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2D CFT blocks for the 4D class $\mathcal{S}_k$ theories

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Abstract: This is the first in a series of papers on the search for the 2D CFT description of a large class of 4D $\mathcal{N}=1$ gauge theories. Here, we identify the 2D CFT symmetry algebra and its representations, namely the conformal blocks of the Virasoro/W-algebra, that underlie the 2D theory and reproduce the Seiberg-Witten curves of the $\mathcal{N}=1$ gauge theories. We find that the blocks corresponding to the SU(N) $\mathcal{S}_k$ gauge theories involve fields in certain non-unitary representations of the $\mathbf{W}_{kN}$ algebra. These conformal blocks give a prediction for the instanton partition functions of the 4D $\mathcal{N}=1$ SCFTs of class $\mathcal{S}_k$.

Keywords: Conformal and W Symmetry, Conformal Field Theory, Duality in Gauge Field Theories, Supersymmetric Gauge Theory

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Contents

1 Introduction

The study of supersymmetric gauge theories was revolutionized by Seiberg and collaborators in the nineties through the use of holomorphicity, symmetries as well as asymptotics (weak coupling behavior) [1]. Building up on these developments, Seiberg and Witten realized [2, 3] that by adding electromagnetic duality (S-duality) to the game, one can obtain the low energy BPS spectrum of $\mathcal{N}=2$ gauge theories by deriving a holomorphic algebraic curve, the so-called Seiberg-Witten (SW) curve, that incorporates all the symmetries (including S-duality) and weak coupling behavior. Soon after, Intriligator and Seiberg [4] obtained the first examples of algebraic curves that compute the low energy coupling constants in the abelian Coulomb phase for $\mathcal{N}=1$ theories.

In the last decade, the most modern developments in the field are based on the deep connection of S-duality in 4D gauge theory with 2D modular invariance. In the prototypical example of the maximally supersymmetric $\mathcal{N}=4$ super Yang-Mills (SYM), the Montonen-Olive $\mathrm{SL}(2,\mathbb{Z})$ duality can be geometrically realized as the modular group of a torus by compactifying the 6D $(2,0)$ SCFT on a torus [5]. Similarly, a large class of 4D $\mathcal{N}=2$ superconformal field theories (SCFTs)s, referred to as class $\mathcal{S}$ [6, 7], can be obtained via compactification of (a twisted version of) the 6D $(2,0)$ SCFT on Riemann surfaces of genus g and with n punctures. The parameter space of the exactly marginal gauge couplings is identified with the complex structure moduli space of the Riemann surface. What is more, the partition function of the 4D $\mathcal{N}=2$ theories on a four sphere[1] [9] are equal to correlation functions of the 2D Liouville/Toda CFT on that Riemann surface [10, 11], which is the core of the celebrated AGT(W) correspondence. The 4D/2D interplay was originally discovered for the $\mathcal{N}=2$ class $\mathcal{S}$ theories in [6] by studying the SW curves and realizing that they arise from the compactification of M5-branes on Riemann surfaces decorated with punctures. See [12, 13] for recent reviews.

Motivated by the above developments for $\mathcal{N}=2$ theories, we wish to explore how much mileage we can get for theories with only $\mathcal{N}=1$ supersymmetry. We begin by recalling that it is not uncommon to find exactly marginal couplings also in $\mathcal{N}=1$ supersymmetric theories [14, 15], with the AdS/CFT correspondence offering a natural route to several examples of $\mathcal{N}=1$ orbifold daughters of $\mathcal{N}=4$ SYM [16, 17]. A very large class of 4D $\mathcal{N}=1$ SCFTs, naturally called $\mathcal{S}_\Gamma$ [18, 19], arise from M5-branes probing the $\mathbb{C}^2/\Gamma$ ADE singularity. Their study was originated in [20], with the $\mathcal{S}_k$ class arising after compactification of $\mathbb{Z}_k$ orbifolds of the (2,0) theory, see also [21, 22] and [18, 23–27]. The SW curves for the class $\mathcal{S}_k$ theories were derived and studied in [28], using Witten's M-theory approach [29].

For $\mathcal{N}=2$ theories, the SW curves completely solve the IR theory. The $\mathcal{N}=2$ supersymmetry and more specifically the $\mathrm{SU}(2)_R$ relates the holomorphic superpotential to the non-holomorphic (in $\mathcal{N}=1$ superspace) Kähler part and thus we can obtain the full prepotential. For theories with only $\mathcal{N}=1$ supersymmetry, we can only hope to fix the holomorphic superpotential part. However, there are $\mathcal{N}=1$ examples for which also the Kähler part can be fixed, see for example [30, 31]. From a field theory point of view this should be a consequence of an extra global symmetry. For the theories in class $\mathcal{S}_\Gamma$, we expect more, than for generic $\mathcal{N}=1$theories, due to their rich global symmetries inherited from the orbifold construction.[2]

[1] Technically [8], on an ellipsoid with deformation parameter $b^2=\frac{\epsilon_1}{\epsilon_2}$, where the ϵ_i are the Ω-background deformation parameters entering the Nekrasov partition functions.

[2] As explained in [20, 28], the $\mathrm{SU}(2)_R$ is broken by the orbifold, but a diagonal $\mathrm{U}(1)_R$ remains. Moreover, instead of the $\mathrm{U}(1)_r$ of $\mathcal{N}=2$, a global symmetry $\mathrm{U}(1)\times\mathrm{U}(1)^{k-1}\times\mathrm{U}(1)^{k-1}$ remains, heavily constraining the theory.

The purpose of this article is to begin the search for the 2D conformal field theories (CFT), whose correlation functions reproduce the partition functions of the 4D $\mathcal{N}=1$ SCFTs of class $\mathcal{S}_k$ and in general of class $\mathcal{S}_\Gamma$. In principle, there is no reason to expect that such a 4D/2D relation exists for $\mathcal{N}=1$ theories. We adopt here a conservative approach - if such a relation exists, then the SW curve of the $\mathcal{S}_k$ theories knows about it and will illuminate the path leading to the symmetry algebra/representations underlying the 2D CFT. Following the $\mathcal{N}=2$ class $\mathcal{S}$ paradigm [10, 32, 33], we first compare the meromorphic differentials ϕ_ℓ of the SW curves derived in [28] with the weighted current correlation functions[3] $\langle\langle\, J_\ell(t)\,\rangle\rangle$ computed on the CFT side

$$\lim_{\epsilon_i\to 0}\langle\langle\, J_\ell(t)\,\rangle\rangle = \phi_\ell(t) \tag{1.1}$$

with the ϵ_i being the Ω-background deformation parameters. Since the CFT primary fields enter in the computation of $\langle\langle\, J_\ell(t)\,\rangle\rangle$, the above identification dictates to us their quantum numbers. In particular, we can learn the form of the CFT representations that the primary fields live in.

We discover that the spectral curves of the 4D SU(N) gauge theories of class $\mathcal{S}_k$ can be reproduced from the 2D CFT weighted current correlation functions of the $\mathbf{W}_{Nk}$ algebra with *non-unitary primary fields*. This is based on the observation that the SW curves of SU(N) class $\mathcal{S}_k$ theories can be obtained from the $\mathcal{N}=2$ SU(Nk) curves by tuning the mass/Coulomb branch parameters appropriately. On the CFT side, one then simply computes the conformal/W-blocks for $\mathbf{W}_{Nk}$ with $Nk=2,3,4,\ldots$ and sets the parameters to appropriate values. In addition, we use the known AGT correspondence for the $\mathcal{N}=2$ SU(Nk) theories to derive a conjecture for the $\mathcal{N}=1$ class $\mathcal{S}_k$ instanton partition functions.

This article is structured as follows. We begin in section 2 by reviewing the construction of the SW curves for the class $\mathcal{S}_k$ theories. We introduce some of their properties and discuss the weak coupling limit and the Gaiotto curve. The next section 3 is concerned with recapitulating some aspects of the AGT correspondence that are essential for our work such as the identifications of the parameters on both sides of the duality and the relationships between the 2D CFT blocks and the 4D instanton partition functions. Since this is a review section, the readers familiar with the AGT correspondence can move directly to the next section 4 in which we present our main results concerning the structures of the CFT representations, the comparisons with the $\mathcal{S}_k$ SW curves and the investigation of the (orbifold) Nekrasov instanton partition functions. We conclude in section 5 where we also overview some potential directions of future research that our article suggests. Most technical computations as well as bulky formulas are stored in the appendices.

[3] We define the $\langle\langle\, J_\ell(t)\,\rangle\rangle$ in section 3.4. For now, it suffices to point out that for the simplest case of three fields they can be computed as a ratio of correlation function

$$\langle\langle\, J_\ell(t)\,\rangle\rangle_3 = \frac{\langle\, J_\ell(t)\mathsf{V}_1(x_1)\mathsf{V}_2(x_2)\mathsf{V}_3(x_3)\,\rangle}{\langle\,\mathsf{V}_1(x_1)\mathsf{V}_2(x_2)\mathsf{V}_3(x_3)\,\rangle},$$

with the V_i being primary fields.

	x^0	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	(x^{10})
M NS5 branes	—	—	—	—	—	—					
N D4-branes	—	—	—	—			—				—
A_{k-1} orbifold					—	—		—	—		

Figure 1. Type IIA brane configuration for the 4D $\mathcal{N} = 1$ theories of class $\mathcal{S}_k$. The A_{k-1} orbifold acts on the 45 and 78 coordinates.

2 The curves

The starting point of our work is the SW curves. By comparing them to the 2D CFT 3 and 4-point blocks, we will discover the algebra and the representations that underly the 2D theory we are looking for. In this section we present the SW curves and provide a short review of their derivation as well as of the important information they contain.

Review of the type IIA/M-theory construction. The class $\mathcal{S}_k$ SW curves (with $k = 1, 2, \ldots$) were derived in [28] following Witten's [29] M-theory construction in which the implementation of the orbifold is very simple. The main points of it we outline here. The SW curves were originally introduced as auxiliary algebraic curves [2, 3]. Using type IIA string theory, $\mathcal{N} = 2$ gauge theories can be realized as world volume theories on D4-branes, which are suspended between NS5-branes. Uplifting this brane setup to M-theory, all the branes can be seen as one single M5-brane with a non-trivial topology. The geometry of this M5-brane is encoded in the SW curve. Therefore, the SW curve can also be derived by studying the minimal surface of the M5-brane [29].

The theories in class $\mathcal{S}_k$ can be realized through the type IIA string theory brane setup of table 1, which was originally considered in [34, 35]. For $k = 1$ there is no orbifold and one obtains the $\mathcal{N} = 2$ theories of class $\mathcal{S}$ [6]. The $\mathrm{SU}(2)_R$ R-symmetry of the $\mathcal{N} = 2$ theories corresponds to the rotation symmetry of the coordinates x^7, x^8 and x^9 which is broken by the orbifold to the $\mathrm{U}(1)_R$ symmetry of x^7, x^8 rotations. The rotation on the x^4, x^5 plane corresponds to the $\mathrm{U}(1)_r$ symmetry of the $\mathcal{N} = 2$ theories and is also lost [6]. The SW curves are derived by uplifting IIA string theory to M-theory and they are functions of the holomorphic coordinates

$$v \equiv x^4 + ix^5\,, \qquad s \equiv x^6 + ix^{10} \qquad \text{and} \qquad t \equiv e^{-\frac{s}{R_{10}}} \tag{2.1}$$

where R_{10} is the M-theory circle. We follow the conventions of [36]. The orbifold action is imposed via the identification

$$v \;\sim\; e^{\frac{2\pi i}{k}} v\,. \tag{2.2}$$

The mass parameters $m_{L,\,i}$ and $m_{R,\,i}$ are given by the asymptotic position of the M5 branes as $t \to 0$ and $t \to \infty$, while the coupling constant q enters the setup via the asymptotic position of the M5 branes for $v \to \infty$, see figure 2 for an illustration.

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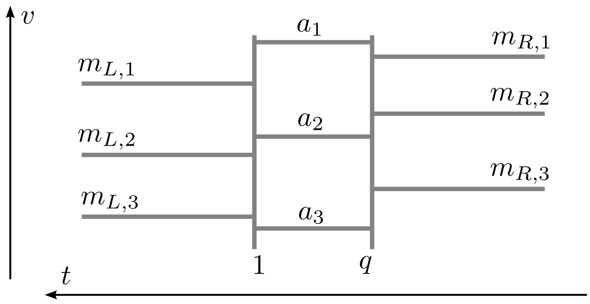


Figure 2. This figure illustrates the position of the branes (horizontal D4s and vertical NS5s) for the case of the $\mathcal{N}=2$ SU(3) gauge theory. In the $\mathcal{N}=1$ case, one needs to introduce an orbifold and image branes as reviewed in [28]. From the equation for the curve (2.3), we see that for $t \to 0/\infty$ the solutions of the curve are $v = m_{L,i}/m_{R,i}$, while for $v \to \infty$ the solutions are $t = 1, q$.

The SCQCD curves. The spectral curve that describes the Coulomb branch of the $\mathbb{Z}_k$ orbifold daughter of $\mathcal{N}=2$ SU(N) SCQCD (SCQCD$_k$) is given by the equation

$$t^2 \prod_{i=1}^{N} \left(v^k - m_{L,i}^k\right) + t\left(-(1+q)v^{Nk} + \sum_{l=1}^{N} u_{lk} v^{(N-l)k}\right) + q \prod_{i=1}^{N}\left(v^k - m_{R,i}^k\right) = 0 \,. \quad (2.3)$$

It is sometimes convenient to group the masses as $m_i = m_{L,i}$ and $m_{N+i} = m_{R,i}$ for $i = 1, \ldots N$. We can rescale the variable v as $v = xt$ and normalize the coefficient of the highest power in x to one.[4] Thus, we can write the equation for the curve as

$$\sum_{\ell=0}^{N} \phi_{k\ell}^{(4)}(t) x^{k(N-\ell)} = 0 \,, \quad (2.4)$$

where the coefficients are given by $\phi_0^{(4)}(t) = 1$ and

$$\phi_{k\ell}^{(4)}(t) = \frac{(-1)^\ell\, \mathfrak{c}_L^{(\ell,k)}\, t^2 + u_{k\ell} t + (-1)^\ell\, \mathfrak{c}_R^{(\ell,k)}\, q}{t^{k\ell}(t-1)(t-q)} \qquad \text{for } \ell = 1, \ldots, N \,. \quad (2.5)$$

In the above, we have used the formula $\prod_{i=1}^{N}(v^k - m_i^k) = \sum_{s=0}^{N} (-1)^s\, \mathfrak{c}^{(s,k)}\, v^{k(N-s)}$ with the Casimirs (let use set for simplicity $\mathfrak{c}^{(s)} \equiv \mathfrak{c}^{(s,1)}$) defined as:

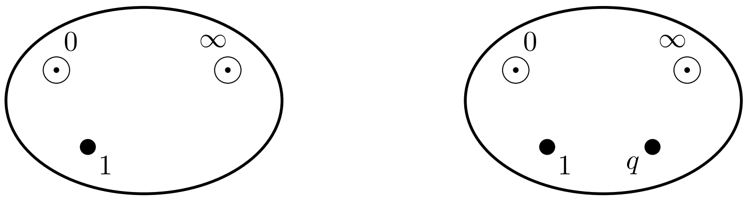
$$\mathfrak{c}^{(s,k)} = \sum_{i_1 < \cdots < i_s = 1}^{N} m_{i_1}^k \cdots m_{i_s}^k \,, \qquad \mathfrak{c}^{(0,k)} = 1 \,. \quad (2.6)$$

For generic values of the masses, the Casimirs $\{\mathfrak{c}^{(s,k)}\}_{\ell=1}^{N}$ are algebraically independent of each other.

Let us now make two remarks.

- One can perform an $SL(2,\mathbb{Z})$ transformation $t \to \frac{\mathsf{a}z+\mathsf{b}}{\mathsf{c}z+\mathsf{d}}$, $x \to (\mathsf{c}z+\mathsf{d})^2 x$ on the curve (2.3) and set $z_1 = -\frac{\mathsf{d}}{\mathsf{c}}$, $z_2 = -\frac{\mathsf{b}-\mathsf{d}}{\mathsf{a}-\mathsf{c}}$, $z_3 = -\frac{\mathsf{b}-\mathsf{d}q}{\mathsf{a}-\mathsf{c}q}$ and $z_4 = -\frac{\mathsf{b}}{\mathsf{a}}$. This sends the singularities at ∞, 1, q and 0 to the generic points z_1, z_2, z_3 and z_4 respectively.

[4] The Seiberg-Witten differential in these coordinates is given by $\lambda_{SW} = x dt$.



Free Trinion $\mathcal{C}_{0,3}^{(k)}$ $\qquad$ $\mathcal{N}=1$ SU(N) SCQCD$_k$

Figure 3. The UV curves of the trinion and of the SCQCD$_k$ theories. They are 3, respectively 4-punctured spheres. The full punctures are depicted by $\odot$ and placed at $t=0$ and $t=\infty$, while the simple punctures $\bullet$ are at $t=1$ and at $t=q$.

- The Coulomb moduli u_{kl} are implicitly functions of the coupling q, of the masses and of the brane positions a_i, see figure 2. They can be computed from the SW curve by evaluating certain period integrals, as we review for the $\mathcal{N}=2$ SU(2) case in appendix F.

The free trinion curves. As explained in [28], the free $\mathcal{C}_{0,3}^{(k)}$ trinion curve can be obtained from the SCQCD$_k$ one by going to the weak coupling regime $q\to 0$ and identifying the Coulomb parameters u_ℓ appropriately with the masses. The resulting equation for the curve reads

$$t\prod_{i=1}^{N}\left(v^k-m_{L,\,i}^k\right)-\prod_{i=1}^{N}\left(v^k-m_{R,\,i}^k\right)=0\,. \tag{2.7}$$

As before, we can rescale $v=xt$ and write the curve as $\sum_{\ell=1}^N\phi_{k\ell}^{(3)}(t)x^{k(N-\ell)}=0$, with the curve coefficients (see (2.6) for the definition of the Casimirs) $\phi_0^{(3)}=1$ and

$$\phi_{k\ell}^{(3)}(t)=(-1)^\ell\frac{\mathfrak{c}_L^{(\ell,k)}\,t-\mathfrak{c}_R^{(\ell,k)}}{t^{k\ell}(t-1)}\qquad\text{for }\ell=1,\dots,N\,. \tag{2.8}$$

The above coefficients can be directly obtained by taking the limit $q\to 0$ in (2.5) and setting

$$u_{k\ell}(q=0)\longrightarrow(-1)^{\ell+1}\,\mathfrak{c}_R^{(\ell,k)}\,. \tag{2.9}$$

The UV curves corresponding to the free trinion and to the SCQCD theories are depicted in figure 3. They are three and respectively four punctured[5] spheres with the punctures at $t=0$ and $t=\infty$ being full punctures $\odot$, while those at $t=1$ and $t=q$ are simple punctures $\bullet$, see [28].

Gaiotto shifts in x for $k=1$. Due to the orbifold relation (2.2), we are allowed to shift the variable x for $k=1$, **but not** for $k>1$. This shift is the consequence of the additional U(1) degrees of freedom that are present for $k=1$ but, as we shall see more in detail later, disappear for $k>1$. For $k=1$, if we go from an equation $\sum_{i=0}^N x^i\phi_i$ to

[5]The UV curves are characterized by the meromorphic differentials $\phi_s^{(n)}$ that have only poles and no branch cuts. The additional punctures $\bigstar$ discussed in [28] will not be relevant for our purposes here.

$\sum_{i=0}^{N} x^i \phi'_i$ by making the tranformation $x \to x - \kappa\phi_1$, then we find

$$\phi'_\ell = \sum_{j=N-\ell}^{N} \binom{j}{N-\ell} \phi_{N-j} (-\kappa\phi_1)^{j+i-N} = \sum_{j=0}^{\ell} \binom{N-j}{N-\ell} \phi_j (-\kappa\phi_1)^{\ell-j} \,. \tag{2.10}$$

We remind that $\phi_0 = 1$ before and after the transformation. It is clear that the shift leaves the 2-form $\Omega_2 = d\lambda_{SW} = dx \wedge dt$ unchanged, however the structure of the poles of λ_{SW} on the various sheets of the curve does change, see [28]. If we put the shift parameter κ equal to $\frac{1}{N}$, then the coefficient ϕ'_1 vanishes - the resulting curve is known as the Gaiotto curve. Let us denote the curve coefficients for the Gaiotto curve by $\tilde{\phi}_\ell^{(n)}$:

$$\tilde{\phi}_\ell^{(n)} = \sum_{j=0}^{\ell} \binom{N-j}{N-\ell} (-1)^{\ell-j} \left(\frac{\phi_1^{(n)}}{N} \right)^{\ell-j} \phi_j^{(n)} \implies \tilde{\phi}_1^{(n)} = 0 \,. \tag{2.11}$$

As we shall review later, their expansion around the poles in t gives the charges of the $\mathbf{W}_N$ algebra. One easily computes

$$\begin{aligned} \tilde{\phi}_\ell^{(3)}(t) &= \frac{-\binom{N}{\ell} \frac{\ell-1}{N^\ell} (M_L - M_R)^\ell}{(t-1)^\ell} + \cdots \,, \\ \tilde{\phi}_\ell^{(4)}(t) &= \frac{-\binom{N}{\ell} \frac{\ell-1}{N^\ell} M_L^\ell}{(t-1)^\ell} + \cdots \,, \qquad \tilde{\phi}_\ell^{(4)}(t) = \frac{-\binom{N}{\ell} \frac{\ell-1}{N^\ell} (-M_R)^\ell}{(t-q)^\ell} + \cdots \,. \end{aligned} \tag{2.12}$$

In the above, we have introduced the left and right center of masses

$$M_L = \sum_{i=1}^{N} m_{L,i} = \mathfrak{c}_L^{(1)} \,, \qquad M_R = \sum_{i=1}^{N} m_{R,i} = \mathfrak{c}_R^{(1)} \,. \tag{2.13}$$

It is useful to furthermore introduce the SU(N) masses

$$\tilde{m}_{L,i} = m_{L,i} - \frac{M_L}{N} \,, \qquad \tilde{m}_{R,i} = m_{R,i} - \frac{M_R}{N} \,, \tag{2.14}$$

which obey $\sum_{i=1}^{N} \tilde{m}_i = 0$. The corresponding Casimirs with the replacement $m \to \tilde{m}$ are denoted by $\tilde{\mathfrak{c}}^{(\ell)}$. Expanding the curve coefficients around $t = 0$ and $t = \infty$ and using (A.2), we find that

$$\tilde{\phi}_\ell^{(n)}(t) = \frac{(-1)^\ell \tilde{\mathfrak{c}}_R^{(\ell)}}{t^\ell} + \cdots \,, \tag{2.15}$$

for $n = 3, 4$. Performing the SL$(2,\mathbb{Z})$ transformation is $t \to -\frac{1}{t}$, we can compute the expansion around $t = \infty$ and we get for $\tilde{\phi}_\ell^{(n)}$ a pole of order ℓ with coefficient $\tilde{\mathfrak{c}}_L^{(\ell)}$.

