Published for SISSA by Springer

Received: May 16, 2017
Revised: June 29, 2017
Accepted: June 30, 2017
Published: August 8, 2017

Three-loop massive tadpoles and polylogarithms through weight six

B.A. Kniehl, A.F. Pikelner[1] and O.L. Veretin

II Institut für Theoretische Physik, Universität Hamburg,
Luruper Chaussee 149, 22761 Hamburg, Germany

E-mail: kniehl@desy.de, andrey.pikelner@desy.de, veretin@mail.desy.de

Abstract: We evaluate the three-loop massive vacuum bubble diagrams in terms of polylogarithms up to weight six. We also construct the basis of irrational constants being harmonic polylgarithms of arguments $e^{ki\pi/3}$.

Keywords: NLO Computations, QCD Phenomenology

ArXiv ePrint: 1705.05136

[1]On leave of absence from Joint Institute for Nuclear Research, 141980 Dubna, Russia.

https://doi.org/10.1007/JHEP08(2017)024

JHEP08(2017)024

Contents

1 Introduction

More than two decades ago, the integration-by-parts relations [1] and asymptotic expansions [2, 3] became common in the Feynman diagram calculus. The combination of these methods provides a powerful tool for the evaluation of multiloop diagrams. In particular, massive propagator diagrams through the three-loop order can be reduced with the help of asymptotic-expansion methods to three-loop massive tadpoles, which can be done, e.g., using the `FORM` [4] package `MATAD` [5] (see also ref. [6]).[1]

There are a lot of physical applications, where the above-mentioned technique was applied. Just to mention but a few examples, it was applied to the evaluation of the three-loop ρ parameter in QCD [7, 8] and the electroweak theory [9], the three-loop QCD corrections to heavy-quark production [10], and many other quantities. Integral topologies with all lines massive find applications in calculations of renormalization group functions [11–13] at the three-loop order and also at higher orders of the epsilon expansion in four-loop [14] and even five-loop [15] calculations.

In his work [16], Broadhurst noticed that all three-loop single-scale vacuum diagrams at order $O((4-d)/2)$ in dimensional regularization can be related to the elements of the

[1] The up-to-date package `MATAD-ng` with full dependence on d can be downloaded from the URL https://github.com/apik/matad-ng and the results of this paper from the direct link https://git.io/mtdw6.

JHEP08(2017)024

algebra of the sixth root of unity. This observation allowed him to evaluate all the three-loop integrals up to their finite parts in terms of a few constants, being polylogarithms of weight four.

In this paper, we proceed by studying three-loop vacuum integrals with a single mass scale at weights five and six. On the one hand, this is a necessary ingredient in evaluations beyond the three-loop approximation, where the three-loop master integrals have to be expanded to higher powers in $d-4$. On the other hand, we would like to test the basis of the algebra of the sixth root of unity through weight six.

2 Notation

We use dimensional regularization with the dimension of space-time being $d = 4 - 2\varepsilon$ in euclidean space. Each loop integration is normalized as follows:

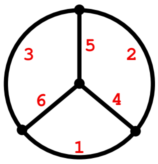
$$\int d[k] \cdots = e^{\gamma\varepsilon} \int \frac{d^d k}{\pi^{d/2}} \cdots \,, \tag{2.1}$$

where $\gamma = 0.577216\ldots$ is the Euler-Mascheroni constant.

Defining

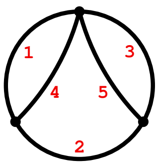
$$C_{i;m} = k_i^2 + m^2, \quad C_{ij;m} = (k_i - k_j)^2 + m^2 \,, \tag{2.2}$$

the general three-loop vacuum bubble diagram with six scalar propagators, where it is implied that the masses either take the values m or zero, may be written as

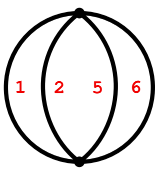


$$= D_{a_1a_2a_3a_4a_5a_6} = \int \frac{d[k_1]d[k_2]d[k_3]}{C_{1;m_1}^{a_1} C_{2;m_2}^{a_2} C_{3;m_3}^{a_3} C_{12;m_4}^{a_4} C_{23;m_5}^{a_5} C_{31;m_6}^{a_6}} \,. \tag{2.3}$$

In addition to diagrams with six lines, we also have three-loop digrams with five and four lines,

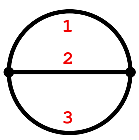


$$= E_{a_1a_2a_3a_4a_5} = \int \frac{d[k_1]d[k_2]d[k_3]}{C_{1;m_1}^{a_1} C_{2;m_2}^{a_2} C_{3;m_3}^{a_3} C_{12;m_4}^{a_4} C_{23;m_5}^{a_5}} \,, \tag{2.4}$$

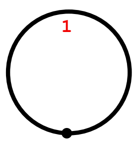


$$= B_{a_1a_2a_5a_6} = \int \frac{d[k_1]d[k_2]d[k_3]}{C_{1;m_1}^{a_1} C_{2;m_2}^{a_2} C_{23;m_5}^{a_5} C_{31;m_6}^{a_6}} \,, \tag{2.5}$$

as well as two-loop and one-loop diagrams,



$$= T_{a_1a_2a_3} = \int \frac{d[k_1]d[k_2]}{C_{1;m_1}^{a_1} C_{2;m_2}^{a_2} C_{12;m_3}^{a_3}} \,,$$



$$= V_{a_1} = \int \frac{d[k_1]}{C_{1;m_1}^{a_1}} \,. \tag{2.6}$$

JHEP08(2017)024

3 Polylogarithms, algebra of the sixth root of unity, and its subalgebras

In our study, the key role is played by multiple polylogarithms [17–19], defined recursively as repeated integrals,

$$G_{a_1 a_2 \ldots a_w}(z) = \int_0^z \frac{dt_1}{t_1 - a_1} \underbrace{\int_0^{t_1} \frac{dt_1}{t_2 - a_2} \cdots \int_0^{t_{w-1}} \frac{dt_w}{t_w - a_w}}_{G_{a_2 \ldots a_w}(t_1)}, \qquad a_w \neq 0\,, \tag{3.1}$$

where $a_1, a_2, \ldots, a_w$ and z are complex numbers. The definition in eq. (3.1) is modified in the case of q trailing zero indices in the following way:

$$G_{a_1 a_2 \ldots a_{w-q} 00 \ldots 0}(z) = \int_0^z \frac{dt_1}{t_1 - a_1} \int_0^{t_1} \frac{dt_1}{t_2 - a_2} \cdots \int_0^{t_{w-q-1}} \frac{dt_{w-q}}{t_{w-q} - a_{w-q}} \frac{\ln^q(t_{w-q})}{q!}\,. \tag{3.2}$$

The integer number w is called the *weight* of the polylogarithm.

The functions G obey the so-called shuffle and stuffle relations. In particular, any product of two G functions with the same argument and weights w_1 and w_2 can be rewritten as a linear combination of G functions of weight $w_1 + w_2$. In other words, polylogarithms form a graded algebra.

The algebra of the sixth root of unity $\mathcal{A}_\omega$ is obtained from general polylogarithms by restricting all a_j to the seven-letter alphabet $\{0, \omega^0, \omega^1, \omega^2 \ldots, \omega^5\}$, where

$$\omega = \exp\left(\frac{i\pi}{3}\right) \tag{3.3}$$

is a primitive sixth root of unity. At arbitrary argument z, such functions include the so-called inverse-binomial-sums functions [20–25] and are related to cyclotomic polylogarithms [26]. At $z = 1$, they represent a set of irrational[2] constants, which is relevant for the description of some single-scale massive diagrams (in particular, three-loop vacuum bubble and two-loop on-shell self-energy diagrams). The complete basis of the algebra of the sixth root of unity $\mathcal{A}_\omega$ through weight 6 has recently been constructed in ref. [27].

In this work, we construct the basis of the subalgebra of $\mathcal{A}_\omega$ formed by the harmonic polylogarithms [28] $H_{n_1 \ldots n_p}(z)$ of arguments $z_k = \omega^k$. We shall call such an algebra $\mathcal{A}_{H(\omega^k)}$. The harmonic polylogarithms are defined similarly to eqs. (3.1)–(3.2), but now the parameters a_j can only take the values $-1, 0, +1$. For historical reasons, there is also a difference in the overall sign. Specifically, the harmonic-polylogarithm functions $H_{n_1 \ldots n_p}(z)$ are related to the generalized polylogarithms G via

$$H_{n_1 n_2 \ldots n_w}(z) = (-1)^{(\text{number of } n_j\, =\, 1)} G_{n_1 n_2 \ldots n_w}(z)\,, \tag{3.4}$$

where $n_j = -1, 0, +1$.

Using the scaling properties of the polylogarithms together with the shuffle relations, it is easy to show that any element of the form $H_{n_1 \ldots n_p}(\omega^k)$ can be rewritten as $G_{\omega^{k_1} \ldots \omega^{k_p}}(1)$

[2]This means irrational up to the precision used for the `PSLQ` reconstruction.

JHEP08(2017)024

w	$\mathrm{Re}\{\mathcal{A}\}_\omega$	$\mathrm{Im}\{\mathcal{A}\}_\omega$	$\mathrm{Re}\{\mathcal{A}\}_{H(\omega)}$	$\mathrm{Im}\{\mathcal{A}\}_{H(\omega)}$
1	2	1	1	1
2	5	3	3	3
3	12	9	8	8
4	30	25	21	21
5	76	68	55	55
6	195	182	144	144

Table 1. Values of $\mathrm{Re}\{\mathcal{A}\}_\omega$, $\mathrm{Im}\{\mathcal{A}\}_\omega$, $\mathrm{Re}\{\mathcal{A}\}_{H(\omega)}$, and $\mathrm{Im}\{\mathcal{A}\}_{H(\omega)}$ for $w = 1, \ldots, 6$.

with some integers $k_j = 0, 1, \ldots, 5$. For example, we have $H_{1,0,-1}(\omega) = G_{1/\omega,0,-1/\omega}(1) = G_{\omega^5,0,\omega^2}(1)$. In other words,

$$\mathcal{A}_{H(\omega^k)} \subset \mathcal{A}_\omega\,, \qquad k = 0, 1, \ldots, 5\,. \tag{3.5}$$

In particular, $\mathcal{A}_{H(\omega^0)} \equiv \mathcal{A}_{H(1)}$ and $\mathcal{A}_{H(\omega^3)} \equiv \mathcal{A}_{H(-1)}$ form the shuffle algebra of the multiple zeta values (see, e.g., ref. [29]). The pairs $(\mathcal{A}_{H(\omega^1)}, \mathcal{A}_{H(\omega^5)})$ and $(\mathcal{A}_{H(\omega^2)}, \mathcal{A}_{H(\omega^4)})$ are related to each other by complex conjugation, since $\overline{\omega^1} = \omega^5$ and $\overline{\omega^2} = \omega^4$. Moreover, all $\mathcal{A}_{H(\omega^k)}$, $k = 1, 2, 4, 5$ are isomorph. We prove this by explicitly constructing the relation between the corresponding bases over $\mathbb{Q}$. Thus, we need to consider only $\mathcal{A}_{H(\omega)}$. We can split each element into its real and imaginary parts,

$$H_{n_1\ldots n_w}(\omega) = \mathrm{Re}\, H_{n_1\ldots n_w}(\omega) + i\, \mathrm{Im}\, H_{n_1\ldots n_w}(\omega)\,. \tag{3.6}$$

Note that $\mathrm{Re}\{\mathcal{A}\}_{H(\omega)}$ can involve products of even numbers of elements from $\mathrm{Im}\{\mathcal{A}\}_{H(\omega)}$, while $\mathrm{Im}\{\mathcal{A}\}_{H(\omega)}$ can involve products of elements from $\mathrm{Re}\{\mathcal{A}\}_{H(\omega)}$.

Using the shuffle relations and the PSLQ algorithm [30], we construct the real and imaginary bases for the weights $w = 1, \ldots, 6$. The numbers of the basis elements at each weight for the algebra of the sixth root of unity $\mathcal{A}_\omega$ and for $\mathcal{A}_{H(\omega)}$ are summarized in table 1. The results for $\mathrm{Re}\{\mathcal{A}\}_\omega$ and $\mathrm{Im}\{\mathcal{A}\}_\omega$ were obtained by explicit calculation in ref. [27]. The results for $\mathrm{Re}\{\mathcal{A}\}_{H(\omega)}$ and $\mathrm{Im}\{\mathcal{A}\}_{H(\omega)}$ are obtained in this work. Unlike $\mathcal{A}_\omega$, the real and imaginary parts of $\mathcal{A}_{H(\omega)}$ have the same numbers of basis elements. The integer sequence $1, 3, 8, 21, 55, 144, \ldots$ corresponds to $\{F_{2w}\}$, $w = 1, 2, 3, 4, 5, 6, \ldots$, where F_j denotes the j-th Fibonacci number.

We shall denote the uniform bases of $\mathrm{Re}\{\mathcal{A}\}_{H(\omega)}$ and $\mathrm{Im}\{\mathcal{A}\}_{H(\omega)}$ for fixed weight w as $\mathrm{Re}\, H_w$ and $\mathrm{Im}\, H_w$, respectively. In the next sections, we apply the constructed bases $\mathrm{Re}\, H_w$ and $\mathrm{Im}\, H_w$ to the evaluation of the three-loop massive vacuum bubble diagrams.

4 Evaluation of the three-loop vacuum bubble integrals

Using integration-by-parts relations, it is possible to reduce any three-loop bubble integral with a single scale to a set of twelve three-loop master integrals, two two-loop integrals, and one one-loop bubble. These diagrams are shown in figures 1 and 2.

JHEP08(2017)024

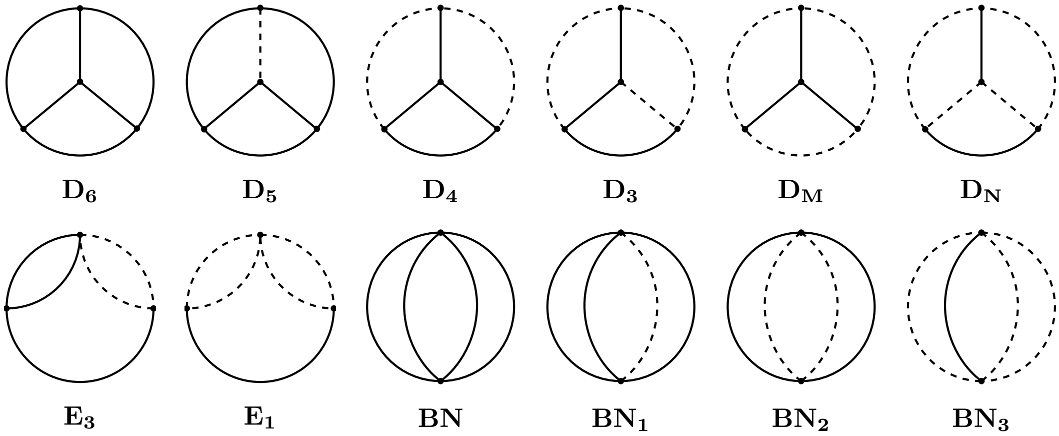


Figure 1. Full set of non-factorizing master integrals. Solid and dashed lines correspond to massive and massless scalar propagators, respectively.

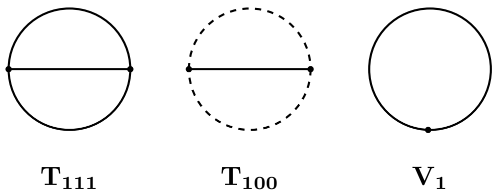


Figure 2. Two-loop and one-loop master integrals. The line coding is the same as in figure 1.

It is the goal of this paper to evaluate these master integrals analytically in terms of polylogarithms through weight six.

In the previous section, we discussed the construction of the bases of $\mathrm{Re}\{\mathcal{A}\}_{H(\omega)}$ and $\mathrm{Im}\{\mathcal{A}\}_{H(\omega)}$. We now use these bases to reconstruct the analytic expressions for the ε expansions of these diagrams using the PSLQ algorithm [30]. For that purpose, we first need a precise numerical value of each diagram.

Specifically, for the fully massive diagrams $\mathbf{D_6}$ and $\mathbf{E_5}$, we make use of the series obtained with the help of the DRA method, based on dimensional recurrence relations and analyticity, presented in ref. [31] and summed with the help of the `SummerTime` package [32]. Within a few hours, we were able to get 20,000 decimal figures of precision for these diagrams.

