The interplay between GUT and flavour symmetries in a Pati-Salam $\times$ S_4 model

Reinier de Adelhart Toorop,^a Federica Bazzocchi^b and Luca Merlo^{c,d}

^a*Nikhef Theory Group,*

Science Park 105, 1098 XG, Amsterdam, The Netherlands

^b*Department of Physics and Astronomy, Vrije Universiteit Amsterdam,*

1081 HV Amsterdam, The Netherlands

^c*Dipartimento di Fisica ‘G. Galilei’, Università di Padova,*

INFN, Sezione di Padova, Via Marzolo 8, I-35131 Padua, Italy

^d*CERN, Department of Physics, Theory Division,*

CH-1211 Geneva 23, Switzerland

E-mail: reintoorop@nikhef.nl, fbazzoc@few.vu.nl, merlo@pd.infn.it

ABSTRACT: Both Grand Unified symmetries and discrete flavour symmetries are appealing ways to describe apparent structures in the gauge and flavour sectors of the Standard Model. Both symmetries put constraints on the high energy behaviour of the theory. This can give rise to unexpected interplay when building models that possess both symmetries. We investigate on the possibility to combine a Pati-Salam model with the discrete flavour symmetry S_4 that gives rise to quark-lepton complementarity. Under appropriate assumptions at the GUT scale, the model reproduces fermion masses and mixings both in the quark and in the lepton sectors. We show that in particular the Higgs sector and the running Yukawa couplings are strongly affected by the combined constraints of the Grand Unified and family symmetries. This in turn reduces the phenomenologically viable parameter space, with high energy mass scales confined to a small region and some parameters in the neutrino sector slightly unnatural. In the allowed regions, we can reproduce the quark masses and the CKM matrix. In the lepton sector, we reproduce the charged lepton masses, including bottom-tau unification and the Georgi-Jarlskog relation as well as the two known angles of the PMNS matrix. The neutrino mass spectrum can present a normal or an inverse hierarchy, and only allowing the neutrino parameters to spread into a range of values between λ^{-2} and λ^2 , with $\lambda \simeq 0.2$. Finally, our model suggests that the reactor mixing angle is close to its current experimental bound.

KEYWORDS: Beyond Standard Model, GUT, Discrete and Finite Symmetries, Global Symmetries

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1 Introduction

Gauge coupling unification suggests that the Standard Model (SM) gauge group is generated when a unified larger gauge group is broken at a very high energy scale compared to the electroweak (EW) one. In this picture SM fermions are accommodated in representations of the unified gauge group G and an appropriate scalar Higgs sector is introduced both to trigger the spontaneous breaking of G down to $SU(3)_C \times U(1)_{\text{em}}$ and to reproduce the fermion mass matrices. Typically Grand Unified Theories (GUTs) are quite constrained and non-trivial relations are obtained among the SM fermion mass matrices. On the other hand it is a common belief that fermion mass and mixing matrix patterns can be explained by appealing to a flavour symmetry G_f and in the last years particular attention has been devoted to the study of discrete flavour symmetries, thanks to their simplicity in recovering realistic lepton mixing patterns.

The present neutrino oscillation data [1–10] are summarised in table 1, where we display the results of two independent global fits. The pattern of the mixings is characterised by two large angles and a small one: θ_{23}^l is compatible with a maximal value, but the accuracy admits relatively large deviations; θ_{12}^l is large, but about 5σ far from the maximal value; θ_{13}^l has only an upper bound. According to the type of the experiments which measured them, the mixing angle θ_{23}^l is called atmospheric, θ_{12}^l solar and θ_{13}^l reactor. We underline that there is a tension among the two global fits presented in table 1 on the central value of the reactor angle: in [1, 2] we can read a suggestion for a positive value of $\sin^2 \theta_{13}^l \simeq 0.016 \pm 0.010$ [1.6σ], while in [3, 4] a best fit value consistent with zero within less than 1σ is found. Therefore we need a direct measurement by the future experiments

parameter	Ref. [1, 2]		Ref. [3, 4]	
	best fit (1σ)	3σ -interval	best fit (1σ)	3σ -interval
$\Delta m_{\text{sol}}^2 [\times 10^{-5} \text{eV}^2]$	$7.67^{+0.16}_{-0.19}$	$7.14 - 8.19$	$7.65^{+0.23}_{-0.20}$	$7.05 - 8.34$
$\Delta m_{\text{atm}}^2 [\times 10^{-3} \text{eV}^2]$	$2.39^{+0.11}_{-0.08}$	$2.06 - 2.81$	$2.40^{+0.12}_{-0.11}$	$2.07 - 2.75$
$\sin^2 \theta_{12}^l$	$0.312^{+0.019}_{-0.018}$	$0.26 - 0.37$	$0.304^{+0.022}_{-0.016}$	$0.25 - 0.37$
$\sin^2 \theta_{23}^l$	$0.466^{+0.073}_{-0.058}$	$0.331 - 0.644$	$0.50^{+0.07}_{-0.06}$	$0.36 - 0.67$
$\sin^2 \theta_{13}^l$	$0.016^{+0.010}_{-0.010}$	≤ 0.046	$0.010^{+0.016}_{-0.011}$	≤ 0.056

Table 1. Neutrino oscillation parameters from two independent global fits [1–4].

like DOUBLE CHOOZ [11, 12], Daya Bay [13, 14], MINOS [15], RENO [16], T2K [17] and NOvA [18].

The closeness of the leptonic atmospheric angle θ_{23}^l to the maximal value gives relevant indications on the flavour symmetry: it is well known [19, 20] that a maximal θ_{23}^l is not achievable with an exact realistic symmetry. This forces to study models based on the breaking of the flavour symmetry and a promising choice is the kind of realizations based on non-Abelian discrete groups which reproduce the lepton mixing matrix of the so-called tribimaximal (TB) pattern [21–27] ($\sin^2 \theta_{12}^{\text{TB}} = 1/3$, $\sin^2 \theta_{23}^{\text{TB}} = 1/2$ and $\sin \theta_{13}^{\text{TB}} = 0$) at leading order. This mixing scheme represents a very good approximation of the experimental measurements: the TB values for the atmospheric and the reactor angles are inside the 1σ error level, while that one for the solar angle is very close to the upper 1σ value. The corrections from the symmetry breaking provide perturbations to the angles and in particular a deviation from zero for the reactor angle, in agreement with the recent indication of a positive value for θ_{13}^l [1, 2]. Despite this success of the TB mixing scheme, there are some complications, in particular when combining the flavour symmetry with GUTs. In the following, we will comment on this and motivate the viability of a competing scheme, the bimaximal (BM) mixing pattern [28–30].

A lot of effort has been put in reproducing the TB pattern by the use of non-Abelian discrete symmetries: the best known groups implemented in the construction of flavour models are A_4 [31–80], S_4 [81–97], T' [98–107] and $\Delta(27)$ [108–110]. A common feature among many of these realizations is to get a spontaneous breaking scheme responsible for the TB mixing by the use of a convenient assignment of the quantum numbers to the SM particles and the introduction of a suitable set of scalar fields, the “flavons”, which, getting non-zero vacuum expectation values (VEVs), are responsible for the symmetry breaking of G_f . A central aspect of the model building is the symmetry breaking chain: the flavour group is broken down to two distinct subgroups, which correspond to the low-energy flavour symmetries of the charged leptons and of neutrinos.

Further studies have been presented in which the SM is extended to a GUT scenario: in our opinion, these analyses emphasize the difficulty in the construction of a flavour GUT model [111–128] which naturally lead to realistic fermion phenomenology and to a fair

explanation of the gauge symmetry breaking chain to get $SU(3)_C \times U(1)_{\text{em}}$. Concerning the case of the flavour GUT models [129–135] that predicts the TB mixing in the lepton sector, the combinations of the constraints arising by the flavour symmetry and by the GUT group lead to wrong predictions for the fermion masses and mixings. The problem is usually avoided by recurring to non-minimal Higgs or flavon field content and by assuming peculiar symmetry breaking patterns of the GUT gauge symmetry and ascribing quite often at type-II See-Saw as the origin of the neutrino masses. Moreover these patterns are often not supported by the study of the scalar Higgs potential, leaving open the question if such peculiar patterns may be actually realized or not.

Moreover, a common feature of flavour models which deal with the TB scheme is a value for the reactor angle very close to zero, in the absence of specific dynamical tricks (see [136] for a model in which such a trick is implemented). However, if the next future neutrino-appearance experiments will find a value for θ_{13}^l close to its present upper bound, about the sine of the Cabibbo angle λ , the TB mixing should be considered an accidental symmetry. In this case a new leading principle would be necessary, for which quark-lepton complementarity [137–157], $\theta_{12}^l + \lambda \sim \pi/4$, would be a good candidate. This has the advantages that it can be naturally related to GUTs and does not necessarily predict a small reactor mixing angle.

This idea was developed in a model [158] based on the S_4 discrete group. In this construction the PMNS matrix coincides with the bimaximal mixing [28–30] ($\sin^2 \theta_{12}^{\text{BM}} = 1/2$, $\sin^2 \theta_{23}^{\text{BM}} = 1/2$ and $\sin \theta_{13}^{\text{BM}} = 0$) in first approximation; since the BM value of the solar angle exceeds the 3σ error, large corrections are needed to make the model agree with the data; the perturbations are naturally constrained to get the “weak” complementarity relation, $\theta_{12}^l + \mathcal{O}(\lambda) \sim \pi/4$, and $\sin \theta_{13}^l \sim \lambda$ in most of the parameter space. The model only deals with the lepton sector and an extension in the quarks sector is lacking. In this paper we aim at revisiting the model in [158] in order to include a realistic description of quarks.

Our starting point lies in the complementarity relations:

$$\theta_{12}^l + \theta_{12}^q \simeq \pi/4 + \mathcal{O}(\lambda^2), \quad \theta_{23}^l + \theta_{23}^q \simeq -\pi/4 + \mathcal{O}(\lambda^2). \quad (1.1)$$

For the third mixing angles we know that in the quark sector it is very small, $\theta_{13}^q = \mathcal{O}(\lambda^3)$, while in the lepton sector, as already mentioned, it has only an upper bound, $\sin \theta_{13}^l \lesssim \lambda$. We will see that in the model described below, we predict it to be $\mathcal{O}(\lambda)$.

In non-GUT contexts, no compelling model leading to the exact complementarity has been produced so far and indeed in [158] a weaker version of eq. (1.1) has been used in which the quark contributions are substituted by similar terms originating from the charged lepton sector. On the other hand, exact complementarity is possible in cases where the flavour symmetry group is combined with a GUT group, as in the Pati-Salam¹ context [159–171], where the charged lepton and the down-quark mass matrices are similar,

$$M_e \sim M_d. \quad (1.2)$$

¹Adopting a Pati-Salam context, we loose the exact gauge coupling unification, but at the same time we avoid several complications which are instead present in other GUT scenarios, as described in the text.

Other popular GUTs are SU(5) and SO(10), which however are less appealing when trying to recover the QL complementarity relations. Indeed, in the minimal SU(5) (for a general review on SU(5), see [172, 173]) one has $M_e \sim M_d^T$ and as result, a correction of order λ to the solar angle does not correspond to the Cabibbo angle of the CKM matrix.

On the other hand a reason to prefer Pati-Salam over SO(10) is related to the type-I and type-II See-Saw mechanisms. In these two GUT contexts, we expect both left-handed (LH) and right-handed (RH) neutrino Majorana mass matrices to be present. As a result, the effective LH neutrino mass matrix will get the contributions through the type-I as well as the type-II See-Saw mechanisms. In general, and this happens also in our proposal, this interplay introduces two mass scales and a highly non-trivial flavour structure for the effective neutrino mass matrix, which difficultly leads to a realistic description of the PMNS matrix. For this reason, a hierarchy between the two contributions is usually assumed. As we will see we can reproduce quark-lepton complementarity in our model, if the type-II See-Saw is dominant.

This possibility has already been investigated in several flavour GUT models, for example in [174–177] in the context of the SO(10) GUT. However in [178] it has been proven that the type-II dominance is not a viable solution in such GUTs. In the Pati-Salam context, there is more freedom and the eventual dominance of one of the two contributions could be realized. In this paper we study the gauge Higgs potential and we verify that a type-II dominance can be justified, even if it puts strong constraints on the model building realization.

After a detailed analysis of the fermion phenomenology, we move to the study on the Higgs gauge sector, which is responsible for the gauge symmetry breaking steps to finally get $SU(3)_C \times U(1)_{em}$. Thus we consider the renormalization group equations (RGEs) both for the gauge couplings and for the fermion masses and mixings from the cutoff of the theory down to the low-energy scale. Here we anticipate that the RGEs analysis is crucial: the model turns out to be viable only in a small region of the parameter space. On one side the gauge coupling RGEs analysis is deeply modified by the non-minimal Higgs field content required by the presence of the flavour symmetry G_f and puts strong constraints on the scales of the model. On the other side the GUT flavour mass matrix structures interfere with each other, through the Yukawa RGEs, with non-negligible consequences for the neutrino phenomenology.

The paper is organized as follows. In section 2 we describe the flavour structure of the fermion mass matrices in order to recover realistic lepton and quark mixing matrices and fermion mass hierarchies. In section 3 we enter in the details of the model building construction, specifying the transformations of all the fields of the model under the gauge and flavour groups and discussing the flavon vacuum misalignment necessary to the flavour symmetry breaking chain. Afterwards we deal in section 4 with the study of the gauge Higgs potential and we analyze the constraints coming from the flavour symmetry on the scalar field content. In section 5 we perform the analysis of the renormalization group running of fermion masses and mixings and of the gauge coupling constants from the GUT scale down to the low-energy scale. In section 6 we report the phenomenological analysis after the running evolution. Finally in section 7 we conclude. In the appendices we report technical details.

2 Outline of the model

In this section we present a general discussion on the flavour structure of fermion masses and mixings resulting from the analysis of the complementarity relations. Eq. (1.1) suggests that the angles in the CKM and PMNS matrices may have a common origin. Looking at their definitions

$$V = V_u^\dagger V_d, \quad U = U_e^\dagger U_\nu, \quad (2.1)$$

where V_u , V_d , U_e , U_ν diagonalize $M_u M_u^\dagger$, $M_d M_d^\dagger$, $M_e M_e^\dagger$ and m_ν respectively, we see that this common origin can be motivated in a GUT context where some relations among the mass matrices are present. In PS models, we have the following expression which links down-quarks and charged leptons

$$U_e \sim V_d; \quad (2.2)$$

we will see in a while how this enters in the model construction.

In [158] the two large lepton angles in the PMNS matrix arise only from the neutrino sector: in the first order approximation, both the solar and the atmospheric angles are maximal and after that some corrections from the charged lepton sectors lower the value of the solar angle in order to accommodate the data. In our realization one maximal angle originates from the charged leptons and the other from the neutrinos: afterwards some corrections are necessary to make the solar angle agree with the measurements and we require that these corrections come from the charged leptons. This is an other possible choice with respect to [158] and in the next section we will show that it is easily realized in our context.

In the approximation of small λ , the lepton mixing matrix can schematically be written as

$$U = R_{23} \left(-\frac{\pi}{4} \right) R_{13}(\lambda) R_{12} \left(\frac{\pi}{4} - \lambda \right) = \underbrace{\left(R_{23} \left(-\frac{\pi}{4} \right) R_{13}(\lambda) R_{12}(-\lambda) \right)}_{U_e^\dagger} \underbrace{R_{12} \left(\frac{\pi}{4} \right)}_{U_\nu}, \quad (2.3)$$

where $R_{ij}(\theta)$ stands for a rotation in the (ij) plane of an angle θ .² As a result, after a suitable commutation of matrices, U_e can naively be written as

$$U_e = R_{23} \left(\frac{\pi}{4} \right) R_{13}(\lambda) R_{12}(\lambda). \quad (2.4)$$

The CKM matrix is given in first approximation as

$$V = R_{12}(\lambda). \quad (2.5)$$

Because of eq. (2.2), V_d has the same structure as U_e in eq. (2.4) and therefore we can obtain the CKM matrix as

$$V = \underbrace{\left(R_{12}(-\alpha\lambda) R_{13}(-\lambda) R_{23} \left(-\frac{\pi}{4} \right) \right)}_{V_u^\dagger} \underbrace{R_{23} \left(\frac{\pi}{4} \right) R_{13}(\lambda) R_{12}(\lambda)}_{V_d}. \quad (2.6)$$

²A coefficient of order one is in general present to multiply each angle, but we do not show them here in order to simply the discussion. We will consider the precise expressions in the following sections.

We see that the angles of the (23) and (13) rotations in $V_u^\dagger$ should be the opposite of those in V_d , while the angles in the (12) sector should be different. We have schematically indicated this via the α coefficient. Analogous to eq. (2.4), we write V_u as

$$V_u = R_{23} \left(\frac{\pi}{4} \right) R_{13}(\lambda) R_{12}(\alpha\lambda) . \quad (2.7)$$

Moving to the explicit form of the mass matrices, the generic effective Majorana neutrino mass matrix m_ν which is diagonalized by U_ν as in eq. (2.3), $U_\nu = R_{12} \left(\frac{\pi}{4} \right)$, through $m_\nu^{\text{diag}} = U_\nu^T m_\nu U_\nu$, is given by

$$m_\nu \sim \begin{pmatrix} a & b & 0 \\ b & a & 0 \\ 0 & 0 & c \end{pmatrix} . \quad (2.8)$$

In the charged lepton sector, we are looking for the most general form of the mass matrix, whose square $M_e M_e^\dagger$ is diagonalized by the action of U_e of eq. (2.4),

$$\left(M_e M_e^\dagger \right)^{\text{diag}} = U_e^\dagger M_e M_e^\dagger U_e . \quad (2.9)$$

Inverting eq. (2.9) we find in the limit $m_e \rightarrow 0$

$$M_e M_e^\dagger \sim \frac{m_\tau^2}{2} \begin{pmatrix} 0 & \lambda & \lambda \\ \lambda & 1 & 1 \\ \lambda & 1 & 1 \end{pmatrix} + \frac{m_\mu^2}{2} \begin{pmatrix} 0 & \lambda & -\lambda \\ \lambda & 1 & -1 \\ -\lambda & -1 & 1 \end{pmatrix} + \dots , \quad (2.10)$$

where the dots stand for suppressed contributions. This can be obtained if M_e is given by

$$M_e \sim \frac{m_\tau}{\sqrt{2}} \begin{pmatrix} 0 & 0 & \lambda \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} + \frac{m_\mu}{\sqrt{2}} \begin{pmatrix} 0 & \lambda & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{pmatrix} + \dots . \quad (2.11)$$

It is interesting to note that, moving to the basis of diagonal charged leptons and considering only the leading order terms, the neutrino mass matrix results to be of the BM type, already presented in the introduction.

As already stated, the PS gauge structure implies the relation $M_e \sim M_d$ and therefore the down-quark matrix has a similar structure as in eq. (2.11). Looking at eq. (2.7) we find that also M_u should have a similar structure, but here there is a slight difference due to the α coefficient.

In order to reproduce the result in eq. (2.6), i.e. exact cancellations in the (23) and (13) sectors but not in the (12) sector, we need to study the origin of the different rotations in (2.7). It is easy to see that in the diagonalization of the mass matrices in eq. (2.10), the (12) rotation originates from the second families, while the (13) rotation comes from the third families. We conclude that the up-quark mass matrix has the same form as (2.11), in which the third column is proportional to that of the down-quarks, while the second column is not.

As already stated in the introduction, we expect contributions on the effective neutrino mass matrix coming from both the type-I and the type-II See-Saw mechanisms. For the moment we just assume that the type-II is the only responsible of eq. (2.8) and we verify in section 4 that it is indeed dominating with respect to the type-I terms.

In the next section we enter in the details of the model building, explaining the origin of the mass matrices displayed above.

3 The flavour model building

In this section we define our framework. The model is based on the PS gauge symmetry, $SU(4)_C \times SU(2)_L \times SU(2)_R$, present at high energy where a supersymmetric context is assumed. The complete flavour group G_f is the same as in [158] given by the product of the following different terms:

$$G_f = S_4 \times Z_4 \times U(1)_{\text{FN}} \times U(1)_R. \quad (3.1)$$

The group S_4 is the permutation group of four distinct objects, isomorphic to the group O which is the symmetry group of a regular octahedron. It has 24 distinct elements filled in five conjugate classes and therefore it has five irreducible representations, two one-dimensional denoted as 1_1 and 1_2 , one two-dimensional labelled as 2 and two three-dimensional written as 3_1 and 3_2 . Here we recall the multiplication rules, while the Clebsch-Gordan coefficients, the explicit structures of the generators, the list of the elements and further details on the group theory of S_4 , are reported in appendix A:

$$\begin{aligned} 1_1 \times R &= R \times 1_1 = R && \text{where } R \text{ stands for any representation} \\ 1_2 \times 1_2 &= 1_1 \\ 1_2 \times 2 &= 2 \\ 1_2 \times 3_1 &= 3_2 \\ 1_2 \times 3_2 &= 3_1 \\ \\ 2 \times 2 &= 1_1 + 1_2 + 2 \\ 2 \times 3_1 &= 3_1 + 3_2 \\ 2 \times 3_2 &= 3_1 + 3_2 \\ \\ 3_1 \times 3_1 &= 3_2 \times 3_2 = 1_1 + 2 + 3_1 + 3_2 \\ 3_1 \times 3_2 &= 1_2 + 2 + 3_1 + 3_2. \end{aligned} \quad (3.2)$$

The spontaneous symmetry breaking of S_4 is responsible for the flavour structure of the matrices in eqs. (2.8), (2.11): S_4 is broken down to two distinct subgroups and it is the presence of this mismatch at the LO which allows to construct eqs. (2.8), (2.11). More in detail, the different subgroups to which S_4 is broken down are the subgroups preserved by the VEVs of the flavons. In order to determine these structures, it is necessary to identify the elements of the group which leave the VEVs of the flavons invariant under their action. Doing so, we find that S_4 is broken down to $Z_2 \times Z_2$ in the neutrino sector, originated

by the elements ST^2S and T^3ST of the classes $\mathcal{C}_2$ and $\mathcal{C}_3$, respectively. In the charged fermion sector, we anticipate that the superpotential is invariant under an accidental Z_2 symmetry and therefore the identification of the residual subgroup must account of this additional term. We find that S_4 is broken down to a $Z_2 \times Z_2$ group, distinct to that one in the neutrino sector, generated by the two elements $-T^2$ and $-TST^2S$ of the classes $\mathcal{C}_2$ and $\mathcal{C}_3$, respectively. One can argue that the group $Z_2 \times Z_2$ implies a degeneracy among two of the charged fermion families, but this does not happen, because the two Z_2 relate different families: the accidental Z_2 exchanges F_2^c and F_3^c , while the Z_2 subgroup of S_4 relates F_1^c and F_2^c .

The other terms in G_f carry out other roles: the Abelian Z_4 symmetry is required to avoid dangerous terms in the superpotential, it helps to keep the different sectors of the model separated, quarks from leptons and neutrinos from charged leptons, and it is also useful to guarantee the flavon vacuum alignment; the continuous Frogatt-Nielsen (FN) Abelian symmetry [179], $U(1)_{FN}$, is introduced to justify charged fermion mass hierarchies; the continuous R -symmetry $U(1)_R$, that contains the usual R -parity as a subgroup, is a common feature of supersymmetric formulations and simplifies the constructions of the scalar potential. It is worth stressing that the supersymmetric context is of great utility in the discussion of the scalar potential and helps in the gauge coupling running, but a similar non-supersymmetric model can be constructed as well. In particular in the present paper we only deal with SM particle and therefore we use the same symbols for a supermultiplet and its even R -parity components.

3.1 The matter, Higgs and flavon content of the model

In the PS context, the five matter multiplets of each family of the SM plus a RH neutrino and their superpartners are unified in only two supermultiplets: a LH and a RH ones as follows,

$$\begin{aligned}
 \text{SU(3)}_C \times \text{SU(2)}_L \times \text{U(1)}_Y & \rightarrow \text{PS} \\
 (3, 2, 1/6)_Q + (1, 2, -1)_L & \rightarrow (4, 2, 1) \\
 (\bar{3}, 1, -2/3)_{uc} + (\bar{3}, 1, 1/3)_{dc} + (1, 1, 1)_{ec} + (1, 1, 0)_{\nu^c} & \rightarrow (\bar{4}, 1, 2)
 \end{aligned} \tag{3.3}$$

The three copies of the LH supermultiplet are combined in the three-dimensional representation 3_1 of S_4 , while the three families of the RH supermultiplet are in 1_2 , 1_2 and 1_1 respectively. The fact that we can put different representations within one family in different representations of the family symmetry group is essential here. Note that this would not be possible in (minimal) $\text{SO}(10)$, where all Standard Model particles are in one sixteen dimensional representation. The first two families are also charged under $U(1)_{FN}$ by two units. This suppresses their masses with respect to the third family ones. Further suppression of the first family with respect to the second is due to their different Z_4 charges.

All the properties of the matter fields are summarized in table 2.

Our model contains five flavon fields: two S_4 triplets (φ and φ') that, because of their Z_4 charge, deal at LO only with the Dirac Yukawa couplings of quarks and leptons, and two

Matter	F_L	F_1^c	F_2^c	F_3^c
PS	$(4, 2, 1)$	$(\bar{4}, 1, 2)$	$(\bar{4}, 1, 2)$	$(\bar{4}, 1, 2)$
S_4	3_1	1_2	1_2	1_1
$U(1)_{FN}$	0	2	2	0
Z_4	1	1	i	$-i$

Table 2. Transformation properties of the matter fields. Notice that the PS assignments should be read in agreement with $SU(4)_C \times SU(2)_L \times SU(2)_R$.

Flavons	θ	φ	φ'	χ	σ
S_4	1_1	3_1	3_2	3_1	1_1
$U(1)_{FN}$	-1	0	0	0	0
Z_4	1	i	i	1	1

Table 3. The flavon field content and their transformation properties under the flavour symmetries. All flavon fields are singlet of the gauge group.

fields, one singlet (σ) and one triplet (χ), that, by Z_4 , deal at LO only with the Majorana masses of neutrinos. The fifth flavon is the Froggatt-Nielsen messenger, which we indicate with θ . Their properties are shown in table 3. Under the continuous R -symmetry, the matter superfields transform as $U(1)_R = 1$, while all the flavons are neutral.

Fermion masses and mixings arise from the spontaneous breaking of the flavour symmetry by means of the flavons which develop VEVs according to the following configuration: at LO we have

$$\langle \varphi \rangle = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} v_\varphi, \quad \langle \varphi' \rangle = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} v_{\varphi'}, \quad (3.4)$$

$$\langle \chi \rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v_\chi, \quad \langle \sigma \rangle = v_\sigma, \quad (3.5)$$

$$\langle \theta \rangle = v_\theta. \quad (3.6)$$

In this section we simply assume this VEV alignment and we will prove it to be a natural solution of the minimization of the scalar potential in section 3.4. Furthermore we assume that the FN messenger and the other flavons have VEVs of the same order of magnitude: it results partly from the minimization procedure and partly from the constraints coming from the comparison with the measured mass hierarchies, as it will be clearer in the following. Without loss of generality it is useful to keep the notation compact and write

$$\frac{VEV}{\Lambda} \approx \lambda, \quad (3.7)$$

where VEV refers to the vacuum expectation value of any flavon of the model and $\lambda \simeq 0.2$

Higgses	ϕ, ϕ'	ρ	Δ_L	Δ_R
PS	$(1, 2, 2)$	$(15, 2, 2)$	$(\overline{10}, 3, 1)$	$(10, 1, 3)$
Z_4	1	-1	1	-1

Table 4. The Higgs fields responsible of generating fermion mass matrices and their transformation under the gauge and the Abelian flavour symmetries. All Higgs fields are singlets under $S_4 \times U(1)_{FN} \times U(1)_R$, while they can transform under the Z_4 factor.

is close to the Cabibbo angle. Λ is the flavon cut-off that we assume to be the largest scale in the model: it corresponds to the scale of the flavon dynamics.

