

First-quantized theory of expanding universe from field quantization in mini-superspace

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ABSTRACT: We propose an improved variant of the third-quantization scheme, for the spatially homogeneous and isotropic cosmological models in Einstein gravity coupled with a neutral massless scalar field. Our strategy is to specify a semi-Riemannian structure on the mini-superspace and to consider the quantum Klein-Gordon field on the mini-superspace. Then, the Hilbert space of this quantum system becomes inseparable, which causes the creation of infinite number of universes. To overcome this issue, we introduce a vector bundle structure on the Hilbert space and the connection of the vector bundle. Then, we can define a consistent unitary time evolution of the quantum universe in terms of the connection field on the vector bundle. By doing this, we are able to treat the quantum dynamics of a single-universe state. We also find an appropriate observable set constituting the CCR-algebra, and obtain the Schrödinger equation for the wave function of the single-universe state. We show that the present quantum theory correctly reproduces the classical solution to the Einstein equation.

KEYWORDS: Models of Quantum Gravity, 2D Gravity

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Contents

1	Introduction	1
2	Preliminaries: mini-superspace model	3
3	Quantum Klein-Gordon field in mini-superspace	4
3.1	Case of flat universe ($K = 0$)	4
3.2	Case of hyperbolic universe ($K = -1$)	7
3.3	Case of elliptic universe ($K = 1$)	12
4	Classical-quantum correspondence in the first-quantized theory	17
5	Concluding remarks	20

1 Introduction

One fundamental problem in quantum cosmology is the issue on the observable set and the quantum state of the space-time. The purpose of this paper is to construct an appropriate set of observables in the mini-superspace model of quantum cosmology and quantum states of the universe with clear classical-quantum correspondence.

While the perturbative quantization of the general relativity confronts the difficulty due to the non-renormalizability of the theory, alternative approaches such as string theory, loop gravity, etc. have been widely discussed for a long time. Needless to say, an important application of the quantum gravity would be in the quantum cosmology, which would give quantum descriptions of the expanding universe at early times and the big-bang singularity. Although we don't have a fully reliable theory of quantum gravity, it will be worthwhile to discuss a quantum theory of the expanding universe, such as the mini-superspace model, defined only on the finite dimensional reduced phase space of the gravitational field.

In the Hamiltonian formulation of the general relativity, we are led, according to the Dirac's prescription [1] for singular Lagrangian systems, to the constrained Hamiltonian system, in which the total Hamiltonian of the gravitational field

$$H_T = \int_{\Sigma_t} d^3x \left(N\Phi_0 + \sum_{i=1}^3 N^i\Phi_i \right)$$

consists of a linear combination of first-class constraint functions $\Phi_\mu \approx 0$ ($\mu = 0, 1, 2, 3$) [2, 3].

The configuration space of the gravitational field is parametrized by the Riemannian 3-metric h_{ij} on the Cauchy surface Σ_t , and the canonical momentum is given by

$$\pi^{ij} = (16\pi G)^{-1} \sqrt{h} (K^{ij} - Kh^{ij}),$$

where $K_{ij} = (1/2)\mathcal{L}_n h_{ij}$ (n is the future pointing unit normal vector field on Σ_t) is the extrinsic curvature of Σ_t . In terms of these canonical variables, the constraints are written as

$$\begin{aligned}\Phi_0 &= G_{ij,kl}\pi^{ij}\pi^{kl} - \frac{\sqrt{h}\mathcal{R}}{16\pi G} \approx 0, \\ \Phi_i &= -2D_k\pi_i^k \approx 0,\end{aligned}$$

where \mathcal{R} and D_k are the scalar curvature and the covariant derivative (for tensor densities) on (Σ_t, h_{ij}) , respectively, and

$$G_{ij,kl} := \frac{8\pi G}{\sqrt{h}} (h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl})$$

determines the semi-Riemannian structure of the configuration space of the 3-metrics.

According to the Dirac's algorithm of quantization, each first-class constraint function Φ_μ is replaced with a corresponding linear operator $\hat{\Phi}_\mu$ in a Hilbert space \mathcal{H} . Then, the observed physical state is required to be a ray in the subspace $\mathcal{H}_{\text{phys}} = \{|\psi\rangle \in \mathcal{H}; \hat{\Phi}_\mu|\psi\rangle = 0\}$ determined by these constraint operators. These requirements: $\hat{\Phi}_\mu|\psi\rangle = 0$, for the physical state, which become functional differential equations in the Schrödinger representation, are well known as the Wheeler-DeWitt equations [4, 5].

This formulation of quantum gravity is purely kinematical, in that the Wheeler-DeWitt equations only make a restriction on the physical Hilbert space, but does not define the dynamics of the wave function, since the Hamiltonian itself becomes a constraint operator: $\hat{H}_T|\psi\rangle = 0$, which is called the problem of time in the literature [6-8]. Furthermore, we don't have a fully satisfactory interpretation of the quantum state subject to the Wheeler-DeWitt equations.

In order to approach such conceptual issues, it will be worthwhile, as a first step, to discuss the mini-superspace model of the quantum cosmology. According to the concept of the geometrodynamics by Wheeler [5], the solution to the Einstein equation can be viewed as the motion of a test particle in the superspace [9, 10], which is the space of the 3-geometries on the Cauchy surface Σ_t , more precisely, it is the configuration space of 3-metrics above modulo the proper subgroup of the self-diffeomorphism group of Σ_t . Since this viewpoint is very useful in our formulation of the quantum theory of cosmology, let us respect it in the mini-superspace model in what follows.

The organization of the paper is as follows. In section 2, we briefly review the Hamiltonian formulation of the classical Einstein gravity applied to the Friedmann-Robertson-Walker (FRW) space-time, where we introduce the semi-Riemannian metric g_C on the mini-superspace. In section 3, we consider the quantum Klein-Gordon field in the mini-superspace with respect to the semi-Riemannian structure given by g_C , where we are faced with the issue regarding the inseparability of the Hilbert space in the cases of the spatially non-flat universe. In section 3.2 and 3.3, we introduce the vector bundle structure in the inseparable Hilbert space to define the consistent quantum theory with the unitary time evolution described by the global section of the vector bundle, where we obtain the first-quantized theory of the expanding universe on the space of 1-particle states of the quantum

Klein-Gordon field. In section 4, we show that our quantum theory correctly reproduces the classical solution to the Einstein equation. In section 5, we give concluding remarks.

2 Preliminaries: mini-superspace model

In the mini-superspace model of the isotropic and homogeneous universe, we consider the class of space-times represented by the FRW metric

$$g = -N(t)^2 dt^2 + a(t)^2 \gamma_K,$$

where γ_K ($K = 0, \pm 1$) denotes the Riemannian metric of the 3-space of the constant sectional curvature K . Let us consider the neutral massless scalar field minimally coupled with the Einstein-Hilbert action, which is provided by the classical action

$$S[g, X] = \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left(R - \frac{1}{2} X_{,\mu} X^{,\mu} \right).$$

