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Neutrino predictions from generalized CP symmetries of charged leptons

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ABSTRACT: We study the implications of generalized CP transformations acting on the mass matrices of charged leptons in a model-independent way. Generalized $e - \mu$, $e - \tau$ and $\mu - \tau$ symmetries are considered in detail. In all cases the physical parameters of the lepton mixing matrix, three mixing angles and three CP phases can be expressed in terms of a restricted set of independent “theory parameters” that characterize a given choice of CP transformation. This leads to implications for neutrino oscillations as well as neutrinoless double beta decay experiments.

KEYWORDS: Neutrino Physics, CP violation

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1 Introduction

Following the discovery of neutrino oscillations [1, 2], particle physics has entered probably its most exciting phase in many decades. The observed pattern of the lepton mixing angles, at odds with those of the quark sector, has added a further challenge to the so-called flavour problem, a long-standing mystery in particle physics. Over the last two decades, neutrino oscillation experiments have made considerable progress in the determination of the neutrino oscillation parameters, four of which are rather well measured. These include the solar and atmospheric mass splittings, as well as two of the lepton mixing angles θ_{12} and θ_{13} [3]. So far the octant of the angle θ_{23} is not yet well determined, nor the phase parameters characterizing CP violation. Concerning CP violation there are three physical phases in the simplest unitary mixing matrix, one Dirac and the two Majorana phases [4, 5]. Although latest results of T2K [6] and NO ν A [7] provide a first positive hint for CP violation, characterized by the Dirac CP phase δ_{CP} around $3\pi/2$, its value is not yet measured with high significance [3]. Finally, negative searches for neutrinoless double beta decay do not allow us to decide whether neutrinos are Majorana or Dirac particles nor determine the two associated Majorana phases.

Understanding the pattern of neutrino mixing and CP violation constitutes a fundamental problem in particle physics. Flavor symmetries provide an attractive framework for explaining the leptonic mixing angles and phases. In particular, non-Abelian discrete flavor symmetries have been widely studied in the literature see, for example, [8–15].

It has been noted in recent years that discrete flavor symmetries can be extended so as to include a CP symmetry [16–24]. The possible lepton mixing patterns which can be obtained from the breaking of flavor and CP symmetry have been widely explored (see,

e.g., [25–34] and references quoted therein). It is remarkable that the observed patterns of quark and lepton flavor mixings can be simultaneously understood from a flavor group and CP symmetry [35]. In this approach the mixing is determined in terms of a few free parameters, so that certain sum rules relate the mixing angles and CP violation phases. These sum rules are sensitive probes to test the discrete flavor symmetry approach in current and future neutrino experiments [36–38].

The imposition of a CP symmetry may allow us to obtain predictions for CP violating phases. A simple and interesting example is the so called $\mu - \tau$ reflection symmetry which exchanges a muon (tau) neutrino with a tau (muon) antineutrino. If the neutrino mass matrix is invariant under the action of $\mu - \tau$ reflection in the charged lepton diagonal basis, both atmospheric mixing angle θ_{23} and Dirac CP phase δ_{CP} are maximal and the Majorana phases are trivial [39–41]. The above prediction for maximal θ_{23} may be at variance with current experiments [6, 7]. A generalized $\mu - \tau$ reflection in the charged lepton diagonal mass basis is suggested in [23]. This can accommodate the observed non-maximal θ_{23} together with maximal δ_{CP} . Moreover, all possible CP transformations can be classified according to the number of zero entries [24].

In this work we shall study three kinds of simple and attractive CP symmetries: generalized $e - \mu$, generalized $e - \tau$ and generalized $\mu - \tau$ symmetries in the neutrino mass diagonal basis. The rest of this paper is organized as follows: in section 2 the symmetric parametrization of the lepton mixing matrix is reviewed. In section 3 we demonstrate how to extract the lepton mixing matrix from an imposed CP transformation, explaining the difference between Dirac and Majorana neutrino cases. In section 4 we perform a detailed study of the generalized $e - \mu$, $e - \tau$ and $\mu - \tau$ symmetries acting on the charged lepton fields. The predictions for lepton mixing parameters are discussed, and the correlations among mixing angles and CP phases are analyzed. In section 5 we discuss the phenomenological implications of our scheme for neutrinoless double beta decay. Further summary and conclusions are given in section 6.

2 Lepton mixing matrix parametrization

Throughout this paper we will use the so-called “symmetric parametrization” of the lepton mixing matrix. In this parametrization [4], the lepton mixing matrix U can be written as [42]¹

$$U = P_0 U_{23} U_{13} U_{12}, \quad (2.1)$$

where $P_0 = e^{\text{diag}(i\delta_1, i\delta_2, i\delta_3)}$ and the U_{ij} are the complex rotation matrices in the ij -axis. For example:

$$U_{13} = \begin{pmatrix} \cos \theta_{13} & 0 & e^{-i\phi_{13}} \sin \theta_{13} \\ 0 & 1 & 0 \\ -e^{i\phi_{13}} \sin \theta_{13} & 0 & \cos \theta_{13} \end{pmatrix} \quad (2.2)$$

¹This is based on Okubo’s parametrization of unitary groups and is specially convenient to describe both quark and lepton mixing matrices in full generality [4].

The mixing angles θ_{12} , θ_{23} and θ_{13} can be limited in the range $0 \leq \theta_{ij} \leq \pi/2$, the CP violation phases ϕ_{12} and ϕ_{13} can take values in the range $0 \leq \phi_{12}, \phi_{13} < \pi$, and the phase ϕ_{23} can take values in the range $0 \leq \phi_{23} < 2\pi$. Moreover, the phases δ_1 , δ_2 and δ_3 can be rotated away by redefinitions of the left-handed charged leptons and are therefore unphysical. In this parametrization, the invariant describing CP violation in conventional neutrino oscillations, takes the form [43]

$$J_{\text{CP}} = \Im(U_{11}U_{33}U_{13}^*U_{31}^*) = \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{13} \sin 2\theta_{23} \cos \theta_{13} \sin(\phi_{13} - \phi_{12} - \phi_{23}). \quad (2.3)$$

Hence the usual Dirac CP phase relevant to neutrino oscillation is simply given by

$$\delta_{\text{CP}} = \phi_{13} - \phi_{12} - \phi_{23}. \quad (2.4)$$

The other two rephasing invariants associated with the Majorana phases are [44–46]

$$I_1 = \Im(U_{12}^2 U_{11}^{*2}), \quad I_2 = \Im(U_{13}^2 U_{11}^{*2}), \quad (2.5)$$

which take the following form in the symmetric parametrization

$$I_1 = -\frac{1}{4} \sin^2 2\theta_{12} \cos^4 \theta_{13} \sin(2\phi_{12}), \quad I_2 = -\frac{1}{4} \sin^2 2\theta_{13} \cos^2 \theta_{12} \sin(2\phi_{13}). \quad (2.6)$$

On the other hand, the relevant parameter characterizing the neutrinoless double beta decay amplitude, i.e. the “effective Majorana mass” parameter, m_{ee} , is given as

$$m_{ee} = \left| m_1 c_{12}^2 c_{13}^2 + m_2 c_{13}^2 s_{12}^2 e^{-2i\phi_{12}} + m_3 s_{13}^2 e^{-2i\phi_{13}} \right|. \quad (2.7)$$

Notice that only the phases ϕ_{12} and ϕ_{13} , but not ϕ_{23} , appear in m_{ee} .