The insertion of flavons in the mass terms, leads to non-renormalizable operators, as we will see in the following section. This means that we can write the superpotential as an expansion in powers of flavon/ Λ and we can stop the expansion after the first orders.

The Higgs fields of our model relevant to build the fermion mass matrices transform under the gauge group and under the Z_4 factor of the flavour symmetry: in table 4 we summarize their transformation rules. The first three fields, ϕ , ϕ' and ρ , deal at LO only with the Dirac Yukawas. Due to the Z_4 charges, ϕ and ϕ' are responsible of the third family and the charm quark masses, while ρ is responsible for the strange and μ masses. The field $\rho \sim (15, 2, 2)$ being in the adjoint of $SU(4)_C$ may develop VEV along the $SU(4)_C$ direction $\text{diag}(-3, 1, 1, 1)$. This implies that the leptons which get mass via this field are a factor 3 heavier than the corresponding quarks and therefore this field is very useful to describe the second family, at least in the down sector, reproducing the well known Georgi-Jarlskog relation [180], $m_\mu \approx 3m_s$, at the high energy scale. As we will see in the next sections, in order to recover the up-quark mass hierarchies the ρ projection along the light doublet Higgses, v_ρ^u and v_ρ^d , has to point only in the down direction: the requirement $v_\rho^u = 0$ can be realized only if the Higgs field content contains two identical copy of the Higgs field $(1, 2, 2)$ and this justify the presence of ϕ and ϕ' .

Finally, as we will see in detail in the following sections, the field Δ_R is necessary to conclude the PS symmetry breaking pattern and to recover the SM gauge group through its spontaneous symmetry breaking VEV. At the same time, when Δ_R develops a VEV, it gives a Majorana mass to the right-handed neutrinos thus contributing to the effective neutrino mass matrix through the usual type-I See-Saw mechanism. A second source for the neutrino mass matrix comes from Δ_L , in terms of the type-II See-Saw mechanism.

In our scheme the neutrino mass matrix is dominated by type-II See-Saw. As already stated in the introduction, the flavour structure of the effective neutrino mass matrix in eq. (2.8) arises from an interplay between the two See-Saw sources. In the PS context $m_D \sim M_u$ and this suggests a hierarchical structure for the type-I contribution, which does not agree with the flavour structure in eq. (2.8). We will show that, given $m_D \sim M_u$, the required flavour structure for the Majorana RH neutrino mass necessary to recover eq. (2.8) is not allowed in our model. This suggests to find a construction in which the type-II contributions are dominating over the type-I and we will show that such a feature puts strong constraints on the model building.

3.2 Fermion mass matrices at leading order

In this section we study the matter superpotential, $\mathcal{W}_Y$, and the resulting mass matrices for all the fermions. The Yukawa superpotential can be written as a sum of two pieces,

$$\mathcal{W}_Y = \mathcal{W}_{Dir} + \mathcal{W}_{Maj} \quad (3.8)$$

containing respectively terms with Dirac and Majorana structures. Furthermore, making a power expansion in terms of flavon/ Λ , we distinguish between leading and subleading couplings. We refer as “leading order” operators to those terms which provide $M_{e,d,u}$ and m_ν as in eqs. (2.8), (2.11) in the limit $\lambda \rightarrow 0$. The subleading orders corresponds to operators of higher dimensions. In this way it is easier to identify each entry of the mass matrices with the corresponding operator in the superpotential and furthermore it underlines the relevance of the subleading contributions.

3.2.1 Dirac mass terms

We first study the Dirac matter superpotential at LO which reads

$$\begin{aligned} \mathcal{W}_{Dir}^{LO} = & y_1 \frac{1}{\Lambda} F_L F_3^c (\phi + \phi') \varphi + \\ & + y_2 \frac{1}{\Lambda^3} F_L F_2^c \theta^2 \rho \varphi' + \sum_{i=1}^4 y_{3,(i)} \frac{1}{\Lambda^5} F_L F_2^c \theta^2 (\phi + \phi') X_i^{(1)} + \\ & + \sum_{i=1}^3 y_{4,(i)} \frac{1}{\Lambda^4} F_L F_1^c \theta^2 \rho X_i^{(2)} + y_5 \frac{1}{\Lambda^5} F_L F_1^c \theta^2 (\phi + \phi') \chi^3 . \end{aligned} \quad (3.9)$$

Here we use a compact notation to avoid the proliferation of coefficients: the term $X_i^{(1,2)}$ represents a list of products defined as

$$\begin{aligned} X_i^{(1)} &\equiv \{\varphi^3, \varphi^2 \varphi', \varphi \varphi'^2, \varphi'^3\} , \\ X_i^{(2)} &\equiv \{\varphi^2, \varphi \varphi', \varphi'^2\} , \end{aligned} \quad (3.10)$$

where each term represents all the different S_4 contractions which can be constructed with those flavons; furthermore we indicate with $y_1(\phi + \phi')$ the combination $y_1^{(1)}\phi + y_1^{(2)}\phi'$ and similarly for $y_{3,(i)}$ and y_5 .

When the flavour symmetry is broken the model describes a non-minimal PS model in which the Yukawa couplings present a well defined structure. Then, when the PS gauge symmetry is broken to the SM gauge group ϕ, ϕ' and the colour singlet component of ρ mix in four light Higgses, two up-type and two down-type, $h_{u,d}$ and $h'_{u,d}$. Thus at the EW scale ϕ, ϕ' and ρ have non-vanishing projections to the light Higgs components that acquire a VEV. We will indicate these components as $v_\phi^{u,d}, v_{\phi'}^{u,d}$ and $v_\rho^{u,d}$. As already said, we need to impose that the ρ field has no projection along the up direction: $v_\rho^u = 0$. This can be realized because in terms of the light Higgs up-VEVs, $v_1^u = \langle h_u \rangle$ and $v_2^u = \langle h'_u \rangle$, v_ρ^u is given by

$$v_\rho^u = U_{13}v_1^u + U_{23}v_2^u , \quad (3.11)$$

where the matrix U is introduced in the appendix B. The constraint $v_2^u = -U_{13}/U_{23}v_1^u$ can be imposed thanks to the freedom we have in the superpotential and in the soft potential.³ Note that we could relax the condition $v_\rho^u = 0$ allowing a mild hierarchy between v_ρ^u and v_ρ^d , for example of order λ^2 , without affecting the final mass hierarchies, but in the following, for simplicity, we work under the assumption that $v_\rho^u = 0$.

The final Dirac fermion mass matrices we get are given by

$$M_e^{LO} = -3 \begin{pmatrix} 0 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & -y_2 & 0 \end{pmatrix} v_\rho^d \lambda^3 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y_1 \\ 0 & 0 & y_1 \end{pmatrix} v_\phi^d \lambda, \quad (3.12)$$

$$M_d^{LO} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & y_2 & 0 \\ 0 & -y_2 & 0 \end{pmatrix} v_\rho^d \lambda^3 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y_1 \\ 0 & 0 & y_1 \end{pmatrix} v_\phi^d \lambda, \quad (3.13)$$

$$M_u^{LO} = m_D^{LO} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & y_3 & 0 \\ 0 & -y_3 & 0 \end{pmatrix} v_\phi^u \lambda^5 + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & y_1 \\ 0 & 0 & y_1 \end{pmatrix} v_\phi^u \lambda, \quad (3.14)$$

where we used the compact notation $y_i v_\phi^{u/d}$ to indicate $y_i^{(1)} v_\phi^{u/d} + y_i^{(2)} v_{\phi'}^{u/d}$ and absorbed all the non-relevant CG coefficients. Note that y_3 is the sum of the different $y_{3,(i)}$ and that, by construction, $y_{1,2,3}$ can be considered complex coefficients with modulus of order 1. Note also that the different numerical coefficients between charged leptons and down-quarks originate from the presence of ρ instead of ϕ (ϕ') in the superpotential. The operators which should give contributions to the first families (those proportional to y_4 and y_5) are vanishing, thanks to the special flavon VEV alignment. As a final comment, we are neglecting at this level of approximation the contributions to $M_{e,d}^{LO}$ coming from the operators proportional to y_3 : these terms, which preserve the anti-alignment of the second and third entries of the second columns, are λ^2 suppressed with respect to the LO ones proportional to y_2 .

In order to identify the mass matrices in eqs. (3.12)–(3.13)–(3.14) with those in eqs. (2.8), (2.11), we need to define the (complex) fermion masses as follows:

$$\begin{aligned} m_\mu &\equiv -3y_2 v_\rho^d \lambda^3, & m_\tau &\equiv y_1 v_\phi^d \lambda, \\ m_s &\equiv y_2 v_\rho^d \lambda^3, & m_b &\equiv y_1 v_\phi^d \lambda, \\ m_c &\equiv y_3 v_\phi^u \lambda^5, & m_t &\equiv y_1 v_\phi^u \lambda. \end{aligned} \quad (3.15)$$

Now we can comment on the masses as well as on the mixing matrices which can be driven at this approximation level. Notice that the top-quark Yukawa does not come

³This requirement, imposed *by hand*, could be motivated by some symmetry argument, but to introduce a mechanism that could explain $v_\rho^u = 0$, it would be necessary deeply modifying the structure of our model. In the present paper we just assume this fine-tuning and we refer to the appendix B.2.1 for further details. We just anticipate that the fine-tuning we introduce in the model is similar to the fine-tuning which is universally accepted whenever the MSSM has to be recovered.

from a renormalizable coupling and it presents the same suppression as the other third family fermion masses: as a result the dominance of the top-quark mass can be justified by the hierarchy between v_ϕ^u and v_ϕ^d . The mass matrices in eqs. (3.12)–(3.13)–(3.14) are diagonalized by a maximal rotation in the sector (23), i.e. $U_e = V_d = V_u = R_{23}(\pi/4)$, while the fermion mass hierarchies are given by

$$\left| \frac{m_\mu}{m_\tau} \right| \sim \left| \frac{m_s}{m_b} \right| \sim \mathcal{O}(\lambda^2), \quad \left| \frac{m_c}{m_t} \right| \sim \mathcal{O}(\lambda^4). \quad (3.16)$$

Furthermore, at the cutoff, we recover some relations among the masses of different fermions: the $b - \tau$ unification and the Georgi-Jarlskog relation [180]

$$|m_\tau| = |m_b|, \quad |m_\mu| = 3|m_s|. \quad (3.17)$$

Finally we should comment of the relative value of the top and of the bottom masses:

$$\left| \frac{m_t}{m_b} \right| = \left| \frac{y_1^{(1)} v_\phi^u + y_1^{(2)} v_{\phi'}^u}{y_1^{(1)} v_\phi^d + y_1^{(2)} v_{\phi'}^d} \right|. \quad (3.18)$$

Notice that usually this ratio is proportional to $\tan \beta$, the ratio between the up- and down-VEVs of the light Higgses, but this is not the case: indeed the light Higgses are combinations of ϕ , ϕ' and ρ and therefore we may define $\tan \beta$ as

$$\tan \beta \equiv \frac{\sqrt{(v_\phi^u)^2 + (v_{\phi'}^u)^2}}{\sqrt{(v_\phi^d)^2 + (v_{\phi'}^d)^2 + (v_\rho^d)^2}} \neq \left| \frac{m_t}{m_b} \right|. \quad (3.19)$$

3.2.2 Majorana mass terms

We now discuss the part of the superpotential which contains the Majorana couplings. At LO⁴ it is given by

$$\mathcal{W}_{Maj}^{LO} = \tilde{k}_0 F_L F_L \Delta_L + \sum_{i=1}^2 \frac{\tilde{k}_{1,(i)}}{\Lambda} F_L F_L \Delta_L X_i^{(3)} + \sum_{i=1}^3 \frac{\tilde{k}_{2,(i)}}{\Lambda^2} F_L F_L \Delta_L X_i^{(4)} + z_1 F_3^c F_3^c \Delta_R, \quad (3.20)$$

where we used the compact notation

$$\begin{aligned} X^{(3)} &\equiv \{\chi, \sigma\}, \\ X^{(4)} &\equiv \{\chi^2, \chi\sigma, \sigma^2\}. \end{aligned} \quad (3.21)$$

This superpotential is responsible for giving the following Majorana LH and RH neutrino mass matrices:

$$M_L = \begin{pmatrix} k_0 & k_1 \lambda & 0 \\ k_1 \lambda & k_0 & 0 \\ 0 & 0 & k_0 + k_2 \lambda^2 \end{pmatrix} v_L, \quad M_R = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z_1 \end{pmatrix} v_R, \quad (3.22)$$

⁴Regarding the terms which contribute to M_R , we consider at the LO only the first non vanishing term in the superpotential.

where $k_{0,1,2}$ and z_1 are coefficients of order 1 and v_L, v_R are the VEVs of $\Delta_{L,R}$ respectively. In particular $k_{0,1,2}$ are defined in terms of $\tilde{k}_0, \tilde{k}_{1,(i)}$ and $\tilde{k}_{2,(i)}$:

$$\begin{aligned} k_0 &\equiv \tilde{k}_0 + \tilde{k}_{1,(2)}\lambda + \tilde{k}_{2,(1)}\lambda^2 + \tilde{k}_{2,(3)}\lambda^2, \\ k_1 &\equiv \tilde{k}_{1,(1)} + \tilde{k}_{2,(2)}\lambda, \\ k_2 &\equiv -\tilde{k}_{2,(1)}. \end{aligned} \quad (3.23)$$

While M_L corresponds to the type-II contribution to the effective neutrino mass matrix, M_R provides a type-I term. Even at this approximation level, we note the tension between the two See-Saw contributions: $m_\nu^{(\text{type-II})} \equiv M_L$ presents a maximal rotation in the (12) sector, while it is easy to verify that $m_\nu^{(\text{type-I})} \equiv m_D M_R^{-1} m_D^T$ shows a democratic structure on the (23) sector which corresponds to a maximal rotation in this sector. To recover the mass matrix in eq. (2.8) it is necessary that the type-II contribution dominates over the type-I terms: in this case it is sufficient to identify a with k_0 , b with $k_1\lambda$ and c with $k_0 + k_2\lambda^2$.

The neutrino masses can be written as

$$|m_{1,2}|^2 = \left(|k_0|^2 \mp 2|k_0||k_1| \cos(\theta_{k_0} - \theta_{k_1})\lambda + |k_1|^2 \lambda^2 \right) v_L^2, \quad (3.24)$$

$$|m_3|^2 = \left(|k_0|^2 + 2|k_0||k_2| \cos(\theta_{k_0} - \theta_{k_2})\lambda^2 \right) v_L^2, \quad (3.25)$$

with θ_{k_i} the argument of the complex number k_i . We need $\cos(\theta_{k_0} - \theta_{k_1}) > 0$ in order to have $|m_1|$ smaller than $|m_2|$. We see that in most of parameter space the spectrum is quasi degenerate, as the term with $|k_0|^2$ that appears in all three masses dominates over the other terms that are λ or λ^2 suppressed.

We also see that in most of the parameter space $|m_3|$ is the central eigenvalue: indeed $|m_1|$ ($|m_2|$) is shifted down (up) from the central value $|k_0|^2 v_L^2$ by a term proportional to λ , while $|m_3|$ stays closer to this central value as it is only shifted by a term proportional to λ^2 . Having $|m_3|$ as the central eigenvalue is obviously in contradiction with experimental data. We conclude that our model is only viable when the term $2|k_0||k_1| \cos(\theta_{k_0} - \theta_{k_1})\lambda$ is suppressed. This is clearly possible if either $|k_0|$ or $|k_1|$ or $\cos(\theta_{k_0} - \theta_{k_1})$ (or a combination of them) is small. In particular, the latter condition means that k_0 and k_1 are almost perpendicular in the complex plane. We now investigate on these different scenarios, calculating the solar and atmospheric mass squared differences: taking as definition of Δm_{atm}^2 the mass squared difference between the heaviest and the lightest neutrinos, we have different results for normal (NO) and inverse (IO) mass ordering as given by

$$\begin{aligned} \Delta m_{sol}^2 &\equiv m_2^2 - m_1^2 \\ &= 4|k_0||k_1|\lambda \cos(\theta_{k_0} - \theta_{k_1})v_L^2, \\ \Delta m_{atm}^2 &\equiv \begin{cases} \Delta m_{atmNO}^2 \equiv (m_3^2 - m_1^2) \\ \Delta m_{atmIO}^2 \equiv (m_2^2 - m_3^2) \end{cases} \\ &= \underbrace{2|k_0||k_1| \cos(\theta_{k_0} - \theta_{k_1})\lambda v_L^2}_{\Delta m_{sol}^2/2} \mp (|k_1|^2 - 2|k_0||k_2| \cos(\theta_{k_0} - \theta_{k_2}))\lambda^2 v_L^2, \end{aligned} \quad (3.26)$$

On the right-hand side of the last equation, the first term is suppressed by λ , while the second one by λ^2 , and therefore we could conclude that it is the dominant contribution: it is however exactly the term which must be suppressed in order to avoid $|m_3|$ as the central eigenvalue. Furthermore, this term is equal to half of the solar mass squared difference that is about 30 times as small as the atmospheric splitting (see table 1). As a result, to recover a value for Δm_{atm}^2 close to the measured one, we need that the second term on the right-hand side of eq. (3.26) is the dominant one. We can estimate the ratio between the two terms, calculating r , the ratio of the two mass squared differences:

$$\begin{aligned}
 (r_{NO,IO})^{-1} &\equiv \frac{\Delta m_{atm,NO,IO}^2}{\Delta m_{sol}^2} \\
 &= \frac{2|k_0||k_1| \cos(\theta_{k_0} - \theta_{k_1}) \lambda \mp (|k_1|^2 - 2|k_0||k_2| \cos(\theta_{k_0} - \theta_{k_2})) \lambda^2}{4|k_0||k_1| \cos(\theta_{k_0} - \theta_{k_1}) \lambda} \quad (3.27) \\
 &= \frac{1}{2} \mp \frac{|k_1|^2 - 2|k_0||k_2| \cos(\theta_{k_0} - \theta_{k_2})}{4|k_0||k_1| \cos(\theta_{k_0} - \theta_{k_1})} \lambda.
 \end{aligned}$$

The natural range of this quantity r^{-1} would be something like $[0.3 - 0.7]$ (central value 0.5 and corrections of order λ). However, measurements give $r = 0.032_{-0.005}^{+0.006} \approx \lambda^2$, or in other words $1/r \sim 30 \sim \lambda^{-2}$. We conclude that

$$\frac{4|k_0||k_1| \cos(\theta_{k_0} - \theta_{k_1})}{|k_1|^2 - 2|k_0||k_2| \cos(\theta_{k_0} - \theta_{k_2})} \approx \lambda^3. \quad (3.28)$$

The most natural explanation may be assuming that $\cos(\theta_{k_0} - \theta_{k_1})$ is very small. In that case the absolute values of all parameters can still be of order one, which was part of the naturalness requirement of the model. Neutrinos present a quasi degenerate (QD) spectrum and both normal and inverse ordering are possible. The typical scale of neutrino masses v_L is given in this case by

$$v_L^2 = \frac{\Delta m_{atm}^2}{\lambda^2 |k_1|^2 - 2|k_0||k_2| \cos(\theta_{k_0} - \theta_{k_2})} \approx (0.1 \text{eV})^2. \quad (3.29)$$

Possible alternative solutions of eq. (3.28) that give a non-QD spectrum can be obtained only by admitting the parameters belong to a larger, but less natural, range, $\lambda^2 - \lambda^{-2}$. In this case when k_0 is of order λ^{-2} and $\cos(\theta_{k_0} - \theta_{k_1}) \sim \lambda$ while $k_{1,2}$ still of order one a inverse hierarchical spectrum can be obtained. Another possibility to get an inverse hierarchical (IH) spectrum is having k_0 and $\cos(\theta_{k_0} - \theta_{k_1})$ of order one, $k_1 \sim \lambda^{-1}$ and $k_2 \sim \lambda^2$. Finally the normal hierarchical (NH) spectrum can be obtained only in the case in which $k_2 \sim \lambda^{-2}$, $\cos(\theta_{k_0} - \theta_{k_1})$ and k_1 of order 1 and $k_0 \sim \lambda^2$.

3.2.3 Mixing matrices at leading order

Apart from the charged fermion masses of the second and third families, mixing angles and masses at leading order do not fit the experimental data: the charged fermion first families are massless, the CKM matrix is the unity matrix and the PMNS matrix can account for

two maximal rotations in the (12) and (23) sectors:

$$V = \mathbb{1}, \quad U = R_{23} \left(-\frac{\pi}{4} \right) R_{12} \left(\frac{\pi}{4} \right) = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/2 & 1/2 & +1/\sqrt{2} \end{pmatrix}. \quad (3.30)$$

At this level of approximation, the PMNS matrix corresponds to the BM pattern discussed in the introduction. As already stated, this mixing scheme does not fit the data, due to the large value of the bimaximal solar angle which is out of about 5σ . Only considering the NLO contributions, which we will study in the next section, the model agrees with the measurements.

3.3 Fermion mass matrices at higher orders

The NLO contributions in the mass matrices originate from two sources: the first are the higher order terms in the superpotential, while the others come from the insertion of the NLO flavon VEVs in the operators in eqs. (3.9), (3.20). In section 3.4 we will show how the flavons develop VEVs and how they are corrected. Here we anticipate the results, reporting the flavon VEVs in a form which is useful for the discussion in this section:

$$\begin{aligned} \langle \varphi \rangle &= \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} v_\varphi + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \delta v_\varphi, & \langle \varphi' \rangle &= \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} v_{\varphi'} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \delta v_{\varphi'}, \\ \langle \chi \rangle &= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v_\chi + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \delta^2 v_\chi, & \langle \sigma \rangle &= v_\sigma. \end{aligned} \quad (3.31)$$

Some comments are noteworthy: the subleading corrections are suppressed with respect the LO terms as $\delta v/v \sim \lambda$ and $\delta^2 v/v \sim \lambda^2$ for each flavon; NLO corrections to the second and third entries of $\langle \varphi \rangle$ and $\langle \varphi' \rangle$ are present, but they present the same structure of the LO terms and can be re-absorbed; similarly, the NLO corrections to the third entry of $\langle \chi \rangle$ and to $\langle \sigma \rangle$ are present, but they can be re-absorbed into the LO terms; the other entries of $\langle \chi \rangle$ do not receive any corrections at NLO. If we consider the NNLO approximation level, i.e. corrections of relative order λ^2 with respect the LO terms, we see that the second and the third entries of $\langle \varphi \rangle$ ($\langle \varphi' \rangle$) are not (anti-)aligned anymore and that the second entry of $\langle \chi \rangle$ is filled in. It is interesting that the first entry of $\langle \chi \rangle$ is still vanishing at this level. We will see the relevance of this structure in a while.

3.3.1 Dirac mass terms

The Dirac matter superpotential at NLO is given by

$$\begin{aligned} \mathcal{W}_{Dir}^{NLO} &= \sum_{i=1}^3 y_{6,(i)} \frac{1}{\Lambda^2} F_L F_3^c (\phi + \phi') X_i^{(5)} + \\ &+ \sum_{i=1}^3 y_{7,(i)} \frac{1}{\Lambda^4} F_L F_2^c \theta^2 \rho X_i^{(6)} + \sum_{i=1}^8 y_{8,(i)} \frac{1}{\Lambda^6} F_L F_2^c \theta^2 (\phi + \phi') X_i^{(7)} + \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^6 y_{9,(i)} \frac{1}{\Lambda^5} F_L F_1^c \theta^2 \rho X_i^{(8)} + \sum_{i=1}^7 y_{10,(i)} \frac{1}{\Lambda^6} F_L F_1^c \theta^2 (\phi + \phi') X_i^{(9)} + \\
 & + \sum_{i=1}^{13} y_{11,(i)} \frac{1}{\Lambda^7} F_L F_1^c \theta^2 (\phi + \phi') X_i^{(10)} ,
 \end{aligned} \tag{3.32}$$

where, as for eq. (3.9), we adopt a compact notation:

$$\begin{aligned}
 X_i^{(5)} & \equiv \{\varphi\sigma, \varphi\chi, \varphi'\chi\} , \\
 X_i^{(6)} & \equiv \{\varphi'\sigma, \varphi\chi, \varphi'\chi\} , \\
 X_i^{(7)} & \equiv \{\varphi^3\chi, \varphi^2\varphi'\chi, \varphi\varphi'^2\chi, \varphi'^3\chi, \varphi^3\sigma, \varphi^2\varphi'\sigma, \varphi\varphi'^2\sigma, \varphi'^3\sigma\} , \\
 X_i^{(8)} & \equiv \{\varphi^2\chi, \varphi\varphi'\chi, \varphi'^2\chi, \varphi^2\sigma, \varphi\varphi'\sigma, \varphi'^2\sigma\} , \\
 X_i^{(9)} & \equiv \{\varphi^4, \varphi^3\varphi', \varphi^2\varphi'^2, \varphi\varphi'^3, \varphi'^4, \chi^4, \chi^3\sigma\} , \\
 X_i^{(10)} & \equiv \{\varphi^4\chi, \varphi^3\varphi'\chi, \varphi^2\varphi'^2\chi, \varphi\varphi'^3\chi, \varphi'^4\chi, \varphi^4\sigma, \varphi^3\varphi'\sigma, \varphi^2\varphi'^2\sigma, \varphi\varphi'^3\sigma, \\
 & \quad \varphi'^4\sigma, \chi^5, \chi^4\sigma, \chi^3\sigma^2\} .
 \end{aligned} \tag{3.33}$$