The general method of calculation which is used in this work and is applicable to all the considered diagrams consists in writing the systems of differential equations for the integrals and solving them later by the Frobenius method. In the first step, instead of a single-scale diagram with mass m, we introduce a similar diagram with two different masses, m_1 and m_2. Then, using integration by parts, we can write the system of differential equations in the mass ratio $z = m_1^2/m_2^2$. In general, these equations cannot be solved analytically. We solve them as series of the form $\sum_n z^n c_n$. The unknown coefficients c_n are to be determined by substituting the series in the differential equations.

We also need the boundary conditions. The easiest choice in our case is the boundary conditions at $z = 0$, which correspond to a single-scale bubble integral with a smaller number of massive lines.

Finally, we set $z = 1$ in the series solution in order to recover the original diagram. The summation of the series is done numerically. In this way, we are able to evaluate integrals to an accuracy of typically 4,000 to 10,000 decimal figures depending on the diagram. For that purpose, we need to sum up to 20,000 terms in the n sum in some cases.

Let us consider as an example the integral $\mathbf{D_N}$. There are two massive lines in this diagram. Instead of two equal masses, we introduce now two different masses, one of which we set to unity. Thus, we set in eq. (2.3) $m_1 = z$, $m_5 = 1$, and $m_2 = m_3 = m_4 = m_6 = 0$. With such masses, we have the following set of master integrals, which depend on z:

$$D_{100111}, D_{101110}, D_{201110}, D_{101011}, D_{011111}, D_{111110}, D_{111111}\,, \tag{4.1}$$

where we use the definition in eq. (2.3).

Let us denote, for brevity, the integrals in eq. (4.1) as $f_1, \ldots, f_7$ in this very order. Then, the functions $f_1(z), \ldots, f_7(z)$ obey a system of linear differential equations in the variable z, which reads:

$$\begin{aligned}
z^2(1+z)f_7' + \frac{1}{2}(d-4)z(1+2z)f_7 &= (d-3)zf_6 + (d-3)zf_5 + (d-2)zf_4 \\
&\qquad -2z(1+z)f_3 - (3d-8)zf_2 - (d-2)f_1\,, \\
zf_6' - \frac{1}{2}(3d-10)f_6 &= 0\,, \\
f_5' &= 0\,, \\
zf_4' - (d-3)f_4 &= 0, \\
zf_2' - zf_3 &= 0\,, \\
z(z-1)f_3' + \frac{1}{2}(d-3)(3d-8)f_2 - \frac{1}{2}(4-d-16z+5dz)f_3 &= 0\,, \\
zf_1' - \frac{1}{2}(d-2)f_1 &= 0\,,
\end{aligned} \tag{4.2}$$

where $f_j' = df_j/dz$.

To solve the system in eq. (4.2), we substitute the following collective ansatz:

$$f_j = \sum_{n=0}^{\infty}\sum_{k=1}^{K} c_{j,n,k}\, z^{\mu_k+n}\,. \tag{4.3}$$

The exponent shifts μ_k are determined as usual in the Frobenius method from the indicial polynomials. Actually, it is easy to establish that μ_j can take the values $0, -\varepsilon, -2\varepsilon, -3\varepsilon$. Therefore, we have, for each value of j, four different solutions $f_j^{(1)}, f_j^{(2)}, f_j^{(3)}, f_j^{(4)}$, corresponding to the different values of μ_k, and the solution we are looking for is the linear combination

$$f_j = \sum_{k=1}^{4} C_{j,k} f_j^{(k)}\,, \tag{4.4}$$

with unknown constants $C_{j,k}$, which should be determined from the boundary conditions at $z = 0$. Thus, for each value $j = 1, \ldots, 7$, we need four boundary conditions, one for each value of μ_k.

The boundary conditions correspond to the expansions of the integrals in eq. (4.1) about $z = 0$. Following the standard rules of the large-mass expansion [2, 3], we should take into account the four hard subgraphs $\{123456\}$, $\{23456\}$, $\{356\} + \{245\}$, and $\{5\}$. These four subgraphs provide the four boundary conditions for eq. (4.4).

5 Results and discussion

We present the terms of the ε expansions analytically in terms of the bases $\mathrm{Re}\, H_j$ and $\mathrm{Im}\, H_j$, $j = 1, \ldots, 6$ in the appendix. In each case, we take the prefactor in such a way that the terms of the expansion are homogeneous in the weights. In some cases, this requires us to evaluate additional integrals (with dots on lines) and to re-express the original integral with the help of integration-by-parts relations. Moreover, we find that, with the suitable choice of prefactors, the elements of the expansion are expressed in each case either through the $\mathrm{Re}\, H$ basis or the $\mathrm{Im}\, H$ basis. This feature is a convenient property which allows us to reduce the length of PSLQ vector. In addition to the analytic expressions, we also give their numerical values accurate to 50 decimal figures.

There is, of course, a certain degree of arbitrariness in the choice of the basis elements. We just use the lexicographical ordering of the three-letter alphabet $\{-1, 0, 1\}$. The sets of basis elements and the transformation between different bases (with arguments ω and ω^2), as well as all analytic results can be found in the attachment in a `Mathematica`-readable form. In addition, we give the numerical values of all basis elements both for $\mathcal{A}_{H(\omega)}$ and $\mathcal{A}_{H(\omega^2)}$ to an accuracy of 20,000 decimal figures.

5.1 Integrals evaluated in terms of Γ functions

Four of the integrals in figure 1 can be evaluated in terms of Γ functions, namely

$$\mathbf{V_1} = e^{\gamma\varepsilon}\Gamma(-1+\varepsilon)\,, \tag{5.1}$$

$$\mathbf{T_{100}} = e^{2\gamma\varepsilon}\frac{\Gamma(-1+2\varepsilon)\Gamma(\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(2-\varepsilon)}\,, \tag{5.2}$$

$$\mathbf{E_1} = e^{3\gamma\varepsilon}\frac{\Gamma(2-3\varepsilon)\Gamma^4(1-\varepsilon)\Gamma^2(\varepsilon)\Gamma(-1+3\varepsilon)}{\Gamma^2(2-2\varepsilon)\Gamma(2-\varepsilon)}\,, \tag{5.3}$$

$$\mathbf{BN_3} = e^{3\gamma\varepsilon}\frac{\Gamma^3(1-\varepsilon)\Gamma(-1+2\varepsilon)\Gamma(-2+3\varepsilon)}{\Gamma(2-\varepsilon)}\,, \tag{5.4}$$

$$\mathbf{BN_2} = e^{3\gamma\varepsilon}\frac{\Gamma^2(1-\varepsilon)\Gamma(\varepsilon)\Gamma^2(-1+2\varepsilon)\Gamma(-2+3\varepsilon)}{\Gamma(2-\varepsilon)\Gamma(-2+4\varepsilon)}\,. \tag{5.5}$$

5.2 Two-loop integral $\mathbf{T_{111}}$

The two-loop integral $\mathbf{T_{111}}$ was considered in ref. [33], where its representation in terms of the hypergeometric function ${}_4F_3$ was given. The construction of its ε expansion was discussed in great detail in ref. [34]. There, the expansion to all orders in ε was found in terms of log-sine integrals.

We find that $\mathbf{T_{111}}$ can be written in terms of our bases as

$$\mathbf{T_{111}} = \frac{e^{2\gamma\varepsilon}\Gamma^2(1+\varepsilon)}{\sqrt{3}(-1+\varepsilon)(1-2\varepsilon)}\left(\frac{3\sqrt{3}}{2\varepsilon^2}+\bar{T}^{(0)}_{111}+\varepsilon\bar{T}^{(1)}_{111}+\varepsilon^2\bar{T}^{(2)}_{111}+\varepsilon^3\bar{T}^{(3)}_{111}+\varepsilon^4\bar{T}^{(4)}_{111}+\dots\right), \qquad (5.6)$$

where $\bar{T}^{(k)}_{111}$ are expressed in terms of the homogeneous bases $\operatorname{Im} H_{k+2}$. They are presented in the appendix.

It should be noted here that, in eq. (5.6), it is necessary to take out the factor $\Gamma^2(1+\varepsilon)$ to avoid the mixing of the $\operatorname{Im} H$ and $\operatorname{Re} H$ bases. This mixing occurs, since $\zeta_k \in \operatorname{Re}\{\mathcal{A}\}_{H(\omega)}$, while $\zeta_k \notin \operatorname{Im}\{\mathcal{A}\}_{H(\omega)}$ and $\sqrt{3}\zeta_k \notin \operatorname{Im}\{\mathcal{A}\}_{H(\omega)}$.

To the accuracy of 50 decimal figures, we have

$$\begin{aligned}
\mathbf{T_{111}} =& -\frac{3}{2\varepsilon^2}-\frac{9}{2\varepsilon}\\
&-9.4515402422381513188062793166606221801853203373564\\
&-24.208928021203592678721338219570948925801493085960\varepsilon\\
&-38.717599744915838872316613641777943709417494962957\varepsilon^2\\
&-101.95399100959711442266247857697577872945662436761\varepsilon^3\\
&-152.48276547467415258599823439719064567392808286431\varepsilon^4+\dots . \qquad (5.7)
\end{aligned}$$

5.3 Diagram BN

The integral **BN** belongs to the class of the so-called 'QED-type' integrals. These are the integrals with an even number of massive lines at each vertex. They have an especially simple structure and were considered in ref. [35] though weight six. We have

$$\begin{aligned}
\mathbf{BN} =& \frac{e^{3\gamma\varepsilon}\Gamma^3(1+\varepsilon)}{(1-\varepsilon)(1-2\varepsilon)(1-3\varepsilon)(2-3\varepsilon)}\\
&\times\left(\frac{4}{\varepsilon^3}-\frac{44}{3\varepsilon^2}+\varepsilon\overline{BN}^{(1)}+\varepsilon^2\overline{BN}^{(2)}+\varepsilon^3\overline{BN}^{(3)}+\varepsilon^4\overline{BN}^{(4)}+\dots\right), \qquad (5.8)
\end{aligned}$$

where $\overline{BN}^{(k)}$ are expressed in terms of the homogeneous bases $\operatorname{Re} H_{k+2}$ and are explicitly given in the appendix.

Numerically, we have

$$\begin{aligned}
\mathbf{BN} =& \frac{2}{\varepsilon^3}+\frac{23}{3\varepsilon^2}\\
&+22.434802200544679309417245499938075567656849703620\frac{1}{\varepsilon}\\
&+39.429294629102082115299964760073056361154617945864\\
&+62.927093755359100705477989920486998916962128778534\varepsilon\\
&-126.33666539901207007224982170333707283310295118164\varepsilon^2\\
&-584.77850194492638360751899973098599972365977374517\varepsilon^3\\
&-4108.8159602165199632668134484734533423368833382544\varepsilon^4+\dots . \qquad (5.9)
\end{aligned}$$

5.4 Diagram $\mathbf{BN_1}$

The diagram $\mathbf{BN_1}$ was previously considered in refs. [36, 37]. There, its explicit representation in terms of the hypergeometric function ${}_QF_P$ of argument $1/4$ was obtained. The corresponding ε expansion, through weight-five polylogarithms, was constructed in terms of log-sine integrals.

In order to keep the property of the weight homogeneity and to separate the real and imaginary bases, we write $\mathbf{BN_1}$ in terms of the additional integrals $\mathbf{BN_1'}$, which is $\mathbf{BN_1}$ with additional dots and $\mathbf{V_1}$,

$$\begin{aligned}\mathbf{BN_1} =& \frac{9\left(\overline{BN}_1^{(0)} + \varepsilon\overline{BN}_1^{(1)} + \varepsilon^2\overline{BN}_1^{(2)} + \varepsilon^3\overline{BN}_1^{(3)} + \varepsilon^4\overline{BN}_1^{(4)} + \dots\right)}{2\sqrt{3}(1-\varepsilon)(-1+2\varepsilon)(-2+3\varepsilon)(-1+3\varepsilon)} \\ &+ \frac{(-1+\varepsilon)^2(-4+15\varepsilon)}{2(-1+2\varepsilon)(-2+3\varepsilon)(-1+3\varepsilon)}\mathbf{V_1}^3 \,. \end{aligned} \tag{5.10}$$

Numerically, we have

$$\begin{aligned}\mathbf{BN_1} =& \frac{1}{\varepsilon^3} + \frac{15}{4\varepsilon^2} + \frac{1}{\varepsilon}\left(\frac{65}{8} + \frac{3}{2}\zeta_2\right) \\ &+ 21.761988509912961923611112300835065104056263673628 \\ &+ 7.8517428364255311757755955915256850308098967304537\varepsilon \\ &- 71.052070175912095038002576965797146234900221815177\varepsilon^2 \\ &- 716.82162754590202527205013899706978778657937882568\varepsilon^3 \\ &- 2486.5823094068232493409812154539302262366876271196\varepsilon^4 + \dots \,. \end{aligned} \tag{5.11}$$

5.5 Diagram $\mathbf{E_3}$

The integral $\mathbf{E_3}$ was also considered in refs. [36, 37], where the same analysis as for $\mathbf{BN_1}$ can be found. Our representation of this integral reads:

$$\begin{aligned}\mathbf{E_3} =& \frac{1}{\sqrt{3}(1-\varepsilon)(1-2\varepsilon)^2}\left(\frac{1}{\varepsilon}\bar{E}_3^{(-1)} + \bar{E}_3^{(0)} + \varepsilon\bar{E}_3^{(1)} + \varepsilon^2\bar{E}_3^{(2)} + \varepsilon^3\bar{E}_3^{(3)} + \dots\right) \\ &+ \frac{e^{3\gamma\varepsilon}\Gamma(1-\varepsilon)\Gamma(1+2\varepsilon)\Gamma^2(1+\varepsilon)}{2\varepsilon^3(1-\varepsilon)(1-2\varepsilon)^2}\left(1 + \frac{1-\varepsilon}{3(1-3\varepsilon)}\frac{\Gamma(1+2\varepsilon)\Gamma(1+3\varepsilon)}{\Gamma(1+\varepsilon)\Gamma(1+4\varepsilon)}\right) \,, \end{aligned} \tag{5.12}$$

where $\bar{E}_3^{(k)}$ are expressed in terms of the homogeneous bases $\operatorname{Im} H_{k+3}$ and are explicitly given in the appendix.

Numerically, we have

$$\begin{aligned}\mathbf{E_3} =& \frac{2}{3\varepsilon^3} + \frac{11}{3\varepsilon^2} \\ &+ 13.774007275662264537042486899983634774794795287960\frac{1}{\varepsilon} \\ &+ 55.659622461206330171361395086424121982538065496717 \\ &+ 151.93523620176745531459840301896170801637068885020\varepsilon \\ &+ 574.65405761296725286615112197885868312967183890132\varepsilon^2 \\ &+ 1417.6830429498080864815903343175553853260055772458\varepsilon^3 + \dots \,. \end{aligned} \tag{5.13}$$

JHEP08(2017)024

5.6 D-type diagrams

For all diagrams of the mercedes type, we have the same representation

$$\mathbf{D}_x = \frac{1}{(1-\varepsilon)(1-2\varepsilon)}\left(\frac{2\zeta_3}{\varepsilon} + \bar{D}_x^{(0)} + \varepsilon\bar{D}_x^{(1)} + \varepsilon^2\bar{D}_x^{(2)} + \dots\right), \tag{5.14}$$

where all $\bar{D}_x^{(k)}$ are expressed in terms of the homogeneous bases $\mathrm{Re}\,H_{k+4}$ and are explicitly given in the appendix.

It should be noted that the integral $\mathbf{D_5}$ is sometimes replaced by the fully massive three-loop integral with 5 lines. The latter integral was considered in ref. [38], where, again, the hypergeometric representation was presented.