3 Review of some aspects of the AGT correspondence

In this section, we wish to review the essentials of the AGT correspondence and especially of the elements that we shall need in the rest of the article. The essential elements are summarized in table 1.

Gauge theory	Toda CFT	Relations
Ω deformation parameters ϵ_1, ϵ_2	Coupling b	$b = \sqrt{\frac{\epsilon_1}{\epsilon_2}}$
$\epsilon \equiv \epsilon_1 + \epsilon_2$	$Q = b + b^{-1}$	$Q = \frac{\epsilon}{\sqrt{\epsilon_1 \epsilon_2}}$
Masses m_i	Charges of the external states $\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_n$	(3.17)–(3.20)
Coulomb moduli u_ℓ	Charges of the intermediate states $\mathbf{w}$	(3.44), (3.46)
Coulomb branch parameters $\mathfrak{a}^{(\ell)}$	Casimirs of the intermediate state $\boldsymbol{\alpha}$ (3.21)	(3.22)
Full punctures $\odot$, see figure 3	Primary fields $\mathsf{V}_\odot$ (3.7), (3.17)	
Simple punctures $\bullet$, see figure 3	Primary fields $\mathsf{V}_\bullet$ (3.7), (3.19)	
Shift $x \to x - \kappa\phi_1$ in the curve (2.10)	Redefinitions of the currents (3.40)	
Instanton partition functions $\mathcal{Z}^{\text{inst}}$ (3.31)	W-blocks $\mathcal{B}$ (3.25)	(3.32)
SW coefficients curve $\phi_\ell^{(n)}$ (2.4)	Ratios of W-blocks $\langle\langle\, J \,\rangle\rangle_n$ (3.35)	(3.13), (3.41)
S^4 partition function	Full correlation function	(3.23)

Table 1. This table presents an overview of the elements of the AGT correspondence that we need as well as the equations where the identifications appear.

We begin with a short introduction of the Toda CFT and its symmetries. We then relate the charges of the Toda currents to the curves of the previous section and thus match the parameters. Following this, we explain how to recover the complete curve coefficients from the CFTs as ratios of conformal/W-blocks and relate the blocks to the instanton partition functions of the gauge theory.

3.1 The Toda CFT

We refer to the appendix B of [37] for our conventions regarding the SU(N) weights h_i, simple roots e_i, fundamental weights ω_i, Weyl vector ρ and scalar product $(\cdot,\cdot)$.

The action (see [38]) of the SU(N) Toda theory in our normalizations reads (we define the $\boldsymbol{\varphi}$ fields below)

$$S_{\text{Toda}} = \int \left(\frac{1}{8\pi} \hat{g}^{mn} \left(\partial_m \boldsymbol{\varphi}, \partial_n \boldsymbol{\varphi}\right) + \frac{(\mathcal{Q}, \boldsymbol{\varphi})}{4\pi} \hat{R} + \mu \sum_{j=1}^{N-1} e^{b(e_j, \boldsymbol{\varphi})} \right) \sqrt{\hat{g}}\, d^2x \,, \tag{3.1}$$

where $\hat{g}_{mn}$ is the background metric and $\hat{R}$ is the corresponding scalar curvature coupling to the background charge Q. One defines $\mathcal{Q} = Q\rho$ and relates Q to the coupling b via $Q = b + b^{-1}$ so that the theory is conformal. The cosmological constant μ is not particularly important and only enters the game through the overall normalization of the 3-point structure constants in the quantum theory. The central charge c of the Toda CFT is given by

$$c = N - 1 + 12\,(\mathcal{Q}, \mathcal{Q}) = (N-1)\left(1 + N(N+1)Q^2\right) \,, \tag{3.2}$$

so that $c = N - 1$ for $Q = 0$. We still have to explain the $N - 1$ component field $\boldsymbol{\varphi}$. In order to introduce some notation for later, we start (in the formal free case where the cosmological constant μ is zero) with the N free fields $\{\varphi_j\}_{j=1}^N$ with the OPE $\varphi_i(z)\varphi_j(w) \sim$

$-\delta_{ij} \log|z-w|^2$. Next, we define the SU(N) field $\boldsymbol{\varphi}$

$$\boldsymbol{\varphi} = \sum_{j=1}^{N-1} \omega_j \tilde{\varphi}_j = \sum_{j=1}^{N-1} \omega_j (\varphi_j - \varphi_{j+1}) = \sum_{i=1}^{N-1} \mathsf{h}_i (\varphi_i - \varphi_N)\,, \tag{3.3}$$

with ω_j being the SU(N) fundamental weights. The above implies that $\tilde{\varphi}_i(z)\tilde{\varphi}_j(w) \sim -\mathbf{car}_{ij} \log|z-w|^2$, where $\mathbf{car}_{ij} = 2$ if $i = j$, -1 if $|i-j| = 1$ and zero otherwise is the SU(N) Cartan matrix. The U(1) free field that decouples from the rest of the Toda action is $\lambda = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} \varphi_j$ with the free field OPE $\lambda(z)\lambda(w) \sim -\log|z-w|^2$. The original φ_j fields can be written as $\varphi_j = \frac{1}{\sqrt{N}}\lambda + (\mathsf{h}_j, \boldsymbol{\varphi})$. Using the field $\boldsymbol{\varphi}$ in the free limit is straightforward since we have the OPE

$$(\boldsymbol{\alpha}, \boldsymbol{\varphi})(z)(\boldsymbol{\beta}, \boldsymbol{\varphi})(w) \sim -(\boldsymbol{\alpha}, \boldsymbol{\beta}) \log|z-w|^2\,, \tag{3.4}$$

which follows from the identity (A.4).

The quantum Miura transform (see for example [39, 40]) relates the currents of the $\mathbf{W}_N$ algebra for the Toda theory in terms of the $N-1$ free fields $\boldsymbol{\varphi}$. One roughly speaking sets $\mu = 0$ in (3.1) and expands the Lax operator $\widetilde{\mathcal{R}}_N$ as

$$\widetilde{\mathcal{R}}_N =: \prod_{j=1}^{N} (Q\partial_z + (\mathsf{h}_j, \partial\boldsymbol{\varphi}(z))) := \sum_{s=0}^{N} \mathcal{W}_s(z)(Q\partial_z)^{N-s}\,, \tag{3.5}$$

where :: denotes normal-ordering. Note that the $\mathcal{W}_s$ coming from the quantum Miura transform are for $s > 2$ in general **not** conformal primaries. They differ from the $\mathbf{W}_N$ currents W_s by terms proportional to Q and hence agree (up to a convention dependent normalization that for us is set to one) for $Q = 0$. We remind that the OPEs of the $\mathbf{W}_N$ currents with a primary field $\mathsf{V}_{\boldsymbol{\alpha}}$ are

$$W_s(z_1)\mathsf{V}_{\boldsymbol{\alpha}}(z_2, \bar{z}_2) \sim \frac{w_s(\boldsymbol{\alpha})}{(z_1 - z_2)^s} \mathsf{V}_{\boldsymbol{\alpha}}(z_2, \bar{z}_2) + \sum_{n=1}^{s-1} \frac{W_{s,-n}\mathsf{V}_{\boldsymbol{\alpha}}(z_2, \bar{z}_2)}{(z_1 - z_2)^{s-n}}\,. \tag{3.6}$$

Here the $W_{s,-n}$ denote the lowering modes of the W_s current. We parametrize the primary fields/ vertex operators in terms of SU(N) weights $\boldsymbol{\alpha}$ as[6]

$$\mathsf{V}_{\boldsymbol{\alpha}}(z) = e^{(\boldsymbol{\alpha}, \boldsymbol{\varphi})}(z)\,. \tag{3.7}$$

From this parametrization of the primary fields, using $(\mathsf{h}_j, \partial\boldsymbol{\varphi})(z)\mathsf{V}_{\boldsymbol{\alpha}}(w) \sim -\frac{(\mathsf{h}_j, \boldsymbol{\alpha})}{z-w}\mathsf{V}_{\boldsymbol{\alpha}}(w)$ as well as the general relation (u and d_j are arbitrary complex parameters)

$$\prod_{j=1}^{N} \left(u\partial_z + \frac{d_j}{z}\right) = \sum_{s=0}^{N} \frac{1}{z^s} \left[\sum_{i_1 < i_2 < \cdots < i_s = 1}^{N} \prod_{m=1}^{s} (d_{i_m} - u(k-m)) \right] (u\partial_z)^{N-s}\,, \tag{3.8}$$

[6]The primary fields V also carry a λ dependent part as we write later in (3.42), but we can ignore that part for now.

we derive the charges of the $\mathcal{W}_{s,0}$ modes to be (see also [41])

$$\begin{aligned}\Delta(\boldsymbol{\alpha}) = w_2'(\boldsymbol{\alpha}) &= \sum_{i<j=1}^{N} (\mathsf{h}_i, \boldsymbol{\alpha})\,(\mathsf{h}_j, \boldsymbol{\alpha}) + Q\sum_{j=2}^{N}(j-1)\,(\mathsf{h}_j, \boldsymbol{\alpha}) = \frac{(2\mathcal{Q}-\boldsymbol{\alpha}, \boldsymbol{\alpha})}{2}\,,\\ w_s'(\boldsymbol{\alpha}) &= (-1)^s \sum_{i_1<i_2<\cdots<i_s=1}^{N} \prod_{j=1}^{s} \left((\mathsf{h}_{i_j}, \boldsymbol{\alpha}) + Q(s-j)\right)\,,\end{aligned} \tag{3.9}$$

where we have used (A.4) and $\mathcal{Q} = Q\rho$ with $\rho = \sum_{j=2}^{N}(j-1)\mathsf{h}_j$. The charges of the primary $\mathbf{W}_s$ fields with modes $W_{s,0}$ with $s > 2$ differ from the above. For example $w_3(\boldsymbol{\alpha}) = w_3'(\boldsymbol{\alpha}) + Q(N-2)w_2'(\boldsymbol{\alpha})$, which can be rewritten as

$$w_3(\boldsymbol{\alpha}) = -\sum_{i_1<i_2<i_3=1}^{3} \prod_{s=1}^{3} (\boldsymbol{\alpha} - \mathcal{Q}, \mathsf{h}_{i_s})\,, \tag{3.10}$$

see also [42] for more details.

The limit $Q \to 0$ is referred to as the "semi-classical" limit[7] and it is defined by the substitution $Q\partial_z \longrightarrow x$ in (3.5). This limit is called semi-classical because it replaces the pair $(Q\partial_z, z)$ that satisfies the Heisenberg commutation relations with the commuting variables (x, z).[8] In that limit, we have $W_s = \mathcal{W}_s$ and hence

$$\begin{aligned}W_s &= \sum_{1\le j_1<j_2<\cdots<j_s\le N} (\mathsf{h}_{j_1}, \partial\boldsymbol{\varphi})\cdots(\mathsf{h}_{j_s}, \partial\boldsymbol{\varphi}) \quad\Longrightarrow\quad T = -\frac{(\partial\boldsymbol{\varphi}, \partial\boldsymbol{\varphi})}{2}\,,\\ \lim_{Q\to 0} w_s(\boldsymbol{\alpha}) &= \lim_{Q\to 0} w_s'(\boldsymbol{\alpha}) = (-1)^s \sum_{1\le j_1<j_2<\cdots<j_s\le N} (\mathsf{h}_{j_1}, \boldsymbol{\alpha})\cdots(\mathsf{h}_{j_s}, \boldsymbol{\alpha})\,.\end{aligned} \tag{3.11}$$

One of the consequences of the AGT correspondence is that the semi-classical limit of the Lax operator reproduces the Seiberg-Witten curve after an x shift to the Gaiotto curve

$$\left\langle\!\left\langle\, \widetilde{\mathcal{R}}(x)\, \right\rangle\!\right\rangle = \sum_{\ell=0}^{N} x^{N-\ell}\,\langle\langle\, W_s(t)\,\rangle\rangle = \sum_{\ell=0}^{N} x^{N-\ell}\tilde{\phi}_\ell(t) = 0\,, \tag{3.12}$$

since as we shall review in section 3.4,

$$\lim_{Q\to 0}\,\langle\langle\, W_s(t)\,\rangle\rangle = \tilde{\phi}_\ell(t)\,. \tag{3.13}$$

We refer to (2.10) and its surrounding paragraph for the definition of the curve coefficients $\tilde{\phi}_\ell(t)$. We shall also see that (3.12) can be made to work also for the case without the shift in x. This requires the reintroduction of the decoupled U(1) degrees of freedom that on the CFT side are contained in the free boson field λ.

[7]This is different from the semi-classical limit $b \to \infty$ of the Toda CFT considered for example in [38].

[8]In order to relate the curve to the CFT, we also need to take the limit $\hbar = \sqrt{\epsilon_1\epsilon_2} \to 0$, as we will describe in the next section.

3.2 Identification of the parameters

In order to make (3.13) precise, we need to first relate the Toda CFT charges $\boldsymbol{\alpha}$ of the primary fields (3.7) with the mass and Coulomb parameters appearing in the curves. We first observe, that the curve contains only parameters with units of mass, while the CFT parameters are massless. In order to resolve the discrepancy, we introduce the parameter $\hbar$ via

$$\hbar = \sqrt{\epsilon_1 \epsilon_2}\,, \tag{3.14}$$

and use it to rescale the curve parameters. Specifically, in all the formulas relating the curve data to the CFT data, one has to make the transformation

$$m \to \frac{m}{\hbar} \tag{3.15}$$

for all the quantities $(m_{L,i}, m_{R,i}, a_i,\ \epsilon = \epsilon_1 + \epsilon_2)$ with units of mass. Since this rescaling is, beyond making the units of mass work, not important for the main arguments of this article, it will be omitted from the formulas and reintroduced only at the essential points. We begin the identification of the parameters by looking at the CFT coupling. It is related to the Ω-background parameters via

$$b = \frac{\epsilon_1}{\hbar} \implies Q = \frac{\epsilon_1 + \epsilon_2}{\hbar}\,. \tag{3.16}$$

From the curves, we have $2N$ mass parameter $m_{L,i}$ and $m_{R,i}$ with $i = 1, \ldots, N$. We defined in (2.14) the SU(N) masses $\tilde{m}_{L,i}$ and $\tilde{m}_{R,i}$ as well as the centers of mass M_L and M_R. After the rescaling (3.15), the masses are related to the weights $\boldsymbol{\alpha}_{\odot}$ of the full punctures $\mathsf{V}_{\odot}$ via

$$\begin{aligned}
\tilde{m}_{L,i} &= -\left(\boldsymbol{\alpha}_{\odot,L} - \mathcal{Q}, \mathsf{h}_i\right)\,, & \tilde{m}_{R,i} &= \left(\boldsymbol{\alpha}_{\odot,R} - \mathcal{Q}, \mathsf{h}_i\right)\,,\\
\boldsymbol{\alpha}_{\odot,L} &= \sum_{i=1}^{N-1} (-\tilde{m}_{L,i} + \tilde{m}_{L,i+1} + Q)\omega_i\,, & \boldsymbol{\alpha}_{\odot,R} &= \sum_{i=1}^{N-1} (\tilde{m}_{R,i} - \tilde{m}_{R,i+1} + Q)\omega_i\,.
\end{aligned} \tag{3.17}$$

Thus, for the case of three points, $\tilde{m}_{L,i} = -(\boldsymbol{\alpha}_1 - \mathcal{Q}, \mathsf{h}_i)$ and $\tilde{m}_{R,i} = (\boldsymbol{\alpha}_3 - \mathcal{Q}, \mathsf{h}_i)$, while for the case of four points the parametrization becomes $\tilde{m}_{L,i} = -(\boldsymbol{\alpha}_1 - \mathcal{Q}, \mathsf{h}_i)$ and $\tilde{m}_{R,i} = (\boldsymbol{\alpha}_4 - \mathcal{Q}, \mathsf{h}_i)$. Equation (3.17) and (3.9) imply for $Q = 0$ that the W_s charges of the full punctures $\mathsf{V}_{\odot}$ are equal to

$$w_s(\boldsymbol{\alpha}_{\odot,L}) = \tilde{\mathfrak{c}}_L^{(s)}\,, \qquad w_s(\boldsymbol{\alpha}_{\odot,R}) = (-1)^s \tilde{\mathfrak{c}}_R^{(s)}\,. \tag{3.18}$$

On the other hand, the weights of the simple punctures $\mathsf{V}_{\bullet}$ are parametrized as[9]

$$\boldsymbol{\alpha}_{\bullet} = -\varkappa \omega_{N-1}\,, \tag{3.19}$$

where $\varkappa$ depends on the puncture. For the three points case, the middle puncture $\boldsymbol{\alpha}_2 = \boldsymbol{\alpha}_{\bullet,2}$ is simple and we have $\boldsymbol{\alpha}_2 = -(M_L - M_R)\omega_{N-1}$. For the four point case, the two middle punctures $\boldsymbol{\alpha}_2$ and $\boldsymbol{\alpha}_3$ are simple and we have $\boldsymbol{\alpha}_2 = -(M_L - \mathfrak{a}^{(1)})\omega_{N-1}$ and $\boldsymbol{\alpha}_3 = (M_R - \mathfrak{a}^{(1)})\omega_{N-1}$. The Casimir $\mathfrak{a}^{(1)} = \sum_{i=1}^{N} a_i$ comes from the intermediate field in the 4-point

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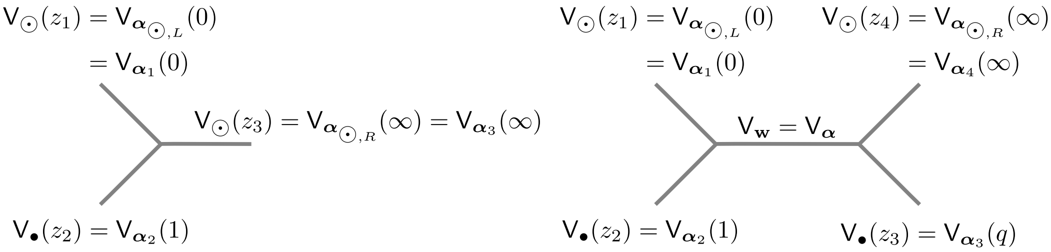

Figure 4. This figure illustrates the parametrization of the primary fields of the Toda CFT for the 3 and 4-point case. It indicates in particular which fields are full and which are simple punctures.

block, see (3.22) below, as well as figure 2. The parametrization of the primary fields is also summarized in figure 4. It follows from (3.19) that the corresponding $\mathbf{W}_N$ charges for $Q = 0$ are given by

$$\begin{aligned} w_s(\boldsymbol{\alpha}_\bullet) &= \varkappa^s \sum_{i_1<\cdots<i_s=1}^{N} (\omega_{N-1}, \mathsf{h}_{i_1}) \cdots (\omega_{N-1}, \mathsf{h}_{i_s}) \\ &= \varkappa^s \left(\sum_{i_1<\cdots<i_s=1}^{N-1} \frac{1}{N^s} + \sum_{i_1<\cdots<i_{s-1}=1}^{N-1} \frac{1}{N^{s-1}} \frac{1-N}{N} \right) \\ &= \frac{\varkappa^s}{N^s} \binom{N-1}{s} \left(1 + \frac{(1-N)s}{N-s}\right) = -\binom{N}{s} \frac{(s-1)\varkappa^s}{N^s}\,, \end{aligned} \tag{3.20}$$

where we have used $(\omega_{N-1}, \mathsf{h}_j) = \frac{1}{N}$ for $j < N$ and $(\omega_{N-1}, \mathsf{h}_N) = \frac{1-N}{N}$.

The last parametrization that we need to discuss is that of the Coulomb moduli u_ℓ of the curves that are related to the intermediate state $\boldsymbol{\alpha}$ in the 4-point block introduced in the next section 3.3, see also figure 4. Similarly to the case of the full punctures (3.17), we put

$$\boldsymbol{\alpha} = \sum_{i=1}^{N-1} (a_i - a_{i+1} + Q)\omega_i \quad \Longleftrightarrow \quad a_i = (\boldsymbol{\alpha} - \mathcal{Q}, \mathsf{h}_i)\,. \tag{3.21}$$

It is useful to define the Casimirs for the parameters a_i as in (2.6), i.e.

$$\mathfrak{a}^{(s,k)} = \sum_{i_1<\cdots<i_s=1}^{N} a_{i_1}^k \cdots a_{i_s}^k\,, \qquad \mathfrak{a}^{(0,k)} = 1\,, \tag{3.22}$$

where again $\mathfrak{a}^{(s)} \equiv \mathfrak{a}^{(s,1)}$. As we shall see in section 3.4, the Coulomb moduli u_ℓ are expressed via the Casimirs $\mathfrak{c}_L$, $\mathfrak{c}_R$ and (3.22). We also define for $k = 1$ the Casimirs $\tilde{\mathfrak{a}}^{(s)}$ obtained by applying the definition (3.22) to the $\tilde{a}_i = a_i - \frac{1}{N}\sum_{j=1}^{N} a_j$. From (3.18) we see that for $Q = 0$ the $\mathbf{W}_N$ charges of the intermediate state are $w_s(\boldsymbol{\alpha}) = (-1)^s \tilde{\mathfrak{a}}^{(s)}$.

3.3 The W-blocks and the instanton partition functions

Overview of the blocks. In any CFT, knowledge of the correlation functions of two (i.e. of the conformal dimensions Δ) and three point functions (i.e. of the structure constants C_{ijk}) completely determines the higher point functions. For ordinary CFTs, it is

[9]These types of weights give rise to semi-degenerate representations of the $\mathbf{W}_N$ algebra.

enough to know the three-point functions of the Virasoro primary fields - the ones involving descendant field being then automatically determined. On the other hand, $\mathbf{W}_N$ symmetry for $N > 2$, while stronger than Virasoro, is not sufficient to determine the correlation functions of all descendant fields just from the knowledge of the correlation functions of the $\mathbf{W}_N$ primaries. Thankfully, for the cases that we consider here, some of the primary fields are short which imposes a sufficient number of extra conditions allowing for the derivation of the 3-point functions and then of the W-blocks.