Note that not all of these terms are non-vanishing when the flavons develop VEV: in particular $X_i^{(9)}$ for any i do not give a contribution when the LO VEVs are considered: only when the corrections to the VEVs are introduced, they contribute to the mass matrices. For this reason also the terms with $X_i^{(10)}$ must be taken into account, even if they are suppressed by an additional Λ . The other terms which are vanishing at this order of approximation are $X_i^{(8)}$ for $i = 4, 5, 6$ and $X_i^{(10)}$ for $i = 6, \dots, 13$. When flavons and Higgs fields develop VEVs, we get the following Dirac mass matrices:

$$M_e^{NLO} = -3 \begin{pmatrix} 0 & 0 & 0 \\ \tilde{y}_4 & 0 & 0 \\ \tilde{y}_9 & 0 & 0 \end{pmatrix} v_\rho^d \lambda^5 - 3 \begin{pmatrix} 0 & \tilde{y}_7 \lambda & 0 \\ 0 & \tilde{y}_2 & 0 \\ 0 & -\tilde{y}_2 & 0 \end{pmatrix} v_\rho^d \lambda^3 + \begin{pmatrix} 0 & 0 & \tilde{y}_6 \lambda \\ 0 & 0 & \tilde{y}_1 \\ 0 & 0 & \tilde{y}_1 \end{pmatrix} v_\phi^d \lambda , \tag{3.34}$$

$$M_d^{NLO} = \begin{pmatrix} 0 & 0 & 0 \\ \tilde{y}_4 & 0 & 0 \\ \tilde{y}_9 & 0 & 0 \end{pmatrix} v_\rho^d \lambda^5 + \begin{pmatrix} 0 & \tilde{y}_7 \lambda & 0 \\ 0 & \tilde{y}_2 & 0 \\ 0 & -\tilde{y}_2 & 0 \end{pmatrix} v_\rho^d \lambda^3 + \begin{pmatrix} 0 & 0 & \tilde{y}_6 \lambda \\ 0 & 0 & \tilde{y}_1 \\ 0 & 0 & \tilde{y}_1 \end{pmatrix} v_\phi^d \lambda , \tag{3.35}$$

$$M_u^{NLO} = \begin{pmatrix} 0 & 0 & 0 \\ \tilde{y}_5 & 0 & 0 \\ \tilde{y}_{10} & 0 & 0 \end{pmatrix} v_\phi^u \lambda^7 + \begin{pmatrix} 0 & \tilde{y}_8 \lambda & 0 \\ 0 & \tilde{y}_3 & 0 \\ 0 & -\tilde{y}_3 & 0 \end{pmatrix} v_\phi^u \lambda^5 + \begin{pmatrix} 0 & 0 & \tilde{y}_6 \lambda \\ 0 & 0 & \tilde{y}_1 \\ 0 & 0 & \tilde{y}_1 \end{pmatrix} v_\phi^u \lambda , \tag{3.36}$$

and $m_D^{NLO} = M_u^{NLO}$, where we used the definitions

$$\begin{aligned}
 \tilde{y}_1 & \equiv y_1 + y_{6,(1)} \lambda , \\
 \tilde{y}_2 & \equiv y_2 + y_{7,(1)} \lambda , \\
 \tilde{y}_3 & \equiv y_3 + \sum_{i=5}^8 y_{6,(i)} \lambda ,
 \end{aligned}$$

$$\begin{aligned}
 \tilde{y}_4 &\equiv \mathcal{F}_1[y_{9,(i)}] + \frac{1}{\lambda} y_4 \left(\frac{\delta v_{\varphi'}}{v_{\varphi'}} - \frac{\delta v_{\varphi}}{v_{\varphi}} \right), \\
 \tilde{y}_5 &\equiv \mathcal{F}_2[y_{11,(i)}] + \frac{1}{\lambda} \mathcal{F}_3 \left[y_{10,(i)}; \frac{\delta v_{\varphi}}{v_{\varphi}}, \frac{\delta v_{\varphi'}}{v_{\varphi'}} \right] + \frac{1}{\lambda^2} \mathcal{F}_4 \left[y_5; \frac{\delta^2 v_{\chi}}{v_{\chi}} \right], \\
 \tilde{y}_6 &\equiv (y_{6,(2)} + y_{6,(3)}) + \frac{1}{\lambda} y_1 \frac{\delta v_{\varphi}}{v_{\varphi}}, \\
 \tilde{y}_7 &\equiv (y_{7,(2)} + y_{7,(3)}) + \frac{1}{\lambda} y_2 \frac{\delta v_{\varphi'}}{v_{\varphi'}}, \\
 \tilde{y}_8 &\equiv \sum_{i=1}^4 y_{8,(i)} + \frac{1}{\lambda} \sum_{i=2}^4 y_{3,(i)} \frac{\delta v_{\varphi'}}{v_{\varphi'}}, \\
 \tilde{y}_9 &\equiv \mathcal{F}_5[y_{9,(i)}] + \frac{1}{\lambda} y_4 \left(\frac{\delta v_{\varphi'}}{v_{\varphi'}} + \frac{\delta v_{\varphi}}{v_{\varphi}} \right), \\
 \tilde{y}_{10} &\equiv \mathcal{F}_6[y_{11,(i)}] + \frac{1}{\lambda} \mathcal{F}_7 \left[y_{10,(i)}; \frac{\delta v_{\varphi}}{v_{\varphi}}, \frac{\delta v_{\varphi'}}{v_{\varphi'}} \right]. \tag{3.37}
 \end{aligned}$$

In the previous definitions we can see that each $\tilde{y}$ is the sum of two pieces: the first refers to the terms in eq. (3.32) when the LO flavon VEVs are considered; the second comes from the terms in eq. (3.9) where the NLO flavon VEVs are introduced. The only exception is $\mathcal{F}_3$ which refers to the term proportional to $X_i^{(9)}$ in eq. (3.32) and that give contribution only when the NLO flavon VEVs are considered. These two parts are of the same order of magnitude, since $\delta v/v \sim \lambda$ and $\delta^2 v/v \sim \lambda^2$. Note that $\mathcal{F}_i$ are distinct linear combinations of the arguments in the squared brackets. The expressions in eqs. (3.34), (3.35), (3.36) are valid at NLO level and note that the (anti-)alignment between the second and third entries of the (second) third families are still preserved. When considering higher order terms, this feature is lost and the (1,1) entry of each mass matrix is filled in.

The values for the charged fermion masses given in eq. (3.15) are modified only by substituting the coefficients y_i with their tilde-versions:

$$y_1 \rightarrow \tilde{y}_1, \quad y_2 \rightarrow \tilde{y}_2, \quad y_3 \rightarrow \tilde{y}_3. \tag{3.38}$$

At this approximation level, the first family masses are not yet well-described, because they are too small.

3.3.2 Majorana mass terms

Moving to the Majorana part of the matter superpotential at NLO, we get

$$\begin{aligned}
 \mathcal{W}_{Maj}^{NLO} &= \sum_{i=1}^3 \frac{k_{3,(i)}}{\Lambda^3} F_L F_L \Delta_L X_i^{(11)} \\
 &\quad + \frac{z_2}{\Lambda^4} F_2^c F_2^c \theta^4 \Delta_R + \frac{z_{3,(1)}}{\Lambda^4} F_2^c F_3^c \theta^2 \Delta_R \varphi \varphi' \\
 &\quad + \sum_{i=2}^3 \frac{z_{3,(i)}}{\Lambda^5} F_2^c F_3^c \theta^2 \Delta_R X_{i-1}^{(12)} + \sum_{i=4}^6 \frac{z_{3,(i)}}{\Lambda^6} F_2^c F_3^c \theta^2 \Delta_R X_{i-3}^{(13)} +
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{i=1}^4 \frac{z_{4,(i)}}{\Lambda^5} F_1^c F_3^c \theta^2 \Delta_R X_i^{(1)} + \sum_{i=5}^{12} \frac{z_{4,(i)}}{\Lambda^6} F_1^c F_3^c \theta^2 \Delta_R X_{i-4}^{(14)} + \\
 & + \frac{z_5}{\Lambda^6} F_1^c F_2^c \theta^4 \Delta_R \varphi \chi + \sum_{i=1}^2 \frac{z_{6,(i)}}{\Lambda^6} F_1^c F_1^c \theta^4 X_i^{(15)}, \quad (3.39)
 \end{aligned}$$

where as usual we used the compact notation with

$$\begin{aligned}
 X_i^{(11)} & \equiv \{\chi^3, \chi^2 \sigma, \sigma^3\}, \\
 X_i^{(12)} & \equiv \{\varphi \varphi' \chi, \varphi \varphi' \sigma\}, \\
 X_i^{(13)} & \equiv \{\varphi^2 \chi^2, \varphi \varphi' \chi^2, \varphi'^2 \chi^2\}, \\
 X_i^{(14)} & \equiv \{\varphi^3 \chi, \varphi^2 \varphi' \chi, \varphi \varphi'^2 \chi, \varphi'^3 \chi, \varphi^3 \sigma, \varphi^2 \varphi' \sigma, \varphi \varphi'^2 \sigma, \varphi'^3 \sigma\}, \\
 X_i^{(15)} & \equiv \{\varphi^2, \varphi'^2\}. \quad (3.40)
 \end{aligned}$$

A few comments are in place. Note that all the terms proportional to k_3 can be reabsorbed by a redefinition of $\tilde{k}_{0,1,2}$ and that the only new structure which corrects $\mathcal{M}_L$ comes from the term $F_L F_L \Delta_L \chi$ when we consider the subleading corrections of $\langle \chi \rangle$: as a result the entries (1,3) and (3,1) of $\mathcal{M}_L$ are filled in by terms proportional to λ^3 . Regarding the contributions to the Majorana mass matrix for the RH neutrinos, it is important to say that the terms proportional to $z_{3,(i)}$ with $i = 1, \dots, 3$ and to $z_{4,(i)}$ with $i = 1, \dots, 4$ are vanishing, due to the particular flavon VEV alignment of the model. As a consequence all the NLO contributions to $\mathcal{M}_R$ are of the order of λ^6 , apart that one to the entry (2,2) which is of the order of λ^4 . Finally, we note that each entry of $\mathcal{M}_R$ is independent from all the others, being F_i^c singlets of the flavour symmetry, and therefore all the z_i are free parameters with modulus of order 1. We listed only the dominant contributions, but the higher order terms would correspond to subleading corrections, which we can safely neglect in the following.

As a result of this analysis the Majorana masses for LH and RH neutrinos are given by

$$M_L^{NLO} = \begin{pmatrix} k'_0 & k'_1 \lambda & k'_3 \lambda^3 \\ k'_1 \lambda & k'_0 & 0 \\ k'_3 \lambda^3 & 0 & k'_0 + k'_2 \lambda^2 \end{pmatrix} v_L, \quad M_R^{NLO} = \begin{pmatrix} z_6 \lambda^6 & z_5 \lambda^6 & z_4 \lambda^6 \\ z_5 \lambda^6 & z_2 \lambda^4 & z_3 \lambda^6 \\ z_4 \lambda^6 & z_3 \lambda^6 & z_1 \end{pmatrix} v_R, \quad (3.41)$$

where with the notation k'_i we account for all the redefinitions done on the parameters. As already stated when discussing the LO mass matrices, the contributions to the effective light neutrino mass matrix come from the type-I and type-II See-Saw mechanisms. The resulting NLO $m_\nu^{(\text{type-I})}$ is given by

$$\begin{aligned}
 m_\nu^{(\text{type-I})} & = m_D^{NLO} (M_R^{NLO})^{-1} (m_D^{NLO})^T \\
 & = \begin{pmatrix} \tilde{y}_6^2 \lambda^2 & \tilde{y}_1 \tilde{y}_6 \lambda & \tilde{y}_1 \tilde{y}_6 \lambda \\ \tilde{y}_1 \tilde{y}_6 \lambda & \tilde{y}_1^2 & \tilde{y}_1^2 \\ \tilde{y}_1 \tilde{y}_6 \lambda & \tilde{y}_1^2 & \tilde{y}_1^2 \end{pmatrix} \frac{(v_\phi^u)^2 \lambda^2}{z_1 v_R}, \quad (3.42)
 \end{aligned}$$

and it is diagonalized by a maximal rotation in the (23) sector and not in the (12) sector as demanded by eq. (2.3). As a result, we need that the type-I See-Saw contribution is at least $\mathcal{O}(\lambda^2)$ the type-II See-Saw one, which we can express using eq. (3.38) as

$$\frac{m_t^2}{z_1 v_R} \leq \lambda^2 k'_0 v_L. \quad (3.43)$$

We remember here that v_L, v_R are the VEVs of Δ_L and Δ_R respectively. In particular v_L is the VEV developed by the SM $(1, 3, 1)$ triplet component of Δ_L and it is induced once the EW symmetry is broken. As we will see in detail in the next sections the physical SM triplet $T \sim (1, 3, 1)$ arises by the mixing between the SM $(1, 3, 1)$ components of Δ_L and of two additional fields, Σ and Σ' transforming under the PS gauge symmetry as $(1, 3, 3)$. The VEV $\langle T \rangle$ of the $SU(2)_L$ triplet T is related to its mass M_T through the following expression

$$\langle T \rangle \simeq \alpha_{ij} \frac{v_i^u v_j^u}{M_T}, \quad (3.44)$$

where α_{ij} are numerical coefficients arising by the details of the scalar potential and $v_1^u = \langle h_u \rangle, v_2^u = \langle h'_u \rangle$ are the VEVs of the two up-type light Higgs doublets needed for the realization of our model (see appendix B for details). Since v_L is the projection of $\langle T \rangle$ along Δ_L neglecting fine tuned cases it holds that

$$v_L \sim \langle T \rangle. \quad (3.45)$$

From eq. (3.43) we need therefore that M_T and v_R satisfy

$$M_T \leq \lambda^2 \left(\frac{k'_0 z_1 \alpha_{ij} v_i^u v_j^u}{m_t^2} \right) v_R. \quad (3.46)$$

At the same time neutrino mass data imply that

$$k'_0 \alpha_{ij} \frac{v_i^u v_j^u}{M_T} \leq \mathcal{O}(1) \text{ eV}. \quad (3.47)$$

Combining the constraints of eqs. (3.46)–(3.47) we see that in the most natural scenario, assuming $\alpha_{ij} = 1$, M_T and v_R satisfy

$$\begin{aligned} v_R &\gtrsim 30 M_T, \\ 10^{12} \text{ GeV} &\lesssim M_T \lesssim 10^{13} \text{ GeV}. \end{aligned} \quad (3.48)$$

Nevertheless if we allow the numerical factors α_{ij} laying in the range $0.1 - 10$, then M_T , and consequently v_R , can be reduced even of two and one orders of magnitude respectively

$$10^{10} \text{ GeV} \lesssim M_T \lesssim 10^{12} \text{ GeV}. \quad (3.49)$$

In the following discussion of lepton mixing and in the phenomenological analysis, we will assume that indeed type-II See-Saw is dominating and we will neglect the type-I contributions. In the section devoted to the study of the scalar potential we will justify and find out the region of the parameters space where indeed type-II See-Saw dominates over type-I.

3.3.3 Mixing angles at the next-to-leading order

Looking at eqs. (3.34)–(3.36)–(3.41) we see that the fermion mass matrices are of the required form as in eqs. (2.8)–(2.11). The resulting mixing matrices are modified with respect to the LO approximation and interesting new features follow. On the quark sector, the CKM matrix receives deviations from the unity and at NLO the angle θ_{12}^q is not vanishing anymore:

$$\theta_{12}^q = \frac{\lambda}{\sqrt{2}} \frac{|\tilde{y}_7 \tilde{y}_3 - \tilde{y}_2 \tilde{y}_8|}{|\tilde{y}_2| |\tilde{y}_3|}. \quad (3.50)$$

Looking at this result, the meaning of the parameter λ is clear: it is defined as the ratio of the flavon VEVs over the cut-off of the theory, but it also determines the order of magnitude of the Cabibbo angle. This justify our initial assumption of $\lambda = 0.2$. We note that this result is in concordance with the discussion below equation (2.11). If the second columns of the up- and down-quark matrices are not proportional to each other, we can generate a non-vanishing Cabibbo angle θ_{12}^q , while the two other angles in the CKM matrix are still vanishing.

In the lepton sector, the PMNS matrix corresponds to the BM scheme which receives the corrections as illustrated in eq. (2.3) and as a result the solar and reactor angles are given by

$$\theta_{13}^l = \frac{\lambda}{2} \left| \frac{\tilde{y}_6}{\tilde{y}_1} - \frac{\tilde{y}_7}{\tilde{y}_2} \right|, \quad (3.51)$$

$$\theta_{12}^l = \frac{\pi}{4} - \frac{\lambda}{2} \left| \frac{\tilde{y}_6}{\tilde{y}_1} + \frac{\tilde{y}_7}{\tilde{y}_2} \right|. \quad (3.52)$$

As it is easy to see, the reactor angle and the deviation from the maximal value of the solar angle are of order λ and therefore the model is now in agreement with the experimental data.⁵ This corresponds to a weaker version of eq. (1.1). At this approximation level, the atmospheric angle remains maximal. As already stated, the forthcoming neutrino appearance experiments will provide a good test for the model, indeed they will be able to verify if θ_{13}^l is close to its present upper bound.

3.3.4 Higher order effects

At the next-to-next-to-leading order (NNLO) and even higher orders, many new terms appear in the superpotential. However, only few of them lead to new terms in the mass matrices, while the rest can be absorbed in redefinitions of the parameters as in eq. (3.37). For this reason we do not report the full list of NNLO contributions, but we just comment on the physical consequences. Three effects are worth mentioning:

- as expected, the masses of the first families are strongly suppressed. We find that the down-quarks and the electron masses are suppressed by a factor of λ^6 and the

⁵Deriving eq. (3.52), we already neglect the corrections which increase the value of the solar angle, instead of decreasing it

up-quark mass by a factor of λ^8 . This leads to the following mass hierarchies:

$$\left| \frac{m_d}{m_b} \right| \sim \left| \frac{m_e}{m_\tau} \right| \sim \lambda^5, \quad \left| \frac{m_u}{m_t} \right| \sim \lambda^7. \quad (3.53)$$

As it is easy to see, the electron and the up-quark masses perfectly fit the experimental values and there is only a small tension in the down-quark sector, where we expected a smaller suppression. However, the large number of parameters, in particular from ϕ , ϕ' as well as ρ couplings, which enter in the definition of the first families allows to correctly fit the down mass, without affecting other observables.

- the (anti-)alignment of the (23) and (33) elements of the Dirac mass matrices in eqs. (3.34)–(3.36) gets broken as the new terms appear. The new elements are λ^2 suppressed with respect to the older terms. As a result, the matrix that diagonalizes $M_i M_i^\dagger$, with $i = e, u, d$, has no longer an exact maximal mixing in the (23) sector. In the lepton sector, this translates to a λ^2 deviation from maximality in the atmospheric angle of the PMNS matrix. In the quark sector, the angle θ_{23}^q becomes of order λ^2 . It is interesting to note that θ_{13}^q remains vanishing at this order. It only appears when even stronger suppressed terms are taken into account and is of order λ^3 , in accordance with the Wolfenstein parametrization [181].
- the third columns of the mass matrices in eqs. (3.34)–(3.35) are proportional to v_ϕ^d , while the second column of M_d^{NLO} is proportional to v_ρ^d and that one of M_e^{NLO} to $-3v_\rho^d$. Therefore, also at NLO, eqs. (3.17) are fulfilled. At the NNLO level, terms proportional to v_ρ^d appear in the third columns of charged lepton and down-type quark matrices and terms proportional to v_ϕ^d in the second columns. The new terms are $\lambda^2 \approx 5\%$ suppressed with respect to the old entries. We thus expect deviations from the relations $|m_\tau| = |m_b|$ and $|m_\mu| = 3|m_s|$ at the 5% level.

3.4 Flavon scalar potential

In this section we comment on the vacuum alignment mechanism which explains the flavon VEVs as in eqs. (3.4)–(3.5)–(3.6). It turns out that our desired alignment is exactly the one presented in [158], once we transform all the fields in our basis. Notice indeed that it is possible to identify each flavon of table 3 with the flavons in [158], by simply comparing the transformation properties under the full flavour group:

$$\varphi \longrightarrow \varphi_l, \quad \varphi' \longrightarrow \chi_l, \quad \chi \longrightarrow \varphi_\nu, \quad \sigma \longrightarrow \xi_\nu. \quad (3.54)$$

Using the following unitary matrix to move from the basis in [158] to our basis,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -i/\sqrt{2} & i/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}, \quad (3.55)$$

Driving	D_R	φ_R	χ_R	σ_R
S_4	2	3_2	3_1	1_1
Z_4	-1	-1	1	1

Table 5. The driving field content and their transformation properties under $S_4 \times Z_4$. They are all singlets under the gauge group and the FN symmetry, while they transform as $U(1)_R = 2$ under the continuous R -symmetry.

we find (up to irrelevant phases) the following flavon VEV alignment.⁶ We will comment in a while about the presence of equivalent solutions.

$$\varphi \propto \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \varphi' \propto \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad \chi \propto \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad (3.56)$$

which correspond to eqs. (3.4)–(3.5).

In [158], a set of driving superfields has been introduced to guarantee the correct flavon vacuum alignment: these new fields are gauge singlets and transform only under the flavour group, but differently from the flavons, they do not develop vacuum expectation values.⁷ Under the continuous R -symmetry all the driving fields transform as $U(1)_R = 2$ and therefore, constructing the superpotential, they appear only linearly. For the same purpose, we introduce in our model a set of driving fields which recall those in [158]. In table 5 we show the driving fields and their transformation properties under $S_4 \times Z_4$.

It is easy to determine the correspondence between our set of driving fields and those of [158]:

$$D_R \longrightarrow \varphi_l^0, \quad \varphi_R \longrightarrow \chi_l^0, \quad \chi_R \longrightarrow \varphi_\nu^0, \quad \sigma_R \longrightarrow \xi_\nu^0. \quad (3.57)$$

We now construct the driving superpotential w_d , which contains only flavons and driving fields and in particular neither matter fields nor Higgses, and look for the conditions that minimize the scalar potential,

$$V = \sum_i \left| \frac{\partial w_d}{\partial \Phi_i} \right|^2 + m_i^2 |\Phi_i|^2 + \dots \quad (3.58)$$

where Φ_i denote collectively all the scalar fields of the theory, m_i^2 are soft masses and dots stand for D -terms for the fields charged under gauge group and possible additional soft breaking terms. Since m_i^2 are expected to be much smaller than the mass scales involved in w_d , it is reasonable to minimize V in the supersymmetric limit and to account for soft breaking effects subsequently.

⁶The VEVs of the fields φ and φ' is recovered by applying the unitary matrix in eq. (3.55) to an equivalent configuration of eq. (51) in [158], resulting from the application of the element $(TS)^2$ to eq. (51) and of the element T to the eq. (18) of [158]

⁷This strictly holds only in the exact supersymmetric phase, while in the broken phase they develop a VEV proportional to the common soft breaking scale [182–184], usually denoted as m_{SUSY} . This could have a relevant impact when discussing on flavour violating processes, as studied in a series of paper [185–190] dealing with a context similar to our model.

Since our flavon and driving field content exactly corresponds to the one in [158], we already know that the VEV alignment in eqs. (3.4)–(3.5) represents an isolate minimum of the scalar potential. We only need to identify the relations which link the VEVs v_φ , $v_{\varphi'}$, v_χ and v_σ among each other in our model. To this purpose we write the driving superpotential, which reads as

$$\begin{aligned} w_d = & f_1 D_R \varphi \varphi + f_2 D_R \varphi' \varphi' + f_3 D_R \varphi \varphi' + f_4 \varphi_R \varphi \varphi' + \\ & + M_1 \Lambda \chi_R \chi + f_5 \chi_R \chi \sigma + f_6 \chi_R \chi \chi + \\ & + M_2^2 \Lambda^2 \sigma_R + M_3 \Lambda \sigma_R \sigma + f_7 \sigma_R \sigma \sigma + f_8 \sigma_R \chi \chi, \end{aligned} \quad (3.59)$$

where the first line deals with only the fields φ and φ' and the other two with χ and σ . As also explained in [158], this leads to the alignment in eqs. (3.4), (3.5) where the VEVs satisfy to

$$\begin{aligned} f_1 v_\varphi^2 + f_2 v_{\varphi'}^2 + \sqrt{3} f_3 v_\varphi v_{\varphi'} &= 0, \\ v_\sigma &= -\frac{M_1}{f_5}, \\ v_\chi^2 &= \frac{f_5^2 M_2^2 - f_5 M_1 M_3 + f_7 M_1^2}{2 f_5^2 f_8}. \end{aligned} \quad (3.60)$$

The solution in eqs. (3.4), (3.5), (3.60) is not unique, but it is possible to introduce a set of soft supersymmetric breaking parameters, which selects this solution as the lowest minimum of the scalar potential.

It is interesting to note the presence of an other source of uncertainty in our solution, which minimizes V . Given the symmetry of w_d and the field configurations of eqs. (3.4), (3.5), (3.60), by acting on them with elements of the flavour symmetry group $S_4 \times Z_4$, we can generate other minima of the scalar potential. These alternative solutions however are physically equivalent to those of the original set and it is not restrictive to analyze the model by choosing as local minimum that one in eqs. (3.4), (3.5), (3.60).

For the FN field θ to acquire a VEV, we assume that the symmetry $U(1)_{FN}$ is gauged such that θ gets its VEV through a D -term. The corresponding potential is of the form:

$$V_{D,FN} = \frac{1}{2} (M_{FI}^2 - g_{FN} |\theta|^2 + \dots)^2 \quad (3.61)$$

where g_{FN} is the gauge coupling constant of $U(1)_{FN}$ and M_{FI}^2 denotes the contribution of the Fayet-Iliopoulos (FI) term. Dots in eq. (3.61) represent e.g. terms involving the fields F_1^c and F_2^c which are charged under $U(1)_{FN}$. These terms are however not relevant to calculate the VEV of the FN field and we omit them in the present discussion. $V_{D,FN}$ leads in the supersymmetric limit to

$$|v_\theta|^2 = |\langle \theta \rangle|^2 = \frac{M_{FI}^2}{g_{FN}}. \quad (3.62)$$

It is relevant to underline that the VEVs in eqs. (3.60), (3.62), depend on Mass parameters: all these mass scales naturally have the same order of magnitude and as a result $VEV_s/\Lambda_f \sim \lambda$. The only exceptions are the VEVs of φ and φ' , which depend on a flat direction. In the model, we simply assume that their VEVs have values of the same order of all the other flavon VEVs.

3.4.1 Higher order contributions

In this section we briefly comment on the corrections which enter in the flavon VEVs, once the higher order contributions are taken into account. We leave all the details to the appendix C.

In the superpotential w_d , the flavons which contribute to the Dirac mass terms, φ and φ' , and those which contribute to the Majorana mass terms, χ and σ , at LO belong to two separated sectors, indeed any mixing term is prevented due to the Z_4 symmetry. This situation is not preserved at NLO, since the fields χ and σ are neutral under the Z_4 symmetry and therefore we can add each of them to all the terms in w_d . This leads to modifications to the LO VEV alignment of eq. (3.4) and it turns out that the first entries of $\langle\varphi\rangle$ and $\langle\varphi'\rangle$ are filled in, while the second and third entries are corrected by terms which can be however absorbed into the LO ones, without spoiling the alignment. Also the VEVs in eq. (3.5) receive some corrections: the first and second entries of $\langle\chi\rangle$ still vanish and the NLO contributions to the third entry can again be absorbed into the LO term. This discussion justifies the results showed in eq. (3.31).

Corrections from the NNLO contributions are without particular alignments: in particular, the second and third entries of $\langle\varphi\rangle$ and $\langle\varphi'\rangle$ are no longer related, and also the second entry of $\langle\chi\rangle$ gets a non-zero value. It is interesting to note the the first entry of $\langle\chi\rangle$ remains zero.

4 Higgs scalar potential

In this section we present the study of the Higgs potential in our model. It is an interesting example of how the introduction of flavour symmetries and the assumptions done to get the correct mass matrices have non-negligible consequences on the Higgs sector. As a result, the study of the Higgs scalar potential and of the gauge and the Yukawa coupling runnings in a general non-flavour PS context does not strictly hold. Notice that even if the following analysis refers to our particular choice of fields and symmetries, our conclusions can be taken as a general hint for a very large class of models that combine a discrete flavour symmetry with a grand unified scenario: indeed our model building strategy shares common features with other constructions. In particular the Higgs fields usually transform under the flavour symmetry G_f and this has direct consequences on the implementation of the grand unified symmetry breaking. Moreover type-II See-Saw dominance and particular patterns of vanishing projections of the heavy Higgs fields on the light Higgs doublets are frequently required to get the correct fermion mass matrices.

In table 6 we list all the Higgs fields which are necessary to reproduce the correct mass matrices and to implement the desired PS symmetry breaking pattern: in the first part of the table we report the Higgses already introduced in table 4, while the second part contains all the additional fields.

