach involves harmonic superspace; see [19–22] and references therein.

In [8], projective superspace was used to formulate models with $(4, 1)$ off-shell supersymmetry. We now adapt this construction for the $(4, 0)$ models discussed in the previous sections. $(4, 0)$ projective superspace has the $(4, 0)$ superspace coordinates x^μ, θ^a together with the complex projective coordinate ζ on $\mathbb{CP}^1$. We consider $(4, 0)$ projective superfields $\eta^i(x, \theta, \zeta), \rho^a(x, \theta, \zeta)$ satisfying

$$\nabla_+ \eta^i = 0, \quad \check{\nabla}_+ \eta^i = 0, \quad \nabla_+ \rho_-^a = 0, \quad \check{\nabla}_+ \rho_-^a = 0, \quad (4.1)$$

where

$$\begin{aligned}\nabla_+ &:= \mathbb{D}_{+1} + \zeta \mathbb{D}_{+2} , \\ \check{\nabla}_+ &:= \bar{\mathbb{D}}_+^1 - \zeta^{-1} \bar{\mathbb{D}}_+^2 .\end{aligned}\tag{4.2}$$

The conjugation acting on meromorphic functions of $f(\zeta)$ by

$$f(\zeta) \rightarrow \check{f}(\zeta)\tag{4.3}$$

is given by the composition of complex conjugation

$$f(\zeta) \rightarrow: f^*(\bar{\zeta}) \equiv (f(\zeta))^*\tag{4.4}$$

and the antipodal map

$$\zeta \rightarrow -\bar{\zeta}^{-1}\tag{4.5}$$

so that

$$\check{f}(\zeta) = f^*(-\zeta^{-1}) .\tag{4.6}$$

The $(4, 0)$ action is $\int[\mathcal{L}_b + \mathcal{L}_f]$, where the bosonic Lagrangian $\mathcal{L}_b$ is formally identical to the bosonic part of the reduced $(4, 1)$ action

$$\int \mathcal{L}_b := i \oint_C \frac{d\zeta}{2\pi i \zeta} \Delta_+ \check{\Delta}_+ \left(\lambda_i(\eta, \check{\eta}) \partial_{=\eta^i} - \check{\lambda}_i(\eta, \check{\eta}) \partial_{=\check{\eta}^i} \right) ,\tag{4.7}$$

and the fermionic Lagrangian is

$$\int \mathcal{L}_f := i \oint_C \frac{d\zeta}{2\pi i \zeta} \Delta_+ \check{\Delta}_+ \left(\rho_-^a h_{ab} \rho_-^b + \rho_-^a h_{a\bar{b}} \check{\rho}_-^{\bar{b}} + \check{\rho}_-^{\bar{a}} \check{h}_{\bar{a}b} \rho_-^b + \check{\rho}_-^{\bar{a}} \check{h}_{\bar{a}\bar{b}} \check{\rho}_-^{\bar{b}} \right) ,\tag{4.8}$$

where h_{ab} and $\check{h}_{\bar{a}\bar{b}}$ are antisymmetric and related by conjugation. The metric $H_{a\bar{b}} := h_{a\bar{b}} - \check{h}_{\bar{b}a}$ is hermitian. The contour is taken to encircle the origin and the measure is formed from two operators orthogonal to the ones in (4.2), and may be replaced by the $(2, 0)$ measure when acting on functions of η and $\bar{\eta}$:

$$\Delta_+ \check{\Delta}_+ \rightarrow \mathbb{D}_+ \bar{\mathbb{D}}_+ .\tag{4.9}$$

For (4.7), (4.8) to give the $(2, 0)$ action (3.1) with extended supersymmetry (3.6) and (3.7), we choose

$$\begin{aligned}\eta^i &= \bar{\phi}^i + \zeta \chi^i , & \bar{\eta}^i &:= \check{\eta}^i = \phi^i - \zeta^{-1} \bar{\chi}^i \\ \rho_-^a &= \bar{\psi}_-^a + \zeta \lambda_-^a , & \bar{\rho}_-^a &:= \check{\rho}_-^a = \psi_-^a - \zeta^{-1} \bar{\lambda}^a\end{aligned}\tag{4.10}$$

In the bosonic sector the metric and B -field is expressible in terms of the vector potentials $k_\alpha = (k_{\phi^i}, k_{\chi^i})$ as in (3.2), (3.3). These are given by

$$\begin{aligned}k_{\phi^i} &= - \oint_C \frac{d\zeta}{2\pi i \zeta} \check{\lambda}_i , & \bar{k}_{\bar{\phi}^i} &= \oint_C \frac{d\zeta}{2\pi i \zeta} \lambda_i \\ k_{\chi^i} &= \oint_C \frac{d\zeta}{2\pi i \zeta} \zeta \lambda_i , & \bar{k}_{\bar{\chi}^i} &= \oint_C \frac{d\zeta}{2\pi i \zeta} \zeta^{-1} \check{\lambda}_i .\end{aligned}\tag{4.11}$$