3 General discussion

In this section we begin with a general discussion of the generalized CP transformations, highlighting key concepts as well as setting up our notation and conventions. Following refs. [20, 23, 24] we start by defining the generalized remnant CP transformations for each fermionic field as follows:

$$\psi \xrightarrow{\text{CP}} iX_\psi \gamma^0 \mathcal{C} \bar{\psi}^T, \quad \psi \in \{\nu_L, \nu_R, l_L, l_R\}. \quad (3.1)$$

Such generalized CP transformations acting on the chiral fermions will be a symmetry of the mass term in the Lagrangian provided they satisfy the following conditions,²

$$X_\psi^T m_\psi X_\psi = m_\psi^*, \quad \text{for Majorana fields}, \quad (3.2)$$

$$X_\psi^\dagger M_\psi^2 X_\psi = M_\psi^{2*}, \quad \text{for Dirac fields, where } M_\psi^2 \equiv m_\psi^\dagger m_\psi, \quad (3.3)$$

²Even though the X-matrix is symmetric, we prefer to use $X^\dagger$ instead of X^* when dealing with Dirac neutrinos.

where m_ψ is written in a basis with left-handed (right-handed) fields on the right-hand (left-hand) side. Note that the mass matrices m_ψ and M_ψ^2 can be diagonalized by a unitary transformation U_ψ ,

$$U_\psi^T m_\psi U_\psi = \text{diag}(m_1, m_2, m_3), \quad \text{for Majorana fields}, \quad (3.4)$$

$$U_\psi^\dagger M_\psi^2 U_\psi = \text{diag}(m_1^2, m_2^2, m_3^2), \quad \text{for Dirac fields}, \quad (3.5)$$

with $m_1 \neq m_2 \neq m_3$. From eq. (3.2), eq. (3.3), eq. (3.4) and eq. (3.5), after straightforward algebra, we find that the unitary transformation U_ψ is subject to the following constraint from the imposed CP symmetry X_ψ ,

$$U_\psi^\dagger X_\psi U_\psi^* \equiv P = \begin{cases} \text{diag}(\pm 1, \pm 1, \pm 1), & \text{for Majorana fields}, \\ \text{diag}(e^{i\delta_e}, e^{i\delta_\mu}, e^{i\delta_\tau}), & \text{for Dirac fields}, \end{cases} \quad (3.6)$$

where δ_e , δ_μ and δ_τ are arbitrary real parameters.³ Because X_ψ is a symmetric matrix, one can use Takagi decomposition (note that this decomposition is not unique) to express X_ψ as

$$X_\psi = \Sigma \cdot \Sigma^T. \quad (3.7)$$

Inserting eq. (3.7) into eq. (3.6), we find that the combination $P^{-\frac{1}{2}} U_\psi^\dagger \Sigma$ is a real orthogonal matrix, i.e.,

$$P^{-\frac{1}{2}} U_\psi^\dagger \Sigma \equiv O_3, \quad (3.8)$$

which implies

$$U_\psi = \Sigma O_3^T P^{-\frac{1}{2}}, \quad (3.9)$$

where O_3 is a generic 3×3 real orthogonal matrix. As we will discuss in detail in section 4, the matrix Σ can be expressed in terms of three independent “CP labels”, namely two of the three phases α, β, γ and the CP angle Θ . These CP labels characterize a given generalized CP transformation. The resulting explicit form of Σ will depend on the generalized CP symmetry under consideration, and will be discussed in detail in section 4. Throughout this paper we will parameterize the orthogonal matrix O_3 as

$$O_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta_1 & \sin \theta_1 \\ 0 & -\sin \theta_1 & \cos \theta_1 \end{pmatrix} \begin{pmatrix} \cos \theta_2 & 0 & \sin \theta_2 \\ 0 & 1 & 0 \\ -\sin \theta_2 & 0 & \cos \theta_2 \end{pmatrix} \begin{pmatrix} \cos \theta_3 & \sin \theta_3 & 0 \\ -\sin \theta_3 & \cos \theta_3 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.10)$$

Notice that the matrix O_3 has the following properties

$$\begin{aligned} O_3(\theta_1 + \pi, \theta_2, \theta_3) &= \text{diag}(1, -1, -1) O_3(\theta_1, \theta_2, \theta_3), \\ O_3(\theta_1, \theta_2 + \pi, \theta_3) &= \text{diag}(-1, -1, 1) O_3(\theta_1, \theta_2, \theta_3), \\ O_3(\theta_1, \theta_2, \theta_3 + \pi) &= \text{diag}(1, -1, -1) O_3(\theta_1, \theta_2, \theta_3), \end{aligned} \quad (3.11)$$

³If neutrinos are Majorana particles and the lightest one is massless (this possibility is still allowed by current experimental data) one “ $\pm$ ” entry would be a complex phase.

where the diagonal matrices can be absorbed into the matrix P . As a consequence, the range of variation of the free parameters $\theta_{1,2,3}$ describing a given theory, can be taken to be $[0, \pi)$. At this point we would like to remark that, the generalized CP symmetries do not impose any constraint on the fermion masses. These can always be chosen to match the required experimental values. The predictive power of generalized CP symmetries lies in the mixing matrix elements and their phases.

Before going to particular cases of generalized CP transformations and their implications we would like to briefly comment about the differences one should expect in case neutrinos are Dirac or Majorana particles. In previous works on generalized CP symmetries [20, 23, 24], the neutrinos were assumed to be Majorana particles but that may not be the case in nature. We now comment on the differences that arise if neutrinos are Dirac particles.

As clear from our previous discussion, and in particular from eq. (3.9), the only difference between Majorana and Dirac neutrino mixing matrices (for a given CP symmetry) is in the diagonal unitary matrix P that appears in the right side of the lepton mixing matrix. For the case of Majorana fields, P is real, while in the Dirac case it is a general diagonal matrix of phases. Using the technique described in [4], it is easy to show that δ_{CP} of a given unitary matrix U is the same as δ_{CP} associated to another unitary matrix given by $U \cdot P$, where P is a diagonal matrix of phases. This implies that for any choice of a given generalized CP symmetry:

- All the mixing parameters characterizing neutrino oscillations, i.e., θ_{12} , θ_{13} , θ_{23} and δ_{CP} are identical both for Majorana or Dirac neutrinos, irrespective of the choice of the set of generalized CP symmetries imposed on either the charged leptons or neutrinos.
- For Majorana neutrinos, the imposition of a given generalized CP symmetry leads to interesting correlations between the Majorana phases, as we will discuss later. In contrast, if neutrinos are Dirac fields, the Majorana phases are unphysical and can be rotated away by appropriate field redefinitions.
- The imposition of generalized CP symmetries for Majorana neutrinos leads to important implications for neutrinoless double beta decays, as we discuss in section 5.⁴ In contrast, if neutrinos are Dirac in nature then neutrinoless double beta decay is simply forbidden.

In what follows we will primarily discuss the implications of generalized CP symmetries acting on charged leptons, assuming neutrinos to be Majorana particles. All of the resulting predictions for neutrino oscillation parameters hold equally well if neutrinos are Dirac-type. In other words, all of the correlations between oscillation observables remain unchanged irrespective of whether neutrinos are Majorana or Dirac-type. In the latter case the Majorana phases are unphysical and neutrinoless double beta decay is forbidden.

⁴Majorana phase correlations from generalized $\mu - \tau$ symmetry acting on neutrinos, along with their implications for neutrinoless double beta decay, have been investigated in ref. [23].

4 Imposing a particular CP symmetry

In this section we consider the implications of the mass matrix having symmetry under certain generalized CP transformations, in the same spirit as the well studied case of generalized $\mu - \tau$ symmetry [23] for the neutrino sector. From now on we will focus in the basis in which the neutrino mass matrix is diagonal and the CP symmetry is imposed in the charged lepton sector.

As already explained, all of our predictions for neutrino oscillation parameters, namely the mixing angles and δ_{CP} , would remain unchanged should neutrinos be Dirac type. Moreover, in the Majorana case we obtain predictions for neutrinoless double beta decay, that would be absent in the case of Dirac neutrinos.