Some of these Higgs fields are already present in the minimal version of PS models [191]: typically a $(15,1,1)$ multiplet — as the fields A and B in table 6 — is used to break $SU(4)_C$ down to $SU(3)_C \times U(1)_{B-L}$ and to induce the VEVs of the couple of $(10,1,3) \oplus (\overline{10},1,3)$ — corresponding to the fields $\Delta_R \oplus \overline{\Delta}_R$ in table 6. The latter VEVs breaks $SU(2)_R \times U(1)_{B-L}$

Higgses	ϕ, ϕ'	ρ	Δ_L	Δ_R
PS	$(1, 2, 2)$	$(15, 2, 2)$	$(\overline{10}, 3, 1)$	$(10, 1, 3)$
Z_4	1	-1	1	-1

Higgses	$\overline{\Delta}_L$	$\overline{\Delta}_R$	A	B	Σ	Σ'	ξ
PS	$(10, 3, 1)$	$(\overline{10}, 1, 3)$	$(15, 1, 1)$	$(15, 1, 1)$	$(1, 3, 3)$	$(1, 3, 3)$	$(1, 1, 1)$
Z_4	1	-1	1	-1	1	-1	-1

Table 6. All the Higgs fields of the model and their transformation properties under the gauge group and under Z_4 . Notice that they are invariant under the other factors of the full flavour symmetry group G_f .

into the SM hypercharge $U(1)_Y$ concluding the symmetry breaking chain from the PS gauge group to the SM $SU(3)_C \times SU(2)_L \times U(1)_Y$. The field that triggers the EW symmetry breaking is usually a bidoublet $(1, 2, 2)$ — as ϕ or ϕ' in table 6. The fields $(\overline{10}, 3, 1) \oplus (10, 3, 1)$ — the fields $\Delta_L \oplus \overline{\Delta}_L$ in table 6 — do not develop VEVs at tree level in the usual minimal PS model, but only when next to leading order terms are taken into account; these are typically suppressed by the Planck scale, as already stated in [191]. For this reason in the minimal PS the type-II See-Saw contributions to the effective neutrino masses are almost negligible.

We identify three main reasons for which the existent studies of the symmetry breaking patterns in the PS context [159–171, 191] have to be modified and this automatically justifies the presence of the new fields in table 6:

- the assumption $v_\rho^u = 0$ necessary to distinguish the up-quark sector from the others and to recover the up-quark mass hierarchies can be realized only if we include two identical copies of bidoublet $(1, 2, 2)$, ϕ and ϕ' (see details in appendix B), and we then impose that four $SU(2)_L$ doublets (2 up-type and 2 down-type) remain light;
- since the fields ρ , Δ_R and $\overline{\Delta}_R$ transform non-trivially under the flavour symmetry Z_4 , it is necessary to introduce two copies of $(15, 1, 1)$ multiplets, A and B , with opposite Z_4 charges, 1 and -1 respectively: A is responsible of inducing the breaking of $SU(2)_R$ through its coupling with Δ_R and $\overline{\Delta}_R$; B allows the coupling of the bidoublets ϕ, ϕ' with the bidoublet ρ . In this way all of these three fields have a non-vanishing projection on the light Higgs $SU(2)_L$ doublets;
- the component of Δ_L which corresponds to the usual SM triplet $(1, 3, 1)$ can develop a VEV once the EW symmetry is broken only in the presence of a trilinear coupling with the $SU(2)_L$ Higgs doublets. This coupling cannot be originated with only the Higgs fields $\Delta_L \oplus \overline{\Delta}_L$ and the field content given in table 6. For this reason we need an additional field which mediate this coupling and the simplest choice would be a bitriplet $\Sigma \sim (1, 3, 3)$ that can couple with the fields ϕ, ϕ' or ρ and at the same time can mixes with Δ_L , when Δ_R develops VEV at the $SU(2)_R$ breaking scale. However, once more, the presence of the Z_4 symmetry obliges the introduction of two distinct

(1,3,3) Higgs fields, Σ and Σ' , with opposite Z_4 charges, 1 and -1 respectively. In this way Σ can couple to the bidoublets ϕ, ϕ' or ρ , while Σ' can mix with Δ_L . Finally, we need a new ingredient that mixes Σ with Σ' : a PS singlet ξ charged -1 under Z_4 can do the job.

The scalar part of the superpotential is then given by⁸

$$\begin{aligned}
 \mathcal{W} = & \frac{1}{2}M_\phi \phi\phi + \frac{1}{2}M_{\phi'} \phi'\phi' + M_{\phi\phi'} \phi\phi' + \frac{1}{2}M_\rho \rho\rho + M_{\Delta_L} \Delta_L \bar{\Delta}_L + \\
 & + M_{\Delta_R} \Delta_R \bar{\Delta}_R + \frac{1}{2}M_A AA + \frac{1}{2}M_B BB + \frac{1}{2}M_\Sigma \Sigma\Sigma + \frac{1}{2}M_{\Sigma'} \Sigma'\Sigma' + \frac{1}{2}M_\xi \xi\xi + \\
 & + \lambda_\xi \Sigma\Sigma'\xi + \lambda_{\xi AB} AB\xi + \lambda_{\phi\rho} \phi B\rho + \lambda_{\phi'\rho} \phi' B\rho + \frac{1}{3}\lambda_A AAA + \\
 & + \frac{1}{2}\lambda_B BBA + \lambda_L \Delta_L \bar{\Delta}_L A + \lambda_R \Delta_R \bar{\Delta}_R A + \\
 & + \frac{1}{2}\lambda_{\phi\Sigma} \phi\phi\Sigma + \frac{1}{2}\lambda_{\phi'\Sigma} \phi'\phi'\Sigma + \lambda_{\phi\phi\Sigma} \phi'\phi\Sigma + \frac{1}{2}\lambda_{\rho\Sigma} \rho\rho\Sigma + \\
 & + \lambda_{\Delta\Sigma'} \Delta_L \bar{\Delta}_R \Sigma' + \bar{\lambda}_{\Delta\Sigma'} \Delta_R \bar{\Delta}_L \Sigma' + \frac{1}{3}\lambda_\Sigma \Sigma\Sigma\Sigma + \frac{1}{2}\lambda_{\Sigma'} \Sigma'\Sigma'\Sigma.
 \end{aligned} \tag{4.1}$$

The vacuum configuration at the GUT breaking scale is given by⁹

$$\langle \Delta_R \rangle = \langle \bar{\Delta}_R \rangle = M_R, \quad \langle A \rangle = M_{C_1}, \quad \langle B \rangle = M_{C_2}, \quad \langle \xi \rangle = V_\xi. \tag{4.2}$$

The VEVs $\langle A \rangle$ and $\langle B \rangle$ break the $SU(4)_C$ to $SU(3)_C \times U(1)_{B-L}$, while $\langle \Delta_R \rangle$ and $\langle \bar{\Delta}_R \rangle$ break $SU(4)_C \times SU(2)_R$ into $SU(3)_C \times U(1)_Y$. Therefore given our field content the colour breaking scale, $M_C = \text{Max}(M_{C_1}, M_{C_2})$, is never smaller than the $SU(2)_R$ breaking scale, M_R . We may expect that $M_{C_1} \sim M_{C_2} \equiv M_C$ but in principle they could be different. Finally V_ξ is expected to be close to the flavour breaking scale, due to the ξ gauge singlet nature. The PS breaking pattern so far sketched is therefore summarized by

$$\begin{aligned}
 & SU(4)_C \times SU(2)_L \times SU(2)_R \\
 & \quad \downarrow \\
 & SU(3)_C \times U(1)_{B-L} \times SU(2)_L \times SU(2)_R \\
 & \quad \downarrow \\
 & SU(3)_C \times SU(2)_L \times U(1)_Y.
 \end{aligned} \tag{4.3}$$

⁸Since all the Higgs fields are neutral under the continuous $U(1)_R$, the scalar superpotential explicitly breaks it, while preserving the usual R-parity. The terms in eq. (4.1) could be generated from a $U(1)_R$ -conserving superpotential in which the breaking is mediated by additional fields, which are $U(1)_R = 2$ and develop non-vanishing VEVs. For instance the mass term $M_\phi \phi\phi$ could originate from a trilinear term $X \phi\phi$, when $\langle X \rangle = M_\phi$. Similarly the trilinear coupling $\lambda_\xi \Sigma\Sigma'\xi$ could originate by the non-renormalizable term $X \Sigma\Sigma'\xi/\Lambda'$, when $\langle X \rangle/\Lambda' = \lambda_\xi$ and Λ' is the energy scale of the dynamics of the field X . In our model we simply assume the existence of the terms in eq. (4.1) in the superpotential and allow for an explicit breaking of the $U(1)_R$ symmetry in this sector.

⁹We redefine $\langle \Delta_R \rangle = v_R$ as $\langle \Delta_R \rangle = M_R$ in order to adapt to the usual notation.

The F-derivative system obtained by the superpotential in eq. (4.1) reads as

$$\begin{aligned}
M_{\Delta_R} + \frac{3}{\sqrt{2}}\lambda_R M_{C_1} &= 0, \\
M_B M_{C_2} \sqrt{2}\lambda_B M_{C_1} M_{C_2} + \lambda_{AB\xi} M_{C_1} V_\xi &= 0, \\
M_A M_{C_1} + \frac{1}{\sqrt{2}}\lambda_B M_{C_2}^2 - \frac{2}{\sqrt{3}}\lambda_R M_R^2 + \sqrt{2}\lambda_A M_{C_1}^2 + \lambda_{AB\xi} M_{C_2} V_\xi &= 0, \\
M_\xi V_\xi + \lambda_{AB\xi} M_{C_1} M_{C_2} &= 0.
\end{aligned} \tag{4.4}$$

By solving the previous equations, we can express the mass parameters that enter in the superpotential in term of the adimensional parameters λ_i and the physical breaking scales. All the details regarding the mass spectrum are reported in the appendix B, but some comments are in place. As in the minimal supersymmetric PS [191] when the singlet component of A develops a VEV, there is an accidental $SU(3)$ symmetry involving Δ_R and $\overline{\Delta}_R$. When the singlet components of these fields acquire a VEV the accidental symmetry is broken to $SU(2)$ giving rise to 5 Goldstone Bosons (GBs). At the same time $SU(2)_R \times U(1)_{B-L}$ is broken down to $U(1)_Y$, eating 3 of the 5 GBs. Therefore 2 of them, namely δ^{++} and $\overline{\delta}^{++}$ are left massless, down to the SUSY soft breaking scale ~ 1 TeV. This is a well known prediction of SUSY PS theories, which can be tested at LHC [192–195]. On the other hand, contrary to the minimal case described in [191], due to the mixing between the Higgs fields A and B , no colour octet is lighter than M_R .

In order to assure type-II dominance and to get the correct PS symmetry breaking pattern, it is necessary that $M_T \leq M_R \leq M_C \leq V_\xi$. We will see in a while the constraints on the parameters which can be derived from this special mass ordering. For the moment, we just assume this scheme and we describe the mass spectrum in terms of the SM gauge group. Starting from the heaviest energy scale

- at V_ξ
 - 2 heavy $SU(2)_L$ triplets given in section B.3.2;
- between V_ξ and M_C
 - all the SM singlets given in section B.1.1 except one, called ξ_0 ,
 - the colour triplets given in section B.1.2,
 - the colour octets given in section B.1.3,
 - the heavy doublets given in section B.2.1,
 - the two heavy couples of $SU(2)_L$ triplets given in section B.3.1;

- at M_C
 - the color scalars originating by $\Delta_R \oplus \overline{\Delta}_R$, ρ and $\Delta_L \oplus \overline{\Delta}_L$:

$$\left\{ \begin{array}{l} \text{the } SU(2)_L \text{ singlets given from section B.1.6 to section B.1.10,} \\ \text{the } SU(2)_L \text{ doublets given from section B.2.2 to section B.2.4,} \\ \text{the } SU(2)_L \text{ triplets given in section B.3.3 and section B.3.4;} \end{array} \right.$$

- at M_R
the singlet ξ_0 ;
- at M_T
the light couple of $SU(2)_L$ triplets given in section B.3.1;
- at M_{SUSY}
the scalar singlets δ^{++} and $\bar{\delta}^{++}$ given in section B.1.5,
the $SU(2)_L$ light doublets given in section B.2.1.

With the Higgs field content given in tables 4 and 6 and the scalar spectrum so far sketched we can plug the gauge coupling runnings and see if the conditions given in eq. (3.48) or those in eq. (3.49) can be satisfied. Furthermore, in the study of the gauge coupling runnings (see appendix D for details) from the M_{GUT} to the EW scale we have to impose the following constraints:

- recover the EW values for α_3, α_2 and α_1 , related to the gauge couplings of the SM gauge group ;
- impose that, at $M_R \leq M_C$, $U(1)_Y$ is originated by the $SU(2)_R \times U(1)_{B-L}$ breaking;
- impose $\alpha_{B-L} = \alpha_C$ at M_C ;
- define the GUT scale as the scale at which the largest $\alpha_i = 1$. In this way we are sure to be in a perturbative regime up to the GUT scale and thus we are allowed to adopt the one-loop renormalization group equations (RGEs).

Even using all these constraints we are left with two more freedoms, the value of the $SU(4)_C$ and $SU(2)_R$ breaking scales, i.e. M_C and M_R respectively.

We adopt two distinct approaches, that we will indicate as the *more constraining* and the *less constraining* ones. In the first case we define M_C the scale at which the largest α_i is equal or smaller than $1/4\pi$. In this way all the gauge coupling at M_C are smaller than 1. In the second case we allow the largest α_i to correspond to a gauge coupling in the range $1 < g_i < 3$. Then M_R should satisfies eq. (3.48) or eq. (3.49), but its exact value is not fixed yet.

A few general comments are needed. The non-minimal PS field content affects the gauge coupling runnings in a non-negligible way. In particular the presence of the charged singlets δ^{++} and $\bar{\delta}^{++}$ down to M_{SUSY} deeply modifies the $U(1)_Y$ and $SU(2)_R$ gauge coupling evolution. It turns out that the largest α_i above the M_R scale is always α_R . Therefore the two approaches we described can be formulated as follows:

- More constraining approach $\iff \alpha_R \leq 1/4\pi$, at M_C
- Less constraining approach $\iff 1 < g_R < 3$, at M_C .

Figure 1. The running of the gauge coupling constants in the more constraining approach. $M_T = 10^{11}$ GeV, $M_R = 10^{12}$ GeV, $M_C = 1.4 \times 10^{12}$ GeV (where $\alpha_R = 1/4\pi$) and $M_{gut} = 4.0 \times 10^{12}$ GeV (where $\alpha_R = 1$). In the dotted figure, we show a detail of the full plot, restricting the energy scale inside the range $10^{10} \div 10^{13}$ GeV.

More constraining approach. In this case there are no solutions neither for the ranges of values of M_T and M_R given in eq. (3.48) nor for those in eq. (3.49). Indeed we find that $M_R \leq 10^{12}$ GeV, as can be seen in figure 1. In other words if we adopt this constraining approach to fix the value of M_C , the type-I and type-II See-Saw scales require Yukawa parameters which are at least 2 order of magnitude far from their natural values to reproduce the correct neutrino mass scale. We cannot be satisfied with such a solution because in this case the type-II See-Saw dominance is obtained by increasing the Yukawa couplings in the right neutrino sector and at the same time reducing the coupling of the left-handed neutrinos with the scalar triplet. Even if this may be considered a solution, our challenge was to provide a justification of type-II See-Saw dominance through the analysis of the Higgs scalar potential and not by tuning the Yukawa parameters. Moreover we introduced a FN Abelian symmetry to explain the small ($\leq 10^{-2}$) Yukawa parameters necessary in the charged fermion sector to reproduce the correct mass hierarchies. The presence of Yukawa parameters of this order in the purely left-handed neutrino sector makes the introduction of the FN symmetry questionable. Notice that in the discussion of the neutrino spectrum we have enlarged the parameter range up to values not smaller than λ^2 .

Figure 2. The running of the gauge coupling constants in the less constraining approach. $M_T = 10^{12}$ GeV, $M_R = 10^{13}$ GeV, $M_C = 1.8 \times 10^{13}$ GeV (where $\alpha_R = 1/3$) and $M_{gut} = 2.2 \times 10^{13}$ GeV (where $\alpha_R = 1$). In the dotted figure, we show a detail of the full plot, restricting the energy scale inside the range $10^{11} \div 10^{14}$ GeV.

Less constraining approach. In this case there are solutions only for the second range of values of M_T and M_R given in eq. (3.49). As can be seen in figure 2, M_R can now reach the value of 10^{13} GeV. However the three scales M_R , M_C and M_{GUT} are compressed in a narrow region around 10^{13} GeV and therefore our model is described by an extended MSSM model almost up to the GUT scale.¹⁰ Nevertheless, its PS origin is reflected in the non-trivial relations between the Yukawa couplings. In conclusion, by admitting the Yukawa parameters span in a range 0.1–10 and switching M_R , M_C and M_{GUT} very close to each other, we have found a narrow region of the parameter space where our model could still give a realistic description of fermion masses and mixings and in which type-II See-Saw dominance is not imposed by hand.

To finally consider our model viable, we should study the stability of the flavour structure of the mass matrices under the RGEs from the GUT scale down to the EW one. The study of the full set of the RGEs of the model presented is beyond the purpose of this paper. For this reason we will neither run the parameters of the scalar superpotential nor include

¹⁰Above this energy scale another gauge structure could be active. We do not take in consideration any high-energy completion of the model, but it is reasonable that larger gauge groups or particular constructions could be present at these energies: for example an SO(10) inspired approach in which fermions do not belong to a unique representation.

and run the parameters of the soft SUSY breaking potential. Under these approximations the EW vacuum expectation values do not change from the GUT scale down to the EW one. However this does not affect our conclusions for what concerns the stability of the mass matrix structures since the EW VEV shifts due to the running factorize out and leave the Yukawa flavour structure unchanged. The study of the Yukawa matrix running will be done in the next section.

5 Yukawa coupling running

In the previous section we analyzed the constraints on the scalar Higgs sector coming from the requirement of type-II See-Saw dominance and from the presence of the flavour symmetry under which the Higgs fields non-trivially transform. We found that the model is viable only in a small region of the parameter space for which M_R , M_C and $M_{GUT} \sim 10^{13}$ GeV are very close to each other. At the same time M_T lies at only one order of magnitude below M_R . For these reasons to study the stability of the flavour structure of the fermion mass matrices at low scale we can consider only the running from M_T onwards, thus neglecting the running from higher energies. Furthermore we also neglect the running from M_{SUSY} to the EW scale, which would introduce only minor corrections. We work under the assumption that type-II See-Saw is dominating over type-I and moreover that the effects from the type-I terms under the RGEs are negligible. Therefore in the studying of the Yukawa coupling running we do not take into account the Weinberg operator originating by integrating out the right-handed neutrinos. The error we introduce in this way is less than λ^2 and we will see in a while that these contributions do not modify our results. Furthermore, we study the stability under the Renormalization Group (RG) running, in the approximation corresponding to the NLO, i.e. considering the mass matrices introduced in eqs. (3.34)–(3.36).

5.1 Yukawa matrices at M_T

Since we start the renormalization group running at M_T , at this scale we integrate out the $SU(2)_L$ scalar triplet T obtaining an effective Weinberg operator responsible of the type-II See-Saw contribution. We recall here the origin of this effective operator. The Majorana parts of the matter superpotential given in eqs. (3.20), (3.39) contain terms with the coupling

$$F_L F_L \Delta_L, \quad (5.1)$$

while the scalar part of the superpotential in eq. (4.1) contains the terms

$$\begin{aligned} & \frac{1}{2} \lambda_{\phi\Sigma} \phi\phi\Sigma + \frac{1}{2} \lambda_{\phi'\Sigma} \phi'\phi'\Sigma + \lambda_{\phi\phi\Sigma} \phi'\phi\Sigma + \frac{1}{2} \lambda_{\rho\Sigma} \rho\rho\Sigma + \\ & + \lambda_{\Delta\Sigma'} \Delta_L \bar{\Delta}_R \Sigma' + \bar{\lambda}_{\Delta\Sigma'} \Delta_R \bar{\Delta}_L \Sigma' + \frac{1}{3} \lambda_{\Sigma\Sigma\Sigma} + \frac{1}{2} \lambda_{\Sigma'\Sigma'\Sigma}, \end{aligned} \quad (5.2)$$

that ensures the mixing between the (1,3,1) ((1,3,-1)) components of Δ_L ($\bar{\Delta}_L$), Σ and Σ' , whose lighter combination is identified with T ($\bar{T}$), and provides the coupling of T ($\bar{T}$)

with the light doublets h_d and h'_d (h_u and h'_u). The effective Weinberg operator at M_T is given by

$$\alpha_{ij} Y_{Lrs} \frac{L_r L_s h_{u_i} h_{u_j}}{M_T} \quad (5.3)$$

where L_i is the $SU(2)_L$ lepton doublets, $h_{u_1} = h_u$, $h_{u_2} = h'_u$ and α_{ij} are coefficients arising by the scalar potential and Y_L is given by

$$Y_L = \begin{pmatrix} k'_0 & k'_1 \lambda & 0 \\ k'_1 \lambda & k'_0 & 0 \\ 0 & 0 & k'_0 + k'_2 \lambda^2 \end{pmatrix}. \quad (5.4)$$

Notice that we are neglecting the λ^3 terms in Y_L , because they would be irrelevant for the following analysis.

For what concerns the charged fermion Yukawa, at M_T the Dirac part of the superpotential is written as

$$Y_u Q U^c h_u + Y'_u Q U^c h'_u + Y_d Q D^c h_d + Y'_d Q D^c h'_d + Y_e L E^c h_d + Y'_e L E^c h'_d, \quad (5.5)$$

with

$$\begin{aligned} Y_u &= \frac{1}{\beta} Y'_u = \begin{pmatrix} 0 & \tilde{y}_8 \lambda^5 & \tilde{y}_6 \lambda \\ \tilde{y}_5 \lambda^6 & \tilde{y}_3 \lambda^4 & \tilde{y}_1 \\ \tilde{y}_{10} \lambda^6 & -\tilde{y}_3 \lambda^4 & \tilde{y}_1 \end{pmatrix}, \\ Y_d &= U_{13} \begin{pmatrix} 0 & \tilde{y}_7 \lambda & 0 \\ \tilde{y}_4 \lambda^2 & \tilde{y}_2 & 0 \\ \tilde{y}_9 \lambda^2 & -\tilde{y}_2 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & 0 & \tilde{y}_6 \lambda \\ 0 & 0 & \tilde{y}_1 \\ 0 & 0 & \tilde{y}_1 \end{pmatrix}, \\ Y'_d &= U_{23} \begin{pmatrix} 0 & \tilde{y}_7 \lambda & 0 \\ \tilde{y}_4 \lambda^2 & \tilde{y}_2 & 0 \\ \tilde{y}_9 \lambda^2 & -\tilde{y}_2 & 0 \end{pmatrix} \lambda^2 + \beta \begin{pmatrix} 0 & 0 & \tilde{y}_6 \lambda \\ 0 & 0 & \tilde{y}_1 \\ 0 & 0 & \tilde{y}_1 \end{pmatrix}, \\ Y_e &= -3U_{13} \begin{pmatrix} 0 & \tilde{y}_7 \lambda & 0 \\ \tilde{y}_4 \lambda^2 & \tilde{y}_2 & 0 \\ \tilde{y}_9 \lambda^2 & -\tilde{y}_2 & 0 \end{pmatrix} \lambda^2 + \begin{pmatrix} 0 & 0 & \tilde{y}_6 \lambda \\ 0 & 0 & \tilde{y}_1 \\ 0 & 0 & \tilde{y}_1 \end{pmatrix}, \\ Y'_e &= -3U_{23} \begin{pmatrix} 0 & \tilde{y}_7 \lambda & 0 \\ \tilde{y}_4 \lambda^2 & \tilde{y}_2 & 0 \\ \tilde{y}_9 \lambda^2 & -\tilde{y}_2 & 0 \end{pmatrix} \lambda^2 + \beta \begin{pmatrix} 0 & 0 & \tilde{y}_6 \lambda \\ 0 & 0 & \tilde{y}_1 \\ 0 & 0 & \tilde{y}_1 \end{pmatrix}. \end{aligned} \quad (5.6)$$

The U matrix defines the light $SU(2)_L$ Higgses in term of the PS Higgs field components, as explicitly written in appendix B, and $\tilde{y}_i$ has to be read as $U_{11} \tilde{y}_i^{(1)} + U_{12} \tilde{y}_i^{(2)}$ while $\beta \tilde{y}_i$ as $U_{21} \tilde{y}_i^{(1)} + U_{22} \tilde{y}_i^{(2)}$.

With respect to the mass matrices given in eqs. (3.34)–(3.36) we have reabsorbed a power of λ .

5.2 Analytical approximations

In the appendix E, we report all the RGEs for the Yukawa matrices, while here we discuss the results. The RGEs present the general compact expressions

$$\begin{aligned}\frac{dY_L}{dt'} &= \mathcal{F}_L [Y_{f'} Y_{f'}^\dagger] Y_L + Y_L \mathcal{F}_L^T [Y_{f'} Y_{f'}^\dagger] + \left[\mathcal{G}_L [\text{Tr} (Y_{f'} Y_{f'}^\dagger)] - \sum_i c_i^\nu g_i^2 \right] Y_L, \\ \frac{dY_f}{dt'} &= \mathcal{F}_f [Y_{f'} Y_{f'}^\dagger] Y_f + \left[\mathcal{G}_f [\text{Tr} (Y_{f'} Y_{f'}^\dagger)] - \sum_i c_i^f g_i^2 \right] Y_f,\end{aligned}\quad (5.7)$$

where the index f runs over $\{e, u, d\}$, the parameter t' is defined as $t' \equiv t/(16\pi^2) \equiv \log \mu/(16\pi^2)$, $\mathcal{F}_X [\dots]$ is a matrix written in terms of the fermion Yukawa matrices $Y_{f'}$, $\mathcal{G}_X [\dots]$ is function of the trace in the flavour space over the Yukawa matrices $Y_{f'}$ and c_i^f are the Casimir coefficients related to the group representations (see appendix E for the details). The generic solutions are given by

$$\begin{aligned}Y_L(\mu) &\sim \prod_i e^{-c_i^\nu \mathcal{I}_i} \times \exp \left[\int_{t'(\mu_0)}^{t'(\mu)} \mathcal{G}_L [\text{Tr} (Y_{f'} Y_{f'}^\dagger)] dt' \right] \times \exp \left[\int_{t'(\mu_0)}^{t'(\mu)} \mathcal{F}_L [Y_{f'} Y_{f'}^\dagger] dt' \right] \times \\ &\quad \times Y_L(\mu_0) \times \exp \left[\int_{t'(\mu_0)}^{t'(\mu)} \mathcal{F}_L^T [Y_{f'} Y_{f'}^\dagger] dt' \right], \\ Y_f(\mu) &= \prod_i e^{-c_i^f \mathcal{I}_i} \times \exp \left[\int_{t'(\mu_0)}^{t'(\mu)} \mathcal{G}_f [\text{Tr} (Y_{f'} Y_{f'}^\dagger)] dt' \right] \times \exp \left[\int_{t'(\mu_0)}^{t'(\mu)} \mathcal{F}_f [Y_{f'} Y_{f'}^\dagger] dt' \right] \times \\ &\quad \times Y_f(\mu_0).\end{aligned}\quad (5.8)$$

where $\mathcal{I}_i = \int_{t'(\mu_0)}^{t'(\mu)} g_i(t')^2 dt'$. When we fix $\mu_0 \sim M_T$ and $\mu \sim M_{\text{SUSY}}$ these formulas can be approximated by

$$\begin{aligned}Y_L(M_{\text{SUSY}}) &\simeq \left(1 + \mathcal{G}_L [\text{Tr} (Y_{f'} Y_{f'}^\dagger)] \Delta t' - \sum_i c_i^\nu \mathcal{I}_i \right) Y_L(M_T) + \\ &\quad + \left(\mathcal{F}_L [Y_{f'} Y_{f'}^\dagger] Y_L(M_T) + Y_L(M_T) \mathcal{F}_L^T [Y_{f'} Y_{f'}^\dagger] \right) \Delta t', \\ Y_f(M_{\text{SUSY}}) &\simeq \left(1 + \mathcal{G}_f [\text{Tr} (Y_{f'} Y_{f'}^\dagger)] \Delta t' - \sum_i c_i^f \mathcal{I}_i \right) Y_f(M_T) + \mathcal{F}_f [Y_{f'} Y_{f'}^\dagger] Y_f(M_T) \Delta t',\end{aligned}\quad (5.9)$$

with $\Delta t' = 1/16\pi^2 \log(M_{\text{SUSY}}/M_T) \approx 0.13$.

