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Carving out the end of the world or (superconformal bootstrap in six dimensions)

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Abstract: We bootstrap $\mathcal{N} = (1, 0)$ superconformal field theories in six dimensions, by analyzing the four-point function of flavor current multiplets. Assuming E_8 flavor group, we present universal bounds on the central charge C_T and the flavor central charge C_J. Based on the numerical data, we conjecture that the rank-one E-string theory saturates the universal lower bound on C_J, and numerically determine the spectrum of long multiplets in the rank-one E-string theory. We comment on the possibility of solving the higher-rank E-string theories by bootstrap and thereby probing M-theory on $AdS_7 \times S^4/\mathbb{Z}_2$.

Keywords: Conformal Field Models in String Theory, Conformal Field Theory, Extended Supersymmetry, M-Theory

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Dedicated to John Schwarz on the occasion of his 75th birthday.

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Contents

1 Introduction and summary **2**

2 Review of superconformal representation theory **3**

3 Four-point function of half-BPS operators **5**

4 Superconformal blocks **7**
- 4.1 Long multiplets 8
- 4.2 Short multiplets 9

5 Flavor symmetry **11**

6 Central charges **12**
- 6.1 Central charge C_T 13
- 6.2 Flavor central charge C_J 15

7 Semidefinite programming **16**

8 Results **19**
- 8.1 Free hypermultiplet: a check 19
- 8.2 E-string theories 19

9 Outlook **24**

A Bosonic conformal blocks **27**
- A.1 Expansion in Jack polynomials 28
- A.2 Recursive representation 29
- A.3 Diagonal limit 30
- A.4 Recursion on transverse derivatives 31

B Superconformal Ward identities **31**

C Crossing equation for $b(u, v)$ **32**

D Relating central charges to OPE coefficients **34**
- D.1 C_T to $\lambda^2_{\mathcal{B}[0]_0}$ 34
- D.2 C_J to $\lambda^2_{\mathcal{D}[2]}$ 35

E The central charge C_T of the three-derivative fermion **36**

F Analytic examples of solutions to crossing **39**
- F.1 Generalized free fields 39
- F.2 Free hypermultiplet 40

G Details on numerics **43**

JHEP08(2017)128

1 Introduction and summary

Conformal field theories in six dimensions parent a plethora of conformal field theories in lower dimensions through compactification. A primal example is the compactification of $\mathcal{N} = (2, 0)$ theories on Riemann surfaces to class $\mathcal{S}$ theories in four dimensions [1, 2]. While no argument exists for the necessity of supersymmetry, all known interacting conformal field theories in six dimensions are in fact superconformal.[1] It follows from representation theory that these interacting theories have neither marginal nor relevant deformations [5–10]. Moreover, no known interacting theory admits a classical limit (hence essentially strongly coupled), or arises in the infrared limit of renormalization group flows from a Lagrangian theory. For these reasons, only a scarcity of tools exists for extracting physical quantities in these theories.

The conformal bootstrap aims to extract physical observables in strongly coupled conformal field theories, using only the basic assumptions: unitarity, (super)conformal symmetry, and the associativity of operator product expansions (OPEs) [11–14]. The past decade has seen substantial developments of numerical bootstrap techniques — most notably the linear functional method — in constraining conformal field theories [15–44]. In particular, the bootstrap has been applied to $\mathcal{N} = (2, 0)$ superconformal symmetry in six dimensions, and substantial evidence was found to support the conjecture that the bootstrap bound on the central charge is saturated by the A_1 theory, which arises in the infrared limit of the worldvolume theory of two coinciding M5 branes [34]. For theories that saturate the bootstrap bounds, the linear functional method determines the scaling dimensions and OPE coefficients of all the operators that contribute to the correlators under analysis [20]. By incorporating more and more correlators, the conformal bootstrap potentially solves these theories completely.[2]

In this paper, we apply the conformal bootstrap to study yet another interesting class of six-dimensional conformal field theories — the E-string theories — which arise in the infrared limit of the worldvolume theory of M5 branes lying inside an "end-of-the-world" M9 brane [45, 46]. These $\mathcal{N} = (1, 0)$ theories have tensionless string excitations charged under an E_8 flavor symmetry, and are related to various lower-dimensional conformal field theories. For instance, upon compactification on a circle with the presence of E_8 Wilson lines, they reduce to Seiberg's E_n theories in five dimensions [47–49]. Compactifying on Riemann surfaces lands us on various $\mathcal{N} = 1$ theories in four dimensions [50, 51].

There is a larger class of $\mathcal{N} = (1, 0)$ theories coming from F-theory constructions that contains the E-string theories as a subclass [52–55]. In order to pinpoint specific theories on the solution space of bootstrap, we need to know the values of certain physical observables. One physical observable that has been computed in known six-dimensional theories is the anomaly polynomial [56–62]. By superconformal symmetry, the anomaly

[1] A class of non-supersymmetric AdS_7 vacua in the massive type IIA supergravity was recently proposed as potential duals to non-supersymmetric 6d CFTs [3] (we thank Xi Yin for pointing this out to us). However, it is unclear whether those vacua are stable. The proposal also violates the strong version of the weak gravity conjecture [4].

[2] The mixed correlator bootstrap refines the constraints on the space of unitary conformal field theories [25, 38, 39, 43].

JHEP08(2017)128

polynomial uniquely fixes both the central charge C_T and flavor central charge C_J, which are in turn related to certain OPE coefficients [63–65]. The precise relation between C_J and the 't Hooft anomaly coefficients should appear in [66], and the relation for C_T was determined in [65, 67, 68].

Employing numerical bootstrap techniques, we analyze the four-point function of scalar superconformal primaries in the E_8 flavor current multiplets. Based on the results, we propose the following conjecture:

Conjecture 1 *The rank-one E-string theory has the minimal flavor central charge* $C_J = 150$ *among all unitary interacting superconformal field theories in six dimensions with an* E_8 *flavor group.*

We emphasize to the reader that the true virtue of this conjecture is *not* that we can compute C_J by bootstrap, but rather the fact that if the rank-one E-string theory indeed saturates the bootstrap bound, then the entire OPEs between the flavor current multiplets can be determined (up to signs) by the linear functional method. This would be invaluable input towards a full solution of the rank-one E-string theory by the conformal bootstrap. We shall comment on the possibility of solving the higher-rank E-string theories and thereby probing the dual M-theory on $\mathrm{AdS}_7 \times \mathrm{S}^4/\mathbb{Z}_2$.

The organization of this paper is as follows. Section 2 reviews the superconformal representation theory of the $\mathcal{N} = (1,0)$ algebra in six dimensions. In sections 3 and 4, we write down the general form of the four-point function involving $\frac{1}{2}$-BPS scalars in flavor current multiplets that solves the superconformal Ward identities, and determine the superconformal blocks. Section 5 explains how to introduce non-abelian flavor symmetry. In section 6, we relate the central charge C_T and flavor central charge C_J to certain coefficients in the OPEs between flavor current multiplet scalars. In section 7, we review the linear functional method which turns the problem of bounding OPE coefficients to a problem in semidefinite programming. Section 8 presents the numerical bounds and their physical implications. Section 9 discusses the future outlook.

2 Review of superconformal representation theory

The six-dimensional $\mathcal{N} = (1,0)$ superconformal algebra is $\mathfrak{osp}(8|2)$, which contains a bosonic subalgebra $\mathfrak{so}(2,6) \times \mathfrak{su}(2)_R$. There are sixteen fermonic generators: eight supercharges Q^A_α and eight superconformal supercharges S^α_A, where $\alpha = 1, \cdots, 4$ and $A = 1, 2$ are the $\mathfrak{so}(6)$ and $\mathfrak{su}(2)_R$ spinor indices, respectively. Superconformal primaries are operators that are annihilated by all the superconformal supercharges S^α_A. A highest weight state of $\mathfrak{osp}(8^*|2)$ is a superconformal primary that is also a highest weight state of the maximal compact subalgebra $\mathfrak{so}(2) \times \mathfrak{so}(6) \times \mathfrak{su}(2)_R$. Representations of the superconformal algebra are generated by successively acting the supercharges Q^A_α and the lowering generators of $\mathfrak{so}(6) \times \mathfrak{su}(2)_R$ on the highest weight states. While some descendants of a highest weight state can appear to have zero norm, in unitary theories, they must be decoupled, and the shortened multiplets are referred to as short multiplets.

JHEP08(2017)128

Each superconformal multiplet can be labeled by the charges $\Delta, h_1, h_2, h_3, J_R$ of its highest weight state under the Cartan generators of $\mathfrak{so}(2) \times \mathfrak{so}(6) \times \mathfrak{su}(2)_R$, where h_1, h_2, h_3 are the charges under the subgroup $\mathfrak{so}(2)^3 \subset \mathfrak{so}(6)$. All the charges are real for unitary representations of the Lorentzian conformal algebra $\mathfrak{so}(2,6) \times \mathfrak{su}(2)_R$. The short representations are classified into $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ types, satisfying the following relations [5, 6, 8, 9],

$$\begin{aligned}
\mathcal{A}: &\quad \Delta = 4J_R + \frac{c_1}{2} + c_2 + \frac{3c_3}{2} + 6, \\
\mathcal{B}: &\quad \Delta = 4J_R + \frac{c_1}{2} + c_2 + 4, && c_3 = 0, \\
\mathcal{C}: &\quad \Delta = 4J_R + \frac{c_1}{2} + 2, && c_2 = c_3 = 0, \\
\mathcal{D}: &\quad \Delta = 4J_R, && c_1 = c_2 = c_3 = 0,
\end{aligned} \tag{2.1}$$

where c_1, c_2 and c_3 are the Dynkin labels of $\mathfrak{su}(4)$ which is related to the h_1, h_2 and h_3 by

$$h_1 = \frac{1}{2}c_1 + c_2 + \frac{1}{2}c_3, \quad h_2 = \frac{1}{2}c_1 + \frac{1}{2}c_3, \quad h_3 = \frac{1}{2}c_1 - \frac{1}{2}c_3. \tag{2.2}$$

The $\mathcal{D}$-type highest weight states are annihilated by the four supercharges with positive R-charge, and are therefore $\frac{1}{2}$-BPS. The $\mathcal{A}$-, $\mathcal{B}$-, and $\mathcal{C}$-type multiplets always contain BPS operators, although their highest weight states are not BPS. The long representations satisfy the inequality

$$\mathcal{L}: \Delta > 4J_R + \frac{c_1}{2} + c_2 + \frac{3c_3}{2} + 6. \tag{2.3}$$

Let us denote the multiplets by[3]

$$\mathcal{X}[\Delta; c_1, c_2, c_3; 2J_R], \quad \mathcal{X} = \mathcal{L}, \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}. \tag{2.4}$$

Due to OPE selection rules, later we only have to consider multiplets whose superconformal primaries are in the symmetric rank-ℓ representation of $\mathfrak{so}(6)$. We denote such representations by

$$\mathcal{X}[2J_R]_{\Delta,\ell} = \mathcal{X}[\Delta; 0, \ell, 0; 2J_R]. \tag{2.5}$$

The Δ, ℓ subscripts for $\mathcal{D}$-type multiplets and the Δ subscript for $\mathcal{B}$-type will be omitted since their values are fixed by (2.1) and (2.5).

Important short multiplets. We give names to certain special short multiplets, some of which contain conserved currents.

- Identity multiplet $\mathcal{D}[0]$: this multiplet contains only the identity operator (vacuum state).
- Hypermultiplet $\mathcal{D}[1]$: contains two complex scalars and one Weyl spinor.
- Flavor current multiplet $\mathcal{D}[2]$: contains conserved currents transforming in the adjoint of a flavor symmetry, and their supertners.

[3] We use $2J_R$ since it is the Dynkin label of $\mathfrak{su}(2)_R$.

JHEP08(2017)128

- Stress tensor multiplet $\mathcal{B}[0]_0$: contains the R-symmetry currents, the stress tensor and their superpartners.
- Higher spin multiplet $\mathcal{B}[0]_\ell$ for $\ell > 0$: contains a spin-$(\ell+2)$ higher spin conserved current and and their superpartners. These multiplets generally live in a decoupled free subsector [69–71].

3 Four-point function of half-BPS operators

In this section, we consider the four-point function of the scalar superconformal primaries in the $\frac{1}{2}$-BPS multiplet $\mathcal{D}[k]$, and review the constraints from superconformal symmetry [72]. The $\frac{1}{2}$-BPS condition implies that this four-point function uniquely fixes the entire set of four-point functions of the (primary or descendant) operators in $\mathcal{D}[k]$.[4] Although we are interested in $\mathcal{N} = (1,0)$ in six dimensions, the setup is the same for superconformal field theories in other dimensions where the R-symmetry is $\mathfrak{su}(2)_R$, namely, $\mathcal{N} = 1$ in five dimensions and $\mathcal{N} = 3$ in three dimensions.[5] Hence we keep the spacetime dimension general and write it as $d = 2(\epsilon+1)$.

The scalar superconformal primaries form a spin-$\frac{k}{2}$ representation of $\mathfrak{su}(2)_R$, and their weight is fixed by the BPS condition $\Delta = \epsilon k$. The scalars can be written as $\mathcal{O}_{A_1\cdots A_k}(x)$, which is a symmetric rank-k tensor of the fundamental representation of $\mathfrak{su}(2)_R$, $A_i = 1, 2$. We can contract the indices with auxiliary variables Y^A to form an operator $\mathcal{O}(x, Y)$ that has homogenous degree $(-\epsilon k, k)$. The four point function of $\mathcal{O}(x, Y)$ is then a homogenous degree $(-4\epsilon k, 4k)$ function, and is polynomial in Y^A. Therefore it must take the form

$$\begin{aligned}\langle \mathcal{O}(x_1,Y_1)\mathcal{O}(x_2,Y_2)\mathcal{O}(x_3,Y_3)\mathcal{O}(x_4,Y_4)\rangle &= \left(\frac{(Y_1\cdot Y_2)(Y_3\cdot Y_4)}{x_{12}^{2\epsilon}x_{34}^{2\epsilon}}\right)^k G(u,v;w),\\ G(u,v;w) &= G_0(u,v) + G_1(u,v)w^{-1} + \cdots + G_k(u,v)w^{-k},\end{aligned} \tag{3.1}$$

where the cross ratios u, v, and w are defined as[6]

$$\begin{aligned} u &= \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \quad w = \frac{(Y_1\cdot Y_2)(Y_3\cdot Y_4)}{(Y_1\cdot Y_4)(Y_2\cdot Y_3)},\\ x_{12}^2 &= (x_1-x_2)^2, \qquad Y_1\cdot Y_2 = Y_1^A Y_2^B \epsilon_{BA}.\end{aligned} \tag{3.2}$$

As all four external scalars are identical, the invariance of (3.1) under $(x_1, Y_1) \leftrightarrow (x_3, Y_3)$ leads to the crossing symmetry constraint

$$G(u,v;w) = \left(\frac{u^\epsilon}{v^\epsilon w}\right)^k G(v,u;w^{-1}). \tag{3.3}$$

[4] The superfield for a $\frac{1}{2}$-BPS multiplet only depends on four fermionic coordinates (half the number of fermionic coordinates in full superspace). The four-point function of such superfields depends on sixteen fermionic coordinates, which is the same as the number of fermionic generators in the superconformal algebra. Hence the four-point function of the superfields can be obtained by supersymmetrizing the four-point function of the superconformal primaries. There is no extra constraint coming from the crossing symmetry of the four-point functions of superconformal descendants.

[5] Our setup does not apply to $\mathcal{N} = 2$ in four dimensions. In particular, such a theory has a protected subsector corresponding to a two-dimensional chiral algebra [29, 72].

[6] The variables Y_i^A satisfy the identity $(Y_1\cdot Y_2)(Y_3\cdot Y_4) - (Y_1\cdot Y_3)(Y_2\cdot Y_4) + (Y_1\cdot Y_4)(Y_2\cdot Y_3) = 0$.

JHEP08(2017)128

Similarly, the invariance of (3.1) under $(x_1, Y_1) \leftrightarrow (x_2, Y_2)$ leads to the constraint

$$G(u,v;w) = G\left(\frac{u}{v}, \frac{1}{v}; -\frac{w}{w+1}\right). \tag{3.4}$$

The four-point function is further constrained by the superconformal Ward identities, which we review in appendix B. They were solved in [72], and the solutions are parametrized by $k-2$ functions $b_n(u,v)$,

$$G(u,v;w) = \sum_{n=0}^{k-2} u^{(n+2)\epsilon} \Delta_\epsilon \left[(v+uw^{-1})(1+w^{-1}) - w^{-1}\right] \left(1+w^{-1}\right)^n b_n(u,v), \tag{3.5}$$

where the differential operator Δ_ϵ is defined as

$$\begin{aligned} D_\epsilon &= u\frac{\partial^2}{\partial u^2} + v\frac{\partial^2}{\partial v^2} + (u+v-1)\frac{\partial^2}{\partial u \partial v} + (1+\epsilon)\left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right), \\ \Delta_\epsilon &= (D_\epsilon)^{\epsilon-1} u^{\epsilon-1}. \end{aligned} \tag{3.6}$$

In even dimensions, Δ_ϵ is a well-defined differential operator, and is invariant under crossing. One approach to solving the crossing equation is to "factor out" $(D_\epsilon)^{\epsilon-1}$ and write down a crossing equation for $b_n(u,v)$ (while carefully taking care of the kernel of $(D_\epsilon)^{\epsilon-1}$), as was the approach of [34]. However, in odd dimensions, the differential operator $(D_\epsilon)^{\epsilon-1}$ is defined only formally on the functional space spanned by Jack polynomials with eigenvalues given in (A.10), and this functional space does not map to itself under crossing $u \leftrightarrow v$.[7] To make our setup easily generalizable to five and three dimensions, we will not study the crossing equation for $b_n(u,v)$, but will instead analyze the crossing equation for $G(u,v;w)$ directly. See appendix C for the setup of the crossing equation for $b_n(u,v)$ in the special case of $\epsilon = k = 2$.

The rest of the paper specializes to the case of $k=2$. Then $G(u,v;w)$ is a second degree polynomial in w^{-1}. By matching the coefficients of the monomials in w, the crossing equation (3.3) can be separated into three equations involving only u and v,

$$\begin{aligned} u^{-2\epsilon} G_2(u,v) &= v^{-2\epsilon} G_0(v,u), \\ u^{-2\epsilon} G_1(u,v) &= v^{-2\epsilon} G_1(v,u), \\ u^{-2\epsilon} G_0(u,v) &= v^{-2\epsilon} G_2(v,u), \end{aligned} \tag{3.7}$$

where G_i are defined in (3.1), and the third equation is trivially equivalent to the first equation. In appendix B, we show that the second equation also follows from the first equation as a consequence of the superconformal Ward identities (B.1). Moreover, the superconformal Ward identities imply an identity (B.8) on the first equation, which is important when we need to identify the independent constraints from the crossing equation in order when applying the linear functional method.

[7] We thank Silviu S. Pufu for a discussion on the subtleties of the differential operator Δ_ϵ.

JHEP08(2017)128

4 Superconformal blocks

The four point function can be expanded in superconformal blocks as

$$G(u,v;w)=\sum_{\mathcal{X}}\lambda_{\mathcal{X}}^2\mathcal{A}^{\mathcal{X}}(u,v;w), \tag{4.1}$$

where $\mathcal{A}^{\mathcal{X}}(u,v;w)$ is the superconformal block of the superconformal multiplet $\mathcal{X}$. The sum is over the superconformal multiplets allowed in the OPE of two $\mathcal{D}[2]$. The selection rule is

$$\mathcal{D}[2]\times\mathcal{D}[2]=\sum_{j=0}^{2}\mathcal{D}[4-2j]+\sum_{j=1}^{2}\sum_{\ell=0}^{\infty}\mathcal{B}[4-2j]_\ell+\sum_{\ell=0}^{\infty}\sum_{\Delta>\ell+2\epsilon(2-j)+6}\mathcal{L}[0]_{\Delta,\ell}, \tag{4.2}$$

as we presently argue. First, a generalization of [73] shows (4.2) with the possible addition of $\mathcal{A}[0]_\ell$ and $\mathcal{C}[0]$.[8] However, no consistent superconformal block satisfying the superconformal Ward identities exists for $\mathcal{A}[0]_\ell$ and $\mathcal{C}[0]$, thereby proving their absence (see footnote 11). The constraint (3.4) imposes an additional selection rule $\ell+J_R\in2\mathbb{Z}$ on the intermediate primary operators.[9] A superconformal block can be expanded in products of bosonic conformal blocks $\mathcal{G}_{\Delta,\ell}$ and $\mathfrak{su}(2)_R$ harmonics,

$$\mathcal{A}^{\mathcal{X}}(u,v;w)=\sum_{(2J_R,\Delta,\ell)\in\mathcal{X}}c_{2J_R,\Delta,\ell}P_{J_R}\left(1+\tfrac{2}{w}\right)\mathcal{G}_{\Delta,\ell}(u,v), \tag{4.3}$$

where $P_{J_R}(x)$ are Legendre polynomials. The summation $\sum_{(2J_R,\Delta,\ell)\in\mathcal{X}}$ is over all primary operators in the superconformal multiplet $\mathcal{X}$ that appear in the OPE, labeled by $(2J_R,\Delta,\ell)$. It is a finite sum as there are only finitely many primary operators contained in each superconformal multiplet. Bosonic conformal blocks are reviewed in appendix A.

The coefficients $c_{2J_R,\Delta,\ell}$ are fixed by the superconformal Ward identities (B.1). The superconformal block expansion (4.1) implies that the functions $b_n(u,v)$ parameterizing solutions to the superconformal Ward identities (see (3.5)) have expansions

$$b_n(u,v)=\sum_{\mathcal{X}}\lambda_{\mathcal{X}}^2b_n^{\mathcal{X}}(u,v). \tag{4.4}$$

Comparing (4.1) and (4.3) with (3.5) gives the relation

$$\sum_{J_R=0}^{2}P_{J_R}\left(1+\tfrac{2}{w}\right)\mathcal{A}^{\mathcal{X}}_{2J_R}(u,v)=u^{2\epsilon}\Delta_\epsilon\left[(v+uw^{-1})(1+w^{-1})-w^{-1}\right]b_0^{\mathcal{X}}(u,v), \tag{4.5}$$

where $\mathcal{A}^{\mathcal{X}}_{2J_R}(u,v)$ is defined as

$$\mathcal{A}^{\mathcal{X}}_{2J_R}(u,v)=\sum_{(\Delta,\ell)\in\mathcal{X}|_{2J_R}}c_{2J_R,\Delta,\ell}\mathcal{G}_{\Delta,\ell}(u,v). \tag{4.6}$$

[8] Consider the three-point function of two superfields of $\mathcal{D}[2]$ with a generic superfield $\mathcal{O}$ in the harmonic superspace. The bottom component of such three-point function takes the form of equation (3.3) in [73] with the obvious modifications. By the arguments of [73], $\mathcal{O}$ must correspond to either a $\mathcal{D}$- or $\mathcal{B}$-type multiplet if $\mathcal{O}$ has $2J_R=2$, and a $\mathcal{D}$-type if $2J_R=4$.