Numerically we have

$$\begin{aligned}
\mathbf{D_6} = \frac{2\zeta_3}{\varepsilon}
&- 10.035278479768789171914700685158900238650333496003 \\
&+ 41.876702083031576174334902670970991466431593917007\varepsilon \\
&- 146.80128953959941603962375965123680914875429640084\varepsilon^2 + \dots,
\end{aligned} \tag{5.15}$$

$$\begin{aligned}
\mathbf{D_5} = \frac{2\zeta_3}{\varepsilon}
&- 8.2168598175087380629133983386010858249695083391726 \\
&+ 36.473684211550968259944718569763658485586503339935\varepsilon \\
&- 122.50284392807361438626452778528813541284319434062\varepsilon^2 + \dots,
\end{aligned} \tag{5.16}$$

$$\begin{aligned}
\mathbf{D_4} = \frac{2\zeta_3}{\varepsilon}
&- 5.9132047838840205304957178925354050268834109915340 \\
&+ 31.793875865203350915027031305982932318904242706405\varepsilon \\
&- 95.531868585060481541530996460991495963507326204506\varepsilon^2 + \dots,
\end{aligned} \tag{5.17}$$

$$\begin{aligned}
\mathbf{D_3} = \frac{2\zeta_3}{\varepsilon}
&- 3.0270094939876520197863747017589572861507417864174 \\
&+ 28.736435119523636809010469958622996315186640636723\varepsilon \\
&- 63.461003588316921857688768719288636938911285487969\varepsilon^2 + \dots,
\end{aligned} \tag{5.18}$$

$$\begin{aligned}
\mathbf{D_M} = \frac{2\zeta_3}{\varepsilon}
&- 2.8608622241393273502727845677732419175614414620201 \\
&+ 29.006674437837759083319026315817224175180046773850\varepsilon \\
&- 62.361396342296251606481070393459830940783578107024\varepsilon^2 + \dots,
\end{aligned} \tag{5.19}$$

$$\begin{aligned}
\mathbf{D_N} = \frac{2\zeta_3}{\varepsilon}
&+ 1.1202483970392420822725165482242095262757766719791 \\
&+ 0.68103527534589055078588298235627556530451079283 8\varepsilon \\
&- 13.303460640858248361888942340988906552580290997554\varepsilon^2 + \dots.
\end{aligned} \tag{5.20}$$

6 Conclusions

In this work, we considered the three-loop massive vacuum bubble diagrams and constructed the pertaining bases of irrational constants through weight six, which are harmonic polylogarithms of argument $\omega = e^{i\pi/3}$. These bases are smaller than the bases of the algebra of the sixth root of unity. Nevertheless, we found by explicit calculation that such reduced bases are large enough to describe all the three-loop single-scale vacuum integrals. We presented the results for all relevant master integrals both numerically and analytically in terms of the introduced constants. Our basis is universal, and its application is not restricted to three-loop tadpoles. As an example, we succeeded in reconstructing[3] all the three-loop integrals contributing to the massive planar form factor through weight six presented in ref. [39]. As expected, these integrals only involve the real parts of the basis $\mathrm{Re}\{\mathcal{A}\}_{H(\omega)}$.

Acknowledgments

We thank Roman Lee for his help with the `SummerTime` package. This work was supported in part by the German Federal Ministry for Education and Research BMBF through Grant No. 05H15GUCC1, by the German Research Foundation DFG through the Collaborative Research Centre No. SFB 676 *Particles, Strings and the Early Universe: the Structure of Matter and Space-Time*, and by the Heisenberg-Landau Programme.

A Master integrals in terms of harmonic polylogarithms of argument $e^{\frac{i\pi}{3}}$

In this appendix, we present the first few terms of the ε expansions of the relevant master integrals.

For the elements of the bases $\mathrm{Re}\, H_k$ and $\mathrm{Im}\, H_k$, we introduce the following short-hand notation:

$$\mathrm{Re}\, H_{n_1 \dots n_w}(\omega) = \mathcal{R}_{n_1 \dots n_w}\,, \tag{A.1}$$

$$\mathrm{Im}\, H_{n_1 \dots n_w}(\omega) = \mathcal{I}_{n_1 \dots n_w}\,. \tag{A.2}$$

Then, the coefficients $\bar{T}$, $\bar{B}$, $\bar{E}$, and $\bar{D}$ introduced in section 5 read:

$$\begin{aligned}\bar{D}_6^{(0)} = & -\frac{72}{11}\mathcal{R}_{1,-1,1,0}+\frac{180}{11}\mathcal{R}_{1,-1,1,1}+\frac{148}{11}\mathcal{R}_{1,0,1,0}-\frac{144}{11}\mathcal{R}_{1,1,-1,0}+\frac{360}{11}\mathcal{R}_{1,1,-1,1}\\ & +\frac{540}{11}\mathcal{R}_{1,1,1,-1}-\frac{33587}{55}\mathcal{R}_{1,1,1,1}\end{aligned} \tag{A.3}$$

$$\begin{aligned}\bar{D}_6^{(1)} = & \,156\mathcal{R}_{1,-1,1,1,0}-16\mathcal{R}_{1,0,-1,1,0}-16\mathcal{R}_{1,0,1,-1,0}-468\mathcal{R}_{1,0,1,1,1}+\frac{7712}{87}\mathcal{R}_{1,1,-1,0,0}\\ & -\frac{7084}{87}\mathcal{R}_{1,1,-1,0,1}+\frac{15884}{87}\mathcal{R}_{1,1,-1,1,0}+\frac{5632}{29}\mathcal{R}_{1,1,-1,1,1}-36\mathcal{R}_{1,1,0,1,-1}\\ & -\frac{13708319\mathcal{R}_{1,1,0,1,1}}{15138}+\frac{9448}{87}\mathcal{R}_{1,1,1,-1,0}+\frac{15640}{29}\mathcal{R}_{1,1,1,-1,1}-108\mathcal{R}_{1,1,1,0,-1}\\ & -\frac{59151355\mathcal{R}_{1,1,1,0,1}}{45414}+992\mathcal{R}_{1,1,1,1,-1}-\frac{51713387\mathcal{R}_{1,1,1,1,0}}{22707}\end{aligned} \tag{A.4}$$

[3] Our results are available online as an attachment to the arXiv version of this paper.

$$
\begin{aligned}
\bar{D}_6^{(2)} = & -624\mathcal{R}_{1,-1,-1,1,1,0}-\frac{8592}{11}\mathcal{R}_{1,-1,1,-1,1,0}+\frac{4320}{11}\mathcal{R}_{1,-1,1,-1,1,1}+\frac{2848}{11}\mathcal{R}_{1,-1,1,0,1,0} \\
& -\frac{10320}{11}\mathcal{R}_{1,-1,1,1,-1,0}+\frac{8640}{11}\mathcal{R}_{1,-1,1,1,-1,1}+144\mathcal{R}_{1,-1,1,1,0,-1}+\frac{12960}{11}\mathcal{R}_{1,-1,1,1,1,-1} \\
& -\frac{391256}{55}\mathcal{R}_{1,-1,1,1,1,1}+32\mathcal{R}_{1,0,-1,-1,1,0}+32\mathcal{R}_{1,0,-1,1,-1,0}+1976\mathcal{R}_{1,0,-1,1,1,1} \\
& +32\mathcal{R}_{1,0,1,-1,-1,1}+32\mathcal{R}_{1,0,1,-1,1,-1}+\frac{5024}{33}\mathcal{R}_{1,0,1,-1,1,0}+\frac{17432}{11}\mathcal{R}_{1,0,1,-1,1,1} \\
& -\frac{53444}{99}\mathcal{R}_{1,0,1,0,1,1}+64\mathcal{R}_{1,0,1,1,-1,-1}+\frac{11104}{33}\mathcal{R}_{1,0,1,1,-1,0}+\frac{12776}{11}\mathcal{R}_{1,0,1,1,-1,1} \\
& -\frac{39451}{99}\mathcal{R}_{1,0,1,1,0,1}+\frac{7768}{11}\mathcal{R}_{1,0,1,1,1,-1}+\frac{96931}{99}\mathcal{R}_{1,0,1,1,1,0}-\frac{16128}{11}\mathcal{R}_{1,1,-1,-1,1,0} \\
& +\frac{6000}{11}\mathcal{R}_{1,1,-1,-1,1,1}-\frac{19584}{11}\mathcal{R}_{1,1,-1,1,-1,0}+\frac{14640}{11}\mathcal{R}_{1,1,-1,1,-1,1}+288\mathcal{R}_{1,1,-1,1,0,-1} \\
& -\frac{5696}{11}\mathcal{R}_{1,1,-1,1,0,1}+\frac{23280}{11}\mathcal{R}_{1,1,-1,1,1,-1}-\frac{2128}{11}\mathcal{R}_{1,1,-1,1,1,0}-\frac{372192}{55}\mathcal{R}_{1,1,-1,1,1,1} \\
& +\frac{12064}{3}\mathcal{R}_{1,1,0,-1,1,1}+208\mathcal{R}_{1,1,0,1,-1,-1}+\frac{1405771580504\mathcal{R}_{1,1,0,1,-1,0}}{52401195} \\
& -\frac{449711597968\mathcal{R}_{1,1,0,1,-1,1}}{17467065}-\frac{820125684944\mathcal{R}_{1,1,0,1,1,-1}}{17467065}-\frac{44309295476\mathcal{R}_{1,1,1,-1,-1,0}}{1940785} \\
& +\frac{42581643956\mathcal{R}_{1,1,1,-1,-1,1}}{1940785}-\frac{5328}{5}\mathcal{R}_{1,1,1,-1,0,-1}+\frac{56916979364\mathcal{R}_{1,1,1,-1,0,0}}{7485885} \\
& -\frac{113890854256\mathcal{R}_{1,1,1,-1,0,1}}{7485885}+\frac{44868241556\mathcal{R}_{1,1,1,-1,1,-1}}{1940785}-\frac{42534744232\mathcal{R}_{1,1,1,-1,1,0}}{52401195} \\
& +\frac{104647550068\mathcal{R}_{1,1,1,-1,1,1}}{17467065}+432\mathcal{R}_{1,1,1,0,-1,-1}+\frac{179716135264\mathcal{R}_{1,1,1,0,-1,0}}{1940785} \\
& -\frac{26026092432\mathcal{R}_{1,1,1,0,-1,1}}{277255}+\frac{37024}{5}\mathcal{R}_{1,1,1,0,0,-1}+\frac{2350091860163\mathcal{R}_{1,1,1,0,0,1}}{471610755} \\
& -\frac{909009749728\mathcal{R}_{1,1,1,0,1,-1}}{5822355}-\frac{1515749729887\mathcal{R}_{1,1,1,0,1,0}}{471610755}+\frac{163974915824\mathcal{R}_{1,1,1,1,-1,-1}}{1940785} \\
& +\frac{551982130556\mathcal{R}_{1,1,1,1,-1,0}}{17467065}+\frac{280123847084\mathcal{R}_{1,1,1,1,-1,1}}{17467065}-\frac{19847388288\mathcal{R}_{1,1,1,1,0,-1}}{55451} \\
& +\frac{8283679238293\mathcal{R}_{1,1,1,1,0,0}}{419209560}-\frac{4590966964\mathcal{R}_{1,1,1,1,1,-1}}{388157}-\frac{14507424807748399\mathcal{R}_{1,1,1,1,1,1}}{2400927480}
\end{aligned} \tag{A.5}
$$

$$
\begin{aligned}
\bar{D}_5^{(0)} = & -26\mathcal{R}_{-1,1,1,0}-26\mathcal{R}_{1,-1,1,0}+\frac{16}{3}\mathcal{R}_{1,0,1,0}-26\mathcal{R}_{1,1,-1,0}+6\mathcal{R}_{1,1,0,-1} \\
& +\frac{616}{15}\mathcal{R}_{1,1,1,1}
\end{aligned} \tag{A.6}
$$

$$
\begin{aligned}
\bar{D}_5^{(1)} = & \, 104\mathcal{R}_{-1,-1,1,1,0}+\frac{1432}{11}\mathcal{R}_{-1,1,-1,1,0}-\frac{720}{11}\mathcal{R}_{-1,1,-1,1,1}-\frac{1424}{33}\mathcal{R}_{-1,1,0,1,0} \\
& +\frac{1720}{11}\mathcal{R}_{-1,1,1,-1,0}-\frac{1440}{11}\mathcal{R}_{-1,1,1,-1,1}-24\mathcal{R}_{-1,1,1,0,-1}-\frac{2160}{11}\mathcal{R}_{-1,1,1,1,-1} \\
& +\frac{195628}{165}\mathcal{R}_{-1,1,1,1,1}+260\mathcal{R}_{1,-1,-1,1,1}+\frac{2572}{11}\mathcal{R}_{1,-1,1,-1,1}+\frac{668}{33}\mathcal{R}_{1,-1,1,0,1} \\
& +\frac{2284}{11}\mathcal{R}_{1,-1,1,1,-1}+\frac{674}{3}\mathcal{R}_{1,-1,1,1,0}+\frac{1076}{11}\mathcal{R}_{1,0,-1,1,0}+\frac{1316}{11}\mathcal{R}_{1,0,1,-1,0} \\
& -\frac{360}{11}\mathcal{R}_{1,0,1,-1,1}-20\mathcal{R}_{1,0,1,0,-1}-\frac{720}{11}\mathcal{R}_{1,0,1,1,-1}-\frac{14198}{33}\mathcal{R}_{1,0,1,1,1} \\
& +\frac{4864}{11}\mathcal{R}_{1,1,-1,-1,1}-\frac{11992}{405}\mathcal{R}_{1,1,-1,0,0}+\frac{759664\mathcal{R}_{1,1,-1,0,1}}{4455}+\frac{5008}{11}\mathcal{R}_{1,1,-1,1,-1} \\
& +\frac{2357524\mathcal{R}_{1,1,-1,1,0}}{4455}-\frac{198584}{135}\mathcal{R}_{1,1,-1,1,1}-24\mathcal{R}_{1,1,0,-1,-1}-\frac{926}{11}\mathcal{R}_{1,1,0,1,-1} \\
& -\frac{559454947\mathcal{R}_{1,1,0,1,1}}{775170}+\frac{6012}{11}\mathcal{R}_{1,1,1,-1,-1}+\frac{850486}{891}\mathcal{R}_{1,1,1,-1,0}-\frac{5116084\mathcal{R}_{1,1,1,-1,1}}{1485} \\
& +\frac{3514}{33}\mathcal{R}_{1,1,1,0,-1}-\frac{44410145\mathcal{R}_{1,1,1,0,1}}{42282}-\frac{9118444\mathcal{R}_{1,1,1,1,-1}}{1485}-\frac{2126716774\mathcal{R}_{1,1,1,1,0}}{1162755}
\end{aligned} \tag{A.7}
$$