For what concerns the quark sector we find the following approximated expressions for the masses of the last two families and the Cabibbo angle

$$\begin{aligned}m_t^2(M_{\text{SUSY}}) &\sim m_t^2 (1 - 2 \sum_i c_i^u \mathcal{I}_i + 14 \gamma \Delta t') , \\ m_c^2(M_{\text{SUSY}}) &\sim m_c^2 (1 - 2 \sum_i c_i^u \mathcal{I}_i + 6 \gamma \Delta t') , \\ m_b^2(M_{\text{SUSY}}) &\sim m_b^2 (1 - 2 \sum_i c_i^d \mathcal{I}_i + 16 \gamma \Delta t') , \\ m_s^2(M_{\text{SUSY}}) &\sim m_s^2 (1 - 2 \sum_i c_i^d \mathcal{I}_i + 8 \gamma \Delta t') , \\ \theta_{12}^q(M_{\text{SUSY}}) &\sim \theta_{12}^q + \frac{1}{6\sqrt{2}} U_{13}^2 \lambda \Delta t' ,\end{aligned}\quad (5.10)$$

where $\gamma = m_t^2/(v_1^u + \beta v_2^u)^2$ and the masses and the Cabibbo angle on the right of the previous expressions are intended at the M_T scale. Note that the demand that m_b^2 is still positive at M_{SUSY} gives an upper bound on γ of 0.7. The charged lepton masses are very similar to the down-quark masses and indeed we have

$$\begin{aligned} m_\tau^2(M_{\text{SUSY}}) &\sim m_\tau^2 (1 - 2 \sum_i c_i^e \mathcal{I}_i + 16 \gamma \Delta t') , \\ m_\mu^2(M_{\text{SUSY}}) &\sim m_\mu^2 (1 - 2 \sum_i c_i^e \mathcal{I}_i + 8 \gamma \Delta t') . \end{aligned} \quad (5.11)$$

We now consider the neutrino sector (see [196–208] for a general approach at RGEs with or without flavour symmetries) and the modification due to the RG running. We remember that our model at the GUT scale naturally predicts the QD spectrum, with both normal and inverse ordering, while choosing a less natural range of the parameters space we may have both NH and IH spectrum. To analyze the effect of the RGEs on the neutrino mass matrix M_ν , it is worth rotate M_ν of a maximal rotation in the (12) sector. Then, at the high scale, M_ν is diagonal and reads as

$$M_\nu = \begin{pmatrix} k'_0 - \lambda k'_1 & 0 & 0 \\ 0 & k'_0 + \lambda k'_1 & 0 \\ 0 & 0 & k'_0 + \lambda^2 k'_2 \end{pmatrix} v_L , \quad (5.12)$$

where without loss of generality k'_0 can be taken real, by a redefinition on the phases. After the running at M_{SUSY} eq. (5.12) gets a correction ΔM_ν that is given by

$$\begin{aligned} \frac{\Delta M_\nu}{v_L} &\sim \left(-k'_0 \sum_i c_i^\nu \mathcal{I}_i + \frac{13}{2} k'_0 \gamma \Delta t' \right) \mathbb{1} + \sum_i c_i^\nu \mathcal{I}_i \begin{pmatrix} \lambda k'_1 & 0 & 0 \\ 0 & -\lambda k'_1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \\ &+ \Delta t' \begin{pmatrix} -2\lambda k'_0 - 13\lambda k'_1 \gamma/2 & -k'_0 \gamma/2 & k'_0 w_- + k'_1 \lambda \gamma/2\sqrt{2} \\ -k'_0 \gamma/2 & 2\lambda k'_0 + 13\lambda k'_1 \gamma/2 & k'_0 w_+ + k'_1 \lambda \gamma/2\sqrt{2} \\ k'_0 w_- + k'_1 \lambda \gamma/2\sqrt{2} & k'_0 w_+ + k'_1 \lambda \gamma/2\sqrt{2} & 7\gamma k'_0 \end{pmatrix} , \end{aligned} \quad (5.13)$$

with $w_\pm = (\pm\gamma/\sqrt{2} + \sqrt{2}\lambda)$, for the QD and IH case when $k'_2 \sim \mathcal{O}(1)$. For the NH case characterized by $k'_2 \sim \lambda^{-2}$, ΔM_ν assumes the following form

$$\begin{aligned} \frac{\Delta M_\nu}{v_L} &\sim \left(-k'_0 \sum_i c_i^\nu \mathcal{I}_i + \frac{13}{2} k'_0 \gamma \Delta t' \right) \mathbb{1} + \sum_i c_i^\nu \mathcal{I}_i \begin{pmatrix} \lambda k'_1 & 0 & 0 \\ 0 & -\lambda k'_1 & 0 \\ 0 & 0 & -\lambda^2 k'_2 \end{pmatrix} + \\ &+ \Delta t' \begin{pmatrix} -2\lambda k'_0 - 13\lambda k'_1 \gamma/2 & -k'_0 \gamma/2 & k'_0 w_- + k'_- \gamma/2\sqrt{2} \\ -k'_0 \gamma/2 & 2\lambda k'_0 + 13\lambda k'_1 \gamma/2 & k'_0 w_+ + k'_+ \gamma/2\sqrt{2} \\ k'_0 w_- + k'_- \gamma/2\sqrt{2} & k'_0 w_+ + k'_+ \gamma/2\sqrt{2} & 7\gamma(k'_0 + k'_2 \lambda^2) \end{pmatrix} , \end{aligned} \quad (5.14)$$

with $k'_\pm = (k'_1 \lambda \pm k'_2 \lambda^2)$.

We can now consider the three different cases QD, NH and IH, which the model accounts for at the GUT scale.

- QD case $\implies k'_0, k'_1, k'_2 \sim \mathcal{O}(1)$.

The correction given by the running induces a rotation in the (23) sector characterized by

$$\tan 2\theta_{23}' \sim -\frac{\sqrt{2}\Delta t'}{\lambda} \frac{k'_0}{|k'_1|} \gamma \sim -2\sqrt{2}\lambda \frac{k'_0}{|k'_1|} \gamma, \quad (5.15)$$

being $\Delta t' \sim 2\lambda^2$. This is a large contribution, which deviates the atmospheric angle from the initial maximal value, spoiling the agreement with the experimental data at 3σ . A possible way-out would be if this large correction is erased by a corresponding large correction in the charged lepton mass matrix. However this is not the case, because the maximal θ_{23}^e in the charged lepton mixing matrix is stable under the RG running. As a result, the QD case is not viable.

- NH case $\implies k'_0 \sim \lambda^2, k'_1 \sim \mathcal{O}(1)$ and $k'_2 \sim \lambda^{-2}$.

The corrections both for the atmospheric and the reactor angles are of order $\Delta t' \gamma / 2\sqrt{2} \sim 3\lambda^3$ and can be safely neglected. Analogously, also the mass splittings receive deviations which can be neglected. On the other hand, the charged lepton mixing matrix is stable under the RG running. As a result the three mixing angles at M_{SUSY} can be well approximated with their initial values at M_T .

- IH case $\implies k'_0 \sim \lambda^2$ and $k'_1, k'_2 \sim \mathcal{O}(1)$.

All the corrections to the neutrino mixing are of order $\gamma \Delta t' / \sqrt{2}$. While the solar mass splitting receives negligible contributions, the atmospheric mass splitting is corrected as follows

$$\Delta m_{atm}^2(M_{\text{SUSY}}) = \Delta m_{atm}^2 \left(1 - 2 \sum_i c_i' \mathcal{I}_i + 13 \gamma \Delta t' \right). \quad (5.16)$$

Combining the neutrino mixing with the charged lepton one we get the following mixing angles for the lepton sector:

$$\begin{aligned} \theta_{12}^l(M_{\text{SUSY}}) &\sim \pi/4 - \theta_{12}^e + \theta_{13}^e \gamma / 2\Delta t', \\ \theta_{23}^l(M_{\text{SUSY}}) &\sim \theta_{23}^e + \gamma \Delta t', \\ \theta_{13}^l(M_{\text{SUSY}}) &\sim \theta_{13}^e - \theta_{12}^e \gamma / 2\Delta t', \end{aligned} \quad (5.17)$$

where the corrections coming from the RG running are proportional to $\Delta t'$. In particular the corrections to $\theta_{12}^l(M_{\text{SUSY}})$ and $\theta_{13}^l(M_{\text{SUSY}})$ come from the charged lepton sector, while that one to $\theta_{23}^l(M_{\text{SUSY}})$ arises only by the neutrino sector. This distinction strictly holds only at this order of approximation, while considering contributions of the order of λ^3 all the angles receive corrections from both the sectors.

- IH case $\implies k'_2 \sim \lambda^2, k'_0 \sim 1$ and $k'_1 \sim \lambda^{-1}$.

With respect to the previous case, the solar mass splitting gets a non-negligible correction, that can be written as

$$\Delta m_{sol}^2(M_{\text{SUSY}}) = \Delta m_{sol}^2 (1 - 2 \sum_i c_i' \mathcal{I}_i + 14 \gamma \Delta t'). \quad (5.18)$$

For what concerns the atmospheric mass splitting and the lepton mixing angles we recover the same shifts as in the previous case.

6 Neutrino phenomenological analysis

In the previous section, it was concluded that the quasi-degenerate spectrum is unstable under the RG running and becomes phenomenologically inviable due to too large corrections to the atmospheric mixing angle. Only the normal and inverse hierarchies were shown to be stable and phenomenologically viable. This is different from [158], where only the normal hierarchy and the quasi-degenerate spectrum with normal ordering were found.

In this section, we will discuss neutrino phenomenology in more detail, focussing on the value of the reactor mixing angle θ_{13}^l and the possibility of neutrinoless double beta decay.

The analytical expression of the reactor mixing angle is given by eq. (3.51) at NLO and it is corrected by the RG running as in eq. (5.17) for the two IH scenarios studied in the previous section. We see that θ_{13}^l is typically of order $\lambda \approx 0.2$, so $\sin^2 \theta_{13}^l \approx 0.04$, which is rather large, but still allowed at the 3σ level as shown in table 1. Here, we complete the study on θ_{13}^l , performing a numerical analysis. As can be seen from eqs. (3.34)–(3.41), the neutrino and the charged lepton mass matrices at NLO are a function of many parameters. However the GUT nature of the model allows us to fix most parameters that occur at LO because they enter in the low energy expressions for quark and charged lepton masses, as can be seen comparing eq. (3.15) with eq. (5.10). Note that all dimensionless parameters, i.e. the $\tilde{y}_i$, can be fixed of order 1.

The other free parameters can be fixed as random numbers of order 1, except for the cases where we have argued in the previous section that they should have slightly larger or smaller values. Because of the use of random numbers, the predictions of our model are no longer single valued.

We plot the reactor angle versus the solar angle in figure 3. At NLO, eq. (3.52), θ_{12}^l is driven away from the maximal value $\pi/4$ by a term proportional to λ (note that we take only the corrections which decrease the value of the solar angle, neglecting those which increase it). We see that this deviation is not for all values of the parameters large enough to bring it in the observed region, although this happens for a significant number of them. As explained above, larger values of $\sin^2 \theta_{13}^l$ are favoured and almost all points are in the sensitive region for experiments.

To study neutrinoless double beta decay, we consider the effective $0\nu\beta\beta$ parameter m_{ee} , defined as

$$m_{ee} = [U \text{diag}(m_1, m_2, m_3) U]_{11}. \quad (6.1)$$

In figure 4 we plot m_{ee} against the lightest neutrino mass, which is m_1 and m_3 in the NH and IH case respectively. The future experiments are expected to reach good sensitivities: 90 meV [210] (GERDA), 20 meV [211] (Majorana), 50 meV [212] (SuperNEMO), 15 meV [213] (CUORE) and 24 meV [214] (EXO). As a result, looking at figure 4, the whole IH band will be tested in the next future and with it the two cases of our model which allow for the IH spectrum.

Figure 3. The solar angle versus the reactor angle. On the upper line the two IH cases of the previous section (on the left the first case and on the right the second one), while on the lower line the NH one. The two vertical lines are the 3σ bounds for $\sin^2 \theta_{12}^l$ according to [1, 2]. The upper horizontal line is the 3σ upper bound for $\sin^2 \theta_{13}^l$ and the middle line is the best fit value. The lower line is the 95 % exclusion confidence level after 3 years of Daya Bay data taking [13, 14].

7 Conclusion

In this paper we have addressed several aspects of the interplay between a GUT based model and a discrete flavour symmetry. The paper should indeed be considered as the combination of two distinct parts: in the first one we mainly discussed the building of the model from the flavour point of view, while in the second one we faced the problem to justify the assumptions made in the first part and to achieve the correct gauge symmetry breaking chain.

More in detail, the symmetry group of our model is $PS \times G_f$, where PS stands for the GUT Pati-Salam gauge group $SU(4)_C \times SU(2)_L \times SU(2)_R$ and G_f for the flavour group $S_4 \times Z_4 \times U(1)_{FN} \times U(1)_R$. Within this GUT context one has the relationship between the down-quark and charged leptons mass matrices, $M_d \sim M_e$, which can easily be used to revise the old idea of quark-lepton complementarity. In the model this is obtained by the use of the non-Abelian discrete flavour symmetry S_4 properly broken through the VEVs of a set of flavon fields, which transform as triplets under S_4 . The additional Abelian symmetries,

Figure 4. Neutrinoless double beta decay plots. On the upper line the two IH cases of the previous section (on the left the first case and on the right the second one), while on the lower line the NH one. The background red (blue) points refer to the allowed region for the NH (IH), taking into account the lepton mixing angle values with their 3σ errors.

which enters in G_f , play different roles: Z_4 keeps quarks separated from leptons and neutrinos from charged leptons and prevents dangerous couplings in the superpotential of the model; $U(1)_{FN}$ helps to justify the charged fermion mass hierarchies; $U(1)_R$ is a common ingredient of supersymmetric flavour models. It contains the discrete R -parity and is useful to build a suitable flavon superpotential that allows the correct S_4 breaking pattern.

Already at the leading order, the model shows nice features: we are able to reproduce the mass hierarchy between the third and the second charged fermion families, the bottom-tau unification, the Georgi-Jarlskog [180] relation $|m_\mu| = 3|m_s|$ and, under the assumption of type-II See-Saw dominance at the GUT scale, a realistic neutrino spectrum. However at this level of approximation, both the CKM and the PMNS mixing matrices are not correct: the quark mixing matrix coincides with the identity matrix, while the lepton one is given by the BM pattern. It is worth to recall here that the BM mixing corresponds to maximal solar and atmospheric angles and to a vanishing reactor angle: only the solar angle is not

in agreement with the data as it deviates from the experimental central value by a quantity close to the Cabibbo angle, $\lambda \sim 0.2$.

At next-to-leading order, the wrong predictions for the fermion mixing angles are corrected: in the CKM matrix, the mixing angle θ_{12}^q receives contributions of the order of λ , fitting the value of the Cabibbo angle; analogously, in the PMNS matrix, the solar angle is corrected by the same amount and we find the nice result that $\theta_{12}^l \sim \pi/4 - \mathcal{O}(\lambda)$. At the same time, also the reactor angle receives significant contributions and indeed at this level of approximation it results $\theta_{13}^l \sim \mathcal{O}(\lambda)$: this is an interesting feature of our model, because this value is close to its present upper bound and it will be tested in the forthcoming neutrino experiments [11–18].

Once we consider the higher order terms, we find the other two CKM angles of the correct order of magnitude, $\theta_{23}^q \sim \mathcal{O}(\lambda^2)$ and $\theta_{13}^q \sim \mathcal{O}(\lambda^3)$, and small corrections are introduced in the PMNS angles: in particular the atmospheric angle becomes $\theta_{23}^l \sim \pi/4 + \mathcal{O}(\lambda^2)$, justifying a small deviation from the maximality. For what concerns the masses, all the fermions are massive and the mass hierarchies fit the experimental observations.

On the other hand, the neutrino spectrum could be either quasi degenerate or normal or inverse hierarchical. Only the first case corresponds to a completely natural choice of the parameters, which, in the absence of an explanation coming from a higher energy theory, should be of order 1: in order to allow the NH and the IH, the parameters should span in a larger range of values, namely $\lambda^{-2} \div \lambda^2$.

In the second part of the paper we have studied the Higgs scalar potential and the running of both the gauge couplings and the Yukawa mass matrices under the RGEs. With this analysis we looked for the constraints which arise to justify the Higgs field VEV pattern used in the flavour section and the assumption of the type-II See-Saw dominance. The presence of the flavour group G_f modifies the Higgs field content necessary to implement the classical breaking pattern of the PS gauge group and as consequence not all the results obtained by studying minimal versions of PS are recovered. In particular we need the presence of two PS multiplets $(15, 1, 1)$, A and B , responsible to break the unified colour symmetry $SU(4)_C$ to $SU(3)_C \times U(1)_{B-L}$. Two copies of the $(1, 2, 2)$ multiplet, ϕ and ϕ' , and one $(15, 2, 2)$ field, ρ , are necessary to implement the condition $v_\rho^u = 0$. Lastly, we need the new fields $\Sigma, \Sigma' \sim (1, 3, 3)$ and $\xi \sim (1, 1, 1)$ to have a type-II See-Saw contribution at tree level. Gauge coupling runnings are affected by the large field content and in particular we found that the requirement of having type-II See-Saw dominance constrains the model in a small region of the parameter space, in which all the heavy mass scales are sandwiched between 10^{11} GeV and 10^{13} GeV. At the same time Yukawa mass matrix running shows that while the CKM Cabibbo angle is stable under the RGEs evolution, the PMNS mixing angles are stable only if neutrinos present a NH or an IH spectrum, ruling out the QD case. As already stated, the QD spectrum would be the most natural and probable case at the GUT scale, but the Yukawa RGEs analysis further reduces the allowed region of the parameter space.

In section 6, we performed a brief phenomenological analysis on neutrino observables considering all the constraints which come from the flavour and the Higgs sectors. We have first considered the value of the reactor angle in terms of the deviations of the solar angle

	n	h	χ_{11}	χ_{12}	χ_2	χ_{31}	χ_{32}	Elements
$\mathcal{C}_1$	1	1	1	1	2	3	3	$\mathbb{K}$
$\mathcal{C}_2$	3	2	1	1	2	-1	-1	T^2, ST^2S, ST^2ST^2
$\mathcal{C}_3$	6	2	1	-1	0	1	-1	$S, T^3ST, TST^3, T^2ST^2, ST^2ST, TST^2S$
$\mathcal{C}_4$	8	3	1	1	-1	0	0	$TS, ST, (TS)^2, (ST)^2, T^2ST, TST^2, T^3ST^2, T^2ST^3$
$\mathcal{C}_5$	6	4	1	-1	0	-1	1	$T, T^3, ST^2, T^2S, STS, TST$

Table 7. Character table of the S_4 discrete group. $\mathcal{C}_i$ are the conjugacy classes, n the number of elements in each class, h the smallest value for which $\chi^h = \mathbb{K}$. In the last column we have reported the elements for each class in terms of the generators of the group.

from the maximal value and the numerical analysis confirmed the analytical results: in our model, θ_{13}^l naturally acquires a value not far from its present upper bound. After this, we studied the neutrinoless double beta decay effective mass, m_{ee} , and we have seen that the next future experiments are expected to reach sufficiently good sensitivities to test our model in the IH regime.

As a final comment, we underline that there is a strong tension in combining a GUT model with a (discrete non-Abelian) flavour symmetry and therefore a parallel study is not only interesting but also recommended to provide a viable model. In the first part of the paper we produced a flavour-GUT model that is quite realistic and viable. Only the study done in the second part, regarding the Higgs sector reveals that the model is restricted into a small region of the parameter space, reducing the freedom in the choice of the parameter values. Even if our results are model dependent, our construction shares many features with other models present in literature, where often a detailed discussion of the Higgs sector is missing. In our opinion, this is a serious drawback and we would suggest to consider the interplay between GUTs and flavour symmetries in this kind of models as well.

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A The symmetric group S_4

In this appendix we report the character table and the Clebsch-Gordan coefficients of the S_4 discrete group in our basis.

The generators, S and T , obey to the following rules

$$T^4 = S^2 = (ST)^3 = (TS)^3 = \mathbb{K} , \quad (\text{A.1})$$

and are of the following form for the five different representations:

$$\begin{aligned}
1_1 : S &= 1, & T &= 1, \\
1_2 : S &= -1, & T &= -1, \\
2 : S &= \frac{1}{2} \begin{pmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix}, & T &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \\
3_1 : S &= \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}, & T &= \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \\
3_2 : S &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & T &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.
\end{aligned}$$

Using these generators we calculate the Clebsch Gordan coefficient for all the Kronecker products. In the following we use α_i to indicate the elements of the first representation of the product and β_i to indicate those of the second representation.

We start with all the multiplication rules which include the 1-dimensional representations:

$$1_1 \otimes \eta = \eta \otimes 1_1 = \eta \quad \text{with } \eta \text{ any representation}$$

$$1_2 \otimes 1_2 = 1_1 \sim \alpha\beta$$

$$1_2 \otimes 2 = 2 \sim \begin{pmatrix} -\alpha\beta_2 \\ \alpha\beta_1 \end{pmatrix}$$

$$1_2 \otimes 3_1 = 3_2 \sim \begin{pmatrix} \alpha\beta_1 \\ \alpha\beta_2 \\ \alpha\beta_3 \end{pmatrix}$$

$$1_2 \otimes 3_2 = 3_1 \sim \begin{pmatrix} \alpha\beta_1 \\ \alpha\beta_2 \\ \alpha\beta_3 \end{pmatrix}$$

The multiplication rules with the 2-dimensional representation are the following:

$$2 \otimes 2 = 1_1 \oplus 1_2 \oplus 2 \quad \text{with} \quad \begin{cases} 1_1 \sim \alpha_1\beta_1 + \alpha_2\beta_2 \\ 1_2 \sim -\alpha_1\beta_2 + \alpha_2\beta_1 \\ 2 \sim \begin{pmatrix} \alpha_1\beta_2 + \alpha_2\beta_1 \\ \alpha_1\beta_1 - \alpha_2\beta_2 \end{pmatrix} \end{cases}$$

$$\begin{aligned}
 2 \otimes 3_1 &= 3_1 \oplus 3_2 & \text{with} & \left\{ \begin{aligned} 3_1 &\sim \begin{pmatrix} \alpha_2 \beta_1 \\ -\frac{1}{2}(\sqrt{3}\alpha_1\beta_2 + \alpha_2\beta_2) \\ \frac{1}{2}(\sqrt{3}\alpha_1\beta_3 - \alpha_2\beta_3) \end{pmatrix} \\ 3_2 &\sim \begin{pmatrix} \alpha_1\beta_1 \\ \frac{1}{2}(\sqrt{3}\alpha_2\beta_2 - \alpha_1\beta_2) \\ -\frac{1}{2}(\sqrt{3}\alpha_2\beta_3 + \alpha_1\beta_3) \end{pmatrix} \end{aligned} \right. \\
 2 \otimes 3_2 &= 3_1 \oplus 3_2 & \text{with} & \left\{ \begin{aligned} 3_1 &\sim \begin{pmatrix} \alpha_1\beta_1 \\ \frac{1}{2}(\sqrt{3}\alpha_2\beta_2 - \alpha_1\beta_2) \\ -\frac{1}{2}(\sqrt{3}\alpha_2\beta_3 + \alpha_1\beta_3) \end{pmatrix} \\ 3_2 &\sim \begin{pmatrix} \alpha_2\beta_1 \\ -\frac{1}{2}(\sqrt{3}\alpha_1\beta_2 + \alpha_2\beta_2) \\ \frac{1}{2}(\sqrt{3}\alpha_1\beta_3 - \alpha_2\beta_3) \end{pmatrix} \end{aligned} \right. \tag{A.2}
 \end{aligned}$$

The multiplication rules with the 3-dimensional representations are the following:

$$\begin{aligned}
 3_1 \otimes 3_1 &= 3_2 \otimes 3_2 = 1_1 \oplus 2 \oplus 3_1 \oplus 3_2 & \text{with} & \left\{ \begin{aligned} 1_1 &\sim \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 \\ 2 &\sim \begin{pmatrix} \frac{1}{\sqrt{2}}(\alpha_2\beta_2 - \alpha_3\beta_3) \\ \frac{1}{\sqrt{6}}(-2\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3) \end{pmatrix} \\ 3_1 &\sim \begin{pmatrix} \alpha_2\beta_3 + \alpha_3\beta_2 \\ \alpha_1\beta_3 + \alpha_3\beta_1 \\ \alpha_1\beta_2 + \alpha_2\beta_1 \end{pmatrix} \\ 3_2 &\sim \begin{pmatrix} \alpha_3\beta_2 - \alpha_2\beta_3 \\ \alpha_1\beta_3 - \alpha_3\beta_1 \\ \alpha_2\beta_1 - \alpha_1\beta_2 \end{pmatrix} \end{aligned} \right. \\
 3_1 \otimes 3_2 &= 1_2 \oplus 2 \oplus 3_1 \oplus 3_2 & \text{with} & \left\{ \begin{aligned} 1_2 &\sim \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 \\ 2 &\sim \begin{pmatrix} \frac{1}{\sqrt{6}}(2\alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3) \\ \frac{1}{\sqrt{2}}(\alpha_2\beta_2 - \alpha_3\beta_3) \end{pmatrix} \\ 3_1 &\sim \begin{pmatrix} \alpha_3\beta_2 - \alpha_2\beta_3 \\ \alpha_1\beta_3 - \alpha_3\beta_1 \\ \alpha_2\beta_1 - \alpha_1\beta_2 \end{pmatrix} \\ 3_2 &\sim \begin{pmatrix} \alpha_2\beta_3 + \alpha_3\beta_2 \\ \alpha_1\beta_3 + \alpha_3\beta_1 \\ \alpha_1\beta_2 + \alpha_2\beta_1 \end{pmatrix} \end{aligned} \right.
 \end{aligned}$$

B Higgs scalar spectrum

We now proceed to presenting the scalar mass matrices for all the fields introduced in section 4 according to the $\text{SM} \sim \text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)_Y$ representations, indicating their origin with respect to the PS and the colour-broken Pati-Salam (CbPS) phase. We recall that in the colour broken phase the symmetry group is given by $\text{SU}(3)_C \times \text{SU}(2)_L \times \text{SU}(2)_R \times \text{U}(1)_{B-L}$. For completeness for each field we also indicate the corresponding T_{3R} value, with T_{3R} the diagonal generator of $\text{SU}(2)_R$. We use the same notation as in [209], in which we write the Dirac scalar mass matrices as they could be read directly from the superpotential of eq. (4.1) at the scale M_R . We label the mass matrices with S , D and T according to the singlet, doublet or triplet representations of the $\text{SU}(2)_L$ gauge symmetry.