In the standard procedure, we are led to the total Hamiltonian

$$\begin{aligned} H_T &= N\Phi_0, \\ \Phi_0 &= \frac{8\pi G}{v} \left(-\frac{p_a^2}{12a} + \frac{p_X^2}{a^3} \right) - \frac{3Kva}{8\pi G}, \end{aligned} \tag{2.1}$$

where we consider the fixed spatial coordinate volume and set $v = \int d^3x \sqrt{\gamma_K}$. The configuration space \mathcal{M} , which is called the mini-superspace, is parametrized by $q^m = (a, X)$, and their conjugate momenta are denoted as $p_m = (p_a, p_X)$. The momentum constraints $\Phi_i \approx 0$ are automatically satisfied in this parametrization. As the Hamiltonian can be written in the form:

$$\begin{aligned} H_T &= N [(g_S)^{mn} p_m p_n + u], \\ g_S &= \frac{v}{8\pi G} (-12ada^2 + a^3 dX^2), \\ u &= -\frac{3Kva}{8\pi G}, \end{aligned}$$

the dynamics of the universe is equivalent to that of a point particle in the curved space-time with the metric g_S and the potential function u . In fact, the equation of motion is given by

$$\begin{aligned} \frac{d^2 q^k}{dt^2} + \Gamma[g_S]_{mn}^k \frac{dq^m}{dt} \frac{dq^n}{dt} &\approx N^{-1} \frac{dN}{dt} \frac{dq^k}{dt} - 2N^2 (g_S)^{kl} \frac{du}{dq^l}, \\ \Phi_0 &= \frac{1}{4N^2} (g_S)_{mn} \frac{dq^m}{dt} \frac{dq^n}{dt} + u \approx 0. \end{aligned}$$

The present system has an invariance under the coupled conformal transformation of the Lorentzian metric g_S and the pointwise scaling of the potential: $(g_S, u) \mapsto (f g_S, f^{-1} u)$ for $f(q^m) \neq 0$. Using this invariance, the system can always be reduced to that of the geodesic particle in \mathcal{M} but with a conformally related metric [9, 10].

In fact, assuming $u \neq 0$, under the conformal transformation

$$(g_S)_{mn} = \frac{C^2}{u} (g_C)_{mn} \tag{2.2}$$

and the reparametrization of the time function

$$t(s) = \int^s ds \frac{C}{2Nu},$$

the equations of motion become

$$\frac{d^2 q^k}{ds^2} + \Gamma[g_C]_{mn}^k \frac{dq^m}{ds} \frac{dq^n}{ds} \approx 0, \tag{2.3}$$

$$\Phi_0 = u \left((g_C)_{mn} \frac{dq^m}{ds} \frac{dq^n}{ds} + 1 \right) \approx 0, \tag{2.4}$$

where C is a constant with the mass dimension 1. The eq. (2.3) is the geodesic equation with respect to the semi-Riemannian metric g_C , and the Hamiltonian constraint (2.4) requires that the new parameter s is the proper time. Hence, the equivalent system is given by the Hamiltonian of the geodesic particle in (\mathcal{M}, g_C) ,

$$H'_T = \lambda ((g_C)^{mn} p_m p_n + 1),$$

where λ is the Lagrange multiplier.

On the other hand, when $u = 0$, we set $g_C = g_S$. Then, the Hamiltonian is simply given by

$$H'_T = N (g_C)^{mn} p_m p_n.$$

This describes the null geodesic particle in (\mathcal{M}, g_C) .

As a quantized system corresponds to the classical system defined by eq. (2.1), we consider the quantum fields in the mini-superspace (\mathcal{M}, g_C) as a semi-Riemannian manifold.

3 Quantum Klein-Gordon field in mini-superspace

Here, we consider the Klein-Gordon field in (\mathcal{M}, g_C) and give a consistent quantized theory. Let us see $K = 0, \pm 1$ cases separately, since they are technically different.

3.1 Case of flat universe ($K = 0$)

The spatially flat case turns out to be the most simple, so that it is appropriate to explain this case first. In this case, the dynamics of the universe is equivalent to that of a null geodesic particle in (\mathcal{M}, g_C) , where the metric is given by

$$g_C = \frac{v}{8\pi G C} (-12ada^2 + a^3 dX^2).$$

We construct the localized 1-particle states of the massless quantum Klein-Gordon field to describe a quantum mechanical counterpart of this classical light-like particle.

Firstly, we introduce the new time function T by $a = a_0 e^{\beta T}$. Then, the metric on \mathcal{M} becomes the conformally flat form

$$g_C = A^2 e^{3\beta T} (-dT^2 + dX^2),$$

where we set

$$\beta = (12)^{-1/2}, \quad A^2 = \frac{va_0^3}{8\pi GC}.$$

The classical action of the massless Klein-Gordon field in (\mathcal{M}, g_C) becomes that in the flat space:

$$S[\phi] = \frac{1}{2} \int dT dX [(\partial_T \phi)^2 - (\partial_X \phi)^2].$$

A solution to the Klein-Gordon equation can be decomposed into a linear combination of the mode functions $\{f(p; T, X), f^*(p; T, X)\}$ ($p \in \mathbf{R}$), where

$$f(p; T, X) = (4\pi|p|)^{-1/2} e^{-i|p|T} e^{ipX}.$$

The quantum Klein-Gordon field is expanded into the form

$$\phi(T, X) = \int_{-\infty}^{\infty} dp [a(p)f(p; T, X) + a^*(p)f^*(p; T, X)],$$

and the conjugate momentum operator is written as $\pi(T, X) = \partial_T \phi(T, X)$. The canonical commutation relations (CCRs)

$$\begin{aligned} [\phi(T, X), \pi(T, X')]_- &= i\delta(X - X')\mathbf{1}, \\ [\phi(T, X), \phi(T, X')]_- &= 0, \quad [\pi(T, X), \pi(T, X')]_- = 0, \end{aligned}$$

are equivalent to

$$[a(p), a^*(p')]_- = \delta(p - p')\mathbf{1}, \quad [a(p), a(p')]_- = 0, \quad [a^*(p), a^*(p')]_- = 0.$$

In terms of these, the quantum Hamiltonian operator is given by

$$H = \int_{-\infty}^{\infty} dp |p| a^*(p) a(p).$$

As usual, the vacuum state $|\Omega\rangle$ is defined by the requirement:

$$a(p)|\Omega\rangle = 0, \quad \text{for all } p \in \mathbf{R},$$

and the Fock space is constructed in the standard procedure. Let $\mathcal{F}_\Omega^{(1)}$ denote the Hilbert space of 1-particle states, which we are mainly concerned with.

Define, for each $p \in \mathbf{R}$, the momentum operator

$$P := \int_{-\infty}^{\infty} dp p a^*(p) a(p),$$

which is a Hermitian operator on $\mathcal{F}_\Omega^{(1)}$. The eigenvectors of P are given by

$$|p\rangle := a^*(p)|\Omega\rangle, \quad \text{for } p \in \mathbf{R},$$

which satisfy

$$P|p\rangle = p|p\rangle.$$

These satisfy the orthogonality condition

$$\langle p|p'\rangle = \delta(p - p')$$

and the completeness condition

$$\int_{-\infty}^{\infty} dp |p\rangle\langle p| = \mathbf{1},$$

where $\mathbf{1}$ denotes the identity operator on $\mathcal{F}_\Omega^{(1)}$.

The position operator is defined by

$$Q := i \int_{-\infty}^{\infty} dp a^*(p) \partial_p a(p),$$

which is a Hermitian operator on $\mathcal{F}_\Omega^{(1)}$. The eigenvector of Q is formally written as a state

$$|X\rangle_Q := \int_{-\infty}^{\infty} dp e^{-ipX} |p\rangle, \quad \text{for } X \in \mathbf{R},$$

and it satisfies

$$Q|X\rangle_Q = X|X\rangle_Q.$$

The vector $|X\rangle_Q$ is the localized state considered by Newton and Wigner [11] long time ago.

The operators P and Q constitute the CCR-algebra

$$[Q, P]_- = i\mathbf{1}.$$

Thus, we obtain the canonical observable set (P, Q) and its Schrödinger representation on $\mathcal{F}_\Omega^{(1)}$.