Once the 2 and 3-point functions are known, the n-point functions can be determined by expanding in conformal/ W-blocks (see for example [43] for a review). The blocks $\mathcal{B}$ are purely kinematic/symmetry quantities that are theory independent - they depend only on the charges $\mathbf{w}$ of the fields (both the external n ones as well as the intermediate ones) on the positions $q_1, \ldots, q_{n-3}$ that are not fixed by conformal symmetry and on the central charge c. The whole theory dependent information is contained in the 3-point structure constants C_{ijk}.

Let us review the 4-point $\mathbf{W}_2$ case of Liouville theory for simplicity. Putting the points $z_1, \ldots, z_4$ to $\infty, 1, q, 0$ respectively, the full 4-point correlation function[10] can be expanded (in the s-channel) as

$$\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)\mathsf{V}_3(q,\bar{q})\mathsf{V}_4(0)\,\rangle = \int d\alpha (C_{12\alpha} H^{-1}_{\alpha\alpha} C_{\alpha 34})\left| q^{\Delta_\alpha - \Delta_3 - \Delta_4} \mathcal{B}_{\Delta_\alpha}(\Delta_1, \Delta_2, \Delta_3, \Delta_4 | q)\right|^2, \tag{3.24}$$

where α labels[11] the intermediate state in the OPE decomposition, and the integral is done over the space of physical Virasoro fields: $\alpha \in \frac{Q}{2} + i\mathbb{R}$ with $\Delta_\alpha = \alpha(Q-\alpha)$. The $H_{\alpha\beta} = \langle \mathsf{V}_\alpha \,|\, \mathsf{V}_\beta \rangle$ is an orthonormalization constant that is zero if $\alpha \neq \beta$ and that can be absorbed in the normalization of the primary fieds.

Having introduced the decomposition of the full 4-point correlation function in terms of blocks in the Liouville case, we now want to concentrate on the blocks $\mathcal{B}$ and to consider them for the general $\mathbf{W}_N$ case. They can be expanded in a power series in q as

$$\mathcal{B}_{\mathbf{w}}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4 | q) = \sum_{\mathbf{Y}, \mathbf{Y}', |\mathbf{Y}| = |\mathbf{Y}'|} q^{|\mathbf{Y}|} \gamma_{12\mathbf{w}}(\mathbf{Y}) \mathsf{Q}^{-1}_{\mathbf{w}}(\mathbf{Y}, \mathbf{Y}') \bar{\gamma}_{\mathbf{w};34}(\mathbf{Y}')\,. \tag{3.25}$$

In order to understand the above, we need to introduce all the ingredients (namely the charges $\mathbf{w}$, the 3-point blocks/vertices $\gamma_{12\mathbf{w}}$ and $\bar{\gamma}_{\mathbf{w};34}$ as well as the Shapovalov form $\mathsf{Q}_{\mathbf{w}}$) which requires some work. We start by reminding that the currents of the $\mathbf{W}_N$ algebra are the $\{W_s(z)\}_{s=2}^N$. The currents are expanded in modes as $W_s(z) = \sum_{n=-\infty}^{\infty} z^{-n-s} W_{s,n}$. We often write $L_n \equiv W_{2,n}$ as well as sometimes $W_n \equiv W_{3,n}$ if confusion can be avoided. Then we can straightforwardly define the elements needed for the blocks (3.25):

[10] Recall that the AGT correspondence identifies that full correlation function with the S^4 partition function:

$$\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)\mathsf{V}_3(q,\bar{q})\mathsf{V}_4(0)\,\rangle \propto \mathcal{Z}^{S^4}\,, \tag{3.23}$$

where the proportionality constant is not important here. For the correlation function (3.24), it is the partition function of the SU(2) SCQCD theory with $N_F = 4$.

[11] For $N = 2$ one sets $\boldsymbol{\alpha} = 2\alpha\omega_1$. In general, the physical Toda fields obey $\mathrm{Re}(\boldsymbol{\alpha}) = \mathcal{Q}$.

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- A highest weight Verma module of the $\mathbf{W}_N$ algebra is spanned by the vectors $W_{-\mathbf{Y}}\mathsf{V}_{\mathbf{w}}$, where
$$\mathbf{w} \stackrel{\text{def}}{=} \{\Delta, w_3, w_4, \ldots, w_N\} \tag{3.26}$$
are the $\mathsf{V}_{\mathbf{w}}$ charges of the $W_{n,0}$ generators and $\mathsf{V}_{\mathbf{w}}$ is annihilated by all the positive mode generators. We use the symbol $\mathsf{V}_{\mathbf{w}}$ both for the state in the Hilbert space and for the vertex operator that corresponds to it. The descendant states are labeled by a set $\mathbf{Y} = \{Y_2; Y_3, \ldots, Y_N\}$ with each $Y_s = \{Y_{s,1}, Y_{s,2}, \ldots\}$ a partition of integers (arranged as $Y_{s,i} \leq Y_{s,i+1}$). The state $W_{-\mathbf{Y}}\mathsf{V}_{\mathbf{w}}$ is explicitly written as
$$\begin{aligned} W_{-\mathbf{Y}}\mathsf{V}_{\mathbf{w}} = &\left(W_{2,-Y_{2,1}}W_{2,-Y_{2,2}}\cdots\right) \\ &\times\left(W_{3,-Y_{3,1}}W_{3,-Y_{3,2}}\cdots\right)\cdots\left(W_{N,-Y_{N,1}}W_{N,-Y_{N,2}}\cdots\right)\mathsf{V}_{\mathbf{w}}\,. \end{aligned} \tag{3.27}$$
For example, for $N=3$, $W_{-\{\{1,1,2\};\{2\}\}}\mathsf{V}_{\mathbf{w}} = L_{-1}^2 L_{-2} W_{-2}\mathsf{V}_{\mathbf{w}}$. The conformal dimension of the state $W_{-\mathbf{Y}}\mathsf{V}_{\mathbf{w}}$ is equal to $\Delta + |\mathbf{Y}|$ with $|\mathbf{Y}| = \sum_{s=2}^N |Y_s|$. The action of the other zero modes $W_{s,0}$ on the descendant states is in general not diagonal.

- The Shapovalov form Q is the scalar product of vectors in the Verma module
$$\mathsf{Q}_{\mathbf{w}}(\mathbf{Y}, \mathbf{Y}') = \left\langle W_{-\mathbf{Y}}\mathsf{V}_{\mathbf{w}} \,|\, W_{-\mathbf{Y}'}\mathsf{V}_{\mathbf{w}}\right\rangle\,, \tag{3.28}$$
where we demand that the scalar product obeys $\langle W_{s,-n}\mathsf{V}_1 \,|\, \mathsf{V}_2\rangle = \langle \mathsf{V}_1 \,|\, W_{s,n}\mathsf{V}_2\rangle$.

- An important object is the 3-point W-block/vertex $\gamma_{12\mathbf{w}}(\mathbf{Y})$. For our purposes, it is defined as the ratio of a 3-point function of two primary fields and one descendant $W_{-\mathbf{Y}}\mathsf{V}_{\mathbf{w}}$ to the 3-point function of just the primary fields:
$$\gamma_{12\mathbf{w}}(\mathbf{Y}) = \frac{\langle\, \mathsf{V}_1(\infty)\mathsf{V}_2(1)\,(W_{-\mathbf{Y}}\mathsf{V}_{\mathbf{w}})\,(0)\,\rangle}{\langle\, \mathsf{V}_1(\infty)\mathsf{V}_2(1)\mathsf{V}_{\mathbf{w}}(0)\,\rangle}\,. \tag{3.29}$$
Of course, it is possible to consider the cases in which V_1 or V_2 are not primary, but we do not need them here.

- A similar object to γ is the vertex
$$\bar{\gamma}_{\mathbf{w};34}(\mathbf{Y}) = \frac{\langle W_{-\mathbf{Y}}\mathsf{V}_{\mathbf{w}} \,|\, \mathsf{V}_3(1)\mathsf{V}_4(0)\rangle}{\langle \mathsf{V}_{\mathbf{w}} \,|\, \mathsf{V}_3(1)\mathsf{V}_4(0)\rangle}\,, \tag{3.30}$$
i.e. the normalized scalar product of a state with the product of two primary fields inserted at 1 and at 0. While for the Virasoro case, there is no need to introduce the $\bar{\gamma}$ since $\bar{\gamma}_{\Delta;34} = \gamma_{43\Delta}$ (see the recursion relations (D.8)), this is not true anymore for the general $\mathbf{W}_N$ algebra.

- It is important to note that all the building blocks Q, γ, $\bar{\gamma}$ and $\mathcal{B}$ are implicitly dependent on the central charge c. Furthermore, while for the Liouville case of $\mathbf{W}_2$ the dependence on c starts appearing only at order q^2 in the four-point block $\mathcal{B}$, for the algebras $\mathbf{W}_N$ with $N > 2$, the central charge appears already at linear order in q.

One can depict the 3 and 4-point blocks graphically as sketched in 5.

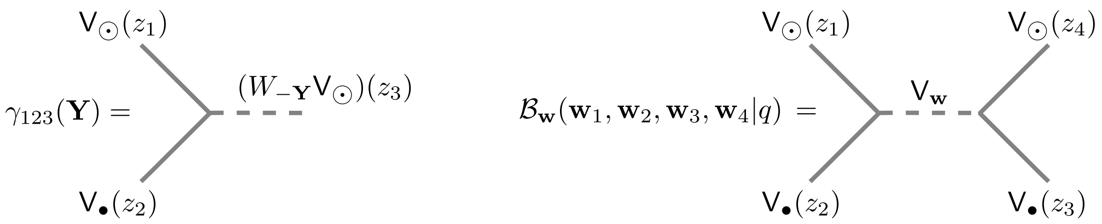

Figure 5. This figure depicts the three and four point W-blocks. Using conformal symmetry, for three points, we set $z_1 = \infty$, $z_2 = 1$ and $z_3 = 0$, while for four points, we put $z_1 = \infty$, $z_2 = 1$, $z_3 = q$ and $z_4 = 0$. The dashed lines indicate descendant fields.

The instanton partition functions and the blocks. The AGT correspondence identifies the Nekrasov instanton partition function $\mathcal{Z}_{\text{inst}}$ to the W-blocks, after an appropriate factor has been removed. In the case that we are dealing with, namely for the $\mathcal{N} = 2$ SU(N) SCQCD with $N_F = 2N$, the instanton partition function reads

$$\mathcal{Z}_{\text{inst}} = \sum_{\mathbf{Y}} q^{|\mathbf{Y}|} \mathcal{Z}_{\text{vec}}(\mathbf{a}, \mathbf{Y}) \prod_{i=1}^{N} \mathcal{Z}_{\text{antifund}}(\mathbf{a}, \mathbf{Y}; -m_{L,i}) \prod_{j=1}^{N} \mathcal{Z}_{\text{fund}}(\mathbf{a}, \mathbf{Y}; m_{R,j}) \,, \tag{3.31}$$

where $\mathbf{a} = (a_1, \ldots, a_N)$ and $\mathbf{Y} = \{Y_1, \ldots, Y_N\}$ is a set of N Young diagrams and the building blocks of $\mathcal{Z}_{\text{inst}}$ are defined in appendix E. The partition function is related to the W-blocks as

$$\mathcal{Z}_{\text{inst}} = \mathcal{B}_{\text{U}(1)} \mathcal{B}_{\mathbf{w}}(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4|q) \,. \tag{3.32}$$

We remark that to relate the CFT data to the 4D Nekrasov partition functions, one should rescale on the CFT side all parameters with dimension of mass as in (3.15).

The $\mathbf{W}_N$ algebra charges $\mathbf{w}_i$ are obtained by using the parametrization for $\boldsymbol{\alpha}_i$ in section 3.2 and using the identities (3.9), (3.10). The U(1) contribution, the 4-point block $\mathcal{B}_{\text{U}(1)}$, is given by the formula (D.3) derived in appendix D.1

$$\mathcal{B}_{\text{U}(1)} = (1-q)^{p_2 p_3} = (1-q)^{\frac{(M_L - \mathfrak{a}^{(1)})(M_R - \mathfrak{a}^{(1)} - N\epsilon)}{N\epsilon_1\epsilon_2}} \tag{3.33}$$

with the charges $p_2 = -i\frac{M_L - \mathfrak{a}^{(1)}}{\sqrt{N\epsilon_1\epsilon_2}}$ and $p_3 = i\frac{M_R - \mathfrak{a}^{(1)} - N\epsilon}{\sqrt{N\epsilon_1\epsilon_2}}$ (compare with (3.45)). In the above, we have used $\sum_{i=1}^{N} a_i = \mathfrak{a}^{(1)}$, see (3.22).

3.4 Comparisons of the curves with the blocks

We now want to compare the curve coefficients ϕ_ℓ with the $\mathbf{W}_N$ blocks, for three and for four points. In order to connect the blocks with the curve, we need to introduce yet another object, namely the *3-point W-block with the insertion of an arbitrary current $J(t)$ at point t*. We write it as

$$\boldsymbol{\gamma}_{12\mathbf{w}}(J(t); \mathbf{Y}) \overset{\text{def}}{=} \frac{\langle\, \mathsf{V}_1(\infty)\mathsf{V}_2(1)J(t)\,(W_{-\mathbf{Y}}\mathsf{V}_{\mathbf{w}})\,(0)\,\rangle}{\langle\, \mathsf{V}_1(\infty)\mathsf{V}_2(1)\mathsf{V}_{\mathbf{w}}(0)\,\rangle} \,. \tag{3.34}$$

The numerator of the above quantity is strictly speaking a 4-point function, but since $J(t)$ is a symmetry current and not an arbitrary object, the dependence of t can be obtained by

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expanding $J(t)$ in modes and using the blocks $\gamma_{12\mathbf{w}}(\mathbf{Y})$. Thus, we refer to $\boldsymbol{\gamma}_{12\mathbf{w}}(J(t);\mathbf{Y})$ as a 3-point block with an insertion of a current.

Armed with that definition, we define the weighted current correlation functions $\langle\langle\, J(t)\,\rangle\rangle$ as the following ratio of blocks:

$$\langle\langle\, J(t)\,\rangle\rangle_n \stackrel{\text{def}}{=} \frac{n\text{-point W-block with insertion of } J(t)}{n\text{-point W-block}}\,, \tag{3.35}$$

where the n-point W-block are computed with for n primary fields. In the cases that concern us, two of the primary fields are full punctures $\mathsf{V}_{\odot}$ placed at z_1 and z_n and the remainig $n-2$ ones are simple punctures $\mathsf{V}_{\bullet}$ at the points $z_2,\ldots,z_{n-1}$. By a conformal transformation, we place $z_1=\infty$, $z_2=1$ and $z_n=0$. In particular, for three points, we have for three primary fields

$$\langle\langle\, J(t)\,\rangle\rangle_3 = \frac{\boldsymbol{\gamma}_{123}(J(t);\emptyset)}{\gamma_{123}(\emptyset)} = \boldsymbol{\gamma}_{123}(J(t);\emptyset) = \frac{\langle\, \mathsf{V}_1(\infty)\mathsf{V}_2(1)J(t)\mathsf{V}_{\mathbf{w}}(0)\,\rangle}{\langle\, \mathsf{V}_1(\infty)\mathsf{V}_2(1)\mathsf{V}_{\mathbf{w}}(0)\,\rangle}\,. \tag{3.36}$$

For four points, we have to specify the representation flowing in the middle with the label $\mathbf{w}$. Labeling the point z_3 by q, the quantity $\langle\langle\, J(t)\,\rangle\rangle_4$ can be written as a power series expansion in q as

$$\langle\langle\, J(t)\,\rangle\rangle_4 = \frac{\sum_{\mathbf{Y},\mathbf{Y}',|\mathbf{Y}|=|\mathbf{Y}'|} q^{|\mathbf{Y}|}\boldsymbol{\gamma}_{12\mathbf{w}}(J(t);\mathbf{Y})\mathsf{Q}_{\mathbf{w}}^{-1}(\mathbf{Y},\mathbf{Y}')\bar{\gamma}_{\mathbf{w};34}(\mathbf{Y}')}{\sum_{\mathbf{Y},\mathbf{Y}',|\mathbf{Y}|=|\mathbf{Y}'|} q^{|\mathbf{Y}|}\gamma_{12\mathbf{w}}(\mathbf{Y})\mathsf{Q}_{\mathbf{w}}^{-1}(\mathbf{Y},\mathbf{Y}')\bar{\gamma}_{\mathbf{w};34}(\mathbf{Y}')}\,. \tag{3.37}$$

We note that in the above, if $J(t)$ is a spin s current, the sum over the partitions $\mathbf{Y}=\{Y_2,\ldots,Y_N\}$ contains only those $\mathbf{Y}$ with $Y_{s+1}=\cdots=Y_N=\emptyset$.

We now want to illustrate how the $\langle\langle\, J_s\,\rangle\rangle_n$ reproduce (see (1.1)) the curve coefficients $\phi_s^{(n)}$ for a few select cases. The comparisons with the curve coefficients in *the rest of this section are all done in the limit* $\epsilon_i\to 0$.

The U(1) current. Before we can make (1.1) precise, we need to discuss how the U(1) degrees of freedom contained in the free boson λ, defined in section 3.1, affect the identification. For $k=1$, i.e. for the $\mathcal{N}=2$ theories, we are allowed to shift $x\to x-\kappa\phi_1$ in the curve. The Gaiotto curve with coefficients $\tilde{\phi}_s$ is obtained for $\kappa=\frac{1}{N}$ and for that curve we have the identification (3.12) between the ratios of blocks with insertions of the Toda currents W_s and the curve coefficients $\tilde{\phi}_s$. We can of course now perform the inverse shift $x\to x+\frac{1}{N}\phi_1$. One might then ask how the currents should be modified in order for the ratio of blocks to give ϕ_s. The answer lies in bringing back to the game the free boson λ. We define $J_1=i\partial\lambda$ be the spin 1 free boson current. We demand that in our normalizations

$$\langle\langle\, J_1(t)\,\rangle\rangle_n \stackrel{!}{=} -i\sqrt{N}\phi_1^{(n)}(t)\,. \tag{3.38}$$

Since λ is completely decoupled from the Toda action, we can simply shift $x\to x-i\frac{1}{\sqrt{N}}J_1$ in (3.12) and get for the Lax operator (remember that $Q\to 0$)

$$\mathcal{R}(x)=\widetilde{\mathcal{R}}(x-i\frac{1}{\sqrt{N}}J_1)=\prod_{j=1}^{N}\left(x+\frac{1}{\sqrt{N}}\partial\lambda+(\mathsf{h}_j,\partial\boldsymbol{\varphi})\right)=\prod_{j=1}^{N}(x+\partial\varphi_j)\ , \tag{3.39}$$

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where we have used $\frac{1}{\sqrt{N}}\lambda + (\mathsf{h}_j, \boldsymbol{\varphi}) = \varphi_j$. The currents $\mathcal{J}_s$ are given by expanding the Lax operator[12] $\mathcal{R}(x)$. We get

$$\mathcal{J}_s = \sum_{\ell=0}^{s} \binom{N-\ell}{N-s} W_\ell \left(\frac{-i}{\sqrt{N}} J_1 \right)^{s-\ell} , \tag{3.40}$$

with $W_0 = 1$ and $W_1 = 0$. In particular, one has $\mathcal{J}_1 = -\frac{i}{\sqrt{N}} J_1$ for the normalized spin one current. One can of course derive the expressions for the currents $\mathcal{J}_s$ for general values of the shift κ, but we don't need them in what follows. The relation between the currents $\mathcal{J}_s$ and the curve coefficients reads

$$\langle\langle\, \mathcal{J}_s(t) \,\rangle\rangle_n = \phi_s^{(n)}(t) \,. \tag{3.41}$$

In order to have (3.38), the primary fields have to also carry a J_1 charge p as

$$\mathsf{V}_{\odot} = e^{\left(\boldsymbol{\alpha}_{\odot}, \boldsymbol{\varphi}\right)} e^{p_{\odot}\lambda} , \qquad \mathsf{V}_{\bullet} = e^{(\boldsymbol{\alpha}_{\bullet}, \boldsymbol{\varphi})} e^{p_{\bullet}\lambda} . \tag{3.42}$$

We can now compare $\langle\langle\, J_1(t) \,\rangle\rangle_n$ with the SW curve coefficient $\phi_1^{(n)}$ to fix the charges $p_{\odot}$ and $p_{\bullet}$. Let us consider the 4-point case. From (2.5) we get for $k = 1$ and any N

$$\phi_1^{(4)}(t) = \frac{qM_R - M_L t^2 + t u_1(q)}{(t-1)t(t-q)} \,. \tag{3.43}$$

In order to make the coefficients of the highest order poles in t independent of q, we need to set

$$u_1(q) = q(M_L + M_R) + \mathfrak{a}^{(1)}(1-q) \,, \tag{3.44}$$

for $\mathfrak{a}^{(1)}$ defined in (3.22), which leads to $\phi_1^{(4)}(t) = \frac{\mathfrak{a}^{(1)} - M_L}{t-1} + \frac{-\mathfrak{a}^{(1)} + M_R}{t-q} - \frac{M_R}{t}$. The U(1) blocks needed for the computation of $\langle\langle\, J_1 \,\rangle\rangle_4$ are found in appendix D.1. The comparison with (D.5) tells us that (3.38) is satisfied if we set the momenta of the vertex operators and intermediate state to

$$\begin{aligned} p_1 &= i\frac{M_L}{\sqrt{N}} \,, & p_2 &= -\,i\frac{M_L - \mathfrak{a}^{(1)}}{\sqrt{N}} \,, & p_3 &= i\frac{M_R - \mathfrak{a}^{(1)}}{\sqrt{N}} \,, \\ p_4 &= -\,i\frac{M_R}{\sqrt{N}} \,, & p &= -\,i\frac{\mathfrak{a}^{(1)}}{\sqrt{N}} \,. \end{aligned} \tag{3.45}$$

The above agrees with (3.33) after the usual rescaling (3.15) and in the limit $Q \to 0$. In the 3-point case, we have $p_1 = i\frac{M_L}{\sqrt{N}}$, $p_2 = -i\frac{M_L - M_R}{\sqrt{N}}$ and $p_3 = -i\frac{M_R}{\sqrt{N}}$.

Comparisons with the curves. We refer to appendix D for the computations of the $\mathbf{W}_2$ and $\mathbf{W}_3$ blocks relevant for the comparison with the curve coefficients and to [43] for an overview of the techniques needed for these computations.

[12]The Toda action (3.1) can be referred to as the SU(N) Toda CFT and the algebra $\mathbf{W}_N$ as the SU(N) W-algebra. Adding the decoupled free boson λ brings us to the U(N) Toda CFT and the currents (3.40) generate the U(N) W-algebra.