4.1 Generalized $e - \mu$ reflection symmetry for charged leptons

In analogy with the generalized $\mu - \tau$ symmetry in the neutrino sector one can consider other possibilities, for example, the case of a $e - \mu$ symmetry imposed on the charged leptons. The CP transformation corresponding to the generalized $e - \mu$ symmetry is given by

$$X_{e\mu} = \begin{pmatrix} e^{i\alpha} \cos \Theta & ie^{\frac{1}{2}i(\alpha+\beta)} \sin \Theta & 0 \\ ie^{\frac{1}{2}i(\alpha+\beta)} \sin \Theta & e^{i\beta} \cos \Theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (4.1)$$

where the three CP labels α , β and Θ characterize a given CP transformation. The phases α and β can take any value between 0 and 2π , and Θ can be limited in the range $0 \leq \Theta \leq \pi$ without loss of generality. Notice that the X_{33} entry can be taken to be real without loss of generality, as a global phase in the CP matrix will be unphysical. The Takagi factorization for $X_{e\mu}$ gives us

$$\Sigma_{e\mu} = \begin{pmatrix} e^{i\frac{\alpha}{2}} & 0 & 0 \\ 0 & e^{i\frac{\beta}{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos \frac{\Theta}{2} & i \sin \frac{\Theta}{2} & 0 \\ i \sin \frac{\Theta}{2} & \cos \frac{\Theta}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (4.2)$$

Given the above assumptions and using eq. (3.8), the charged lepton mixing matrix takes the form

$$U_l = \Sigma_{e\mu} O_3^T P^{-\frac{1}{2}}. \quad (4.3)$$

Since we are in the diagonal neutrino mass basis, the lepton mixing matrix U arises solely from the charged lepton sector, and it is simply given by

$$U = U_l^\dagger = P^{\frac{1}{2}} O_3 \Sigma_{e\mu}^\dagger, \quad (4.4)$$

where the phases in the matrix P can be absorbed into the charged lepton fields. It follows that, for a given CP transformation $X_{e\mu}$, the lepton mixing matrix i.e. all mixing angles and CP phases depend on three free parameters $\theta_{1,2,3}$ plus the three CP labels. We first discuss the interesting subclass of generalized $e - \mu$ symmetry in which $\alpha = \beta = 0$, leaving only

one label, Θ . Using the explicit form of the lepton mixing matrix in eq. (4.4), the mixing and phase parameters characterizing the lepton mixing matrix can be extracted as follows

$$\sin^2 \theta_{13} = \sin^2 \theta_2, \quad \sin^2 \theta_{12} = \frac{1}{2} (1 - \cos 2\theta_3 \cos \Theta), \quad \sin^2 \theta_{23} = \sin^2 \theta_1, \quad (4.5)$$

and for the CP invariants we get

$$J_{\text{CP}} = -\frac{1}{4} \sin 2\theta_1 \sin \theta_2 \cos^2 \theta_2 \sin \Theta, \quad (4.6)$$

$$I_1 = -\frac{1}{4} \cos^4 \theta_2 \sin 4\theta_3 \sin \Theta, \quad (4.7)$$

$$I_2 = \frac{1}{8} \sin^2 2\theta_2 \sin 2\theta_3 \sin \Theta, \quad (4.8)$$

which implies the following CP violation phases

$$\tan \delta_{\text{CP}} = -\csc 2\theta_3 \tan \Theta, \quad (4.9)$$

$$\sin 2\phi_{12} = -\frac{\sin 4\theta_3 \sin \Theta}{\cos^2 2\theta_3 \cos^2 \Theta - 1}, \quad (4.10)$$

$$\sin 2\phi_{13} = -\frac{\sin 2\theta_3 \sin \Theta}{\cos 2\theta_3 \cos \Theta + 1}. \quad (4.11)$$

For the most general case, in which $\alpha \neq \beta \neq 0$, the Majorana CP phases ϕ_{12} and ϕ_{13} would change into $\phi_{12} + (\alpha - \beta)/2$ and $\phi_{13} + \alpha/2$, respectively, while all the mixing angles as well as the Dirac CP phase δ_{CP} would remain intact.

In the simplifying case of $\alpha = \beta = 0$, different models are characterized by a single label, namely Θ . Different values of Θ correspond to different models of this class. In the remaining part of the discussion we shall treat the generalized CP label Θ as a free parameter of the theory, however it should be kept in mind that it is really a label and any given model will have a given fixed Θ value.

Contrasting the current ranges for the lepton mixing parameters obtained from general neutrino oscillation global fits [3] with the predicted relations in eq. (4.5)–eq. (4.11), one can obtain the allowed ranges for the CP label Θ and the parameters $\theta_1, \theta_2, \theta_3$. This can in turn be used to obtain predictions for the values of the Majorana CP phases ϕ_{12}, ϕ_{13} determining the neutrinoless double beta decay amplitude.

For example, taking the current best fit values of the lepton mixing angles of [3], the resulting values for the CP label Θ and the parameters $\theta_1, \theta_2, \theta_3$ and the Majorana CP phases ϕ_{12}, ϕ_{13} are shown in table 1, for both normal (NO) as well as inverted (IO) mass ordering. In particular, if Θ is a rational angle, i.e. a rational multiple of π , the experimental data on lepton flavor mixing can be accommodated as well. For example, for $\Theta = 3\pi/8$, we have

$$\begin{aligned} \theta_1 &\simeq 41.0^\circ, & \theta_2 &\simeq 8.4^\circ \text{ or } 171.6^\circ, & \theta_3 &\simeq 10.3^\circ, \\ \theta_{12} &\simeq 34.5^\circ, & \theta_{13} &\simeq 8.44^\circ, & \theta_{23} &\simeq 41.0^\circ, \\ \delta_{\text{CP}} &\simeq 278.3^\circ, & \phi_{12} &\simeq 67.8^\circ, & \phi_{13} &\simeq 173.0^\circ, \end{aligned} \quad (4.12)$$

and

$$\begin{aligned} \theta_1 &\simeq 41.0^\circ, & \theta_2 &\simeq 8.4^\circ \text{ or } 171.6^\circ, & \theta_3 &\simeq 169.7^\circ, \\ \theta_{12} &\simeq 34.5^\circ, & \theta_{13} &\simeq 8.44^\circ, & \theta_{23} &\simeq 41.0^\circ, \\ \delta_{\text{CP}} &\simeq 261.7^\circ, & \phi_{12} &\simeq 112.2^\circ, & \phi_{13} &\simeq 7.0^\circ. \end{aligned} \quad (4.13)$$

numerical benchmark for generalized $e - \mu$ reflection							
	Θ	θ_1	θ_2	θ_3	δ_{CP}	ϕ_{12}	ϕ_{13}
NO	69.0°	41.0°	8.4 or 171.6°	0.0°	270.0°	90.0°	0.0°
	111.0°	41.0°	8.4 or 171.6°	90.0°	270.0°	90.0°	90.0°
IO	69.0°	50.5°	8.4 or 171.6°	0.0°	270.0°	90.0°	0.0°
	111.0°	50.5°	8.4 or 171.6°	90.0°	270.0°	90.0°	90.0°

Table 1. Numerical examples for the generalized $e - \mu$ reflection, where $\alpha = \beta = 0$ is assumed. The values of Θ and $\theta_{1,2,3}$ are fixed by the best fit value of mixing angles and the favored value $\delta_{\text{CP}} = 3\pi/2$ [6, 7]. Notice that the two Majorana CP phases are *predicted*, as shown in the table.