We regard the quantum state of the universe as being described by a 1-particle state of the Fock space. Since the Hamiltonian of the Klein-Gordon field acts on $\mathcal{F}_\Omega^{(1)}$, any 1-particle state remains in $\mathcal{F}_\Omega^{(1)}$ in the course of the unitary time evolution. In fact, the Hamiltonian restricted on $\mathcal{F}_\Omega^{(1)}$ can be written as

$$H^{(1)} = |P|.$$

The expansion of the 1-particle state at the time T

$$|\psi(T)\rangle = \int_{-\infty}^{\infty} dp \psi(p, T) |p\rangle$$

defines the wave function $\psi(p, T)$ of the universe in the momentum representation. Then, the time evolution of the 1-particle state is determined by the Schrödinger equation

$$i\partial_T\psi(p, X) = \langle p|H^{(1)}|\psi(T)\rangle = |p|\psi(p, T),$$

according to the unitary time evolution of the quantum Klein-Gordon field. In this way, we obtain a quantum Hamiltonian for the wave function of the universe without operator ordering ambiguity [12], which clearly describes the massless particles in (\mathcal{M}, g_C) , besides the well-defined canonical observable set (P, Q) .

3.2 Case of hyperbolic universe ($K = -1$)

In this case, we are led to the massive field in the expanding chart of the Milne space. Then, according to the standard approach, the quantum field theory in this background geometry describes the continuous pair creations of scalar particles, which is hard to interpret in the context of the quantum cosmology. This occurs due to the inseparability of the Hilbert space involved. Nevertheless, we show that it is possible to construct a consistent quantum theory which describes the dynamics of a 1-particle state with a correct classical interpretation.

The semi-Riemannian metric on \mathcal{M} in this case is given by [See eq. (2.2)]

$$g_C = \frac{3v^2}{(8\pi G)^2 C^2} (-12a^2 da^2 + a^4 dX^2).$$

Introducing the new time function T by $a = a_0 e^{\beta T}$, the metric is written as

$$g_C = A^2 e^{4\beta T} (-dT^2 + dX^2),$$

where we set

$$\beta = (12)^{-1/2}, \quad A = \left| \frac{\sqrt{3}va_0^2}{8\pi GC} \right|.$$

Now, we start with the action of the massive Klein-Gordon field

$$S[\phi] = \frac{1}{2} \int dT dX [(\partial_T \phi)^2 - (\partial_X \phi)^2 - A^2 m^2 e^{4\beta T} \phi^2],$$

of the mass m . The complete set of the solutions to the Klein-Gordon equation is given by $\{f(p; T, X), f^*(p; T, X)\}$ ($p \in \mathbf{R}$), where

$$f(p; T, X) := (4\pi|p|)^{-1/2} F(|p|; T) e^{ipX},$$

$$F(|p|; T) := \Gamma(1 - i(2\beta)^{-1}|p|) \left(\frac{Am}{4\beta} \right)^{i(2\beta)^{-1}|p|} J_{-i(2\beta)^{-1}|p|} \left(\frac{Am e^{2\beta T}}{2\beta} \right),$$

and $\Gamma(x)$ and $J_\alpha(x)$ denote the Gamma and Bessel functions, respectively. The mode functions have been normalized in terms of the Klein-Gordon product:

$$(f, g)_{\text{KG}} := i \int_{-\infty}^{\infty} dX [f^* \partial_T g - (\partial_T f^*) g],$$

such that

$$\begin{aligned} (f(p; T, X), f(p'; T, X))_{\text{KG}} &= -(f^*(p; T, X), f^*(p'; T, X))_{\text{KG}} = \delta(p - p'), \\ (f(p; T, X), f^*(p'; T, X))_{\text{KG}} &= 0 \end{aligned}$$

hold. Since the mode function $f(p; T, X)$ approximates the plane wave in the Minkowski space-time in the limit $T \rightarrow -\infty$:

$$f(p; T, X) = (4\pi|p|)^{-1/2} e^{-i|p|T} e^{ipX} + O(e^{4\beta T}),$$

we take $\{f(p; T, X)\}$ as the positive frequency solutions and expand the quantum Klein-Gordon field in the form

$$\phi = \int_{-\infty}^{\infty} dp [a(p)f(p; T, X) + a^*(p)f^*(p; T, X)].$$

The Fock space \mathcal{F}_Ω is constructed from the vacuum state $|\Omega\rangle$ defined by the requirement:

$$a(p)|\Omega\rangle = 0, \quad (\text{for all } p \in \mathbf{R}),$$

according to the standard procedure.

The computation of the quantum Hamiltonian operator for the Klein-Gordon field in this case leads to

$$H(T) = \int dp \left[\sigma(|p|; T) a^*(p) a(p) + \frac{\tau(|p|; T)}{2} a(p) a(-p) + \frac{\tau^*(|p|; T)}{2} a^*(p) a^*(-p) \right],$$

where

$$\begin{aligned} \sigma(|p|; T) &:= |\Gamma(1 - i(2\beta)^{-1}|p|)|^2 \\ &\times \left[\frac{A^2 m^2 e^{4\beta T}}{8|p|} \left| J_{i(2\beta)^{-1}|p|-1} \left(\frac{Ame^{2\beta T}}{2\beta} \right) - J_{i(2\beta)^{-1}|p|+1} \left(\frac{Ame^{2\beta T}}{2\beta} \right) \right|^2 \right. \\ &\quad \left. + \frac{1}{2} \left(|p| + \frac{A^2 m^2 e^{4\beta T}}{|p|} \right) \left| J_{i(2\beta)^{-1}|p|} \left(\frac{Ame^{2\beta T}}{2\beta} \right) \right|^2 \right], \\ \tau(|p|; T) &:= \Gamma(1 - i(2\beta)^{-1}|p|)^2 \left(\frac{Am}{4\beta} \right)^{i\beta^{-1}|p|} \\ &\times \left\{ \frac{A^2 m^2 e^{4\beta T}}{8|p|} \left[J_{-i(2\beta)^{-1}|p|-1} \left(\frac{Ame^{2\beta T}}{2\beta} \right) - J_{-i(2\beta)^{-1}|p|+1} \left(\frac{Ame^{2\beta T}}{2\beta} \right) \right]^2 \right. \\ &\quad \left. + \frac{1}{2} \left(|p| + \frac{A^2 m^2 e^{4\beta T}}{|p|} \right) \left[J_{-i(2\beta)^{-1}|p|} \left(\frac{Ame^{2\beta T}}{2\beta} \right) \right]^2 \right\}. \end{aligned}$$

The Hamiltonian can be put into the diagonal form by the transformation [13, 14]

$$a(p; T) = a(p) \cosh \theta(|p|; T) + a^*(-p) e^{i\gamma(|p|; T)} \sinh \theta(|p|; T), \quad (3.1)$$

where the real functions $\theta(|p|; T)$ and $\gamma(|p|; T)$ are determined by

$$\begin{aligned} e^{2\theta(|p|; T)} &= \sqrt{\frac{\sigma(|p|; T) + |\tau(|p|; T)|}{\sigma(|p|; T) - |\tau(|p|; T)|}}, \\ e^{-i\gamma(|p|; T)} &= \frac{\tau(|p|; T)}{|\tau(|p|; T)|}. \end{aligned}$$

Then, the Hamiltonian becomes

$$H = \int dp \omega(|p|; T) a^*(p; T) a(p; T),$$

where

$$\omega(|p|; T) := \sqrt{\sigma(|p|; T)^2 - |\tau(|p|; T)|^2}.$$

Taking the number operators $N(p; T) := a^*(p; T) a(p; T)$ as the observables at the time T , the evaluation of $\langle \Omega | N(p; T) | \Omega \rangle$ gives a non-zero value. Then, it is argued that it describes the creation of scalar particles in the expanding universe, which is the approach frequently taken in the context of the quantum field theory in the FRW universe. Here, we seek for an appropriate formulation of the quantum cosmology assuming that we can observe only a 1-particle state of the universe as a guiding principle.