For the stress-energy tensor, we compute $\langle\langle T(t)\rangle\rangle_3$ in (D.7) and $\langle\langle T(t)\rangle\rangle_4$ to quadratic order in q in (D.14). Comparing them with $\tilde{\phi}_2^{(n)} = \phi_2^{(n)} - \frac{N-1}{2N}(\phi_1^{(n)})^2$, with the $\phi_s^{(n)}$ from (2.5), (2.8), leads to a perfect agreement if one sets the Coulomb branch parameter u_2 to be equal to[13]

$$
\begin{aligned}
u_2(q) = & -\mathfrak{a}^{(2)} + \frac{q}{\tilde{\mathfrak{a}}^{(2)}}\Bigg[-\frac{\mathfrak{c}_L^{(2)}\,\mathfrak{c}_R^{(2)}}{2} + \frac{(N-1)\,\mathfrak{a}^{(1)}(M_L\,\mathfrak{c}_R^{(2)} + \mathfrak{c}_L^{(2)}\,M_R)}{2N} \\
& - \mathfrak{a}^{(2)}\left(\frac{N-1}{N}M_LM_R + \frac{\mathfrak{c}_L^{(2)}}{2} + \frac{\mathfrak{c}_R^{(2)}}{2}\right) \\
& + \frac{(N-1)\,\mathfrak{a}^{(1)}\,\mathfrak{a}^{(2)}(M_L+M_R)}{2N} + \mathfrak{a}^{(2)}\left(\frac{\mathfrak{a}^{(2)}}{2} - \frac{N-1}{2N}(\mathfrak{a}^{(1)})^2\right)\Bigg] + \mathcal{O}(q^2)\,.
\end{aligned}
\tag{3.46}
$$

For simplicity, we have truncated the expansion to linear order in q. For the Liouville case, the central charge c makes an appearance at order q^2. Since it is possible to compute $u_2(q)$ from the curve alone,[14] as we do in appendix F, one might think that this gives one a way to fix the CFT central charge from the SW curve. However, we show in that appendix that we cannot fix the central charge from the curve because we need to also take the limit $\hbar \to 0$, in which case u_2 becomes insensitive to c.

In a similar fashion, $\langle\langle W_3(t)\rangle\rangle_3$ is to be found in (D.18) and $\langle\langle W_3(t)\rangle\rangle_4$ can be computed to linear order in q with the tools provided in appendix D.3. We compare them with $\tilde{\phi}_3^{(n)}$, where

$$
\tilde{\phi}_3^{(n)} = \phi_3^{(n)} - \frac{(N-2)}{N}\phi_1^{(n)}\phi_2^{(n)} + \frac{(N-2)(N-1)}{3N^2}(\phi_1^{(n)})^3\,. \tag{3.47}
$$

The comparison works perfectly if we use the parameter identification of section 3.2 and if we express u_3 as a function of q, of the $\mathfrak{a}^{(s)}$ and of the mass parameters, just like we did for u_2 in (3.46). One can even perform the comparison for $\mathbf{W}_4$, see [44] for the relevant commutation relations, but the computations become very tedious and we omit them.

4 The AGT correspondence for the $\mathcal{S}_k$ theories

Having reviewed in the last section some essential elements of the AGT correspondence, we can now apply them to the $\mathcal{S}_k$ theories. *The main principle guiding us is the observation that the class $\mathcal{S}_k$ curves for $SU(N)$ can be obtained from the $\mathcal{N}=2$ $\mathcal{S}$ curves for $SU(Nk)$.*

In order to see that, we introduce a map that takes the SU(Nk) curve and sets the mass/Coulomb parameters to special values. Let us write this map as $\pi_{N,k}$ and define its action on the SU(Nk) masses and Coulomb parameters as follows

$$
m_{L,\,j+Ns}^{\mathrm{SU}(Nk)} \longmapsto m_{L,\,j}\,\mathrm{e}^{\frac{2\pi i}{k}s}\,, \qquad m_{R,\,j+Ns}^{\mathrm{SU}(Nk)} \longmapsto m_{R,\,j}\,\mathrm{e}^{\frac{2\pi i}{k}s}\,, \qquad a_{j+Ns}^{\mathrm{SU}(Nk)} \longmapsto a_j\,\mathrm{e}^{\frac{2\pi i}{k}s}\,, \tag{4.1}
$$

where the indices run as $j=1,\ldots,N$, $s=0,\ldots,k-1$. The parameters on the right hand side of (4.1) are those of the class $\mathcal{S}_k$ SU(N) theory. Since $\prod_{s=0}^{k-1}\left(v - m\,\mathrm{e}^{\frac{2\pi i}{k}s}\right) = v^k - m^k$,

[13] Observe that the transition from the SCQCD curve to the free trinion one makes us set $a_i = m_{R,\,i}$, which puts $u_\ell(q=0) = (-1)^{\ell+1}\,\mathfrak{c}_R^{(\ell)}$, see (2.9), (3.44) and (3.46).

[14] Using the so-called inverse mirror map.

it is clear from the curve equations (2.3) and (2.7) that $\pi_{N,k}$ maps the $\mathcal{N}=2$ SU(Nk) curve with $k=1$ to the $\mathcal{N}=1$ $\mathcal{S}_k$ SU(N) curve. Furthermore, it is clear that $\pi_{N,k}$ maps the sums of all the left/right masses to zero. This generalizes to the following action on the Casimirs:

$$\pi_{N,k}\left(\mathfrak{c}^{(k\ell),\mathrm{SU}(Nk)}\right) = (-1)^{\ell(k+1)}\,\mathfrak{c}^{(\ell,k)}\,. \tag{4.2}$$

and $\pi_{N,k}\left(\mathfrak{c}^{(s),\mathrm{SU}(Nk)}\right)=0$ if $s\neq k\ell$. The above is proved in appendix A, see equation (A.5). The action (4.2) together with the expression for the u_ℓ as functions of the Casimirs (for example (3.44) and (3.46)) implies that for $Q=0$ we have

$$u_s^{\mathrm{SU}(Nk)} \longmapsto \begin{cases} u_s & \text{if } s \bmod k = 0 \\ 0 & \text{otherwise} \end{cases} \qquad \text{with } s=1,\dots,Nk\,. \tag{4.3}$$

Our guiding principle can now be stated as follows: since the map (4.1) sends the $\mathcal{N}=2$ SU(Nk) curve to the $\mathcal{N}=1$ SU(N) class $\mathcal{S}_k$ curve, we can expect that $\pi_{N,k}$ would preserve the aspects of the AGT correspondence of section 3, namely the identification of blocks and instanton partition functions as well as the correspondence between the curves and the ratios of the blocks with current insertions.

In this section, we shall study the consequences of this principle. We begin with some $\mathbf{W}_N$ representation theory and show in particular that the simple punctures are mapped by $\pi_{N,k}$ to non-unitary representations. Following that, we look at the structure of the corresponding 3 and 4-point blocks and study the Ward identities. Finally, we compute the corresponding $\langle\langle W_s\rangle\rangle_n$ in the limit $Q\to 0$ and recover the $\mathcal{S}_k$ curves (2.5) and (2.8), thus providing a check of the proposal. We remark that, as illustrated in appendix F, the curve gives no information on the CFT central charge. For now, following the principle stated in the preceding paragraph and since $\pi_{N,k}$ does not act on the ϵ_i, we assume that the central charges of the putative $\mathcal{S}_k$ SU(N) CFTs are given by the central charges of the class $\mathcal{S}$ SU(kN) CFTs. In particular, for the $Q=0$ case, this implies $c=kN-1$. Since our computations lead to a conjecture for the $\mathcal{S}_k$ instanton partition functions, a direct computation of these instantons will lead to a computation of the central charge.

4.1 The structure of the punctures

Let us now study the consequences of the map (4.1) on the punctures. For $k=1$, the full punctures $\mathsf{V}_\odot$ are generic $\mathbf{W}_N$ representations with no special properties, while the simple ones $\mathsf{V}_\bullet$ are representations with $\frac{(N-2)(N-1)}{2}$ null vectors, which allows us to compute the three and four point W-blocks. Both the simple and the full punctures are unitary representations of $\mathbf{W}_N$.

The simple punctures. For $k>1$, all the charges of the simple punctures vanish, i.e. $\mathbf{w}_\bullet=\{0,\dots,0\}$. This follows from the fact that, see (3.19), the parameter $\varkappa$ determining $\boldsymbol{\alpha}_\bullet$ is given by the sum of all the left/right masses which are mapped by $\pi_{N,k}$ to zero. *However, the* $\mathsf{V}_\bullet$ *are still different from the identity field* $\mathbf{I}$! The first and most important difference is that $L_{-1}\mathbf{I}=0$ but $L_{-1}\mathsf{V}_\bullet\neq 0$, because otherwise, the W-block would not depend on the insertion point of the simple puncture, which would prevent us from

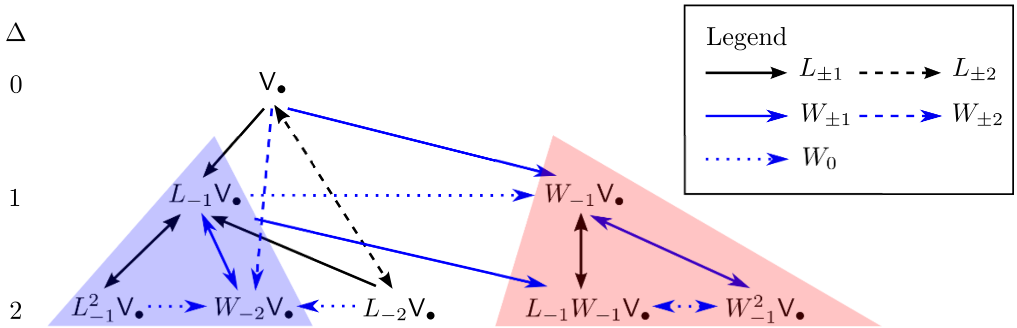

Figure 6. Structure of the first 3 levels of the simple puncture for $N = 1$ and $k = 3$ which implies $\Delta = w = 0$. For $c = 2$, one quotients out the submodule (shaded in red) generated by $W_{-1}\mathsf{V}_\bullet$. For $c \neq 2$, i.e. for $Q \neq 0$, one should quotient out the submodule generated by the vector $(W_{-1} + \frac{Q}{2}L_{-1})\mathsf{V}_\bullet$ instead. We remark that the singular vector $L_{-1}\mathsf{V}_\bullet$ generates an indecomposable submodule, shaded in blue, whose elements all have zero norm. If we were to quotient out the zero norm states as well, then we would obtain the identity representation. The color and type of the of the arrows indicates which generators are acting, as depicted in the legend.

recovering the curve coefficients from $\langle\langle W_s \rangle\rangle_n$. Of course, the norm of the state $L_{-1}\mathsf{V}_\bullet$ for $k > 1$ must be zero, since $\langle L_{-1}\mathsf{V}_\bullet \,|\, L_{-1}\mathsf{V}_\bullet\rangle = 2\Delta_\bullet \langle \mathsf{V}_\bullet \,|\, \mathsf{V}_\bullet\rangle$ and $\Delta_\bullet$ is zero. Since we have non-zero states with zero norm, the CFT that we need to consider for the $\mathcal{S}_k$ AGT correspondence **is non-unitary**. One should not conflate non-vanishing null vectors with non-unitarity. While the former implies the latter, the converse is not true. Unitarity plays no role in the usual $\mathcal{N} = 2$ AGT correspondence, for which the CFT is only unitary if Q is real.[15] The difference here is that non-unitarity seems unavoidable, since the simple punctures have to be present.

We can now look at the null states in the simple punctures. First, let us consider the case $Q = 0$, which allows us to learn from the Seiberg-Witten curve. We see that the curve coefficients (2.5) have only simple poles at $t = 1$ and $t = q$. For $k = 1$, this is due to the presence of the U(1) factors. In that case, we can shift $x \to x - \frac{1}{N}\phi_1$ and then obtain curve coefficients $\tilde{\phi}_\ell$ that have poles of order ℓ at $t = 1$ and $t = q$ whose coefficients are related to the action of the modes $W_{\ell,-n}$ by (3.6). For $k > 1$, we are not allowed to shift in x anymore[16] and therefore, we have to conclude that

$$W_{s,-n}\mathsf{V}_\bullet = 0 \qquad \text{for } n = 0, 1, \ldots, s-2\,, \tag{4.4}$$

for all $s = 2, \ldots, Nk$. This of course confirms that the charges $\mathbf{w}$ of the simple puncture vanish and implies there are

$$\sum_{s=2}^{Nk}(s-2) = \frac{(Nk-2)(Nk-1)}{2} \tag{4.5}$$

[15] We remark that the Liouville CFT with $c \leq 1$ (known also as "timelike" Liouville theory in the literature) also has a field of zero conformal dimension that is not the identity, see [45].

[16] By (3.45) the U(1) charges are zero since $\pi(\mathfrak{a}^{(1)}) = \pi(M_L) = \pi(M_R) = 0$. Hence the U(1) contribution is zero and is not responsible for the fact that the poles at the simple puncture are only first order.

	$\mathsf{V}_\bullet$	$\mathsf{V}_\odot$
Δ for $k=2$	0	$\frac{N(4N^2-1)}{12}Q^2 - \sum_{i=1}^N m_i^2$
Δ for $k>2$	0	$\frac{Nk((Nk)^2-1)}{24}Q^2$
Higher charges	0	$\neq 0$
Null states for $Q=0$	$W_{s,-n}\mathsf{V}_\bullet = 0$ for $n=0,1,\dots,s-2$ and $s=2,\dots,Nk$	None

Table 2. This table contains an overview of the main properties of the punctures for the SU(N) $\mathcal{S}_k$ theory for $k>1$.

null vectors. Hence, the number of null vectors for the simple punctures of the SU(N) $\mathcal{S}_k$ theories is the same as for the $\mathcal{N}=2$ SU(Nk) theories. Hence we conjecture that the null vectors are inherited from the $\mathcal{N}=2$ theory, i.e. obtained from it by mapping the parameters with $\pi_{N,k}$. Let us check this for the case $Nk=3$, where we write for simplicity $W_n \equiv W_{3,n}$ for the modes. For general Q and k, we can use (3.9), (3.10) and (3.19) to compute for the simple puncture $\Delta_\bullet = \frac{1}{3}\varkappa(3Q-\varkappa)$, $w_\bullet = -\frac{1}{27}\varkappa(3Q-\varkappa)(3Q-2\varkappa)$. Hence, the null vector is

$$\left(W_{-1} - \frac{3w_\bullet}{2\Delta_\bullet}L_{-1}\right)\mathsf{V}_\bullet = \left(W_{-1} + \frac{3Q-2\varkappa}{6}L_{-1}\right)\mathsf{V}_\bullet = 0\,. \tag{4.6}$$

For $k>1$, $\pi_{N,k}$ maps the parameter $\varkappa$ to zero and we have $\Delta_\bullet = w_\bullet = 0$. By (4.6) the limit $\varkappa \to 0$ of the ratio $\frac{w_\bullet}{\Delta_\bullet}$ is non-zero, leading to the null vector $\left(W_{-1} + \frac{Q}{2}L_{-1}\right)\mathsf{V}_\bullet = 0$. For $Q=0$, this gives (just like the curves do, see (4.4)) the condition $W_{-1}\mathsf{V}_\bullet = 0$, confirming the conjecture that the null vectors are inherited from the $\mathcal{N}=2$ case.

Let us now show the structure of the simple puncture $\mathsf{V}_\bullet$ in more detail, again taking the $\mathbf{W}_3$ algebra case for simplicity. For further simplicity, we set $Q=0$ so that the null vector is $W_{-1}\mathsf{V}_\bullet$. The structure of the first three levels of the representation is depicted in figure 6. It is important to remark that the structure shown in figure 6 holds only for $c=2$, i.e. for $Q=0$. Otherwise, there are generators that act on the states like $W_{-1}^2\mathsf{V}_\bullet$, that have to be set to zero, but don't give zero, meaning that the quotient is only well defined if $c=2$, i.e. for $Q=0$. This is to be expected, since the null vector for $Q\neq 0$ is $\left(W_{-1} + \frac{Q}{2}L_{-1}\right)\mathsf{V}_\bullet$. We remark that, unlike for generic $\mathbf{W}_N$ Verma modules, the action of the $W_{s,0}$ modes with $s>2$ on the simple punctures will not be diagonalizable.

The full punctures. For $k>1$ and $Q=0$, the curve coefficients (2.5) imply that some of the charges of the full punctures $\mathsf{V}_\odot$ become zero as well. Specifically, only the $w_{k\ell}$ with $\ell=1,\dots,N$ are non-zero. For $k>2$, this implies that for $Q=0$ the conformal dimension of the full punctures vanishes, i.e. $\Delta_\odot = 0$. However, we do not want the full punctures to become the identity field and hence, as for the simple punctures, we require that $L_{-1}\mathsf{V}_\odot \neq 0$. Thus, they generically correspond to non-unitary representations as well, only without null-states. The main properties of the punctures are summarized for the reader's convenience in table 2.

We wish to finish this section with a remark. In the Toda theory, the primary fields, both those corresponding to the full punctures as well as those corresponding to the simple

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ones are obtained as $\mathsf{V} = e^{(\boldsymbol{\alpha},\boldsymbol{\varphi})}$ for some appropriate $\boldsymbol{\alpha}$. In the CFTs that ought to be dual to the $\mathcal{N}=1$ class $\mathcal{S}_k$ theories, this is still true for the full punctures, *but cannot be true for the simple ones* since for them the exponent is mapped by $\pi_{N,k}$ to zero and $e^0 = \mathbf{I}$ is the identity field. It is unclear whether it is possible to write the simple punctures by using the Toda fields $\boldsymbol{\varphi}$ at all.

4.2 The 3-point blocks with one simple puncture

Let us now take the general considerations of the previous subsections and use them to compute the 3-point W-blocks. We perform the computations in the limit $Q \to 0$ that is needed for the comparison with the curves. Let us denote by $\widehat{\mathsf{V}}_{\mathbf{w}}$ an arbitrary descendant of the primary $\mathsf{V}_{\mathbf{w}}$. We compute using standard CFT techniques the recursion relations (each contour integral comes equipped with a factor of $\frac{1}{2\pi i}$ that we omit)

$$\begin{aligned}\Big\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)(W_{s,-n}\widehat{\mathsf{V}}_{\mathbf{w}})(0)\Big\rangle &= \oint_0 \frac{dz}{z^{n-s+1}}\Big\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)W_s(z)\widehat{\mathsf{V}}_{\mathbf{w}}(0)\Big\rangle \\ &= -\sum_{k=-\infty}^{\infty}\oint_1 \frac{dz}{z^{n-s+1}(z-1)^{k+s}}\Big\langle \mathsf{V}_1(\infty)(W_{s,k}\mathsf{V}_2)(1)\widehat{\mathsf{V}}_{\mathbf{w}}(0)\Big\rangle \\ &\quad + (-1)^s\sum_{k=-\infty}^{\infty}\oint_\infty dz\frac{z^{k-s}}{z^{n-s+1}}\Big\langle (W_{s,k}\mathsf{V}_1)(\infty)\mathsf{V}_2(1)\widehat{\mathsf{V}}_{\mathbf{w}}(0)\Big\rangle\,,\end{aligned} \tag{4.7}$$

where in the last line we have used (for a primary field) the relation $W_s(z^{-1}) = (-z^{-2})^s W_s(z)$ and also the fact that the contour had to be oriented the other way. Computing the residues, we find for $n \geq 0$

$$\begin{aligned}\Big\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)(W_{s,-n}\widehat{\mathsf{V}}_{\mathbf{w}})(0)\Big\rangle =& (-1)^s\Big\langle (W_{s,n}\mathsf{V}_1)(\infty)\mathsf{V}_2(1)\widehat{\mathsf{V}}_{\mathbf{w}}(0)\Big\rangle \\ &- \Big\langle \mathsf{V}_1(\infty)(W_{s,-s+1}\mathsf{V}_2)(1)\widehat{\mathsf{V}}_{\mathbf{w}}(0)\Big\rangle \\ &- \binom{-n+s-1}{s-1}\Big\langle \mathsf{V}_1(\infty)(W_{s,0}\mathsf{V}_2)(1)\widehat{\mathsf{V}}_{\mathbf{w}}(0)\Big\rangle\,,\end{aligned} \tag{4.8}$$

where we have used (4.4) following from the fact that V_2 is a simple puncture. At this point, there is a distinction between the case $k=1$ (i.e. for $\mathcal{N}=2$ gauge theories) in which $W_{s,-n}\mathsf{V}_2$ can be expressed through the $L_{-m}\mathsf{V}_2$ and the case $k>1$ (i.e. $\mathcal{N}=1$ gauge theories) in which $W_{s,-n}\mathsf{V}_2 = 0$. We only consider the latter case here and write the recursion relations for $k=1$ and $N=2,3$ in appendix D. Plugging $n=0$ in (4.8), we find the relation

$$\begin{aligned}\Big\langle \mathsf{V}_1(\infty)(W_{s,-s+1}\mathsf{V}_2)(1)\widehat{\mathsf{V}}_{\mathbf{w}}(0)\Big\rangle =& \big((-1)^s w_{s;1} - w_{s;2}\big)\Big\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)\widehat{\mathsf{V}}_{\mathbf{w}}(0)\Big\rangle \\ &- \Big\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)(W_{s,0}\widehat{\mathsf{V}}_{\mathbf{w}})(0)\Big\rangle\,.\end{aligned} \tag{4.9}$$

In the above, we denote by $w_{s;i}$ the charge of W_s when acting on the primary V_i. Remark that the action of $W_{s,0}$ on descendant states does not need to be diagonal, unlike the action of L_0. Plugging (4.9) into (4.8), we obtain for $n>1$ the expression

$$\begin{aligned}\left\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)(W_{s,-n}\widehat{\mathsf{V}}_{\mathbf{w}})(0)\right\rangle =& \left\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)(W_{s,0}\widehat{\mathsf{V}}_{\mathbf{w}})(0)\right\rangle \\ &+\left[\left(1-\binom{-n+s-1}{s-1}\right)w_{s;2}-(-1)^s w_{s;1}\right] \\ &\times\left\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)\widehat{\mathsf{V}}_{\mathbf{w}}(0)\right\rangle .\end{aligned} \tag{4.10}$$

For the computation of 4-point blocks, we also need the recursion relations for the $\bar{\gamma}$ vertices. Using the same tools, we can derive the following relation for $n>1$

$$\begin{aligned}\left\langle W_{s,-n}\widehat{\mathsf{V}}\,|\,\mathsf{V}_3(1)\mathsf{V}_4(0)\right\rangle =& \left\langle W_{s,0}\widehat{\mathsf{V}}\,|\,\mathsf{V}_3(1)\mathsf{V}_4(0)\right\rangle \\ &+\left[\left(\binom{n+s-1}{s-1}-1\right)w_{s;3}-w_{s;4}\right]\left\langle\widehat{\mathsf{V}}\,|\,\mathsf{V}_3(1)\mathsf{V}_4(0)\right\rangle .\end{aligned} \tag{4.11}$$

The action of $W_{s,0}$ on descendant fields needs to be computed using the appropriate **W**-algebra commutation relation, which then together with (4.11) allows us to compute the $\bar{\gamma}$ vertices.