Moreover, eq. (4.5) implies that the solar neutrino mixing angle θ_{12} depends on the generalized CP label Θ and the angle θ_3 of the rotation matrix O_3 . Using the current experimental 3σ range of the angle θ_{12} from [3], one can obtain the allowed regions for the Θ and θ_3 , as shown in figure 1. To be more specific, from eq. (4.5) we have

$$\cos \Theta \cos 2\theta_3 = \cos 2\theta_{12} . \quad (4.14)$$

Thus, the value of Θ cannot be arbitrary and is restricted by the experimental measurement of θ_{12} . Inputting the 3σ experiment range $0.273 < \sin^2 \theta_{12} < 0.379$ given in [3], we find that $\cos \Theta$ is constrained to be

$$0.242 < |\cos \Theta| \leq 1 . \quad (4.15)$$

Notice that a residual CP symmetry characterized by $\Theta = \pi/2$ is disfavored by current oscillation data. Since the Dirac phase δ_{CP} depends on the CP label Θ and the parameter θ_3 as well, as shown in eq. (4.9), we display the contour plot of $|\sin \delta_{\text{CP}}|$ in the $\theta_3 - \Theta$ plane in figure 1, where only the values of $|\sin \delta_{\text{CP}}|$ in the phenomenologically viable regions are shown.

Notice also that the above relations eq. (4.5)–eq. (4.11) lead to several important correlations among the various neutrino mixing and CP violation parameters, e.g.

- $\theta_{12} - \delta_{\text{CP}}$ correlation

$$\sin^2 2\theta_{12} \sin^2 \delta_{\text{CP}} = \sin^2 \Theta . \quad (4.16)$$

- $\phi_{12} - \delta_{\text{CP}} - \theta_{12}$ correlation

$$\sin 2\phi_{12} = -\frac{\cos 2\theta_{12} \sin 2\delta_{\text{CP}}}{1 - \sin^2 2\theta_{12} \sin^2 \delta_{\text{CP}}} , \quad \cos 2\phi_{12} = \frac{2 \cos^2 \delta_{\text{CP}}}{1 - \sin^2 2\theta_{12} \sin^2 \delta_{\text{CP}}} - 1 . \quad (4.17)$$

- $\phi_{13} - \delta_{\text{CP}} - \theta_{12}$ correlation

$$\frac{\sin 2\delta_{\text{CP}} \sin^2 \theta_{12}}{\sin 2\phi_{13}} = \cos \Theta , \quad \text{or} \quad \sin^2 2\phi_{13} = \frac{\sin^2 2\delta_{\text{CP}} \sin^4 \theta_{12}}{1 - \sin^2 2\theta_{12} \sin^2 \delta_{\text{CP}}} . \quad (4.18)$$

Figure 1. Regions in the $\theta_3 - \Theta$ plane allowed by current neutrino oscillation data on $\sin^2 \theta_{12}$ at 3σ level [3]. The values of $|\sin \delta_{CP}|$ are indicated by the color shadings.

Figure 2. Correlation between δ_{CP} and θ_{12} for various Θ values, where Θ labels the residual CP symmetry. Cyan, blue, black, pink, green and red dashed lines correspond to $\pi/12$, $\pi/6$, $\pi/4$, $\pi/3$, $3\pi/8$ and $2\pi/5$ respectively. The orange and yellow bands indicate the allowed 1σ and 3σ regions (NO case), respectively. The best fit point of θ_{12} and δ_{CP} is indicated with a star. The gray region corresponding to $0 \leq |\cos \Theta| \leq 0.242$ is excluded at the 3σ level [3].

The correlation in eq. (4.16) implies that for the generalized $e - \mu$ symmetry in the charged lepton sector, the mixing angle θ_{12} and the CP violating phase δ_{CP} are correlated with each other, for a given choice of the CP label Θ , as shown in figure 2.

We see that the measurement of δ_{CP} in next generation long baseline experiments, together with high precision measurement of θ_{12} , could help us to determine the CP label

Figure 3. Correlations among the CP violation phases δ_{CP} , ϕ_{12} and ϕ_{13} for the generalized $e - \mu$ reflection in the charged lepton sector, taking $\alpha = \beta = 0$. The light blue and green areas correspond to $\phi_{12} - \delta_{\text{CP}}$ and $\phi_{13} - \delta_{\text{CP}}$ respectively.

Θ characterizing the residual CP symmetry. Notice that this prediction is analogous to the prediction for the atmospheric mixing angle θ_{23} and the Dirac phase δ_{CP} obtained if we impose a generalized $\mu - \tau$ reflection in the neutrino sector [23].

Given the current experimental range of θ_{12} , the other two correlations in eq. (4.17) and eq. (4.18) can be used to obtain the allowed ranges for the Dirac CP phase δ_{CP} and the two Majorana phases ϕ_{12} and ϕ_{13} . These relations together also imply correlations between the two Majorana phases ϕ_{12} and ϕ_{13} .

Notice that all three parameters $\theta_{1,2,3}$ and the CP label Θ are varied randomly between 0 and π , keeping only the points for which the lepton mixing angles are consistent with experimental data at 3σ level. The resulting predictions for the three CP phases δ_{CP} , ϕ_{12} and ϕ_{13} are displayed in figure 3. We notice that the values of ϕ_{13} is around 0, $\pi/2$ and π . The implications of the correlations between the Majorana phases and other mixing parameters for neutrinoless double beta decay will be further discussed in section 5.

4.2 Generalized $e - \tau$ reflection symmetry for charged leptons

In this section we look at the implications of imposing a generalized $e - \tau$ symmetry on charged leptons. The corresponding CP transformation matrix $X_{e\tau}$ is of the form⁵

$$X_{e\tau} = \begin{pmatrix} e^{i\alpha} \cos \Theta & 0 & ie^{\frac{1}{2}i(\alpha+\gamma)} \sin \Theta \\ 0 & 1 & 0 \\ ie^{\frac{1}{2}i(\alpha+\gamma)} \sin \Theta & 0 & e^{i\gamma} \cos \Theta \end{pmatrix}, \quad (4.19)$$

where the two phase labels α and γ take values in the range $0 \leq \alpha, \gamma \leq 2\pi$. The angle Θ labeling the residual CP transformation lies in $[0, \pi]$. The X_{22} element is taken as real

⁵Note that, although the CP label Θ characterizing the $e - \tau$ symmetry differs from that of the $e - \mu$ case, we denote it by the same symbol (similarly also for the $\mu - \tau$ case, discussed in subsection 4.3). However, they have different ranges of variation, see eqs. (4.15), (4.39) and (4.56).

without loss of generality, due to the freedom to fix a global phase without changing the physics. We find that the Takagi factorization of $X_{e\tau}$ is given by

$$\Sigma_{e\tau} = \begin{pmatrix} e^{i\frac{\alpha}{2}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} \begin{pmatrix} \cos \frac{\Theta}{2} & 0 & i \sin \frac{\Theta}{2} \\ 0 & 1 & 0 \\ i \sin \frac{\Theta}{2} & 0 & \cos \frac{\Theta}{2} \end{pmatrix}. \quad (4.20)$$

As before, we use our master formula, eq. (3.9), to extract the total lepton mixing matrix U , which is the hermitian conjugate of the charged lepton mixing matrix in the diagonal neutrino basis, and takes the following form

$$U = P^{\frac{1}{2}} O_3 \Sigma_{e\tau}^\dagger. \quad (4.21)$$

In this case the mixing angles are found to be given as

$$\sin^2 \theta_{13} = \cos^2 \theta_2 \cos^2 \theta_3 \sin^2 \frac{\Theta}{2} + \sin^2 \theta_2 \cos^2 \frac{\Theta}{2}, \quad (4.22)$$

$$\sin^2 \theta_{12} = \frac{\sin^2 \theta_3 \cos^2 \theta_2}{1 - \cos^2 \theta_2 \cos^2 \theta_3 \sin^2 \frac{\Theta}{2} - \sin^2 \theta_2 \cos^2 \frac{\Theta}{2}}, \quad (4.23)$$

$$\sin^2 \theta_{23} = \frac{\sin^2 \frac{\Theta}{2} (\sin \theta_1 \sin \theta_2 \cos \theta_3 + \sin \theta_3 \cos \theta_1)^2 + \sin^2 \theta_1 \cos^2 \theta_2 \cos^2 \frac{\Theta}{2}}{1 - \cos^2 \theta_2 \cos^2 \theta_3 \sin^2 \frac{\Theta}{2} - \sin^2 \theta_2 \cos^2 \frac{\Theta}{2}}. \quad (4.24)$$