[9] Note the the bosonic conformal blocks satisfy $\mathcal{G}_{\Delta,\ell}(u,v)=(-1)^\ell\mathcal{G}_{\Delta,\ell}(u/v,1/v)$.

JHEP08(2017)128

The relation (4.5) can then be written as

$$\begin{aligned}
\mathcal{A}_0^{\mathcal{X}}(u,v) &= u^{2\epsilon}\Delta_\epsilon\left(\frac{1}{2}-\frac{1}{6}u+\frac{1}{2}v\right)b^{\mathcal{X}}(u,v),\\
\mathcal{A}_2^{\mathcal{X}}(u,v) &= \frac{1}{2}u^{2\epsilon}\Delta_\epsilon(v-1)b^{\mathcal{X}}(u,v),\\
\mathcal{A}_4^{\mathcal{X}}(u,v) &= \frac{1}{6}u^{2\epsilon}\Delta_\epsilon u b^{\mathcal{X}}(u,v),
\end{aligned} \tag{4.7}$$

where we abbreviate b_0 as b since there is no other b_n.

In the following subsections, we give explicit expressions for the superconformal blocks by solving (4.7). The bosonic conformal blocks are normalized such that in the limit of $u = v \ll 1$, the leading term in the u expansion is u^Δ. The superconformal blocks are normalized such that in the same limit, the leading term is $(-)^{J_R}u^\Delta P_{J_R}\left(1+\frac{2}{w}\right)$.

4.1 Long multiplets

Inside the superconformal multiplet $\mathcal{L}[0]_{\Delta,\ell}$, there is a unique conformal primary of dimension $\Delta+2$, spin ℓ, and transforming in the $\mathfrak{su}(2)_R$ representation with $2J_R = 4$ [8, 9]. Thus we can solve for b using the last line of (4.7):

$$b_{\mathcal{L}[0]_{\Delta,\ell}}(u,v) = 6c_{4,\Delta,\ell}u^{-1}\Delta_\epsilon^{-1}u^{-2\epsilon}\mathcal{G}_{\Delta+2,\ell}(u,v), \tag{4.8}$$

where

$$c_{4,\Delta,\ell} = \frac{(\Delta+\ell)(\Delta-\ell-2\epsilon)}{6(\Delta+\ell-2\epsilon+2)(\Delta-\ell-4\epsilon+2)}. \tag{4.9}$$

Using the formulae in appendix D of [24], we obtain the explicit decomposition of the long multiplet superconformal blocks into bosonic conformal blocks, as follows:

$$\begin{aligned}
\mathcal{A}_0^{\mathcal{L}[0]_{\Delta,\ell}} = \mathcal{G}_{\Delta,\ell}&\\
&+\frac{(\ell-1)\ell(\Delta-\ell)(\Delta-\ell-2\epsilon)(\Delta-\ell-2\epsilon+2)}{16(\ell+\epsilon-1)(\ell+\epsilon)(\Delta-\ell-4\epsilon+2)(\Delta-\ell-2\epsilon+1)(\Delta-\ell-2\epsilon+3)}\mathcal{G}_{\Delta+2,\ell-2}\\
&+\frac{(\Delta+\ell)(\Delta+\ell+2)(\ell+2\epsilon)(\ell+2\epsilon+1)(\Delta+\ell+2\epsilon)}{16(\Delta+\ell+1)(\Delta+\ell+3)(\ell+\epsilon)(\ell+\epsilon+1)(\Delta+\ell-2\epsilon+2)}\mathcal{G}_{\Delta+2,\ell+2}\\
&+\frac{(\Delta+1)(\Delta+2)(\Delta-2\epsilon+2)(\Delta-2\epsilon+3)(\Delta-\ell)(\Delta+\ell)(\Delta+\ell+2)}{256(\Delta-\epsilon+1)(\Delta-\epsilon+2)^2(\Delta-\epsilon+3)(\Delta+\ell+1)(\Delta+\ell+3)(\Delta-\ell-4\epsilon+2)}\\
&\times\frac{(\Delta-\ell-2\epsilon)(\Delta-\ell-2\epsilon+2)(\Delta+\ell+2\epsilon)}{(\Delta-\ell-2\epsilon+1)(\Delta-\ell-2\epsilon+3)(\Delta+\ell-2\epsilon+2)}\mathcal{G}_{\Delta+4,\ell}\\
&+\frac{(\Delta+\ell)(\Delta-\ell-2\epsilon)\left(1-\frac{3(\epsilon-1)\epsilon\left(\Delta^2-2\Delta(\epsilon-1)+\ell^2+2(\ell+3)\epsilon-2\epsilon^2-4\right)}{2(\Delta-\epsilon)(\Delta-\epsilon+2)(\ell+\epsilon-1)(\ell+\epsilon+1)}\right)}{12(\Delta+\ell-2\epsilon+2)(\Delta-\ell-4\epsilon+2)}\mathcal{G}_{\Delta+2,\ell},
\end{aligned} \tag{4.10}$$

JHEP08(2017)128

$$
\begin{aligned}
\mathcal{A}_2^{\mathcal{L}[0]_{\Delta,\ell}} =& -\frac{\ell(\Delta-\ell-2\epsilon)}{2(\ell+\epsilon)(\Delta-\ell-4\epsilon+2)}\mathcal{G}_{\Delta+1,\ell-1}\\
&-\frac{(\Delta+\ell)(\ell+2\epsilon)}{2(\ell+\epsilon)(\Delta+\ell-2\epsilon+2)}\mathcal{G}_{\Delta+1,\ell+1}\\
&-\frac{(\Delta+1)\ell(\Delta-2\epsilon+2)(\Delta-\ell)}{32(\Delta-\epsilon+1)(\Delta-\epsilon+2)(\ell+\epsilon)(\Delta-\ell-4\epsilon+2)}\\
&\times\frac{(\Delta+\ell)(\Delta-\ell-2\epsilon)(\Delta-\ell-2\epsilon+2)}{(\Delta-\ell-2\epsilon+1)(\Delta-\ell-2\epsilon+3)(\Delta+\ell-2\epsilon+2)}\mathcal{G}_{\Delta+3,\ell-1}\\
&-\frac{(\Delta+1)(\Delta-2\epsilon+2)(\Delta+\ell)(\Delta+\ell+2)}{32(\Delta-\epsilon+1)(\Delta-\epsilon+2)(\Delta+\ell+1)(\Delta+\ell+3)}\\
&\times\frac{(\ell+2\epsilon)(\Delta-\ell-2\epsilon)(\Delta+\ell+2\epsilon)G(\Delta+3,\ell+1)}{(\ell+\epsilon)(\Delta-\ell-4\epsilon+2)(\Delta+\ell-2\epsilon+2)}\mathcal{G}_{\Delta+3,\ell+1},\\
\mathcal{A}_4^{\mathcal{L}[0]_{\Delta,\ell}} =& \frac{(\Delta+\ell)(\Delta-\ell-2\epsilon)}{6(\Delta+\ell-2\epsilon+2)(\Delta-\ell-4\epsilon+2)}\mathcal{G}_{\Delta+2,\ell}.
\end{aligned}
\tag{4.11}
$$

4.2 Short multiplets

The superconformal blocks for the short multiplets can be obtained by taking limits of the superconformal block for $\mathcal{L}[0]_{\Delta,\ell}$, as follows:

$$
\begin{aligned}
\mathcal{A}^{\mathcal{B}[2]_\ell}(u,v;w) &= \frac{\ell+\epsilon+1}{(\ell+1)(\epsilon-1)}\lim_{\Delta\to\ell+4\epsilon-1}(\Delta-\ell-4\epsilon+1)\mathcal{A}^{\mathcal{L}[0]_{\Delta,\ell+1}}(u,v;w),\\
\mathcal{A}^{\mathcal{B}[0]_\ell}(u,v;w) &= \lim_{\Delta\to\ell+2\epsilon}(\Delta-\ell-2\epsilon)\mathcal{A}^{\mathcal{L}[0]_{\Delta,\ell}}(u,v;w),\\
\mathcal{A}^{\mathcal{D}[4]}(u,v;w) &= \frac{3\epsilon}{(\epsilon-1)(2\epsilon-1)}\lim_{\Delta\to4\epsilon-2}(\Delta-4\epsilon+2)\mathcal{A}^{\mathcal{L}[0]_{\Delta,0}}(u,v;w),\\
\mathcal{A}^{\mathcal{D}[2]}(u,v;w) &= \frac{1}{(2\epsilon-1)}\lim_{\Delta\to2\epsilon-1}(\Delta-2\epsilon+1)\mathcal{A}^{\mathcal{L}[0]_{\Delta,-1}}(u,v;w),
\end{aligned}
\tag{4.12}
$$

where the first and third equations follow from the recombination rules at the unitary bound.[10] In the second and forth equations, we need to analytically continue the superconformal block $\mathcal{A}^{\mathcal{L}[0]_{\Delta,\ell}}$ to Δ below the unitarity bound (2.3), so the limits should be regarded as mere tricks to generate solutions to the superconformal Ward identities. One can explicitly check that the superconformal blocks for short multiplets obtained this way indeed have the correct decompositions into bosonic conformal blocks. One can also show that given the content of each multiplet, (4.10) or (4.12) is the unique combination of bosonic conformal blocks that solves the superconformal Ward identities. In fact, as mentioned earlier, the lack of a solution for $\mathcal{A}_\ell[0]$ and $\mathcal{C}[0]$ proves their absence in the selection rule (4.2).[11]

[10] See (4.4) in [8] or (2.63) in [9].

[11] All the bosonic component fields in $\mathcal{C}[0]$ are R-symmetry neutral, hence the superconformal Ward identities reduce to

$$
\partial_\chi G(u,v;w)|_{w\to\chi}=0,\quad \partial_{\bar\chi}G(u,v;w)|_{w\to\bar\chi}=0,
\tag{4.13}
$$

JHEP08(2017)128

The superconformal block for $\mathcal{B}[2]_\ell$ is given by

$$\begin{aligned}
\mathcal{A}_0^{\mathcal{B}[2]_\ell} &= \frac{\ell\epsilon}{2(2\epsilon+1)(\ell+\epsilon)}\mathcal{G}_{\ell+4\epsilon+1,\ell-1} + \frac{(\ell+2\epsilon)(\ell+4\epsilon)}{6(\ell+\epsilon)(\ell+3\epsilon+1)}\mathcal{G}_{\ell+4\epsilon+1,\ell+1} \\
&\quad + \frac{\epsilon(\ell+2\epsilon)(\ell+2\epsilon+1)^2(\ell+2\epsilon+2)(\ell+4\epsilon)(\ell+4\epsilon+1)}{8(\ell+1)(2\epsilon+1)(\ell+3\epsilon+1)^2(\ell+3\epsilon+2)(2\ell+4\epsilon+1)(2\ell+4\epsilon+3)}\mathcal{G}_{\ell+4\epsilon+3,\ell+1}, \\
\mathcal{A}_2^{\mathcal{B}[2]_\ell} &= -\mathcal{G}_{\ell+4\epsilon,\ell} - \frac{\epsilon(\ell+2\epsilon)(\ell+2\epsilon+1)(\ell+4\epsilon)}{4(2\epsilon+1)(\ell+\epsilon+1)(\ell+3\epsilon)(\ell+3\epsilon+1)}\mathcal{G}_{\ell+4\epsilon+2,\ell} \\
&\quad - \frac{(\ell+2\epsilon)(\ell+2\epsilon+1)^3(\ell+4\epsilon)}{4(\ell+1)(\ell+\epsilon+1)(\ell+3\epsilon+1)(2\ell+4\epsilon+1)(2\ell+4\epsilon+3)}\mathcal{G}_{\ell+4\epsilon+2,\ell+2}, \\
\mathcal{A}_4^{\mathcal{B}[2]_\ell} &= \frac{\ell+2\epsilon}{3(\ell+1)}\mathcal{G}_{\ell+4\epsilon+1,\ell+1}.
\end{aligned} \tag{4.16}$$

The superconformal block for $\mathcal{B}[0]_\ell$ is given by

$$\begin{aligned}
\mathcal{A}_0^{\mathcal{B}[0]_\ell} &= \mathcal{G}_{\ell+2\epsilon,\ell} + \frac{(\ell+2\epsilon)^2(\ell+2\epsilon+1)}{4(\ell+1)(2\ell+2\epsilon+1)(2\ell+2\epsilon+3)}\mathcal{G}_{\ell+2\epsilon+2,\ell+2}, \\
\mathcal{A}_2^{\mathcal{B}[0]_\ell} &= -\frac{\ell+2\epsilon}{2(\ell+1)}\mathcal{G}_{\ell+2\epsilon+1,\ell+1}.
\end{aligned} \tag{4.17}$$

The superconformal block for $\mathcal{D}[4]$ is given by

$$\begin{aligned}
\mathcal{A}_0^{\mathcal{D}[4]} &= \frac{2\epsilon^2}{3(4\epsilon+1)(3\epsilon+1)}\mathcal{G}_{4\epsilon+2,0}, \\
\mathcal{A}_2^{\mathcal{D}[4]} &= -\frac{2\epsilon}{4\epsilon+1}\mathcal{G}_{4\epsilon+1,1}, \\
\mathcal{A}_4^{\mathcal{D}[4]} &= \mathcal{G}_{4\epsilon,0}.
\end{aligned} \tag{4.18}$$

The superconformal block for $\mathcal{D}[2]$ is given by

$$\begin{aligned}
\mathcal{A}_0^{\mathcal{D}[2]} &= \frac{\epsilon}{2\epsilon+1}\mathcal{G}_{2\epsilon+1,1}, \\
\mathcal{A}_2^{\mathcal{D}[2]} &= -\mathcal{G}_{2\epsilon,0}.
\end{aligned} \tag{4.19}$$

The superconformal block for $\mathcal{D}[0]$ is given by

$$\mathcal{A}_0^{\mathcal{D}[0]} = 1. \tag{4.20}$$

---

which cannot be satisfied by any non-vacuum block. The superconformal block for $\mathcal{A}[0]_\ell$ must take the form

$$\mathcal{A}^{\mathcal{A}[0]_\ell}(\rho,\theta;w) = a\left[\mathcal{G}_{\ell+6,\ell}(\rho,\theta) + \mathcal{O}(\rho^{\ell+7})\right] + b\left[\mathcal{G}_{\ell+7,\ell+1}(\rho,\theta) + \mathcal{O}(\rho^{\ell+8})\right] P_1(1+\tfrac{2}{w}), \tag{4.14}$$

where ρ and θ are defined by $\chi = \rho e^{i\theta}$ and $\bar{\chi} = \rho e^{-i\theta}$. By

$$\begin{aligned}
&\lim_{\theta\to 0}(\partial_\chi + 2\partial_w)\mathcal{A}^{\mathcal{A}[0]_\ell}(\rho,\theta;w)\Big|_{w\to\chi} = \frac{1}{2}\left[a(\ell+6) + 2b(\ell+3)\right]\rho^{\ell+5} + \mathcal{O}(\rho^{\ell+6}), \\
&\lim_{\theta\to\frac{\pi}{2}}(\partial_\chi + 2\partial_w)\mathcal{A}^{\mathcal{A}[0]_\ell}(\rho,\theta;w)\Big|_{w\to\chi} \\
&\qquad = \frac{3}{2}\left[\frac{a(\ell+4)+2b(\ell+3)}{(\ell+2)(\ell+4)}\sin\frac{\pi\ell}{2} - i\frac{a(\ell+6)+2b(\ell+1)}{(\ell+1)(\ell+3)}\cos\frac{\pi\ell}{2}\right]\rho^{\ell+5} + \mathcal{O}(\rho^{\ell+6}),
\end{aligned} \tag{4.15}$$

it is clear that (4.14) cannot satisfy the superconformal Ward identities unless $a = b = 0$.

JHEP08(2017)128

A different derivation of the blocks using the superconformal Casimir equations appears in [74].[12]

5 Flavor symmetry

We want to consider theories with non-abelian flavor symmetry. Since flavor currents are contained in the $\mathcal{D}[2]$ multiplets, the superconformal primaries $\mathcal{O}_a(x_i, Y_i)$ transform in the adjoint representation of the flavor symmetry group G_F, where a is the adjoint index. The four-point function of $\mathcal{O}_a(x_i, Y_i)$ takes the form

$$\langle \mathcal{O}_a(x_1, Y_1)\mathcal{O}_b(x_2, Y_2)\mathcal{O}_c(x_3, Y_3)\mathcal{O}_d(x_4, Y_4)\rangle = \frac{(Y_1 \cdot Y_2)^2 (Y_3 \cdot Y_4)^2}{x_{12}^{4\epsilon} x_{34}^{4\epsilon}} G_{abcd}(u, v; w), \tag{5.1}$$

and $G_{abcd}(u, v; w)$ admits a decomposition into superconformal blocks as in section 4. The operators that appear in the OPE of $\mathcal{O}_a(x_1, Y_1)$ and $\mathcal{O}_b(x_2, Y_2)$ transform in the tensor product representation $\mathbf{adj} \otimes \mathbf{adj}$, which can further be decomposed into irreducible representations $\mathcal{R}_i$. The decomposition of $G_{abcd}(u, v; w)$ takes the form

$$\begin{aligned} G_{abcd}(u, v; w) &= \sum_{\mathcal{R}_i \in \mathbf{adj}\otimes\mathbf{adj}} P_i^{abcd} G_i(u, v; w), \\ G_i(u, v; w) &= \sum_{\mathcal{X}} \lambda_{\mathcal{X},i}^2 \mathcal{A}^{\mathcal{X}}(u, v; w), \end{aligned} \tag{5.2}$$

where P_i^{abcd} is the projection matrix that projects onto the contributions of operators in the OPE that transform in the representation $\mathcal{R}_i$. They satisfy [75]

$$P_i^{abcd} P_j^{dcef} = \delta_{ij} P_i^{abef}, \quad P_i^{abba} = \dim(\mathcal{R}_i). \tag{5.3}$$

The projection matrices of the trivial representation and the adjoint representation are

$$P_{\mathbf{1}}^{abcd} = \frac{1}{\dim(G_F)} \delta^{ab}\delta^{cd}, \quad P_{\mathbf{adj}}^{abcd} = \frac{1}{\psi^2 h^\vee} f^{abe} f^{edc}, \tag{5.4}$$

where $h^\vee$ is the dual Coxeter number and $\psi^2 = 2$ is the length squared of the longest root of the flavor group.

The identity operator and the stress tensor multiplet $\mathcal{B}[0]_0$ can only transform in the trivial representation $\mathbf{1}$ of the flavor group, while the flavor current multiplet $\mathcal{D}[2]$ can only be in the adjoint representation $\mathbf{adj}$. Their OPE coefficients satisfy

$$\lambda^2_{\mathcal{D}[0],i} = \dim(G_F)\delta_{i,\mathbf{1}}, \quad \lambda^2_{\mathcal{B}[0]_0,i} = \lambda^2_{\mathcal{B}[0]} \dim(G_F)\delta_{i,\mathbf{1}}, \quad \lambda^2_{\mathcal{D}[2],i} = \lambda^2_{\mathcal{D}[2]}\delta_{i,\mathbf{adj}}. \tag{5.5}$$

In section 6, we will relate the coefficients $\lambda^2_{\mathcal{B}[0]_0}$ and $\lambda^2_{\mathcal{D}[2]}$ to the central charge C_T and flavor central charge C_J, which are in turn related to the anomaly coefficients and can be determined through other methods.

[12] The paper [74] points out typos in (4.16) and (4.18) in the early versions of this paper. However, the correct formulae for the blocks were used in the actual bootstrap implementation in all versions.

JHEP08(2017)128

G_F	$h^\vee$	**adj** $\otimes_S$ **adj**	**adj** $\otimes_A$ **adj**	F
SU(2)	2	**1** + **5**	**3**	$\begin{pmatrix} \frac{1}{3} & \frac{5}{3} & 1 \\ \frac{1}{3} & \frac{1}{6} & -\frac{1}{2} \\ \frac{1}{3} & -\frac{5}{6} & \frac{1}{2} \end{pmatrix}$
E_8	30	**1** + **3875** + **27000**	**248** + **30380**	$\begin{pmatrix} \frac{1}{248} & \frac{125}{8} & \frac{3375}{31} & 1 & \frac{245}{2} \\ \frac{1}{248} & -\frac{3}{8} & \frac{27}{31} & \frac{1}{5} & -\frac{7}{10} \\ \frac{1}{248} & \frac{1}{8} & \frac{23}{62} & -\frac{1}{30} & -\frac{7}{15} \\ \frac{1}{248} & \frac{25}{8} & -\frac{225}{62} & \frac{1}{2} & 0 \\ \frac{1}{248} & -\frac{5}{56} & -\frac{90}{217} & 0 & \frac{1}{2} \end{pmatrix}$

Table 1. The decomposition of **adj** $\otimes$ **adj** into irreducible representations and the crossing matrices for SU(2) and E_8 flavor groups. The basis of representations in the crossing matrix are in the order shown in columns **adj** $\otimes_S$ **adj** and then **adj** $\otimes_A$ **adj**.

Because all four external scalars are identical, the four-point function (5.1) is invariant under $(x_1, Y_1, a) \leftrightarrow (x_3, Y_3, c)$, leading to the crossing symmetry constraint

$$F_i{}^j G_j(u, v; w) = \frac{u^{2\epsilon}}{v^{2\epsilon} w^2} G_i(v, u; w^{-1}), \tag{5.6}$$

where the crossing matrix $F_i{}^j$ is defined as

$$F_i{}^j = \frac{1}{\dim(\mathcal{R}_i)} P_i^{dabc} P_j^{abcd}. \tag{5.7}$$

Similarly, the invariance under $(x_1, Y_1, a) \leftrightarrow (x_2, Y_2, b)$ leads to the constraint

$$G_i(u, v; w) = (-1)^{|\mathcal{R}_i|} G_i\left(\frac{u}{v}, \frac{1}{v}; -\frac{w}{w+1}\right), \tag{5.8}$$

where $|\mathcal{R}_i| = 0$ for $\mathcal{R}_i$ appearing in the symmetric tensor product of two adjoint representations, and $|\mathcal{R}_i| = 1$ for $\mathcal{R}_i$ appearing in the anti-symmetric tensor product. The constraint (5.8) amounts to imposing the selection rule $\ell + J_R + |\mathcal{R}_i| \in 2\mathbb{Z}$ on the intermediate primary operators.