Since the defining equation (3.1) of $a(p; T)$ has the form of the Bogoliubov transformation, it can be also written as

$$a(p; T) = U(T) a(p) U^*(T),$$

$$U(T) := \exp \left\{ \frac{1}{2} \int dp \theta(|p|; T) [e^{-i\gamma(|p|; T)} a(p) a(-p) - e^{i\gamma(|p|; T)} a^*(p) a^*(-p)] \right\}.$$

The unitary operator $U(T)$ is improper ([15], section 3) in the sense that it is not a unitary operator on the Fock space \mathcal{F}_Ω . The Fock space \mathcal{F}_T ($T \in \mathbf{R}$) can be built from the “ T -vacuum”

$$|\Omega; T\rangle := U(T) |\Omega\rangle,$$

subject to $a(p; T) |\Omega; T\rangle = 0$ ($p \in \mathbf{R}$), by applying all polynomials of $a^*(p; T)$ according to the standard procedure for the Fock representation. In this way, the T -vacuum $|\Omega; T\rangle$ is regarded as the instantaneous vacuum of the Klein-Gordon field at the finite time T , and we have a continuum of mutually improperly equivalent Fock spaces parametrized by T .

Since the Hamiltonian $H(T)$ is a Hermitian operator in \mathcal{F}_T , a finite particle state in \mathcal{F}_T remains in a finite particle state under the action of the unitary operator $\exp(-iH(T)dT)$ in \mathcal{F}_T . This is of course not the state in \mathcal{F}_{T+dT} . Instead, it is physically interpreted as the infinite particle state in \mathcal{F}_{T+dT} . This implies that the Klein-Gordon Hamiltonian $H(T)$ does not define a unitary time evolution in a separable Hilbert space. In this sense, this theory deviates from the framework of the standard quantum theory.

This leads to a paradoxical conclusion that an infinitely many universes are continuously created. While in the context of the quantum field theory in the FRW background, such divergence of the particle number might not be regarded as so problematic, since its physical meaning is clearly interpretable. In fact, this kind of divergence comes from the infinite spatial volume of the background geometry, and the expectation value of the appropriate number density operator remains finite. Nevertheless, in the present context of the quantum cosmology, it is hard to see the correspondence to the classical solution to the

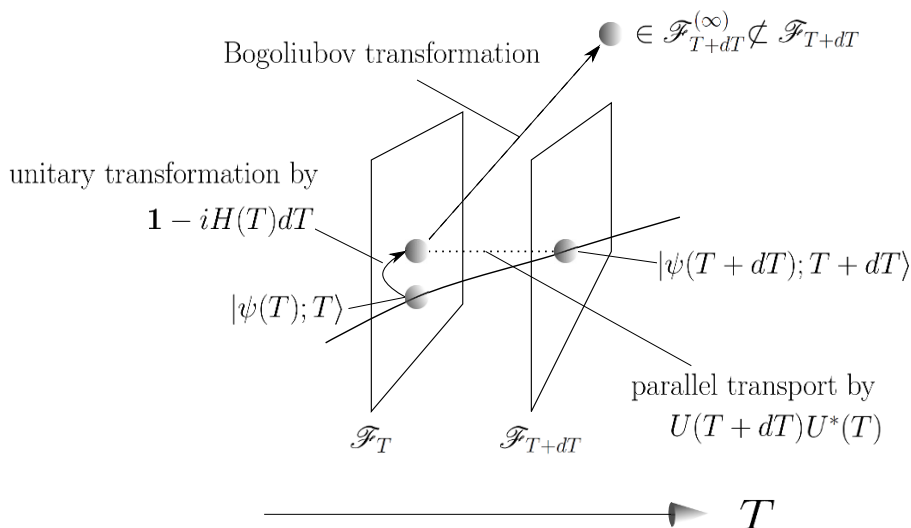


Figure 1. Schematic picture of the vector bundle structure in the continuum of Fock spaces. The dynamics of the quantum state is given by the unitary transformation $\exp[-iH(T)dt]$ in \mathcal{F}_T followed by the parallel transport defined by $U(T+dT)U^*(T)$.

Einstein equation with such an interpretation, because we always observe only a single-universe state in any case. Hence, an alternative framework for the quantum dynamics is required.

In order to provide a natural notion of the unitary time evolution, we need to define a time derivative between states belonging to mutually different Fock spaces. A natural way to realize this would be provided by considering the fibre bundle structure [16] in the continuum of the Fock spaces by

$$\pi : \bigcup_{T \in \mathbf{R}} \mathcal{F}_T =: F \rightarrow \mathbf{R}; |\psi; T\rangle \mapsto T,$$

and the local trivialization of this vector bundle is supposed to be given by

$$\varphi : \mathbf{R} \times \mathcal{F}_\Omega \rightarrow F; (T, |\psi; \Omega\rangle) \mapsto |\psi; T\rangle := U(T)|\psi; \Omega\rangle.$$

In this setting, the dynamics of the quantum state can be described by a global section

$$\psi : \mathbf{R} \rightarrow F; T \mapsto |\psi(T); T\rangle$$

of F (see figure 1).

In order to formulate the quantum dynamics, we have to define the time derivative of the motion $|\psi(T); T\rangle$, where a problem is that d/dT is not an anti-Hermitian operator on the Fock space \mathcal{F}_T . To make the time derivative of the quantum state a meaningful operation, we need to introduce the notion of parallel transport of the quantum state from \mathcal{F}_T to \mathcal{F}_{T+dT} . Then, our vector bundle structure provides a natural framework to define the parallel transport in terms of the connection of the vector bundle. The most natural choice of the parallel transport would be given by

$$u(T', T) := U(T')U^*(T) : \mathcal{F}_T \rightarrow \mathcal{F}_{T'}.$$

Now, we define the covariant time derivative of the quantum state with respect to this parallel transport by

$$D_T := \frac{d}{dT} + U(T)(\partial_T U^*(T)),$$

which is an anti-Hermitian operator on \mathcal{F}_T . The second term appeared in D_T is, in a sense, the gauge field, which belongs to the Lie algebra of the structure group: the group of all unitary operators on the Fock space \mathcal{F}_Ω . In terms of this covariant time derivative operator, we postulate that the unitary time evolution of the quantum state is determined by the covariant Schrödinger equation¹

$$iD_T|\psi(T); T\rangle = H(T)|\psi(T); T\rangle.$$

In terms of the local coordinates in F , this is equivalent to the Schrödinger equation

$$\begin{aligned} i\partial_T|\psi(T); \Omega\rangle &= H_\Omega(T)|\psi(T); \Omega\rangle, \\ H_\Omega(T) &:= \int dp \omega(|p|; T)a^*(p)a(p), \end{aligned}$$

in \mathcal{F}_Ω . Thus, in this formulation, everything is described in the language of a single separable Hilbert space \mathcal{F}_Ω .

Our quantum dynamics has an advantage that any 1-particle state remains in a 1-particle state, which would be particularly suitable for the quantum cosmology. So, let us introduce the canonical observable set on the 1-particle Fock spaces $\mathcal{F}_T^{(1)}$, along the line of the case of the flat universe.

Firstly, define the Hermitian operators

$$\begin{aligned} P(T) &:= \int_{-\infty}^{\infty} dp pa^*(p; T)a(p; T), \\ Q(T) &:= i \int_{-\infty}^{\infty} dp a^*(p; T)\partial_p a(p; T), \end{aligned}$$

on $\mathcal{F}_T^{(1)}$ are regarded as the momentum and position operators, respectively. In fact, these realize the representation of the CCR-algebra

$$[Q(T), P(T)]_- = i\mathbf{1},$$

on the 1-particle Fock space $\mathcal{F}_T^{(1)}$.