The general properties of a function $f(\eta, \check{\eta})$,

$$\begin{aligned} f_{,\phi^i} &= f_{,\bar{\eta}^i}, & f_{,\chi^i} &= \zeta f_{,\eta^i} \\ f_{,\bar{\phi}^i} &= f_{,\eta^i}, & f_{,\bar{\chi}^i} &= -\zeta^{-1} f_{,\bar{\eta}^i}, \end{aligned} \quad (4.12)$$

implies that the potentials satisfy

$$\begin{aligned} k_{\phi^i, \bar{\phi}^j} + \bar{k}_{\bar{\chi}^i, \chi^j} &= 0, \\ k_{\phi^i, \bar{\chi}^j} - \bar{k}_{\bar{\chi}^i, \phi^j} &= 0, \end{aligned} \quad (4.13)$$

which is sufficient for, but does not imply, the relations (3.11) and (3.12). Geometrically, these conditions, together with the hermiticity (3.11), imply that for each I the complex structure $\mathbb{I}^{(I)}$ is compatible with the B -field

$$(\mathbb{I}^{(I)})^t B^{(1,1)} \mathbb{I}^{(I)} = B^{(1,1)}, \quad (4.14)$$

where $B^{(1,1)}$

$$B_{\alpha\bar{\alpha}}^{(1,1)} = i (\bar{k}_{\bar{\alpha}, \alpha} + k_{\alpha, \bar{\alpha}}). \quad (4.15)$$

and is related to $B^{(2,0)} + B^{(0,2)}$ by a gauge transformation. It follows that

$$(\mathbb{I}^{(I)})^t E \mathbb{I}^{(I)} = E, \quad (4.16)$$

where

$$E := g + B^{(1,1)}. \quad (4.17)$$

Turning now to the fermionic sector, we find that the fibre metric and e -field are given by

$$\begin{aligned} G_{\psi^a \bar{\psi}^b} &= - \oint_C \frac{d\zeta}{2\pi i \zeta} H_{\bar{a}b}, & G_{\psi^a \bar{\chi}^b} &= -2 \oint_C \frac{d\zeta}{2\pi i \zeta} \zeta^{-1} h_{\bar{a}b}, \\ G_{\lambda^a \bar{\psi}^b} &= 2 \oint_C \frac{d\zeta}{2\pi i \zeta} \zeta h_{ab}, & G_{\lambda^a \bar{\chi}^b} &= - \oint_C \frac{d\zeta}{2\pi i \zeta} H_{\bar{a}b}, \\ e_{\psi^a \psi^b} &= \oint_C \frac{d\zeta}{2\pi i \zeta} h_{\bar{a}b}, & e_{\lambda^a \psi^b} &= \frac{1}{2} \oint_C \frac{d\zeta}{2\pi i \zeta} \zeta H_{\bar{a}b}, \\ e_{\lambda^a \lambda^b} &= \oint_C \frac{d\zeta}{2\pi i \zeta} \zeta^2 h_{ab}, & e_{\bar{\psi}^a \bar{\psi}^b} &= \oint_C \frac{d\zeta}{2\pi i \zeta} h_{ab}, \\ e_{\bar{\psi}^a \bar{\chi}^b} &= -\frac{1}{2} \oint_C \frac{d\zeta}{2\pi i \zeta} \zeta^{-1} H_{\bar{a}b}, & e_{\bar{\chi}^a \bar{\chi}^b} &= \oint_C \frac{d\zeta}{2\pi i \zeta} \zeta^{-2} h_{\bar{a}b}. \end{aligned} \quad (4.18)$$

The expressions in (4.18) satisfy all the relations in (3.13) and (3.14). Proving this requires use of derivatives of the relations for a function $f(\eta, \check{\eta})$ in (4.12):

$$\begin{aligned} f_{,\phi^i \bar{\phi}^j} &= f_{,\bar{\eta}^i \eta^j}, & f_{,\chi^i \bar{\chi}^j} &= -f_{,\eta^i \bar{\eta}^j} \\ f_{,\chi^i \bar{\phi}^j} &= \zeta f_{,\eta^i \bar{\eta}^j}, & f_{,\phi^i \bar{\chi}^j} &= -\zeta^{-1} f_{,\bar{\eta}^i \eta^j}. \end{aligned} \quad (4.19)$$

This relation also means that all the geometric fields $U = (G_{\mu\bar{\nu}}, e_{\mu\nu}, \bar{e}_{\bar{\mu}\bar{\nu}})$ in (4.18) will obey

$$U_{,\phi^i\bar{\phi}^j} + U_{,\chi^j\bar{\chi}^i} = 0, \quad (4.20)$$

in addition to (3.14). It serves as a check that, in the special case when the vector bundle is the tangent bundle and model has $(4,1)$ supersymmetry, the equation (4.20) follows from (4.13).