In the limit $\alpha = \gamma = 0$, the leptonic CP invariants are given by

$$J_{\text{CP}} = \frac{1}{4} \sin \theta_3 \cos \theta_2 \sin \Theta (\sin 2\theta_1 (\cos^2 \theta_3 - \sin^2 \theta_2 \sin^2 \theta_3) + \sin \theta_2 \sin 2\theta_3 \cos 2\theta_1), \quad (4.25)$$

$$I_1 = \sin^2 \theta_3 \cos^2 \theta_2 \sin \theta_2 \cos \theta_2 \cos \theta_3 \sin \Theta, \quad (4.26)$$

$$I_2 = \frac{1}{4} [4 \sin \theta_2 \cos^3 \theta_2 \cos^3 \theta_3 \sin \Theta + 4 \sin^3 \theta_2 \cos \theta_2 \cos \theta_3 \sin \Theta]. \quad (4.27)$$

For the most general case $\alpha \neq \gamma \neq 0$, the Majorana CP phases satisfy

$$\sin 2\phi'_{12} = -\frac{\sin 2\theta_2 \cos \theta_3 \sin \Theta}{2 \sin^2 \theta_2 \sin^2 \frac{\Theta}{2} + 2 \cos^2 \theta_2 \cos^2 \theta_3 \cos^2 \frac{\Theta}{2}}, \quad (4.28)$$

$$\sin 2\phi'_{13} = \frac{4 \sin 2\theta_2 \cos \theta_3 \sin \Theta (\cos^2 \theta_2 \cos^2 \theta_3 - \sin^2 \theta_2)}{2 \cos^4 \theta_2 \cos^4 \theta_3 \sin^2 \Theta + 2 \sin^4 \theta_2 \sin^2 \Theta + \sin^2 2\theta_2 \cos^2 \theta_3 (1 + \cos^2 \Theta)}, \quad (4.29)$$

where

$$2\phi'_{12} = 2\phi_{12} + \alpha, \quad 2\phi'_{13} = 2\phi_{13} + \alpha - \gamma. \quad (4.30)$$

The effect of non-zero α and γ is to only shift the Majorana phases ϕ_{12} and ϕ_{13} by an amount. As in section 4.1, taking the current best fit values of the leptonic mixing parameters in [3], the values of the CP label Θ characterizing the CP transformation and the θ_i can be determined, so that the Majorana CP phases ϕ_{12}, ϕ_{13} can be predicted if both α and γ are set to be zero.

The results are summarized in table 2. As an example, for the representative value $\Theta = \pi/9$, the best fit values of the three lepton mixing angles [3] can be reproduced and

numerical benchmark for generalized $e - \tau$ reflection							
	Θ	θ_1	θ_2	θ_3	δ_{CP}	ϕ_{12}	ϕ_{13}
NO	20.4°	40.9°	179.8°	34.1°	270.0°	0.1°	91.7°
	20.4°	139.1°	0.2°	145.9°	270.0°	0.1°	91.7°
	159.6°	49.5°	55.9°	89.6°	270.0°	90.1°	91.7°
	159.6°	130.5°	124.1°	90.4°	270.0°	90.1°	91.7°
IO	20.3°	50.7°	179.7°	145.9°	270.0°	179.9°	87.6°
	20.3°	129.3°	0.3°	34.1°	270.0°	179.9°	87.6°
	159.7°	38.8°	55.9°	90.6°	270.0°	89.9°	87.6°
	159.7°	141.2°	124.1°	89.4°	270.0°	89.9°	87.6°

Table 2. Numerical examples for the generalized $e - \tau$ reflection, assuming $\alpha = \gamma = 0$. The values of Θ and $\theta_{1,2,3}$ are determined from the best fit values of neutrino mixing angles including $\delta_{\text{CP}} = 3\pi/2$ [6, 7]. Notice that the two Majorana CP phases are *predicted*, as shown in the table.

we have

$$\begin{aligned}
 \theta_1 &\simeq 40.9^\circ, & \theta_2 &\simeq 178.3^\circ, & \theta_3 &\simeq 34.1^\circ, \\
 \theta_{12} &\simeq 34.5^\circ, & \theta_{13} &\simeq 8.44^\circ, & \theta_{23} &\simeq 41.0^\circ, \\
 \delta_{\text{CP}} &\simeq 280.0^\circ, & \phi_{12} &\simeq 0.4^\circ, & \phi_{13} &\simeq 102.1^\circ,
 \end{aligned} \tag{4.31}$$

and

$$\begin{aligned}
 \theta_1 &\simeq 40.9^\circ, & \theta_2 &\simeq 178.3^\circ, & \theta_3 &\simeq 145.9^\circ, \\
 \theta_{12} &\simeq 34.5^\circ, & \theta_{13} &\simeq 8.44^\circ, & \theta_{23} &\simeq 41.0^\circ, \\
 \delta_{\text{CP}} &\simeq 256.7^\circ, & \phi_{12} &\simeq 179.6^\circ, & \phi_{13} &\simeq 77.9^\circ.
 \end{aligned} \tag{4.32}$$

Since the lepton mixing matrix U depends on three free rotation angles $\theta_{1,2,3}$ besides the CP label Θ , the mixing angles plus CP phases are correlated with each other. After tedious algebra calculations, we find the following relations:

$$\sin^2 \Theta = \frac{4s_{13}^2 c_{13}^2 s_{23}^2 c_{23}^2 c_{12}^2 \sin^2 \delta_{\text{CP}}}{s_{23}^2 c_{23}^2 (c_{12}^2 + s_{12}^2 s_{13}^2)^2 \sin^2 \delta_{\text{CP}} + (s_{23} c_{23} (c_{12}^2 - s_{12}^2 s_{13}^2) \cos \delta_{\text{CP}} + s_{13} s_{12} c_{12} \cos 2\theta_{23})^2}, \tag{4.33}$$

$$\begin{aligned}
 \sin^2 2\phi_{12} &= 4s_{13}^4 s_{23}^2 c_{23}^2 (s_{23} c_{23} (s_{12}^2 s_{13}^2 - c_{12}^2) \cos \delta_{\text{CP}} - s_{12} c_{12} s_{13} \cos 2\theta_{23})^2 \sin^2 \delta_{\text{CP}} / \\
 &\quad \left\{ \left[s_{23}^2 c_{23}^2 (c_{12}^2 + s_{12}^2 s_{13}^2)^2 \sin^2 \delta_{\text{CP}} + (s_{23} c_{23} (c_{12}^2 - s_{12}^2 s_{13}^2) \cos \delta_{\text{CP}} + s_{13} s_{12} c_{12} \cos 2\theta_{23})^2 \right] \right. \\
 &\quad \times \left. \left[s_{23}^2 c_{23}^2 (s_{13}^2 - c_{12}^2 c_{13}^2)^2 \sin^2 \delta_{\text{CP}} + (s_{23} c_{23} (c_{12}^2 - s_{12}^2 s_{13}^2) \cos \delta_{\text{CP}} + s_{12} c_{12} s_{13} \cos 2\theta_{23})^2 \right] \right\},
 \end{aligned} \tag{4.34}$$