The eigenvector of $P(T)$ is written as

$$|p; T\rangle := a^*(p; T)|\Omega; T\rangle,$$

corresponding to the eigenvalue $p \in \mathbf{R}$, and the eigenvector of $Q(T)$ is formally written as

$$|X; T\rangle_Q := (2\pi)^{-1/2} \int_{-\infty}^{\infty} dp e^{-ipX}|p; T\rangle,$$

¹The general form of the covariant Schrödinger equation is possibly given by $(iD_T + A(T))|\psi(T); T\rangle = H(T)|\psi(T); T\rangle$ in terms of a gauge field $A(T)$. Here we simply consider the case: $A(T) = 0$, because an additional structure to determine $A(T) \neq 0$ is not equipped with the present settings.

which corresponds to the eigenvalue $X \in \mathbf{R}$. Hence, $|X; T\rangle_Q$ can be regarded as the 1-particle state localized at the position X at time T .

Since $|X; T'\rangle_Q = u(T', T)|X; T\rangle_Q$ holds, the localized 1-particle state $|X; T'\rangle_Q \in \mathcal{F}_{T'}^{(1)}$ is the parallel transport of $|X; T\rangle_Q \in \mathcal{F}_T^{(1)}$ with the same position. It gives justification for our choice of the connection of the vector bundle.

From the completeness of the 1-particle states

$$\int_{-\infty}^{\infty} dp |p; T\rangle\langle p; T| = \mathbf{1},$$

in $\mathcal{F}_T^{(1)}$, we can expand the 1-particle state $|\psi(T); T\rangle$ as

$$|\psi(T); T\rangle = \int_{-\infty}^{\infty} dp \psi(p, T)|p; T\rangle,$$

and the coefficient $\psi(p, T)$ is the wave function in the momentum representation. The Hamiltonian operator $H(T)$ restricted on the 1-particle Fock space $\mathcal{F}_T^{(1)}$ can be written as

$$H^{(1)}(T) = \int_{-\infty}^{\infty} dp \omega(|p|, T)a^*(p; T)a(p; T).$$

By calculating its matrix element, we obtain the Schrödinger equation

$$\begin{aligned} i\partial_T\psi(p, T) &= \int_{-\infty}^{\infty} dp' \langle p; T|H^{(1)}(T)|p'; T\rangle\psi(p', T) \\ &= \omega(|p|, T)\psi(p, T). \end{aligned} \tag{3.2}$$

in the momentum representation on $\mathcal{F}_T^{(1)}$. We show in section 4 that this reproduces a correct classical dynamics.

3.3 Case of elliptic universe ($K = 1$)

The semi-Riemannian metric on \mathcal{M} in this case is given by [See eq. (2.2)]

$$g_C = \frac{3v^2}{(8\pi G)^2 C^2} (12a^2 da^2 - a^4 dX^2).$$

Introducing T by $a = a_0 e^{\beta T}$, this becomes

$$g_C = A^2 e^{4\beta T} (dT^2 - dX^2),$$

where

$$\beta = (12)^{-1/2}, \quad A = \left| \frac{\sqrt{3}va_0^2}{8\pi GC} \right|.$$

Since the classical Hamiltonian is given by

$$H'_T = \lambda((g_C)^{mn} p_m p_n + 1),$$

a classical motion corresponds to a space-like geodesic in \mathcal{M} , if we take T as the time function.

As a quantum version of this classical system, we take the tachyonic quantum Klein-Gordon field [17] described by the action

$$S[\phi] = \frac{1}{2} \int dT dX [(\partial_T \phi)^2 - (\partial_X \phi)^2 + A^2 m^2 e^{4\beta T} \phi^2].$$

The mode functions $\{f(p; T, X), f^*(p; T, X)\}$ for the Klein-Gordon equation in this case can be written as

$$f(p; T, X) := (4\pi|p|)^{-1/2} F(|p; T) e^{ipX},$$

$$F(|p; T) := \Gamma(1 - i(2\beta)^{-1}|p|) \left(\frac{Am}{4\beta}\right)^{i(2\beta)^{-1}|p|} I_{-i(2\beta)^{-1}|p|} \left(\frac{Am e^{2\beta T}}{2\beta}\right),$$

where $I_\alpha(x)$ denotes the modified Bessel function of the first kind. Since the mode function $f(p; T, X)$ becomes the plane wave

$$f(p; T, X) = (4\pi|p|)^{-1/2} e^{-i|p|T} e^{ipX} + O(e^{4\beta T}),$$

as $T \rightarrow -\infty$, we regard $\{f(p; T, X); p \in \mathbf{R}\}$ as the positive frequency solutions.

The subsequent procedure to obtain the first-quantized theory is parallel to the $K = -1$ case. However, we need to specify the functional space of the wave functions to define the unitary dynamics of the 1-particle quantum state of the tachyonic field.

Firstly, quantize the Klein-Gordon field

$$\phi(T, X) = \int dp [a(p)f(p; T, X) + a^*(p)f^*(p; T, X)],$$

with the operators $\{a(p), a^*(p); p \in \mathbf{R}\}$. In the case of the tachyonic scalar field in the Minkowski background, it is known that the CCRs for the scalar field is not compatible with the harmonic-oscillator commutation relations for the creation and annihilation operators [18]. It however turns out that we don't encounter this kind of difficulties in the present background geometry. Here, the CCRs for $\{\phi(T, X), \partial_T \phi(T, X); X \in \mathbf{R}\}$ are equivalent to the harmonic-oscillator commutation relations

$$[a(p), a^*(p')]_- = \delta(p - p') \mathbf{1}, \quad [a(p), a(p')]_- = 0, \quad [a^*(p), a^*(p')]_- = 0,$$

for the operators $\{a(p), a^*(p); p \in \mathbf{R}\}$. The vacuum state is defined by $a(p)|\Omega\rangle = 0$ ($p \in \mathbf{R}$) and we obtain the Fock representation on \mathcal{F}_Ω in the standard procedure.

Then, the Hamiltonian operator becomes

$$H(T) = \int dp h(p; T),$$

where

$$\begin{aligned}
 h(p; T) &:= \sigma(|p; T) a^*(p) a(p) + \frac{\tau(|p; T)}{2} a(p) a(-p) + \frac{\tau^*(|p; T)}{2} a^*(p) a^*(-p), \\
 \sigma(|p; T) &:= |\Gamma(1 - i(2\beta)^{-1}|p||^2 \\
 &\quad \times \left[\frac{A^2 m^2 e^{4\beta T}}{8|p|} \left| I_{i(2\beta)^{-1}|p|-1} \left(\frac{Ame^{2\beta T}}{2\beta} \right) + I_{i(2\beta)^{-1}|p|+1} \left(\frac{Ame^{2\beta T}}{2\beta} \right) \right|^2 \right. \\
 &\quad \left. + \frac{1}{2} \left(|p| - \frac{A^2 m^2 e^{4\beta T}}{|p|} \right) \left| I_{i(2\beta)^{-1}|p|} \left(\frac{Ame^{2\beta T}}{2\beta} \right) \right|^2 \right], \\
 \tau(|p; T) &:= \Gamma(1 - i(2\beta)^{-1}|p|)^2 \left(\frac{Am}{4\beta} \right)^{i\beta^{-1}|p|} \\
 &\quad \times \left\{ \frac{A^2 m^2 e^{4\beta T}}{8|p|} \left[I_{-i(2\beta)^{-1}|p|-1} \left(\frac{Ame^{2\beta T}}{2\beta} \right) + I_{-i(2\beta)^{-1}|p|+1} \left(\frac{Ame^{2\beta T}}{2\beta} \right) \right]^2 \right. \\
 &\quad \left. + \frac{1}{2} \left(|p| - \frac{A^2 m^2 e^{4\beta T}}{|p|} \right) \left[I_{-i(2\beta)^{-1}|p|} \left(\frac{Ame^{2\beta T}}{2\beta} \right) \right]^2 \right\}.
 \end{aligned}$$