B.1 $\text{SU}(2)_L$ singlets

B.1.1 Singlets (1,1,0)

$$M_{S1} = \begin{pmatrix} 0 & 0 & \frac{3}{\sqrt{2}}\lambda_R M_R & 0 & 0 \\ 0 & 0 & \frac{3}{\sqrt{2}}\lambda_R M_R & 0 & 0 \\ \frac{3}{\sqrt{2}}\lambda_R M_R & \frac{3}{\sqrt{2}}\lambda_R M_R & \sqrt{2}M_{C1}\lambda_A + x & \lambda_{AB\xi}V_\xi + \frac{1}{\sqrt{2}}\lambda_B M_{C2} & \frac{1}{\sqrt{2}}\lambda_{AB\xi}M_{C2} \\ 0 & 0 & \lambda_{AB\xi}V_\xi + \frac{1}{\sqrt{2}}\lambda_B M_{C2} & \frac{1}{\sqrt{2}}\lambda_{B\xi}M_{C1} - y & \frac{1}{\sqrt{2}}\lambda_{AB\xi}M_{C2} \\ 0 & 0 & \frac{1}{\sqrt{2}}\lambda_{AB\xi}M_{C2} & \frac{1}{\sqrt{2}}\lambda_{AB\xi}M_{C1} & -\frac{M_{C1}M_{C2}}{V_\xi\sqrt{2}}\lambda_{AB\xi} \end{pmatrix} \quad (\text{B.1})$$

with

$$\begin{aligned} x &= \frac{1}{2\sqrt{2}M_{C1}}(-4\lambda_A M_{C1}^2 - 2\sqrt{2}\lambda_{AB\xi}M_{C2}V_\xi - 2\lambda_B M_{C2}^2 - 8\lambda_R M_R^2) \\ y &= \frac{M_{C1}}{\sqrt{2}M_{C2}}(\sqrt{2}\lambda_{AB\xi}V_\xi + 2\lambda_B M_{C2}). \end{aligned} \quad (\text{B.2})$$

M_{S1} has a vanishing eigenvalue that is eaten by the corresponding gauge boson. Moreover it can be checked that one of the singlets, which we call ξ_0 , has a mass $\sim M_R$ while all the others masses appear as combination of M_{C1} , M_{C2} and V_ξ .

		PS	CbPS	T_{3R}
C_{1,R_1}	Δ_R	$(10, 1, 3)$	$(1, 1, 3, -2)$	1
C_{2,R_2}	$\overline{\Delta}_R$	$(\overline{10}, 1, 3)$	$(1, 1, 3, 2)$	-1
C_{3,R_3}	A	$(15, 1, 1)$	$(1, 1, 1, 0)$	0
C_{4,R_4}	B	$(15, 1, 1)$	$(1, 1, 1, 0)$	0
C_{5,R_5}	ξ	$(1, 1, 1)$	$(1, 1, 1, 0)$	0

B.1.2 Singlets $(3, 1, 2/3) \oplus (\bar{3}, 1, -2/3)$

$$M_{S2} = \begin{pmatrix} \sqrt{2}\lambda_A M_{C1} + x & \lambda_{AB\xi} V_\xi + \frac{1}{\sqrt{2}}\lambda_B M_{C2} & 2\lambda_R M_R \\ \lambda_{AB\xi} V_\xi + \frac{1}{\sqrt{2}}\lambda_B M_{C2} & \frac{1}{\sqrt{2}}\lambda_B M_{C1} - y & 0 \\ 2\lambda_R M_R & 0 & -\sqrt{2}\lambda_R M_{C1} \end{pmatrix} \quad (\text{B.3})$$

M_{S2} has a vanishing eigenvalue: it corresponds to the massless GBs $(3, 1, 2/3) \oplus (\bar{3}, 1, -2/3)$ eaten by the gauge bosons.

		PS	CbPS	T_{3R}
C_1	A	$(15, 1, 1)$	$(3, 1, 1, 4/3)$	0
C_2	B	$(15, 1, 1)$	$(3, 1, 1, 4/3)$	0
C_3	Δ_R	$(10, 1, 3)$	$(3, 1, 3, -2/3)$	1
R_1	A	$(15, 1, 1)$	$(\bar{3}, 1, 1, -4/3)$	0
R_2	B	$(15, 1, 1)$	$(\bar{3}, 1, 1, -4/3)$	0
R_3	$\bar{\Delta}_R$	$(\bar{10}, 1, 3)$	$(\bar{3}, 1, 3, 2/3)$	-1

B.1.3 Singlets $(8, 1, 0)$

$$M_{S3} = \begin{pmatrix} -\sqrt{2}M_{C1}\lambda_A + x & V_\xi\lambda_{AB\xi} - \frac{1}{\sqrt{2}}\lambda_B M_{C2} \\ V_\xi\lambda_{AB\xi} - \frac{1}{\sqrt{2}}\lambda_B M_{C2} & -\frac{1}{\sqrt{2}}M_{C1}\lambda_B - y \end{pmatrix} \quad (\text{B.4})$$

		PS	CbPS	T_{3R}
C_1, R_1	A	$(15, 1, 1)$	$(8, 1, 1, 0)$	0
C_2, R_2	B	$(15, 1, 1)$	$(8, 1, 1, 0)$	0

B.1.4 Singlets $(1, 1, \pm 1)$

These states correspond to the last two massless GBs eaten by the respective gauge bosons. Indeed with these last two massless states the total amount of GBs eaten is $9 = 1 + 2 \times 3 + 2$, as it should be for the breaking $\text{SU}(4)_C \times \text{SU}(2)_L \times \text{SU}(2)_R \rightarrow \text{SU}(3)_C \times \text{SU}(2)_L \times \text{U}(1)_Y$.

		PS	CbPS	T_{3R}
C_1	Δ_R	$(10, 1, 3)$	$(1, 1, 3, -2)$	0
R_1	$\bar{\Delta}_R$	$(\bar{10}, 1, 3)$	$(1, 1, 3, 2)$	0

B.1.5 Singlets $(1, 1, \pm 2)$

Two massless charged singlet (δ^{++} and $\bar{\delta}^{++}$).

		PS	CbPS	T_{3R}
C_1	Δ_R	$(10, 1, 3)$	$(1, 1, 3, -2)$	-1
R_1	$\bar{\Delta}_R$	$(\bar{10}, 1, 3)$	$(1, 1, 3, 2)$	1

B.1.6 Singlets $(3, 1, -1/3) \oplus (\bar{3}, 1, 1/3)$

$$-\sqrt{2}M_{C1}\lambda_R$$

		PS	CbPS	T_{3R}
C_1	Δ_R	$(10, 1, 3)$	$(3, 1, 3, -2/3)$	0
R_1	$\bar{\Delta}_R$	$(\bar{10}, 1, 3)$	$(\bar{3}, 1, 3, 2/3)$	0

B.1.7 Singlets $(3, 1, -4/3) \oplus (\bar{3}, 1, 4/3)$

$$-\sqrt{2}M_{C1}\lambda_R$$

		PS	CbPS	T_{3R}
C_1	Δ_R	$(10, 1, 3)$	$(3, 1, 3, -2/3)$	-1
R_1	$\bar{\Delta}_R$	$(\bar{10}, 1, 3)$	$(\bar{3}, 1, 3, 2/3)$	1

B.1.8 Singlets $(6, 1, 4/3) \oplus (\bar{6}, 1, -4/3)$

$$-2\sqrt{2}M_{C1}\lambda_R$$

		PS	CbPS	T_{3R}
C_1	Δ_R	$(10, 1, 3)$	$(6, 1, 3, 2/3)$	1
R_1	$\bar{\Delta}_R$	$(\bar{10}, 1, 3)$	$(\bar{6}, 1, 3, -2/3)$	-1

B.1.9 Singlets $(6, 1, 1/3) \oplus (\bar{6}, 1, -1/3)$

$$-2\sqrt{2}M_{C1}\lambda_R$$

		PS	CbPS	T_{3R}
C_1	Δ_R	$(10, 1, 3)$	$(6, 1, 3, 2/3)$	0
R_1	$\bar{\Delta}_R$	$(\bar{10}, 1, 3)$	$(\bar{6}, 1, 3, -2/3)$	0

B.1.10 Singlets $(6, 1, -2/3) \oplus (\bar{6}, 1, 2/3)$

$$-2\sqrt{2}M_{C1}\lambda_R$$

		PS	CbPS	T_{3R}
C_1	Δ_R	$(10, 1, 3)$	$(6, 1, 3, 2/3)$	-1
R_1	$\bar{\Delta}_R$	$(\bar{10}, 1, 3)$	$(\bar{6}, 1, 3, -2/3)$	1

B.2 $SU(2)_L$ doublets
B.2.1 Doublets $(1, 2, \pm 1/2)$

$$M_{D1} = \begin{pmatrix} M_\phi & M_{\phi\phi'} & \frac{1}{\sqrt{2}}M_{C2}\lambda_{\phi\rho} \\ M_{\phi\phi'} & M_{\phi'} & \frac{1}{\sqrt{2}}M_{C2}\lambda_{\phi'\rho} \\ \frac{1}{\sqrt{2}}M_{C2}\lambda_{\phi\rho} & \frac{1}{\sqrt{2}}M_{C2}\lambda_{\phi'\rho} & M_\rho + \frac{1}{\sqrt{2}}\lambda_{\rho A}M_{C1} \end{pmatrix} \quad (\text{B.5})$$

When M_{D1}^2 is diagonalized according to

$$U^T \cdot M_{D1}^2 \cdot U = \hat{M}_{D1} \hat{M}_{D1} \quad (\text{B.6})$$

we get three up-type (down-type) Higgs doublets h_u, h'_u, H_u (h_d, h'_d, H_d).

$$\begin{pmatrix} \phi_{u,d} \\ \phi'_{u,d} \\ \rho_{u,d} \end{pmatrix} = U^T \begin{pmatrix} h_{u,d} \\ h'_{u,d} \\ H_{u,d} \end{pmatrix} \quad (\text{B.7})$$

Therefore for the up (down) projections we have

$$\begin{aligned} v_\phi^{u,d} &= \langle \phi_{u,d} \rangle = U_{11}v_1^{u,d} + U_{21}v_2^{u,d} + U_{31}v_3^{u,d}, \\ v_{\phi'}^{u,d} &= \langle \phi'_{u,d} \rangle = U_{12}v_1^{u,d} + U_{22}v_2^{u,d} + U_{32}v_3^{u,d}, \\ v_\rho^{u,d} &= \langle \rho_{u,d} \rangle = U_{13}v_1^{u,d} + U_{23}v_2^{u,d} + U_{33}v_3^{u,d}, \end{aligned} \quad (\text{B.8})$$

where $v_1^{u,d} = \langle h_{u,d} \rangle$, $v_2^{u,d} = \langle h'_{u,d} \rangle$, $v_3^{u,d} = \langle H_{u,d} \rangle$. In general a light doublet, i.e. massless at the M_C scale, gets a mass term at M_{SUSY} and its VEV, at the EW scale, is of order of the EW scale $\sim v_W$. On the contrary for a heavy doublet of mass M , its induced VEV at the EW scale is $\sim v_W^2/M$, that for $M \sim M_C$ is completely negligible with respect to v_W . Consider now the condition $\langle \rho_u \rangle = 0$ that we imposed to get the correct fermion mass matrices. In the standard case we would have only one up- and one down-type light Higgs doublets, being all the other doublets heavy. Assume now that $h_{u,d}$ in eq. (B.7) are the up- and down-type light doublets at M_C . Then we have $v_1^{u,d} \sim v_W$ while $v_2^{u,d} \sim v_3^{u,d} \sim 0$. From eq. (B.8) we see that in this case it would be impossible to make vanishing the ρ projection along the up-direction (that implies $U_{13} = 0$) still maintaining a non-vanishing $\langle \rho_d \rangle$.¹¹ For

¹¹It is worth to say that even the condition $U_{13} = 0$ is not natural and easily realized. Therefore in the most general case with only one up-type (down-type) light doublet, the condition $\langle \rho_u \rangle = 0$ implies that all the other VEVs given in eq. (B.8) vanish.

this reason the condition $\langle \rho_u \rangle = 0$ implies a non-standard scenario and the presence of two light doublets of up-type at the M_C scale, namely h_u and h'_u . However the symmetric nature of M_{D1} ensures that as consequence we are also left with two down-type light Higgs doublets, h_d and h'_d . Nevertheless this does not imply that $\langle \rho_d \rangle$ vanishes because we have

$$\begin{aligned}\langle \rho_u \rangle &= U_{13}v_1^u + U_{23}v_2^u = 0, \\ \langle \rho_d \rangle &= U_{13}v_1^d + U_{23}v_2^d \neq 0,\end{aligned}\tag{B.9}$$

since $v_i^{u,d}$ depend on the soft terms.

In conclusion we have to impose two constraints on the free parameters that enter in M_{D1} corresponding to require that M_{D1} has two vanishing eigenvalues. Notice that the condition of having M_{D1} of rank 1 is fine-tuned but it is not more fine-tuned than imposing M_{D1} of rank 2, which is universally accepted whenever the MSSM has to be recovered. In our case the fermion mass matrix structures impose a slightly different condition $-M_{D1}$ of rank 1 — but both the requirements are satisfied by fine-tuning the parameters that enter in the mass matrix.

		PS	CbPS	T_{3R}
C_1	ϕ	$(1, 2, 2)$	$(1, 2, 2, 0)$	$1/2$
C_2	ϕ'	$(1, 2, 2)$	$(1, 2, 2, 0)$	$1/2$
C_3	ρ	$(15, 2, 2)$	$(1, 2, 2, 0)$	$1/2$
R_1	ϕ	$(1, 2, 2)$	$(1, 2, 2, 0)$	$-1/2$
R_2	ϕ'	$(1, 2, 2)$	$(1, 2, 2, 0)$	$-1/2$
R_3	ρ	$(15, 2, 2)$	$(1, 2, 2, 0)$	$-1/2$

B.2.2 Doublets $(3, 2, 7/6) \oplus (\bar{3}, 2, -7/6)$

$$M_\rho + \frac{1}{\sqrt{2}}\lambda_{\rho A}M_{C_1}$$

		PS	CbPS	T_{3R}
C_1	ρ	$(15, 2, 2)$	$(3, 2, 2, 4/3)$	$1/2$
R_1	ρ	$(15, 2, 2)$	$(\bar{3}, 2, 2, -4/3)$	$-1/2$

B.2.3 Doublets $(3, 2, 1/6) \oplus (\bar{3}, 2, -1/6)$

$$M_\rho + \frac{1}{\sqrt{2}}\lambda_{\rho A}M_{C_1}$$

		PS	CbPS	T_{3R}
C_1	ρ	$(15, 2, 2)$	$(3, 2, 2, 4/3)$	$-1/2$
R_1	ρ	$(15, 2, 2)$	$(\bar{3}, 2, 2, -4/3)$	$1/2$

B.2.4 Doublets $(8, 2, 1/2) \oplus (8, 2, -1/2)$

$$M_\rho - \frac{1}{\sqrt{2}}\lambda_{\rho A}M_{C_1}$$

		PS	CbPS	T_{3R}
C_1	ρ	$(15, 2, 2)$	$(8, 2, 2, 0)$	$1/2$
R_1	ρ	$(15, 2, 2)$	$(8, 2, 2, 0)$	$-1/2$

B.3 $SU(2)_L$ triplets

B.3.1 Triplets $(1, 3, 1) \oplus (1, 3, -1)$

$$M_{T1} = \begin{pmatrix} M_\Sigma & \lambda_\xi V_\xi & 0 \\ \lambda_\xi V_\xi & M_{\Sigma'} & \frac{1}{\sqrt{2}}\lambda M_R \\ 0 & \frac{1}{\sqrt{2}}\bar{\lambda} M_R & M_{\Delta L} + \frac{3}{\sqrt{2}}\lambda_L M_{C_1} \end{pmatrix} \quad (\text{B.10})$$

with

		PS	CbPS	T_{3R}
C_1	Σ	$(1, 3, 3)$	$(1, 3, 3, 0)$	1
C_2	Σ'	$(1, 3, 3)$	$(1, 3, 3, 0)$	1
C_3	Δ_L	$(\bar{10}, 3, 1)$	$(1, 3, 1, 2)$	0
R_1	Σ	$(1, 3, 3)$	$(1, 3, 3, 0)$	-1
R_2	Σ'	$(1, 3, 3)$	$(1, 3, 3, 0)$	-1
R_3	$\bar{\Delta}_L$	$(10, 3, 1)$	$(1, 3, 1, -2)$	0

B.3.2 Triplets $(1, 3, 0)$

$$M_{T2} = \begin{pmatrix} M_\Sigma & \lambda_\xi V_\xi \\ \lambda_\xi V_\xi & M_{\Sigma'} \end{pmatrix} \quad (\text{B.11})$$

with

		PS	CbPS	T_{3R}
C_1, R_1	Σ	$(1, 3, 3)$	$(1, 3, 3, 0)$	0
C_2, R_2	Σ'	$(1, 3, 3)$	$(1, 3, 3, 0)$	0

B.3.3 Triplets $(3, 3, -1/3) \oplus (\bar{3}, 3, 1/3)$

$$M_{\Delta_L} + \frac{1}{\sqrt{2}} \lambda_L M_{C1}$$

with

		PS	CbPS	T_{3R}
C_1	Δ_L	$(\bar{10}, 3, 1)$	$(3, 3, 1, 2/3)$	0
R_1	$\bar{\Delta}_L$	$(10, 3, 1)$	$(\bar{3}, 3, 1, -2/3)$	0

B.3.4 Triplets $(6, 3, 1/3) \oplus (\bar{6}, 3, -1/3)$

$$M_{\Delta_L} - \frac{1}{\sqrt{2}} \lambda_L M_{C1}$$

with

		PS	CbPS	T_{3R}
C_1	Δ_L	$(\bar{10}, 3, 1)$	$(6, 3, 1, -2/3)$	0
R_1	$\bar{\Delta}_L$	$(10, 3, 1)$	$(\bar{6}, 3, 1, 2/3)$	0

C NLO contributions to the flavon scalar potential

The superpotential w_d , linear in the driving fields D_R , φ_R , χ_R and σ_R , is modified into:

$$w_d = w_d^0 + \delta w_d. \quad (\text{C.1})$$

where δw_d contains the NLO contributions, suppressed by one power of $1/\Lambda$ with respect to w_d . The corrective term δw_d is given by the most general quartic, $S_4 \times Z_4$ -invariant polynomial linear in the driving fields, and can be obtained by inserting an additional flavon field in all the LO terms. The Z_4 -charges prevent any addition of the flavons φ and φ' at NLO, while a factor of σ or χ can be added to all the LO terms. The full expression of δw_d is the following:

$$\delta w_d = \frac{1}{\Lambda} \left(\sum_{i=1}^3 x_i I_i^{\sigma_R} + \sum_{i=1}^5 w_i I_i^{\chi_R} + \sum_{i=1}^6 s_i I_i^{D_R} + \sum_{i=1}^5 v_i I_i^{\varphi_R} \right) \quad (\text{C.2})$$

where x_i , w_i , s_i and v_i are coefficients and $\{I_i^{\sigma_R}, I_i^{\chi_R}, I_i^{D_R}, I_i^{\varphi_R}\}$ represent a basis of independent quartic invariants:

$$\begin{aligned} I_1^{\sigma_R} &= \sigma_R \sigma \sigma \sigma & I_3^{\sigma_R} &= \sigma_R \sigma (\chi \chi) \\ I_2^{\sigma_R} &= \sigma_R (\chi (\chi \chi)_{3_1}) & I_4^{\chi_R} &= (\chi_R (\chi \chi)_{3_1}) \sigma \\ I_1^{\chi_R} &= (\chi_R \chi) (\chi \chi) & I_5^{\chi_R} &= (\chi_R \chi) \sigma \sigma \\ I_2^{\chi_R} &= ((\chi_R \chi)_2 (\chi \chi)_2) & & \\ I_3^{\chi_R} &= ((\chi_R \chi)_{3_1} (\chi \chi)_{3_1}) & & \end{aligned}$$

$$\begin{aligned}
 I_1^{DR} &= ((D_R \chi)_{3_1} (\varphi \varphi')_{3_1}) & I_4^{DR} &= (D_R (\varphi \varphi)_2) \sigma \\
 I_2^{DR} &= ((D_R \chi)_{3_2} (\varphi \varphi')_{3_2}) & I_5^{DR} &= (D_R (\varphi' \varphi')_2) \sigma \\
 I_3^{DR} &= ((D_R \chi)_{3_1} (\varphi' \varphi')_{3_1}) & I_6^{DR} &= (D_R (\varphi \varphi')_2) \sigma \\
 I_1^{\varphi R} &= (\varphi_R \chi)' (\varphi \varphi')' & I_4^{\varphi R} &= ((\varphi_R \chi)_{3_2} (\varphi \varphi')_{3_2}) \\
 I_2^{\varphi R} &= ((\varphi_R \chi)_2 (\varphi \varphi')_2) & I_5^{\varphi R} &= (\varphi_R (\varphi \varphi')_{3_2}) \sigma . \\
 I_3^{\varphi R} &= ((\varphi_R \chi)_{3_1} (\varphi \varphi')_{3_1})
 \end{aligned}$$

In the previous terms we indicate with (...) the singlet 1_1 , with (...)’ the singlet 1_2 and with (...) $_R$ ($R = 2, 3_1, 3_2$) the representation R .

The NLO flavon VEVs are obtained by imposing the vanishing of the first derivative of $w_d + \delta w_d$ with respect to the driving fields σ_R , χ_R , D_R and φ_R . We look for a solution that perturbs eqs. (3.4) and (3.5) to first order in the $1/\Lambda$ expansion: for all components of the flavons $\Phi = (\sigma, \chi, \varphi, \varphi')$, we denote the shifted VEV’s by

$$\langle \Phi \rangle = \langle \Phi \rangle_{LO} + \delta \Phi \quad (\text{C.3})$$

where $\langle \Phi \rangle_{LO}$ are given by eqs. (3.4) and (3.5).

It is straightforward to verify the following results. In the Majorana mass sector the shifts $\delta \sigma$, $\delta \chi$ turn out to be proportional to the LO VEV’s $\langle \Phi \rangle_{LO}$ and can be absorbed in a redefinition of the parameters v_χ and v_σ . Instead, in the Dirac mass sector, the shifts $\delta \varphi$, $\delta \varphi'$ have a non-trivial structure, so that the LO texture is modified:

$$\langle \varphi \rangle = \begin{pmatrix} \delta v_\varphi \\ v'_\varphi \\ v'_\varphi \end{pmatrix} \quad \langle \varphi' \rangle = \begin{pmatrix} \delta v_{\varphi'} \\ v'_{\varphi'} \\ -v'_{\varphi'} \end{pmatrix} \quad (\text{C.4})$$

where v'_φ and $v'_{\varphi'}$ satisfy a relation similar to that in eq. (3.60) and the shifts δv_φ and $\delta v_{\varphi'}$ are suppressed by a factor λ with respect to the LO entries v'_φ and $v'_{\varphi'}$, respectively.

D Gauge coupling running

In this appendix, we provide the coefficients of the β -functions for the gauge coupling running in the different regimes. The complete matter fields run from the GUT scale down to the M_{SUSY} scale, where the SUSY partners decouple. For what concerns the scalar fields in section 4, we have already outlined the scalar spectrum according to the different scale at which the fields decouple. As a result the computation of the β -functions is straightforward. Calling μ the generic scale, we have

- for $M_C < \mu < M_{GUT}$: all matter is in the left and right handed multiplets $(4, 2, 1)$ and $(\bar{4}, 1, 2)$ as mentioned in table 2. In the Higgs sector, we have all the fields mentioned in table 6. This leads to the coefficients

$$\beta_{\text{SU}(4)_C} = 54, \quad \beta_{\text{SU}(2)_L} = 69, \quad \beta_{\text{SU}(2)_R} = 69. \quad (\text{D.1})$$

Due to the large matter content these coefficients are very large and the β -functions are very steep. As consequence the theory is in the Pati-Salam regime only for a very small range of energies, as can indeed be seen in figure 2. Almost after passing the scale M_C , the $SU(2)_R$ coupling constant enters the non-perturbative regime.

- $M_R < \mu < M_C$: the theory undergoes to the $SU(3)_C \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ symmetry. Considering the matter (left and right handed doublets of quarks and leptons characterized by different $U(1)_{B-L}$ charges) and the scalar fields, we find the coefficients

$$\beta_{SU(3)_C} = -3, \quad \beta_{SU(2)_L} = 18, \quad \beta_{SU(2)_R} = 18, \quad \beta_{U(1)_{B-L}} = 24. \quad (D.2)$$

- $M_T < \mu < M_R$: in this regime we have all the usual MSSM matter particles, four light higgs doublets (two up-type and two down-type), a couple of SM triplets $(1, 3, 1) \oplus (1, 3, -1)$ and two extra charged singlets (δ^{++} and $\bar{\delta}^{++}$). The coefficients of the β -functions are

$$\beta_{SU(3)_C} = -3, \quad \beta_{SU(2)_L} = 4, \quad \beta_{U(1)_Y} = 69/5. \quad (D.3)$$

The hypercharge that appears in the last term is related to $SU(2)_R$ and the $B - L$ charges in the previous regime by

$$Y = T_{3R} + \frac{B - L}{2}.$$

- $M_{SUSY} < \mu < M_T$: in this regime we have all the usual MSSM matter particles, four light higgs doublets (two up-type and two down-type) and two extra charged singlets (δ^{++} and $\bar{\delta}^{++}$). The β -function coefficients are

$$\beta_{SU(3)_C} = -3, \quad \beta_{SU(2)_L} = 2, \quad \beta_{U(1)_Y} = 12. \quad (D.4)$$

This should be compared with the $(-3, 1, 33/5)$ coefficients of the ordinary MSSM.

- $v_W < \mu < M_{SUSY}$: we have the particle content of the standard model, with the exception that there are four Higgs doublets. We have therefore the following β -function coefficients

$$\beta_{SU(3)_C} = -7, \quad \beta_{SU(2)_L} = -8/3, \quad \beta_{U(1)_Y} = 22/5. \quad (D.5)$$

This should be compared with the $(-7, -19/6, 41/10)$ coefficients of the ordinary SM.

E Yukawa running

As we already reported in section 5, because of the closeness of the intermediate scales between M_{GUT} and M_T we can consider only the running between M_T and M_{SUSY} to provide analytical approximations for the evolution of fermion masses and mixing under

the RGEs effect. At M_T the scalar $SU(2)_L$ triplet has been already integrated out giving rise to the effective Weinberg operator given by

$$\alpha_{ij} Y_{Lrs} \frac{L_r L_s h_{u_i} h_{u_j}}{M_T}, \quad (\text{E.1})$$

with $h_{u_1} = h_u, h_{u_2} = h'_u$ and α_{ij} coefficients arising by the scalar potential and Y_L given in eq. (5.4).