$$\sin 2\phi_{13} = -\frac{2s_{23} c_{23} (c_{12}^2 c_{13}^2 - s_{13}^2) (s_{23} c_{23} (s_{12}^2 s_{13}^2 - c_{12}^2) \cos \delta_{\text{CP}} - s_{12} c_{12} s_{13} \cos 2\theta_{23}) \sin \delta_{\text{CP}}}{s_{23}^2 c_{23}^2 (s_{13}^2 - c_{12}^2 c_{13}^2)^2 \sin^2 \delta_{\text{CP}} + (s_{23} c_{23} (c_{12}^2 - s_{12}^2 s_{13}^2) \cos \delta_{\text{CP}} + s_{12} c_{12} s_{13} \cos 2\theta_{23})^2}, \tag{4.35}$$

where $c_{ij} = \cos \theta_{ij}$ and $s_{ij} = \sin \theta_{ij}$. These expressions are not very illuminating. However one can extract simple approximations which provide a rough insight into what is going on. By expanding the right-handed side of these equations in terms of the small reactor angle θ_{13} , one obtains

$$\sin^2 \Theta \simeq \frac{4 \sin^2 \theta_{13} \sin^2 \delta_{\text{CP}}}{\cos^2 \theta_{12}} [1 - 4 \sin \theta_{13} \tan \theta_{12} \cot 2\theta_{23} \cos \delta_{\text{CP}}] , \quad (4.36)$$

$$\sin^2 2\phi_{12} \simeq \frac{2 \sin^4 \theta_{13} \sin 2\delta_{\text{CP}} \sin \delta_{\text{CP}}}{\cos^4 \theta_{12}} [\cos \delta_{\text{CP}} - 4 \sin \theta_{13} \tan \theta_{12} \cot 2\theta_{23} \cos 2\delta_{\text{CP}}] \quad (4.37)$$

$$\sin 2\phi_{13} \simeq \sin 2\delta_{\text{CP}} - 4 \sin \theta_{13} \tan \theta_{12} \cot 2\theta_{23} \sin \delta_{\text{CP}} \cos 2\delta_{\text{CP}} . \quad (4.38)$$

One sees from eq. (4.36) that $\sin^2 \Theta$ should be quite small in this case as it is proportional to $\sin^2 \theta_{13}$ at leading order. Detailed numerical analysis shows that the phenomenologically viable ranges of Θ are

$$\Theta \in [0, 0.12\pi] \cup [0.88\pi, \pi] . \quad (4.39)$$

Note that all three CP phases would be trivial for $\Theta = 0, \pi$. Moreover, one sees that the solar mixing angle θ_{12} and the Dirac CP phase δ_{CP} are strongly correlated. For example, taking $\Theta = \pi/12$ and $\Theta = \pi/9$, the allowed regions of δ_{CP} and θ_{12} are determined as shown in figure 4, where both θ_{13} and θ_{23} are required to lie within their current 3σ global ranges [3].

Concerning the Majorana CP phase ϕ_{12} , from eq. (4.37) one sees that in this case $\sin 2\phi_{12}$ is of order $\sin^2 \theta_{13}$, so that ϕ_{12} lies close to 0, $\pi/2$ or π . Moreover, eq. (4.38) indicates that the Majorana phase ϕ_{13} is the same as δ_{CP} up to π at leading order. The strong correlation between ϕ_{13} and δ_{CP} is displayed in figure 5, as expected from the approximation of eq. (4.38). The important implications of these correlations for neutrinoless double beta decay experiments will be discussed in section 5.

4.3 Generalized $\mu - \tau$ reflection symmetry for charged leptons

In this section we proceed to study the implications of imposing generalized $\mu - \tau$ symmetry on the charged lepton sector. The analogous case of generalized $\mu - \tau$ symmetry acting on neutrinos has been studied previously in [23]. The CP transformation corresponding to the generalized $\mu - \tau$ symmetry, $X_{\mu\tau}$ is given by

$$X_{\mu\tau} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\beta} \cos \Theta & ie^{\frac{1}{2}i(\beta+\gamma)} \sin \Theta \\ 0 & ie^{\frac{1}{2}i(\beta+\gamma)} \sin \Theta & e^{i\gamma} \cos \Theta \end{pmatrix} , \quad (4.40)$$

where the phase labels β and γ can take any value between 0 and 2π , and the fundamental range for the CP label Θ is $[0, \pi]$. The Takagi factorization of $X_{\mu\tau}$ is found to be

$$\Sigma_{\mu\tau} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\frac{\beta}{2}} & 0 \\ 0 & 0 & e^{i\frac{\gamma}{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \frac{\Theta}{2} & i \sin \frac{\Theta}{2} \\ 0 & i \sin \frac{\Theta}{2} & \cos \frac{\Theta}{2} \end{pmatrix} . \quad (4.41)$$

Figure 4. Predicted correlation between the Dirac CP phase δ_{CP} and the solar mixing angle θ_{12} in the case of generalized $e - \tau$ reflection, when θ_{13} and θ_{23} are in the 3σ ranges given in [3]. The vertical orange and yellow bands are the currently allowed 1σ and 3σ θ_{12} and δ_{CP} regions respectively for normal ordering, while the star denotes the best fit point. The green and light blue hatched regions correspond to $\Theta = \pi/12$ and $\Theta = \pi/9$ respectively.

Figure 5. Strong correlation between ϕ_{13} and δ_{CP} for the case of generalized $e - \tau$ reflection in the charged lepton sector, assuming $\alpha = \gamma = 0$ and the three mixing angles in the experimentally preferred 3σ regions [3].

As a result, the total lepton mixing matrix U in the diagonal neutrino basis is simply given by

$$U = P^{\frac{1}{2}} O_3 \Sigma_{\mu\tau}^\dagger. \quad (4.42)$$

Notice that the matrix $\Sigma_{\mu\tau}$ is related to $\Sigma_{e\tau}$ as follows,

$$\Sigma_{\mu\tau} = P_{12} \Sigma_{e\tau} (\alpha \rightarrow \beta) P_{12}, \quad (4.43)$$

where $\alpha \rightarrow \beta$ implies replacement of phase α by β in eq. (4.20) and P_{12} is a permutation matrix given by

$$P_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (4.44)$$

Thus it is straightforward to see that the mixing parameters predicted by the generalized $e - \tau$ and $\mu - \tau$ symmetries obey the following relations

$$\theta_{13}^{\mu\tau} = \theta_{13}^{e\tau}(\theta_3 \rightarrow \theta_3 - \pi/2, \alpha \rightarrow \beta), \quad (4.45)$$

$$\theta_{23}^{\mu\tau} = \theta_{23}^{e\tau}(\theta_3 \rightarrow \theta_3 - \pi/2, \alpha \rightarrow \beta), \quad (4.46)$$

$$\theta_{12}^{\mu\tau} = \pi/2 - \theta_{12}^{e\tau}(\theta_3 \rightarrow \theta_3 - \pi/2, \alpha \rightarrow \beta), \quad (4.47)$$

$$\delta_{\text{CP}}^{\mu\tau} = \delta_{\text{CP}}^{e\tau}(\theta_3 \rightarrow \theta_3 - \pi/2, \alpha \rightarrow \beta) + \pi, \quad (4.48)$$

$$\phi_{12}^{\mu\tau} = -\phi_{12}^{e\tau}(\theta_3 \rightarrow \theta_3 - \pi/2, \alpha \rightarrow \beta), \quad (4.49)$$

$$\phi_{13}^{\mu\tau} = \phi_{13}^{e\tau}(\theta_3 \rightarrow \theta_3 - \pi/2, \alpha \rightarrow \beta) - \phi_{12}^{e\tau}(\theta_3 \rightarrow \theta_3 - \pi/2, \alpha \rightarrow \beta). \quad (4.50)$$

From the predictions of generalized $e - \tau$ symmetry in previous section, the above relations can be used to obtain the values of the mixing angles and CP phases corresponding the generalized $\mu - \tau$ symmetry. As before, the values of the parameters $\theta_{1,2,3}$ and the CP label Θ can be determined from the measured values of three lepton mixing angles and the favored $\delta_{\text{CP}} \simeq 3\pi/2$ value [6, 7].