Finally, for two full and one simple puncture (hence with $w_{s;2}=0$), we can use (4.10) and obtain the W-block with insertion of the current

$$\begin{aligned}\boldsymbol{\gamma}_{12\mathbf{w}}(W_s(t);\emptyset) &= \sum_{n=-\infty}^{0} t^{-n-s}\frac{\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)(W_{s;n}\mathsf{V}_{\mathbf{w}})(0)\rangle}{\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)\mathsf{V}_{\mathbf{w}}(0)\rangle} \\ &= t^{-s}w_{s;\mathbf{w}}+\sum_{n=1}^{\infty}(w_{s;\mathbf{w}}-(-1)^s w_{s;1})t^{n-s} = \frac{(-1)^s w_{s;1}t-w_{s;\mathbf{w}}}{t^s(t-1)} .\end{aligned} \tag{4.12}$$

We can immediately compare the above with the curve coefficients[17] $\phi_s^{(3)}$ of (2.8). We see that for $s=k\ell$, we have to have $w_{s;1}=(-1)^{\ell(k+1)}\,\mathfrak{c}_L^{(\ell,k)}$ and $w_{s;\mathbf{w}}=(-1)^{\ell}\,\mathfrak{c}_R^{(\ell,k)}$, while for $s\neq k\ell$ the charges have to vanish. This is in complete agreement with the parametrization (3.18) (we can omit the tilde, since the sum of the left/right masses is zero for $k>1$) of the SU(Nk) theory with the action (4.2) of the projection on the Casimirs.

Hence, we conclude that the 3-point blocks of two full and one simple puncture with insertion of the W_s current do reproduce the curve coefficients of the orbifold gauge theories if one uses the punctures of section 4.1, i.e. the punctures inherited from the SU(Nk) theory that have been acted upon by the projection $\pi_{N,k}$.

Ward identities. We can recover the formula (4.12) also using Ward identities. For a current W_s of spin sl, we have the following Ward identities

$$\begin{aligned}\sum_{i=1}^{n}&\left(\frac{W_{s,0;i}}{(t-z_i)^s}+\frac{W_{s,-1;i}}{(t-z_i)^{s-1}}+\cdots\frac{W_{s,-s+1;i}}{t-z_i}\right) \\ &\times\langle \mathsf{V}_1(z_1)\ldots\mathsf{V}_n(z_n)\rangle = \langle W_s(t)\mathsf{V}_1(z_1)\ldots\mathsf{V}_n(z_n)\rangle ,\end{aligned} \tag{4.13}$$

[17] Remember that for $k>1$, we cannot do a shift in x, $\phi_1=0$ and hence there is no difference between ϕ_s and $\tilde{\phi}_s$.

where $W_{s,-m;i}$ is the mode $W_{s,m}$ acting on the i^{th} field. Since we demand that $W_s(t)$ goes like t^{-2s} at infinity, multiplying (4.13) with t^j with $j = 0, \ldots, 2s-2$ and doing a contour integral around the insertion points of all the primary fields gives us $2s-1$ global Ward identities. We note that the $W_{s,0;i}$ act diagonally on the vertex operators, i.e. they just give the charges $w_{s;i}$. Let us summarize the counting of unknowns and constraints:

1. We have $2s-1$ independent Ward identities for an n-point function. The number is the same for any n.

2. For an n-point function, we have $n(s-1)$ unknowns that we need to determine in order to compute the ratio $\langle\, W(t) \cdots \rangle \,/\, \langle \cdots \rangle$ from (4.13). Each unknown corresponds to an insertion of a lowering operator $W_{s,-m}$ at the point z_i in the correlation function, where $i \in \{1, 2, \ldots, n\}$ and $m = 1, \ldots, s-1$.

3. Since for the n-point function will have $n-2$ simple punctures, this gives through (4.4) exactly $(n-2)(s-2)$ conditions.

In total, for an n-point function, we are left with

$$n(s-1) - (2s-1) - (n-2)(s-2) = n-3 \tag{4.14}$$

unknowns. Thus, for $n = 3$, we can compute the weighted correlation function with an insertion of the current just by using the Ward identities. We just need to insert the solutions for the unknowns in (4.13). Doing so, we obtain the same result as (4.12):

$$\frac{\langle\, \mathsf{V}_1(\infty)\mathsf{V}_2(1)W_s(t)\mathsf{V}_{\mathbf{w}}(0)\,\rangle}{\langle\, \mathsf{V}_1(\infty)\mathsf{V}_2(1)\mathsf{V}_{\mathbf{w}}(0)\,\rangle} = \boldsymbol{\gamma}_{12\mathbf{w}}(W_s(t); \emptyset)\,. \tag{4.15}$$

Thus, the comparison between the free trinion curve and the CFT data is trivial - it follows only from the assumptions for the full/simple punctures, their charges and the existence of the currents of appropriate spin. The appropriate form of the algebra becomes noticeable only at four points.

4.3 Four point blocks and the instanton partition functions

Having seen that the proposal we introduced at the beginning of the current section for the relationship between the CFT blocks and the orbifold $\mathcal{S}_k$ curves works wonderfully for the case of three points, we now want to turn to the 4-point blocks.

In the present section, we shall check our proposal by computing $\langle\langle\, T(t)\,\rangle\rangle_4 \equiv \langle\langle\, W_2(t)\,\rangle\rangle_4$ for quadratic order in q and $\langle\langle\, W(t)\,\rangle\rangle_4 \equiv \langle\langle\, W_3(t)\,\rangle\rangle_4$ to linear order in q for $k \geq 2$ and comparing to the curves.

4.3.1 The four point blocks

In this section, we use (3.37) to compute $\langle\langle\, W_s(t)\,\rangle\rangle_4$. The relevant γ and $\bar{\gamma}$ vertices are given either in the previous subsection 4.2 or in appendix D.

The stress-energy tensor. Let us consider first the case of the spin two current and compute $\langle\langle\, T(t)\,\rangle\rangle_4$ for the theories with $k \geq 2$. For $k = 2$, we can simply take the general computation (D.14) done in the appendix and set (use (3.18), $\boldsymbol{\alpha}_\bullet = 0$ and (4.2))

$$\begin{aligned}
\Delta_1 &= -\,\mathfrak{c}_L^{(1,2)} = -\sum_{i=1}^N m_{L,\,i}^2\,, & \Delta_2 &= \Delta_3 = 0\,,\\
\Delta_4 &= -\,\mathfrak{c}_R^{(1,2)} = -\sum_{i=1}^N m_{R,\,i}^2\,, & \Delta &= -\,\mathfrak{a}^{(1,2)} = -\sum_{i=1}^N a_i^2\,.
\end{aligned} \tag{4.16}$$

Plugging this in (D.14), we get the cumbersome expression for $\langle\langle\, T(t)\,\rangle\rangle_4$ up to quadratic order in q

$$\begin{aligned}
\langle\langle\, T(t)\,\rangle\rangle_4 =& \frac{\mathfrak{a}^{(1,2)} - t\,\mathfrak{c}_L^{(1,2)}}{(t-1)t^2} - q\frac{(\mathfrak{a}^{(1,2)} - \mathfrak{c}_R^{(1,2)})((t-2)\,\mathfrak{a}^{(1,2)} + t\,\mathfrak{c}_L^{(1,2)})}{2(t-1)t^3\,\mathfrak{a}^{(1,2)}}\\
&- q^2\frac{\mathfrak{a}^{(1,2)} - \mathfrak{c}_R^{(1,2)}}{(t-1)t^4(2\,\mathfrak{a}^{(1,2)})^2\left(c(1-2\,\mathfrak{a}^{(1,2)}) + 2\,\mathfrak{a}^{(1,2)}(8\,\mathfrak{a}^{(1,2)}+5)\right)}\\
&\times\Bigg\{c\Big[-(\mathfrak{a}^{(1,2)})^2\left(4t\,\mathfrak{c}_L^{(1,2)} + t^2\,\mathfrak{c}_R^{(1,2)} - 2t + 4\right)\\
&\quad + t\,\mathfrak{a}^{(1,2)}\,\mathfrak{c}_L^{(1,2)}(t\,\mathfrak{c}_L^{(1,2)}+2) - \left(t^2+4t-8\right)(\mathfrak{a}^{(1,2)})^3 + t^2(\mathfrak{c}_L^{(1,2)})^2\,\mathfrak{c}_R^{(1,2)}\Big]\\
&\quad + 2\,\mathfrak{a}^{(1,2)}\Big[(\mathfrak{a}^{(1,2)})^2\left(t^2(-2\,\mathfrak{c}_L^{(1,2)} + \mathfrak{c}_R^{(1,2)} + 2) + 2t(8\,\mathfrak{c}_L^{(1,2)}+5) - 20\right)\\
&\quad - t\,\mathfrak{a}^{(1,2)}\,\mathfrak{c}_L^{(1,2)}(t(\mathfrak{c}_L^{(1,2)} + 6\,\mathfrak{c}_R^{(1,2)} + 2) - 10) + \left(3t^2+16t-32\right)(\mathfrak{a}^{(1,2)})^3\\
&\quad + 5t^2(\mathfrak{c}_L^{(1,2)})^2\,\mathfrak{c}_R^{(1,2)}\Big]\Bigg\} + \mathcal{O}\left(q^3\right)\,,
\end{aligned} \tag{4.17}$$

where $c = 2N - 1$ is the central charge of the SU$(2N)$ theory for $Q = 0$. Comparing with $\phi_2^{(4)}(t)$ (for $k = 2$ and N general) of (2.5), we get a perfect agreement if the Coulomb modulus $u_2(q)$ takes the form

$$u_2(q) = \mathfrak{a}^{(1,2)} + \frac{q}{2}\left[\frac{\mathfrak{c}_L^{(1,2)}\,\mathfrak{c}_R^{(1,2)}}{\mathfrak{a}^{(1,2)}} + (\mathfrak{c}_L^{(1,2)} + \mathfrak{c}_R^{(1,2)}) - \mathfrak{a}^{(1,2)}\right] + \mathcal{O}(q^2)\,. \tag{4.18}$$

Compare this result for $u_2(q)$ with the $k = 1$ case of (3.46), while keeping the action (4.2) in mind. In the above calculation, we computed $\langle\langle\, T(t)\,\rangle\rangle_4$ by doing the computation in the SU$(2N)$ theory and then projecting using $\pi_{N,2}$. Alternatively, we can straightforwardly use the tools of the previous subsection 4.2 and obtain the same result.

Since our proposal reproduces the curves, we are given hope that the blocks would give the $\mathcal{S}_k$ instanton partition functions, even for $Q \neq 0$. In particular, for $N = 1$, the full algebra of the theory is $\mathbf{W}_2$ and hence (D.15) gives the full 4-point block. To first order in q, this reads

$$\mathcal{B}_\Delta(\Delta_1, \Delta_2, \Delta_3, \Delta_4|q) = 1 - q\frac{2(a^2 - M_L^2)(a^2 - M_R^2)}{4a^2 - Q^2} + \mathcal{O}(q^2)\,, \tag{4.19}$$

since for $N = 1$ we have $\Delta = -a^2 + \frac{Q^2}{4}$, $\Delta_1 = -M_L^2 + \frac{Q^2}{4}$ and $\Delta_4 = -M_R^2 + \frac{Q^2}{4}$, compare with table 2.

Computing $\langle\langle\, T(t)\,\rangle\rangle_4$ in the case $k > 2$ is slightly trickier since for $Q = 0$, the conformal dimension Δ of the exchanged operator vanishes and one would need to divide by zero to compute the blocks. Hence, the correct approach is to perform the computation for $Q \neq 0$ such that $\Delta = \frac{Nk((Nk)^2-1)}{24} Q^2$ (see table 2) and to then take the limit $Q \to 0$. This computation is well defined and it is straightforward to then check that $\lim_{Q\to 0} \langle\langle\, T(t)\,\rangle\rangle_4 = 0 = \phi_2^{(4)}(t)$, in agreement with (2.5).

The spin three current. The case of the W_3 current is straightforward too. For $k = 3$ and N general, the recursion relations of section 4.2 give us after some straightforward computations

$$\begin{aligned} \boldsymbol{\gamma}_{12\mathbf{w}}(W(t);\{\emptyset,\emptyset\}) &= \frac{-w_1 t - w_{\mathbf{w}}}{(t-1)t^3}\,, \qquad \boldsymbol{\gamma}_{12\mathbf{w}}(W(t);\{\{1\},\emptyset\}) = \frac{(t-3)w_{\mathbf{w}} - 2tw_1}{(t-1)t^4}\,, \\ \boldsymbol{\gamma}_{12\mathbf{w}}(W(t);\{\emptyset,\{1\}\}) &= -\frac{(w_{\mathbf{w}}+w_1)(tw_1+w_{\mathbf{w}})}{(t-1)t^3}\,. \end{aligned} \tag{4.20}$$

Combined with $\bar{\gamma}_{12\mathbf{w}}(\{\{1\};\emptyset\}) = 0$, $\bar{\gamma}_{12\mathbf{w}}(\{\emptyset;\{1\}\}) = w_{\mathbf{w}} - w_4$ and (B.2) with $\Delta_{\mathbf{w}} = \Delta_i = 0$, we can calculate $\langle\langle\, W(t)\,\rangle\rangle_4$ to linear order in q. Since $w_1 = \mathfrak{c}_L^{(1,3)}$, $w_{\mathbf{w}} = -\,\mathfrak{a}^{(1,3)}$ and $w_4 = -\,\mathfrak{c}_R^{(1,3)}$, we find

$$\begin{aligned} \langle\langle\, W(t)\,\rangle\rangle_4 = \frac{1}{1+0\cdot q}\Bigg[&\frac{-\,\mathfrak{c}_L^{(1,3)}\,t + \mathfrak{a}^{(1,3)}}{(t-1)t^3} \\ &+ q\frac{1}{-3\,\mathfrak{a}^{(1,3)}}\frac{(3-t)\,\mathfrak{a}^{(1,3)} - 2t\,\mathfrak{c}_L^{(1,3)}}{(t-1)t^4}\left(-\,\mathfrak{a}^{(1,3)} + \mathfrak{c}_R^{(1,3)}\right)\Bigg] + \mathcal{O}(q^2)\,. \end{aligned} \tag{4.21}$$

The above agrees perfectly with the curve coefficient $\phi_3^{(4)}(t)$ in (2.5) for $k = 3$ if we set the Coulomb modulus to the value

$$u_3(q) = \mathfrak{a}^{(1,3)} + \frac{q}{3}\left[\frac{2\,\mathfrak{c}_L^{(1,3)}\,\mathfrak{c}_R^{(1,3)}}{\mathfrak{a}^{(1,3)}} + \left(\mathfrak{c}_L^{(1,3)} + \mathfrak{c}_R^{(1,3)}\right) - \mathfrak{a}^{(1,3)}\right] + \mathcal{O}(q^2)\,. \tag{4.22}$$

Hence, our proposal agrees with the first non-trivial $\mathcal{S}_3$ curve coefficient.

We can also compute (for $N = 1$ and $k = 3$) the 4-point block $\mathcal{B}$ for general Q. The non-trivial $\mathbf{W}_3$ charges are $\mathbf{w}_1 = \{Q^2, M_L^3\}$, $\mathbf{w}_4 = \{Q^2, -M_R^3\}$ and $\mathbf{w} = \{Q^2, -a^3\}$ for the intermediate state. From (B.2), we find after putting $c = 2(1 + 12Q^2)$ for the first level Shapovalov form

$$\mathsf{Q}_{\mathbf{w}}^{(1)} = \begin{pmatrix} 2Q^2 & -3a^3 \\ -3a^3 & -\frac{Q^4}{6} \end{pmatrix}\,. \tag{4.23}$$

Since $\gamma_{12\mathbf{w}}(\{\{1\};\emptyset\}) = \Delta + \Delta_2 - \Delta_1 = Q^2 - Q^2 = 0$ and (see (D.19)) $\gamma_{12\mathbf{w}}(\{\emptyset;\{1\}\}) = w_1 + w_{\mathbf{w}} = -a^3 + M_L^3$. Similarly, see (4.11), $\bar{\gamma}_{12\mathbf{w}}(\{\{1\};\emptyset\}) = 0$ and $\bar{\gamma}_{12\mathbf{w}}(\{\emptyset;\{1\}\}) = -a^3 + M_R^3$.

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Hence, inverting (4.23), we find that the $\mathbf{W}_3$ block up to level 1 is

$$\begin{aligned}\mathcal{B}_{\mathbf{w}}(\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3,\mathbf{w}_4|q) &= 1+q\left(-\frac{6Q^2}{27a^6+Q^6}\right)(-a^3+M_L^3)(-a^3+M_R^3)+\mathcal{O}(q^2)\\ &= 1-q\frac{6Q^2(a^3-M_L^3)(a^3-M_R^3)}{27a^6+Q^6}+\mathcal{O}(q^2)\,.\end{aligned}\tag{4.24}$$

In addition to the computations for $\langle\langle\, W_2\,\rangle\rangle$ and $\langle\langle\, W_3\,\rangle\rangle$ that we have shown here, we have performed additional checks - for $\langle\langle\, W_4\,\rangle\rangle$ and for higher orders in q.

4.3.2 The instanton partition function of the orbifold theories

Having checked in the previous subsection that our proposal reproduces the curves, we now want to investigate the instanton partition functions. Since the AGT correspondence holds in $\mathcal{N}=2$ case, it is trivial that the correspondence between the four-point blocks $\mathcal{B}$ of section 4.3.1 will agree with the Nekrasov partition functions projected with $\pi_{N,k}$. Still, it is worth looking at the way the projection $\pi_{N,k}$ acts to see what we can learn from it about the class $\mathcal{S}_k$ theories.

The image of the Nekrasov instanton partition function $\mathcal{Z}^{(Nk,1)}_{\text{inst}}$ of the SU(Nk) $\mathcal{N}=2$ SCQCD (3.31) under the map $\pi_{N,k}$ can be easily obtained. We can use $\prod_{r=0}^{k-1}\big(a-m\,\mathrm{e}^{\frac{2\pi i}{k}r}\big)=a^k-m^k$ to write

$$\begin{aligned}\mathcal{Z}^{(N,k)}_{\text{inst}} &= \pi_{N,k}(\mathcal{Z}^{(Nk,1)}_{\text{inst}}) \stackrel{\text{def}}{=} \sum_{\mathbf{Y}=\{Y_1,\dots,Y_{Nk}\}} q^{|\mathbf{Y}|}\tilde{z}^{(N,k)}_{\text{inst}}(\mathbf{Y})\\ &= \sum_{\mathbf{Y}=\{Y_1,\dots,Y_{Nk}\}} q^{|\mathbf{Y}|}\prod_{u=1}^{N}\prod_{i=1}^{N}\prod_{r=0}^{k-1}\prod_{(\mu,\nu)\in Y_{i+Nr}}\left[\left(\epsilon-a_i\,\mathrm{e}^{\frac{2\pi i}{k}r}-\epsilon_1\mu-\epsilon_2\nu\right)^k-m^k_{L,u}\right]\\ &\quad\times\prod_{u=1}^{N}\prod_{i=1}^{N}\prod_{r=0}^{k-1}\prod_{(\mu,\nu)\in Y_{i+Nr}}\left[\left(a_i\,\mathrm{e}^{\frac{2\pi i}{k}r}+\epsilon_1\mu+\epsilon_2\nu\right)^k-m^k_{R,u}\right]\\ &\quad\times\Bigg\{\prod_{i,j=1}^{N}\prod_{r,s=0}^{k-1}\prod_{(\mu,\nu)\in Y_{i+Nr}}\Big[a_i\,\mathrm{e}^{\frac{2\pi i}{k}r}-a_j\,\mathrm{e}^{\frac{2\pi i}{k}s}-\epsilon_1 L_{Y_{j+Ns}}(\mu,\nu)\\ &\qquad\qquad+\epsilon_2\left(A_{Y_{i+Nr}}(\mu,\nu)+1\right)\Big]\\ &\quad\times\prod_{(\mu',\nu')\in Y_{j+Ns}}\Big[\epsilon+a_i\,\mathrm{e}^{\frac{2\pi i}{k}r}-a_j\,\mathrm{e}^{\frac{2\pi i}{k}s}+\epsilon_1 L_{Y_{i+Nr}}(\mu',\nu')\\ &\qquad\qquad-\epsilon_2\left(A_{Y_{j+Ns}}(\mu',\nu')+1\right)\Big]\Bigg\}^{-1}.\end{aligned}\tag{4.25}$$

The resulting sum is still full of phases which lead to many cancellations when the sums over the partitions are performed. It is useful to split the sum over the partitions $\mathbf{Y}$ into orbits of the orbifold group $\mathbb{Z}_N$, where the action of that group on $\mathbf{Y}$ is defined via the

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elementary cyclic shift

$$\begin{aligned}&\{Y_1,\dots,Y_N,Y_{N+1},\dots,Y_{2N},\dots,Y_{(k-1)N+1},\dots,Y_{kN}\}\longmapsto\\&\qquad\longmapsto\{Y_{(k-1)N+1},\dots,Y_{kN},Y_1,\dots,Y_N,\dots,Y_{(k-2)N+1},\dots,Y_{(k-1)N}\}\,.\end{aligned}\tag{4.26}$$

Thus, we can rewrite the instanton partition function with the summands expressed as sums over the cyclic permutations:

$$\mathcal{Z}^{(N,k)}_{\text{inst}}=\sum_{\mathbf{Y}=\{Y_1,\dots,Y_{Nk}\}}q^{|\mathbf{Y}|}\tilde{z}^{(N,k)}_{\text{inst}}(\mathbf{Y})=\sum_{[\mathbf{Y}]\in\{Y_1,\dots,Y_{Nk}\}/\mathbb{Z}_k}q^{|\mathbf{Y}|}\underbrace{\sum_{\sigma\in\mathbb{Z}_k}\tilde{z}^{(N,k)}_{\text{inst}}(\sigma\cdot\mathbf{Y})}_{\stackrel{\text{def}}{=}z^{(N,k)}_{\text{inst}}([\mathbf{Y}])}\,.\tag{4.27}$$

It seems quite non-trivial to obtain closed analytic expressions for the $z^{(N,k)}_{\text{inst}}([\mathbf{Y}])$ for general N, k and equivalence class $[\mathbf{Y}]$. For the simplest case of $N=1$ and k general, one finds for the first non-trivial equivalence class $[\{\{1\},\emptyset,\dots,\emptyset\}]$ the result

$$z^{(1,k)}_{\text{inst}}([\{\{1\},\emptyset,\dots,\emptyset\}])=-\frac{\epsilon\left(a^k-M_L^k\right)}{\epsilon_1\epsilon_2ka^{k-1}}\sum_{s=0}^{k-1}\mathrm{e}^{\frac{2\pi i}{k}s}\,\frac{\left(\epsilon+a\,\mathrm{e}^{\frac{2\pi i}{k}s}\right)^k-M_R^k}{\left(\epsilon+a\,\mathrm{e}^{\frac{2\pi i}{k}s}\right)^k-a^k}\,.\tag{4.28}$$

The first few cases of $z^{(1,k)}_{\text{inst}}\equiv z^{(1,k)}_{\text{inst}}\left([\{\{1\},\emptyset,\dots,\emptyset\}]\right)$ with $k>1$ can be simplified to

$$\begin{aligned}z^{(1,2)}_{\text{inst}}&=-\frac{2\left(a^2-M_L^2\right)\left(a^2-M_R^2\right)}{\epsilon_1\epsilon_2(4a^2-\epsilon^2)}\,,\\z^{(1,3)}_{\text{inst}}&=-\frac{6\epsilon^2\left(a^3-M_L^3\right)\left(a^3-M_R^3\right)}{\epsilon_1\epsilon_2(27a^6+\epsilon^6)}\,,\\z^{(1,4)}_{\text{inst}}&=\frac{20\epsilon^2\left(a^4-M_L^4\right)\left(a^4-M_R^4\right)}{\epsilon_1\epsilon_2(-64a^8-12a^4\epsilon^4+\epsilon^8)}\,,\end{aligned}\tag{4.29}$$

$$z^{(1,5)}_{\text{inst}}=-\frac{10\epsilon^2\left(125a^{10}+7\epsilon^{10}\right)\left(a^5-M_L^5\right)\left(a^5-M_R^5\right)}{\epsilon_1\epsilon_2(3125a^{20}+625a^{10}\epsilon^{10}+\epsilon^{20})}\,.\tag{4.30}$$

The above clearly agrees with (4.19) and (4.24). We have checked for higher k that for $k>1$ equation (4.28) is equal to $\frac{1}{\epsilon_1\epsilon_2}\frac{P_k(\epsilon,a)}{P'_k(\epsilon,a)}\left(a^k-M_L^k\right)\left(a^k-M_R^k\right)$, where P_k and P'_k are homogeneous polynomials in ϵ and a with $\deg P'_k-\deg P_k=2(k-1)$.