For what concerns the charged fermion Yukawa, at M_T the Dirac part of the superpotential is written as

$$Y_u Q U^c h_u + Y'_u Q U^c h'_u + Y_d Q D^c h_d + Y'_d Q D^c h'_d + Y_e L E^c h_d + Y'_e L E^c h'_d, \quad (\text{E.2})$$

with the Yukawa mass matrices given in eq. (5.6).

The Yukawa matrices RGEs are therefore given by

$$\begin{aligned} 16\pi^2 \frac{dY_u}{dt} &= \left[3 Y_u Y_u^\dagger + Y_d Y_d^\dagger + 3 Y'_u Y_u'^\dagger + Y'_d Y_d'^\dagger + 3 \text{Tr}(Y_u Y_u^\dagger) - \Sigma_i c_i^u g_i^2 \right] Y_u, \\ 16\pi^2 \frac{dY'_u}{dt} &= \left[3 Y_u Y_u^\dagger + Y_d Y_d^\dagger + 3 Y'_u Y_u'^\dagger + Y'_d Y_d'^\dagger + 3 \text{Tr}(Y'_u Y_u'^\dagger) - \Sigma_i c_i^u g_i^2 \right] Y'_u, \\ 16\pi^2 \frac{dY_d}{dt} &= \left[Y_u Y_u^\dagger + 3 Y_d Y_d^\dagger + Y'_u Y_u'^\dagger + 3 Y'_d Y_d'^\dagger + 3 \text{Tr}(Y_d Y_d^\dagger) + \text{Tr}(Y_e Y_e^\dagger) - \Sigma_i c_i^d g_i^2 \right] Y_d, \\ 16\pi^2 \frac{dY'_d}{dt} &= \left[Y_u Y_u^\dagger + 3 Y_d Y_d^\dagger + Y'_u Y_u'^\dagger + 3 Y'_d Y_d'^\dagger + 3 \text{Tr}(Y'_d Y_d'^\dagger) + \text{Tr}(Y'_e Y_e'^\dagger) - \Sigma_i c_i^d g_i^2 \right] Y'_d, \\ 16\pi^2 \frac{dY_e}{dt} &= \left[3 Y_e Y_e^\dagger + 3 Y'_e Y_e'^\dagger + 3 \text{Tr}(Y_d Y_d^\dagger) + \text{Tr}(Y_e Y_e^\dagger) - \Sigma_i c_i^e g_i^2 \right] Y_e, \\ 16\pi^2 \frac{dY'_e}{dt} &= \left[3 Y_e Y_e^\dagger + 3 Y'_e Y_e'^\dagger + 3 \text{Tr}(Y'_d Y_d'^\dagger) + \text{Tr}(Y'_e Y_e'^\dagger) - \Sigma_i c_i^e g_i^2 \right] Y'_e, \\ 16\pi^2 \frac{dY_L}{dt} &= \left[(Y_e Y_e^\dagger + Y'_e Y_e'^\dagger) Y_L + Y_L (Y_e Y_e^\dagger + Y'_e Y_e'^\dagger)^T - \Sigma_i c_i^\nu g_i^2 Y_L \right] (\alpha_{11} + \alpha_{12} + \alpha_{22}) \\ &\quad + \left[6 \text{Tr}(Y_u Y_u^\dagger) \alpha_{11} + 3 \text{Tr}(Y_u Y_u^\dagger) \alpha_{12} + 3 \text{Tr}(Y'_u Y_u'^\dagger) \alpha_{12} + 6 \text{Tr}(Y'_u Y_u'^\dagger) \alpha_{22} \right] Y_L, \end{aligned} \quad (\text{E.3})$$

with

$$\begin{aligned} c_1^u &= \frac{13}{15}, & c_2^u &= 3, & c_3^u &= \frac{16}{3}, & c_1^d &= \frac{7}{15}, & c_2^d &= 3, & c_3^d &= \frac{16}{3}, \\ c_1^e &= \frac{9}{5}, & c_2^e &= 3, & c_3^e &= 0, & c_1^\nu &= \frac{6}{5}, & c_2^\nu &= 6, & c_3^\nu &= 0. \end{aligned} \quad (\text{E.4})$$

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References

- [1] G.L. Fogli, E. Lisi, A. Marrone, A. Palazzo and A.M. Rotunno, *What we (would like to) know about the neutrino mass*, [arXiv:0809.2936](#) [[SPIRES](#)].

- [2] G.L. Fogli, E. Lisi, A. Marrone, A. Palazzo and A.M. Rotunno, *Hints of $\theta_{13} > 0$ from global neutrino data analysis*, *Phys. Rev. Lett.* **101** (2008) 141801 [[arXiv:0806.2649](#)] [[SPIRES](#)].
- [3] T. Schwetz, M.A. Tortola and J.W.F. Valle, *Three-flavour neutrino oscillation update*, *New J. Phys.* **10** (2008) 113011 [[arXiv:0808.2016](#)] [[SPIRES](#)].
- [4] M. Maltoni and T. Schwetz, *Three-flavour neutrino oscillation update and comments on possible hints for a non-zero θ_{13}* , *PoS(IDM2008)072* [[arXiv:0812.3161](#)] [[SPIRES](#)].
- [5] A. Strumia and F. Vissani, *Neutrino masses and mixings and...*, [hep-ph/0606054](#) [[SPIRES](#)].
- [6] G.L. Fogli et al., *Neutrino mass and mixing: 2006 status*, *Nucl. Phys. Proc. Suppl.* **168** (2007) 341 [[SPIRES](#)].
- [7] M.C. Gonzalez-Garcia and M. Maltoni, *Phenomenology with massive neutrinos*, *Phys. Rept.* **460** (2008) 1 [[arXiv:0704.1800](#)] [[SPIRES](#)].
- [8] T. Schwetz, *Neutrino oscillations: present status and outlook*, *AIP Conf. Proc.* **981** (2008) 8 [[arXiv:0710.5027](#)] [[SPIRES](#)].
- [9] M.C. Gonzalez-Garcia and M. Maltoni, *Status of oscillation plus decay of atmospheric and long-baseline neutrinos*, *Phys. Lett. B* **663** (2008) 405 [[arXiv:0802.3699](#)] [[SPIRES](#)].
- [10] A. Bandyopadhyay, S. Choubey, S. Goswami, S.T. Petcov and D.P. Roy, *Neutrino oscillation parameters after high statistics KamLAND results*, [arXiv:0804.4857](#) [[SPIRES](#)].
- [11] F. Ardellier et al., *Letter of intent for double-CHOOZ: a search for the mixing angle θ_{13}* , [hep-ex/0405032](#) [[SPIRES](#)].
- [12] DOUBLE CHOOZ collaboration, F. Ardellier et al., *Double CHOOZ: a search for the neutrino mixing angle θ_{13}* , [hep-ex/0606025](#) [[SPIRES](#)].
- [13] Y.-f. Wang, *Measuring $\sin^2(2\theta_{13})$ with the Daya Bay nuclear reactors*, [hep-ex/0610024](#) [[SPIRES](#)].
- [14] DAYA-BAY collaboration, X. Guo et al., *A precision measurement of the neutrino mixing angle θ_{13} using reactor antineutrinos at Daya Bay*, [hep-ex/0701029](#) [[SPIRES](#)].
- [15] A.B. Pereira e Sousa, *Studies of $\nu_\mu \rightarrow \nu_e$ oscillation appearance in the MINOS experiment*, [FERMILAB-THESIS-2005-67](#) [[SPIRES](#)].
- [16] RENO collaboration, S.B. Kim, *RENO: Reactor experiment for neutrino oscillation at Yonggwang*, *AIP Conf. Proc.* **981** (2008) 205 [*J. Phys. Conf. Ser.* **120** (2008) 052025] [[SPIRES](#)].
- [17] THE T2K collaboration, Y. Itow et al., *The JHF-Kamioka neutrino project*, [hep-ex/0106019](#) [[SPIRES](#)].
- [18] NOvA collaboration, D.S. Ayres et al., *NOvA proposal to build a 30-kiloton off-axis detector to study neutrino oscillations in the Fermilab NuMI beamline*, [hep-ex/0503053](#) [[SPIRES](#)].
- [19] C.I. Low and R.R. Volkas, *Tri-bimaximal mixing, discrete family symmetries and a conjecture connecting the quark and lepton mixing matrices*, *Phys. Rev. D* **68** (2003) 033007 [[hep-ph/0305243](#)] [[SPIRES](#)].
- [20] F. Feruglio, *Models of neutrino masses and mixings*, *Nucl. Phys. Proc. Suppl.* **143** (2005) 184 [*Nucl. Phys. Proc. Suppl.* **145** (2005) 225] [[hep-ph/0410131](#)] [[SPIRES](#)].
- [21] P.F. Harrison, D.H. Perkins and W.G. Scott, *Tri-bimaximal mixing and the neutrino oscillation data*, *Phys. Lett. B* **530** (2002) 167 [[hep-ph/0202074](#)] [[SPIRES](#)].

- [22] P.F. Harrison and W.G. Scott, *Symmetries and generalisations of tri-bimaximal neutrino mixing*, *Phys. Lett. B* **535** (2002) 163 [[hep-ph/0203209](#)] [[SPIRES](#)].
- [23] Z.-z. Xing, *Nearly tri-bimaximal neutrino mixing and CP-violation*, *Phys. Lett. B* **533** (2002) 85 [[hep-ph/0204049](#)] [[SPIRES](#)].
- [24] P.F. Harrison and W.G. Scott, *μ - τ reflection symmetry in lepton mixing and neutrino oscillations*, *Phys. Lett. B* **547** (2002) 219 [[hep-ph/0210197](#)] [[SPIRES](#)].
- [25] P.F. Harrison and W.G. Scott, *Permutation symmetry, tri-bimaximal neutrino mixing and the S_3 group characters*, *Phys. Lett. B* **557** (2003) 76 [[hep-ph/0302025](#)] [[SPIRES](#)].
- [26] P.F. Harrison and W.G. Scott, *Status of tri-/bi-maximal neutrino mixing*, [hep-ph/0402006](#) [[SPIRES](#)].
- [27] P.F. Harrison and W.G. Scott, *The simplest neutrino mass matrix*, *Phys. Lett. B* **594** (2004) 324 [[hep-ph/0403278](#)] [[SPIRES](#)].
- [28] V.D. Barger, S. Pakvasa, T.J. Weiler and K. Whisnant, *Bi-maximal mixing of three neutrinos*, *Phys. Lett. B* **437** (1998) 107 [[hep-ph/9806387](#)] [[SPIRES](#)].
- [29] Y. Nomura and T. Yanagida, *Bi-maximal neutrino mixing in $SO(10)(GUT)$* , *Phys. Rev. D* **59** (1999) 017303 [[hep-ph/9807325](#)] [[SPIRES](#)].
- [30] G. Altarelli and F. Feruglio, *Neutrino mass textures from oscillations with maximal mixing*, *Phys. Lett. B* **439** (1998) 112 [[hep-ph/9807353](#)] [[SPIRES](#)].
- [31] E. Ma and G. Rajasekaran, *Softly broken A_4 symmetry for nearly degenerate neutrino masses*, *Phys. Rev. D* **64** (2001) 113012 [[hep-ph/0106291](#)] [[SPIRES](#)].
- [32] E. Ma, *Quark mass matrices in the A_4 model*, *Mod. Phys. Lett. A* **17** (2002) 627 [[hep-ph/0203238](#)] [[SPIRES](#)].
- [33] K.S. Babu, E. Ma and J.W.F. Valle, *Underlying A_4 symmetry for the neutrino mass matrix and the quark mixing matrix*, *Phys. Lett. B* **552** (2003) 207 [[hep-ph/0206292](#)] [[SPIRES](#)].
- [34] M. Hirsch, J.C. Romao, S. Skadhauge, J.W.F. Valle and A. Villanova del Moral, *Degenerate neutrinos from a supersymmetric A_4 model*, [hep-ph/0312244](#) [[SPIRES](#)].
- [35] M. Hirsch, J.C. Romao, S. Skadhauge, J.W.F. Valle and A. Villanova del Moral, *Phenomenological tests of supersymmetric A_4 family symmetry model of neutrino mass*, *Phys. Rev. D* **69** (2004) 093006 [[hep-ph/0312265](#)] [[SPIRES](#)].
- [36] E. Ma, *A_4 origin of the neutrino mass matrix*, *Phys. Rev. D* **70** (2004) 031901 [[hep-ph/0404199](#)] [[SPIRES](#)].
- [37] E. Ma, *A_4 origin of the neutrino mass matrix*, *Phys. Rev. D* **70** (2004) 031901 [[hep-ph/0404199](#)] [[SPIRES](#)].
- [38] E. Ma, *Non-Abelian discrete symmetries and neutrino masses: two examples*, *New J. Phys.* **6** (2004) 104 [[hep-ph/0405152](#)] [[SPIRES](#)].
- [39] E. Ma, *Non-Abelian discrete family symmetries of leptons and quarks*, [hep-ph/0409075](#) [[SPIRES](#)].
- [40] G. Altarelli and F. Feruglio, *Tri-bimaximal neutrino mixing from discrete symmetry in extra dimensions*, *Nucl. Phys. B* **720** (2005) 64 [[hep-ph/0504165](#)] [[SPIRES](#)].
- [41] S.-L. Chen, M. Frigerio and E. Ma, *Hybrid seesaw neutrino masses with A_4 family symmetry*, *Nucl. Phys. B* **724** (2005) 423 [[hep-ph/0504181](#)] [[SPIRES](#)].
- [42] E. Ma, *Aspects of the tetrahedral neutrino mass matrix*, *Phys. Rev. D* **72** (2005) 037301 [[hep-ph/0505209](#)] [[SPIRES](#)].

- [43] M. Hirsch, A. Villanova del Moral, J.W.F. Valle and E. Ma, *Predicting neutrinoless double beta decay*, *Phys. Rev. D* **72** (2005) 091301 [Erratum *ibid.* **72** (2005) 119904] [[hep-ph/0507148](#)] [[SPIRES](#)].
- [44] K.S. Babu and X.-G. He, *Model of geometric neutrino mixing*, [hep-ph/0507217](#) [[SPIRES](#)].
- [45] E. Ma, *Tetrahedral family symmetry and the neutrino mixing matrix*, *Mod. Phys. Lett. A* **20** (2005) 2601 [[hep-ph/0508099](#)] [[SPIRES](#)].
- [46] A. Zee, *Obtaining the neutrino mixing matrix with the tetrahedral group*, *Phys. Lett. B* **630** (2005) 58 [[hep-ph/0508278](#)] [[SPIRES](#)].
- [47] E. Ma, *Tribimaximal neutrino mixing from a supersymmetric model with A_4 family symmetry*, *Phys. Rev. D* **73** (2006) 057304 [[hep-ph/0511133](#)] [[SPIRES](#)].
- [48] G. Altarelli and F. Feruglio, *Tri-bimaximal neutrino mixing, A_4 and the modular symmetry*, *Nucl. Phys. B* **741** (2006) 215 [[hep-ph/0512103](#)] [[SPIRES](#)].
- [49] X.-G. He, Y.-Y. Keum and R.R. Volkas, *A_4 flavour symmetry breaking scheme for understanding quark and neutrino mixing angles*, *JHEP* **04** (2006) 039 [[hep-ph/0601001](#)] [[SPIRES](#)].
- [50] B. Adhikary, B. Brahmachari, A. Ghosal, E. Ma and M.K. Parida, *A_4 symmetry and prediction of $U(e3)$ in a modified Altarelli-Feruglio model*, *Phys. Lett. B* **638** (2006) 345 [[hep-ph/0603059](#)] [[SPIRES](#)].
- [51] E. Ma, *Suitability of A_4 as a family symmetry in grand unification*, *Mod. Phys. Lett. A* **21** (2006) 2931 [[hep-ph/0607190](#)] [[SPIRES](#)].
- [52] E. Ma, *Supersymmetric $A_4 \times Z_3$ and A_4 realizations of neutrino tribimaximal mixing without and with corrections*, *Mod. Phys. Lett. A* **22** (2007) 101 [[hep-ph/0610342](#)] [[SPIRES](#)].
- [53] L. Lavoura and H. Kuhbock, *Predictions of an A_4 model with a five-parameter neutrino mass matrix*, *Mod. Phys. Lett. A* **22** (2007) 181 [[hep-ph/0610050](#)] [[SPIRES](#)].
- [54] G. Altarelli, F. Feruglio and Y. Lin, *Tri-bimaximal neutrino mixing from orbifolding*, *Nucl. Phys. B* **775** (2007) 31 [[hep-ph/0610165](#)] [[SPIRES](#)].
- [55] X.-G. He, *A_4 group and tri-bimaximal neutrino mixing: A renormalizable model*, *Nucl. Phys. Proc. Suppl.* **168** (2007) 350 [[hep-ph/0612080](#)] [[SPIRES](#)].
- [56] Y. Koide, *A_4 symmetry and lepton masses and mixing*, *Eur. Phys. J. C* **52** (2007) 617 [[hep-ph/0701018](#)] [[SPIRES](#)].
- [57] M. Hirsch, A.S. Joshipura, S. Kaneko and J.W.F. Valle, *Predictive flavour symmetries of the neutrino mass matrix*, *Phys. Rev. Lett.* **99** (2007) 151802 [[hep-ph/0703046](#)] [[SPIRES](#)].
- [58] F. Yin, *Neutrino mixing matrix in the 3-3-1 model with heavy leptons and A_4 symmetry*, *Phys. Rev. D* **75** (2007) 073010 [[arXiv:0704.3827](#)] [[SPIRES](#)].
- [59] F. Bazzocchi, S. Kaneko and S. Morisi, *A SUSY A_4 model for fermion masses and mixings*, *JHEP* **03** (2008) 063 [[arXiv:0707.3032](#)] [[SPIRES](#)].
- [60] F. Bazzocchi, S. Morisi and M. Picariello, *Embedding A_4 into left-right flavor symmetry: tribimaximal neutrino mixing and fermion hierarchy*, *Phys. Lett. B* **659** (2008) 628 [[arXiv:0710.2928](#)] [[SPIRES](#)].
- [61] G. Altarelli, *Lectures on models of neutrino masses and mixings*, [arXiv:0711.0161](#) [[SPIRES](#)].
- [62] A.C.B. Machado and V. Pleitez, *The flavor problem and discrete symmetries*, *Phys. Lett. B* **674** (2009) 223 [[arXiv:0712.0781](#)] [[SPIRES](#)].

- [63] M. Honda and M. Tanimoto, *Deviation from tri-bimaximal neutrino mixing in A_4 flavor symmetry*, *Prog. Theor. Phys.* **119** (2008) 583 [[arXiv:0801.0181](#)] [[SPIRES](#)].
- [64] B. Brahmachari, S. Choubey and M. Mitra, *The A_4 flavor symmetry and neutrino phenomenology*, *Phys. Rev. D* **77** (2008) 073008 [Erratum *ibid.* **D 77** (2008) 119901] [[arXiv:0801.3554](#)] [[SPIRES](#)].
- [65] B. Adhikary and A. Ghosal, *Nonzero U_{e3} , CP-violation and leptogenesis in a see-saw type softly broken A_4 symmetric model*, *Phys. Rev. D* **78** (2008) 073007 [[arXiv:0803.3582](#)] [[SPIRES](#)].
- [66] M. Hirsch, S. Morisi and J.W.F. Valle, *Tri-bimaximal neutrino mixing and neutrinoless double beta decay*, *Phys. Rev. D* **78** (2008) 093007 [[arXiv:0804.1521](#)] [[SPIRES](#)].
- [67] Y. Lin, *A predictive A_4 model, charged lepton hierarchy and tri-bimaximal sum rule*, *Nucl. Phys. B* **813** (2009) 91 [[arXiv:0804.2867](#)] [[SPIRES](#)].
- [68] C. Csáki, C. Delaunay, C. Grojean and Y. Grossman, *A model of lepton masses from a warped extra dimension*, *JHEP* **10** (2008) 055 [[arXiv:0806.0356](#)] [[SPIRES](#)].
- [69] P.H. Frampton and S. Matsuzaki, *Renormalizable A_4 model for lepton sector*, [arXiv:0806.4592](#) [[SPIRES](#)].
- [70] E.E. Jenkins and A.V. Manohar, *Tribimaximal mixing, leptogenesis and θ_{13}* , *Phys. Lett. B* **668** (2008) 210 [[arXiv:0807.4176](#)] [[SPIRES](#)].
- [71] Riazuddin, *Tribimaximal mixing and leptogenesis in a seesaw model*, [arXiv:0809.3648](#) [[SPIRES](#)].
- [72] W. Grimus and L. Lavoura, *Tri-bimaximal lepton mixing from symmetry only*, *JHEP* **04** (2009) 013 [[arXiv:0811.4766](#)] [[SPIRES](#)].
- [73] S. Morisi, *Tri-Bimaximal lepton mixing with A_4 semidirect product $Z_2 \times Z_2 \times Z_2$* , *Phys. Rev. D* **79** (2009) 033008 [[arXiv:0901.1080](#)] [[SPIRES](#)].
- [74] M.-C. Chen and S.F. King, *A_4 see-saw models and form dominance*, *JHEP* **06** (2009) 072 [[arXiv:0903.0125](#)] [[SPIRES](#)].
- [75] G.C. Branco, R. Gonzalez Felipe, M.N. Rebelo and H. Serodio, *Resonant leptogenesis and tribimaximal leptonic mixing with A_4 symmetry*, *Phys. Rev. D* **79** (2009) 093008 [[arXiv:0904.3076](#)] [[SPIRES](#)].
- [76] G. Altarelli and D. Meloni, *A simplest A_4 model for tri-bimaximal neutrino mixing*, *J. Phys. G* **36** (2009) 085005 [[arXiv:0905.0620](#)] [[SPIRES](#)].
- [77] M. Hirsch, S. Morisi and J.W.F. Valle, *A_4 -based tri-bimaximal mixing within inverse and linear seesaw schemes*, *Phys. Lett. B* **679** (2009) 454 [[arXiv:0905.3056](#)] [[SPIRES](#)].
- [78] T.J. Burrows and S.F. King, *A_4 Family Symmetry from $SU(5)$ SUSY GUTs in 6d*, *Nucl. Phys. B* **835** (2010) 174 [[arXiv:0909.1433](#)] [[SPIRES](#)].
- [79] S. Morisi and E. Peinado, *An A_4 model for lepton masses and mixings*, *Phys. Rev. D* **80** (2009) 113011 [[arXiv:0910.4389](#)] [[SPIRES](#)].
- [80] G.-J. Ding and J.-F. Liu, *Lepton flavor violation in models with A_4 and S_4 flavor symmetries*, *JHEP* **05** (2010) 029 [[arXiv:0911.4799](#)] [[SPIRES](#)].
- [81] S. Pakvasa and H. Sugawara, *Mass of the t quark in $SU(2) \times U(1)$* , *Phys. Lett. B* **82** (1979) 105 [[SPIRES](#)].
- [82] T. Brown, N. Deshpande, S. Pakvasa and H. Sugawara, *CP nonconservation and rare processes in S^4 model of permutation symmetry*, *Phys. Lett. B* **141** (1984) 95 [[SPIRES](#)].

- [83] T. Brown, S. Pakvasa, H. Sugawara and Y. Yamanaka, *Neutrino masses, mixing and oscillations in S^4 model of permutation symmetry*, *Phys. Rev. D* **30** (1984) 255 [SPIRES].
- [84] D.-G. Lee and R.N. Mohapatra, *An $SO(10) \times S^4$ scenario for naturally degenerate neutrinos*, *Phys. Lett. B* **329** (1994) 463 [hep-ph/9403201] [SPIRES].
- [85] R.N. Mohapatra, M.K. Parida and G. Rajasekaran, *High scale mixing unification and large neutrino mixing angles*, *Phys. Rev. D* **69** (2004) 053007 [hep-ph/0301234] [SPIRES].
- [86] E. Ma, *Neutrino mass matrix from S^4 symmetry*, *Phys. Lett. B* **632** (2006) 352 [hep-ph/0508231] [SPIRES].
- [87] C. Hagedorn, M. Lindner and R.N. Mohapatra, *S^4 flavor symmetry and fermion masses: towards a grand unified theory of flavor*, *JHEP* **06** (2006) 042 [hep-ph/0602244] [SPIRES].
- [88] Y. Cai and H.-B. Yu, *An $SO(10)$ GUT model with S^4 flavor symmetry*, *Phys. Rev. D* **74** (2006) 115005 [hep-ph/0608022] [SPIRES].
- [89] F. Caravaglios and S. Morisi, *Gauge boson families in grand unified theories of fermion masses: $E_6^4 \times S_4$* , *Int. J. Mod. Phys. A* **22** (2007) 2469 [hep-ph/0611078] [SPIRES].
- [90] H. Zhang, *Flavor $S^4 \times Z(2)$ symmetry and neutrino mixing*, *Phys. Lett. B* **655** (2007) 132 [hep-ph/0612214] [SPIRES].
- [91] Y. Koide, *S_4 flavor symmetry embedded into $SU(3)$ and lepton masses and mixing*, *JHEP* **08** (2007) 086 [arXiv:0705.2275] [SPIRES].
- [92] M.K. Parida, *Intermediate left-right gauge symmetry, unification of couplings and fermion masses in $SUSY SO(10) \times S_4$* , *Phys. Rev. D* **78** (2008) 053004 [arXiv:0804.4571] [SPIRES].
- [93] F. Bazzocchi and S. Morisi, *S^4 as a natural flavor symmetry for lepton mixing*, *Phys. Rev. D* **80** (2009) 096005 [arXiv:0811.0345] [SPIRES].
- [94] H. Ishimori, Y. Shimizu and M. Tanimoto, *S^4 flavor symmetry of quarks and leptons in $SU(5)$ GUT*, *Prog. Theor. Phys.* **121** (2009) 769 [arXiv:0812.5031] [SPIRES].
- [95] F. Bazzocchi, L. Merlo and S. Morisi, *Fermion masses and mixings in a S^4 -based model*, *Nucl. Phys. B* **816** (2009) 204 [arXiv:0901.2086] [SPIRES].
- [96] F. Bazzocchi, L. Merlo and S. Morisi, *Phenomenological consequences of see-saw in S^4 based models*, *Phys. Rev. D* **80** (2009) 053003 [arXiv:0902.2849] [SPIRES].
- [97] G.-J. Ding, *Fermion masses and flavor mixings in a model with S_4 flavor symmetry*, *Nucl. Phys. B* **827** (2010) 82 [arXiv:0909.2210] [SPIRES].
- [98] A. Aranda, C.D. Carone and R.F. Lebed, *$U(2)$ flavor physics without $U(2)$ symmetry*, *Phys. Lett. B* **474** (2000) 170 [hep-ph/9910392] [SPIRES].
- [99] A. Aranda, C.D. Carone and R.F. Lebed, *Maximal neutrino mixing from a minimal flavor symmetry*, *Phys. Rev. D* **62** (2000) 016009 [hep-ph/0002044] [SPIRES].
- [100] F. Feruglio, C. Hagedorn, Y. Lin and L. Merlo, *Tri-bimaximal neutrino mixing and quark masses from a discrete flavour symmetry*, *Nucl. Phys. B* **775** (2007) 120 [hep-ph/0702194] [SPIRES].
- [101] M.-C. Chen and K.T. Mahanthappa, *CKM and tri-bimaximal MNS matrices in a $SU(5) \times^{(d)} T$ model*, *Phys. Lett. B* **652** (2007) 34 [arXiv:0705.0714] [SPIRES].
- [102] P.H. Frampton and T.W. Kephart, *Flavor symmetry for quarks and leptons*, *JHEP* **09** (2007) 110 [arXiv:0706.1186] [SPIRES].
- [103] A. Aranda, *Neutrino mixing from the double tetrahedral group T'* , *Phys. Rev. D* **76** (2007) 111301 [arXiv:0707.3661] [SPIRES].