We stress that the conditions (4.20), although reminiscent of the conditions (4.13) in the bosonic sector, are not all needed for the invariance of the fermionic part of the action, (3.13) and (3.14): only (4.20) for $U = e_{\mu\nu}$ is required for invariance. In section 3, we constructed the general sigma model for our off-shell $(4,0)$ multiplets. The projective superspace action given here only gives a special subclass of these models. Finally, we note that we made a particular choice of projective superfield with the ansatz (4.10), and other choices with other ζ dependence give a wider class of models, typically involving left- or right-moving multiplets, and/or auxiliary fields.

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A $(1,0)$ superspace form of $(4,0)$ transformations

The $(4,0)$ multiplet (2.1) consists of a pair of $(4,0)$ superfields ϕ, χ satisfying the constraints

$$\begin{aligned} \bar{\mathbb{D}}_+^1 \phi &= 0 = \mathbb{D}_{+2} \phi, & \bar{\mathbb{D}}_+^1 \chi &= 0 = \mathbb{D}_{+2} \chi, \\ \bar{\mathbb{D}}_+^2 \chi &= -i \bar{\mathbb{D}}_+^1 \bar{\phi}, & \bar{\mathbb{D}}_+^2 \phi &= i \bar{\mathbb{D}}_+^1 \bar{\chi}. \end{aligned} \quad (A.1)$$

The supersymmetry transformations can be put into the form (1.6) by expanding in $(1,0)$ superspace. The $(4,0)$ multiplet in (A.1) can be formulated in $(1,0)$ superspace by defining

$$\phi|_{\theta_2^+=0, \theta_1^+=\bar{\theta}_1^+} = \tilde{\phi}, \quad \chi|_{\theta_2^+=0, \theta_1^+=\bar{\theta}_1^+} = \tilde{\chi} \quad (A.2)$$

The constraints (A.1) then determine the terms in ϕ, χ of higher order in $\theta_2, \theta_1^+ - \bar{\theta}_1^+$ in terms of $\tilde{\phi}, \tilde{\chi}$ and give the supersymmetry transformations under the non-manifest supersymmetries. We define four real $(4,0)$ superspace spinor derivatives D_+ and $\check{D}_+^{(A)}$, $A = 1, 2, 3$ by

$$\begin{aligned} \mathbb{D}_{+1} &=: D_+ - i \check{D}_+^{(1)} \\ \mathbb{D}_{+2} &=: \check{D}_+^{(2)} - \check{D}_+^{(3)}, \end{aligned} \quad (A.3)$$

Then D_+ is the $(1,0)$ superspace spinor derivative and the three differential operators $\check{D}_+^{(A)}$, $A = 1, 2, 3$ determine the generators of nonmanifest supersymmetries $Q_+^{(A)}$ via the

constraint (A.1)

$$\check{D}_+^{(A)} \phi \Big|_{\theta_1^+ = \bar{\theta}_1^+, \theta_2^+ = 0} = Q_+^{(A)} \tilde{\phi}, \quad (\text{A.4})$$

$$\check{D}_+^{(A)} \chi \Big|_{\theta_1^+ = \bar{\theta}_1^+, \theta_2^+ = 0} = Q_+^{(A)} \tilde{\chi}, \quad (\text{A.5})$$

resulting in the following relation for the extended supersymmetries

$$Q_+^{(A)} \begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \\ \bar{\phi} \\ \bar{\chi} \end{pmatrix} =: \mathbb{J}^{(A)} D_+ \begin{pmatrix} \tilde{\phi} \\ \tilde{\chi} \\ \bar{\phi} \\ \bar{\chi} \end{pmatrix}. \quad (\text{A.6})$$

where the complex structures

$$\mathbb{J}^{(A)} = \mathbb{I}^{(A)} \otimes \mathbb{1}_{d \times d} \quad (\text{A.7})$$

with

$$\mathbb{I}^{(1)} = \begin{pmatrix} i\mathbb{1} & 0 \\ 0 & -i\mathbb{1} \end{pmatrix}, \quad \mathbb{I}^{(2)} = \begin{pmatrix} 0 & i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}, \quad \mathbb{I}^{(3)} = \begin{pmatrix} 0 & -\sigma_2 \\ \sigma_2 & 0 \end{pmatrix} \quad (\text{A.8})$$

are constant in this coordinate system and satisfy the quaternion algebra

$$\mathbb{J}^{(A)} \mathbb{J}^{(B)} = -\delta^{AB} + \epsilon^{ABC} \mathbb{J}^{(C)}. \quad (\text{A.9})$$

Then this gives transformations for $\tilde{\phi}, \tilde{\chi}$ of the form (1.6). The (1,0) superspace formulation of the fermionic superfields can be found similarly.

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