We will be interested in the SU(2) and E_8 flavor groups. The **adj**$\otimes$**adj** decompositions and crossing matrices are summarized in table 1.[13]

6 Central charges

In this section, we review the definitions of the central charge C_T and the flavor central charge C_J, and derive their relations to the OPE coefficients $\lambda^2_{\mathcal{B}[0]_0}$, $\lambda^2_{\mathcal{D}[2]}$.

[13] We compute the crossing matrices following the methods explained in [75].

JHEP08(2017)128

6.1 Central charge C_T

Conformal symmetry fixes the two-point function of the stress tensor up to an overall coefficient. Since the stress tensor has a canonical normalization, this coefficient is physical and is referred to in the literature is as the central charge C_T. More precisely [76],

$$\langle T_{\mu\nu}(x)T_{\sigma\rho}(0)\rangle = \frac{C_T}{V^2_{\widehat{\mathrm{S}}^{d-1}}}\frac{\mathcal{I}_{\mu\nu,\sigma\rho}(x)}{x^{2d}}, \tag{6.1}$$

where $V_{\widehat{\mathrm{S}}^{d-1}} = 2\pi^{\frac{d}{2}}/\Gamma\left(\frac{d}{2}\right)$ is the volume of a unit $(d-1)$-sphere, and the conformal structure $\mathcal{I}_{\mu\nu,\sigma\rho}(x)$ is given by

$$\begin{aligned}\mathcal{I}_{\mu\nu,\sigma\rho}(x) &= \frac{1}{2}\left[I_{\mu\sigma}(x)I_{\nu\rho}(x) + I_{\mu\rho}(x)I_{\nu\sigma}(x)\right] - \frac{1}{d}\delta_{\mu\nu}\delta_{\sigma\rho},\\ I_{\mu\nu}(x) &= \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}.\end{aligned} \tag{6.2}$$

In appendix D.1, we review how the contribution of the stress tensor multiplet to the four-point function of identical scalars is fully determined by the value of C_T. Assuming that there is a unique flavor-singlet stress tensor multiplet $\mathcal{B}[0]_0$, the relation between the OPE coefficient $\lambda_{\mathcal{B}[0]_0}$ and the central charge C_T is

$$\lambda^2_{\mathcal{B}[0]_0} = \frac{4(2\epsilon+2)(2\epsilon+3)}{2\epsilon+1}\frac{1}{C_T}. \tag{6.3}$$

To later compare with numerical bounds, we present here the values of C_T for six-dimensional superconformal field theories of interest, by relating C_T to a Weyl anomaly coefficient. The Weyl anomaly in six-dimensional conformal field theories takes the form [77–79]

$$\mathcal{A}_{6d} = (4\pi)^3\langle T^\mu_\mu\rangle = -aE_6 + c_1I_1 + c_2I_2 + c_3I_3 + \text{scheme dependent}, \tag{6.4}$$

where E_6 is the Euler density and $I_{1,2,3}$ are certain Weyl invariants. I_3 is normalized as $I_3 = C_{\mu\nu\sigma\rho}\nabla^2C^{\mu\nu\sigma\rho} + \cdots$, $C_{\mu\nu\sigma\rho}$ being the Weyl tensor (see [79] for the precise definition of I_3). The a-coefficient appears in the stress tensor four-point function, c_1 and c_2 in the stress tensor three-point function, and c_3 in the stress tensor two-point function. The relation between c_3 and C_T is

$$C_T = 3024c_3. \tag{6.5}$$

In theories with supersymmetry, the Weyl anomaly coefficients are linearly related to the 't Hooft anomaly coefficients [63–65], which appear in the anomaly polynomial involving gravitational and R-symmetry anomalies (see [65] for precise definitions and normalizations)

$$\mathcal{I}_8 = \frac{1}{4!}\left(\alpha c_2(R)^2 + \beta c_2(R)p_1(T) + \gamma p_1(T)^2 + \delta p_2(T)\right). \tag{6.6}$$

In [65], the authors proposed that the coefficients appearing in the linear relations can be fixed by computing the values of α, β, γ, δ and a, c_1, c_2, c_3 in free theories, e.g., the free

JHEP08(2017)128

hypermultiplet, the free tensor multiplet, and a class of non-unitary free theories. The relation between c_3 and α, β, γ, δ was determined up to an unfixed parameter ξ,

$$c_3 = \frac{1}{9}\alpha - \frac{1}{14}(5+3\xi)\beta + \frac{1}{21}(2-3\xi)\gamma - \frac{1}{12}(2+3\xi)\delta. \tag{6.7}$$

The value of ξ can be further fixed by considering a superconformal vector multiplet $V^{(1,0)}$, which has the same field content as the flavor current multiplet, but whose component fields have higher-derivative kinetic terms. More explicitly, the multiplet consists of a four-derivative vector, a three-derivative Weyl fermion, and three standard two-derivative scalars. The anomaly coefficients are [65]

$$(\alpha, \beta, \gamma, \delta) = \left(-1, -\frac{1}{2}, -\frac{7}{240}, \frac{1}{60}\right). \tag{6.8}$$

Thus the constant ξ can be determined by

$$\xi = \frac{C_T(V^{(1,0)})}{324} - \frac{26}{45}. \tag{6.9}$$

Since the theory is free, the C_T of $V^{(1,0)}$ is simply the sum of that of its component fields. The C_T of a free scalar is known from [76],

$$C_T = \frac{6}{5} \quad \text{(standard scalar)}, \tag{6.10}$$

and that of a free four-derivative vector was computed in [80, 81] to be

$$C_T = -90 \quad \text{(four-derivative vector)}. \tag{6.11}$$

In [68], the authors computed the C_T for a three-derivative Weyl fermion by studying the partition function on $S^1 \times \mathbb{H}^5$, and found

$$C_T = -\frac{72}{5} \quad \text{(three-derivative Weyl fermion)}. \tag{6.12}$$

In appendix E, we verify this answer by explicitly constructing the stress tensor for the three-derivative fermion and computing its two-point function. Thus

$$C_T(V^{(1,0)}) = 3 \times \frac{6}{5} - \frac{72}{5} - 90 = -\frac{504}{5}, \tag{6.13}$$

and they concluded that

$$\xi = -\frac{8}{9}, \tag{6.14}$$

which corroborates with what was first found in [67] via a different method.[14] In [68], the conformal anomaly coefficients for an infinite family of free, non-unitary, higher-derivative $\mathcal{N} = (1,0)$ superconformal multiplets were also computed, and indeed found to satisfy the linear relation (6.7) with this value of ξ.

[14] We thank Matteo Beccaria, Arkady A. Tseytlin, and Yang Zhou for sharing this result before publication.

JHEP08(2017)128

There are various techniques for inferring the values of 't Hooft anomaly coefficients in superconformal field theories, even when the theory is strongly interacting and direct handles are lacking. For instance, if a construction within string theory or M theory exists, the 't Hooft anomaly coefficients can be computed by anomaly inflow [56, 58]. Another approach is anomaly matching by going onto the tensor branch or the Higgs branch [57, 59, 60, 62].

In the following, we present the values of C_T for the free hypermultiplet and the E-string theories.

Free hypermultiplet. The C_T for each free scalar ϕ and each free Dirac spinor ψ are [76]

$$C_T^\phi = \frac{(2\epsilon+2)}{(2\epsilon+1)}, \quad C_T^\psi = 2^{\lfloor \epsilon \rfloor}(2\epsilon+2). \tag{6.15}$$

Thus the C_T for a free hypermultiplet is

$$C_T^{\text{hyper}} = 4C_T^\phi + \frac{1}{2}C_T^\psi = \frac{84}{5}. \tag{6.16}$$

E-string theories. The rank-N E-string theory is realized by stacking N M5 branes inside an end-of-the-world M9 brane [45, 46]. The flavor symmetry is E_8 for rank-one and $E_8 \times \mathrm{SU}(2)$ for higher ranks. The 't Hooft anomaly coefficients and the conformal anomaly coefficient c_3 are given by (including the free hypermultiplet describing the center-of-mass degrees of freedom parallel to the M9 brane)

$$\begin{aligned} &\alpha = N(4N^2+6N+3), \qquad \beta = -\frac{N}{2}(6N+5), \quad \gamma = \frac{7N}{8}, \quad \delta = -\frac{N}{2}, \\ &c_3 = \frac{4}{9}N^3 + \frac{7}{6}N^2 + \frac{11}{12}N, \qquad C_T = 84N(16N^2+42N+33). \end{aligned} \tag{6.17}$$

The minimal central charge is achieved in the $N = 1$ case, which after decoupling the free hypermultiplet is

$$C_T = 7644 - \frac{84}{5} = \frac{38136}{5}. \tag{6.18}$$

6.2 Flavor central charge C_J

We can perform a similar analysis for the flavor currents J_μ^a, which are canonically normalized in the following way. In radial quantization, the non-abelian charge of a state on the cylinder which corresponds to an operator inserted at the origin $x^\mu = 0$ is measured by

$$Q^a = \int_{\mathrm{S}^{d-1}} J_\mu^a(x)\hat{r}^\mu dS, \tag{6.19}$$

where $\hat{r}^\mu = x^\mu/|x|$ is the radial unit vector, and the integral is over an S^{d-1} surrounding the origin. If we consider a state $\left|J_\mu^b\right\rangle$ that corresponds to the current J_μ^b, then the non-abelian charge of this state is given by the structure constants,

$$Q^a\left|J_\mu^b\right\rangle = f^{ab}{}_c\left|J_\mu^c\right\rangle. \tag{6.20}$$

JHEP08(2017)128

We can normalize the structure constants by

$$\frac{1}{\psi^2 h^\vee} f^{ab}{}_c f^{dc}{}_b = \delta^{ad}, \tag{6.21}$$

where $h^\vee$ is the dual Coxeter number and $\psi^2 = 2$ is the length squared of the longest root of the flavor group. This then endows the currents with a normalization.

Conformal symmetry constrains the two point function of the flavor currents J^a_μ up to an overall coefficient, which is called the flavor central charge C_J [76],

$$\left\langle J^a_\mu(x) J^b_\nu(0) \right\rangle = \frac{C_J}{V^2_{\mathbb{S}^{d-1}}} \frac{\delta_{ab} I_{\mu\nu}(x)}{x^{2(d-1)}}. \tag{6.22}$$

The contribution of the flavor current multiplet to the four-point function of identical scalars is fully determined by the value of C_J. In appendix D.2, we derive the relation between the OPE coefficient $\lambda_{\mathcal{D}[2]}$ and the central charge C_J,

$$\lambda^2_{\mathcal{D}[2]} = \frac{2(2\epsilon+1)}{2\epsilon} \frac{\psi^2 h^\vee}{C_J}. \tag{6.23}$$

Similar to the central charge C_T, the flavor central charge C_J can be linearly related to 't Hooft anomaly coefficients [66]. We list the values of C_J for the theories of interest.

Free hypermultiplet. The flavor central charge of a single free hypermultiplet can be determined by (6.23) and (F.23), giving

$$C_J = \frac{5}{2}. \tag{6.24}$$

E-string theories. The C_J of the E_8 flavor group of E-string theories is

$$C_J = 60N^2 + 90N. \tag{6.25}$$

For rank one, $C_J = 150$.

7 Semidefinite programming

We proceed by employing the linear functional method [15] to exploit the crossing symmetry constraint (3.3) (setting $\epsilon = k = 2$), as well as the non-negativity of the coefficients in the superconformal block expansion (4.1), where $\mathcal{X}$ is summed over the multiplets (4.2) allowed by selection rules. To keep the discussion simple, we only display formulae for U(1) flavor symmetry. Also recall from that $G(u, v; w)$ has an expansion in w^{-1} as shown in (3.1). Putting these together, we have

$$\begin{aligned} G(u,v;w) &= \left(\frac{u^2}{v^2 w}\right)^2 G(v,u;w^{-1}), \\ G(u,v;w) &= \sum_{\mathcal{X}} \lambda^2_{\mathcal{X}} \mathcal{A}^{\mathcal{X}}(u,v;w), \\ G(u,v;w) &= G_0(u,v) + G_1(u,v) w^{-1} + G_2(u,v) w^{-2}, \end{aligned} \tag{7.1}$$

JHEP08(2017)128

where each superconformal block $\mathcal{A}^{\mathcal{X}}(u,v;w)$ also has an expansion in w^{-1} that terminates at quadratic order,[15]

$$\mathcal{A}^{\mathcal{X}}(u,v;w) = \widetilde{A}_0^{\mathcal{X}}(u,v) + \widetilde{A}_1^{\mathcal{X}}(u,v)w^{-1} + \widetilde{A}_2^{\mathcal{X}}(u,v)w^{-2}. \tag{7.2}$$

The precise formulae for these superconformal blocks are detailed in section 4.

As explained in the final paragraph of section 3, the superconformal Ward identities imply that the independent constraints from crossing symmetry are contained in the equation

$$v^4 G_2(u,v) = u^4 G_0(v,u). \tag{7.3}$$

Putting things together compactly, the constraints we need to analyze are[16]

$$\begin{aligned} 0 &= \sum_{\mathcal{X}\in\mathcal{I}\cup\{\mathcal{D}[0]\}} \lambda_{\mathcal{X}}^2 \mathcal{K}^{\mathcal{X}}(u,v), \quad \mathcal{K}^{\mathcal{X}}(u,v) \equiv v^4 \widetilde{A}_2^{\mathcal{X}}(u,v) - u^4 \widetilde{A}_0^{\mathcal{X}}(v,u), \\ \lambda_{\mathcal{D}[0]}^2 &= 1, \quad \lambda_{\mathcal{X}}^2 \geq 0 \text{ for } \mathcal{X}\in\mathcal{I}, \end{aligned} \tag{7.4}$$

where $\mathcal{I}$, the putative spectrum of superconformal multiplets with the identity multiplet excluded, contains a subset of

$$\mathcal{L}[0]_{\Delta,\ell}, \quad \mathcal{B}[2]_\ell, \quad \mathcal{B}[0]_\ell, \quad \mathcal{D}[4], \quad \mathcal{D}[2]. \tag{7.5}$$

It is a subset because there are further restrictions on the set of $\mathcal{X}$ over which we sum:

- With abelian flavor symmetry, there is a further selection rule that requires $\ell + J_R$ to be even.
- With non-abelian flavor symmetry, the selection rule allows symmetric representations in $\mathbf{adj}\times\mathbf{adj}$ for $\ell + J_R$ even and anti-symmetric ones for $\ell + J_R$ odd.
- $\mathcal{D}[0]$ only appears in the trivial representation of the flavor group.
- $\mathcal{D}[2]$ can only appear in the adjoint representation of the flavor group since these multiplets contain flavor currents (hence $\mathcal{D}[2]$ are absent for abelian flavor).
- In interacting theories with a unique stress tensor, $\mathcal{B}[0]_0$ only exists in the trivial representation, and $\mathcal{B}[0]_\ell$ for $\ell > 0$ do not exist since these multiplets contain higher spin conserved currents.[17,18]

Our goal is to put bounds on the central charges C_T and C_J, which are inversely proportional to $\lambda^2_{\mathcal{B}[0]_0}$ and $\lambda^2_{\mathcal{D}[2]}$ via (6.3) and (6.23). We presently explain how to put a universal lower bound on C_T, or equivalently an upper bound on $\lambda^2_{\mathcal{B}[0]_0}$, using the linear

[15] Notice that $\widetilde{A}_i^{\mathcal{X}}(u,v)$ are different from the $\mathcal{A}_i^{\mathcal{X}}(u,v)$ defined in (4.6), which are the coefficients in the expansion of superconformal blocks $\mathcal{A}^{\mathcal{X}}(u,v;w)$ in Legendre polynomials rather than in monomials in w^{-1}.

[16] Recall from (5.5) that when the flavor group is non-abelian, the normalization is $\lambda^2_{\mathcal{D}[0],i} = \dim(G_F)\delta_{i,\mathbf{1}}$.

[17] Later when we mention "interacting theories", we always assume that the stress tensor is unique.

[18] We thank the JHEP referee for pointing out a mistake in our draft, where we wrongly assumed that even in free theories, $\mathcal{B}[0]_\ell$ for all ℓ can only transform in the trivial or adjoint representation of the flavor group.

JHEP08(2017)128

functional method. Simple modifications of the following setup allow us to further bound theories to within a finite region in the $C_T^{-1} - C_J^{-1}$ plane.

Consider the space of linear functionals on functions of u, v. Suppose we can find a linear functional α that satisfies

$$\alpha[\mathcal{K}^{\mathcal{D}[0]}] = -1, \quad \alpha[\mathcal{K}^{\mathcal{X}}] \geq 0 \text{ for } \mathcal{X} \in \mathcal{I}, \tag{7.6}$$

then these constraints combined with the constraints (7.4) imply an upper bound on $\lambda^2_{\mathcal{D}[2]}$,

$$\lambda^2_{\mathcal{D}[2]} = \frac{\lambda^2_{\mathcal{D}[2]}}{\sum_{\mathcal{X}\in\mathcal{I}} \lambda^2_{\mathcal{X}} \alpha[\mathcal{K}^{\mathcal{X}}]} \leq \frac{1}{\alpha[\mathcal{K}^{\mathcal{D}[2]}]}. \tag{7.7}$$

The optimal upper bound is obtained by maximizing $\alpha[\mathcal{K}^{\mathcal{D}[2]}]$ within the space of linear functionals satisfying (7.6). The resulting functional is referred to as the *extremal functional*, which we denote by α_E [20]. Thus the linear functional method turns the problem of putting an upper bound on $\lambda^2_{\mathcal{D}[2]}$ to a problem in semidefinite programming.

There is a unique four-point function saturating (7.7), called the *extremal four-point function* [20, 23]. This four-point function satisfies

$$0 = \sum_{\mathcal{X}\in\mathcal{I}\setminus\{\mathcal{D}[2]\}} \lambda^2_{\mathcal{X}}\, \alpha_E[\mathcal{K}^{\mathcal{X}}], \tag{7.8}$$

which, given (7.6), means that the long multiplets that can contribute to this extremal four-point function must have Δ, ℓ at which $\alpha_E[\mathcal{K}^{\mathcal{L}[0]_{\Delta,\ell}}]$ vanishes.

In practice, we can only perform the above minimization procedure within a finite-dimensional subspace of linear functionals, with the constraints (7.6) imposed on a finite number of multiplets. We achieve the latter by restricting to multiplets with spins no larger than a certain maximum ℓ_{max}, and estimate how the bound weakens with increasing ℓ_{max}. Empirically we find that the amount of weakening is roughly inversely proportional to ℓ_{max}, and so we can estimate the errors by extrapolations. This issue is examined further in appendix G. As for truncating the linear functionals, a convenient subspace is given by the following. Define variables $z, \bar{z}$ by

$$u = z\bar{z}, \quad v = (1-z)(1-\bar{z}), \tag{7.9}$$

such that crossing $u \leftrightarrow v$ amounts to $(z, \bar{z}) \leftrightarrow (1-z, 1-\bar{z})$. Consider the expansion of linear functionals in the basis of taking derivatives with respect to ∂_z and $\partial_{\bar{z}}$ and evaluating at the crossing symmetric point $z = \bar{z} = \frac{1}{2}$. Our subspace is simply the truncation of these derivatives to having total degree no larger than Λ, namely,

$$\alpha = \sum_{m,n=0}^{\Lambda} \alpha_{m,n} \partial_z^m \partial_{\bar{z}}^n \big|_{z=\bar{z}=\frac{1}{2}}. \tag{7.10}$$

Bosonic conformal blocks and their derivatives evaluated at the crossing symmetric point are computed by utilizing the recursive representation [82], the diagonal limit [19, 83], and a recursion relation on transverse derivatives [19] that follows from the conformal Casimir equation. The computations are described in appendix A. We use the SDPB package [31] to perform the semidefinite programming procedure. Details on the numerical implementations are discussed in appendix G.

JHEP08(2017)128

8 Results

8.1 Free hypermultiplet: a check

In the semidefinite programming approach to constraining superconformal field theories, free theories differ from interacting theories by the presence of multiplets that contain higher spin conserved currents, $\mathcal{B}[0]_\ell$ with $\ell > 0$. This means that the functional α acted on these multiplets must also be non-negative, leading to weaker constraints than the interacting case.

A single free hypermultiplet has SU(2) flavor symmetry. In particular, the SO(4) that rotates the four real scalars is the combination of the flavor SU(2) and R-symmetry $\mathrm{SU}(2)_R$. The superconformal primaries of the $\mathcal{D}[2]$ multiplets are scalar bilinears, and their four-point function can be computed explicitly by Wick contractions. We refer the reader to appendix F.2 for the explicit form of this four-point function and its decomposition into superconformal blocks. An important property is the absence of $\mathcal{B}[0]_\ell$ in the **5** representation, an additional condition that we impose in the bootstrap analysis. We also note that the long multiplets appearing in the **1** channel have lowest scaling dimension $\Delta = 8$, and in the **5** channel have lowest $\Delta = 10$.