$$
\begin{aligned}
\bar{D}_5^{(2)} = &-416\mathcal{R}_{-1,-1,-1,1,1,0}-\frac{5728}{11}\mathcal{R}_{-1,-1,1,-1,1,0}+\frac{2880}{11}\mathcal{R}_{-1,-1,1,-1,1,1}+\frac{5696}{33}\mathcal{R}_{-1,-1,1,0,1,0}\\
&-\frac{6880}{11}\mathcal{R}_{-1,-1,1,1,-1,0}+\frac{5760}{11}\mathcal{R}_{-1,-1,1,1,-1,1}+96\mathcal{R}_{-1,-1,1,1,0,-1}+\frac{8640}{11}\mathcal{R}_{-1,-1,1,1,1,-1}\\
&-\frac{782512}{165}\mathcal{R}_{-1,-1,1,1,1,1}-1040\mathcal{R}_{-1,1,-1,-1,1,1}-\frac{10288}{11}\mathcal{R}_{-1,1,-1,1,-1,1}-\frac{2672}{33}\mathcal{R}_{-1,1,-1,1,0,1}\\
&-\frac{9136}{11}\mathcal{R}_{-1,1,-1,1,1,-1}-\frac{2696}{3}\mathcal{R}_{-1,1,-1,1,1,0}-\frac{4304}{11}\mathcal{R}_{-1,1,0,-1,1,0}-\frac{5264}{11}\mathcal{R}_{-1,1,0,1,-1,0}\\
&+\frac{1440}{11}\mathcal{R}_{-1,1,0,1,-1,1}+80\mathcal{R}_{-1,1,0,1,0,-1}+\frac{2880}{11}\mathcal{R}_{-1,1,0,1,1,-1}+\frac{56792}{33}\mathcal{R}_{-1,1,0,1,1,1}\\
&-\frac{19456}{11}\mathcal{R}_{-1,1,1,-1,-1,1}+\frac{3257312\mathcal{R}_{-1,1,1,-1,0,0}}{11745}-\frac{117630944\mathcal{R}_{-1,1,1,-1,0,1}}{129195}\\
&-\frac{20032}{11}\mathcal{R}_{-1,1,1,-1,1,-1}-\frac{302982704\mathcal{R}_{-1,1,1,-1,1,0}}{129195}+\frac{24202144\mathcal{R}_{-1,1,1,-1,1,1}}{3915}\\
&+96\mathcal{R}_{-1,1,1,0,-1,-1}+\frac{3704}{11}\mathcal{R}_{-1,1,1,0,1,-1}+\frac{38639896921\mathcal{R}_{-1,1,1,0,1,1}}{11239965}\\
&-\frac{24048}{11}\mathcal{R}_{-1,1,1,1,-1,-1}-\frac{123547352\mathcal{R}_{-1,1,1,1,-1,0}}{25839}+\frac{649919504\mathcal{R}_{-1,1,1,1,-1,1}}{43065}\\
&-\frac{14056}{33}\mathcal{R}_{-1,1,1,1,0,-1}+\frac{3586576799\mathcal{R}_{-1,1,1,1,0,1}}{613089}+\frac{41605936\mathcal{R}_{-1,1,1,1,1,-1}}{1485}\\
&+\frac{357300160814\mathcal{R}_{-1,1,1,1,1,0}}{33719895}-1040\mathcal{R}_{1,-1,-1,-1,1,1}-\frac{10288}{11}\mathcal{R}_{1,-1,-1,1,-1,1}\\
&-\frac{2672}{33}\mathcal{R}_{1,-1,-1,1,0,1}-\frac{9136}{11}\mathcal{R}_{1,-1,-1,1,1,-1}-\frac{2696}{3}\mathcal{R}_{1,-1,-1,1,1,0}\\
&-\frac{576}{11}\mathcal{R}_{1,-1,1,-1,1,-1}-\frac{387824}{121}\mathcal{R}_{1,-1,1,-1,1,0}+\frac{13241792\mathcal{R}_{1,-1,1,-1,1,1}}{1815}\\
&-\frac{35584}{33}\mathcal{R}_{1,-1,1,0,-1,1}-\frac{28792}{33}\mathcal{R}_{1,-1,1,0,1,-1}+\frac{2296288\mathcal{R}_{1,-1,1,0,1,0}}{1089}\\
&+\frac{14864}{11}\mathcal{R}_{1,-1,1,1,-1,-1}-\frac{5392193704\mathcal{R}_{1,-1,1,1,-1,0}}{1421145}+\frac{5442900016\mathcal{R}_{1,-1,1,1,-1,1}}{473715}\\
&-\frac{32280}{11}\mathcal{R}_{1,-1,1,1,0,-1}+\frac{6570705136\mathcal{R}_{1,-1,1,1,1,-1}}{473715}+\frac{25996220167837\mathcal{R}_{1,-1,1,1,1,1}}{1854594225}\\
&+\frac{64}{3}\mathcal{R}_{1,0,-1,-1,1,0}+\frac{64}{3}\mathcal{R}_{1,0,-1,1,-1,0}-\frac{208}{3}\mathcal{R}_{1,0,-1,1,1,1}-\frac{47936}{33}\mathcal{R}_{1,0,1,-1,-1,1}\\
&-\frac{47936}{33}\mathcal{R}_{1,0,1,-1,1,-1}+\frac{540736}{363}\mathcal{R}_{1,0,1,-1,1,0}-\frac{345216128\mathcal{R}_{1,0,1,-1,1,1}}{94743}\\
&+80\mathcal{R}_{1,0,1,0,-1,-1}+\frac{49280509019\mathcal{R}_{1,0,1,0,1,1}}{148367538}-\frac{95872}{33}\mathcal{R}_{1,0,1,1,-1,-1}\\
&+\frac{634929920\mathcal{R}_{1,0,1,1,-1,0}}{852687}-\frac{676898224\mathcal{R}_{1,0,1,1,-1,1}}{284229}+\frac{387546944255\mathcal{R}_{1,0,1,1,0,1}}{222551307}\\
&+\frac{352181312\mathcal{R}_{1,0,1,1,1,-1}}{94743}+\frac{4241416264027\mathcal{R}_{1,0,1,1,1,0}}{445102614}+\frac{19456}{11}\mathcal{R}_{1,1,-1,-1,-1,1}\\
&+\frac{18880}{11}\mathcal{R}_{1,1,-1,-1,1,-1}-\frac{1901200}{363}\mathcal{R}_{1,1,-1,-1,1,0}+\frac{19253152\mathcal{R}_{1,1,-1,-1,1,1}}{1815}\\
&+\frac{824}{11}\mathcal{R}_{1,1,-1,0,1,-1}+\frac{53776}{11}\mathcal{R}_{1,1,-1,1,-1,-1}-\frac{5754526792\mathcal{R}_{1,1,-1,1,-1,0}}{1421145}\\
&+\frac{965522608\mathcal{R}_{1,1,-1,1,-1,1}}{94743}-\frac{34744}{11}\mathcal{R}_{1,1,-1,1,0,-1}-\frac{1791826416874\mathcal{R}_{1,1,-1,1,0,1}}{370918845}\\
&+\frac{201931024\mathcal{R}_{1,1,-1,1,1,-1}}{94743}-\frac{5946278372713\mathcal{R}_{1,1,-1,1,1,0}}{370918845}+\frac{181225331611139\mathcal{R}_{1,1,-1,1,1,1}}{1854594225}\\
&+96\mathcal{R}_{1,1,0,-1,-1,-1}+\frac{824}{11}\mathcal{R}_{1,1,0,-1,1,-1}-\frac{61094644559\mathcal{R}_{1,1,0,-1,1,1}}{11239965}-\frac{93400}{33}\mathcal{R}_{1,1,0,1,-1,-1}\\
&+\frac{7167353497811919 1589\mathcal{R}_{1,1,0,1,-1,0}}{588987597758175}-\frac{27778395373685590393\mathcal{R}_{1,1,0,1,-1,1}}{196329199252725}\\
&-\frac{39808709428404512639\mathcal{R}_{1,1,0,1,1,-1}}{196329199252725}+\frac{92688}{11}\mathcal{R}_{1,1,1,-1,-1,-1}\\
&-\frac{49767882477121799173\mathcal{R}_{1,1,1,-1,-1,0}}{392658398505450}+\frac{50254587600763374613\mathcal{R}_{1,1,1,-1,-1,1}}{392658398505450}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{191463192286\mathcal{R}_{1,1,1,-1,0,-1}}{18733275}+\frac{30945325345674631\mathcal{R}_{1,1,1,-1,0,0}}{692519221350}\\
&-\frac{15249629583930690367\mathcal{R}_{1,1,1,-1,0,1}}{168282170788050}+\frac{43031534692552744213\mathcal{R}_{1,1,1,-1,1,-1}}{392658398505450}\\
&-\frac{41886306721880707259\mathcal{R}_{1,1,1,-1,1,0}}{1177975195516350}+\frac{10879108295448464191 1\mathcal{R}_{1,1,1,-1,1,1}}{392658398505450}\\
&-\frac{14056}{33}\mathcal{R}_{1,1,1,0,-1,-1}+\frac{8965779447734965424 6\mathcal{R}_{1,1,1,0,-1,0}}{196329199252725}\\
&-\frac{4767577759273275116\mathcal{R}_{1,1,1,0,-1,1}}{9349009488225}+\frac{1559414393279\mathcal{R}_{1,1,1,0,0,-1}}{33719895}\\
&+\frac{13690397558977227041\mathcal{R}_{1,1,1,0,0,1}}{848142140771772}-\frac{15533573816964260686\mathcal{R}_{1,1,1,0,1,-1}}{21814355472525}\\
&-\frac{88230948250575031858 9\mathcal{R}_{1,1,1,0,1,0}}{21203553519294300}+\frac{8894454380015822402 6\mathcal{R}_{1,1,1,1,-1,-1}}{196329199252725}\\
&+\frac{54362172151172329627\mathcal{R}_{1,1,1,1,-1,0}}{392658398505450}+\frac{18794468744404263205 3\mathcal{R}_{1,1,1,1,-1,1}}{392658398505450}\\
&-\frac{5578118784574135554\mathcal{R}_{1,1,1,1,0,-1}}{322379637525}-\frac{35564368167206265810 7\mathcal{R}_{1,1,1,1,0,0}}{18847603128261600}\\
&+\frac{13865776375815235211\mathcal{R}_{1,1,1,1,1,-1}}{26177226567030}-\frac{3656464529968891203697567\mathcal{R}_{1,1,1,1,1,1}}{107945363370952800}
\end{aligned}
\tag{A.8}
$$

$$
\bar{D}_4^{(0)}=12\mathcal{R}_{1,0,1,0}-\frac{2742}{5}\mathcal{R}_{1,1,1,1}
\tag{A.9}
$$

$$
\begin{aligned}
\bar{D}_4^{(1)}=&-56\mathcal{R}_{1,0,-1,1,0}-56\mathcal{R}_{1,0,1,-1,0}+198\mathcal{R}_{1,0,1,1,1}+\frac{15539}{174}\mathcal{R}_{1,1,0,1,1}\\
&-\frac{152353}{522}\mathcal{R}_{1,1,1,0,1}-\frac{219164}{261}\mathcal{R}_{1,1,1,1,0}
\end{aligned}
\tag{A.10}
$$

$$
\begin{aligned}
\bar{D}_4^{(2)}=&\,304\mathcal{R}_{1,0,-1,-1,1,0}+304\mathcal{R}_{1,0,-1,1,-1,0}-1020\mathcal{R}_{1,0,-1,1,1,1}+304\mathcal{R}_{1,0,1,-1,-1,1}\\
&+304\mathcal{R}_{1,0,1,-1,1,-1}+\frac{608}{3}\mathcal{R}_{1,0,1,-1,1,0}-1628\mathcal{R}_{1,0,1,-1,1,1}+\frac{2602}{9}\mathcal{R}_{1,0,1,0,1,1}\\
&+608\mathcal{R}_{1,0,1,1,-1,-1}+\frac{2128}{3}\mathcal{R}_{1,0,1,1,-1,0}-2540\mathcal{R}_{1,0,1,1,-1,1}+\frac{3733}{18}\mathcal{R}_{1,0,1,1,0,1}\\
&-3756\mathcal{R}_{1,0,1,1,1,-1}-\frac{18773}{18}\mathcal{R}_{1,0,1,1,1,0}-\frac{4144}{3}\mathcal{R}_{1,1,0,-1,1,1}+608\mathcal{R}_{1,1,0,1,-1,-1}\\
&-\frac{84256336\mathcal{R}_{1,1,0,1,-1,0}}{136107}-\frac{69730720\mathcal{R}_{1,1,0,1,-1,1}}{45369}-\frac{104376080\mathcal{R}_{1,1,0,1,1,-1}}{45369}\\
&+\frac{177554 24\mathcal{R}_{1,1,1,-1,-1,0}}{45369}-\frac{17755424\mathcal{R}_{1,1,1,-1,-1,1}}{45369}-\frac{21428960\mathcal{R}_{1,1,1,-1,0,0}}{136107}\\
&+\frac{4285792\mathcal{R}_{1,1,1,-1,0,1}}{15123}-\frac{17755424\mathcal{R}_{1,1,1,-1,1,-1}}{45369}+\frac{3061280\mathcal{R}_{1,1,1,-1,1,0}}{136107}\\
&-\frac{6734816\mathcal{R}_{1,1,1,-1,1,1}}{45369}-\frac{20198672\mathcal{R}_{1,1,1,0,-1,0}}{5041}+\frac{129077548\mathcal{R}_{1,1,1,0,-1,1}}{45369}\\
&-\frac{1187408470\mathcal{R}_{1,1,1,0,0,1}}{1224963}+\frac{190647580\mathcal{R}_{1,1,1,0,1,-1}}{45369}+\frac{3293614283\mathcal{R}_{1,1,1,0,1,0}}{1224963}\\
&-\frac{71021696\mathcal{R}_{1,1,1,1,-1,-1}}{45369}-\frac{26327008\mathcal{R}_{1,1,1,1,-1,0}}{45369}-\frac{7959328\mathcal{R}_{1,1,1,1,-1,1}}{45369}\\
&+\frac{707222944\mathcal{R}_{1,1,1,1,0,-1}}{45369}+\frac{4181188831\mathcal{R}_{1,1,1,1,0,0}}{1088856}+\frac{3061280\mathcal{R}_{1,1,1,1,1,-1}}{5041}\\
&+\frac{25498178108029\mathcal{R}_{1,1,1,1,1,1}}{68597928}
\end{aligned}
\tag{A.11}
$$

$$
\bar{D}_3^{(0)}=12\mathcal{R}_{1,0,1,0}-\frac{2454}{5}\mathcal{R}_{1,1,1,1}
\tag{A.12}
$$

$$
\begin{aligned}
\bar{D}_3^{(1)}=&-72\mathcal{R}_{1,0,-1,1,0}-72\mathcal{R}_{1,0,1,-1,0}+234\mathcal{R}_{1,0,1,1,1}+\frac{8565}{58}\mathcal{R}_{1,1,0,1,1}\\
&-\frac{12469}{58}\mathcal{R}_{1,1,1,0,1}-\frac{21182}{29}\mathcal{R}_{1,1,1,1,0}
\end{aligned}
\tag{A.13}
$$

$$
\begin{aligned}
\bar{D}_3^{(2)} = {} & 432\mathcal{R}_{1,0,-1,-1,1,0} + 432\mathcal{R}_{1,0,-1,1,-1,0} - 1404\mathcal{R}_{1,0,-1,1,1,1} + 432\mathcal{R}_{1,0,1,-1,-1,1} \\
& + 432\mathcal{R}_{1,0,1,-1,1,-1} + 288\mathcal{R}_{1,0,1,-1,1,0} - 2268\mathcal{R}_{1,0,1,-1,1,1} + 426\mathcal{R}_{1,0,1,0,1,1} \\
& + 864\mathcal{R}_{1,0,1,1,-1,-1} + 1008\mathcal{R}_{1,0,1,1,-1,0} - 3564\mathcal{R}_{1,0,1,1,-1,1} + \frac{597}{2}\mathcal{R}_{1,0,1,1,0,1} \\
& - 5292\mathcal{R}_{1,0,1,1,1,-1} - \frac{2997}{2}\mathcal{R}_{1,0,1,1,1,0} - 1872\mathcal{R}_{1,1,0,-1,1,1} + 864\mathcal{R}_{1,1,0,1,-1,-1} \\
& - \frac{44345544\mathcal{R}_{1,1,0,1,-1,0}}{5041} - \frac{10551648\mathcal{R}_{1,1,0,1,-1,1}}{5041} - \frac{16021968\mathcal{R}_{1,1,0,1,1,-1}}{5041} \\
& + \frac{2803488\mathcal{R}_{1,1,1,-1,-1,0}}{5041} - \frac{2803488\mathcal{R}_{1,1,1,-1,-1,1}}{5041} - \frac{1127840\mathcal{R}_{1,1,1,-1,0,0}}{5041} \\
& + \frac{2030112\mathcal{R}_{1,1,1,-1,0,1}}{5041} - \frac{2803488\mathcal{R}_{1,1,1,-1,1,-1}}{5041} + \frac{161120\mathcal{R}_{1,1,1,-1,1,0}}{5041} \\
& - \frac{1063392\mathcal{R}_{1,1,1,-1,1,1}}{5041} - \frac{28703376\mathcal{R}_{1,1,1,0,-1,0}}{5041} + \frac{21068364\mathcal{R}_{1,1,1,0,-1,1}}{5041} \\
& - \frac{64500970\mathcal{R}_{1,1,1,0,0,1}}{45369} + \frac{30789948\mathcal{R}_{1,1,1,0,1,-1}}{5041} + \frac{171036689\mathcal{R}_{1,1,1,0,1,0}}{45369} \\
& - \frac{11213952\mathcal{R}_{1,1,1,1,-1,-1}}{5041} - \frac{4156896\mathcal{R}_{1,1,1,1,-1,0}}{5041} - \frac{1256736\mathcal{R}_{1,1,1,1,-1,1}}{5041} \\
& + \frac{112583712\mathcal{R}_{1,1,1,1,0,-1}}{5041} + \frac{216348149\mathcal{R}_{1,1,1,1,0,0}}{40328} + \frac{4350240\mathcal{R}_{1,1,1,1,1,-1}}{5041} \\
& + \frac{1158178961791\mathcal{R}_{1,1,1,1,1,1}}{2540664}
\end{aligned} \tag{A.14}
$$