The Hamiltonian at the time T can be made in a diagonalized form in the integration range $|p| > p_0(T)$, where the function $p_0(T)$ is the smallest positive solution of the equation

$$\sigma(p_0(T); T) = |\tau(p_0(T); T)|,$$

that is given by

$$p_0(T) = Ame^{2\beta T}.$$

Keeping this in mind, we separate the Hamiltonian into two parts as

$$H(T) = \left(\int_{|p|>\xi} + \int_{|p|\leq\xi} \right) dp h(p; T),$$

in terms of an arbitrarily fixed positive parameter ξ . The first part of the Hamiltonian can be diagonalized via the Bogoliubov transformation

$$\begin{aligned}
 a(p; T) &= U(T) a(p) U^*(T), \\
 U(T) &:= \exp \left\{ \frac{1}{2} \int_{|p|>\xi} dp \theta(|p; T) [e^{-i\gamma(|p; T)} a(p) a(-p) - e^{i\gamma(|p; T)} a^*(p) a^*(-p)] \right\}, \\
 e^{2\theta(|p; T)} &:= \sqrt{\frac{\sigma(|p; T) + |\tau(|p; T)|}{\sigma(|p; T) - |\tau(|p; T)|}}, \\
 e^{-i\gamma(|p; T)} &:= \frac{\tau(|p; T)}{|\tau(|p; T)|}.
 \end{aligned}$$

The above expressions are valid for the range $T \in (-\infty, T_1)$, where T_1 is determined by $\xi = p_0(T_1)$, i.e. $T_1 = (2\beta)^{-1} \log(\xi/Am)$.

In this way, we have a partly diagonalized form of the Hamiltonian

$$\begin{aligned}
 H(T) &= \int_{|p|>\xi} dp \omega(|p; T) a^*(p; T) a(p; T) + \int_{|p|\leq\xi} dp h(p; T), \\
 \omega(|p; T) &:= \sqrt{\sigma(|p; T)^2 - |\tau(|p; T)|^2}.
 \end{aligned}$$

where $\omega(|p|; T)$ is a real function, hence the first diagonal part is a Hermitian operator on \mathcal{F}_T for $T < T_1$.

Accordingly, as shown in the following, we can at best obtain the unitary theory of the 1-particle states within the time duration $T \in (-\infty, T_1)$ for $K = 1$ universe. This would correspond to the fact that each classical solution describes a recollapsing universe, and hence the scale factor $a = a_0 e^{\beta T}$ always has an upper bound.

To describe this system, we regard the Fock space \mathcal{F}_Ω as the tensor product space $\mathcal{A} \otimes \mathcal{B}$ of the subsystems \mathcal{A} and \mathcal{B} , where \mathcal{A} (resp. \mathcal{B}) is the Fock space with respect to the creation and annihilation operators $\{a(p), a^*(p); |p| > \xi\}$ (resp. $\{a(q), a^*(q); |q| \leq \xi\}$). The T -vacuum is defined by

$$\begin{aligned} a(p; T)|\Omega; T\rangle &= 0, \quad \text{for } |p| > \xi \\ a(q)|\Omega; T\rangle &= 0, \quad \text{for } |q| \leq \xi \end{aligned}$$

and the Fock space \mathcal{F}_T is constructed with respect to this vacuum by applying polynomials of $\{a^*(p; T), a^*(q); |p| > \xi, |q| \leq \xi\}$. The Fock space \mathcal{F}_T can be regarded as the tensor product of a pair of Fock spaces \mathcal{A}_T and \mathcal{B} , \mathcal{A}_T (resp. \mathcal{B}) being the Fock representation space of the algebra generated by $\{a(p; T), a^*(p; T); |p| > \xi\}$ (resp. $\{a(q), a^*(q); |q| \leq \xi\}$). In other words, \mathcal{F}_T is spanned by the vectors in the form

$$|n_{p_1}, n_{p_2}, \dots, n_{p_r}; T\rangle \otimes |n_{q_1}, n_{q_2}, \dots, n_{q_s}\rangle,$$

where $|n_{p_1}, n_{p_2}, \dots, n_{p_r}; T\rangle \in \mathcal{A}_T$ (resp. $|n_{q_1}, n_{q_2}, \dots, n_{q_s}\rangle \in \mathcal{B}$) denotes the simultaneous eigenvector of the number operators $a^*(p; T)a(p; T)$ ($|p| > \xi$) [resp. $a^*(q)a(q)$ ($|q| \leq \xi$)]. Accordingly, the Hamiltonian $H(T)$ is regarded as the sum

$$\begin{aligned} H(T) &= H_\xi(T) \otimes \mathbf{1} + \mathbf{1} \otimes \int_{|p| \leq \xi} dp h(p; T), \\ H_\xi(T) &:= \int_{|p| > \xi} dp \omega(|p|; T) a^*(p; T) a(p; T), \end{aligned}$$

of the operator on \mathcal{A}_T and the operator on \mathcal{B} . Hence, for a vector state $|\psi\rangle \otimes |\psi'\rangle \in \mathcal{A}_T \otimes \mathcal{B}$, each vector $|\psi\rangle$ (resp. $|\psi'\rangle$) independently undergoes the unitary time evolution in the inseparable Fock space $F_\xi := \bigcup_{T < T_1} \mathcal{A}_T$ (resp. \mathcal{B}). Hence, we can consistently assume that our observable set consists only of the Hermitian operators of the form $O(T) \otimes \mathbf{1}$. In other words, we can restrict ourselves to the dynamics of the subsystem F_ξ .

As in the case of $K = -1$, the Hilbert spaces \mathcal{A}_T and $\mathcal{A}_{T'}$ are improperly unitary equivalent, when $T \neq T'$. In a similar procedure to the $K = -1$ case, we introduce the vector bundle structure by the projection

$$\pi : \bigcup_{T < T_1} \mathcal{A}_T =: F_\xi \rightarrow (-\infty, T_1); |\psi; T\rangle \mapsto T,$$

and the local trivialization

$$\varphi : (-\infty, T_1) \times \mathcal{A}_\Omega \rightarrow F; (T, |\psi; \Omega\rangle) \mapsto |\psi; T\rangle := U(T)|\psi; \Omega\rangle.$$

Our Hilbert space describing the 1-particle state of the universe at the time T is assumed to be the subspace $\mathcal{A}_T^{(1)} \otimes \mathcal{B}$, where $\mathcal{A}_T^{(1)}$ denotes the space of 1-particle states in \mathcal{A}_T . Each such state is projected onto the space of quantum state on \mathcal{A}_T via the partial trace over the subsystem \mathcal{B} , which may produce a mixed state on \mathcal{A}_T . Since the structure of the Hamiltonian ensures that the operation of the partial trace and the unitary time evolution by the Hamiltonian commute, the quantum dynamics is reduced to that of a separable state, which can be represented as a vector state consisting of a single term $|\psi; T\rangle \otimes |\psi'\rangle$. Such a separable state simply corresponds to the vector state $|\psi; T\rangle \in \mathcal{A}_T$, and its time evolution is assumed to be described by the Schrödinger equation

$$iD_T|\psi(T); T\rangle = H_\xi(T)|\psi(T); T\rangle,$$

$$D_T := \frac{d}{dT} + U(T)(\partial_T U(T))^*,$$

for $T \in (-\infty, T_1)$.