- [104] P.D. Carr and P.H. Frampton, *Group theoretic bases for tribimaximal mixing*, [hep-ph/0701034](#) [[SPIRES](#)].
- [105] G.-J. Ding, *Fermion mass hierarchies and flavor mixing from T' symmetry*, *Phys. Rev. D* **78** (2008) 036011 [[arXiv:0803.2278](#)] [[SPIRES](#)].
- [106] P.H. Frampton and S. Matsuzaki, *T' predictions of PMNS and CKM angles*, *Phys. Lett. B* **679** (2009) 347 [[arXiv:0902.1140](#)] [[SPIRES](#)].
- [107] M.-C. Chen and K.T. Mahanthappa, *Group theoretical origin of CP-violation*, *Phys. Lett. B* **681** (2009) 444 [[arXiv:0904.1721](#)] [[SPIRES](#)].
- [108] E. Ma, *Neutrino mass matrix from Δ_{27} symmetry*, *Mod. Phys. Lett. A* **21** (2006) 1917 [[hep-ph/0607056](#)] [[SPIRES](#)].
- [109] I. de Medeiros Varzielas, S.F. King and G.G. Ross, *Neutrino tri-bi-maximal mixing from a non-Abelian discrete family symmetry*, *Phys. Lett. B* **648** (2007) 201 [[hep-ph/0607045](#)] [[SPIRES](#)].
- [110] I. de Medeiros Varzielas, S.F. King and G.G. Ross, *Tri-bimaximal neutrino mixing from discrete subgroups of SU(3) and SO(3) family symmetry*, *Phys. Lett. B* **644** (2007) 153 [[hep-ph/0512313](#)] [[SPIRES](#)].
- [111] Z. Berezhiani, *Problem of flavour in SUSY GUT and horizontal symmetry*, *Nucl. Phys. Proc. Suppl.* **52A** (1997) 153 [[hep-ph/9607363](#)] [[SPIRES](#)].
- [112] C.H. Albright, *SO(10) SUSY GUT model with a $U(1) \times Z(2) \times Z(2)$ flavor symmetry*, in the proceedings of the *APS/DPF/DPB Summer Study on the Future of Particle Physics*, June 30–July 21, Snowmass, Colorado (2001), [hep-ph/0110259](#) [[SPIRES](#)].
- [113] I. de Medeiros Varzielas and G.G. Ross, *SU(3) family symmetry and neutrino bi-tri-maximal mixing*, *Nucl. Phys. B* **733** (2006) 31 [[hep-ph/0507176](#)] [[SPIRES](#)].
- [114] S. Morisi, *Fermion mass in E_6 GUT with discrete family permutation symmetry S^3* , [hep-ph/0605167](#) [[SPIRES](#)].
- [115] F. Caravaglios and S. Morisi, *Fermion mass in $E(6)$ GUT with discrete family permutation symmetry S^3* , prepared for the *IFAE2006*, April 19–21, Pavia, Italy (2006) [[SPIRES](#)].
- [116] I. de Medeiros Varzielas, S.F. King and G.G. Ross, *Neutrino tri-bi-maximal mixing from a non-Abelian discrete family symmetry*, *Phys. Lett. B* **648** (2007) 201 [[hep-ph/0607045](#)] [[SPIRES](#)].
- [117] S.F. King and M. Malinsky, *Towards a complete theory of fermion masses and mixings with SO(3) family symmetry and 5D SO(10) unification*, *JHEP* **11** (2006) 071 [[hep-ph/0608021](#)] [[SPIRES](#)].
- [118] Y. Cai and H.-B. Yu, *An SO(10) GUT model with S_4 flavor symmetry*, *Phys. Rev. D* **74** (2006) 115005 [[hep-ph/0608022](#)] [[SPIRES](#)].
- [119] F. Caravaglios and S. Morisi, *Gauge boson families in grand unified theories of fermion masses: $E_6^4 \times S_4$* , *Int. J. Mod. Phys. A* **22** (2007) 2469 [[hep-ph/0611078](#)] [[SPIRES](#)].
- [120] J.R. Ellis, M.E. Gomez and S. Lola, *CP and lepton-number violation in GUT neutrino models with abelian flavour symmetries*, *JHEP* **07** (2007) 052 [[hep-ph/0612292](#)] [[SPIRES](#)].
- [121] M. Albrecht, W. Altmannshofer, A.J. Buras, D. Guadagnoli and D.M. Straub, *Challenging SO(10) SUSY GUTs with family symmetries through FCNC processes*, *JHEP* **10** (2007) 055 [[arXiv:0707.3954](#)] [[SPIRES](#)].
- [122] W. Altmannshofer, *Flavor changing neutral current processes in a SO(10) SUSY GUT with family symmetry*, [arXiv:0710.1488](#) [[SPIRES](#)].

- [123] F. Bazzocchi, S. Morisi and M. Picariello, *Embedding A_4 into left-right flavor symmetry: tribimaximal neutrino mixing and fermion hierarchy*, *Phys. Lett. B* **659** (2008) 628 [[arXiv:0710.2928](#)] [[SPIRES](#)].
- [124] F. Bazzocchi, S. Morisi, M. Picariello and E. Torrente-Lujan, *Embedding A_4 into $SU(3) \times U(1)$ flavor symmetry: large neutrino mixing and fermion mass hierarchy in $SO(10)$ GUT*, *J. Phys. G* **36** (2009) 015002 [[arXiv:0802.1693](#)] [[SPIRES](#)].
- [125] C. Hagedorn, M.A. Schmidt and A.Y. Smirnov, *Lepton mixing and cancellation of the Dirac mass hierarchy in $SO(10)$ GUTs with flavor symmetries T^7 and Σ_{81}* , *Phys. Rev. D* **79** (2009) 036002 [[arXiv:0811.2955](#)] [[SPIRES](#)].
- [126] H. Ishimori, Y. Shimizu and M. Tanimoto, *S^4 flavor symmetry of quarks and leptons in $SU(5)$ GUT*, *Prog. Theor. Phys.* **121** (2009) 769 [[arXiv:0812.5031](#)] [[SPIRES](#)].
- [127] F. Bazzocchi and I. de Medeiros Varzielas, *Tri-bi-maximal mixing in viable family symmetry unified model with extended seesaw*, *Phys. Rev. D* **79** (2009) 093001 [[arXiv:0902.3250](#)] [[SPIRES](#)].
- [128] M. Ishiduki, S.G. Kim, N. Maekawa and K. Sakurai, *Spontaneous CP-violation in E^6 SUSY GUT with $SU(2)$ flavor and anomalous $U(1)$ symmetries*, *Phys. Rev. D* **80** (2009) 115011 [*Erratum ibid.* **D 81** (2010) 039901] [[arXiv:0910.1336](#)] [[SPIRES](#)].
- [129] S.F. King and M. Malinsky, *A_4 family symmetry and quark-lepton unification*, *Phys. Lett. B* **645** (2007) 351 [[hep-ph/0610250](#)] [[SPIRES](#)].
- [130] S. Morisi, M. Picariello and E. Torrente-Lujan, *A model for fermion masses and lepton mixing in $SO(10) \times A_4$* , *Phys. Rev. D* **75** (2007) 075015 [[hep-ph/0702034](#)] [[SPIRES](#)].
- [131] G. Altarelli, F. Feruglio and C. Hagedorn, *A SUSY $SU(5)$ grand unified model of tri-bimaximal mixing from A_4* , *JHEP* **03** (2008) 052 [[arXiv:0802.0090](#)] [[SPIRES](#)].
- [132] F. Bazzocchi, S. Morisi, M. Picariello and E. Torrente-Lujan, *Embedding A_4 into $SU(3) \times U(1)$ flavor symmetry: large neutrino mixing and fermion mass hierarchy in $SO(10)$ GUT*, *J. Phys. G* **36** (2009) 015002 [[arXiv:0802.1693](#)] [[SPIRES](#)].
- [133] F. Bazzocchi, M. Frigerio and S. Morisi, *Fermion masses and mixing in models with $SO(10) \times A_4$ symmetry*, *Phys. Rev. D* **78** (2008) 116018 [[arXiv:0809.3573](#)] [[SPIRES](#)].
- [134] P. Ciafaloni, M. Picariello, E. Torrente-Lujan and A. Urbano, *Neutrino masses and tribimaximal mixing in minimal renormalizable SUSY $SU(5)$ grand unified model with A_4 flavor symmetry*, *Phys. Rev. D* **79** (2009) 116010 [[arXiv:0901.2236](#)] [[SPIRES](#)].
- [135] C. Hagedorn, S.F. King and C. Luhn, *A SUSY GUT of Flavour with $S^4 \times SU(5)$ to NLO*, *JHEP* **06** (2010) 048 [[arXiv:1003.4249](#)] [[SPIRES](#)].
- [136] Y. Lin, *Tri-bimaximal Neutrino Mixing from A_4 and $\theta_{13} \sim \theta_C$* , *Nucl. Phys. B* **824** (2010) 95 [[arXiv:0905.3534](#)] [[SPIRES](#)].
- [137] P.H. Frampton, S.T. Petcov and W. Rodejohann, *On deviations from bimaximal neutrino mixing*, *Nucl. Phys. B* **687** (2004) 31 [[hep-ph/0401206](#)] [[SPIRES](#)].
- [138] G. Altarelli, F. Feruglio and I. Masina, *Can neutrino mixings arise from the charged lepton sector?*, *Nucl. Phys. B* **689** (2004) 157 [[hep-ph/0402155](#)] [[SPIRES](#)].
- [139] M. Raidal, *Relation between the neutrino and quark mixing angles and grand unification*, *Phys. Rev. Lett.* **93** (2004) 161801 [[hep-ph/0404046](#)] [[SPIRES](#)].
- [140] H. Minakata and A.Y. Smirnov, *Neutrino mixing and quark-lepton complementarity*, *Phys. Rev. D* **70** (2004) 073009 [[hep-ph/0405088](#)] [[SPIRES](#)].

- [141] P.H. Frampton and R.N. Mohapatra, *Possible gauge theoretic origin for quark-lepton complementarity*, *JHEP* **01** (2005) 025 [[hep-ph/0407139](#)] [[SPIRES](#)].
- [142] J. Ferrandis and S. Pakvasa, *QLC relation and neutrino mass hierarchy*, *Phys. Rev. D* **71** (2005) 033004 [[hep-ph/0412038](#)] [[SPIRES](#)].
- [143] S.K. Kang, C.S. Kim and J. Lee, *Quark-lepton complementarity with renormalization effects through threshold corrections*, *Phys. Lett. B* **619** (2005) 129 [[hep-ph/0501029](#)] [[SPIRES](#)].
- [144] N. Li and B.-Q. Ma, *Unified parametrization of quark and lepton mixing matrices*, *Phys. Rev. D* **71** (2005) 097301 [[hep-ph/0501226](#)] [[SPIRES](#)].
- [145] K. Cheung, S.K. Kang, C.S. Kim and J. Lee, *Lepton flavor violation as a probe of quark-lepton unification*, *Phys. Rev. D* **72** (2005) 036003 [[hep-ph/0503122](#)] [[SPIRES](#)].
- [146] Z.-z. Xing, *Nontrivial correlation between the CKM and MNS matrices*, *Phys. Lett. B* **618** (2005) 141 [[hep-ph/0503200](#)] [[SPIRES](#)].
- [147] A. Datta, L. Everett and P. Ramond, *Cabibbo haze in lepton mixing*, *Phys. Lett. B* **620** (2005) 42 [[hep-ph/0503222](#)] [[SPIRES](#)].
- [148] S. Antusch, S.F. King and R.N. Mohapatra, *Quark lepton complementarity in unified theories*, *Phys. Lett. B* **618** (2005) 150 [[hep-ph/0504007](#)] [[SPIRES](#)].
- [149] M. Lindner, M.A. Schmidt and A.Y. Smirnov, *Screening of Dirac flavor structure in the seesaw and neutrino mixing*, *JHEP* **07** (2005) 048 [[hep-ph/0505067](#)] [[SPIRES](#)].
- [150] H. Minakata, *Quark-lepton complementarity: a review*, [hep-ph/0505262](#) [[SPIRES](#)].
- [151] T. Ohlsson, *Bimaximal fermion mixing from the quark and leptonic mixing matrices*, *Phys. Lett. B* **622** (2005) 159 [[hep-ph/0506094](#)] [[SPIRES](#)].
- [152] A. Dighe, S. Goswami and P. Roy, *Quark-lepton complementarity with quasidegenerate Majorana neutrinos*, *Phys. Rev. D* **73** (2006) 071301 [[hep-ph/0602062](#)] [[SPIRES](#)].
- [153] B.C. Chauhan, M. Picariello, J. Pulido and E. Torrente-Lujan, *Quark-lepton complementarity, neutrino and standard model data predict $(\theta_{13}^{PMNS} = 9_{-2}^{+1})^\circ$* , *Eur. Phys. J. C* **50** (2007) 573 [[hep-ph/0605032](#)] [[SPIRES](#)].
- [154] K.A. Hochmuth and W. Rodejohann, *Low and high energy phenomenology of quark-lepton complementarity scenarios*, *Phys. Rev. D* **75** (2007) 073001 [[hep-ph/0607103](#)] [[SPIRES](#)].
- [155] M.A. Schmidt and A.Y. Smirnov, *Quark lepton complementarity and renormalization group effects*, *Phys. Rev. D* **74** (2006) 113003 [[hep-ph/0607232](#)] [[SPIRES](#)].
- [156] F. Plentinger, G. Seidl and W. Winter, *Systematic parameter space search of extended quark-lepton complementarity*, *Nucl. Phys. B* **791** (2008) 60 [[hep-ph/0612169](#)] [[SPIRES](#)].
- [157] F. Plentinger, G. Seidl and W. Winter, *The seesaw mechanism in quark-lepton complementarity*, *Phys. Rev. D* **76** (2007) 113003 [[arXiv:0707.2379](#)] [[SPIRES](#)].
- [158] G. Altarelli, F. Feruglio and L. Merlo, *Revisiting bimaximal neutrino mixing in a model with S_4 discrete symmetry*, *JHEP* **05** (2009) 020 [[arXiv:0903.1940](#)] [[SPIRES](#)].
- [159] J.C. Pati and A. Salam, *Lepton number as the fourth color*, *Phys. Rev. D* **10** (1974) 275 [*Erratum ibid.* **D 11** (1975) 703] [[SPIRES](#)].
- [160] V. Elias and A.R. Swift, *Generalization of the Pati-Salam model*, *Phys. Rev. D* **13** (1976) 2083 [[SPIRES](#)].
- [161] V. Elias, *Coupling constant renormalization in unified gauge theories containing the Pati-Salam model*, *Phys. Rev. D* **14** (1976) 1896 [[SPIRES](#)].

- [162] V. Elias, *Gauge coupling constant magnitudes in the Pati-Salam model*, *Phys. Rev. D* **16** (1977) 1586 [SPIRES].
- [163] D.M. Capper, *Pseudo-Goldstone bosons in the Pati-Salam model*, *J. Phys. G* **3** (1977) 1317 [SPIRES].
- [164] A. Schorr, *Improved Pati-Salam models*, *Phys. Rev. D* **18** (1978) 863 [SPIRES].
- [165] B. Bhuyan and B.B. Deo, *Symmetry breaking hierarchies in the Pati-Salam group induced by supergravity*, *Phys. Rev. D* **36** (1987) 966 [SPIRES].
- [166] G.K. Leontaris and J. Rizos, *A Pati-Salam model from branes*, *Phys. Lett. B* **510** (2001) 295 [hep-ph/0012255] [SPIRES].
- [167] T. Blazek, S.F. King and J.K. Parry, *Global analysis of a supersymmetric Pati-Salam model*, *JHEP* **05** (2003) 016 [hep-ph/0303192] [SPIRES].
- [168] A. Prikas and N.D. Tracas, *Gauge unification and quark masses in a Pati-Salam model from branes*, *New J. Phys.* **5** (2003) 144 [hep-ph/0303258] [SPIRES].
- [169] C.S. Aulakh and A. Girdhar, *MSGUT a la Pati-Salam: from futility to precision*, *Nucl. Phys. B* **711** (2005) 275 [hep-ph/0405074] [SPIRES].
- [170] J.B. Dent and T.W. Kephart, *Minimal Pati-Salam model from string theory unification*, *Phys. Rev. D* **77** (2008) 115008 [arXiv:0705.1995] [SPIRES].
- [171] F. Braam, J. Reuter and D. Wiesler, *The Pati-Salam supersymmetric SM (PSSSM)*, *AIP Conf. Proc.* **1200** (2010) 458 [arXiv:0909.3081] [SPIRES].
- [172] C. Kounnas, A. Masiero, D.V. Nanopoulos and K.A. Olive, *Grand unification with and without supersymmetry and cosmological implications*, World Scientific, Singapore (1984), pag. 425 [SPIRES].
- [173] G.G. Ross, *Grand unified theories*, Frontiers in Physics volume 60, Benjamin and Cummings, Reading U.S.A. (1984), pag. 497 [SPIRES].
- [174] A.S. Joshipura, B.P. Kodrani and K.M. Patel, *Fermion masses and mixings in a μ - τ symmetric SO(10)*, *Phys. Rev. D* **79** (2009) 115017 [arXiv:0903.2161] [SPIRES].
- [175] B. Dutta, Y. Mimura and R.N. Mohapatra, *Origin of quark-lepton flavor in SO(10) with type II seesaw*, *Phys. Rev. D* **80** (2009) 095021 [arXiv:0910.1043] [SPIRES].
- [176] B. Dutta, Y. Mimura and R.N. Mohapatra, *An SO(10) Grand Unified Theory of flavor*, *JHEP* **05** (2010) 034 [arXiv:0911.2242] [SPIRES].
- [177] S.F. King and C. Luhn, *A supersymmetric Grand Unified Theory of flavour with $\text{PSL}(2,7) \times \text{SO}(10)$* , *Nucl. Phys. B* **832** (2010) 414 [arXiv:0912.1344] [SPIRES].
- [178] S. Bertolini, T. Schwetz and M. Malinsky, *Fermion masses and mixings in SO(10) models and the neutrino challenge to SUSY GUTs*, *Phys. Rev. D* **73** (2006) 115012 [hep-ph/0605006] [SPIRES].
- [179] C.D. Froggatt and H.B. Nielsen, *Hierarchy of Quark Masses, Cabibbo Angles and CP-violation*, *Nucl. Phys. B* **147** (1979) 277 [SPIRES].
- [180] H. Georgi and C. Jarlskog, *A new lepton-quark mass relation in a unified theory*, *Phys. Lett. B* **86** (1979) 297 [SPIRES].
- [181] L. Wolfenstein, *Parametrization of the Kobayashi-Maskawa matrix*, *Phys. Rev. Lett.* **51** (1983) 1945 [SPIRES].
- [182] G.G. Ross and O. Vives, *Yukawa structure, flavour and CP-violation in supergravity*, *Phys. Rev. D* **67** (2003) 095013 [hep-ph/0211279] [SPIRES].

- [183] G.G. Ross, L. Velasco-Sevilla and O. Vives, *Spontaneous CP-violation and non-Abelian family symmetry in SUSY*, *Nucl. Phys. B* **692** (2004) 50 [[hep-ph/0401064](#)] [[SPIRES](#)].
- [184] F. Feruglio, C. Hagedorn and L. Merlo, *Vacuum alignment in SUSY A_4 models*, *JHEP* **03** (2010) 084 [[arXiv:0910.4058](#)] [[SPIRES](#)].
- [185] F. Feruglio, C. Hagedorn, Y. Lin and L. Merlo, *Lepton flavour violation in models with A_4 flavour symmetry*, *Nucl. Phys. B* **809** (2009) 218 [[arXiv:0807.3160](#)] [[SPIRES](#)].
- [186] H. Ishimori, T. Kobayashi, Y. Omura and M. Tanimoto, *Soft supersymmetry breaking terms from A_4 lepton flavor symmetry*, *JHEP* **12** (2008) 082 [[arXiv:0807.4625](#)] [[SPIRES](#)].
- [187] F. Feruglio, C. Hagedorn, Y. Lin and L. Merlo, *Theory of the neutrino mass*, in the proceedings of the 4th *International Workshop on Neutrino Oscillations in Venice (NO – VE2008)*, April 15–18, Venice, Italy (2008), [arXiv:0808.0812](#) [[SPIRES](#)].
- [188] A. Hayakawa, H. Ishimori, Y. Shimizu and M. Tanimoto, *Deviation from tri-bimaximal mixing and flavor symmetry breaking in a seesaw type A_4 model*, *Phys. Lett. B* **680** (2009) 334 [[arXiv:0904.3820](#)] [[SPIRES](#)].
- [189] C. Hagedorn, E. Molinaro and S.T. Petcov, *Charged lepton flavour violating radiative decays $\ell_i \rightarrow \ell_j + \gamma$ in see-saw models with A_4 symmetry*, *JHEP* **02** (2010) 047 [[arXiv:0911.3605](#)] [[SPIRES](#)].
- [190] F. Feruglio, C. Hagedorn, Y. Lin and L. Merlo, *Lepton flavour violation in a supersymmetric model with A_4 flavour symmetry*, [arXiv:0911.3874](#) [[SPIRES](#)].
- [191] A. Melfo and G. Senjanović, *Minimal supersymmetric Pati-Salam theory: determination of physical scales*, *Phys. Rev. D* **68** (2003) 035013 [[hep-ph/0302216](#)] [[SPIRES](#)].
- [192] Z. Chacko and R.N. Mohapatra, *Supersymmetric left-right models and light doubly charged Higgs bosons and Higgsinos*, *Phys. Rev. D* **58** (1998) 015003 [[hep-ph/9712359](#)] [[SPIRES](#)].
- [193] M. Frank, K. Huitu and S.K. Rai, *Single production of doubly charged higgsinos at linear e^-e^- colliders*, *Phys. Rev. D* **77** (2008) 015006 [[arXiv:0710.2415](#)] [[SPIRES](#)].
- [194] D.A. Demir, M. Frank, K. Huitu, S.K. Rai and I. Turan, *Signals of doubly-charged higgsinos at the CERN Large Hadron Collider*, *Phys. Rev. D* **78** (2008) 035013 [[arXiv:0805.4202](#)] [[SPIRES](#)].
- [195] D.A. Demir et al., *Doubly charged higgsinos at Tevatron*, *Phys. Rev. D* **79** (2009) 095006 [[arXiv:0903.3955](#)] [[SPIRES](#)].
- [196] S. Antusch, J. Kersten, M. Lindner and M. Ratz, *The LMA solution from bimaximal lepton mixing at the GUT scale by renormalization group running*, *Phys. Lett. B* **544** (2002) 1 [[hep-ph/0206078](#)] [[SPIRES](#)].
- [197] S. Antusch, J. Kersten, M. Lindner and M. Ratz, *Running neutrino masses, mixings and CP phases: analytical results and phenomenological consequences*, *Nucl. Phys. B* **674** (2003) 401 [[hep-ph/0305273](#)] [[SPIRES](#)].
- [198] T. Miura, T. Shindou and E. Takasugi, *The renormalization group effect to the bi-maximal mixing*, *Phys. Rev. D* **68** (2003) 093009 [[hep-ph/0308109](#)] [[SPIRES](#)].
- [199] S. Antusch, P. Huber, J. Kersten, T. Schwetz and W. Winter, *Is there maximal mixing in the lepton sector?*, *Phys. Rev. D* **70** (2004) 097302 [[hep-ph/0404268](#)] [[SPIRES](#)].
- [200] S. Antusch, J. Kersten, M. Lindner, M. Ratz and M.A. Schmidt, *Running neutrino mass parameters in see-saw scenarios*, *JHEP* **03** (2005) 024 [[hep-ph/0501272](#)] [[SPIRES](#)].
- [201] J.-w. Mei, *Running neutrino masses, leptonic mixing angles and CP-violating phases: from $M(Z)$ to Λ_{GUT}* , *Phys. Rev. D* **71** (2005) 073012 [[hep-ph/0502015](#)] [[SPIRES](#)].

- [202] M. Lindner, M.A. Schmidt and A.Y. Smirnov, *Screening of Dirac flavor structure in the seesaw and neutrino mixing*, *JHEP* **07** (2005) 048 [[hep-ph/0505067](#)] [[SPIRES](#)].
- [203] J.R. Ellis, A. Hektor, M. Kadastik, K. Kannike and M. Raidal, *Running of low-energy neutrino masses, mixing angles and CP-violation*, *Phys. Lett. B* **631** (2005) 32 [[hep-ph/0506122](#)] [[SPIRES](#)].
- [204] A. Dighe, S. Goswami and P. Roy, *Quark-lepton complementarity with quasidegenerate Majorana neutrinos*, *Phys. Rev. D* **73** (2006) 071301 [[hep-ph/0602062](#)] [[SPIRES](#)].
- [205] A. Dighe, S. Goswami and W. Rodejohann, *Corrections to tri-bimaximal neutrino mixing: renormalization and Planck scale effects*, *Phys. Rev. D* **75** (2007) 073023 [[hep-ph/0612328](#)] [[SPIRES](#)].
- [206] A. Dighe, S. Goswami and P. Roy, *Radiatively broken symmetries of nonhierarchical neutrinos*, *Phys. Rev. D* **76** (2007) 096005 [[arXiv:0704.3735](#)] [[SPIRES](#)].
- [207] S. Boudjemaa and S.F. King, *Deviations from tri-bimaximal mixing: charged lepton corrections and renormalization group running*, *Phys. Rev. D* **79** (2009) 033001 [[arXiv:0808.2782](#)] [[SPIRES](#)].
- [208] Y. Lin, L. Merlo and A. Paris, *Running effects on lepton mixing angles in flavour models with type I seesaw*, *Nucl. Phys. B* **835** (2010) 238 [[arXiv:0911.3037](#)] [[SPIRES](#)].
- [209] M. Malinsky, *Higgs sector of the next-to-minimal renormalizable SUSY SO(10)*, [arXiv:0807.0591](#) [[SPIRES](#)].
- [210] GERDA collaboration, A.A. Smolnikov, *Status of the GERDA experiment aimed to search for neutrinoless double beta decay of ^{76}Ge* , [arXiv:0812.4194](#) [[SPIRES](#)].
- [211] MAJORANA collaboration, V.E. Guiseppe et al., *The Majorana neutrinoless double-beta decay experiment*, [arXiv:0811.2446](#) [[SPIRES](#)].
- [212] NEMO and SUPERNEMO collaborations, H. Ohsumi, *SuperNEMO project*, *J. Phys. Conf. Ser.* **120** (2008) 052054 [[SPIRES](#)].
- [213] CUORE collaboration, A. Giuliani, *From Cuoricino to CUORE: investigating the inverted hierarchy region of neutrino mass*, *J. Phys. Conf. Ser.* **120** (2008) 052051 [[SPIRES](#)].
- [214] M. Danilov et al., *Detection of very small neutrino masses in double-beta decay using laser tagging*, *Phys. Lett. B* **480** (2000) 12 [[hep-ex/0002003](#)] [[SPIRES](#)].