In conclusion, we see that the Nekrasov partition function (4.27) does indeed reproduce the CFT blocks with non-unitary fields. It still remains to determine closed formulas for the summands $z^{(N,k)}_{\text{inst}}([\mathbf{Y}])$ that do not depend on the phases introduced by $\pi_{N,k}$.

5 Conclusion and outlook

In this article, we showed that the Seiberg-Witten curves of the SU(N) class $\mathcal{S}_k$ gauge theories derived in [28] can be obtained from the weighted current correlation functions $\langle\langle\, W_s(t)\,\rangle\rangle$ of the $\mathbf{W}_{Nk}$ algebra once the mass parameters of the SU(Nk) theory have been properly identified under the $\mathbb{Z}_k$ orbifold condition. To do this, we first found the quantum numbers of the vertex operators $\mathsf{V}_{\odot}$ and $\mathsf{V}_{\bullet}$ of the full and the simple punctures respectively, and observed that in general the punctures correspond to non-unitary representations of $\mathbf{W}_{Nk}$. We then argued that the null vectors of the simple punctures are inherited from the SU(Nk) and spelled out consequences of our conjecture by computing $\langle\langle W_s(t)\rangle\rangle_n$ for $s = 2, 3$ and both $n = 3$ and $n = 4$ points and comparing with the meromorphic differentials of the Seiberg-Witten curve. We furthermore conjectured that the SU(Nk) Nekrasov instanton partition functions with the orbifold values of the masses and the Coulomb branch parameters (4.25) give the instanton contributions of the SU(N) class $\mathcal{S}_k$ gauge theories. Moreover, it is natural to further conjecture that the algebra, the blocks and the instanton partition functions of any theory in class $\mathcal{S}_\Gamma$ is also obtained in this way, with the masses and the Coulomb branch parameters identified under the $\Gamma\in$ADE orbifold condition.

It seems natural to think that the full extend of the AGT correspondence applies to the class $\mathcal{S}_\Gamma$ gauge theories. A necessary first step involves the computation of the full 3-point functions of two full and one simple puncture, which can then be used through a block decomposition à la (3.24) to compute the full 4-point CFT correlation function. This correlation function should correspond to the S^4 partition function of the SU(N) class $\mathcal{S}_k$ theories. For the 3-point functions of two full punctures and one simple one, the appropriate 4D theory is a free one, namely the orbifold of the free trinion:

$$\mathcal{Z}^{S^4}_{\text{free trinion}} = \left\langle \mathsf{V}_{\odot}(\infty)\mathsf{V}_{\bullet}(1)\mathsf{V}_{\odot}(0) \right\rangle . \tag{5.1}$$

Since we are dealing with a free theory, the S^4 partition function can be straightforwardly computed by counting the eigenvalues of Dirac and Laplace operators. This is work in progress [46]. Once these 3-point correlation functions have been computed, one also needs to check that the 4-point function satisfies the CFT crossing relations.

For $\mathcal{N} = 2$ gauge theories in 4D, the S^4 partition function is not scheme independent [47] and the scheme dependence is understood as transformations of the Kähler potential of the conformal manifold. For theories with only $\mathcal{N} = 1$ supersymmetry, the ability to control this ambiguity is lost[18] [47]. However, for theories in the class $\mathcal{S}_\Gamma$ at the orbifold point we expect that to not be the case. Our expectations stem from the AdS/CFT correspondence, the inheritance arguments of [50, 51] and our large experience from the study of $\mathcal{N} = 2$ orbifold daughters of $\mathcal{N} = 4$ SYM [52–58]. When all the coupling constants are equal to each other (i.e. at the orbifold point), certain observables in the untwisted sector are equal to the $\mathcal{N} = 4$ ones. Since, the theories in class $\mathcal{S}_k$ are also orbifolds of $\mathcal{N} = 4$ SYM, the inheritance arguments apply to them. In addition, they are by definition orbifolds of the $\mathcal{N} = 2$ class $\mathcal{S}$ theories and we are studying the case with all the coupling constants equal. Hence, we expect certain observables to be equal to the corresponding $\mathcal{N} = 2$ ones as well and conjecture that the partition function on S^4 is well defined.

Our results so far suggest, with a bit of optimism, that for any supersymmetric theory with a Lagrangian description and an *abelian Coulomb phase*, we should be able to *guess*

[18] Despite these ambiguities, the partition functions still contain well defined physical information. For example, certain derivatives of the free energy are scheme independent [48, 49].

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the dual 2D CFT, just by knowing: 1) *the Seiberg-Witten curve* from which one extracts the symmetry algebra, the representations and then the instanton partition functions and 2) *the free trinion partition function*. Once these two are known, it should be possible to compute the complete 3-point functions and to check that the 4-point function satisfies the crossing equations.

Beyond this point, there are still many questions left open. Some of them concern exploring the nature of the CFTs dual to the $\mathcal{N} = 1$ class $\mathcal{S}_k$ theories and, in particular, their marginal deformations. In a work in progress [59], the SW curves away from the orbifold point are investigated. It would be very important to find the 2D CFT operation that is dual to adding a marginal deformation to the orbifold point Lagrangian.

In addition, it would be instructive to try to repeat for the $\mathcal{N} = 1$ theories of class $\mathcal{S}_\Gamma$ the strategy of [60], who starting from the (2,0) theory in 6D where able to obtain a direct derivation of the AGT correspondence. In particular, it would be interesting to see what is the orbifolded version of the intermediate complex Chern-Simons theory in this approach.

Since we conjectured in section 4 that the instanton partition functions of the class $\mathcal{S}_k$ theories are obtained from the $\mathcal{N} = 2$ ones after specializing the parameters, it would be very important to compute these instanton contributions from first principles following [61]. Alternatively, one could try to adapt Nekrasov localization techniques [62, 63] and especially their most modern incarnation [64]. The comparison of these direct instanton computations with our conjecture would allow one to fix the 2D CFT central charge.

In this article, we studied the effect of performing a $\mathbb{Z}_k$ orbifold on the transverse directions of the M5 branes that breaks the supersymmetry of the gauge theory down to $\mathcal{N} = 1$. This should be distinguished from quotienting out a $\mathbb{Z}_r$ on space time directions and considering the $\mathcal{N} = 2$ theory on $\mathbb{R}^4/\mathbb{Z}_r$. In the latter case, the dual CFT is a coset model (parafermionic Toda CFTs) and the correspondence has been studied in [65–71] among others. It would be interesting to do both quotients, i.e. to investigate the AGT correspondence for the class $\mathcal{S}_k$ theories on $\mathbb{R}^4/\mathbb{Z}_r$.

One is also interested in more general correlation and partition functions. For the $\mathcal{N} = 2$ theories, the free trinion partition function only gives the 3-point correlation functions (i.e. the 3-point structure constants) with one simple puncture, which is a semi-degenerate field. In order to compute the correlation functions of three generic fields, dual to the partition function of the full trinion T_N, we used the refined topological string vertex in [37, 72, 73]. It would be important to develop the refined topological vertex for D-brane configurations subjected to the orbifold identification (2.2), for it would give us a path towards the 3-point correlation functions of arbitrary primary fields.

Another potential direction of investigation concerns supersymmetric line and surface operators/defects. It would be important to classify them for the class $\mathcal{S}_\Gamma$ gauge theories and to understand precisely how they are realized in the 2D CFT side, following closely the work of [74] for the $\mathcal{N} = 2$ case. See also the more recent reviews [75, 76] and references therein. It seems very possible that the results of the present paper will immediately apply. Furthermore, it would be important to make contact with the recent works of [77–79] based on the superconformal index.

Lastly, we would like to state that the existence of a dual CFT whose correlation functions reproduce the partition functions gives one hope that a generalization of Pestun's localization to some $\mathcal{N} = 1$ theories on $\mathbb{S}^4$ or the ellipsoid should be possible. This is currently being researched [80].

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A Summation identities

The Casimirs are defined as (We write $\mathfrak{c}^{(s)} \equiv \mathfrak{c}^{(s,1)}$)

$$\mathfrak{c}^{(s,k)} = \sum_{i_1<\cdots<i_s=1}^{N} m_{i_1}^k \cdots m_{i_s}^k \,, \qquad \mathfrak{c}^{(0,k)} = 1 \,. \tag{A.1}$$

For $k = 1$, they obey the important identity allowing to express the Casimirs of SU(N) in terms of the U(N) ones:

$$\sum_{j=0}^{i} \frac{(-1)^{i-j}}{N^{i-j}} \binom{N-j}{N-i} \mathfrak{c}^{(j)} (\mathfrak{c}^{(1)})^{i-j} = \mathfrak{c}^{(i)}\big|_{m_a \to \tilde{m}_a} \stackrel{\text{def}}{=} \tilde{\mathfrak{c}}^{(i)} \,. \tag{A.2}$$

We remind that $\tilde{m}_a = m_a - \frac{M}{N}$ with $M = \mathfrak{c}^{(1)} = \sum_{a=1}^{N} m_a$. It is clear from the definition that $\tilde{\mathfrak{c}}^{(1)} = 0$.

We have ($\mathbf{car}_{ij}$ is the SU(N) Cartan matrix) the following formulas for contractions involving the Cartan matrix and the fundamental weights

$$\sum_{i_1,i_2=1}^{N-1} (\omega_{i_1}, \omega_{i_2}) \mathbf{car}_{i_1,i_2} = N - 1 \,, \qquad \sum_{i_1,i_2,i_3,i_4=1}^{N-1} (\omega_{i_1}, \omega_{i_2})(\omega_{i_3}, \omega_{i_4}) \mathbf{car}_{i_1,i_3} \mathbf{car}_{i_2,i_4} = N - 1 \,. \tag{A.3}$$

The second identity follows from the first one if we also apply the first of the formulas

$$\sum_{i,j=1}^{N-1} (\boldsymbol{\alpha}, \omega_i)(\boldsymbol{\beta}, \omega_j) \mathbf{car}_{i,j} = (\boldsymbol{\alpha}, \boldsymbol{\beta}) \,, \qquad \sum_{i<j=1}^{N} (\boldsymbol{\alpha}, \mathsf{h}_i)(\boldsymbol{\beta}, \mathsf{h}_j) = -\frac{1}{2}(\boldsymbol{\alpha}, \boldsymbol{\beta}) \,. \tag{A.4}$$

Finally, we have the following summation identity

$$\sum_{(i_1,n_1)<\cdots<(i_\ell,n_\ell)} m_{i_1} e^{\frac{2\pi i n_1}{k}} \cdots m_{i_\ell} e^{\frac{2\pi i n_\ell}{k}} = (-1)^{(k+1)s} \sum_{i_1<\cdots<i_s=1}^{N} m_{i_1}^k \cdots m_{i_s}^k \,, \tag{A.5}$$

if $\ell = ks$ with $s = 0, 1, \ldots$ and is zero otherwise. In the sum, the indices i_j run over $1, \ldots, N$ and n_j over $1, \ldots, k$ with the inequality $(i, h) < (i', n')$ iff $i < i'$ or $i = i'$

and $n < n'$. Equation (A.5) is proven by expanding the left hand side of the identity $\prod_{n=1}^{k}\left(x - e^{\frac{2\pi i n}{k}}\right) = x^k - 1$ in powers of x, which leads to the formula

$$\sum_{n_1<n_2<\cdots<n_l}^{k} e^{\frac{2\pi i}{k}(n_1+\cdots+n_l)} = \begin{cases} 0 & \text{if } l \neq k \\ (-1)^{k+1} & \text{if } l = k \end{cases} . \tag{A.6}$$

It hence follows that in the sum of (A.5) only those terms remain for which the i_j's clump into bunches of size k for which the sum over the n's gives a factor of $(-1)^{k+1}$. This completes the proof of (A.5).

B Shapovalov forms

The Virasoro case. The Shapovalov form for the first 3 levels reads $\mathsf{Q}_{\Delta}^{(0)} = (1)$, $\mathsf{Q}_{\Delta}^{(1)} = (2\Delta)$ as well as

$$\begin{aligned} \mathsf{Q}_{\Delta}^{(2)} &= \begin{pmatrix} \frac{1}{2}(c+8\Delta) & 6\Delta \\ 6\Delta & 4\Delta(2\Delta+1) \end{pmatrix} , \\ \mathsf{Q}_{\Delta}^{(3)} &= \begin{pmatrix} 2(c+3\Delta) & 2(c+8\Delta) & 24\Delta \\ 2(c+8\Delta) & c(\Delta+2)+2\Delta(4\Delta+17) & 36\Delta(\Delta+1) \\ 24\Delta & 36\Delta(\Delta+1) & 24\Delta\left(2\Delta^2+3\Delta+1\right) \end{pmatrix} . \end{aligned} \tag{B.1}$$

The last matrix is wrt. to the basis $\{3\}, \{1,2\}, \{1,1,1\}$, where $\{1,2\}$ stands for $L_{-1}L_{-2}\mathsf{V}_{\Delta}$. We remind that the generators in the algebra are ordered as $L_{-n_1}^{m_1}\cdots L_{-n_s}^{m_s}\mathsf{V}_{\Delta}$ with $n_i < n_{i+1}$.

The $\mathbf{W_3}$ case. For the $\mathbf{W}_3$, using the commutation relations of appendix C, the first non-trivial Shapovalov form reads

$$\mathsf{Q}_{\Delta,w}^{(1)} = \begin{pmatrix} 2\Delta & 3w \\ 3w & \frac{1}{48}(c-32\Delta-2)\Delta \end{pmatrix} , \tag{B.2}$$

in the basis $\{\{1\};\emptyset\} \equiv L_{-1}\mathsf{V}_{\Delta,w}$ and $\{\emptyset;\{1\}\} \equiv W_{-1}\mathsf{V}_{\Delta,w}$. Similarly, in the basis $\{\{2\};\emptyset\}$, $\{\{1,1\};\emptyset\}$, $\{\emptyset;\{2\}\}$ $\{\emptyset;\{1,1\}\}$, $\{\{1\};\{1\}\}$ we find at level 2

$$\mathsf{Q}_{\Delta,w}^{(2)} = \begin{pmatrix} \frac{1}{2}(c+8\Delta) & 6\Delta & 6w & \frac{5}{48}(c-32\Delta-2)\Delta & 9w \\ 6\Delta & 4\Delta(2\Delta+1) & 12w & 18w^2+\frac{1}{8}\Delta(c-32\Delta-2) & 6(2\Delta w+w) \\ 6w & 12w & -\frac{1}{6}\Delta(c+8\Delta+6) & -\frac{1}{8}w(c+48\Delta+14) & \frac{1}{12}(c-32\Delta-2)\Delta \\ \frac{5}{48}(c-32\Delta-2)\Delta & 18w^2+\frac{1}{8}\Delta(c-32\Delta-2) & -\frac{1}{8}w(c+48\Delta+14) & \frac{(c-32\Delta-2)\Delta\left(-64\Delta^2+2(c-34)\Delta+c-34\right)-27648w^2}{2304} & \frac{1}{16}w(c-32\Delta-2)(2\Delta+3) \\ 9w & 6(2\Delta w+w) & \frac{1}{12}(c-32\Delta-2)\Delta & \frac{1}{16}w(c-32\Delta-2)(2\Delta+3) & \frac{1}{24}\left(216w^2+\Delta(\Delta+1)(c-32\Delta-2)\right) \end{pmatrix} . \tag{B.3}$$

C The $\mathbf{W}_3$ algebra

We have $c = 2(1 + 12Q^2)$ and introduce the parameter $\beta = \frac{16}{22+5c} = \frac{2}{4+15Q^2}$. The commutation relations of the modes are

$$
\begin{aligned}
[L_m\,,\,L_n] &= \frac{c}{12}m(m^2-1)\delta_{m+n,0} + (m-n)L_{m+n} \\
[L_m\,,\,W_n] &= (2m-n)W_{m+n} \\
[W_m\,,\,W_n] &= -\frac{1}{3\beta}\frac{c}{3\cdot 5!}m(m^2-1)(m^2-4)\delta_{m+n,0} \\
&\quad -\frac{(m-n)}{3\beta}\left(\frac{(m+n+3)(m+n+2)}{15} - \frac{(m+2)(n+2)}{6}\right)L_{m+n} \\
&\quad -\frac{(m-n)}{3}\Lambda_{m+n}\,,
\end{aligned}
\tag{C.1}
$$

where the spin four field $\Lambda(z) = (TT)(z) - \frac{3}{10}\partial^2 T$ has the mode expansion

$$
\Lambda_m = \sum_{p=-\infty}^{-2} L_p L_{m-p} + \sum_{p=-1}^{\infty} L_{m-p}L_p - \frac{3}{10}(m+2)(m+3)L_m\,. \tag{C.2}
$$

Compared to the commutation relations given in [81], we have rescaled $W \to iW$. The conformal dimension and w charge are given by in terms of SU(3) weights through

$$
\Delta(\boldsymbol{\alpha}) = \frac{(2\mathcal{Q}-\boldsymbol{\alpha},\boldsymbol{\alpha})}{2}\,, \qquad w(\boldsymbol{\alpha}) = -\left(\boldsymbol{\alpha}-\mathcal{Q},\mathsf{h}_1\right)\left(\boldsymbol{\alpha}-\mathcal{Q},\mathsf{h}_2\right)\left(\boldsymbol{\alpha}-\mathcal{Q},\mathsf{h}_3\right)\,. \tag{C.3}
$$

D Blocks computations

In this appendix, we summarize the essentials for the computations of the U(1), $\mathbf{W}_2$ and $\mathbf{W}_3$ 3 and 4-point blocks as well as for the calculations of the blocks with insertions of the currents.

D.1 The U(1) blocks

We can define U(1) blocks in a fashion similar to the $\mathbf{W}$ algebra case. The charge conservation seems built into the system. The current is $J_1(z) = i\partial\lambda$, which has a mode expansion

$$
J_1(z) = \sum_{n=-\infty}^{\infty} z^{-n-1}\mathsf{a}_n\,, \qquad \text{with} \qquad [\mathsf{a}_n\,,\,\mathsf{a}_m] = n\delta_{n+m,0}\,. \tag{D.1}
$$

The modes a_n form the $\hat{\mathfrak{u}}_1$ affine algebra. We create representations by starting with V_p annihilated by all a_n with $n > 0$ that obeys $\mathsf{a}_0\mathsf{V}_p = p\mathsf{V}_p$. We are as generally in this article, denoting the vertex operator and the state it creates by the same symbol. Using the standard rule for the adjoint, we can define a Shapovalov form and find that the norm of the state $\mathsf{a}_{-1}^{n_1}\ldots\mathsf{a}_{-m}^{n_m}\mathsf{V}_p$ is given by $\prod_{j=1}^m n_j! j^{n_j}$. The numbers n_j are related to the Young diagram $Y = \{Y_1,\ldots,Y_s\}$ as follows: the number Y_j is the number of boxes of the j^{th} row (drawn from the bottom upwards) of the Young diagram Y, while n_r is number of rows in Y of exactly r boxes. For example, for $Y = \{1, 1, 2, 4\}$ we have $n_1 = 2$, $n_2 = 1$, $n_3 = 0$ and $n_4 = 1$.