The canonical observable set $\{Q(T), P(T)\}$ on $\mathcal{A}_T^{(1)}$ are also constructed in a similar manner. Firstly, we define the momentum operator by

$$P(T) := \int_{|p|>\xi} dp p a^*(p; T)a(p; T),$$

which is a Hermitian operator on $\mathcal{A}_T^{(1)}$. The eigenstate of $P(T)$ is given by

$$|p; T\rangle := a^*(p; T)|\Omega_{\mathcal{A}}; T\rangle,$$

which satisfies

$$P(T)|p; T\rangle = p|p; T\rangle,$$

where $|\Omega_{\mathcal{A}}; T\rangle$ denotes the vacuum state in \mathcal{A}_T . Next, the position operator is defined by

$$Q(T) := i \int_{|p|>\xi} dp a^*(p; T)\partial_p a(p; T),$$

which is a Hermitian operator on $\mathcal{A}_T^{(1)}$. These constitute the CCR-algebra

$$[Q(T), P(T)]_- = i\mathbf{1},$$

where we abbreviate the identity operator on $\mathcal{A}_T^{(1)}$ as

$$\mathbf{1} := \int_{|p|>\xi} dp a^*(p; T)a(p; T).$$

The eigenvector of $Q(T)$ can be written as

$$|X; T\rangle_Q := (2\pi)^{-1/2} \int_{|p|>\xi} dp e^{-ipX} |p; T\rangle,$$

for $X \in \mathbf{R}$, and it satisfies

$$Q(T)|X; T\rangle_Q = X|X; T\rangle_Q.$$

Unlike the $K = -1$ case, $\{|X; T\rangle_Q; X \in \mathbf{R}\}$ does not satisfy the orthogonality condition, i.e. it holds

$${}_Q\langle X; T|X'; T\rangle_Q \neq \delta(X - X'),$$

in the present case. However, since for the operator $F_X(T) := |X; T\rangle_Q{}_Q\langle X; T|$, it hold

$$\begin{aligned} \langle \psi; T|F_X(T)|\psi; T\rangle &> 0, \quad \text{for all } |\psi; T\rangle \in \mathcal{A}_T^{(1)} \\ \int_{-\infty}^{\infty} dX F_X(T) &= \mathbf{1}, \end{aligned}$$

the measurement of the position operator

$$Q(T) = \int dX X F_X(T)$$

can be regarded as a POVM measurement, though not a von Neumann measurement.

In the momentum representation on the 1-particle Fock space $\mathcal{A}_T^{(1)}$, the 1-particle state is described by the wave function $\psi(p, T) := \langle p; T|\psi(T); T\rangle$, and it is subject to the Schrödinger equation

$$i\partial_T\psi(p, T) = \omega(|p|, T)\psi(p, T).$$

We see in the next section that the quantum system described here reproduces the correct classical theory.

4 Classical-quantum correspondence in the first-quantized theory

Here, we show that the first-quantized theory obtained in the previous section correctly reproduces the classical Einstein gravity.

In the previous section, we have obtained a quantum theory of the expanding universe described by the wave function $\psi(p, T)$ of the universe subject to the Schrödinger equation. The expression of $\omega(|p|; T)$ includes the Bessel functions when $K = \pm 1$, which makes it difficult to solve the Schrödinger equation strictly. Nevertheless we can solve it approximately in terms of asymptotic forms of the Bessel functions. The Bessel function and the modified Bessel function in the mode functions are expanded around their argument $Ame^{2\beta T} = 0$ as

$$\begin{aligned} J_{-i(2\beta)^{-1}|p|} \left(\frac{Ame^{2\beta T}}{2\beta} \right) &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n - i(2\beta)^{-1}|p| + 1)} \left(\frac{Ame^{2\beta T}}{4\beta} \right)^{2n - i(2\beta)^{-1}|p|}, \\ I_{-i(2\beta)^{-1}|p|} \left(\frac{Ame^{2\beta T}}{2\beta} \right) &= \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n - i(2\beta)^{-1}|p| + 1)} \left(\frac{Ame^{2\beta T}}{4\beta} \right)^{2n - i(2\beta)^{-1}|p|}. \end{aligned}$$

From these, it can be seen that the series expansion of $\omega(|p|; T)$ becomes

$$\omega(|p|; T) = \sqrt{p^2 - KA^2m^2e^{4\beta T}} + K \frac{3(16 + 19p^2 - 45p^4 + 18p^6)}{32|p|^5(1 + 3p^2)^3(4 + 3p^2)} A^6 m^6 e^{12\beta T} + O(e^{16\beta T}).$$

For $K = -1$, this expression is valid for $p \neq 0$, and for $K = 1$, it is valid for $|p| > Ame^{2\beta T}$. Thus, our Hamiltonian operator $\omega(|p|, T)$ well approximates the naive Hamiltonian

$$H_c := \sqrt{p^2 - KA^2m^2e^{4\beta T}},$$

for $e^{2\beta T} \ll (Am)^{-1}$. The classical counterpart of this naive Hamiltonian exactly gives the equation of motion for the classical trajectory.

This correspondence between our Hamiltonian $\omega(|p|, T)$ and the naive Hamiltonian H_c holds wider range of T . To see this, we next consider the late time behavior of $\omega(|p|; T)$. For $e^{2\beta T} \gg (Am)^{-1}$ and $p \neq 0$, the Bessel function in the mode function behaves as

$$\begin{aligned} J_{-i(2\beta)^{-1}|p|} \left(\frac{Ame^{2\beta T}}{2\beta} \right) &= \sqrt{\frac{4\beta}{\pi Am}} e^{-\beta T} \left[\left\{ 1 - \frac{3(1 + 12p^2)(3 + 4p^2)}{128A^2m^2} e^{-4\beta T} \right\} \right. \\ &\quad \times \cos \left(\frac{Am}{2\beta} e^{2\beta T} - \frac{\pi}{4} + \frac{i\pi|p|}{4\beta} \right) \\ &\quad \left. + \frac{\beta(1 + 12p^2)}{4Am} e^{-2\beta T} \sin \left(\frac{Am}{2\beta} e^{2\beta T} - \frac{\pi}{4} + \frac{i\pi|p|}{4\beta} \right) \right] + O(e^{-7\beta T}), \end{aligned}$$

which yields the behavior of $\omega(|p|; T)$ for $K = -1$,

$$\begin{aligned} \omega(|p|; T) &= Ame^{2\beta T} + \frac{1}{2} \frac{e^{-2\beta T}}{Am \sinh^2(\pi|p|/(2\beta))} \left[\frac{(1 - 12p^2)}{12} \right. \\ &\quad \left. + \frac{5 + 48p^4}{32} \left\{ \cos \left(\frac{Am}{\beta} e^{2\beta T} - \frac{\pi}{2} \right) \cosh \frac{\pi|p|}{2\beta} - 1 \right\} \right. \\ &\quad \left. + 2p^2 \cos \left(\frac{Am}{\beta} e^{2\beta T} - \frac{\pi}{2} \right) \cosh \frac{\pi|p|}{2\beta} \right] + O(e^{-4\beta T}). \end{aligned}$$

On the other hand, the naive Hamiltonian H_c for $K = -1$ is expanded for $e^{2\beta T} \gg (Am)^{-1}$ as

$$\begin{aligned} H_c &= \sqrt{p^2 + A^2m^2e^{4\beta T}} \\ &= Ame^{2\beta T} + \frac{p^2}{2A^2m^2} e^{-2\beta T} + O(e^{-6\beta T}). \end{aligned}$$

This shows $\omega(|p|; T)$ approximates H_c in the limit $T \rightarrow \infty$,

$$\frac{\omega(|p|; T) - H_c}{H_c} = O(e^{-2\beta T}),$$

for $p \neq 0$.

For the intermediate range of T , i.e. for $Am e^{2\beta T} = O(1)$, we can see the good numerical coincidence between $\omega(|p|, T)$ and H_c . A similar argument also holds for the closeness between $\partial_p \omega(|p|, T)$ and $\partial_p H_c$. Thus, we conclude that

$$\begin{aligned}\omega(|p|; T) &\approx \sqrt{p^2 - KA^2 m^2 e^{4\beta T}}, \\ \partial_p \omega(|p|; T) &\approx \frac{p}{\sqrt{p^2 - KA^2 m^2 e^{4\beta T}}},\end{aligned}$$

holds for $K = 0, \pm 1$, and for all ranges of T in consideration.