Assuming SU(2) flavor symmetry and the existence of higher spin conserved currents in the trivial **1** or adjoint **3** representation, figure 1 shows the universal lower bounds on C_T and C_J at various derivative orders Λ, as well as extrapolations to $\Lambda \to \infty$ using the quadratic ansatz

$$\min C_{T/J} = a + \frac{b}{\Lambda} + \frac{c}{\Lambda^2}, \quad b < 0, \quad \Lambda \geq 24, 28, 32. \tag{8.1}$$

We see that both $\min C_T$ and $\min C_J$ tend towards the values for a single free hypermultiplet. The left side of figure 2 shows the extremal functional optimizing the lower bound on C_J acted on the contribution of the spin-zero long multiplet to the crossing equation, $\alpha_E[\mathcal{K}^{\mathcal{L}[0]_{\Delta,0}}]$, in the **1** and **5** channels of the SU(2) flavor. We can read off the low-lying spectrum of long multiplets from the zeroes.[19] The right side of figure 2 shows how the lowest Δ in each channel varies with increasing Λ and tends towards $\Delta = 8$ and $\Delta = 10$. Also shown are extrapolations to infinite Λ using the ansatz

$$\Delta_{\text{gap}}^{\mathcal{L}[0]} = a + b \exp \frac{c}{\Lambda}, \quad \Lambda \geq 24. \tag{8.2}$$

Due to the oscillatory behavior of the data points, we perform separate extrapolations for $\Lambda \in 4\mathbb{Z}$ and $\Lambda \in 4\mathbb{Z} + 2$, for both $\min C_{T/J}$ and $\Delta_{\text{gap}}^{\mathcal{L}[0]}$. These results suggest that a free hypermultiplet saturates the lower bounds on both C_T and C_J.

8.2 E-string theories

Let us now turn our attention to the E-string theories. We first present universal lower bounds on C_T and C_J for theories whose flavor group contains E_8 as a subgroup. Figure 3 shows the bounds on C_T and C_J at different derivative orders Λ, and extrapolations to infinite Λ using the quadratic ansatz (8.1).

[19] The results are almost identical to those using the extremal functional obtained by minimizing C_T, $\alpha_E^{\mathcal{B}[0]_0}[\mathcal{K}^{\mathcal{L}[0]_{\Delta,0}}]$.

JHEP08(2017)128

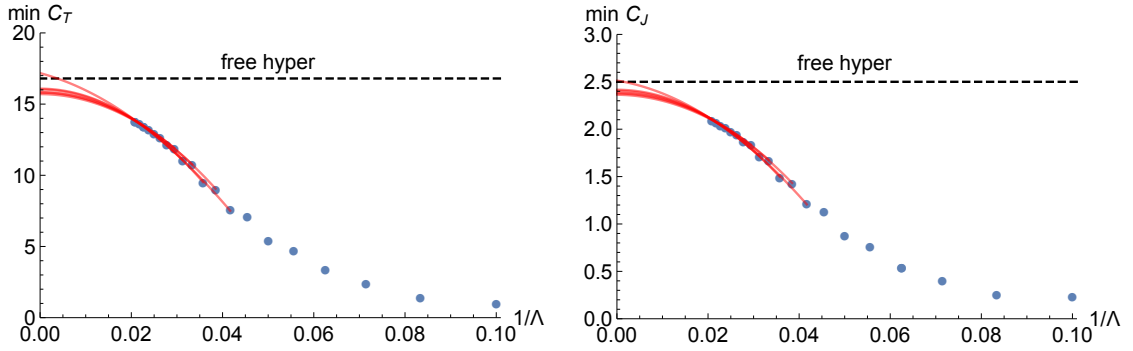

Figure 1. The lower bounds on C_T and C_J at different derivative orders Λ, assuming SU(2) flavor group and allowing higher spin conserved currents in the trivial or adjoint representation. Also shown are the values for a free hypermultiplet, $C_T = \frac{84}{5}$ and $C_J = \frac{5}{2}$. Also shown are the extrapolations to $\Lambda \to \infty$ using the ansatz (8.2), for $\Lambda \in 4\mathbb{Z}$ and $\Lambda \in 4\mathbb{Z}+2$, separately.

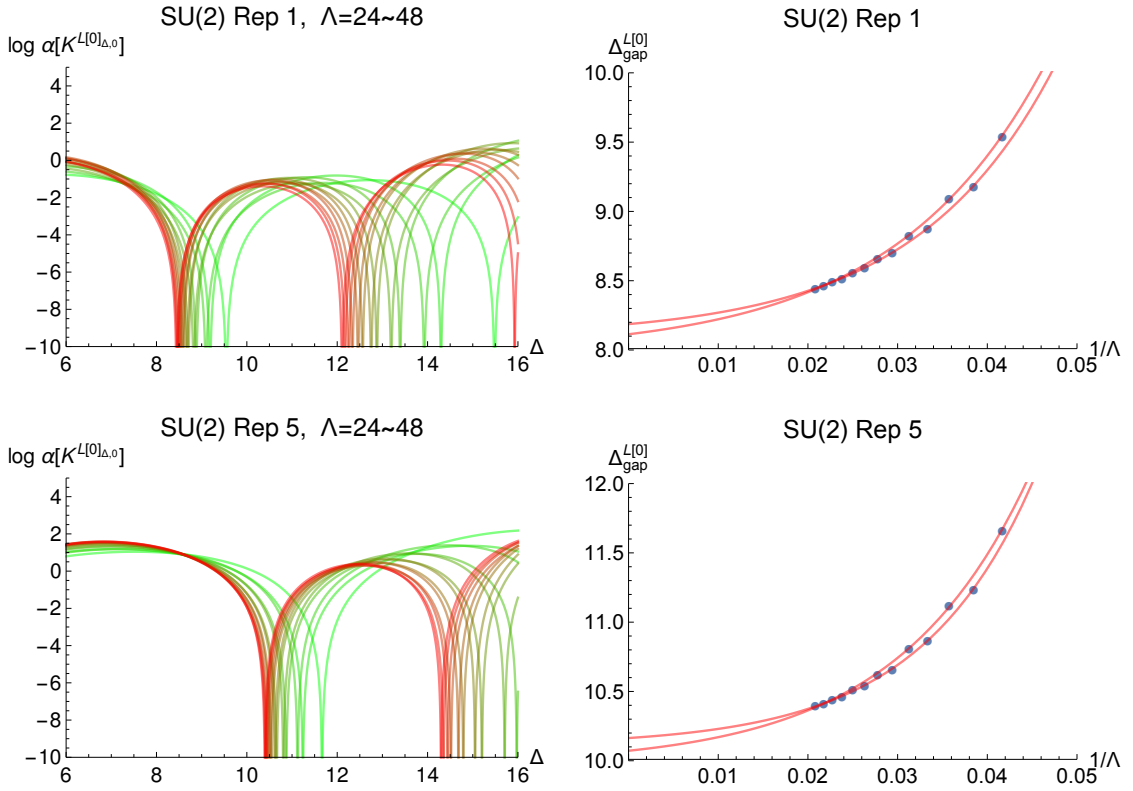

Figure 2. *Left:* the extremal functional optimizing the lower bound on C_J acted on the contribution of the spin-zero long multiplet to the crossing equation, $\alpha_E[\mathcal{K}^{\mathcal{L}[0]_{\Delta,0}}]$, in the **1** and **5** channels of the SU(2) flavor, plotted in logarithmic scale. Increasing derivative orders $\Lambda = 24, 26, \ldots, 48$ are shown from green to red. *Right:* the gap (lowest scaling dimension) in the spectrum of long multiplets in each channel at different Λ. Also shown are the extrapolations to $\Lambda \to \infty$ using the ansatz (8.1), for $\Lambda \in 4\mathbb{Z}$ and $\Lambda \in 4\mathbb{Z}+2$, separately.

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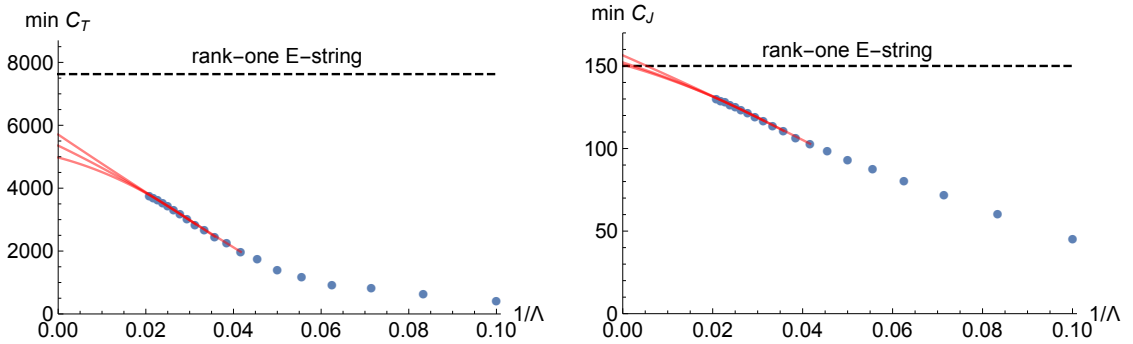

Figure 3. The lower bounds on C_T and C_J at different derivative orders Λ for interacting theories with E_8 flavor group. Also shown are the extrapolations to infinite derivative order using the quadratic ansatz (8.1), as well as the values in the rank-one E-string theory.

	min C_T	min C_J
$\Lambda = 48$	3.78×10^3	1.30×10^2
Extrapolations		
$\Lambda \geq 24$	5.71×10^3	1.56×10^2
$\Lambda \geq 28$	5.35×10^3	1.52×10^2
$\Lambda \geq 32$	4.98×10^3	1.51×10^2
Rank-one E-string	$\frac{38136}{5} \approx 7.63 \times 10^3$	150

Table 2. The lower bounds on C_T and C_J for interacting theories with E_8 flavor symmetry. Presented are the bounds at the highest derivative order computed ($\Lambda = 48$), as well as the extrapolations to infinite Λ using the quadratic ansatz (8.1).

Table 2 summarizes the results of the extrapolations, as well as the C_T and C_J values in the rank-one E-string theory. Notice that the extrapolated lower bound on C_J sits close to the rank-one E-string value, while that on C_T is still some distance away. The former observation motivates Conjecture 1 stated in the introduction.

To supply further evidence for Conjecture 1, we perform a full survey of the range of allowed (C_J, C_T). Figure 4 shows the allowed region in the $C_T^{-1} - C_J^{-1}$ plane for derivative orders $\Lambda = 24, 28, \ldots, 40$. Notice that the point of minimal C_J has a value of C_T that sits close to the value of C_T in the rank-one E-string theory. To quantify this observation more precisely, we show in figure 5 how the value of C_T at min C_J tends to the rank-one E-string value with increasing derivative order. The value appears to be rather stable between derivative orders 24 and 48, and although it is somewhat smaller than the rank-one E-string value, a closer examination shows a trend of potential convergence to the rank-one E-string at higher derivative orders.[20]

While our data do not permit a reliable extrapolation of the entire allowed region to infinite derivative order, we comment on some of the features. First, given any two unitary

[20]The deviation of C_T at min C_J from the rank-one E-string value ($\sim 7\%$) is larger than the estimated error due to the truncation on spins ($\lesssim 2\%$). See appendix G.

JHEP08(2017)128

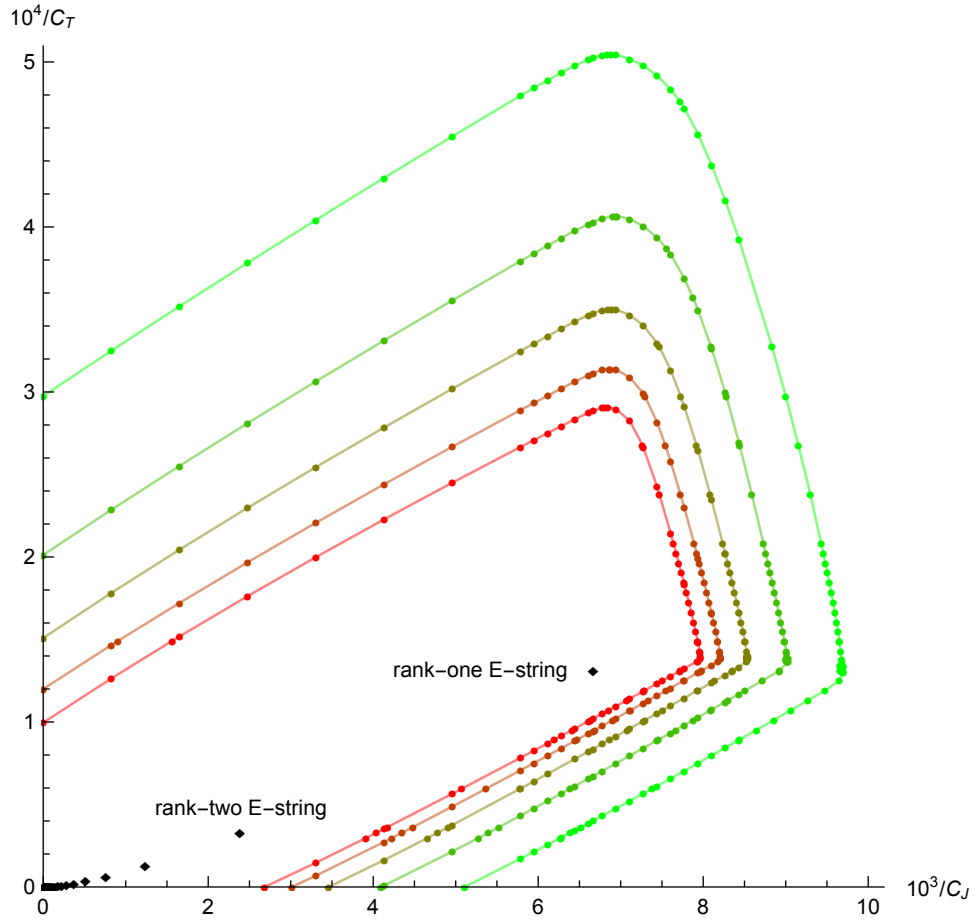

Figure 4. The allowed region in the $C_T^{-1} - C_J^{-1}$ plane for interacting theories with E_8 flavor group, at derivative orders $\Lambda = 24, 28, \ldots, 40$, shown from green to red. Also plotted are the points corresponding to the E-string theories.

solutions to crossing, $G_1(u, v; w)$ and $G_2(u, v; w)$, we can construct a family of unitary solutions $\alpha G_1(u, v; w) + (1 - \alpha)G_2(u, v; w)$ for $0 \leq \alpha \leq 1$ that populate the line segment between the two points corresponding to $G_1(u, v; w)$ and $G_2(u, v; w)$ on the $C_T^{-1} - C_J^{-1}$ plane. This means that the allowed region is convex.[21] Second, there seem to be two kinks, one corresponding to the rank-one E-string theory, and another with a C_J value close to that of the rank-one E-string, but with a smaller C_T.[22] A third feature is that the lower boundary appears to approach the locus of points corresponding to the higher rank E-string theories. We discuss the last feature more in section 9.

[21] Unitary solutions to crossing that populate the boundary of the allowed region can be explicitly constructed using the extremal functional method.

[22] We do not know what to make of the proximity of C_J at min C_T to the rank-one E-string value, as shown in figure 6, or are aware of any candidate theory that sits at this second kink; one logical possibility is that min C_T changes trend at very high derivative orders and becomes saturated by the rank-one E-string theory.

JHEP08(2017)128

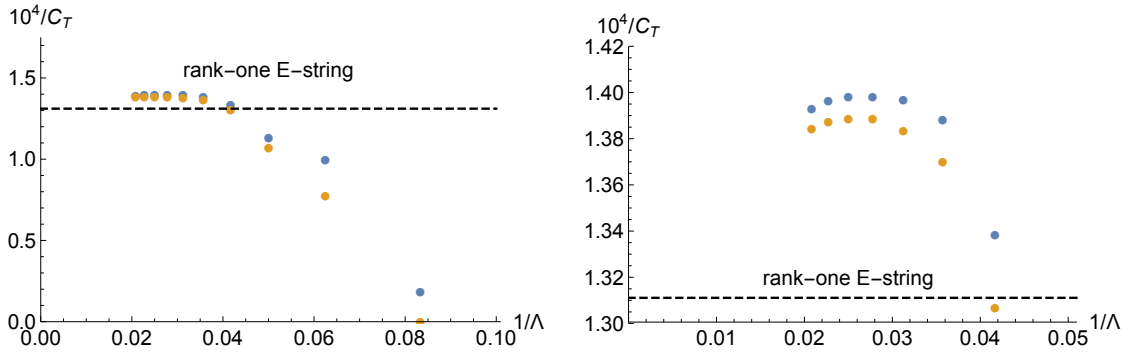

Figure 5. *Left:* the upper and lowers bounds on the inverse of the central charge C_T^{-1} when the value of the flavor central charge C_J is set close to saturating the lower bound, $C_J = (1 + 10^{-4}) \min C_J$, at different derivative orders Λ, for interacting theories with E_8 flavor group. Also shown is the value for the rank-one E-string theory. *Right:* the same plot zoomed in on high derivative orders, showing a trend that the value of C_T at min C_J potential approaches the rank-one E-string value as $\Lambda \to \infty$.

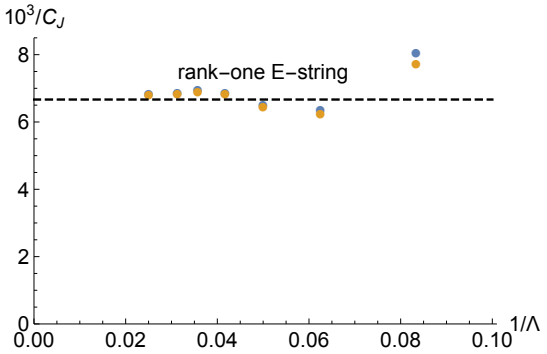

Figure 6. The upper and lowers bounds on the inverse of the flavor central charge C_J^{-1} when the value of the central charge C_T is set close to saturating the lower bound, $C_T = (1 + 10^{-4}) \min C_T$, at different derivative orders Λ, for interacting theories with E_8 flavor group. Also shown is the value for the rank-one E-string theory.

A further check of Conjecture 1 is the following. The Higgs branch of the rank-one E-string theory is the one-instanton moduli space of the flavor group E_8, which is isomorphic to the minimal nilpotent orbit of E_8 [45, 84, 85]. The minimal nilpotent orbit can be defined by quadratic polynomial equations in the complexified $\mathfrak{e}_8$ Lie algebra. More explicitly, for $\mathbf{r} \in \mathfrak{e}_8$, the defining equation for the minimal nilpotent orbit is

$$(\mathbf{r} \otimes \mathbf{r})\big|_{\mathbf{1} \oplus \mathbf{3875}} = 0. \tag{8.3}$$

The Higgs branch chiral ring is isomorphic to the coordinate ring of the Higgs branch [84, 86–88]. The latter admits a description as the polynomial ring generated by the E_8 moment maps (the superconformal primaries of the $\mathcal{D}[2]$ multiplets), quotient by the Joseph ideal generated by the superconformal primaries of the $\mathcal{D}[4]$ multiplets in the representations **1** and **3875** [84, 85]. In other words, in the rank-one E-string theory, the $\mathcal{D}[4]$ multiplets in the representations **1** and **3875** do not appear in the OPE of two $\mathcal{D}[2]$

JHEP08(2017)128

Representation	$\alpha_E[\mathcal{K}^{\mathcal{D}[4]}]$bb
1	1.1×10^{1}
3875	5.7×10^{2}
27000	1.1×10^{-11}

Table 3. The extremal functional optimizing the lower bound on C_J for interacting theories with E_8 flavor group, at derivative order $\Lambda = 48$, acted on the contributions of the $\mathcal{D}[4]$ multiplets to the crossing equation in each channel.

Rep of E_8	Extrapolated $\Delta_{\text{gap}}^{\mathcal{L}[0]}$
1	7.0
3875	6.7
27000	8.2

Table 4. The estimated gaps in the scaling dimensions of long multiplets transforming in the **1**, **3875**, **27000** of E_8 in the OPEs of flavor current multiplets, in the rank-one E-string theory.

multiplets, while those in the **27000** do.[23] In accordance with this expectation, table 3 shows the extremal functional optimizing the lower bound on C_J acted on the contributions of the $\mathcal{D}[4]$ multiplets to the crossing equation in each channel.

Assuming that Conjecture 1 is true, we can determine various physical properties of the rank-one E-string theory, such as the spectrum of long multiplets. The left side of figure 7 shows the extremal functional acted on the contribution of the spin-zero long multiplet to the crossing equation, namely, $\alpha_E[\mathcal{K}^{\mathcal{L}[0]_{\Delta,0}}(u,v)]$, in the **1**, **3875**, **27000** channels of E_8. The right side shows how the lowest Δ in each channel varies with increasing derivative order Λ, as well as an extrapolation to infinite Λ using the ansatz (8.2), for $\Lambda \in 4\mathbb{Z}$ and $\Lambda \in 4\mathbb{Z}+2$, separately. The results motivate the next conjecture.

Conjecture 2 *In the $\mathcal{D}[2] \times \mathcal{D}[2]$ OPEs of flavor current multiplets in the rank-one E-string theory, the lightest long multiplet transforms in the* **3875** *of E_8. The estimated gaps in the scaling dimensions of long multiplets in the* **1**, **3875**, **27000** *in $\mathcal{D}[2] \times \mathcal{D}[2]$ are given in table 4.*

9 Outlook

Based on our observations on figure 4, we put forward an optimistic conjecture.

Conjecture 3 *The E-string theories of all ranks sit at the boundary of the space of unitary solutions to crossing.*

As a piece of supporting evidence, figure 8 shows the lower bound on C_J assuming the value of $C_T = \frac{151956}{5}$ in the rank-two E-string theory, where we see that the extrapolated C_J sits close to the rank-two E-string value $C_J = 420$. There is actually more we can do.

[23] We thank Yifan Wang for explaining this fact to us.