We can compute as usual the recursion relations for the 3-point blocks

$$\begin{aligned} \Big\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)(\mathsf{a}_{-n}\widehat{\mathsf{V}}_p)(0) \Big\rangle &= -(\delta_{n,0}p_1 + p_2) \Big\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)\widehat{\mathsf{V}}_p(0) \Big\rangle , \\ \Big\langle \mathsf{a}_{-n}\widehat{\mathsf{V}}_p \,|\, \mathsf{V}_3(1)\mathsf{V}_4(0) \Big\rangle &= (p_3 + \delta_{n,0}p_4) \Big\langle \widehat{\mathsf{V}}_p \,|\, \mathsf{V}_3(1)\mathsf{V}_4(0) \Big\rangle , \end{aligned} \tag{D.2}$$

where $n \geq 0$. We remark that setting $n = 0$ in the above, we obtain the charge conservation relations $p = -p_1 - p_2$ for the first correlator and $p = p_3 + p_4$ for the second. In general, we find that the 3-point blocks are given by $\gamma_{12p}(\mathsf{a}_{-1}^{n_1} \dots \mathsf{a}_{-m}^{n_m}\mathsf{V}_p) = (-p_2)^{n_1+\dots+n_m}$ and $\bar{\gamma}_{p;34}(\mathsf{a}_{-1}^{n_1} \dots \mathsf{a}_{-m}^{n_m}\mathsf{V}_p) = p_3^{n_1+\dots+n_m}$. It follows from the above discussion that the computation of the 4-point blocks factorizes leading to

$$\begin{aligned} \mathcal{B}_{\mathrm{U}(1)} \equiv \mathcal{B}_p(p_1, p_2, p_3, p_4|q) &= \sum_{n_1,n_2,\dots=0}^{\infty} q^{\sum_{j=1}^{\infty} jn_j} \frac{(-p_2p_3)^{\sum_{r=1}^{\infty} n_r}}{\prod_{s=1}^{\infty} n_s! s^{n_s}} \\ &= \prod_{j=1}^{\infty} \sum_{n=0}^{\infty} \frac{q^{jn}(-p_2p_3)^n}{n! j^n} = \prod_{j=1}^{\infty} e^{-\frac{p_2p_3q^j}{j}} = e^{\log(1-q)p_2p_3} = (1-q)^{p_2p_3} . \end{aligned} \tag{D.3}$$

We can now compute some conformal blocks with insertions of the current J_1. We obtain after a short computation

$$\boldsymbol{\gamma}_{12p}(J_1(t); \mathsf{a}_{-1}^{n_1} \cdots \mathsf{a}_{-m}^{n_m}\mathsf{V}_p) = \left\{ \frac{p_2}{t-1} + \frac{p}{t} - \sum_{r=1}^{m} \frac{rn_r}{p_2} \frac{1}{t^{r+1}} \right\} (-p_2)^{n_1+\dots+n_m} . \tag{D.4}$$

After some computations, one finds from (3.37) the formula

$$\begin{aligned} \langle\langle\, J_1(t) \,\rangle\rangle_4 &= \frac{p_2}{t-1} + \frac{p}{t} - \frac{1}{p_2} \frac{(-p_2p_3)q}{t(t-q)} = \frac{p_2}{t-1} + \frac{p_3}{t-q} + \frac{p_4}{t} \\ &= \frac{\langle\, J_1(t)\mathsf{V}_{p_1}(\infty)\mathsf{V}_{p_2}(1)\mathsf{V}_{p_3}(q)\mathsf{V}_{p_4}(0) \,\rangle}{\langle\, \mathsf{V}_{p_1}(\infty)\mathsf{V}_{p_2}(1)\mathsf{V}_{p_3}(q)\mathsf{V}_{p_4}(0) \,\rangle} , \end{aligned} \tag{D.5}$$

where we remind that $p = p_3 + p_4$. We remark that $\langle\langle\, J_1(t) \,\rangle\rangle_4$ is equal to the ratio of the full correlation functions only for the U(1) case because in that case we have charge conservation! This means that only one primary propagates in the four point function and therefore the structure constants cancel in the ratio.

D.2 The Virasoro blocks

Three points. The case of the 3-point W-blocks is almost trivial since the $\langle\langle\, W_s \,\rangle\rangle_3$ are completely fixed by the $\mathbf{W}_N$ Ward identities and the shortening properties of the simple punctures. For the Liouville case, since (we ignore the anti-holomorphic pieces),

$$\begin{aligned} \langle\, \mathsf{V}_1(z_1)\mathsf{V}_2(z_2)\mathsf{V}_3(z_3) \,\rangle &= z_{12}^{\Delta_3-\Delta_1-\Delta_2} z_{13}^{\Delta_2-\Delta_1-\Delta_3} z_{23}^{\Delta_1-\Delta_2-\Delta_3} , \\ \langle\, T(t)\mathsf{V}_1(z_1)\mathsf{V}_2(z_2)\mathsf{V}_3(z_3) \,\rangle &= \sum_{i=1}^{3} \left(\frac{\Delta_i}{(t-z_i)^2} + \frac{\partial_{z_i}}{t-z_i} \right) \langle\, \mathsf{V}_1(z_1)\mathsf{V}_2(z_2)\mathsf{V}_3(z_3) \,\rangle , \end{aligned} \tag{D.6}$$

we find after setting $z_1 \to \infty, z_2 \to 1, z_3 \to 0$

$$\boldsymbol{\gamma}_{123}(T(t);\emptyset) = \langle\langle T(t) \rangle\rangle_3 = \frac{\langle T(t)\mathsf{V}_1(z_1)\mathsf{V}_2(z_2)\mathsf{V}_3(z_3) \rangle}{\langle \mathsf{V}_1(z_1)\mathsf{V}_2(z_2)\mathsf{V}_3(z_3) \rangle} = \frac{\Delta_1 t(t-1) + \Delta_2 t + \Delta_3(1-t)}{t^2(t-1)^2} . \tag{D.7}$$

In general, we have the recursion relations

$$\begin{aligned} \left\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)(L_{-n}\widehat{\mathsf{V}}_\Delta)(0) \right\rangle &= (\Delta + n\Delta_2 - (1-\delta_{n,0})\Delta_1) \left\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)\widehat{\mathsf{V}}_\Delta(0) \right\rangle , \\ \left\langle L_{-n}\widehat{\mathsf{V}}_\Delta \,|\, \mathsf{V}_3(1)\mathsf{V}_4(0) \right\rangle &= (\Delta + n\Delta_3 - (1-\delta_{n,0})\Delta_4) \left\langle \widehat{\mathsf{V}}_\Delta \,|\, \mathsf{V}_3(1)\mathsf{V}_4(0) \right\rangle . \end{aligned} \tag{D.8}$$

We also occasionally need the relations

$$\begin{aligned} \left\langle \mathsf{V}_1(\infty)(L_{-1}\mathsf{V}_2)(1)\widehat{\mathsf{V}}_\Delta(0) \right\rangle &= (\Delta_1 - \Delta_2 - \Delta) \left\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)\widehat{\mathsf{V}}_\Delta(0) \right\rangle \\ \left\langle \widehat{\mathsf{V}}_\Delta \,|\, (L_{-1}\mathsf{V}_3)(1)\mathsf{V}_4(0) \right\rangle &= (\Delta - \Delta_3 - \Delta_4) \left\langle \widehat{\mathsf{V}} \,|\, \mathsf{V}_3(1)\mathsf{V}_4(0) \right\rangle . \end{aligned} \tag{D.9}$$

Four points. Let us compute $\langle\langle T(t) \rangle\rangle_4$ up to quadratic order in q. In the formula (3.34), we have $\mathbf{Y} = \{Y\}$. If Y is the empty partition, we reproduce (D.7) by using (D.8)

$$\begin{aligned} \boldsymbol{\gamma}_{12\Delta}(T(t);\emptyset) &= \frac{1}{\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)\mathsf{V}_\Delta(0) \rangle} \sum_{n=-\infty}^{0} t^{-n-2} \langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)(L_n\mathsf{V}_\Delta)(0) \rangle \\ &= t^{-2}\Delta + \sum_{n=1}^{\infty} t^{n-2}(\Delta + n\Delta_2 - \Delta_1) = \frac{\Delta_1(t-1)t + \Delta_2 t - \Delta(t-1)}{(t-1)^2 t^2} , \end{aligned} \tag{D.10}$$

where we have made use of (D.8). Similarly, we compute

$$\begin{aligned} \boldsymbol{\gamma}_{12\Delta}(T(t);\{1\}) &= \frac{1}{\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)\mathsf{V}_\Delta(0) \rangle} \sum_{n=-\infty}^{1} t^{-n-2} \langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)(L_nL_{-1}\mathsf{V}_\Delta)(0) \rangle \\ &= t^{-3}2\Delta + t^{-2}(1+\Delta)\frac{\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)(L_{-1}\mathsf{V}_\Delta)(0) \rangle}{\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)\mathsf{V}_\Delta(0) \rangle} \\ &\quad + \sum_{n=1}^{\infty} t^{n-2}\frac{\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)(L_{-n}L_{-1}\mathsf{V}_\Delta)(0) \rangle}{\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)\mathsf{V}_\Delta(0) \rangle} \\ &= \frac{2\Delta}{t^3} + \frac{(\Delta+1)(\Delta-\Delta_1+\Delta_2)}{t^2} \\ &\quad + \frac{\Delta_2(\Delta-\Delta_1+\Delta_2)}{(t-1)^2} + \frac{(\Delta-\Delta_1+\Delta_2)(\Delta-\Delta_1+\Delta_2+1)}{t} \\ &\quad - \frac{(\Delta-\Delta_1+\Delta_2)(\Delta-\Delta_1+\Delta_2+1)}{t-1} . \end{aligned} \tag{D.11}$$

In the above we have used the commutation relations $[L_n \,,\, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0}$. We compute in a similar fashion

$$\begin{aligned} \boldsymbol{\gamma}_{12\Delta}(T(t);\{2\}) &= \frac{c+8\Delta}{2t^4} + \frac{3(\Delta-\Delta_1+\Delta_2)}{t^3} + \frac{(\Delta+2)(\Delta-\Delta_1+2\Delta_2)}{t^2} \\ &\quad + \frac{\Delta_2(\Delta-\Delta_1+2\Delta_2)}{(t-1)^2} + \frac{(\Delta-\Delta_1+\Delta_2+2)(\Delta-\Delta_1+2\Delta_2)}{t} \\ &\quad - \frac{(\Delta-\Delta_1+\Delta_2+2)(\Delta-\Delta_1+2\Delta_2)}{t-1} , \end{aligned} \tag{D.12}$$

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as well as

$$\boldsymbol{\gamma}_{12\Delta}(T(t);\{1,1\}) = \frac{6\Delta}{t^4} + \frac{2(2\Delta+1)(\Delta-\Delta_1+\Delta_2)}{t^3} + \frac{(\Delta+2)(\Delta-\Delta_1+\Delta_2)(\Delta-\Delta_1+\Delta_2+1)}{t^2} + \frac{\Delta_2(\Delta-\Delta_1+\Delta_2)(\Delta-\Delta_1+\Delta_2+1)}{(t-1)^2} + \frac{(\Delta-\Delta_1+\Delta_2)(\Delta-\Delta_1+\Delta_2+1)(\Delta-\Delta_1+\Delta_2+2)}{t} - \frac{(\Delta-\Delta_1+\Delta_2)(\Delta-\Delta_1+\Delta_2+1)(\Delta-\Delta_1+\Delta_2+2)}{t-1}. \tag{D.13}$$

Putting everything together, we get

$$\langle\langle T(t)\rangle\rangle_4 = \frac{1}{\mathcal{B}_\Delta(\Delta_1,\Delta_2,\Delta_3,\Delta_4|q)}\Bigg[\boldsymbol{\gamma}_{12\Delta}(T(t);\emptyset) + q\boldsymbol{\gamma}_{12\Delta}(T(t);\{1\})(\mathsf{Q}_\Delta^{(1)})^{-1}\bar{\gamma}_{\alpha;34}(\{1\}) + q^2\big(\boldsymbol{\gamma}_{12\Delta}(T(t);\{2\}),\boldsymbol{\gamma}_{12\Delta}(T(t);\{2\})\big)(\mathsf{Q}_\Delta^{(2)})^{-1}\begin{pmatrix}\bar{\gamma}_{\alpha;34}(\{2\})\\ \bar{\gamma}_{\alpha;34}(\{1,1\})\end{pmatrix}\Bigg] + \mathcal{O}(q^3), \tag{D.14}$$

where the Shapovalov form is to be found in (B.1). Comparison of (D.14) with the curve coefficient $\tilde{\phi}_2^{(4)}$ (see (2.5) and (2.10)) for $N > 2$ shows a perfect agreement if the parameter identifications of section 3.2 are taken into account. The block in the denominator is easily computed by taking the definition (3.25) and using (D.8). It reads

$$\begin{aligned}&\mathcal{B}_\Delta(\Delta_1,\Delta_2,\Delta_3,\Delta_4|q)\\ &= 1 + \frac{q(\Delta-\Delta_1+\Delta_2)(\Delta+\Delta_3-\Delta_4)}{2\Delta}\\ &\quad + q^2\Bigg[(\Delta{+}\Delta_3{-}\Delta_4)(\Delta{+}\Delta_3{-}\Delta_4{+}1)\Bigg(\frac{\left(\frac{c}{2}+4\Delta\right)(\Delta-\Delta_1+\Delta_2)(\Delta-\Delta_1+\Delta_2+1)}{4c\Delta^2+2c\Delta+32\Delta^3-20\Delta^2}\\ &\quad - \frac{6\Delta(\Delta-\Delta_1+2\Delta_2)}{4c\Delta^2+2c\Delta+32\Delta^3-20\Delta^2}\Bigg)\\ &\quad + (\Delta+2\Delta_3-\Delta_4)\Bigg(\frac{\left(4\Delta^2+2\left(2\Delta^2+2\Delta\right)\right)(\Delta-\Delta_1+2\Delta_2)}{4c\Delta^2+2c\Delta+32\Delta^3-20\Delta^2}\\ &\quad - \frac{6\Delta(\Delta-\Delta_1+\Delta_2)(\Delta-\Delta_1+\Delta_2+1)}{4c\Delta^2+2c\Delta+32\Delta^3-20\Delta^2}\Bigg)\Bigg] + \mathcal{O}(q^3).\end{aligned} \tag{D.15}$$

D.3 The $\mathbf{W}_3$-blocks

Ward identities. In the $\mathbf{W}_3$ case, we have to use the shortening condition for V_2 in order to use the Ward identities to compute the 3-point block with an insertion of $W_3(t) \equiv W(t)$.

The Ward identity that we want to use is (see 2.4 of [38])

$$\langle\, W(t)\mathsf{V}_1(z_1)\cdots \mathsf{V}_n(z_n)\,\rangle = \sum_{k=1}^{n}\left(\frac{w_k}{(t-z_k)^3}+\frac{W_{-1;k}}{(t-z_k)^2}+\frac{W_{-2;k}}{t-z_k}\right)\langle\, \mathsf{V}_1(z_1)\cdots \mathsf{V}_n(z_n)\,\rangle\ , \tag{D.16}$$

where $w_k \equiv w_3(\boldsymbol{\alpha}_k)$ with the charge $w_3(\boldsymbol{\alpha})$ defined in (C.3). The action of W_{-1} and W_{-2} cannot in general be expressed via simple differential operators. Taking (D.16), multiplying with z^m, $m = 0,\ldots,4$, integrating in z over a contour encircling all the insertion points and using the fact that $W(t) \propto \frac{1}{t^6}$ for $t\to\infty$ gives five global Ward identities (see for example [81] starting from eq. (2.18) there). Thus, for the 3-point function, we have 5 identities and 6 unknowns, namely the correlation functions $\langle\, W_{-1}\mathsf{V}_1\mathsf{V}_2\mathsf{V}_3\,\rangle$ $\langle\, \mathsf{V}_1 W_{-1}\mathsf{V}_2\mathsf{V}_3\,\rangle$, $\langle\, \mathsf{V}_1\mathsf{V}_2 W_{-1}\mathsf{V}_3\,\rangle$ and similarly another three with insertions of W_{-2} instead. We can thus solve for all of them except for $\langle\, \mathsf{V}_1 W_{-1}\mathsf{V}_2\mathsf{V}_3\,\rangle$. We can then get rid of $\langle\, \mathsf{V}_1 W_{-1}\mathsf{V}_2\mathsf{V}_3\,\rangle$ by using the fact that the primary field V_2 is semi-degenerate and that it has the null-vector $\big(W_{-1}-\frac{3w(\boldsymbol{\alpha}_2)}{2\Delta(\boldsymbol{\alpha}_2)}L_{-1}\big)\mathsf{V}_2 = 0$, so that

$$\begin{aligned}\langle\langle\, \mathsf{V}_1(z_1)(W_{-1}\mathsf{V}_2)(z_2)\mathsf{V}_3(z_3)\,\rangle\rangle &= \frac{3w_2}{2\Delta_2}\frac{\partial}{\partial z_2}\log[\langle\, \mathsf{V}_1(z_1)\mathsf{V}_2(z_2)\mathsf{V}_3(z_3)\,\rangle]\\ &\longrightarrow \frac{3w_2(\Delta_1-\Delta_2-\Delta_3)}{2\Delta_2}\,,\end{aligned} \tag{D.17}$$

after setting z_1, z_2, z_3 to $\infty, 1, 0$. Therefore using the Ward identities, (D.16) and the null vector, we find

$$\begin{aligned}\langle\langle\, W(t)\,\rangle\rangle_3 =&\frac{w_3}{t^3}+\frac{2\Delta_2(w_1+w_3)+w_2(3\Delta_1-\Delta_2-3\Delta_3)}{2\Delta_2 t^2}\\ &+\frac{\Delta_2(w_1+w_3)+w_2(3\Delta_1-2\Delta_2-3\Delta_3)}{\Delta_2 t}\\ &+\frac{w_2(2\Delta_2+3\Delta_3-3\Delta_1)-\Delta_2(w_1+w_3)}{\Delta_2(t-1)}-\frac{3w_2(\Delta_2+\Delta_3-\Delta_1)}{2\Delta_2(t-1)^2}+\frac{w_2}{(t-1)^3}\,.\end{aligned} \tag{D.18}$$

3-point blocks. We can derive recursion relations like (4.10) for more general simple punctures with $W_{-1}\mathsf{V}_2 = uL_{-1}\mathsf{V}_2$ for some parameter u. We find for $n > 0$ the identity

$$\begin{aligned}\Big\langle\, \mathsf{V}_1(\infty)\mathsf{V}_2(1)(W_{-n}\widehat{\mathsf{V}}_{\mathbf{w}})(0)\,\Big\rangle =& \Big\langle\, \mathsf{V}_1(\infty)\mathsf{V}_2(1)(W_0\widehat{\mathsf{V}}_{\mathbf{w}})(0)\,\Big\rangle\\ &+\left[w_1-\frac{n(n-3)}{2}w_2+nu(\Delta_1-\Delta_2-\Delta_{\mathbf{w}})\right]\\ &\times\Big\langle\, \mathsf{V}_1(\infty)\mathsf{V}_2(1)\widehat{\mathsf{V}}_{\mathbf{w}}(0)\,\Big\rangle\,.\end{aligned} \tag{D.19}$$

The last element that we need are the $\bar{\gamma}$ vertices. They can be computed through the following general relation for $n > 0$

$$\begin{aligned}\Big\langle W_{-n}\widehat{\mathsf{V}}_{\mathbf{w}}\,|\,\mathsf{V}_3(1)\mathsf{V}_4(0)\Big\rangle =& \left(\frac{n(n+3)}{2}w_3-w_4\right)\Big\langle \widehat{\mathsf{V}}_{\mathbf{w}}\,|\,\mathsf{V}_3(1)\mathsf{V}_4(0)\Big\rangle\\ &+\Big\langle W_0\widehat{\mathsf{V}}_{\mathbf{w}}\,|\,\mathsf{V}_3(1)\mathsf{V}_4(0)\Big\rangle\\ &+n\Big\langle \widehat{\mathsf{V}}_{\mathbf{w}}\,|\,(W_{-1}\mathsf{V}_3)(1)\mathsf{V}_4(0)\Big\rangle\,.\end{aligned} \tag{D.20}$$

If V_3 is a special puncture, we can use $W_{-1}\mathsf{V}_3 = \frac{3w_3}{2\Delta_3} L_{-1}\mathsf{V}_3$ and the relation (D.9) to compute the $\bar{\gamma}$ vertices iteratively.

The blocks with insertion of currents can be computed with the recursion relations (D.19) and (D.20). If $\widehat{\mathsf{V}}_{\mathbf{w}} = \mathsf{V}_3$ is a primary field (a full puncture for the 3-point case) and if $u = \frac{3w_2}{2\Delta_2}$ (i.e. if V_2 is the standard simple puncture), we find by using (D.19) for the 3-point $\mathbf{W}_3$-block with an insertion of the current $W(z)$ the expression

$$\boldsymbol{\gamma}_{123}(W(t);\emptyset) = \sum_{n=0}^{\infty} t^{n-3} \frac{\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)(W_{-n}\mathsf{V}_3)(0)\rangle}{\langle \mathsf{V}_1(\infty)\mathsf{V}_2(1)\mathsf{V}_3(0)\rangle} = \langle\langle W(t)\rangle\rangle_3 \tag{D.21}$$

where $\langle\langle W(t)\rangle\rangle_3$ was computed via the Ward identities in (D.18).