The predicted Majorana CP phases are given in table 3. As an example, taking $\Theta = \pi/6$, we have

$$\begin{aligned} \theta_1 &\simeq 40.6^\circ, & \theta_2 &\simeq 0^\circ, & \theta_3 &\simeq 145.1^\circ, \\ \theta_{12} &\simeq 33.9^\circ, & \theta_{13} &\simeq 8.51^\circ, & \theta_{23} &\simeq 41.0^\circ, \\ \delta_{\text{CP}} &\simeq 273.7^\circ, & \phi_{12} &\simeq 0^\circ, & \phi_{13} &\simeq 90^\circ, \end{aligned} \quad (4.51)$$

and

$$\begin{aligned} \theta_1 &\simeq 40.6^\circ, & \theta_2 &\simeq 180^\circ, & \theta_3 &\simeq 34.9^\circ, \\ \theta_{12} &\simeq 33.9^\circ, & \theta_{13} &\simeq 8.51^\circ, & \theta_{23} &\simeq 41.0^\circ, \\ \delta_{\text{CP}} &\simeq 273.7^\circ, & \phi_{12} &\simeq 180^\circ, & \phi_{13} &\simeq 90^\circ, \end{aligned} \quad (4.52)$$

As in the previous two cases, here we also find that the relations eq. (4.45)–eq. (4.50) lead to correlations between the neutrino oscillation parameters. The exact results are

numerical benchmark for generalized $\mu - \tau$ reflection							
	Θ	θ_1	θ_2	θ_3	δ_{CP}	ϕ_{12}	ϕ_{13}
NO	29.3°	40.6°	0.5 or 179.5°	144.6°	270.0°	0.2°	86.5°
	29.3°	139.4°	0.5 or 179.5°	35.4°	270.0°	0.2°	86.5°
	150.7°	49.8°	144.6 or 35.4°	179.3°	270.0°	90.2°	176.5°
	150.7°	130.2°	144.6 or 35.4°	0.7°	270.0°	90.2°	176.5°
IO	29.2°	51.1°	179.3 or 0.7°	35.4°	270.0°	179.7°	94.9°
	29.2°	128.9°	179.3 or 0.7°	144.6°	270.0°	179.7°	94.9°
	150.8°	38.4°	35.4 or 144.6°	0.9°	270.0°	89.7°	4.9°
	150.8°	141.6°	35.4 or 144.6°	179.1°	270.0°	89.7°	4.9°

Table 3. Numerical examples for the generalized $\mu - \tau$ reflection, where we assume $\beta = \gamma = 0$. The values of Θ and $\theta_{1,2,3}$ are fixed by the best fitted value of mixing angles and experimentally favored value $\delta_{\text{CP}} = 3\pi/2$ [6, 7]. Notice that the two Majorana CP phases are *predicted*, as shown in the table.

lengthy and not very illuminating. By expanding in the small quantity $\sin \theta_{13}$, one can display the predictions in a simple form

$$\sin^2 \Theta \simeq \frac{4 \sin^2 \theta_{13} \sin^2 \delta_{\text{CP}}}{\sin^2 \theta_{12}} [1 + 4 \sin \theta_{13} \cot \theta_{12} \cot 2\theta_{23} \cos \delta_{\text{CP}}] , \quad (4.53)$$

$$\sin^2 2\phi_{12} \simeq \frac{2 \sin^4 \theta_{13} \sin 2\delta_{\text{CP}} \sin \delta_{\text{CP}}}{\sin^4 \theta_{12}} [\cos \delta_{\text{CP}} + 4 \sin \theta_{13} \cot \theta_{12} \cot 2\theta_{23} \cos 2\delta_{\text{CP}}] , \quad (4.54)$$

$$\sin 2(\phi_{13} - \phi_{12}) \simeq \sin 2\delta_{\text{CP}} + 4 \sin \theta_{13} \cot \theta_{12} \cot 2\theta_{23} \sin \delta_{\text{CP}} \cos 2\delta_{\text{CP}} . \quad (4.55)$$

Taking into account the current allowed 3σ range from [3], eq. (4.53) gives rise to the following viable range of Θ ,

$$\Theta \in [0, 0.18\pi] \cup [0.82\pi, \pi] . \quad (4.56)$$

Notice that the relation between mixing angles and δ_{CP} in eq. (4.53) is different from those in eq. (4.16) and eq. (4.36) corresponding to the cases of generalized $e - \mu$ and $e - \tau$ reflection, respectively. Taking, for example, $\Theta = \pi/9$ and $\Theta = \pi/6$, the possible values of δ_{CP} as a function of the solar mixing angle θ_{12} are shown in figure 6. The measurement of δ_{CP} in future neutrino oscillation experiments can help us to fix the value of the CP label Θ . In addition, eq. (4.54) implies that $\sin 2\phi_{12}$ is quite small, so that ϕ_{12} lies close to 0, $\pi/2$ and π . In addition, one can see from eq. (4.55) that the Majorana phase ϕ_{13} is close to $\delta_{\text{CP}} \pmod{\pi}$ or $\delta_{\text{CP}} + \pi/2 \pmod{\pi}$, up to higher order terms in $\sin \theta_{13}$. The predicted correlation between ϕ_{13} and δ_{CP} is shown in figure 7.

To conclude this section we stress that the imposition of generalized CP symmetries implies correlations involving the lepton mixing angles, the Dirac CP violating phase relevant for neutrino oscillations. We have already discussed some of the phenomenological

Figure 6. Predicted correlation between the Dirac CP phase δ_{CP} as a function of the solar mixing angle θ_{12} in the case of generalized $\mu - \tau$ reflection, when θ_{13} and θ_{23} are required to lie in their 3σ ranges. The vertical orange and yellow bands are the currently allowed 1σ and 3σ (NO case) respectively and the star denotes the best fit point [3]. The green and light blue hatched regions correspond to $\Theta = \pi/9$ and $\Theta = \pi/6$ respectively.

Figure 7. The correlation between ϕ_{13} and δ_{CP} for the generalized $\mu - \tau$ reflection acting on the charged lepton sector, taking $\beta = \gamma = 0$ and the three mixing angles in their 3σ regions [3].

implications of these correlations. Here we mention the generalized CP symmetry predictions for the CP violation phase δ_{CP} .

Figure 8 shows how the correlations between the Dirac phase δ_{CP} characterizing neutrino oscillations and the CP label Θ . Note that the predictions for this phase do not depend on the values of the phase labels α , β and γ . This is as expected, since oscillation probabilities can not depend on Majorana phases [5]. In principle, by fixing a given CP

Figure 8. Predictions for the Dirac CP violation phase δ_{CP} as a function of the CP label Θ associated with generalized CP symmetries. The light blue, magenta and green regions correspond to the cases of generalized $e - \mu$, $e - \tau$ and $\mu - \tau$ symmetries, respectively.

symmetry or Θ value one can make predictions for the upcoming generation of neutrino oscillation experiments such as DUNE [47, 48]. This exercise has been carried out in the analogous case of generalized CP symmetries acting on neutrinos in refs. [23, 24]. In the remaining of this paper we prefer to focus on the rather significant implications of the above predictions for the upcoming neutrinoless double beta decay experiments, as discussed in the next section.

5 Implications for neutrinoless double beta decay

We shall proceed to investigate the implications of generalized CP symmetries of charged leptons in theories with Majorana neutrinos, assuming the basis in which their mass matrix is diagonal.⁶ As in previous sections, we shall set $\alpha = \beta = \gamma = 0$ in this section.