JHEP08(2017)128

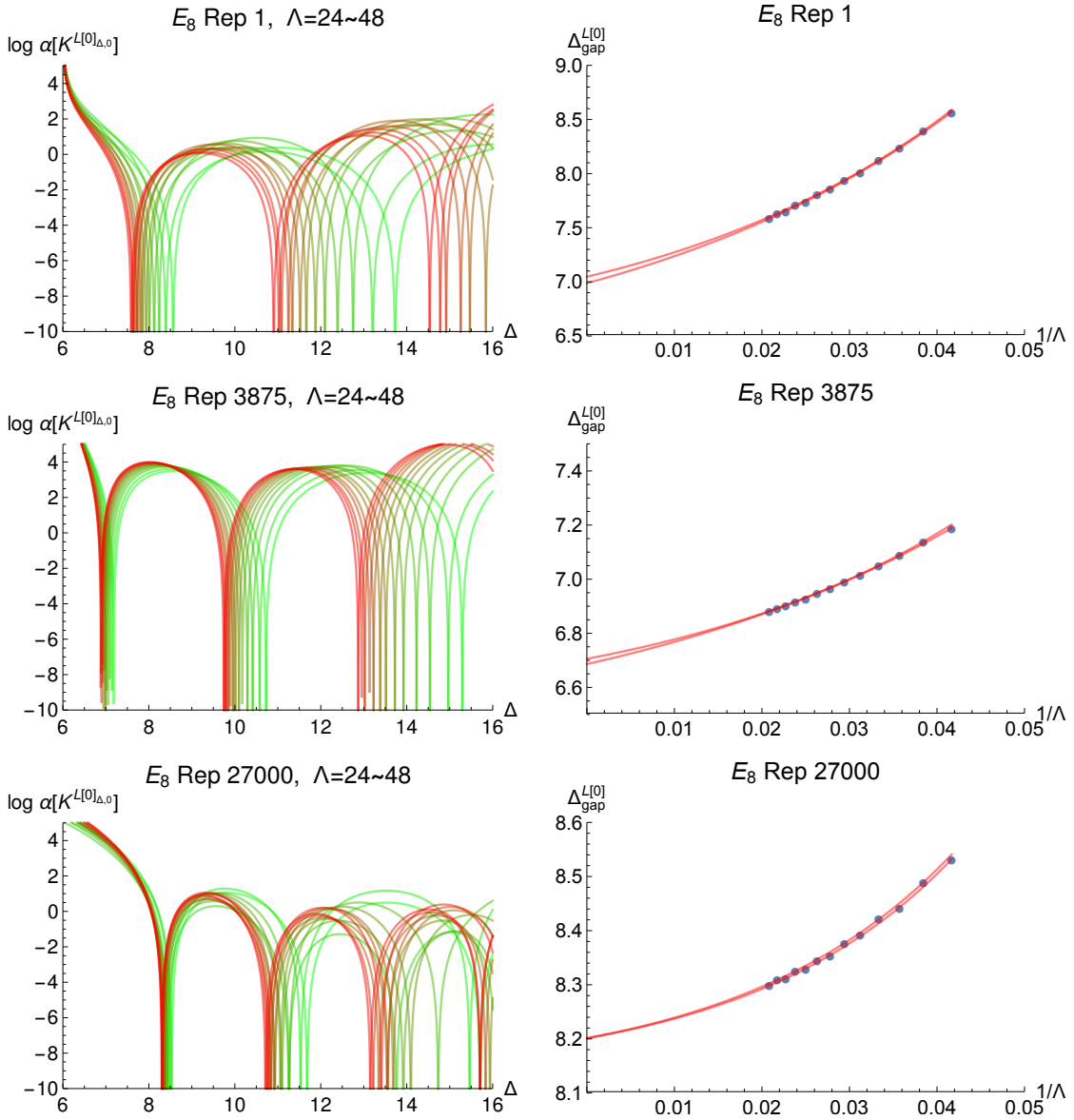

Figure 7. *Left:* the extremal functional optimizing the lower bound on C_J, acted on the contribution of the spin-zero long multiplet to the crossing equation, $\alpha_E[\mathcal{K}^{\mathcal{L}[0]_{\Delta,0}}(u,v)]$, in the **1**, **3875**, **27000** channels of E_8, plotted in logarithmic scale. Increasing derivative orders $\Lambda = 24, 26, \ldots, 48$ are shown from green to red. *Right:* the gap (lowest scaling dimension) in each channel at different Λ, and an extrapolation to $\Lambda \to \infty$ using the ansatz (8.2), for $\Lambda \in 4\mathbb{Z}$ and $\Lambda \in 4\mathbb{Z}+2$, separately.

For $N > 1$, the E-string theories have a larger flavor group $E_8 \times \mathrm{SU}(2)$, and the SU(2) flavor central charge is given by

$$C_J = \frac{5}{2}\left(16N^3 + 6N^2 - 21N - 1\right). \tag{9.1}$$

This additional input may be necessary to put the higher-rank E-string theories on the boundary of the space of unitary solutions to crossing.

JHEP08(2017)128

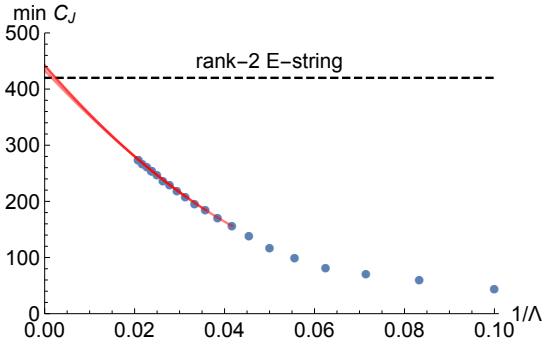

Figure 8. The lower bounds on C_J at different derivative orders Λ, for interacting theories with E_8 flavor group and assuming $C_T = \frac{151956}{5}$, which is the value in the rank-two E-string theory. Also shown is an extrapolation to infinite derivative order using the quadratic ansatz (8.1) with $\Lambda \geq 24$.

If Conjecture 3 is true, then the conformal bootstrap can potentially solve the E-string theories of arbitrary rank N. We can then consider the large N regime, and study the dual M-theory on $\mathrm{AdS}_7 \times \mathrm{S}^4/\mathbb{Z}_2$ beyond the supergravity limit. On the M-theory side, the low energy excitations consist of a supergravity multiplet in the eleven-dimensional bulk and an $\mathcal{N} = 1$ E_8 vector multiplet supported on a ten-dimensional locus, $\mathrm{AdS}_7 \times \mathrm{S}^3$ that is fixed by $\mathbb{Z}_2$. With enough computational power, we can collect information about the non-BPS spectra in the E-string theories of large N, filter out the operators dual to multi-particle excitations of the bulk supergravity and E_8 vector multiplets, and determine for instance the scaling dimension of the operator that corresponds to the first M-brane excitation.[24] The scaling dimension of this operator should behave as

$$\Delta = aN^b \tag{9.2}$$

to leading order at large N. The knowledge of a and b would be an important step towards understanding the quantum nature of M-branes.

We are also exploring other flavor groups. For instance, the $\mathrm{Sp}(4)_R$ R-symmetry in $\mathcal{N} = (2,0)$ theories breaks up into R-symmetry and flavor symmetry parts, $\mathrm{Sp}(2)_R \times \mathrm{Sp}(2)$, when interpreted as $\mathcal{N} = (1,0)$ theories. For the A_{N-1} theory, which is the infrared fixed point of the world-volume theory on a stack of N M5 branes, the central charge and flavor central charge are

$$C_T = 84(4N^3 - 3N - 1), \quad C_J = \frac{5}{2}(4N^3 - 3N - 1). \tag{9.3}$$

Other $\mathcal{N} = (1,0)$ theories include the large class of theories constructed in F-theory [52–55], whose C_T and C_J can be computed by using the anomaly polynomials given in [59, 61]. Finally, a particularly interesting example is a conjectural theory that has SU(3) flavor symmetry, and whose Higgs branch is given by the one-instanton moduli space of SU(3),

[24] Such an operator is analogous to the Konishi operator in $\mathcal{N} = 4$ SYM, whose dimension to leading order at large N is $2g_{\mathrm{YM}}^{1/2}N^{1/4}$ at strong coupling [89, 90] and $3g_{YM}^2 N/4\pi^2$ at weak coupling [91].

JHEP08(2017)128

recently proposed in [62]. It has central charge and flavor central charge

$$C_T = \frac{19488}{5}, \quad C_J = \frac{195}{2}. \tag{9.4}$$

This theory does not seem to appear in the F-theoretic "classification" of $\mathcal{N} = (1,0)$ theories [52–55].[25] The conformal bootstrap can provide evidence for the existence or non-existence of this theory.

The system of equations studied in this paper has straightforward generalizations to superconformal field theories in lower spacetime dimensions, $\mathcal{N} = 1$ in five and $\mathcal{N} = 3$ in three dimensions, which have $\mathrm{SU}(2)_R$ R-symmetry [92]. The C_T of such theories can be computed by taking the second derivative of the squashed three- or five-sphere partition function with respect to the squashing parameter [93–104].[26] In five dimensions, there is another distinguished class of superconformal field theories — Seiberg's E_n theories [47, 48]. If an analog of Conjecture 3 is true for these theories, then we can study the type I' string theory on a fibration of AdS_6 over S^4 [105]. In three dimensions, the Chern-Simons-Matter theories provide many examples of $\mathcal{N} = 3$ superconformal field theories [106–108]. It would be interesting if the conformal bootstrap predicts new $\mathcal{N} = 3$ theories that do not admit Chern-Simons-Matter constructions.

Acknowledgments

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A Bosonic conformal blocks

This appendix reviews properties of bosonic conformal blocks for the four-point function of scalar primaries with scaling dimensions $\Delta_1, \Delta_2, \Delta_3, \Delta_4$ in $d = 2\epsilon + 2$ spacetime dimensions. The conformal blocks depend on the external scaling dimensions only through the

[25] We thank Tom Rudelius for a discussion on this point.

[26] We thank Hee-Cheol Kim for a discussion on this point.

JHEP08(2017)128

differences $\Delta_{12} \equiv \Delta_1 - \Delta_2$ and $\Delta_{34} \equiv \Delta_3 - \Delta_4$, and will be denoted by $\mathcal{G}_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}$. In section A.1, we keep Δ_{12} and Δ_{34} arbitrary since blocks with nonzero Δ_{34} will be needed in appendix C, but for later sections we set $\Delta_{12} = \Delta_{34} = 0$. For notationally simplicity, we abbrevaite

$$\mathcal{G}_{\Delta,\ell}^{0,0} \to \mathcal{G}_{\Delta,\ell}. \tag{A.1}$$

The standard conformal cross ratios u, v are defined in terms of the positions of operators as

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}, \quad x_{12}^2 = (x_1 - x_2)^2. \tag{A.2}$$

We also introduce the variables z, $\bar{z}$ and χ, $\bar{\chi}$ as alternative ways to parameterize the cross ratios,[27]

$$u = z\bar{z} = \frac{\chi\bar{\chi}}{(1+\chi)(1+\bar{\chi})}, \qquad v = (1-z)(1-\bar{z}) = \frac{1}{(1+\chi)(1+\bar{\chi})}. \tag{A.3}$$

Radial coordinates r and η, defined as [109]

$$r(\eta + i\sqrt{1-\eta^2}) = \frac{z}{(1+\sqrt{1-z})^2}, \quad r(\eta - i\sqrt{1-\eta^2}) = \frac{\bar{z}}{(1+\sqrt{1-\bar{z}})^2}, \tag{A.4}$$

will be the variables in which we expand the conformal block in the recursive representation.

A.1 Expansion in Jack polynomials

The conformal block can be expanded in Jack polynomials [110],

$$\mathcal{G}_{\Delta,\ell}^{\Delta_{12},\Delta_{34}}(z,\bar{z}) = \sum_{m,n\geq 0} r_{mn}(\Delta_{12},\Delta_{34},\Delta,\ell) P^{(\epsilon)}_{\frac{1}{2}(\Delta+\ell)+m,\frac{1}{2}(\Delta-\ell)+n}(z,\bar{z}), \tag{A.5}$$

where the expansion coefficients r_{mn} are given by

$$\begin{aligned} r_{mn} = &\left(\frac{1}{2}(\Delta+\ell-\Delta_{12})\right)_m \left(\frac{1}{2}(\Delta+\ell+\Delta_{34})\right)_m \\ &\times \left(\frac{1}{2}(\Delta-\ell-\Delta_{12})-\epsilon\right)_n \left(\frac{1}{2}(\Delta-\ell+\Delta_{34})-\epsilon\right)_n \widehat{r}_{mn}, \end{aligned} \tag{A.6}$$

and $\widehat{r}_{mn}$ are defined recursively via

$$\begin{aligned} &(m(m+\Delta+\ell-1)+n(n+\Delta-\ell-2\epsilon-1))\widehat{r}_{mn} \\ &\quad = \frac{\ell+m-n-1+2\epsilon}{\ell+m-n-1+\epsilon}\widehat{r}_{m-1,n} + \frac{\ell+m-n+1}{\ell+m-n+1+\epsilon}\widehat{r}_{m,n-1}, \end{aligned} \tag{A.7}$$

with the initial condition $r_{00} = 1$.

Jack polynomials can be defined in terms of Gegenbauer polynomials

$$P^{(\epsilon)}_{\lambda_1,\lambda_2}(z,\bar{z}) = \frac{(\lambda_1-\lambda_2)!}{(2\epsilon)_{\lambda_1-\lambda_2}} (z\bar{z})^{\frac{1}{2}(\lambda_1+\lambda_2)} C^{(\epsilon)}_{\lambda_1-\lambda_2}\left(\frac{z+\bar{z}}{2(z\bar{z})^{1/2}}\right), \tag{A.8}$$

[27] The reader should be careful when comparing with [72], as we have swapped what they called z and χ.

JHEP08(2017)128

n_i	Δ_i	ℓ_i	range of k
k	$1-\ell-k$	$\ell+k$	$k=1,2,\ldots$
$2k$	$1+\epsilon-k$	ℓ	$k=1,2,\ldots$
k	$1+\ell+2\epsilon-k$	$\ell-k$	$k=1,2,\ldots,\ell$

Table 5. Three families of degenerate primaries that can appear in the OPE of two scalars, labeled by their scaling dimension Δ_i, spin ℓ_i, and level n_i of the first null descendant.

which satisfy the orthogonality condition

$$\int_{-1}^{1} C_m^{(\epsilon)}(x) C_n^{(\epsilon)}(x)(1-x^2)^{\epsilon-1/2}dx = \delta_{m,n}\frac{2^{1-2\epsilon}\pi\Gamma(m+2\epsilon)}{m!(m+\epsilon)\Gamma(\epsilon)^2}. \tag{A.9}$$

Jack polynomials are eigenfunctions of the differential operator Δ_ϵ defined in (3.6), with eigenvalues

$$\Delta_\epsilon P^{(\epsilon)}_{\lambda_1,\lambda_2}(z,\bar{z}) = \mathcal{E}^{(\epsilon)}_{\lambda_1,\lambda_2} P^{(\epsilon)}_{\lambda_1,\lambda_2}(z,\bar{z}), \quad \mathcal{E}^{(\epsilon)}_{\lambda_1,\lambda_2} = (\lambda_1+1+\epsilon)_{\epsilon-1}(\lambda_2+1)_{\epsilon-1}. \tag{A.10}$$

They also satisfy the relations

$$\begin{aligned}
(z\bar{z})^n P^{(\epsilon)}_{\lambda_1,\lambda_2}(z,\bar{z}) &= P^{(\epsilon)}_{\lambda_1+n,\lambda_2+n}(z,\bar{z}),\\
(z+\bar{z}) P^{(\epsilon)}_{\lambda_1,\lambda_2}(z,\bar{z}) &= \frac{\lambda_1-\lambda_2+2\epsilon}{\lambda_1-\lambda_2+\epsilon} P^{(\epsilon)}_{\lambda_1+1,\lambda_2}(z,\bar{z}) + \frac{\lambda_1-\lambda_2}{\lambda_1-\lambda_2+\epsilon} P^{(\epsilon)}_{\lambda_1,\lambda_2+1}(z,\bar{z}),\\
P^{(\epsilon)}_{\lambda_1,\lambda_2}(z,\bar{z}) &= P^{(\epsilon)}_{\lambda_2-\epsilon,\lambda_1+\epsilon}(z,\bar{z}).
\end{aligned} \tag{A.11}$$

A.2 Recursive representation

From now on we only consider the conformal blocks for the four-point function of identical scalar primaries, and set $\Delta_{12}=\Delta_{34}=0$.

When the scaling dimension of the internal primary is taken to values where a descendant becomes null, the conformal block encounters a simple pole whose residue is again another conformal block. This fact was first used in [111, 112] to write down a recursion formula for Virasoro blocks. The generalization to higher dimensions was obtained in [82], where the authors found that when the external operators are scalars, the degenerate primaries come in three classes, as we list in table 5. Then the conformal blocks admit the following recursive representation

$$\begin{aligned}
\mathcal{G}_{\Delta,\ell}(r,\eta) &= (-)^\ell (4r)^\Delta h_{\Delta,\ell}(r,\eta),\\
h_{\Delta,\ell}(r,\eta) &= \tilde{h}_\ell(r,\eta) + \sum_{i=1}^{3}\sum_k \frac{c_i(k)}{\Delta-\Delta_i(k)} r^{n_i(k)} h_{\Delta_i(k)+n_i(k),\ell_i(k)}(r,\eta),\\
\tilde{h}_\ell(r,\eta) &= \frac{\ell!}{(2\epsilon)_\ell}\frac{(-1)^\ell C_\ell^{(\epsilon)}(\eta)}{(1-r^2)^\epsilon (1+r^2+2r\eta)^{\frac{1}{2}}(1+r^2-2r\eta)^{\frac{1}{2}}},
\end{aligned} \tag{A.12}$$

JHEP08(2017)128

where $C_\ell^{(\epsilon)}(\eta)$ is the Gegenbauer polynomial. The coefficients $c_i(k)$ for the three types of degenerate weights are

$$\begin{aligned}
c_1(k) =& -\frac{4^k k(-1)^k}{(k!)^2}\frac{(\ell+2\epsilon)_k(\frac{1}{2}(1-k))_k(\frac{1}{2}(1-k))_k}{(\ell+\epsilon)_k},\\
c_2(k) =& -\frac{4^{2k} k(-1)^k}{(k!)^2}\frac{(\epsilon-k)_{2k}}{(\ell+\epsilon-k)_{2k}(\ell+\epsilon+1-k)_{2k}}\\
&\times\left(\frac{1}{2}(1-k+\ell+\epsilon)\right)_k\left(\frac{1}{2}(1-k+\ell+\epsilon)\right)_k\\
&\times\left(\frac{1}{2}(1-k+\ell+\epsilon)\right)_k\left(\frac{1}{2}(1-k+\ell+\epsilon)\right)_k,\\
c_3(k) =& -\frac{4^k k(-1)^k}{(k!)^2}\frac{(\ell+1-k)_k(\frac{1}{2}(1-k))_k(\frac{1}{2}(1-k))_k}{(\ell+\epsilon+1-k)_k}.
\end{aligned} \tag{A.13}$$

The virtue of this recursive representation is not only its computational efficiency. Firstly, the expansion in r converges better than the z expansion, as $r = 3-2\sqrt{2}\approx 0.17$ at the crossing symmetric point. Secondly, to a fixed order in r, the truncated conformal block with the $(4r)^\Delta$ prefactor stripped off is a rational function of Δ, whose poles are at values of Δ below the unitarity bound. This latter fact is crucial because semidefinite programming is much more efficient when the inputs are polynomials (for the sake of imposing non-negativity, we can strip off manifestly positive factors from the truncated conformal block); in fact, the SDPB package [31] only allows polynomial input.

For the purpose of computing derivatives of conformal blocks evaluated at the crossing symmetric point, we find it most efficient to — instead of implementing the above recursion relation — expand closed form expressions for conformal blocks in the diagonal limit $\bar{z}\to z$ to a fixed order in r ($\eta = 1$ on the diagonal), take the diagonal derivatives at the crossing symmetric point, and then apply a further recursion relation to obtain the transverse derivatives [19]. The closed form expressions and the recursion on transverse derivatives are reviewed in the next two sections.

A.3 Diagonal limit

When all external scalars have the same scaling dimension, the conformal blocks admit closed form expressions in the diagonal limit $\bar{z}\to z$, defined via a recursion relation [19]

$$\begin{aligned}
&(\ell+d-3)(2\Delta+2-d)\mathcal{G}_{\Delta,\ell}(z,z)\\
&=(d-2)(\Delta+\ell-1)\mathcal{G}_{\Delta,\ell-2}(z,z)+\frac{2-z}{2z}(2\ell+d-4)(\Delta-d+2)\mathcal{G}_{\Delta+1,\ell-1}(z,z)\\
&\quad-\frac{\Delta(2\ell+d-4)(\Delta+2-d)(\Delta+3-d)(\Delta-\ell-d+4)^2}{16(\Delta+1-\frac{d}{2})(\Delta-\frac{d}{2}+2)(\ell-\Delta+d-5)(\ell-\Delta+d-3)}\mathcal{G}_{\Delta+2,\ell-2}(z,z),
\end{aligned} \tag{A.14}$$

starting with seeds

$$\begin{aligned}
\mathcal{G}_{\Delta,0}(z,z) &= \left(\frac{z^2}{1-z}\right)^{\frac{\Delta}{2}}{}_3F_2\left(\tfrac{\Delta}{2},\tfrac{\Delta}{2},\tfrac{\Delta}{2}-\epsilon;\tfrac{\Delta+1}{2},\Delta-\epsilon;\frac{z^2}{4(z-1)}\right),\\
\mathcal{G}_{\Delta,1}(z,z) &= \frac{2-z}{2z}\left(\frac{z^2}{1-z}\right)^{\frac{\Delta+1}{2}}{}_3F_2\left(\tfrac{\Delta+1}{2},\tfrac{\Delta+1}{2},\tfrac{\Delta+1}{2}-\epsilon;\tfrac{\Delta}{2}+1,\Delta-\epsilon;\frac{z^2}{4(z-1)}\right).
\end{aligned} \tag{A.15}$$

JHEP08(2017)128

A.4 Recursion on transverse derivatives

Define

$$z = \frac{1+a+\sqrt{b}}{2}, \quad \bar{z} = \frac{1+a-\sqrt{b}}{2}, \tag{A.16}$$

and denote $\partial_a^m \partial_b^n \mathcal{G}_{\Delta,\ell}|_{a=b=0}$ by $h_{m,n}$. Given the diagonal limit of the conformal block, we can compute $h_{m,0}$ for all $m \geq 0$. The transverse derivatives can then be obtained by the following recursion relation [19],

$$\begin{aligned}
&2(d+2n-3)h_{m,n} \\
&= 2m(d+2n-3)[-h_{m-1,n} + (m-1)h_{m-2,n} + (m-1)(m-2)h_{m-3,n}] \\
&\quad - h_{m+2,n-1} + (d-m-4n+4)h_{m+1,n-1} \\
&\quad + [2C_{\Delta,\ell} + 2d(m+n-1) + m^2 + 8mn - 9m + 4n^2 - 6n + 2]h_{m,n-1} \\
&\quad + m[d(m-2n+1) + m^2 + 12mn - 15m + 12n^2 - 30n + 20]h_{m-1,n-1} \\
&\quad + (n-1)[h_{m+2,n-2} - (d-3m-4n+4)h_{m+1,n-2}].
\end{aligned} \tag{A.17}$$