Next, we show that the approximate Schrödinger equation

$$i\partial_T \psi(p; T) = \sqrt{p^2 - KA^2 m^2 e^{4\beta T}} \psi(p; T) \tag{4.1}$$

applied to the wave packet state, reproduces the geodesic motion in the mini-superspace, which corresponds to the classical solution to the Einstein equation.

Firstly, we note that the Ehrenfest-type theorem can be applied in the present case. The expectation values of the momentum and position operators for the normalized state $\psi(p, T)$ are given by

$$\begin{aligned}\langle P \rangle &= \int dp p \psi^*(p, T) \psi(p, T), \\ \langle Q \rangle &= i \int dp \psi^*(p, T) \partial_p \psi(p, T),\end{aligned}$$

where the range of integration is $p \in \mathbf{R}$ for $K = 0, -1$, and $|p| > \xi$ for $K = 1$. The time derivative of these expectation values are readily obtained as

$$\begin{aligned}\frac{d}{dT} \langle P \rangle &= 0, \\ \frac{d}{dT} \langle Q \rangle &= \left\langle \frac{\partial H_c}{\partial p} \right\rangle := \int dp \frac{\partial H_c}{\partial p} \psi^*(p, T) \psi(p, T).\end{aligned}$$

This shows formal, though not strict, correspondence between quantum and classical dynamics.

Next, we consider the dynamics of the wave packet state. The solution of eq. (4.1) can be written as

$$\psi_k(p, T) = \exp\left(-i \int^T ds \sqrt{k^2 - KA^2 m^2 e^{2\beta s}}\right) \delta(p - k),$$

where the solutions are labeled by the simultaneous eigenvalue k of the momentum operator subject to $k \in \mathbf{R}$ for $K = 0, -1$, and $|k| > \xi$ for $K = 1$. The general solution is written as

$$\psi(p, T) = \int dk c(k) \psi_k(p, T),$$

in terms of a coefficient $c(k)$ subject to the normalization condition

$$\int dk |c(k)|^2 = 1.$$

Now, we consider the position representation of the wave function. This is given by

$$\begin{aligned}\tilde{\psi}(X, T) &:= {}_Q\langle X; T | \psi(T); T \rangle \\ &= (2\pi)^{-1/2} \int dp e^{ipX} \psi(p, T),\end{aligned}$$

so that it is generally written as

$$\tilde{\psi}(X, T) = (2\pi)^{-1/2} \int dk c(k) \exp\left(-i \int^T ds \sqrt{k^2 - KA^2 m^2 e^{4\beta s}}\right) e^{ikX}. \quad (4.2)$$

The Born rule here could be stated as that $|\tilde{\psi}(X, T)|^2$ gives the probability distribution that a measurement of the position operator at time T yields the value $X \in \mathbf{R}$.

A wave packet state corresponding to a geodesic particle in mini-superspace is obtained if we take $c(k)$ as a bell curve, say a Gaussian-like function, centered at $k = k_0 \neq 0$, with the appropriately broad width σ^{-1} . Then, $\tilde{\psi}(X, T)$ describes a wave packet with the width σ .

From eq. (4.2), we can read off the time dependence of the dispersion relation

$$\omega'(k, T) = \sqrt{k^2 - KA^2 m^2 e^{4\beta T}},$$

between the wave-number k and the instantaneous angular frequency ω' . Then, we readily find that the group velocity $v_q(k_0, T)$ of a wave packet with the central wave-number $k_0 \neq 0$ is given by

$$\begin{aligned}v_q(k_0, T) &= \left. \frac{\partial \omega'(k, T)}{\partial k} \right|_{k=k_0} \\ &= \frac{k_0}{\sqrt{k_0^2 - KA^2 m^2 e^{4\beta T}}}.\end{aligned}$$

The corresponding quantity can be obtained from the classical theory. From the Hamiltonian constraint

$$p_T^2 - p_X^2 + KA^2 e^{4\beta T} \approx 0,$$

we get the coordinate velocity of the geodesic particle as

$$v_c = \frac{dX/ds}{dT/ds} = -\frac{p_X}{p_T} \approx \frac{p_X}{\sqrt{p_X^2 - KA^2 e^{4\beta T}}}.$$

This shows good agreement between the classical and quantum predictions if we identify the dimensionless momentum mp_X of the geodesic motion with the central wave-number k_0 of the wave packet state.

5 Concluding remarks

The quantum mechanical model of the cosmology studied here is conceptually similar to the third-quantization model, but technically different in that we respect the specific semi-Riemannian structure (\mathcal{M}, g_C) on the mini-superspace \mathcal{M} , where the classical solution to

the Einstein equation is given by the geodesic motion. In the case of the Einstein gravity coupled with a massless scalar field in the FRW background, the mini-superspace as a semi-Riemannian manifold becomes a two-dimensional expanding universe. Then, the classical solution to the Einstein equation is given by a null (resp. time-like, space-like) geodesic in the mini-superspace for the flat (resp. hyperbolic, elliptic) FRW space-time. Hence, we consider the quantum Klein-Gordon field in the mini-superspace, which becomes massless (resp. massive, tachyonic) for the flat (resp. hyperbolic, elliptic) universe, as a quantized model of the geodesic motion.

In the case of the massive or tachyonic field, we are faced with a known problem in the quantum field theory in a dynamical space-time that an inseparable Hilbert space is required to describe the quantum states of the field, where the Klein-Gordon Hamiltonian at different time belongs to a different Fock space. This continuum of the Fock spaces \mathcal{F}_T parametrized by $T \in \mathbf{R}$ constitutes the inseparable Hilbert space, which is too big. By introducing the vector bundle structure in the continuum of the Fock space and the connection of the vector bundle, we define the covariant time derivative for the quantum states, in which the Klein-Gordon Hamiltonian can be regarded as a Hermitian operator in a separable Hilbert space. Furthermore, this framework gives a unitary time evolution of 1-particle states, which would be preferable in the context of the quantum cosmology. However, in the case of elliptic universe, the momentum space for the Hilbert space of 1-particle states has to be restricted. Then, we obtain the unitary theory for a restricted range of the time parameter $T \in (-\infty, T_1)$. This time parameter corresponds to the scale factor of the elliptic universe, which represents the recollapsing universe in the classical theory, so that this limitation of the time parameter would correspond to the existence of the upper bound for the scale factor. It would not imply that the closed universe does not recollapse, but rather it should be taken as an indication of the limit of applicability of our model. This might be a common issue of the quantum cosmological model based on the minisuperspace, where the time function is given by the scale factor of the universe.

We construct the Hilbert space for the 1-particle states and the canonical observable set constituting the CCR-algebra on the space of 1-particle states. Accordingly, we obtain the Schrödinger equation for the wave function, which is the Schrödinger representation of a 1-particle state. The present quantization scheme is free from the operator ordering ambiguities and the problem of time unlike the Wheeler-DeWitt quantization scheme. We find that the Hamiltonian is close to the naive Hamiltonian predicted from the Klein-Gordon equation but with a small correction in the non-flat background cases. We see that this ensures that our quantum theory correctly reproduces the Einstein equation.

A possible advantage of the present formalism over other third-quantized models is that it is applicable to the fermionic field by considering the quantum Dirac field on (\mathcal{M}, g_C) . Another good point is that we can treat the dynamics of the single-universe state, while in typical third-quantization models [19–24], we always suffer from creation of infinite number of universes [25–27]. However, we have to introduce a mass scale of the quantum field, which is a disadvantage of the theory. In fact, we have left an unknown dimensionless parameter Am in our formulation, which should be determined by a more fundamental theory.

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