As we already saw at length, the imposition of generalized $e - \mu$, $e - \tau$ and $\mu - \tau$ symmetries leads to correlations amongst the lepton mixing angles, the Dirac CP violating phase δ_{CP} , and also the two Majorana phases ϕ_{12} , ϕ_{13} . We have already discussed some features of these correlations. Here we focus on the generalized CP symmetry predictions involving the Majorana CP violation phases.

By treating $\theta_{1,2,3}$ and the CP label Θ as random numbers in the range of 0 and π we perform a comprehensive numerical scan of the parameter space. The three lepton mixing angles are required to lie within their 3σ ranges, as determined in [3]. In figure 9 we show the correlations between the Majorana CP phases ϕ_{12} , ϕ_{13} with the CP label Θ characterizing the three generalized CP symmetries discussed in previous section. As clear from figure 9, the Majorana phases ϕ_{12} and ϕ_{13} are strongly correlated with the generalized

⁶For Dirac neutrinos the Majorana phases are unphysical, as they can be eliminated by field redefinitions [4, 5]. Likewise, in this case neutrinoless double beta decay is always forbidden.

Figure 9. Predictions for the Majorana CP violation phases ϕ_{12} and ϕ_{13} as a function of the CP label Θ . The light blue, magenta and green regions are for the generalized $e - \mu$, $e - \tau$ and $\mu - \tau$ symmetries, respectively. Here we choose $\alpha = \beta = \gamma = 0$.

Figure 10. Correlations between the two Majorana phases ϕ_{12} and ϕ_{13} of the lepton mixing matrix. The light blue, magenta and green regions are for the generalized $e - \mu$, $e - \tau$ and $\mu - \tau$ symmetries respectively. The magenta regions are quite narrow and hardly visible, and they are closely distributed around $\phi_{12} \sim 0$, $\phi_{12} \sim \pi/2$ and $\phi_{12} \sim \pi$. Here we take $\alpha = \beta = \gamma = 0$ for illustration.

CP label Θ . This in turn implies that they are correlated with other mixing parameters as discussed in previous section, as well as mutually correlated, as shown in figure 10.

These correlations have interesting consequences for the effective Majorana mass m_{ee} characterizing the neutrinoless double beta decay amplitude, as we discuss next. We first consider the case of generalized $e - \mu$ reflection. The relevant results and correlations for this case have already been discussed in section 4.1. The effective Majorana mass m_{ee} characterizing the neutrinoless double beta decay amplitude for case of generalized $e - \mu$ symmetry is shown in figure 11.

Figure 11. Attainable values of the effective Majorana mass m_{ee} as a function of the lightest neutrino mass in the generalized $e - \mu$ reflection scheme. In the left panel the CP label Θ is varied over all its values, while we fix $\Theta = 2\pi/5$ and $\Theta = 3\pi/5$ in the right panel. Notice that the expected m_{ee} for $0 \leq \Theta < \pi/2$ and $\pi/2 \leq \Theta \leq \pi$ almost coincide in the case of IO. The red (blue) dashed lines delimit the most general allowed regions for IO (NO) neutrino mass spectrum obtained by varying the mixing parameters over their 3σ ranges [3]. The present most stringent upper limits $m_{ee} < 0.061$ eV from KamLAND-ZEN [49] and EXO-200 [50] is shown by horizontal grey band. The vertical grey band indicates the current sensitivity of cosmological data from the Planck collaboration [51].

As is clear from this figure, almost all the 3σ range values of the effective mass m_{ee} can be reproduced if the variation of the CP label Θ is taken into account. However, the allowed range of m_{ee} for a given fixed Θ is quite restricted, for both NO and IO. We note that due to the constraints on mixing angles and CP phases, the range of m_{lightest} in which m_{ee} can be very small for NO is considerably reduced. Along with the predictions for the neutrino oscillation parameters, the predicted range for m_{ee} can also be used to test the generalized $e - \mu$ reflection. We also indicate by the vertical grey band the sensitivity $\sum m_i < 0.230$ eV at 95% C.L. limit claimed by the Planck collaboration [51].

Similar neutrinoless double beta decay predictions can also be obtained for the cases of $e - \tau$ and $\mu - \tau$ symmetries, as shown in figure 12 and figure 13 respectively. One sees that the generalized CP symmetry allows only a restricted range for m_{ee} , and thus can be used to test the predictions of the generalized CP symmetries. In particular, if we fix the CP label Θ , then the allowed range for m_{ee} becomes much narrower for both mass orderings, and also for both cases of $e - \tau$ and $\mu - \tau$ symmetries. Future neutrinoless double beta decay experiments will probe almost all of the IO region, thus allowing to distinguish between $0 \leq \Theta < \pi/2$ and $\pi/2 \leq \Theta \leq \pi$.

Figure 12. The same as figure 11 for the generalized $e - \tau$ reflection. The CP label Θ is free on the left panel, and we fix $\Theta = \pi/9$ and $\Theta = 8\pi/9$ on the right panel. Notice that the IO: $0 \leq \Theta \leq \frac{\pi}{2}$ region (cyan) is very narrow and is close to the maximum m_{ee} for IO. This region can be tested very soon in neutrinoless double beta decay experiments.

Figure 13. The same as figure 11 for the generalized $\mu - \tau$ reflection. The CP label Θ is free on the left panel, and we fix $\Theta = \pi/6$ and $\Theta = 5\pi/6$ on the right panel. Notice that the IO: $0 \leq \Theta \leq \frac{\pi}{2}$ region (cyan) is very narrow and is close to the maximum m_{ee} for IO. This region can be tested very soon in neutrinoless double beta decay experiments.

6 Conclusion

The imposition of generalized CP symmetries provides a powerful framework for predicting the lepton mixing angles and phases. We have investigated the theory of generalized CP transformations acting on the mass matrices of charged leptons. We have considered in detail the case of generalized $e - \mu$, $e - \tau$ and $\mu - \tau$ symmetries. The basic tool is the Takagi factorization which is used to express the physical parameters of the lepton mixing matrix, three mixing angles, and three CP phases, in terms of a restricted set of independent “theory parameters” (labels) associated with a given choice of the CP transformation. Current best fit values of the mixing angles and the favored value for $\delta_{\text{CP}} = 3\pi/2$ constrain the allowed “theory” values of the labels characterizing the CP transformation, for example Θ . In each case we have obtained strong correlations involving the mixing angles and CP phases, valid both for Majorana and Dirac neutrinos see, for example, in eqs. (4.16), (4.36), (4.53). Specific benchmark model examples were given in the tables. Our predictions for the leptonic CP violating phase δ_{CP} , summarized in figure 8, provides model-independent probes of our underlying generalized CP symmetry approach at upcoming long baseline oscillation experiments, such as DUNE, aimed at the measurement of CP violation. For the case of Majorana neutrinos, our predictions for the neutrino mixing angles and phases also include the two Majorana phases, as seen in figures 9 and 10. We have also derived the resulting predictions for the effective mass parameter m_{ee} characterizing the neutrinoless double beta decay rates. Predicted ranges for m_{ee} in each case can be used to test the residual CP symmetry hypothesis at the upcoming generation of sensitive experiments, such as KamLAND-ZEN, CUORE, LEGEND, nEXO and NEXT. We would like to remind the readers that the effect of the CP labels α , β and γ is to shift the Majorana phases ϕ_{12} and ϕ_{13} . The numerical results in figures 11, 12 and 13 for m_{ee} are obtained under the assumption of $\alpha = \beta = \gamma = 0$.

Notice that, although we have treated in full generality the implications of generalized CP symmetries, we have not attempted to obtain a rationale for their possible origin. In fact this is an interesting open issue that deserves a dedicated study, which goes far beyond the scope of this present paper and that will be tackled elsewhere.

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