B Superconformal Ward identities

The superconformal Ward identities read [72]

$$(\partial_\chi + \epsilon\partial_w)G(u,v;w)\Big|_{w\to\chi} = (\partial_{\bar{\chi}} + \epsilon\partial_w)G(u,v;w)\Big|_{w\to\bar{\chi}} = 0, \tag{B.1}$$

where the variables χ and $\bar{\chi}$ are related to u and v by

$$u = \frac{\chi\bar{\chi}}{(1+\chi)(1+\bar{\chi})}, \quad v = \frac{1}{(1+\chi)(1+\bar{\chi})}. \tag{B.2}$$

We presently show in the case of $k = 2$ that the second equation in (3.7) follows from the first as a consequence of the first superconformal Ward identity in (B.1), which explicitly reads

$$\chi^2\partial_\chi G_0(u,v) + \chi\partial_\chi G_1(u,v) + \partial_\chi G_2(u,v) = \epsilon G_1(u,v) + \frac{2\epsilon}{\chi}G_2(u,v). \tag{B.3}$$

It can be rewritten as

$$\partial_\chi(\chi\bar{\chi})^{-\epsilon}G_1(u,v) = -\chi^{-\epsilon+1}\bar{\chi}^{-\epsilon}\partial_\chi G_0(u,v) - \chi^{\epsilon-1}\bar{\chi}^{\epsilon}\partial_\chi(\chi\bar{\chi})^{-2\epsilon}G_2(u,v). \tag{B.4}$$

Applying $u \leftrightarrow v$ or equivalently $(\chi,\bar{\chi}) \leftrightarrow (\chi^{-1},\bar{\chi}^{-1})$, the above equation becomes

$$\partial_\chi(\chi\bar{\chi})^{\epsilon}G_1(v,u) = -\chi^{\epsilon-1}\bar{\chi}^{\epsilon}\partial_\chi G_0(v,u) - \chi^{-\epsilon+1}\bar{\chi}^{-\epsilon}\partial_\chi(\chi\bar{\chi})^{2\epsilon}G_2(v,u). \tag{B.5}$$

The difference of the two equations gives

$$\begin{aligned}
&\partial_\chi\left[(\chi\bar{\chi})^{-\epsilon}G_1(u,v) - (\chi\bar{\chi})^{\epsilon}G_1(v,u)\right] \\
&= -\chi^{-\epsilon+1}\bar{\chi}^{-\epsilon}\partial_\chi\left[G_0(u,v) - (\chi\bar{\chi})^{2\epsilon}G_2(v,u)\right] - \chi^{\epsilon-1}\bar{\chi}^{\epsilon}\partial_\chi\left[(\chi\bar{\chi})^{-2\epsilon}G_2(u,v) - G_0(v,u)\right].
\end{aligned} \tag{B.6}$$

JHEP08(2017)128

Similarly, with χ replaced by $\bar{\chi}$, we have

$$\begin{aligned}&\partial_{\bar{\chi}}\left[(\chi\bar{\chi})^{-\epsilon}G_1(u,v)-(\chi\bar{\chi})^{\epsilon}G_1(v,u)\right]\\&=-\chi^{-\epsilon}\bar{\chi}^{-\epsilon+1}\partial_{\bar{\chi}}\left[G_0(u,v)-(\chi\bar{\chi})^{2\epsilon}G_2(v,u)\right]-\chi^{\epsilon}\bar{\chi}^{\epsilon-1}\partial_{\bar{\chi}}\left[(\chi\bar{\chi})^{-2\epsilon}G_2(u,v)-G_0(v,u)\right].\end{aligned}\tag{B.7}$$

From (B.6) and (B.7), we see that the first and third equations of (3.7) imply the second equation of (3.7) up to a constant. This constant can be fixed by considering the case of $u=v$.

Let us define $K(u,v)=v^{2\epsilon}G_0(u,v)-u^{2\epsilon}G_2(v,u)$. The compatibility between (B.6) and (B.7) gives the identity

$$\begin{aligned}&\partial_{\bar{\chi}}\left[\chi^{-\epsilon+1}\bar{\chi}^{-\epsilon}\partial_{\chi}v^{-2\epsilon}K(u,v)-\chi^{\epsilon-1}\bar{\chi}^{\epsilon}\partial_{\chi}u^{-2\epsilon}K(v,u)\right]\\&=\partial_{\chi}\left[\chi^{-\epsilon}\bar{\chi}^{-\epsilon+1}\partial_{\bar{\chi}}v^{-2\epsilon}K(u,v)-\chi^{\epsilon}\bar{\chi}^{\epsilon-1}\partial_{\bar{\chi}}u^{-2\epsilon}K(v,u)\right],\end{aligned}\tag{B.8}$$

which is important when we want to identity the independent constraints from the crossing equation.

C Crossing equation for $b(u,v)$

Specializing to $\epsilon=k=2$, let us substitute the solution (3.5) into the superconformal Ward identity into the crossing equation (3.3),

$$D_2(1-z-zw^{-1})(1-\bar{z}-\bar{z}w^{-1})\left[z\bar{z}b(z,\bar{z})-(1-z)(1-\bar{z})b(1-z,1-\bar{z})\right]=0.\tag{C.1}$$

Defining $H(z,\bar{z})=z\bar{z}b(z,\bar{z})-(1-z)(1-\bar{z})b(1-z,1-\bar{z})$, the above equation is equivalent to

$$D_2H(z,\bar{z})=0,\quad D_2(z+\bar{z})H(z,\bar{z})=0,\quad D_2z\bar{z}H(z,\bar{z})=0.\tag{C.2}$$

The general solution to the first equation is[28]

$$H(z,\bar{z})=\sum_n a_nP^{(2)}_{n,0}(z,\bar{z}).\tag{C.3}$$

We also have

$$D_2z\bar{z}H=\sum_n a_n(n+3)P^{(2)}_{n,0}(z,\bar{z}),\quad D_2(z+\bar{z})H=\sum_n a_nnP^{(2)}_{n-1,0}(z,\bar{z}).\tag{C.4}$$

Using the fact that $P^{(2)}_{n,0}(z,\bar{z})$'s are orthogonal polynomials for non-negative integers n, one can argue that (C.2) has no non-trivial solution if we restrict to such n. However, the orthogonality condition fails if we allow n to take negative integer values, and indeed (C.2) has an unique solution

$$H(z,\bar{z})=\frac{1}{(z-\bar{z})^3}.\tag{C.5}$$

[28]There may appear to be another class of solutions $P^{(2)}_{-2,n+2}(z,\bar{z})$, but they are related to $P^{(2)}_{n,0}(z,\bar{z})$ by (A.11).

JHEP08(2017)128

Therefore, the original crossing equation is equivalent to

$$z\bar{z}b(z,\bar{z}) - (1-z)(1-\bar{z})b(1-z,1-\bar{z}) = \frac{c}{(z-\bar{z})^3}, \tag{C.6}$$

where c is an unphysical constant. For example, the generalized free field solution (F.2) corresponds to (up to the unphysical term)

$$b_{\text{gff}}(u,v) = \frac{1}{3}u^{-1}(1+u^{-3}+v^{-3}), \tag{C.7}$$

which solves (C.6) with $c=0$.

A function $b(u,v)$ that gives rise to a physical four-point function $G(u,v;w)$ (via (3.5)) also admits a decomposition into blocks with non-negative coefficients. The blocks $b^{\mathcal{X}}(u,v)$ for the superconformal multiplets (4.2) can be expressed (up to the unphysical term on the r.h.s. of (C.6)) in terms of bosonic conformal blocks with $\Delta_{12}=0$ and $\Delta_{34}=-2$,

$$\begin{aligned}
b^{\mathcal{L}[0]_{\Delta,\ell}}(u,v) &= \left(\frac{1}{2}(\Delta+\ell)\right)_{-1}\left(\frac{1}{2}(\Delta-\ell)-2\right)_{-1} u^{-5}\mathcal{G}^{0,-2}_{\Delta+2,\ell}(u,v),\\
b^{\mathcal{B}[2]_\ell}(u,v) &= \frac{2(\ell+4)}{(\ell+1)}\,(\ell+5)_{-1}\,u^{-5}\mathcal{G}^{0,-2}_{\ell+9,\ell+1}(u,v),\\
b^{\mathcal{B}[0]_\ell}(u,v) &= -\frac{1}{\ell+1}u^{-5}\mathcal{G}^{0,-2}_{\ell+6,\ell}(u,v),\\
b^{\mathcal{D}[4]}(u,v) &= 2u^{-5}\mathcal{G}^{0,-2}_{8,0}(u,v),\\
b^{\mathcal{D}[2]}(u,v) &= -\frac{2}{3}u^{-5}\mathcal{G}^{0,-2}_{5,-1}(u,v),\\
b^{\mathcal{D}[0]}(u,v) &= \frac{1}{3}u^{-4},
\end{aligned} \tag{C.8}$$

where the unphysical $\mathcal{G}^{0,-2}_{5,-1}(u,v)$ is formally defined by its expansion into Jack polynomials. Explicitly, $b^{\mathcal{D}[2]}(u,v)$ can be written as

$$\begin{aligned}
b^{\mathcal{D}[2]}(u,v) = &-\frac{4(z+\bar{z}-z\bar{z})}{z^3\bar{z}^3(z-\bar{z})^2} + \frac{4(\log z-\log\bar{z})}{z\bar{z}(z-\bar{z})^3}\\
&+\frac{4\bar{z}^2(\bar{z}-3z+3z^2-z^3\bar{z})\log(1-z) - 4z^2(z-3\bar{z}+3\bar{z}^2-z\bar{z}^3)\log(1-\bar{z})}{z^4\bar{z}^4(z-\bar{z})^3},
\end{aligned} \tag{C.9}$$

which has a branch point at the origin of the z-plane, and the monodromy around it is

$$b^{\mathcal{D}[2]}(u,v)\Big|_{(z,\bar{z})\to(e^{2\pi i}z,e^{-2\pi i}\bar{z})} = b^{\mathcal{D}[2]}(u,v) + \frac{16i\pi}{z\bar{z}(z-\bar{z})^3}. \tag{C.10}$$

This monodromy can be absorbed into a shift of the constant c,

$$c \to c + 32i\pi n. \tag{C.11}$$

We can therefore restrict to the zeroth sheet, where $b^{\mathcal{D}[2]}(u,v)$ along with other $b^{\mathcal{X}}(u,v)$ are all real functions in $z,\bar{z}$. Moreover, on this sheet, $b^{\mathcal{X}}(u,v)$ are regular as $\bar{z}\to z$, whereas the term on the right hand side of (C.6) is not. Hence, the constant c must vanish for a solution to (C.6) to also admit an expansion into blocks.

JHEP08(2017)128

D Relating central charges to OPE coefficients

D.1 C_T to $\lambda^2_{\mathcal{B}[0]_0}$

Conformal symmetry fixes the three-point function of the stress tensor with two identical scalars $\mathcal{O}$ to be of the form [76]

$$\langle T_{\mu\nu}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\rangle = \frac{C_{\mathcal{O}\mathcal{O}T}}{x_{12}^d x_{23}^{2\Delta-d} x_{13}^d} t_{\mu\nu}(X_{23}), \tag{D.1}$$

where the conformal structure $t_{\mu\nu}$ is given by

$$t_{\mu\nu}(X) = \frac{X_\mu X_\nu}{X^2} - \frac{1}{d}\delta_{\mu\nu}, \quad X_{23}^\mu = \frac{x_{21}^\mu}{x_{21}^2} - \frac{x_{31}^\mu}{x_{31}^2}, \quad X_{23}^2 = \frac{x_{23}^2}{x_{21}^2 x_{31}^2}. \tag{D.2}$$

The OPE coefficient $C_{\mathcal{O}\mathcal{O}T}$ is fixed by the conformal Ward identity to be [17, 113]

$$C_{\mathcal{O}\mathcal{O}T} = -\frac{d\Delta}{(d-1)V_{\mathbb{S}^{d-1}}}, \quad V_{\mathbb{S}^{d-1}} = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}. \tag{D.3}$$

For later use, we note that the tensor structures $I_{\mu\nu}(x)$, X^μ, $\mathcal{I}_{\mu\nu,\sigma\rho}$ and $t_{\mu\nu}$, defined in (6.2) and (D.2), satisfy the identities

$$\begin{aligned} \mathcal{I}_{\mu\nu,\alpha\beta}(x)\mathcal{I}_{\alpha\beta,\sigma\rho}(x) &= \frac{1}{2}(\delta_{\mu\sigma}\delta_{\nu\rho} + \delta_{\mu\rho}\delta_{\nu\sigma}) - \frac{1}{d}\delta_{\mu\nu}\delta_{\sigma\rho}, \\ I_{\mu\alpha}(x_{13})X_{23}^\alpha &= \frac{x_{23}^2}{x_{12}^2} X_{21}^\mu, \\ \mathcal{I}_{\mu\nu,\sigma\rho}(x_{13})t_{\sigma\rho}(X_{23}) &= t_{\sigma\rho}(X_{12}). \end{aligned} \tag{D.4}$$

From the three-point function (D.1), and using the identities (D.4), we can deduce that the OPE of two identical scalars contains

$$\mathcal{O}(x_1)\mathcal{O}(x_2) \sim \frac{1}{x_{12}^{2\Delta}} + \frac{C_{\mathcal{O}\mathcal{O}T}}{C_T}\frac{V^2_{\mathbb{S}^{d-1}}}{x_{12}^{2\Delta-d}} t_{\mu\nu}(x_{12})T^{\mu\nu}(x_2). \tag{D.5}$$

Now, consider the four-point function of four identical scalars $\mathcal{O}$, which reduces to a sum over three-point functions by taking the OPE (D.5) of the operators $\mathcal{O}(x_1)$ and $\mathcal{O}(x_2)$. Using the formula (D.1) for the three-point function, we obtain

$$\begin{aligned} &\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle \\ &\sim \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}}\left[1 + \frac{C^2_{\mathcal{O}\mathcal{O}T}}{C_T}\frac{1}{x_{12}^{-d}} t_{\mu\nu}(x_{12})\frac{V^2_{\mathbb{S}^{d-1}}}{x_{23}^d x_{34}^{-d} x_{24}^d} t_{\mu\nu}(X_{34})\right] \\ &= \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}}\left[1 + \frac{C^2_{\mathcal{O}\mathcal{O}T}}{C_T}\frac{V^2_{\mathbb{S}^{d-1}}}{x_{12}^{-d}x_{23}^d x_{34}^{-d}x_{24}^d}\left(\frac{(x_{12}^2x_{24}^2 + x_{14}^2x_{23}^2 - x_{13}^2x_{24}^2 - x_{12}^2x_{23}^2)^2}{4x_{12}^2x_{34}^2x_{23}^2x_{24}^2} - \frac{1}{d}\right)\right], \end{aligned} \tag{D.6}$$

which can be written in terms of the cross ratios u and v as

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle \sim \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}}\left[1 + \frac{C^2_{\mathcal{O}\mathcal{O}T}}{C_T}\frac{V^2_{\mathbb{S}^{d-1}}}{u^{-\frac{d}{2}}v^{\frac{d}{2}}}\left(\frac{(u+v-1)^2}{4uv} - \frac{1}{d}\right)\right]. \tag{D.7}$$

JHEP08(2017)128

Comparing (D.7) with the conformal block expansion, we determine the coefficient that sits in front of the bosonic stress-tensor block $\mathcal{G}_{d,2}(u,v)$,

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle \supset \mathcal{G}_{0,0}(u,v) + \frac{d}{(d-1)}\frac{\Delta^2}{C_T}\mathcal{G}_{d,2}(u,v). \tag{D.8}$$

The bosonic conformal block $\mathcal{G}_{d,2}(u,v)$ sits inside the $\mathcal{B}[0]_0$ superconformal block with the coefficient given in (4.17). We thus obtain the relation (6.3) between the OPE coefficient $\lambda_{\mathcal{B}[0]_0}$ and the central charge C_T.

D.2 C_J to $\lambda^2_{\mathcal{D}[2]}$

Consider the three-point function of one flavor current with two scalars transforming in representation $\mathcal{R}$ of the flavor group. Conformal symmetry fixes this three-point function to be[29]

$$\left\langle J^a_\mu(x_1)\mathcal{O}^i(x_2)\mathcal{O}^j(x_3)\right\rangle = g^{ik}(T^a_{\mathcal{R}})^j{}_k \frac{1}{V_{\mathbb{S}^{d-1}}}\frac{X^\mu_{23}}{x_{13}^{d-2}x_{12}^{d-2}x_{23}^{2\Delta-d+2}}, \tag{D.10}$$

where i,j are the indices for representation $\mathcal{R}$, $T^a_{\mathcal{R}}$ are the generators of the flavor group in the representation $\mathcal{R}$, and the two point functions of the scalars are normalized as $\left\langle \mathcal{O}^i(x_1)\mathcal{O}^j(x_2)\right\rangle = g^{ij}/x_{12}^{2\Delta}$. We are particularly interested in external scalars that transform in the adjoint representation, in which case $(T^a)^b{}_c = f^{ab}{}_c$.

From the three-point function (D.10), and using the identities (D.4), we obtain the OPE of two scalars in the adjoint representation,

$$\mathcal{O}^a(x_1)\mathcal{O}^b(x_2) \sim \frac{\delta^{ab}}{x_{12}^{2\Delta}} - f^{ab}{}_c\frac{V_{\mathbb{S}^{d-1}}}{C_J}\frac{x^\mu_{12}}{x_{12}^{2\Delta-d+2}}J^c_\mu(x_2). \tag{D.11}$$

Now consider the four-point function of four scalars $\mathcal{O}^a$. Using the OPE (D.11) and the three-point function (D.10), we find

$$\begin{aligned}
&\left\langle \mathcal{O}^a(x_1)\mathcal{O}^b(x_2)\mathcal{O}^c(x_3)\mathcal{O}^d(x_4)\right\rangle \\
&\sim \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}}\left[\delta^{ab}\delta^{cd} + \frac{f^{abe}f^{ecd}}{C_J}\frac{x_{12}X_{34}}{x_{12}^{-d+2}x_{24}^{d-2}x_{23}^{d-2}x_{34}^{-d+2}}\right] \\
&= \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}}\left[\dim(G_F)P^{abcd}_{\mathbf{1}} + \frac{\psi^2 h^\vee P^{abcd}_{\mathbf{adj}}}{2C_J}\frac{x_{12}^{d-2}x_{34}^{d-2}}{x_{24}^d x_{23}^d}\left((x_{12}^2-x_{13}^2)x_{24}^2-(x_{12}^2-x_{14}^2)x_{23}^2\right)\right]
\end{aligned} \tag{D.12}$$

which can be expressed in terms of the cross ratios u and v as

$$\begin{aligned}
&\left\langle \mathcal{O}^a(x_1)\mathcal{O}^b(x_2)\mathcal{O}^c(x_3)\mathcal{O}^d(x_4)\right\rangle \\
&\quad\sim \frac{1}{x_{12}^{2\Delta}x_{34}^{2\Delta}}\left[\dim(G_F)P^{abcd}_{\mathbf{1}} + \frac{\psi^2 h^\vee P^{abcd}_{\mathbf{adj}}}{2C_J}\frac{u^{\frac{d}{2}-1}}{v^{\frac{d}{2}}}(u+v-1)\right].
\end{aligned} \tag{D.13}$$

[29] Acting the charge (6.19) on the scalar $\mathcal{O}^j(0)$ gives

$$Q^a\mathcal{O}^j(0) = (T^a_{\mathcal{R}})^j{}_k\mathcal{O}^k(0), \tag{D.9}$$

which fixes the overall coefficient of the three-point function (D.10).

JHEP08(2017)128

By comparing (D.13) with the conformal block expansion, we can determine the coefficient sitting in front of the bosonic conformal block $\mathcal{G}_{d-1,1}(u,v)$ of the flavor current,

$$\Big\langle \mathcal{O}^a(x_1)\mathcal{O}^b(x_2)\mathcal{O}^c(x_3)\mathcal{O}^d(x_4)\Big\rangle \supset \dim(G_F) P_{\mathbf{1}}^{abcd}\mathcal{G}_{0,0}(u,v) - \frac{\psi^2 h^\vee}{C_J} P_{\mathbf{adj}}^{abcd}\mathcal{G}_{d-1,1}(u,v). \quad \text{(D.14)}$$

The bosonic conformal block $\mathcal{G}_{d-1,1}(u,v)$ sits inside the $\mathcal{D}[2]$ superconformal block with the coefficient given in (4.19). We thus obtain the relation (6.23) between the OPE coefficient $\lambda_{D[2]}$ and the flavor central charge C_J.

E The central charge C_T of the three-derivative fermion

The C_T of a free three-derivative Weyl fermion was recently computed in [68] as the second derivative of the partition function on $\mathrm{S}^1 \times \mathbb{H}^5$ with respect to the S^1 radius. In this appendix, we verify their answer by explicitly constructing the stress tensor for a three-derivative Dirac fermion, and computing its two-point function. The C_T of a Weyl fermion is simply half that of a Dirac fermion. Since the three-derivative Dirac fermion exists in arbitrary d spacetime dimensions, we keep $d = 2\epsilon + 2$ general. The two-point function of a free Dirac fermion with scaling dimension Δ_ψ is

$$\big\langle \psi(x_1)\bar{\psi}(x_2)\big\rangle = \frac{\not{x}_{12}}{x_{12}^{2\Delta_\psi+1}}, \quad \text{(E.1)}$$

where $\not{x} = x^\mu \Gamma_\mu$, and Γ^μ are $2^{\lfloor\epsilon\rfloor+1}\times 2^{\lfloor\epsilon\rfloor+1}$ matrices obeying the Clifford algebra $\{\Gamma^\mu,\Gamma^\nu\} = 2\delta^{\mu\nu}\mathbb{1}$. For a three-derivative fermion, $\Delta_\psi = \epsilon - \frac{1}{2}$.