Four points. Let us compute the first few order of $\langle\langle W(t)\rangle\rangle_4 \equiv \langle\langle W_3(t)\rangle\rangle_4$. The $\mathbf{W}_3$ algebra is presented in appendix C. Together with the recursion relations it is straightforward to use a computer algebra program to compute

$$\begin{aligned}
&\boldsymbol{\gamma}_{12\mathbf{w}}(W(t);\{\emptyset,\{1\}\}) \\
&\quad= \frac{\Delta^2}{t^4} + \frac{2\Delta\Delta_2(\Delta-\Delta_1+\Delta_2)+2\Delta_2 w^2 + w(2\Delta_2 w_1 - w_2(3\Delta-3\Delta_1+\Delta_2))}{2\Delta_2 t^3} \\
&\qquad + \frac{1}{4\Delta_2^2 t^2}\Big[4\Delta_2^2\left(\Delta(\Delta-\Delta_1+\Delta_2)+(w+w_1)^2\right) \\
&\qquad\qquad - 2\Delta_2 w_2(w+w_1)(6\Delta-6\Delta_1+2\Delta_2+3) \\
&\qquad\qquad + w_2^2(3\Delta-3\Delta_1+\Delta_2)(3\Delta-3\Delta_1+\Delta_2+3)\Big] \\
&\qquad + \frac{1}{2\Delta_2^2 t}\Big[2\Delta_2^2\left(\Delta(\Delta-\Delta_1+\Delta_2)+(w+w_1)^2\right) \\
&\qquad\qquad - \Delta_2 w_2(w+w_1)(9\Delta-9\Delta_1+5\Delta_2+6) \\
&\qquad\qquad + w_2^2(3\Delta-3\Delta_1+\Delta_2)(3\Delta-3\Delta_1+2\Delta_2+3)\Big] \\
&\qquad + \frac{1}{2\Delta_2^2(t-1)}\Big[-2\Delta_2^2\left(\Delta(\Delta-\Delta_1+\Delta_2)+(w+w_1)^2\right) \\
&\qquad\qquad + \Delta_2 w_2(w+w_1)(9\Delta-9\Delta_1+5\Delta_2+6) \\
&\qquad\qquad - w_2^2(3\Delta-3\Delta_1+\Delta_2)(3\Delta-3\Delta_1+2\Delta_2+3)\Big] \\
&\qquad + \frac{3w_2(\Delta-\Delta_1+\Delta_2+1)(w_2(3\Delta-3\Delta_1+\Delta_2)-2\Delta_2(w+w_1))}{4\Delta_2^2(t-1)^2} \\
&\qquad - \frac{w_2(w_2(3\Delta-3\Delta_1+\Delta_2)-2\Delta_2(w+w_1))}{2\Delta_2(t-1)^3},
\end{aligned} \tag{D.22}$$

as well as

$$\begin{aligned}
\boldsymbol{\gamma}_{12\mathbf{w}}(W(t);\{\{1\},\emptyset\}) &= \frac{3\Delta}{t^4} + \frac{\Delta_2(\Delta(\Delta-\Delta_1+\Delta_2+2)+2w_1)-w_2(3\Delta-3\Delta_1+\Delta_2)}{\Delta_2 t^3} \\
&+ \frac{2\Delta_2(\Delta+w_1)(\Delta-\Delta_1+\Delta_2+2)-w_2\left(\Delta_2(4\Delta-4\Delta_1+5)+3(\Delta-\Delta_1)(\Delta-\Delta_1+3)+\Delta_2^2\right)}{2\Delta_2 t^2}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(\Delta-\Delta_1+\Delta_2+2)(\Delta_2(\Delta+w_1)+w_2(-3\Delta+3\Delta_1-2\Delta_2))}{\Delta_2 t}+\frac{w_2(\Delta-\Delta_1+\Delta_2)}{(t-1)^3}\\
&+\frac{(\Delta-\Delta_1+\Delta_2+2)(w_2(3\Delta-3\Delta_1+2\Delta_2)-\Delta_2(w+w_1))}{\Delta_2(t-1)}\\
&-\frac{3w_2(\Delta-\Delta_1+\Delta_2)(\Delta-\Delta_1+\Delta_2+1)}{2\Delta_2(t-1)^2}\,.
\end{aligned}
\tag{D.23}
$$

In (D.22) and (D.23), we have put for simplicity $Q=0$ from which follows $c=2$ and $\beta=\frac{1}{2}$.

The four point block $\mathcal{B}$ to linear order in q (for $Q\neq 0$) can be obtained quite straightforwardly by inverting (B.2) and using $\gamma_{12\mathbf{w}}(\{\{1\},\emptyset\})=\Delta-\Delta_1+\Delta_2$, $\gamma_{12\mathbf{w}}(\{\emptyset,\{1\}\})=w+w_1+\frac{3w_2(-\Delta+\Delta_1-\Delta_2)}{2\Delta_2}+w_2$, $\bar{\gamma}_{\mathbf{w};34}(\{\{1\},\emptyset\})=\Delta+\Delta_3-\Delta_4$, $\bar{\gamma}_{\mathbf{w};34}(\{\emptyset,\{1\}\})=w+\frac{3w_3(\Delta-\Delta_3-\Delta_4)}{2\Delta_3}+2w_3-w_4$. Thus, the 4-point $\mathbf{W}_3$-block reads

$$
\begin{aligned}
\mathcal{B}_{\mathbf{w}}(\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3,\mathbf{w}_4|q)=1+{}&\frac{q}{2\Delta_2\Delta_3\left(\Delta^2(-c+32\Delta+2)+216w^2\right)}\times\\
&\times\Big\{\Delta_2\Big[\Delta(\Delta_3(-c+32\Delta+2)(\Delta-\Delta_1+\Delta_2)(\Delta+\Delta_3-\Delta_4)\\
&\quad-48w_1(w_3(3\Delta+\Delta_3-3\Delta_4)-2\Delta_3w_4))\\
&\quad+48\Delta_3w^2(4\Delta-3(\Delta_1-\Delta_2-\Delta_3+\Delta_4))\\
&\quad+24w(2\Delta_3(w_1(\Delta+3\Delta_3-3\Delta_4)-w_4(\Delta-3\Delta_1+3\Delta_2))\\
&\quad+w_3(\Delta-3\Delta_1+3\Delta_2)(3\Delta+\Delta_3-3\Delta_4))\Big]\\
&\quad+24w_2(3\Delta-3\Delta_1+\Delta_2)(\Delta w_3(3\Delta+\Delta_3-3\Delta_4)\\
&\quad-\Delta_3(w(\Delta+3\Delta_3-3\Delta_4)+2\Delta w_4))\Big\}+\mathcal{O}(q^2)\,.
\end{aligned}
\tag{D.24}
$$

We remark that Δ_2,Δ_3,w_2 and w_3 all also depend on the central charge via

$$
\begin{aligned}
\Delta_{2,3}&=-\frac{1}{12}\varkappa_{2,3}\left(\sqrt{6}\sqrt{c-2}+4\varkappa_{2,3}\right)\,,\\
w_{2,3}&=-\frac{1}{432}\varkappa_{2,3}\left(\sqrt{6}\sqrt{c-2}+4\varkappa_{2,3}\right)\left(\sqrt{6}\sqrt{c-2}+8\varkappa_{2,3}\right)\,,
\end{aligned}
\tag{D.25}
$$

with $\varkappa_2=(M_L-\mathfrak{a}^{(1)})$ and $\varkappa_3=-(M_R-\mathfrak{a}^{(1)})$, see section 3.2 for more details.

The higher orders corrections in q of $\mathcal{B}$ are computed similarly. Combining the block $\mathcal{B}$ with (D.22) and (D.23), one can easily compute $\langle\langle\, W(t)\,\rangle\rangle_4$ to linear order in q.

E Nekrasov instanton partition functions

For the $\mathcal{N}=2$, SU(N) Nekrasov instanton partition functions, we define[19] $\epsilon=\epsilon_1+\epsilon_2$ and consider first the matter contributions to the instanton partition function:

$$
\mathcal{Z}_{\text{fund}}(\mathbf{a},\mathbf{Y};m)=\prod_{s=1}^{N}\prod_{(i,j)\in Y_s}\Big[a_s+\epsilon_1 i+\epsilon_2 j-m\Big]\,,
$$

[19] See [82] for a review. Our definition of the antifundamental partition function differs by a sign.

$$\mathcal{Z}_{\text{antifund}}(\mathbf{a},\mathbf{Y};m) = \prod_{s=1}^{N}\prod_{(i,j)\in Y_s}\left[\epsilon - m - a_s - \epsilon_1 i - \epsilon_2 j\right],$$

$$\begin{aligned}\mathcal{Z}_{\text{bifund}}(\mathbf{a},\mathbf{Y};\mathbf{a}',\mathbf{Y}';m) = &\prod_{s,s'=1}^{N}\prod_{(i,j)\in Y_s}\left[a_s - a'_{s'} - \epsilon_1 L_{Y'_{s'}}(i,j) + \epsilon_2\left(A_{Y_s}(i,j)+1\right) - m\right]\\ &\times\prod_{(i',j')\in Y'_{s'}}\left[\epsilon + a_s - a'_{s'} + \epsilon_1 L_{Y_s}(i',j') - \epsilon_2\left(A_{Y'_{s'}}(i',j')+1\right) - m\right],\end{aligned} \tag{E.1}$$

where we define the arm and leg lengths as

$$A_Y(i,j) = Y_i - j\,, \qquad L_Y(i,j) = Y^t_j - i\,. \tag{E.2}$$

Finally, we have the vector multiplet contribution

$$\mathcal{Z}_{\text{vec}}(\mathbf{a},\mathbf{Y}) = \frac{1}{\mathcal{Z}_{\text{bifund}}(\mathbf{a},\mathbf{Y};\mathbf{a},\mathbf{Y};0)}\,. \tag{E.3}$$

Specializations of the bifundamental contribution lead to the following identities

$$\begin{aligned}\mathcal{Z}_{\text{bifund}}(\mathbf{a},\mathbf{Y};\mathbf{b},\boldsymbol{\emptyset};m) &= \prod_{s=1}^{N}\mathcal{Z}_{\text{fund}}(\mathbf{a},\mathbf{Y};m+b_s)\,,\\ \mathcal{Z}_{\text{bifund}}(\mathbf{b},\boldsymbol{\emptyset};\mathbf{a},\mathbf{Y};m) &= \prod_{s=1}^{N}\mathcal{Z}_{\text{antifund}}(\mathbf{a},\mathbf{Y};m-b_s)\,.\end{aligned} \tag{E.4}$$

F The inverse mirror map

In this appendix, we show how to compute the inverse mirror map $u_2(a)$ for the $\mathcal{N}=2$ case with gauge group SU(2) and four flavors. We use [32] as our guide and detail our computations for the reader's convenience.[20] Our strategy in this appendix goes as follows. We first introduce two auxiliary ingredients: we explain how to perform a specific contour integral that we need for the computation of the inverse mirror map $u_2(a)$ and we introduce a cubic polynomial that simplifies some expressions. Once this is done, we apply these two ingredients directly to the computation of $u_2(a)$ from the SW curve.

The α-cycle integral. First, let us show how to perform an α-cycle integral. In general, let $P_4(t) = \prod_{i=1}^{4}(t-r_i)$ be a normalized quartic polynomial. Let C be the contour that encircles in a counterclockwise fashion the points r_1 and r_2. We want to compute the contour integral $\oint_C \frac{dt}{\sqrt{P_4(t)}}$, where the square root has been defined so that the branch cuts lie between the roots r_1 and r_2 on the one side and between r_3 and r_4 on the other. We use a Möbius transformation $f(t)$ to map $r_1,\ldots,r_4$ to $0,\lambda,1,\infty$, where $\lambda = \frac{r_{12}r_{34}}{r_{13}r_{24}}$ with $r_{ij} = r_i - r_j$. Specifically, $f(t) = \frac{\mathsf{a}t+\mathsf{b}}{\mathsf{c}t+\mathsf{d}}$ with $\mathsf{ad}-\mathsf{bc}=1$ and

$$r_1 = -\frac{\mathsf{b}}{\mathsf{a}}\,, \qquad r_2 = -\frac{\mathsf{b}-\mathsf{d}\lambda}{\mathsf{a}-\mathsf{c}\lambda}\,, \qquad r_3 = -\frac{\mathsf{b}-\mathsf{d}}{\mathsf{a}-\mathsf{c}}\,, \qquad r_4 = -\frac{\mathsf{d}}{\mathsf{c}} \qquad . \tag{F.1}$$

[20]We are grateful to Sara Pasquetti for giving us her Mathematica files on this computation.

Then, setting $t = f^{-1}(z)$, we find (setting the branch cuts appropriately)

$$\begin{aligned} \oint_C \frac{dt}{\sqrt{P_4(t)}} &= \oint_{f(C)} \frac{dz}{(\mathsf{a}-\mathsf{c}z)^2} \frac{1}{\sqrt{P_4(f^{-1}(z))}} \\ &= -2i \int_0^\lambda \frac{dz}{\sqrt{r_{13}r_{24}}\sqrt{z}\sqrt{z-\lambda}\sqrt{1-z}} = -4i \frac{K(\lambda)}{\sqrt{r_{13}r_{24}}}\,, \end{aligned} \tag{F.2}$$

where $K(m)$ is the complete elliptic integral of the first kind. Going from the second to the third step, we have used

$$f^{-1}(z) - r = \frac{1}{\mathsf{a}-\mathsf{c}z} \times \begin{cases} -(\mathsf{b}+\mathsf{a}r) & \text{if } \mathsf{d}+\mathsf{c}r = 0 \\ (\mathsf{d}+\mathsf{c}r)(z-f(r)) & \text{if } \mathsf{d}+\mathsf{c}r \neq 0 \end{cases}, \tag{F.3}$$

as well as $(\mathsf{d}+\mathsf{c}r_1)(\mathsf{d}+\mathsf{c}r_2)(\mathsf{d}+\mathsf{c}r_3)(-\mathsf{b}-r_4\mathsf{a}) = \frac{1}{\mathsf{a}\,\mathsf{c}(\mathsf{a}-\mathsf{c})(\mathsf{a}-\mathsf{c}\lambda)} = -r_{13}r_{24}$.

An auxiliary polynomial for the roots. Now we come to a construction involving an auxiliary polynomial Q_3. Its purpose is to give us a simple way of expressing the cross-ratios of the roots of P_4 in terms of the coefficients of P_4. Normally, we are not given directly the roots r_i of the quartic polynomial $P_4(t)$ but rather its coefficients and the expressions relating them can be cumbersome. Let us write $P(t) = t^4 - s_1 t^3 + s_2 t^2 - s_3 t + s_4$, so that the coefficients s_a are expressed using the roots as $s_a = \sum_{j_1<\cdots<j_a=1}^4 r_{j_1}\cdots r_{j_a}$. We want to find convenient expressions for λ and $r_{13}r_{24}$ in term of the s_a. Define first the following linear combinations of the roots

$$\begin{aligned} t_1 &= \frac{1}{2}(r_1+r_2+r_3+r_4)\,, & t_2 &= \frac{1}{2}(r_1-r_2+r_3-r_4)\,, \\ t_3 &= \frac{1}{2}(r_1+r_2-r_3-r_4)\,, & t_4 &= \frac{1}{2}(r_1-r_2-r_3+r_4)\,, \end{aligned} \tag{F.4}$$

as well as the auxiliary cubic polynomial $Q_3(y) = (y-t_2^2)(y-t_3^2)(y-t_4^2)$. We easily check that $Q_3(y) = y^3 + c_2 y^2 + c_1 y + c_0$ with

$$\begin{aligned} c_2 &= 2s_2 - \frac{3s_1^2}{4}\,, \qquad c_1 = \frac{3s_1^4}{16} - s_1^2 s_2 + s_1 s_3 + s_2^2 - 4s_4\,, \\ c_0 &= -\frac{s_1^6}{64} + \frac{s_1^4 s_2}{8} - \frac{s_1^3 s_3}{4} - \frac{s_1^2 s_2^2}{4} + s_1 s_2 s_3 - s_3^2\,. \end{aligned} \tag{F.5}$$

Hence, we have an auxiliary cubic polynomial $Q_3(y)$ with coefficients c_a easily expressed from the original coefficients s_a. The roots of $Q_3(y)$ are easily computed. They are the t_2^2, t_3^2 and t_4^2 and we find $\lambda = \frac{t_2^2-t_4^2}{t_3^2-t_4^2}$ as well as $r_{13}r_{24} = t_3^2 - t_4^2$. There is an S_3 ambiguity in ordering the roots of Q_3 as t_2^2, t_3^2, t_4^2. Under this permutation group, the cross-ratio of the roots λ goes over the values $\left\{\lambda, \frac{\lambda}{\lambda-1}, \frac{1}{\lambda}, \frac{1}{1-\lambda}, \frac{\lambda-1}{\lambda}, 1-\lambda\right\}$. This ambiguity is related to the ambiguity of choosing a canonical set of cycles on the SW curve and we solve it by choosing the one solution that can be expanded for small q.

The inverse mirror map. Let us now apply all this machinery to the computation of a α-cycle integral for the $\mathcal{N}=2$ theory with gauge group SU(2) and four flavors. The SW differential[21] $\lambda_{\rm SW}=xdt$ is explicitly given by

$$\lambda_{\rm SW}=\frac{m_1-m_2}{2}\frac{\sqrt{P_4(t)}}{t(t-1)(t-q)}dt\,, \tag{F.6}$$

where $P_4(t)=t^4-s_1t^3+s_2t^2-s_3t+s_4$ with (we use $m_1=m_{L,1}, m_2=m_{L,2}, m_3=m_{R,1}$ and $m_4=m_{R,2}$)

$$\begin{aligned} s_1&=\frac{2\left(q\left[m_1^2+m_2^2+(m_1+m_2)(m_3+m_4)\right]-2m_1m_2+2u_2\right)}{(m_1-m_2)^2}\,,\\ s_2&=\frac{q\left[q(m_1{+}m_2{+}m_3{+}m_4)^2+2((m_1{+}m_2)(m_3{+}m_4){-}2m_1m_2{-}2m_3m_4)\right]{+}4(q{+}1)u_2}{(m_1-m_2)^2}\,,\\ s_3&=\frac{2q\left(q\left[m_3^2+m_4^2+(m_1+m_2)(m_3+m_4)\right]-2m_3m_4+2u_2\right)}{(m_1-m_2)^2}\qquad s_4=\frac{q^2(m_3-m_4)^2}{(m_1-m_2)^2}\,. \end{aligned} \tag{F.7}$$

Since $A\equiv a_1-a_2=2a=\frac{1}{2\pi i}\oint_{r_1,r_2}\lambda_{\rm SW}$, it follows that

$$\frac{dA}{du_2}=\frac{1}{2\pi i}\oint_{r_1,r_2}\frac{d\lambda_{\rm SW}}{du_2}=\frac{1}{2\pi i}\oint_{r_1,r_2}\frac{2}{n_0}\frac{dt}{\sqrt{P_4(t)}}=-\frac{4}{\pi n_0}\frac{K(\lambda)}{\sqrt{r_{13}r_{24}}}\,. \tag{F.8}$$

We have to assign the roots of the cubic polynomial in such a way that $t_2^2-t_4^2$ vanishes when $q=0$. We thus have an expansion $t_2^2-t_3^2=qN_1+q^2N_2+\cdots$ and $t_3^2-t_4^2=D_0+qD_1+q^2D_2+\cdots$ which we plug into (F.8). The coefficients N_a and D_a are rather complicated functions of the s_a. We now make a substitution $u_2=\frac{v^2}{4}$ from which follows $\frac{dA}{du_2}=\frac{2}{v}\frac{dA}{dv}$. It follows that one obtains (thanks to a computer algebra program) the rather simple expression

$$\begin{aligned} \frac{dA}{dv}=&-1+q\left[-\frac{12(m_1m_2m_3m_4)}{v^4}+\frac{\sum_{i_1<i_2=1}^4 m_{i_1}m_{i_2}}{v^2}-\frac{1}{4}\right]\\ &+q^2\left[-\frac{420\,(m_1m_2m_3m_4)^2}{v^8}\right.\\ &\qquad+\frac{15}{v^6}\left(\sum_{i_1<i_2<i_3=1}^4 m_{i_1}^2m_{i_2}^2m_{i_3}^2+4m_1m_2m_3m_4\sum_{i_1<i_2=1}^4 m_{i_1}m_{i_2}\right)\\ &\qquad-\frac{3}{4v^4}\left(\sum_{i_1<i_2}m_{i_1}^2m_{i_2}^2+2\left(\sum_{i_1<i_2}m_{i_1}m_{i_2}\right)^2-4m_1m_2m_3m_4\right)\\ &\qquad\left.+\frac{-2\sum_{i=1}^4 m_i^2+8\sum_{i_1<i_2=1}^4 m_{i_1}m_{i_2}}{32v^2}-\frac{9}{64}\right]+\mathcal{O}\left(q^3\right)\,. \end{aligned} \tag{F.9}$$

[21] With $x^2=-\tilde\phi_2^{(4)}(t)$, see (2.5) and (2.10) with $\kappa=\frac{1}{2}$. Furthermore, u_1 is given by (3.44) and we have set $\mathfrak{a}^{(1)}=a_1+a_2=0$ for simplicity.

Integrating the above (the integration constant is zero) we get after inverting the expression a formula for v as a function of A. Putting this formula in $u_2 = \frac{v^2}{4}$ and replacing $A = 2a$ leads to the inverse mirror map:

$$
\begin{aligned}
u_2 = a^2 + q\left[\frac{m_1 m_2 m_3 m_4}{2a^2} - \frac{1}{2}\sum_{i_1<i_2=1}^{4} m_{i_1} m_{i_2} - \frac{a^2}{2}\right] + q^2\Bigg[\frac{5(m_1 m_2 m_3 m_4)^2}{32a^6} \\
- \frac{3\sum_{i_1<i_2<i_3=1}^{4} m_{i_1}^2 m_{i_2}^2 m_{i_3}^2}{32a^4} + \frac{\sum_{i_1<i_2=1}^{4} m_{i_1}^2 m_{i_2}^2}{32a^2} + \frac{\sum_{i=1}^{4} m_i^2}{32} - \frac{3a^2}{32}\Bigg] + \mathcal{O}\left(q^3\right) .
\end{aligned}
\tag{F.10}
$$

From the curve/block comparison on the other hand, we get[22]

$$
\begin{aligned}
u_2 = {} & a^2 + q\left[\frac{m_1 m_2 m_3 m_4}{2a^2} - \frac{1}{2}\sum_{i_1<i_2=1}^{4} m_{i_1} m_{i_2} - \frac{a^2}{2}\right] \\
& + q^2 \frac{1}{4a^4\left(16a^4 - 2a^2(c-5) + c\right)}\Bigg[-6a^{10} + a^8\left[c + 2\left(\sum_{i=1}^{4} m_i^2 - 2\right)\right] \\
& + 2a^6\left(\sum_{i=1}^{4} m_i^2 + \sum_{i_1<i_2=1}^{4} m_{i_1}^2 m_{i_2}^2\right) \\
& - a^4\left[c\left(m_1^2 m_2^2 + m_3^2 m_4^2\right) + \left(m_1^2 + m_2^2\right)\left(m_3^2 + m_4^2\right) + 6\sum_{i_1<i_2<i_3=1}^{4} m_{i_1}^2 m_{i_2}^2 m_{i_3}^2\right] \\
& + 10a^2 m_1^2 m_2^2 m_3^2 m_4^2 + c m_1^2 m_2^2 m_3^2 m_4^2\Bigg] + \mathcal{O}\left(q^3\right)
\end{aligned}
\tag{F.11}
$$

and the quadratic terms in q do not agree for any value of c. Specifically, we see that the q^2 term of (F.10) has a Laurent expansion in a that terminates, while (F.11) does not. Furthermore, all the terms in (F.10) are homogeneous of degree 2 under the rescaling of all parameters with units of mass. However, if we rescale, both in (F.10) and in (F.11), all parameters with units of mass as $m \to m/\hbar$ and take the limit $\hbar \to 0$, then the leading order terms (scaling like $\hbar^{-2}$) agree. The central charge $c = 1 + 6Q^2 = 1$ (since $Q = 0$) does not scale at all with $\hbar$ and cannot be seen in that limit.

Hence, we conclude that the agreement between the curve and the blocks is only fully valid for $\epsilon_1 + \epsilon_2 = 0$ and $\sqrt{\epsilon_1 \epsilon_2} = \hbar \to 0$. Furthermore, we see that we cannot determine the CFT central charge from the curve.

[22] We have taken (F.11) from the same computation that led to (3.46) and we remind that we have put $N = 2$ and set for simplicity $a_1 = a$ and $a_2 = -a$.

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