$$
\bar{D}_M^{(0)} = 8\mathcal{R}_{1,0,1,0} - 392\mathcal{R}_{1,1,1,1} \tag{A.15}
$$

$$
\begin{aligned}
\bar{D}_M^{(1)} = {} & -16\mathcal{R}_{1,0,-1,1,0} - 16\mathcal{R}_{1,0,1,-1,0} + 4\mathcal{R}_{1,0,1,1,1} - \frac{21209}{174}\mathcal{R}_{1,1,0,1,1} \\
& - \frac{192041}{522}\mathcal{R}_{1,1,1,0,1} - \frac{178807}{261}\mathcal{R}_{1,1,1,1,0}
\end{aligned} \tag{A.16}
$$

$$
\begin{aligned}
\bar{D}_M^{(2)} = {} & 32\mathcal{R}_{1,0,-1,-1,1,0} + 32\mathcal{R}_{1,0,-1,1,-1,0} - 8\mathcal{R}_{1,0,-1,1,1,1} + 32\mathcal{R}_{1,0,1,-1,-1,1} \\
& + 32\mathcal{R}_{1,0,1,-1,1,-1} + \frac{64}{3}\mathcal{R}_{1,0,1,-1,1,0} - 72\mathcal{R}_{1,0,1,-1,1,1} - \frac{148}{9}\mathcal{R}_{1,0,1,0,1,1} \\
& + 64\mathcal{R}_{1,0,1,1,-1,-1} + \frac{224}{3}\mathcal{R}_{1,0,1,1,-1,0} - 168\mathcal{R}_{1,0,1,1,-1,1} - \frac{1529}{9}\mathcal{R}_{1,0,1,1,0,1} \\
& - 296\mathcal{R}_{1,0,1,1,1,-1} - 479\mathcal{R}_{1,0,1,1,1,0} + \frac{160}{3}\mathcal{R}_{1,1,0,-1,1,1} + 64\mathcal{R}_{1,1,0,1,-1,-1} \\
& - \frac{8869088\mathcal{R}_{1,1,0,1,-1,0}}{136107} + \frac{1676416\mathcal{R}_{1,1,0,1,-1,1}}{45369} - \frac{1970464\mathcal{R}_{1,1,0,1,1,-1}}{45369} \\
& + \frac{1868992\mathcal{R}_{1,1,1,-1,-1,0}}{45369} - \frac{1868992\mathcal{R}_{1,1,1,-1,-1,1}}{45369} - \frac{2255680\mathcal{R}_{1,1,1,-1,0,0}}{136107} \\
& + \frac{451136\mathcal{R}_{1,1,1,-1,0,1}}{15123} - \frac{1868992\mathcal{R}_{1,1,1,-1,1,-1}}{45369} + \frac{322240\mathcal{R}_{1,1,1,-1,1,0}}{136107} \\
& - \frac{708928\mathcal{R}_{1,1,1,-1,1,1}}{45369} - \frac{2126176\mathcal{R}_{1,1,1,0,-1,0}}{5041} + \frac{27111848\mathcal{R}_{1,1,1,0,-1,1}}{45369} \\
& + \frac{262986220\mathcal{R}_{1,1,1,0,0,1}}{1224963} + \frac{33592904\mathcal{R}_{1,1,1,0,1,-1}}{45369} + \frac{1398989602\mathcal{R}_{1,1,1,0,1,0}}{1224963} \\
& - \frac{7475968\mathcal{R}_{1,1,1,1,-1,-1}}{45369} - \frac{2771264\mathcal{R}_{1,1,1,1,-1,0}}{45369} - \frac{837824\mathcal{R}_{1,1,1,1,-1,1}}{45369} \\
& + \frac{92477504\mathcal{R}_{1,1,1,1,0,-1}}{45369} + \frac{49177327\mathcal{R}_{1,1,1,1,0,0}}{20164} + \frac{322240\mathcal{R}_{1,1,1,1,1,-1}}{5041} \\
& + \frac{5318919501463\mathcal{R}_{1,1,1,1,1,1}}{34298964}
\end{aligned} \tag{A.17}
$$

$$
\begin{aligned}
\bar{D}_N^{(0)} = {} & -\frac{72}{11}\mathcal{R}_{1,-1,1,0} + \frac{180}{11}\mathcal{R}_{1,-1,1,1} + \frac{60}{11}\mathcal{R}_{1,0,1,0} \\
& - \frac{144}{11}\mathcal{R}_{1,1,-1,0} + \frac{360}{11}\mathcal{R}_{1,1,-1,1} + \frac{540}{11}\mathcal{R}_{1,1,1,-1} - \frac{10839}{55}\mathcal{R}_{1,1,1,1}
\end{aligned} \tag{A.18}
$$

$$\bar{D}_N^{(1)} = -\frac{328}{29}\mathcal{R}_{1,1,-1,0,0} + \frac{428}{29}\mathcal{R}_{1,1,-1,0,1} + \frac{428}{29}\mathcal{R}_{1,1,-1,1,0} - \frac{528}{29}\mathcal{R}_{1,1,-1,1,1} - \frac{451079\mathcal{R}_{1,1,0,1,1}}{5046} + \frac{1684}{29}\mathcal{R}_{1,1,1,-1,0} - \frac{2184}{29}\mathcal{R}_{1,1,1,-1,1} - \frac{4056811\mathcal{R}_{1,1,1,0,1}}{15138} - 192\mathcal{R}_{1,1,1,1,-1} - \frac{3533174\mathcal{R}_{1,1,1,1,0}}{7569} \tag{A.19}$$

$$\bar{D}_N^{(2)} = -\frac{232020872\mathcal{R}_{1,1,0,1,-1,0}}{35287} + \frac{242178240\mathcal{R}_{1,1,0,1,-1,1}}{35287} + \frac{484356480\mathcal{R}_{1,1,0,1,1,-1}}{35287} + \frac{110839452\mathcal{R}_{1,1,1,-1,-1,0}}{35287} - \frac{110839452\mathcal{R}_{1,1,1,-1,-1,1}}{35287} - \frac{6313068\mathcal{R}_{1,1,1,-1,0,0}}{5041} + \frac{11391752\mathcal{R}_{1,1,1,-1,0,1}}{5041} - \frac{110839452\mathcal{R}_{1,1,1,-1,1,-1}}{35287} + \frac{5849296\mathcal{R}_{1,1,1,-1,1,0}}{35287} - \frac{414000084\mathcal{R}_{1,1,1,-1,1,1}}{35287} - \frac{691680096\mathcal{R}_{1,1,1,0,-1,0}}{35287} + \frac{103164600\mathcal{R}_{1,1,1,0,-1,1}}{5041} - \frac{321226763\mathcal{R}_{1,1,1,0,0,1}}{317583} + \frac{1448686920\mathcal{R}_{1,1,1,0,1,-1}}{35287} - \frac{347521883\mathcal{R}_{1,1,1,0,1,0}}{317583} - \frac{443357808\mathcal{R}_{1,1,1,1,-1,-1}}{35287} - \frac{166918020\mathcal{R}_{1,1,1,1,-1,0}}{35287} - \frac{46386708\mathcal{R}_{1,1,1,1,-1,1}}{35287} + \frac{412658400\mathcal{R}_{1,1,1,1,0,-1}}{5041} - \frac{3381227415\mathcal{R}_{1,1,1,1,0,0}}{282296} + \frac{182067300\mathcal{R}_{1,1,1,1,1,-1}}{35287} + \frac{21952587521807\mathcal{R}_{1,1,1,1,1,1}}{17784648} \tag{A.20}$$

$$\bar{E}_3^{(-1)} = 6\mathcal{I}_{1,0} \tag{A.21}$$

$$\bar{E}_3^{(0)} = -12\mathcal{I}_{-1,1,0} - 12\mathcal{I}_{1,-1,0} + 3\mathcal{I}_{1,1,1} \tag{A.22}$$

$$\bar{E}_3^{(1)} = 24\mathcal{I}_{-1,-1,1,0} + 24\mathcal{I}_{-1,1,-1,0} - 6\mathcal{I}_{-1,1,1,1} + 24\mathcal{I}_{1,-1,-1,1} + 24\mathcal{I}_{1,-1,1,-1} + 16\mathcal{I}_{1,-1,1,0} - 54\mathcal{I}_{1,-1,1,1} - \frac{37}{3}\mathcal{I}_{1,0,1,1} + 48\mathcal{I}_{1,1,-1,-1} + 56\mathcal{I}_{1,1,-1,0} - 126\mathcal{I}_{1,1,-1,1} - \frac{1529}{12}\mathcal{I}_{1,1,0,1} - 222\mathcal{I}_{1,1,1,-1} - \frac{1437}{4}\mathcal{I}_{1,1,1,0} \tag{A.23}$$

$$\bar{E}_3^{(2)} = -48\mathcal{I}_{-1,-1,-1,1,0} - 48\mathcal{I}_{-1,-1,1,-1,0} + 12\mathcal{I}_{-1,-1,1,1,1} - 48\mathcal{I}_{-1,1,-1,-1,1} - 48\mathcal{I}_{-1,1,-1,1,-1} - 32\mathcal{I}_{-1,1,-1,1,0} + 108\mathcal{I}_{-1,1,-1,1,1} + \frac{74}{3}\mathcal{I}_{-1,1,0,1,1} - 96\mathcal{I}_{-1,1,1,-1,-1} - 112\mathcal{I}_{-1,1,1,-1,0} + 252\mathcal{I}_{-1,1,1,-1,1} + \frac{1529}{6}\mathcal{I}_{-1,1,1,0,1} + 444\mathcal{I}_{-1,1,1,1,-1} + \frac{1437}{2}\mathcal{I}_{-1,1,1,1,0} - 48\mathcal{I}_{1,-1,-1,-1,1} - 48\mathcal{I}_{1,-1,-1,1,-1} - 32\mathcal{I}_{1,-1,-1,1,0} - 112\mathcal{I}_{1,-1,1,-1,0} + 96\mathcal{I}_{1,-1,1,-1,1} - 80\mathcal{I}_{1,-1,1,0,-1} + 144\mathcal{I}_{1,-1,1,1,-1} + \frac{11841}{4}\mathcal{I}_{1,-1,1,1,1} + 90\mathcal{I}_{1,0,-1,1,1} - 80\mathcal{I}_{1,0,1,-1,-1} + 130\mathcal{I}_{1,0,1,-1,1} + \frac{8105}{36}\mathcal{I}_{1,0,1,0,1} + 250\mathcal{I}_{1,0,1,1,-1} + \frac{21503}{36}\mathcal{I}_{1,0,1,1,0} + 96\mathcal{I}_{1,1,-1,-1,-1} - 172\mathcal{I}_{1,1,-1,-1,1} + \frac{1725367\mathcal{I}_{1,1,-1,0,0}}{4161} - \frac{412061\mathcal{I}_{1,1,-1,0,1}}{8322} - 268\mathcal{I}_{1,1,-1,1,-1} - \frac{1334797\mathcal{I}_{1,1,-1,1,0}}{2774} + \frac{14659325\mathcal{I}_{1,1,-1,1,1}}{2774} + \frac{771}{2}\mathcal{I}_{1,1,0,-1,1} - \frac{5253406\mathcal{I}_{1,1,0,0,-1}}{12483} + \frac{1081050\mathcal{I}_{1,1,0,1,-1}}{1387} - 1112\mathcal{I}_{1,1,1,-1,-1} - \frac{1316324\mathcal{I}_{1,1,1,-1,0}}{1387} + \frac{408159881\mathcal{I}_{1,1,1,-1,1}}{5548} + \frac{1357934}{657}\mathcal{I}_{1,1,1,0,-1} - \frac{1768682807\mathcal{I}_{1,1,1,0,0}}{898776} + \frac{144459841\mathcal{I}_{1,1,1,1,-1}}{1387} - \frac{3591211175\mathcal{I}_{1,1,1,1,1}}{112347} \tag{A.24}$$

$$
\begin{aligned}
\bar{E}_3^{(3)} = {} & 96\mathcal{I}_{-1,-1,-1,-1,1,0}+96\mathcal{I}_{-1,-1,-1,1,-1,0}-24\mathcal{I}_{-1,-1,-1,1,1,1}+96\mathcal{I}_{-1,-1,1,-1,-1,1} \\
& +96\mathcal{I}_{-1,-1,1,-1,1,-1}+64\mathcal{I}_{-1,-1,1,-1,1,0}-216\mathcal{I}_{-1,-1,1,-1,1,1}-\frac{148}{3}\mathcal{I}_{-1,-1,1,0,1,1} \\
& +192\mathcal{I}_{-1,-1,1,1,-1,-1}+224\mathcal{I}_{-1,-1,1,1,-1,0}-504\mathcal{I}_{-1,-1,1,1,-1,1}-\frac{1529}{3}\mathcal{I}_{-1,-1,1,1,0,1} \\
& -888\mathcal{I}_{-1,-1,1,1,1,-1}-1437\mathcal{I}_{-1,-1,1,1,1,0}+96\mathcal{I}_{-1,1,-1,-1,-1,1}+96\mathcal{I}_{-1,1,-1,-1,1,-1} \\
& +64\mathcal{I}_{-1,1,-1,-1,1,0}+224\mathcal{I}_{-1,1,-1,1,-1,0}-192\mathcal{I}_{-1,1,-1,1,-1,1}+160\mathcal{I}_{-1,1,-1,1,0,-1} \\
& -288\mathcal{I}_{-1,1,-1,1,1,-1}-\frac{11841}{2}\mathcal{I}_{-1,1,-1,1,1,1}-180\mathcal{I}_{-1,1,0,-1,1,1}+160\mathcal{I}_{-1,1,0,1,-1,-1} \\
& -260\mathcal{I}_{-1,1,0,1,-1,1}-\frac{8105}{18}\mathcal{I}_{-1,1,0,1,0,1}-500\mathcal{I}_{-1,1,0,1,1,-1}-\frac{21503}{18}\mathcal{I}_{-1,1,0,1,1,0} \\
& -192\mathcal{I}_{-1,1,1,-1,-1,-1}+344\mathcal{I}_{-1,1,1,-1,-1,1}-\frac{3450734\mathcal{I}_{-1,1,1,-1,0,0}}{4161}+\frac{412061\mathcal{I}_{-1,1,1,-1,0,1}}{4161} \\
& +536\mathcal{I}_{-1,1,1,-1,1,-1}+\frac{1334797\mathcal{I}_{-1,1,1,-1,1,0}}{1387}-\frac{14659325\mathcal{I}_{-1,1,1,-1,1,1}}{1387}-771\mathcal{I}_{-1,1,1,0,-1,1} \\
& +\frac{10506812\mathcal{I}_{-1,1,1,0,0,-1}}{12483}-\frac{2162100\mathcal{I}_{-1,1,1,0,1,-1}}{1387}+2224\mathcal{I}_{-1,1,1,1,-1,-1} \\
& +\frac{2632648\mathcal{I}_{-1,1,1,1,-1,0}}{1387}-\frac{40815981\mathcal{I}_{-1,1,1,1,-1,1}}{2774}-\frac{2715868}{657}\mathcal{I}_{-1,1,1,1,0,-1} \\
& +\frac{1768682807\mathcal{I}_{-1,1,1,1,0,0}}{449388}-\frac{28919682\mathcal{I}_{-1,1,1,1,1,-1}}{1387}+\frac{7182422350\mathcal{I}_{-1,1,1,1,1,1}}{112347} \\
& +96\mathcal{I}_{1,-1,-1,-1,-1,1}+96\mathcal{I}_{1,-1,-1,-1,1,-1}+64\mathcal{I}_{1,-1,-1,-1,1,0}+224\mathcal{I}_{1,-1,-1,1,-1,0} \\
& -192\mathcal{I}_{1,-1,-1,1,-1,1}+160\mathcal{I}_{1,-1,-1,1,0,-1}-288\mathcal{I}_{1,-1,-1,1,1,-1}-\frac{11841}{2}\mathcal{I}_{1,-1,-1,1,1,1} \\
& +248\mathcal{I}_{1,-1,1,-1,-1,1}+440\mathcal{I}_{1,-1,1,-1,1,-1}+\frac{2590}{3}\mathcal{I}_{1,-1,1,-1,1,0}-10608\mathcal{I}_{1,-1,1,-1,1,1} \\
& -320\mathcal{I}_{1,-1,1,0,-1,-1}-\frac{759843\mathcal{I}_{1,-1,1,0,1,-1}}{1387}+\frac{13919645\mathcal{I}_{1,-1,1,0,1,1}}{12483}+1520\mathcal{I}_{1,-1,1,1,-1,-1} \\
& +\frac{7073822\mathcal{I}_{1,-1,1,1,-1,0}}{4161}-\frac{40321721\mathcal{I}_{1,-1,1,1,-1,1}}{2774}+\frac{6271329955\mathcal{I}_{1,-1,1,1,0,1}}{898776} \\
& -\frac{26862395\mathcal{I}_{1,-1,1,1,1,-1}}{1387}+\frac{6038799457\mathcal{I}_{1,-1,1,1,1,0}}{299592}-160\mathcal{I}_{1,0,1,-1,-1,-1} \\
& -\frac{277814945\mathcal{I}_{1,0,1,-1,1,0}}{137313}+\frac{212581060\mathcal{I}_{1,0,1,-1,1,1}}{45771}-\frac{585}{2}\mathcal{I}_{1,0,1,0,-1,1} \\
& -\frac{585}{2}\mathcal{I}_{1,0,1,0,1,-1}+\frac{2272287250\mathcal{I}_{1,0,1,0,1,0}}{1235817}+\frac{1280}{3}\mathcal{I}_{1,0,1,1,-1,-1} \\
& -\frac{221375860\mathcal{I}_{1,0,1,1,-1,0}}{137313}+\frac{4419865}{803}\mathcal{I}_{1,0,1,1,-1,1}-\frac{2624160\mathcal{I}_{1,0,1,1,0,-1}}{1387} \\
& +\frac{86319175\mathcal{I}_{1,0,1,1,1,-1}}{45771}-\frac{7230866663213\mathcal{I}_{1,0,1,1,1,1}}{74149020}+192\mathcal{I}_{1,1,-1,-1,-1,-1} \\
& +152\mathcal{I}_{1,1,-1,-1,-1,1}+344\mathcal{I}_{1,1,-1,-1,1,-1}+\frac{2590}{3}\mathcal{I}_{1,1,-1,-1,1,0}-\frac{30479}{3}\mathcal{I}_{1,1,-1,-1,1,1} \\
& -\frac{426963\mathcal{I}_{1,1,-1,0,1,-1}}{1387}+944\mathcal{I}_{1,1,-1,1,-1,-1}+1601\mathcal{I}_{1,1,-1,1,-1,0}-\frac{83605}{6}\mathcal{I}_{1,1,-1,1,-1,1} \\
& +\frac{2419561}{657}\mathcal{I}_{1,1,-1,1,0,-1}+\frac{837579887\mathcal{I}_{1,1,-1,1,0,1}}{112347}-\frac{73453112\mathcal{I}_{1,1,-1,1,1,-1}}{4161} \\
& +\frac{24602317907\mathcal{I}_{1,1,-1,1,1,0}}{898776}-\frac{2981940259\mathcal{I}_{1,1,-1,1,1,1}}{224694}-571\mathcal{I}_{1,1,0,-1,-1,1} \\
& -\frac{1218940\mathcal{I}_{1,1,0,-1,1,-1}}{1387}-\frac{1284799676\mathcal{I}_{1,1,0,-1,1,0}}{411939}+\frac{172606885810943423\mathcal{I}_{1,1,0,1,-1,-1}}{31412461173192} \\
& +\frac{330844112035157\mathcal{I}_{1,1,0,1,-1,0}}{387808162632}-\frac{335938955928166355\mathcal{I}_{1,1,0,1,-1,1}}{62824922346384} \\
& -\frac{9666835700761579\mathcal{I}_{1,1,0,1,1,-1}}{296343973332}-784\mathcal{I}_{1,1,1,-1,-1,-1}+\frac{32163144510925889\mathcal{I}_{1,1,1,-1,-1,0}}{1903785525648} \\
& -\frac{52473680391301577\mathcal{I}_{1,1,1,-1,-1,1}}{1903785525648}+\frac{910517365657338137\mathcal{I}_{1,1,1,-1,0,-1}}{62824922346384}
\end{aligned}
$$