Our approach is to work in flat space, write down the most general symmetric traceless spin-two primary operator of scaling dimension d, imposed current conservation, and identify the stress tensor by demanding that it has the correct OPE with the fundamental fermion [35],

$$T_{\mu\nu}(x)\psi(0) \sim -\frac{\Delta_\psi}{(d-1)V_{\widehat{\mathrm{S}}^{d-1}}}\frac{\delta_{\mu\nu}x^2 - d x_\mu x_\nu}{|x|^{d+2}}\psi(0) + \frac{d}{2V_{\widehat{\mathrm{S}}^{d-1}}}\frac{x_{(\underline{\mu}}x^\rho\Gamma_{\rho\underline{\nu})}\psi(0)}{|x|^{d+2}} + \cdots . \quad \text{(E.2)}$$

Let us first list all the symmetric traceless spin-two operators of scaling dimension d constructed as fermion bilinears,

$$\begin{aligned}
T^1_{\mu\nu} &= \bar{\psi}\not{\partial}\partial_\mu\partial_\nu\psi - \frac{1}{d}\delta_{\mu\nu}\bar{\psi}\not{\partial}\partial^2\psi, & T^2_{\mu\nu} &= \partial_{(\mu}\bar{\psi}\not{\partial}\partial_{\nu)}\psi - \frac{1}{d}\delta_{\mu\nu}\partial_\rho\bar{\psi}\not{\partial}\partial_\rho\psi,\\
T^3_{\mu\nu} &= \partial_\mu\partial_\nu\bar{\psi}\not{\partial}\psi - \frac{1}{d}\delta_{\mu\nu}\partial^2\bar{\psi}\not{\partial}\psi, & T^4_{\mu\nu} &= \bar{\psi}\Gamma_{(\mu}\partial_{\nu)}\partial^2\psi - \frac{1}{d}\delta_{\mu\nu}\bar{\psi}\not{\partial}\partial^2\psi,\\
T^5_{\mu\nu} &= \partial_\rho\bar{\psi}\Gamma_{(\mu}\partial_{\nu)}\partial_\rho\psi - \frac{1}{d}\delta_{\mu\nu}\partial_\rho\bar{\psi}\not{\partial}\partial_\rho\psi, & T^6_{\mu\nu} &= \partial^2\bar{\psi}\Gamma_{(\mu}\partial_{\nu)}\psi - \frac{1}{d}\delta_{\mu\nu}\partial^2\bar{\psi}\not{\partial}\psi,\\
T^7_{\mu\nu} &= \partial_{(\mu}\bar{\psi}\Gamma_{\nu)}\partial^2\psi - \frac{1}{d}\delta_{\mu\nu}\partial_\rho\bar{\psi}\Gamma_\rho\partial^2\psi, & T^8_{\mu\nu} &= \partial_\rho\partial_{(\mu}\bar{\psi}\Gamma_{\nu)}\partial_\rho\psi - \frac{1}{d}\delta_{\mu\nu}\partial_\rho\partial_\sigma\bar{\psi}\Gamma_\sigma\partial_\rho\psi,\\
T^9_{\mu\nu} &= \partial^2\partial_{(\mu}\bar{\psi}\Gamma_{\nu)}\psi - \frac{1}{d}\delta_{\mu\nu}\partial^2\partial_\rho\bar{\psi}\Gamma_\rho\psi, & T^{10}_{\mu\nu} &= \partial_\rho\bar{\psi}\Gamma_\rho\partial_\mu\partial_\nu\psi - \frac{1}{d}\delta_{\mu\nu}\partial_\rho\bar{\psi}\Gamma_\rho\partial^2\psi,\\
T^{11}_{\mu\nu} &= \partial_{(\mu}\partial_\rho\bar{\psi}\Gamma_\rho\partial_{\nu)}\psi - \frac{1}{d}\delta_{\mu\nu}\partial_\sigma\partial_\rho\bar{\psi}\Gamma_\rho\partial_\sigma\psi, & T^{12}_{\mu\nu} &= \partial_\mu\partial_\nu\partial_\rho\bar{\psi}\Gamma_\rho\psi - \frac{1}{d}\delta_{\mu\nu}\partial^2\partial_\rho\bar{\psi}\Gamma_\rho\psi,\\
T^{13}_{\mu\nu} &= \partial_\rho\bar{\psi}\partial_\sigma\partial_{(\underline{\mu}}\Gamma_{\underline{\nu})\rho\sigma}\psi, & T^{14}_{\mu\nu} &= \partial_\rho\partial_{(\underline{\mu}}\bar{\psi}\partial_\sigma\Gamma_{\underline{\nu})\rho\sigma}\psi,
\end{aligned} \quad \text{(E.3)}$$

JHEP08(2017)128

Eleven linearly independent combinations out of the fourteen $T^i_{\mu\nu}$ are descendants (total derivatives),

$$\begin{aligned}&T^1_{\mu\nu}+T^2_{\mu\nu},\quad T^2_{\mu\nu}+T^3_{\mu\nu},\quad T^4_{\mu\nu}+T^5_{\mu\nu},\quad T^5_{\mu\nu}+T^6_{\mu\nu},\quad T^7_{\mu\nu}+T^8_{\mu\nu},\\&T^8_{\mu\nu}+T^9_{\mu\nu},\quad T^{10}_{\mu\nu}+T^{11}_{\mu\nu},\quad T^{11}_{\mu\nu}+T^{12}_{\mu\nu},\quad T^5_{\mu\nu}+T^8_{\mu\nu},\quad T^2_{\mu\nu}+T^{11}_{\mu\nu},\quad T^{13}_{\mu\nu}+T^{14}_{\mu\nu}.\end{aligned}\tag{E.4}$$

Hence there are three linearly independent combinations of $T^i_{\mu\nu}$ that are primary operators, which by conformal symmetry must have vanishing two-point function with all the descendant operators (E.4). To find the correct linear combinations, we consider the two-point functions involving all fourteen $T^i_{\mu\nu}$,

$$\langle T^i_{\mu\nu}(x)T^j_{\rho\sigma}(0)\rangle.\tag{E.5}$$

We outline the intermediate steps for this computation. First, we compute the four-point functions,

$$\begin{aligned}\langle :\bar\psi(x_1)\Gamma_\mu\psi(x_2)::\bar\psi(x_3)\Gamma_\nu\psi(x_4):\rangle&=2^{\lfloor\epsilon\rfloor+1}\frac{\left(2x_{14,(\mu}x_{23,\nu)}-\delta_{\mu\nu}(x_{14}\cdot x_{23})\right)}{x_{23}^{2\epsilon}x_{14}^{2\epsilon}},\\\langle :\bar\psi(x_1)\Gamma_\mu\psi(x_2)::\bar\psi(x_3)\Gamma_{\nu\rho\sigma}\psi(x_4):\rangle&=2^{\lfloor\epsilon\rfloor+1}\frac{6\delta_{\mu[\nu}x_{14,\rho}x_{23,\sigma]}}{x_{23}^{2\epsilon}x_{14}^{2\epsilon}},\\\langle :\bar\psi(x_1)\Gamma_{\nu_1\rho_1\sigma_1}\psi(x_2)::\bar\psi(x_3)\Gamma_{\nu_2\rho_2\sigma_2}\psi(x_4):\rangle&=\frac{\mathrm{Tr}\left(\Gamma_{\nu_1\rho_1\sigma_1}\not{x}_{23}\Gamma_{\nu_2\rho_2\sigma_2}\not{x}_{14}\right)}{x_{23}^{2\epsilon}x_{14}^{2\epsilon}}.\end{aligned}\tag{E.6}$$

Then the two-point functions (E.5) can be obtained by taking derivatives on (E.6), followed by the limit $x_1,x_2\to x$ and $x_3,x_4\to 0$. For $i,j=13,14$, it is convenient to define

$$\begin{aligned}K_{\nu_1\nu_2}&=\frac{\partial^4}{\partial x_{1,\rho_1}\partial x_{2,\sigma_1}\partial x_{3,\rho_2}\partial x_{4,\sigma_2}}\frac{\mathrm{Tr}\left(\Gamma_{\nu_1\rho_1\sigma_1}\not{x}_{23}\Gamma_{\nu_2\rho_2\sigma_2}\not{x}_{14}\right)}{x_{23}^{2\epsilon}x_{14}^{2\epsilon}}\\&=2^{\lfloor\epsilon\rfloor+1}\times\Bigg\{8\epsilon^2\delta_{\nu_1\nu_2}\left[-3\epsilon\frac{x_{23}\cdot x_{14}}{x_{23}^{2\epsilon+2}x_{14}^{2\epsilon+2}}+2(\epsilon+1)^2\frac{(x_{23}\cdot x_{14})^3}{x_{23}^{2\epsilon+4}x_{14}^{2\epsilon+4}}-2\frac{(x_{23}\cdot x_{14})}{x_{23}^{2\epsilon+2}x_{14}^{2\epsilon+2}}\right]\\&\quad+2(\partial_1)_{(\nu_1}(\partial_2)_{\nu_2)}\left[\frac{2(\epsilon-1)}{x_{23}^{2\epsilon}x_{14}^{2\epsilon}}-4\epsilon^2\frac{(x_{23}\cdot x_{14})^2}{x_{23}^{2\epsilon+2}x_{14}^{2\epsilon+2}}\right]\\&\quad-4\epsilon\left[(\partial_2)_{\nu_1}(\partial_2)_{\nu_2}\frac{x_{23}\cdot x_{14}}{x_{23}^{2\epsilon}x_{14}^{2\epsilon+2}}+(\partial_1)_{\nu_1}(\partial_1)_{\nu_2}\frac{x_{23}\cdot x_{14}}{x_{23}^{2\epsilon+2}x_{14}^{2\epsilon}}\right]\Bigg\}.\end{aligned}\tag{E.7}$$

Then we have

$$\begin{aligned}\langle T^{13}_{\mu_1\nu_1}T^{13}_{\mu_2\nu_2}\rangle&=(\partial_2)_{(\underline{\mu_1}}(\partial_1)_{(\mu_2}K_{\nu_2)\underline{\nu_1)}},\quad\langle T^{13}_{\mu_1\nu_1}T^{14}_{\mu_2\nu_2}\rangle=(\partial_2)_{(\underline{\mu_1}}(\partial_2)_{(\mu_2}K_{\nu_2)\underline{\nu_1)}},\\\langle T^{14}_{\mu_1\nu_1}T^{13}_{\mu_2\nu_2}\rangle&=(\partial_1)_{(\underline{\mu_1}}(\partial_1)_{(\mu_2}K_{\nu_2)\underline{\nu_1)}},\quad\langle T^{14}_{\mu_1\nu_1}T^{14}_{\mu_2\nu_2}\rangle=(\partial_1)_{(\underline{\mu_1}}(\partial_2)_{(\mu_2}K_{\nu_2)\underline{\nu_1)}}.\end{aligned}\tag{E.8}$$

The two-point functions (E.5) allow us to identity the three-dimensional space of primary operators as the space orthogonal to the descendants. In unitary theories, a primary operator with scaling dimension saturating the unitarity bound must be conserved, but this is false in non-unitary theories. Indeed, using the explicit two-point functions (E.5), we find that there are two conserved spin-two primaries and one non-conserved spin-two

JHEP08(2017)128

primary. The stress tensor is a particular linear combination of the two conserved spin-two primaries that satisfies the $T_{\mu\nu}\psi$ OPE (E.2). A consequence of this OPE is that in the large x_2 limit,

$$\begin{aligned} &x_1^\mu x_2^\nu \big\langle T_{\mu\nu}(x_1)\bar\psi(x_2)\not{x}_1\psi(0)\big\rangle \\ &\quad = \frac{2^{\lfloor\epsilon\rfloor+1}\left(\Delta_\psi-\frac{d}{4}\right)(x_1\cdot x_2)^2}{V_{\widehat{\mathbb{S}}^{d-1}}x_1^d x_2^{2\Delta_\psi+1}} + \frac{2^{\lfloor\epsilon\rfloor+1}d}{4V_{\widehat{\mathbb{S}}^{d-1}}x_1^{d-2}x_2^{2\Delta_\psi-1}} + \mathcal{O}(x_2^{-2\Delta_\psi}). \end{aligned} \tag{E.9}$$

To find the stress tensor, we compute the three-point functions

$$x_1^\mu x_2^\nu \big\langle T^i_{\mu\nu}(x_1)\bar\psi(x_2)\not{x}_1\psi(0)\big\rangle, \tag{E.10}$$

and identify the correct linear combination of conserved primaries to match with (E.9). This computation can be done by taking derivatives on the three-point functions

$$\begin{aligned} &\big\langle :\bar\psi(x_4)\Gamma_\mu\psi(x_1):\bar\psi(x_2)\not{x}_3\psi(0)\big\rangle \\ &\quad = -2^{\lfloor\epsilon\rfloor+1}\frac{x_{4,\mu}(x_2\cdot x_3)-x_{3,\mu}(x_2\cdot x_4)+x_{2,\mu}(x_3\cdot x_4)}{x_2^{2\epsilon}x_4^{2\epsilon}} + \mathcal{O}(x_2^{-2\epsilon}), \\ &\big\langle :\bar\psi(x_4)\Gamma_{\mu\nu\rho}\psi(x_1):\bar\psi(x_2)\not{x}_3\psi(0)\big\rangle = 2^{\lfloor\epsilon\rfloor+1}\frac{6x_{2,[\underline{\mu}}x_{3,\underline{\nu}}x_{4,\underline{\rho}]}}{x_2^{2\epsilon}x_4^{2\epsilon}} + \mathcal{O}(x_2^{-2\epsilon}), \end{aligned} \tag{E.11}$$

followed by the limit $x_3, x_4 \to x_1$. For example, we have

$$\begin{aligned} x_1^\mu x_2^\nu \big\langle T^{13}_{\mu\nu}(x_1)\bar\psi(x_2)\not{x}_1\psi(0)\big\rangle &= 0 + \mathcal{O}(x_2^{-2\epsilon+1}), \\ x_1^\mu x_2^\nu \big\langle T^{14}_{\mu\nu}(x_1)\bar\psi(x_2)\not{x}_1\psi(0)\big\rangle &= 0 + \mathcal{O}(x_2^{-2\epsilon+1}). \end{aligned} \tag{E.12}$$

In four spacetime dimensions, the stress tensor $T_{\mu\nu}$, the spin-two conserved primary $\widetilde{T}_{\mu\nu}$ orthogonal to $T_{\mu\nu}$, and the spin-two non-conserved primary $\Theta_{\mu\nu}$ orthogonal to both $T_{\mu\nu}$ and $\widetilde{T}_{\mu\nu}$ are

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{48\pi^2}\big(-2T^1_{\mu\nu}-T^2_{\mu\nu}+7T^3_{\mu\nu}+3T^4_{\mu\nu}+9T^5_{\mu\nu}+9T^6_{\mu\nu}-9T^7_{\mu\nu} \\ &\quad -9T^8_{\mu\nu}-3T^9_{\mu\nu}-7T^{10}_{\mu\nu}+T^{11}_{\mu\nu}+2T^{12}_{\mu\nu}+3T^{13}_{\mu\nu}-3T^{14}_{\mu\nu}\big), \\ \widetilde{T}_{\mu\nu} &= 5T^1_{\mu\nu}+2T^2_{\mu\nu}-17T^3_{\mu\nu}-7T^4_{\mu\nu}-22T^5_{\mu\nu}-21T^6_{\mu\nu}+21T^7_{\mu\nu} \\ &\quad +22T^8_{\mu\nu}+7T^9_{\mu\nu}+17T^{10}_{\mu\nu}-2T^{11}_{\mu\nu}-5T^{12}_{\mu\nu}-8T^{13}_{\mu\nu}+8T^{14}_{\mu\nu}, \\ \Theta_{\mu\nu} &= 2T^1_{\mu\nu}-3T^2_{\mu\nu}-5T^3_{\mu\nu}-2T^4_{\mu\nu}-9T^5_{\mu\nu}-7T^6_{\mu\nu}+7T^7_{\mu\nu} \\ &\quad +9T^8_{\mu\nu}+2T^9_{\mu\nu}+5T^{10}_{\mu\nu}+3T^{11}_{\mu\nu}-2T^{12}_{\mu\nu}-3T^{13}_{\mu\nu}+3T^{14}_{\mu\nu}. \end{aligned} \tag{E.13}$$

In six spacetime dimensions, they are

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{80\pi^3}\big(-2T^1_{\mu\nu}-4T^2_{\mu\nu}+8T^3_{\mu\nu}+5T^4_{\mu\nu}+10T^5_{\mu\nu}+10T^6_{\mu\nu}-10T^7_{\mu\nu} \\ &\quad -10T^8_{\mu\nu}-5T^9_{\mu\nu}-8T^{10}_{\mu\nu}+4T^{11}_{\mu\nu}+2T^{12}_{\mu\nu}+5T^{13}_{\mu\nu}-5T^{14}_{\mu\nu}\big), \\ \widetilde{T}_{\mu\nu} &= 2T^1_{\mu\nu}+2T^2_{\mu\nu}-6T^3_{\mu\nu}-3T^4_{\mu\nu}-8T^5_{\mu\nu}-6T^6_{\mu\nu}+6T^7_{\mu\nu} \\ &\quad +8T^8_{\mu\nu}+3T^9_{\mu\nu}+6T^{10}_{\mu\nu}-2T^{11}_{\mu\nu}-2T^{12}_{\mu\nu}-5T^{13}_{\mu\nu}+5T^{14}_{\mu\nu}, \\ \Theta_{\mu\nu} &= 8T^1_{\mu\nu}-5T^2_{\mu\nu}-13T^3_{\mu\nu}-8T^4_{\mu\nu}-25T^5_{\mu\nu}-17T^6_{\mu\nu}+17T^7_{\mu\nu} \\ &\quad +25T^8_{\mu\nu}+8T^9_{\mu\nu}+13T^{10}_{\mu\nu}+5T^{11}_{\mu\nu}-8T^{12}_{\mu\nu}-15T^{13}_{\mu\nu}+15T^{14}_{\mu\nu}. \end{aligned} \tag{E.14}$$

JHEP08(2017)128

We can read off the central charge C_T from the two-point function (6.1) of the stress tensor $T_{\mu\nu}$. In four spacetime dimensions, we find

$$C_T = -\frac{8}{3} \quad \text{(4d three-derivative Dirac fermion)}. \tag{E.15}$$

In six spacetime dimensions, we find

$$C_T = -\frac{144}{5} \quad \text{(6d three-derivative Dirac fermion)}. \tag{E.16}$$

These values are in agreement with [68, 114, 115].

F Analytic examples of solutions to crossing

We write down two analytic solutions to the superconformal crossing equation (3.3), using first generalized free fields (mean field theory) and second a free hypermultiplet. Since these solutions exist in arbitrary spacetime dimensions, we keep $d = 2\epsilon + 2$ general.

F.1 Generalized free fields

The four-point function of generalized free fields is the sum over factorized two-point functions in the three channels (mean field theory),

$$\langle \mathcal{O}(X_1, Y_1)\mathcal{O}(X_2, Y_2)\mathcal{O}(X_3, Y_3)\mathcal{O}(X_4, Y_4)\rangle = \left(\frac{(Y_1 \cdot Y_2)(Y_3 \cdot Y_4)}{x_{12}^{2\epsilon} x_{34}^{2\epsilon}}\right)^k G_{\text{gff}}(u, v; w), \tag{F.1}$$

where $G_{\text{gff}}(u, v; w)$ is given by

$$G_{\text{gff}}(u, v; w) = 1 + \left(\frac{(1+w)u^\epsilon}{w}\right)^k + \left(\frac{u^\epsilon}{wv^\epsilon}\right)^k. \tag{F.2}$$

It satisfies the crossing equations (3.3) and (3.4) by construction. Let us focus on $k = 2$, and decompose $G_{\text{gff}}(u, v; w)$ into $\mathfrak{su}(2)_R$ harmonics,

$$\begin{aligned} G_{\text{gff}}(u, v; w) = &\left[1 + \frac{1}{3}\left(u^{2\epsilon} + \frac{u^{2\epsilon}}{v^{2\epsilon}}\right)\right] + \frac{1}{2}\left[u^{2\epsilon} - \frac{u^{2\epsilon}}{v^{2\epsilon}}\right] P_1(1 + \tfrac{2}{w}) \\ &+ \frac{1}{6}\left[u^{2\epsilon} + \frac{u^{2\epsilon}}{v^{2\epsilon}}\right] P_2(1 + \tfrac{2}{w}). \end{aligned} \tag{F.3}$$

The factors multiplying $P_{J_R}(1 + \frac{2}{w})$ can be decomposed into bosonic conformal blocks,

$$\begin{aligned} u^{2\epsilon} + \frac{u^{2\epsilon}}{v^{2\epsilon}} &= \sum_{\ell \text{ even}, \ell \geq 0} \sum_{n=0}^{\infty} P_{n,\ell} \mathcal{G}_{\ell+4\epsilon+2n,\ell}(u, v), \\ u^{2\epsilon} - \frac{u^{2\epsilon}}{v^{2\epsilon}} &= \sum_{\ell \text{ odd}, \ell \geq 1} \sum_{n=0}^{\infty} P_{n,\ell} \mathcal{G}_{\ell+4\epsilon+2n,\ell}(u, v), \end{aligned} \tag{F.4}$$

JHEP08(2017)128

where the coefficients $P_{n,\ell}$ are given in [116]

$$P_{n,\ell} = \frac{(\ell+\epsilon)_\epsilon}{(\epsilon)_\epsilon} \times \frac{2(-1)^\ell(\epsilon)_n^2(2\epsilon)_{n+\ell}^2}{\Gamma(\ell+1)\Gamma(n+1)(\ell+\epsilon+1)_n(2\epsilon+n-1)_n(4\epsilon+2n+\ell-1)_\ell(3\epsilon+n+\ell-1)_n}. \tag{F.5}$$

$G_{\text{gff}}(u,v;w)$ can also be decomposed into superconformal blocks,

$$\begin{aligned} G_{\text{gff}}(u,v;w) = 1 + \lambda^2_{\mathcal{D}[4]}\mathcal{A}_{\mathcal{D}[4]} + \sum_{\ell\,\text{odd},\,\ell\geq 1} \lambda^2_{\mathcal{B}[2]_\ell}\mathcal{A}_{\mathcal{B}[2]_\ell} \\ + \sum_{\ell\,\text{even},\,\ell\geq 0}\sum_{n=0}^{\infty} \lambda^2_{\mathcal{L}[0]_{\ell+4\epsilon+2n,\ell}}\mathcal{A}_{\mathcal{L}[0]_{\ell+4\epsilon+2n,\ell}}, \end{aligned} \tag{F.6}$$

where the OPE coefficients are given by

$$\begin{aligned} \lambda^2_{\mathcal{D}[4]} &= \frac{P_{0,0}}{6} = \frac{1}{3}, \quad \lambda^2_{\mathcal{B}[2]_\ell} = \frac{\ell+1}{8(\ell+2\epsilon)}P_{0,\ell+1}, \\ \lambda^2_{\mathcal{L}[0]_{\ell+4\epsilon+2n,\ell}} &= \frac{(n+1)(n+L+\epsilon+1)}{16(n+\epsilon)(n+L+2\epsilon)}P_{n+1,\ell}. \end{aligned} \tag{F.7}$$

We note that the decomposition of the generalized free field solution is void of any conserved current.

F.2 Free hypermultiplet

A free hypermultiplet consists of a pair of complex scalars transforming in the fundamental representation of $\mathfrak{su}(2)_R$, and a fermion singlet. The fermion could be Dirac, Majorana, or Weyl depending on the number of spacetime dimensions; in six dimensions, it is a Weyl fermion. Let us denote the complex scalar doublet by ϕ_A, and $\bar{\phi}^A$ its complex conjugate $\bar{\phi}^A = (\phi_A)^*$. They are normalized by the two-point function

$$\left\langle \phi_{A_1}(x_1)\bar{\phi}^{A_2}(x_2)\right\rangle = \frac{\delta^{A_2}_{A_1}}{x_{12}^{2\epsilon}}. \tag{F.8}$$

The superconformal primaries of a $D[2]$ superconformal multiplet have scaling dimension 2ϵ, and can be constructed as scalar bilinears

$$\bar{\phi}^A(\sigma^a)_A{}^B\phi_B, \tag{F.9}$$

where $A,B=1,2$, $a=1,2,3$, and $(\sigma^a)_A{}^B$ are the Pauli matrices. To keep track of $\mathfrak{su}(2)_R$, we can contract the scalars with auxiliary variables Y^A, and consider the four-point function of $\mathcal{O}\equiv i\phi_A\bar{\phi}^BY^AY_B$,[30]

$$\langle\mathcal{O}(x_1,Y_1)\mathcal{O}(x_2,Y_2)\mathcal{O}(x_3,Y_3)\mathcal{O}(x_4,Y_4)\rangle = \left(\frac{(Y_1\cdot Y_2)(Y_3\cdot Y_4)}{x_{12}^{2\epsilon}x_{34}^{2\epsilon}}\right)^2 G_{\text{hyper}}(u,v,w), \tag{F.10}$$

[30]The indices can be raised and lowered by $Y_A=\epsilon_{AB}Y^B$ and $Y^A=Y_B\epsilon^{BA}$.

JHEP08(2017)128

where $G_{\text{hyper}}(u,v,w)$ is given by

$$\begin{aligned} G_{\text{hyper}}(u,v,w) &= G_{\text{gff}}(u,v,w) + G_{\text{extra}}(u,v,w), \\ G_{\text{extra}}(u,v,w) &= -\frac{2}{w}\left(\frac{u}{v}\right)^{\epsilon} + 2\left(1+\frac{1}{w}\right)u^{\epsilon} + \frac{2}{w}\left(1+\frac{1}{w}\right)\left(\frac{u^2}{v}\right)^{\epsilon}. \end{aligned} \tag{F.11}$$

A single free hypermultiplet has SU(2) flavor symmetry.[31] We can construct a triplet of $D[2]$ superconformal primaries,

$$\mathcal{O}_1 = \frac{i}{2}(\mathcal{O}_+ - \mathcal{O}_-), \quad \mathcal{O}_3 = -\frac{1}{2}(\mathcal{O}_+ + \mathcal{O}_-), \quad \mathcal{O}_2 \equiv \mathcal{O} = i\phi_A \bar{\phi}^B Y^A Y_B, \tag{F.12}$$

where $\mathcal{O}_+ = \phi_A \phi_B Y^A Y^B$ and $\mathcal{O}_- = \bar{\phi}^A \bar{\phi}^B Y_A Y_B$. The SU(2) flavor symmetry can be made manifest by introducing new auxiliary variables $\widetilde{Y}_A$, and defining

$$\mathcal{O}(x,Y,\widetilde{Y}) = i(\sigma^a)_{\dot{A}}{}^{\dot{B}} \widetilde{Y}^{\dot{A}} \widetilde{Y}_{\dot{B}} \mathcal{O}_a(x,Y). \tag{F.13}$$

One can write

$$\mathcal{O}(x,Y,\widetilde{Y}) = (\phi_{\dot{A}A}\widetilde{Y}^{\dot{A}}Y^A)^2, \tag{F.14}$$

where ϕ_A and $\bar{\phi}_A$ are expressed in terms of $\phi_{\dot{A}A}$ as[32]

$$\phi_A = \frac{1}{\sqrt{2}}(\phi_{1A} + i\phi_{2A}), \quad \bar{\phi}_A = \frac{1}{\sqrt{2}}(-\phi_{2A} - i\phi_{1A}). \tag{F.15}$$

It is now straightforward to compute the four-point function,

$$\begin{aligned} &\langle \mathcal{O}(x_1,Y_1,\widetilde{Y}_1)\mathcal{O}(x_2,Y_2,\widetilde{Y}_2)\mathcal{O}(x_3,Y_3,\widetilde{Y}_3)\mathcal{O}(x_4,Y_4,\widetilde{Y}_4)\rangle \\ &= \frac{(Y_1\cdot Y_2)^2(Y_3\cdot Y_4)^2(\widetilde{Y}_1\cdot\widetilde{Y}_2)^2(\widetilde{Y}_3\cdot\widetilde{Y}_4)^2}{x_{12}^{4\epsilon}x_{34}^{4\epsilon}} G_{\text{hyper}}(u,v,w,\widetilde{w}), \end{aligned} \tag{F.16}$$

where $\widetilde{w}$ is defined the same way as w, and $G_{\text{hyper}}(u,v,w,\widetilde{w})$ is given by

$$\begin{aligned} G_{\text{hyper}}(u,v,w,\widetilde{w}) = 4 &+ 4\left(\frac{(1+w)(1+\widetilde{w})u^{\epsilon}}{w\widetilde{w}}\right)^2 + 4\left(\frac{u^{\epsilon}}{w\widetilde{w}v^{\epsilon}}\right)^2 \\ &+ \frac{16}{w\widetilde{w}}\left(\frac{u}{v}\right)^{\epsilon} + 16\frac{(1+w)(1+\widetilde{w})}{w\widetilde{w}}u^{\epsilon} + 16\frac{(1+w)(1+\widetilde{w})}{w^2\widetilde{w}^2}\left(\frac{u^2}{v}\right)^{\epsilon}. \end{aligned} \tag{F.17}$$

[31] The two complex scalars, regarded as four real scalars, can be rotated by an SO(4) action which is a direct sum of the $\text{SU}(2)_R$ R-symmetry and the SU(2) flavor symmetry. The Weyl spinor in six dimensions admits a quaternionic structure, and also transforms as a doublet under the SU(2) flavor symmetry.

[32] The scalar $\phi_{\dot{A}A}$ satisfies the reality condition $(\phi_{\dot{A}A})^* = \phi^{\dot{A}A}$.

JHEP08(2017)128

The four-point function $G_{\text{hyper}}(u,v,w,\widetilde{w})$ can be further decomposed into Legendre polynomials,

$$
\begin{aligned}
&G_{\text{hyper}}(u,v,w,\widetilde{w})\\
&=4\Big\{P_0(1+\tfrac{2}{\widetilde{w}})P_0(1+\tfrac{2}{w})\left[1+\frac{1}{9}\left(u^{2\epsilon}+\left(\frac{u}{v}\right)^{2\epsilon}\right)+\frac{1}{9}\left(9u^{\epsilon}+9\left(\frac{u}{v}\right)^{\epsilon}+\left(\frac{u^2}{v}\right)^{\epsilon}\right)\right]\\
&\quad+P_1(1+\tfrac{2}{\widetilde{w}})P_1(1+\tfrac{2}{w})\left[\frac{1}{4}\left(u^{2\epsilon}+\left(\frac{u}{v}\right)^{2\epsilon}\right)+\left(u^{\epsilon}+\left(\frac{u}{v}\right)^{\epsilon}\right)\right]\\
&\quad+P_2(1+\tfrac{2}{\widetilde{w}})P_2(1+\tfrac{2}{w})\left[\frac{1}{36}\left(u^{2\epsilon}+\left(\frac{u}{v}\right)^{2\epsilon}\right)+\frac{1}{9}\left(\frac{u^2}{v}\right)^{\epsilon}\right]\Big\}\\
&\quad+4\Big\{P_0(1+\tfrac{2}{\widetilde{w}})P_1(1+\tfrac{2}{w})\left[\frac{1}{6}\left(u^{2\epsilon}-\left(\frac{u}{v}\right)^{2\epsilon}\right)+\left(u^{\epsilon}-\left(\frac{u}{v}\right)^{\epsilon}\right)\right]\\
&\quad+P_0(1+\tfrac{2}{\widetilde{w}})P_2(1+\tfrac{2}{w})\left[\frac{1}{18}\left(u^{2\epsilon}+\left(\frac{u}{v}\right)^{2\epsilon}\right)-\frac{1}{9}\left(\frac{u^2}{v}\right)^{\epsilon}\right]\\
&\quad+P_1(1+\tfrac{2}{\widetilde{w}})P_2(1+\tfrac{2}{w})\left[\frac{1}{12}\left(u^{2\epsilon}-\left(\frac{u}{v}\right)^{2\epsilon}\right)\right]+(w\leftrightarrow\widetilde{w})\Big\}.
\end{aligned}
\tag{F.18}
$$

If we define

$$
G_{\text{hyper}}(u,v,w,\widetilde{w})=G_{\text{hyper}}^{a_1a_2a_3a_4}(u,v,w)\mathbf{Y}_{1,a_1}\mathbf{Y}_{2,a_2}\mathbf{Y}_{3,a_3}\mathbf{Y}_{4,a_4},
\tag{F.19}
$$

where $\mathbf{Y}^a\equiv i(\sigma^a)_{\dot{A}}{}^{\dot{B}}\widetilde{Y}^{\dot{A}}\widetilde{Y}_{\dot{B}}$, then $G_{\text{hyper}}^{a_1a_2a_3a_4}(u,v,w)$ admits a superconformal block decomposition,[33]

$$
G_{\text{hyper}}^{a_1a_2a_3a_4}(u,v,w)=\sum_{i\in\{\mathbf{1},\mathbf{3},\mathbf{5}\}}P_i^{a_1a_2a_3a_4}\lambda_{\mathcal{X},i}^2\mathcal{A}_{\mathcal{X}}(u,v,w).
\tag{F.22}
$$

[33]The SU(2) projection matrices are

$$
P_{\mathbf{1}}^{abcd}=\frac{1}{3}\delta^{ab}\delta^{cd},\quad P_{\mathbf{3}}^{abcd}=\frac{1}{2}(\delta^{ad}\delta^{bc}-\delta^{ac}\delta^{bd}),\quad P_{\mathbf{5}}^{abcd}=\frac{1}{2}(\delta^{ad}\delta^{bc}+\delta^{ac}\delta^{bd})-\frac{1}{3}\delta^{ab}\delta^{cd},
\tag{F.20}
$$

which are related to Legendre polynomials by

$$
\begin{aligned}
P_{\mathbf{1}}^{a_1a_2a_3a_4}\mathbf{Y}_{1,a_1}\mathbf{Y}_{2,a_2}\mathbf{Y}_{3,a_3}\mathbf{Y}_{4,a_4}&=\frac{4}{3}(\widetilde{Y}_1\cdot\widetilde{Y}_2)^2(\widetilde{Y}_3\cdot\widetilde{Y}_4)^2P_0(1+\tfrac{2}{\widetilde{w}}),\\
P_{\mathbf{3}}^{a_1a_2a_3a_4}\mathbf{Y}_{1,a_1}\mathbf{Y}_{2,a_2}\mathbf{Y}_{3,a_3}\mathbf{Y}_{4,a_4}&=-2(\widetilde{Y}_1\cdot\widetilde{Y}_2)^2(\widetilde{Y}_3\cdot\widetilde{Y}_4)^2P_1(1+\tfrac{2}{\widetilde{w}}),\\
P_{\mathbf{5}}^{a_1a_2a_3a_4}\mathbf{Y}_{1,a_1}\mathbf{Y}_{2,a_2}\mathbf{Y}_{3,a_3}\mathbf{Y}_{4,a_4}&=\frac{2}{3}(\widetilde{Y}_1\cdot\widetilde{Y}_2)^2(\widetilde{Y}_3\cdot\widetilde{Y}_4)^2P_2(1+\tfrac{2}{\widetilde{w}}).
\end{aligned}
\tag{F.21}
$$

JHEP08(2017)128

In six dimensions, the low-lying nonzero OPE coefficients are

$$\begin{aligned}
\mathbf{1}: \quad & \lambda^2_{\mathcal{D}[0],\mathbf{1}} = 3, \quad \lambda^2_{\mathcal{B}[0]_0,\mathbf{1}} = 6, \quad \lambda^2_{\mathcal{B}[0]_2,\mathbf{1}} = \frac{60}{7}, \quad \lambda^2_{\mathcal{B}[0]_4,\mathbf{1}} = \frac{35}{11}, \\
& \lambda^2_{\mathcal{B}[2]_1,\mathbf{1}} = \frac{8}{3}, \quad \lambda^2_{\mathcal{B}[2]_3,\mathbf{1}} = \frac{62}{13}, \quad \lambda^2_{\mathcal{B}[2]_5,\mathbf{1}} = \frac{256}{85}, \\
& \lambda^2_{\mathcal{L}[0]_{8,0},\mathbf{1}} = 1, \quad \lambda^2_{\mathcal{L}[0]_{10,0},\mathbf{1}} = \frac{4}{15}, \quad \lambda^2_{\mathcal{L}[0]_{10,2},\mathbf{1}} = \frac{265}{66}, \ \cdots, \\
\mathbf{3}: \quad & \lambda^2_{\mathcal{D}[2],\mathbf{3}} = 4, \quad \lambda^2_{\mathcal{B}[0]_1,\mathbf{3}} = \frac{32}{5}, \quad \lambda^2_{\mathcal{B}[0]_3,\mathbf{3}} = \frac{80}{21}, \quad \lambda^2_{\mathcal{B}[0]_5,\mathbf{3}} = \frac{448}{429}, \\
& \lambda^2_{\mathcal{B}[2]_0,\mathbf{3}} = 1, \quad \lambda^2_{\mathcal{B}[2]_2,\mathbf{3}} = \frac{50}{11}, \quad \lambda^2_{\mathcal{B}[2]_4,\mathbf{3}} = \frac{490}{117}, \\
& \lambda^2_{\mathcal{L}[0]_{9,1},\mathbf{3}} = \frac{16}{7}, \quad \lambda^2_{\mathcal{L}[0]_{11,1},\mathbf{3}} = \frac{10}{9}, \quad \lambda^2_{\mathcal{L}[0]_{11,3},\mathbf{3}} = \frac{500}{117}, \ \cdots, \\
\mathbf{5}: \quad & \lambda^2_{\mathcal{D}[4],\mathbf{5}} = 1, \quad \lambda^2_{\mathcal{B}[2]_1,\mathbf{5}} = \frac{32}{9}, \quad \lambda^2_{\mathcal{L}[0]_{10,0},\mathbf{5}} = \frac{64}{105}, \quad \lambda^2_{\mathcal{L}[0]_{10,2},\mathbf{5}} = \frac{10}{3}, \ \cdots.
\end{aligned} \tag{F.23}$$

An observation on the above decomposition is the absence of $\mathcal{B}[0]_{\ell,\mathbf{5}}$, which are allowed in general free theories. This property of the hypermultiplet can be explained as follows. In order to satisfy the relation between Δ and ℓ, the superconformal primary of a $\mathcal{B}[0]_\ell$ multiplet must take the form of a scalar bilinear. Since a scalar transforms in **2**, scalar bilinears can only transform in **1** and **3**.

G Details on numerics

We comment on the parameter settings in the practical implementation of the linear functional method. The most relevant parameters include the derivative order Λ, the truncation on spins $\ell_{\max}$, and the order n_r to which the r expansion of the superconformal blocks (see appendix A.2) are truncated. For fixed Λ, we in principle need to extrapolate to infinite n_r and $\ell_{\max}$ to obtain rigorous bounds. However, in practice, we find that if we set $n_r \geq 2\Lambda$, then the bounds are stable to within numerical precision against further increases in n_r. The numerical bounds in this paper are obtained using $\ell_{\max} = 64$, $n_r = 80$ for $\Lambda \leq 40$ and $n_r = 96$ for $40 < \Lambda \leq 48$. The relevant parameter settings for the SDPB package are

```
precision = 1024,
initialMatrixScalePrimal = initialMatrixScaleDual = 1e20,
dualityGapThreshold = 1e-10.
```
(G.1)

In the past, the weakening of the bounds with increasing $\ell_{\max}$ has been handled by imposing non-negativity conditions on functionals acted on a few blocks of very high spin (such as $\ell = 1000, 1001$ in [18]), in addition to blocks below some $\ell_{\max}$.[34] We find that

[34] In the case of bootstrapping conformal field theories in two spacetime dimensions with an infinite dimensional chiral algebra, it was found that the bounds are stable against increasing $\ell_{\max}$ to within numerical precision, as soon as $\ell_{\max}$ exceeds some threshold, say 2Λ [37, 40, 41, 44]. Some comments on the issue of spin stabilization in higher spacetime dimensions can be found in [26].

JHEP08(2017)128

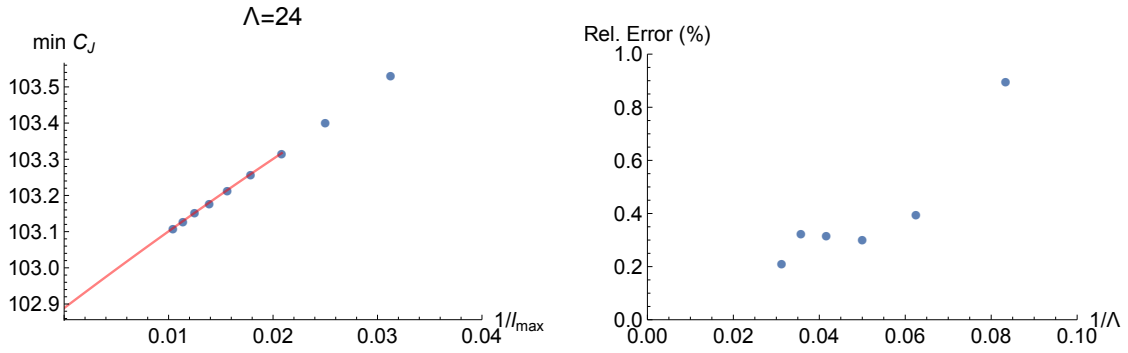

Figure 9. *Left:* the lower bounds on C_J for interacting theories with E_8 flavor group, at derivative order $\Lambda = 24$ and across a range of spin truncations $\ell_{\max}$. Also shown is an extrapolation to $\ell_{\max} \to \infty$ using the quadratic ansatz (G.2). *Right:* the relative errors between $\ell_{\max} = 64$ and the extrapolations to $\ell_{\max} \to \infty$, at different Λ.

this approach does not make our bounds stable against increasing $\ell_{\max}$. But numerical extrapolations to infinite $\ell_{\max}$ require data with a large range of $\ell_{\max}$ for each derivative order, which is computationally intensive and impractical.[35] Our strategy is to use $\ell_{\max} = 64$, and estimate the errors by performing the extrapolations to infinite $\ell_{\max}$ in simpler cases. We shall consider E_8 flavor in the absence of higher spin conserved currents. The left side of figure 9 shows the extrapolations for the lower bound on C_J at derivative order $\Lambda = 24$, and the right side shows the relative error between $\ell_{\max} = 64$ and extrapolations to $\ell_{\max} \to \infty$ using the quadratic ansatz

$$\min C_T = a + \frac{b}{\ell_{\max}} + \frac{c}{\ell_{\max}^2}, \quad 48 \leq \ell_{\max} \leq 96, \tag{G.2}$$

obtained at various derivative orders. We see that the relative error decreases to below 0.5% as we go to high enough derivative orders.

In light of the slight discrepancy between the value of C_T at min C_J and the rank-one E-string, as shown in figure 5, we estimate its error due to spin truncation. Figure 10 shows the upper and lower bounds on C_T, when the value of the flavor central charge C_J is set close to saturating the lower bound, $C_J = (1+10^{-4}) \min C_J$, at derivative order $\Lambda = 24, 32$ and across a range of spin truncations $\ell_{\max}$. The data appears less regular than that for min C_J, and extrapolations using the ansatz (G.2) do not look reliable, but we estimate that the error due to truncating spins to $\ell_{\max} = 64$ is less than 2% for $\Lambda \geq 24$. Similar to min C_J, this error decreases with increasing derivative order.

[35] When the flavor group has a large crossing matrix, to run jobs at very high derivative order Λ and very high spin truncation $\ell_{\max}$, the required RAM for running SDPB exceeds the limitations of the machines we have at hand.

JHEP08(2017)128

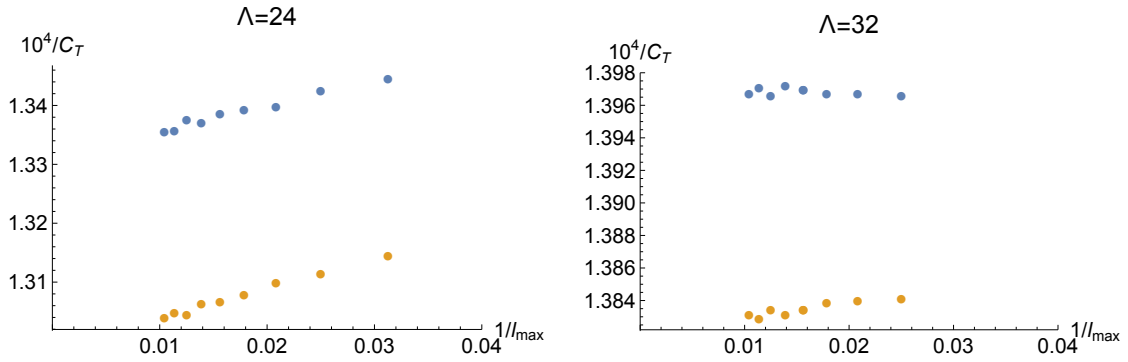

Figure 10. The upper and lower bounds on the inverse central charge C_T^{-1} when the value of the flavor central charge C_J is set close to saturating the lower bound, $C_J = (1 + 10^{-4}) \min C_J$, for interacting theories with E_8 flavor group, at derivative orders $\Lambda = 24, 32$ and across a range of spin truncations $\ell_{\max}$.

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JHEP08(2017)128