JHEP08(2017)024

$$
\begin{aligned}
&+\frac{263100146907443657\mathcal{I}_{1,1,1,-1,0,0}}{17134069730832}-\frac{462801345882267057\mathcal{I}_{1,1,1,-1,0,1}}{188474767039152}\\
&-\frac{57799520399301857\mathcal{I}_{1,1,1,-1,1,-1}}{1903785525648}+\frac{9315305053815793 07\mathcal{I}_{1,1,1,-1,1,0}}{62824922346384}\\
&+\frac{1778265423341603573\mathcal{I}_{1,1,1,-1,1,1}}{62824922346384}+\frac{161624158645205183\mathcal{I}_{1,1,1,0,-1,-1}}{10470820391064}\\
&-\frac{677380848630031 55\mathcal{I}_{1,1,1,0,-1,0}}{5235410195532}-\frac{257096369771474 41\mathcal{I}_{1,1,1,0,-1,1}}{2326848975792}\\
&-\frac{1201750627073139241\mathcal{I}_{1,1,1,0,0,-1}}{205608836769984}\\
&+\frac{3712346122368778903945 9901\mathcal{I}_{1,1,1,0,0,0}}{4896122108236274995 20}-\frac{15065768752177753\mathcal{I}_{1,1,1,0,1,-1}}{183698603352}\\
&+\frac{589072561141981882557 8737\mathcal{I}_{1,1,1,0,1,1}}{257690637275593420 80}-\frac{264179249945215580 3\mathcal{I}_{1,1,1,1,-1,-1}}{31412461173192}\\
&-\frac{232062550415237737\mathcal{I}_{1,1,1,1,-1,0}}{3306574860336}+\frac{709822530971820472\mathcal{I}_{1,1,1,1,-1,1}}{3926557646649}\\
&-\frac{1140348451560297014 11\mathcal{I}_{1,1,1,1,0,-1}}{1130848602234912}+\frac{4971084687572627049459 11459\mathcal{I}_{1,1,1,1,0,1}}{979224421647254999040}\\
&+\frac{190725783964515931 55\mathcal{I}_{1,1,1,1,1,-1}}{31412461173192}+\frac{160662586785011672453801383\mathcal{I}_{1,1,1,1,1,0}}{326408140549084999680}
\end{aligned}
\tag{A.25}
$$

$$
B\bar{N}^{(1)}=112\mathcal{R}_{1,1,0}
\tag{A.26}
$$

$$
\begin{aligned}
B\bar{N}^{(2)}=&-\frac{2304}{11}\mathcal{R}_{1,-1,1,0}+\frac{5760}{11}\mathcal{R}_{1,-1,1,1}+\frac{1920}{11}\mathcal{R}_{1,0,1,0}-\frac{4608}{11}\mathcal{R}_{1,1,-1,0}\\
&+\frac{11520}{11}\mathcal{R}_{1,1,-1,1}+\frac{17280}{11}\mathcal{R}_{1,1,1,-1}-\frac{593952}{55}\mathcal{R}_{1,1,1,1}
\end{aligned}
\tag{A.27}
$$

$$
\begin{aligned}
B\bar{N}^{(3)}=&-\frac{36864}{29}\mathcal{R}_{1,1,-1,0,0}+\frac{52992}{29}\mathcal{R}_{1,1,-1,0,1}+\frac{52992}{29}\mathcal{R}_{1,1,-1,1,0}-\frac{69120}{29}\mathcal{R}_{1,1,-1,1,1}\\
&-\frac{3282240}{841}\mathcal{R}_{1,1,0,1,1}+\frac{223488}{29}\mathcal{R}_{1,1,1,-1,0}-\frac{304128}{29}\mathcal{R}_{1,1,1,-1,1}-\frac{9820736}{841}\mathcal{R}_{1,1,1,0,1}\\
&-27648\mathcal{R}_{1,1,1,1,-1}-\frac{19524608}{841}\mathcal{R}_{1,1,1,1,0}n
\end{aligned}
\tag{A.28}
$$

$$
\begin{aligned}
B\bar{N}^{(4)}=&-\frac{1303119360\mathcal{R}_{1,1,0,1,-1,0}}{5041}+\frac{1354682880\mathcal{R}_{1,1,0,1,-1,1}}{5041}+\frac{2709365760\mathcal{R}_{1,1,0,1,1,-1}}{5041}\\
&+\frac{613018368\mathcal{R}_{1,1,1,-1,-1,0}}{5041}-\frac{613018368\mathcal{R}_{1,1,1,-1,-1,1}}{5041}-\frac{208168704\mathcal{R}_{1,1,1,-1,0,0}}{5041}\\
&+\frac{388641024\mathcal{R}_{1,1,1,-1,0,1}}{5041}-\frac{613018368\mathcal{R}_{1,1,1,-1,1,-1}}{5041}-\frac{20037888\mathcal{R}_{1,1,1,-1,1,0}}{5041}\\
&-\frac{160434432\mathcal{R}_{1,1,1,-1,1,1}}{5041}-\frac{3867098112\mathcal{R}_{1,1,1,0,-1,0}}{5041}+\frac{4021788672\mathcal{R}_{1,1,1,0,-1,1}}{5041}\\
&-\frac{181188480\mathcal{R}_{1,1,1,0,0,1}}{5041}+\frac{8085837312\mathcal{R}_{1,1,1,0,1,-1}}{5041}-\frac{209361792\mathcal{R}_{1,1,1,0,1,0}}{5041}\\
&-\frac{2452073472\mathcal{R}_{1,1,1,1,-1,-1}}{5041}-\frac{1197317376\mathcal{R}_{1,1,1,1,-1,0}}{5041}+\frac{114483456\mathcal{R}_{1,1,1,1,-1,1}}{5041}\\
&+\frac{16087154688\mathcal{R}_{1,1,1,1,0,-1}}{5041}-\frac{2378746336\mathcal{R}_{1,1,1,1,0,0}}{5041}+\frac{2176761600\mathcal{R}_{1,1,1,1,1,-1}}{5041}\\
&+\frac{249395302624\mathcal{R}_{1,1,1,1,1,1}}{5041}
\end{aligned}
\tag{A.29}
$$

$$
B\bar{N}_1^{(0)}=4\mathcal{I}_{1,0}
\tag{A.30}
$$

$$
B\bar{N}_1^{(1)}=-24\mathcal{I}_{-1,1,0}-24\mathcal{I}_{1,-1,0}+78\mathcal{I}_{1,1,1}
\tag{A.31}
$$

$$
\begin{aligned}
B\bar{N}_1^{(2)}=&\,144\mathcal{I}_{-1,-1,1,0}+144\mathcal{I}_{-1,1,-1,0}-468\mathcal{I}_{-1,1,1,1}+144\mathcal{I}_{1,-1,-1,1}+144\mathcal{I}_{1,-1,1,-1}\\
&+96\mathcal{I}_{1,-1,1,0}-756\mathcal{I}_{1,-1,1,1}+142\mathcal{I}_{1,0,1,1}+288\mathcal{I}_{1,1,-1,-1}+336\mathcal{I}_{1,1,-1,0}\\
&-1188\mathcal{I}_{1,1,-1,1}+\frac{199}{2}\mathcal{I}_{1,1,0,1}-1764\mathcal{I}_{1,1,1,-1}-\frac{999}{2}\mathcal{I}_{1,1,1,0}
\end{aligned}
\tag{A.32}
$$

$$
\begin{aligned}
\bar{BN}_1^{(3)} = & -864\mathcal{I}_{-1,-1,-1,1,0}-864\mathcal{I}_{-1,-1,1,-1,0}+2808\mathcal{I}_{-1,-1,1,1,1}-864\mathcal{I}_{-1,1,-1,-1,1} \\
& -864\mathcal{I}_{-1,1,-1,1,-1}-576\mathcal{I}_{-1,1,-1,1,0}+4536\mathcal{I}_{-1,1,-1,1,1}-852\mathcal{I}_{-1,1,0,1,1} \\
& -1728\mathcal{I}_{-1,1,1,-1,-1}-2016\mathcal{I}_{-1,1,1,-1,0}+7128\mathcal{I}_{-1,1,1,-1,1}-597\mathcal{I}_{-1,1,1,0,1} \\
& +10584\mathcal{I}_{-1,1,1,1,-1}+2997\mathcal{I}_{-1,1,1,1,0}-864\mathcal{I}_{1,-1,-1,-1,1}-864\mathcal{I}_{1,-1,-1,1,-1} \\
& -576\mathcal{I}_{1,-1,-1,1,0}-2016\mathcal{I}_{1,-1,1,-1,0}+1728\mathcal{I}_{1,-1,1,-1,1}-1440\mathcal{I}_{1,-1,1,0,-1} \\
& +2592\mathcal{I}_{1,-1,1,1,-1}+\frac{41337}{2}\mathcal{I}_{1,-1,1,1,1}+3780\mathcal{I}_{1,0,-1,1,1}-1440\mathcal{I}_{1,0,1,-1,-1} \\
& +4500\mathcal{I}_{1,0,1,-1,1}+\frac{1625}{2}\mathcal{I}_{1,0,1,0,1}+6660\mathcal{I}_{1,0,1,1,-1}+\frac{5903}{2}\mathcal{I}_{1,0,1,1,0} \\
& +1728\mathcal{I}_{1,1,-1,-1,-1}-5688\mathcal{I}_{1,1,-1,-1,1}+\frac{3566346\mathcal{I}_{1,1,-1,0,0}}{1387}+\frac{2763273\mathcal{I}_{1,1,-1,0,1}}{1387} \\
& -7416\mathcal{I}_{1,1,-1,1,-1}-\frac{1422693\mathcal{I}_{1,1,-1,1,0}}{1387}+\frac{54624501\mathcal{I}_{1,1,-1,1,1}}{1387}+8667\mathcal{I}_{1,1,0,-1,1} \\
& -\frac{5422460\mathcal{I}_{1,1,0,0,-1}}{1387}+\frac{17237556\mathcal{I}_{1,1,0,1,-1}}{1387}-25200\mathcal{I}_{1,1,1,-1,-1} \\
& -\frac{5703624\mathcal{I}_{1,1,1,-1,0}}{1387}+\frac{174917637\mathcal{I}_{1,1,1,-1,1}}{2774}+\frac{1624492}{73}\mathcal{I}_{1,1,1,0,-1} \\
& -\frac{400772087\mathcal{I}_{1,1,1,0,0}}{49932}+\frac{158327730\mathcal{I}_{1,1,1,1,-1}}{1387}-\frac{3279409000\mathcal{I}_{1,1,1,1,1}}{12483}
\end{aligned} \tag{A.33}
$$

$$
\begin{aligned}
\bar{BN}_1^{(4)} = & \,5184\mathcal{I}_{-1,-1,-1,-1,1,0}+5184\mathcal{I}_{-1,-1,-1,1,-1,0}-16848\mathcal{I}_{-1,-1,-1,1,1,1} \\
& +5184\mathcal{I}_{-1,-1,1,-1,-1,1}+5184\mathcal{I}_{-1,-1,1,-1,1,-1}+3456\mathcal{I}_{-1,-1,1,-1,1,0} \\
& -27216\mathcal{I}_{-1,-1,1,-1,1,1}+5112\mathcal{I}_{-1,-1,1,0,1,1}+10368\mathcal{I}_{-1,-1,1,1,-1,-1} \\
& +12096\mathcal{I}_{-1,-1,1,1,-1,0}-42768\mathcal{I}_{-1,-1,1,1,-1,1}+3582\mathcal{I}_{-1,-1,1,1,0,1} \\
& -63504\mathcal{I}_{-1,-1,1,1,1,-1}-17982\mathcal{I}_{-1,-1,1,1,1,0}+5184\mathcal{I}_{-1,1,-1,-1,-1,1} \\
& +5184\mathcal{I}_{-1,1,-1,-1,1,-1}+3456\mathcal{I}_{-1,1,-1,-1,1,0}+12096\mathcal{I}_{-1,1,-1,1,-1,0} \\
& -10368\mathcal{I}_{-1,1,-1,1,-1,1}+8640\mathcal{I}_{-1,1,-1,1,0,-1}-15552\mathcal{I}_{-1,1,-1,1,1,-1} \\
& -124011\mathcal{I}_{-1,1,-1,1,1,1}-22680\mathcal{I}_{-1,1,0,-1,1,1}+8640\mathcal{I}_{-1,1,0,1,-1,-1} \\
& -27000\mathcal{I}_{-1,1,0,1,-1,1}-4875\mathcal{I}_{-1,1,0,1,0,1}-39960\mathcal{I}_{-1,1,0,1,1,-1} \\
& -17709\mathcal{I}_{-1,1,0,1,1,0}-10368\mathcal{I}_{-1,1,1,-1,-1,-1}+34128\mathcal{I}_{-1,1,1,-1,-1,1} \\
& -\frac{21398076\mathcal{I}_{-1,1,1,-1,0,0}}{1387}-\frac{16579638\mathcal{I}_{-1,1,1,-1,0,1}}{1387}+44496\mathcal{I}_{-1,1,1,-1,1,-1} \\
& +\frac{8536158\mathcal{I}_{-1,1,1,-1,1,0}}{1387}-\frac{327747006\mathcal{I}_{-1,1,1,-1,1,1}}{1387}-52002\mathcal{I}_{-1,1,1,0,-1,1} \\
& +\frac{32534760\mathcal{I}_{-1,1,1,0,0,-1}}{1387}-\frac{103425336\mathcal{I}_{-1,1,1,0,1,-1}}{1387}+151200\mathcal{I}_{-1,1,1,1,-1,-1} \\
& +\frac{342221744\mathcal{I}_{-1,1,1,1,-1,0}}{1387}-\frac{524752911\mathcal{I}_{-1,1,1,1,-1,1}}{1387}-\frac{9746952}{73}\mathcal{I}_{-1,1,1,1,0,-1} \\
& +\frac{400772087\mathcal{I}_{-1,1,1,1,0,0}}{8322}-\frac{949966380\mathcal{I}_{-1,1,1,1,1,-1}}{1387}+\frac{6558818000\mathcal{I}_{-1,1,1,1,1,1}}{4161} \\
& +5184\mathcal{I}_{1,-1,-1,-1,-1,1}+5184\mathcal{I}_{1,-1,-1,-1,1,-1}+3456\mathcal{I}_{1,-1,-1,-1,1,0} \\
& +12096\mathcal{I}_{1,-1,-1,1,-1,0}-10368\mathcal{I}_{1,-1,-1,1,-1,1}+8640\mathcal{I}_{1,-1,-1,1,0,-1} \\
& -15552\mathcal{I}_{1,-1,-1,1,1,-1}-124011\mathcal{I}_{1,-1,-1,1,1,1}+28944\mathcal{I}_{1,-1,1,-1,-1,1} \\
& +39312\mathcal{I}_{1,-1,1,-1,1,-1}+18108\mathcal{I}_{1,-1,1,-1,1,0}-290304\mathcal{I}_{1,-1,1,-1,1,1} \\
& -17280\mathcal{I}_{1,-1,1,0,-1,-1}-\frac{13323042\mathcal{I}_{1,-1,1,0,1,-1}}{1387}+\frac{34731966\mathcal{I}_{1,-1,1,0,1,1}}{1387}
\end{aligned}
$$

JHEP08(2017)024

$$
\begin{aligned}
&+113184\mathcal{I}_{1,-1,1,1,-1,-1}+\frac{673381020\mathcal{I}_{1,-1,1,1,-1,0}}{1387}-\frac{655388307\mathcal{I}_{1,-1,1,1,-1,1}}{1387}\\
&+\frac{1961738995\mathcal{I}_{1,-1,1,1,0,1}}{16644}-\frac{1054843506\mathcal{I}_{1,-1,1,1,1,-1}}{1387}+\frac{1636490353\mathcal{I}_{1,-1,1,1,1,0}}{5548}\\
&-8640\mathcal{I}_{1,0,1,-1,-1,-1}-\frac{1057484550\mathcal{I}_{1,0,1,-1,1,0}}{15257}+\frac{2896138440\mathcal{I}_{1,0,1,-1,1,1}}{15257}\\
&-13635\mathcal{I}_{1,0,1,0,-1,1}-13635\mathcal{I}_{1,0,1,0,1,-1}+\frac{2536761620\mathcal{I}_{1,0,1,0,1,0}}{45771}\\
&+23040\mathcal{I}_{1,0,1,1,-1,-1}-\frac{722452920\mathcal{I}_{1,0,1,1,-1,0}}{15257}+\frac{189139590}{803}\mathcal{I}_{1,0,1,1,-1,1}\\
&-\frac{738887840\mathcal{I}_{1,0,1,1,0,-1}}{1387}+\frac{1521319230\mathcal{I}_{1,0,1,1,1,-1}}{15257}-\frac{1739984020721\mathcal{I}_{1,0,1,1,1,1}}{1373130}\\
&+10368\mathcal{I}_{1,1,-1,-1,-1,-1}+23760\mathcal{I}_{1,1,-1,-1,-1,1}+34128\mathcal{I}_{1,1,-1,-1,1,-1}\\
&+18108\mathcal{I}_{1,1,-1,-1,1,0}-323118\mathcal{I}_{1,1,-1,-1,1,1}+\frac{4652478\mathcal{I}_{1,1,-1,0,1,-1}}{1387}\\
&+82080\mathcal{I}_{1,1,-1,1,-1,-1}+60534\mathcal{I}_{1,1,-1,1,-1,0}-548973\mathcal{I}_{1,1,-1,1,-1,1}\\
&+\frac{7212246}{73}\mathcal{I}_{1,1,-1,1,0,-1}+\frac{499549318\mathcal{I}_{1,1,-1,1,0,1}}{4161}-\frac{1149502896\mathcal{I}_{1,1,-1,1,1,-1}}{1387}\\
&+\frac{6314678771\mathcal{I}_{1,1,-1,1,1,0}}{16644}-\frac{3100825687\mathcal{I}_{1,1,-1,1,1,1}}{4161}-15282\mathcal{I}_{1,1,0,-1,-1,1}\\
&-\frac{165543656\mathcal{I}_{1,1,0,-1,1,-1}}{1387}-\frac{1961940664\mathcal{I}_{1,1,0,-1,1,0}}{15257}+\frac{648957016685944431\mathcal{I}_{1,1,0,1,-1,-1}}{581712243948}\\
&-\frac{1378601274793377\mathcal{I}_{1,1,0,1,-1,0}}{21544897924}+\frac{125263181337378061\mathcal{I}_{1,1,0,1,-1,1}}{1163424487896}\\
&-\frac{1211150098669531\mathcal{I}_{1,1,0,1,1,-1}}{5487851358}-11232\mathcal{I}_{1,1,1,-1,-1,-1}\\
&+\frac{2392665851955041\mathcal{I}_{1,1,1,-1,-1,0}}{35255287512}-\frac{2046434995416868 1\mathcal{I}_{1,1,1,-1,-1,1}}{35255287512}\\
&+\frac{189472199905473497\mathcal{I}_{1,1,1,-1,0,-1}}{1163424487896}+\frac{47207116920801641\mathcal{I}_{1,1,1,-1,0,0}}{317297587608}\\
&-\frac{298649341166732401\mathcal{I}_{1,1,1,-1,0,1}}{3490273463688}-\frac{30404966076339713\mathcal{I}_{1,1,1,-1,1,-1}}{35255287512}\\
&+\frac{181778915331736427\mathcal{I}_{1,1,1,-1,1,0}}{1163424487896}-\frac{366128024895147883\mathcal{I}_{1,1,1,-1,1,1}}{1163424487896}\\
&+\frac{5391297450285619 1\mathcal{I}_{1,1,1,0,-1,-1}}{193904081316}-\frac{10304731594730675\mathcal{I}_{1,1,1,0,-1,0}}{96952040658}\\
&-\frac{16995388457766673\mathcal{I}_{1,1,1,0,-1,1}}{43089795848}+\frac{186908888125819415\mathcal{I}_{1,1,1,0,0,-1}}{3807571051296}\\
&+\frac{102767984436002386103743 97\mathcal{I}_{1,1,1,0,0,0}}{9066892793030138880}-\frac{3306662120409625\mathcal{I}_{1,1,1,0,1,-1}}{3401825988}\\
&+\frac{1299407259007733189665969\mathcal{I}_{1,1,1,0,1,1}}{4772048838436915 20}-\frac{384757595682873755\mathcal{I}_{1,1,1,1,-1,-1}}{581712243948}\\
&-\frac{30219787275121801\mathcal{I}_{1,1,1,1,-1,0}}{61232867784}+\frac{22858349690960734 4\mathcal{I}_{1,1,1,1,-1,1}}{145428060987}\\
&-\frac{444092538547948265 95\mathcal{I}_{1,1,1,1,0,-1}}{20941640782128}+\frac{111015775290234750628966883\mathcal{I}_{1,1,1,1,0,1}}{18133785586060277760}\\
&+\frac{2811281629788118067\mathcal{I}_{1,1,1,1,1,-1}}{581712243948}+\frac{3572537948131760111881 6231\mathcal{I}_{1,1,1,1,1,0}}{6044595195353425920}
\end{aligned}
\tag{A.34}
$$

$$\bar{T}_1^{(0)}=6\mathcal{I}_{1,0} \tag{A.35}$$

$$\bar{T}_1^{(1)}=-12\mathcal{I}_{-1,1,0}-12\mathcal{I}_{1,-1,0}+39\mathcal{I}_{1,1,1} \tag{A.36}$$

$$
\begin{aligned}
\bar{T}_1^{(2)} = {} & 24\mathcal{I}_{-1,-1,1,0}+24\mathcal{I}_{-1,1,-1,0}-78\mathcal{I}_{-1,1,1,1}+24\mathcal{I}_{1,-1,-1,1}+24\mathcal{I}_{1,-1,1,-1} \\
& +16\mathcal{I}_{1,-1,1,0}-126\mathcal{I}_{1,-1,1,1}+\frac{80}{3}\mathcal{I}_{1,0,1,1}+48\mathcal{I}_{1,1,-1,-1}+56\mathcal{I}_{1,1,-1,0} \\
& -198\mathcal{I}_{1,1,-1,1}+\frac{127}{12}\mathcal{I}_{1,1,0,1}-294\mathcal{I}_{1,1,1,-1}-\frac{409}{4}\mathcal{I}_{1,1,1,0}
\end{aligned}
\tag{A.37}
$$

$$
\begin{aligned}
\bar{T}_1^{(3)} = {} & -48\mathcal{I}_{-1,-1,-1,1,0}-48\mathcal{I}_{-1,-1,1,-1,0}+156\mathcal{I}_{-1,-1,1,1,1}-48\mathcal{I}_{-1,1,-1,-1,1} \\
& -48\mathcal{I}_{-1,1,-1,1,-1}-32\mathcal{I}_{-1,1,-1,1,0}+252\mathcal{I}_{-1,1,-1,1,1}-\frac{160}{3}\mathcal{I}_{-1,1,0,1,1} \\
& -96\mathcal{I}_{-1,1,1,-1,-1}-112\mathcal{I}_{-1,1,1,-1,0}+396\mathcal{I}_{-1,1,1,-1,1}-\frac{127}{6}\mathcal{I}_{-1,1,1,0,1} \\
& +588\mathcal{I}_{-1,1,1,1,-1}+\frac{409}{2}\mathcal{I}_{-1,1,1,1,0}-48\mathcal{I}_{1,-1,-1,-1,1}-48\mathcal{I}_{1,-1,-1,1,-1} \\
& -32\mathcal{I}_{1,-1,-1,1,0}-112\mathcal{I}_{1,-1,1,-1,0}+96\mathcal{I}_{1,-1,1,-1,1}-80\mathcal{I}_{1,-1,1,0,-1} \\
& +144\mathcal{I}_{1,-1,1,1,-1}+\frac{5189}{4}\mathcal{I}_{1,-1,1,1,1}+210\mathcal{I}_{1,0,-1,1,1}-80\mathcal{I}_{1,0,1,-1,-1} \\
& +250\mathcal{I}_{1,0,1,-1,1}+\frac{2165}{36}\mathcal{I}_{1,0,1,0,1}+370\mathcal{I}_{1,0,1,1,-1}+\frac{7235}{36}\mathcal{I}_{1,0,1,1,0} \\
& +96\mathcal{I}_{1,1,-1,-1,-1}-316\mathcal{I}_{1,1,-1,-1,1}+\frac{634711\mathcal{I}_{1,1,-1,0,0}}{4161}+\frac{1001731\mathcal{I}_{1,1,-1,0,1}}{8322} \\
& -412\mathcal{I}_{1,1,-1,1,-1}-\frac{203321\mathcal{I}_{1,1,-1,1,0}}{2774}+\frac{6715537\mathcal{I}_{1,1,-1,1,1}}{2774}+\frac{1011}{2}\mathcal{I}_{1,1,0,-1,1} \\
& -\frac{2899054\mathcal{I}_{1,1,0,0,-1}}{12483}+\frac{991354\mathcal{I}_{1,1,0,1,-1}}{1387}-1400\mathcal{I}_{1,1,1,-1,-1}-\frac{332028\mathcal{I}_{1,1,1,-1,0}}{1387} \\
& +\frac{20832225\mathcal{I}_{1,1,1,-1,1}}{5548}+\frac{870038}{657}\mathcal{I}_{1,1,1,0,-1}-\frac{502923119\mathcal{I}_{1,1,1,0,0}}{898776}+\frac{9081125\mathcal{I}_{1,1,1,1,-1}}{1387} \\
& -\frac{7166073467\mathcal{I}_{1,1,1,1,1}}{449388}
\end{aligned}
\tag{A.38}
$$

$$
\begin{aligned}
\bar{T}_1^{(4)} = {} & 96\mathcal{I}_{-1,-1,-1,-1,1,0}+96\mathcal{I}_{-1,-1,-1,1,-1,0}-312\mathcal{I}_{-1,-1,-1,1,1,1}+96\mathcal{I}_{-1,-1,1,-1,-1,1} \\
& +96\mathcal{I}_{-1,-1,1,-1,1,-1}+64\mathcal{I}_{-1,-1,1,-1,1,0}-504\mathcal{I}_{-1,-1,1,-1,1,1}+\frac{320}{3}\mathcal{I}_{-1,-1,1,0,1,1} \\
& +192\mathcal{I}_{-1,-1,1,1,-1,-1}+224\mathcal{I}_{-1,-1,1,1,-1,0}-792\mathcal{I}_{-1,-1,1,1,-1,1}+\frac{127}{3}\mathcal{I}_{-1,-1,1,1,0,1} \\
& -1176\mathcal{I}_{-1,-1,1,1,1,-1}-409\mathcal{I}_{-1,-1,1,1,1,0}+96\mathcal{I}_{-1,1,-1,-1,-1,1}+96\mathcal{I}_{-1,1,-1,-1,1,-1} \\
& +64\mathcal{I}_{-1,1,-1,-1,1,0}+224\mathcal{I}_{-1,1,-1,1,-1,0}-192\mathcal{I}_{-1,1,-1,1,-1,1}+160\mathcal{I}_{-1,1,-1,1,0,-1} \\
& -288\mathcal{I}_{-1,1,-1,1,1,-1}-\frac{5189}{2}\mathcal{I}_{-1,1,-1,1,1,1}-420\mathcal{I}_{-1,1,0,-1,1,1}+160\mathcal{I}_{-1,1,0,1,-1,-1} \\
& -500\mathcal{I}_{-1,1,0,1,-1,1}-\frac{2165}{18}\mathcal{I}_{-1,1,0,1,0,1}-740\mathcal{I}_{-1,1,0,1,1,-1}-\frac{7235}{18}\mathcal{I}_{-1,1,0,1,1,0} \\
& -192\mathcal{I}_{-1,1,1,-1,-1,-1}+632\mathcal{I}_{-1,1,1,-1,-1,1}-\frac{1269422\mathcal{I}_{-1,1,1,-1,0,0}}{4161}-\frac{1001731\mathcal{I}_{-1,1,1,-1,0,1}}{4161} \\
& +824\mathcal{I}_{-1,1,1,-1,1,-1}+\frac{203321\mathcal{I}_{-1,1,1,-1,1,0}}{1387}-\frac{6715537\mathcal{I}_{-1,1,1,-1,1,1}}{1387}-1011\mathcal{I}_{-1,1,1,0,-1,1} \\
& +\frac{5798108\mathcal{I}_{-1,1,1,0,0,-1}}{12483}-\frac{1982708\mathcal{I}_{-1,1,1,0,1,-1}}{1387}+2800\mathcal{I}_{-1,1,1,1,-1,-1}+\frac{664056\mathcal{I}_{-1,1,1,1,-1,0}}{1387} \\
& -\frac{20832225\mathcal{I}_{-1,1,1,1,-1,1}}{2774}-\frac{1740076}{657}\mathcal{I}_{-1,1,1,1,0,-1}+\frac{502923119\mathcal{I}_{-1,1,1,1,0,0}}{449388} \\
& -\frac{18162250\mathcal{I}_{-1,1,1,1,1,-1}}{1387}+\frac{7166073467\mathcal{I}_{-1,1,1,1,1,1}}{224694}+96\mathcal{I}_{1,-1,-1,-1,-1,1} \\
& +96\mathcal{I}_{1,-1,-1,-1,1,-1}+64\mathcal{I}_{1,-1,-1,-1,1,0}+224\mathcal{I}_{1,-1,-1,1,-1,0}-192\mathcal{I}_{1,-1,-1,1,-1,1} \\
& +160\mathcal{I}_{1,-1,-1,1,0,-1}-288\mathcal{I}_{1,-1,-1,1,1,-1}-\frac{5189}{2}\mathcal{I}_{1,-1,-1,1,1,1}+536\mathcal{I}_{1,-1,1,-1,-1,1} \\
& +728\mathcal{I}_{1,-1,1,-1,1,-1}+\frac{1162}{3}\mathcal{I}_{1,-1,1,-1,1,0}-5900\mathcal{I}_{1,-1,1,-1,1,1}-320\mathcal{I}_{1,-1,1,0,-1,-1}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{247571\mathcal{I}_{1,-1,1,0,1,-1}}{1387}+\frac{6935939\mathcal{I}_{1,-1,1,0,1,1}}{12483}+2096\mathcal{I}_{1,-1,1,1,-1,-1}\\
&+\frac{3995630\mathcal{I}_{1,-1,1,1,-1,0}}{4161}-\frac{25993133\mathcal{I}_{1,-1,1,1,-1,1}}{2774}+\frac{2339932507\mathcal{I}_{1,-1,1,1,0,1}}{898776}\\
&-\frac{20346339\mathcal{I}_{1,-1,1,1,1,-1}}{1387}+\frac{1987391897\mathcal{I}_{1,-1,1,1,1,0}}{299592}-160\mathcal{I}_{1,0,1,-1,-1,-1}\\
&-\frac{190762625\mathcal{I}_{1,0,1,-1,1,0}}{137313}+\frac{172992580\mathcal{I}_{1,0,1,-1,1,1}}{45771}-\frac{585}{2}\mathcal{I}_{1,0,1,0,-1,1}-\frac{585}{2}\mathcal{I}_{1,0,1,0,1,-1}\\
&+\frac{1401897460\mathcal{I}_{1,0,1,0,1,0}}{1235817}+\frac{1280}{3}\mathcal{I}_{1,0,1,1,-1,-1}-\frac{131698420\mathcal{I}_{1,0,1,1,-1,0}}{137313}\\
&+\frac{3771385}{803}\mathcal{I}_{1,0,1,1,-1,1}-\frac{135186640\mathcal{I}_{1,0,1,1,0,-1}}{12483}+\frac{94194535\mathcal{I}_{1,0,1,1,1,-1}}{45771}\\
&-\frac{224239650581\mathcal{I}_{1,0,1,1,1,1}}{7414902}+192\mathcal{I}_{1,1,-1,-1,-1,-1}+440\mathcal{I}_{1,1,-1,-1,-1,1}\\
&+632\mathcal{I}_{1,1,-1,-1,1,-1}+\frac{1162}{3}\mathcal{I}_{1,1,-1,-1,1,0}-\frac{19523}{3}\mathcal{I}_{1,1,-1,-1,1,1}\\
&+\frac{85309\mathcal{I}_{1,1,-1,0,1,-1}}{1387}+1520\mathcal{I}_{1,1,-1,1,-1,-1}+1201\mathcal{I}_{1,1,-1,1,-1,0}\\
&-\frac{65065}{6}\mathcal{I}_{1,1,-1,1,-1,1}+\frac{1286089}{657}\mathcal{I}_{1,1,-1,1,0,-1}+\frac{294745883\mathcal{I}_{1,1,-1,1,0,1}}{112347}\\
&-\frac{667817712\mathcal{I}_{1,1,-1,1,1,-1}}{4161}+\frac{7672706315\mathcal{I}_{1,1,-1,1,1,0}}{898776}-\frac{1500994997\mathcal{I}_{1,1,-1,1,1,1}}{112347}\\
&-331\mathcal{I}_{1,1,0,-1,-1,1}-\frac{373788\mathcal{I}_{1,1,0,-1,1,-1}}{1387}-\frac{1040075156\mathcal{I}_{1,1,0,-1,1,0}}{411939}\\
&+\frac{710360086250224455\mathcal{I}_{1,1,0,1,-1,-1}}{31412461173192}-\frac{408164510625851\mathcal{I}_{1,1,0,1,-1,0}}{387808162632}\\
&+\frac{1122406123131169253\mathcal{I}_{1,1,0,1,-1,1}}{62824922346384}-\frac{16710798348435 31\mathcal{I}_{1,1,0,1,1,-1}}{296343973332}\\
&-208\mathcal{I}_{1,1,1,-1,-1,-1}+\frac{4630766069502545\mathcal{I}_{1,1,1,-1,-1,0}}{1903785525648}-\frac{23840913916053689\mathcal{I}_{1,1,1,-1,-1,1}}{1903785525648}\\
&+\frac{2367362875520280065\mathcal{I}_{1,1,1,-1,0,-1}}{62824922346384}+\frac{652369324501278 65\mathcal{I}_{1,1,1,-1,0,0}}{17134069730832}\\
&-\frac{3247350788220982 09\mathcal{I}_{1,1,1,-1,0,1}}{188474767039152}-\frac{33937640451327857\mathcal{I}_{1,1,1,-1,1,-1}}{1903785525648}\\
&+\frac{2178901036887715 79\mathcal{I}_{1,1,1,-1,1,0}}{62824922346384}-\frac{980989077384117 07\mathcal{I}_{1,1,1,-1,1,1}}{62824922346384}\\
&+\frac{600532814592842 15\mathcal{I}_{1,1,1,0,-1,-1}}{10470820391064}-\frac{142763098650694 91\mathcal{I}_{1,1,1,0,-1,0}}{5235410195532}\\
&-\frac{176082776697502 97\mathcal{I}_{1,1,1,0,-1,1}}{2326848975792}+\frac{113893834864667471\mathcal{I}_{1,1,1,0,0,-1}}{205608836769984}\\
&+\frac{25788642067550620923 40321\mathcal{I}_{1,1,1,0,0,0}}{97922442164725499904}-\frac{40281579880842 97\mathcal{I}_{1,1,1,0,1,-1}}{183698603352}\\
&+\frac{3423106133760075812558 45\mathcal{I}_{1,1,1,0,1,1}}{5153812745511868416}-\frac{58049937665304 9347\mathcal{I}_{1,1,1,1,-1,-1}}{31412461173192}\\
&-\frac{47970936019094809\mathcal{I}_{1,1,1,1,-1,0}}{3306574860336}+\frac{17536788230298694 0\mathcal{I}_{1,1,1,1,-1,1}}{3926557646649}\\
&-\frac{48827026671130603771\mathcal{I}_{1,1,1,1,0,-1}}{1130848602234912}+\frac{293013290029723347718853 11\mathcal{I}_{1,1,1,1,0,1}}{195844884329450999808}\\
&+\frac{4274072449929751295\mathcal{I}_{1,1,1,1,1,-1}}{31412461173192}+\frac{9649675317039589744890323\mathcal{I}_{1,1,1,1,1,0}}{65281628109816999936}
\end{aligned}
\tag{A.39}
$$

References

[1] K.G. Chetyrkin and F.V. Tkachov, *Integration by Parts: The Algorithm to Calculate β-functions in 4 Loops*, *Nucl. Phys.* **B 192** (1981) 159 [INSPIRE].

[2] V.A. Smirnov, *Asymptotic expansions in limits of large momenta and masses*, *Commun. Math. Phys.* **134** (1990) 109 [INSPIRE].

[3] V.A. Smirnov, *Asymptotic expansions in momenta and masses and calculation of Feynman diagrams*, *Mod. Phys. Lett.* **A 10** (1995) 1485 [hep-th/9412063] [INSPIRE].

[4] J.A.M. Vermaseren, *The Symbolic manipulation program FORM*, KEK-TH-326, KEK-PREPRINT-92-1, (1992).

[5] M. Steinhauser, *MATAD: A program package for the computation of MAssive TADpoles*, *Comput. Phys. Commun.* **134** (2001) 335 [hep-ph/0009029] [INSPIRE].

[6] L.V. Avdeev, *Recurrence relations for three loop prototypes of bubble diagrams with a mass*, *Comput. Phys. Commun.* **98** (1996) 15 [hep-ph/9512442] [INSPIRE].

[7] L. Avdeev, J. Fleischer, S. Mikhailov and O. Tarasov, $0(\alpha\alpha_s^2)$ *correction to the electroweak ρ parameter*, *Phys. Lett.* **B 336** (1994) 560 [*Erratum ibid.* **B 349** (1995) 597] [hep-ph/9406363] [INSPIRE].

[8] K.G. Chetyrkin, J.H. Kuhn and M. Steinhauser, *Corrections of order $\mathcal{O}(G_F M_t^2 \alpha_s^2)$ to the ρ parameter*, *Phys. Lett.* **B 351** (1995) 331 [hep-ph/9502291] [INSPIRE].

[9] M. Faisst, J.H. Kuhn, T. Seidensticker and O. Veretin, *Three loop top quark contributions to the rho parameter*, *Nucl. Phys.* **B 665** (2003) 649 [hep-ph/0302275] [INSPIRE].

[10] K.G. Chetyrkin, J.H. Kuhn and M. Steinhauser, *Three loop polarization function and $O(\alpha S^2)$ corrections to the production of heavy quarks*, *Nucl. Phys.* **B 482** (1996) 213 [hep-ph/9606230] [INSPIRE].

[11] K.G. Chetyrkin, M. Misiak and M. Münz, *β-functions and anomalous dimensions up to three loops*, *Nucl. Phys.* **B 518** (1998) 473 [hep-ph/9711266] [INSPIRE].

[12] A.V. Bednyakov, A.F. Pikelner and V.N. Velizhanin, *Higgs self-coupling β-function in the Standard Model at three loops*, *Nucl. Phys.* **B 875** (2013) 552 [arXiv:1303.4364] [INSPIRE].

[13] K.G. Chetyrkin and M.F. Zoller, *β-function for the Higgs self-interaction in the Standard Model at three-loop level*, *JHEP* **04** (2013) 091 [*Erratum ibid.* **1309** (2013) 155] [arXiv:1303.2890] [INSPIRE].

[14] M. Czakon, *The four-loop QCD β-function and anomalous dimensions*, *Nucl. Phys.* **B 710** (2005) 485 [hep-ph/0411261] [INSPIRE].

[15] T. Luthe, A. Maier, P. Marquard and Y. Schröder, *Complete renormalization of QCD at five loops*, *JHEP* **03** (2017) 020 [arXiv:1701.07068] [INSPIRE].

[16] D.J. Broadhurst, *Massive three - loop Feynman diagrams reducible to SC* primitives of algebras of the sixth root of unity*, *Eur. Phys. J.* **C 8** (1999) 311 [hep-th/9803091] [INSPIRE].

[17] E. Kummer, *Ueber die transcendenten, welche aus wiederholten integrationen rationaler formeln entstehen*, *Journal für die reine und angewandte Mathematik* **21** (1840) 74.

[18] J. Lappo-Danilevsky, *Mémoire sur la théorie des systémes des Ãl'quation difféertielles linéaries*, *Chelsea reprint* (1953).

[19] A.B. Goncharov, *Multiple polylogarithms, cyclotomy and modular complexes*, Math. Res. Lett. **5** (1998) 497 [arXiv:1105.2076] [INSPIRE].

[20] M. Yu. Kalmykov and O. Veretin, *Single scale diagrams and multiple binomial sums*, Phys. Lett. **B 483** (2000) 315 [hep-th/0004010] [INSPIRE].

[21] A.I. Davydychev and M. Yu. Kalmykov, *Massive Feynman diagrams and inverse binomial sums*, Nucl. Phys. **B 699** (2004) 3 [hep-th/0303162] [INSPIRE].

[22] J. Fleischer, A.V. Kotikov and O.L. Veretin, *Analytic two loop results for selfenergy type and vertex type diagrams with one nonzero mass*, Nucl. Phys. **B 547** (1999) 343 [hep-ph/9808242] [INSPIRE].

[23] S. Weinzierl, *Expansion around half integer values, binomial sums and inverse binomial sums*, J. Math. Phys. **45** (2004) 2656 [hep-ph/0402131] [INSPIRE].

[24] M. Yu. Kalmykov and B.A. Kniehl, *'Sixth root of unity' and Feynman diagrams: Hypergeometric function approach point of view*, Nucl. Phys. Proc. Suppl. **205-206** (2010) 129 [arXiv:1007.2373] [INSPIRE].

[25] J. Ablinger, J. Blümlein, C.G. Raab and C. Schneider, *Iterated Binomial Sums and their Associated Iterated Integrals*, J. Math. Phys. **55** (2014) 112301 [arXiv:1407.1822] [INSPIRE].

[26] J. Ablinger, J. Blümlein and C. Schneider, *Harmonic Sums and Polylogarithms Generated by Cyclotomic Polynomials*, J. Math. Phys. **52** (2011) 102301 [arXiv:1105.6063] [INSPIRE].

[27] J.M. Henn, A.V. Smirnov and V.A. Smirnov, *Evaluating Multiple Polylogarithm Values at Sixth Roots of Unity up to Weight Six*, Nucl. Phys. **B 919** (2017) 315 [arXiv:1512.08389] [INSPIRE].

[28] E. Remiddi and J.A.M. Vermaseren, *Harmonic polylogarithms*, Int. J. Mod. Phys. **A 15** (2000) 725 [hep-ph/9905237] [INSPIRE].

[29] J.M. Borwein, D.M. Bradley, D.J. Broadhurst and P. Lisonek, *Combinatorial aspects of multiple zeta values*, math/9812020.

[30] H.R.P. Ferguson, D.H. Bailey and S. Arno, *Analysis of PSLQ, an integer relation finding algorithm*, Math. Comput. **68** (1999) 351.

[31] R.N. Lee, *DRA method: Powerful tool for the calculation of the loop integrals*, J. Phys. Conf. Ser. **368** (2012) 012050 [arXiv:1203.4868] [INSPIRE].

[32] R.N. Lee and K.T. Mingulov, *Introducing SummerTime: a package for high-precision computation of sums appearing in DRA method*, Comput. Phys. Commun. **203** (2016) 255 [arXiv:1507.04256] [INSPIRE].

[33] A.I. Davydychev and J.B. Tausk, *Two loop selfenergy diagrams with different masses and the momentum expansion*, Nucl. Phys. **B 397** (1993) 123 [INSPIRE].

[34] A.I. Davydychev and M. Yu. Kalmykov, *Some remarks on the ϵ-expansion of dimensionally regulated Feynman diagrams*, Nucl. Phys. Proc. Suppl. **89** (2000) 283 [hep-th/0005287] [INSPIRE].

[35] R.N. Lee and I.S. Terekhov, *Application of the DRA method to the calculation of the four-loop QED-type tadpoles*, JHEP **01** (2011) 068 [arXiv:1010.6117] [INSPIRE].

[36] J. Fleischer and M. Yu. Kalmykov, *Single mass scale diagrams: Construction of a basis for the ϵ-expansion*, Phys. Lett. **B 470** (1999) 168 [hep-ph/9910223] [INSPIRE].

JHEP08(2017)024

[37] A.I. Davydychev and M. Yu. Kalmykov, *New results for the ϵ-expansion of certain one, two and three loop Feynman diagrams*, Nucl. Phys. **B 605** (2001) 266 [hep-th/0012189] [INSPIRE].

[38] M. Yu. Kalmykov, *About higher order ϵ-expansion of some massive two- and three-loop master-integrals*, Nucl. Phys. **B 718** (2005) 276 [hep-ph/0503070] [INSPIRE].

[39] J.M. Henn, A.V. Smirnov and V.A. Smirnov, *Analytic results for planar three-loop integrals for massive form factors*, JHEP **12** (2016) 144 [arXiv:1611.06523] [INSPIRE].