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Scattering of gravitational and electromagnetic waves off $AdS_2 \times S^2$ in extreme Reissner-Nordstrom

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ABSTRACT: The direct product of two-dimensional anti-de Sitter spacetime with a two-sphere is an exact solution of four-dimensional Einstein-Maxwell theory without a cosmological constant. In this paper, we analytically solve the coupled gravitational and electromagnetic perturbation equations of $AdS_2 \times S^2$ in Einstein-Maxwell theory. On the other hand, $AdS_2 \times S^2$ also describes the near-horizon region of the extreme Reissner-Nordstrom (ERN) black hole. We therefore also solve the connection problem: we show how the $AdS_2 \times S^2$ perturbation equations arise from an appropriate near-horizon approximation of the corresponding equations for the ERN and then, using matched asymptotic expansions, we analytically extend the $AdS_2 \times S^2$ solutions away from the near-horizon region connecting them with solutions in the far asymptotically flat region. From the point of view of ERN our results may be thought of as computing the classical scattering matrix for gravitational and electromagnetic waves which probe the near-horizon region of the black hole.

KEYWORDS: 2D Gravity, AdS-CFT Correspondence, Black Holes

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1 Introduction

AdS_2 is currently the focus of much activity in the context of two-dimensional quantum gravity in asymptotically anti-de Sitter spacetimes; see e.g. [1–3] and references therein. At the same time, AdS_2 has two major advantages over its higher dimensional counterparts in the AdS/CFT correspondence. The first advantage is that a direct product with a two-sphere produces an exact solution of four-dimensional pure Einstein-Maxwell theory without a cosmological constant, the so-called Bertotti-Robinson universe [4, 5]. The second advantage is that the same solution also describes the near-horizon geometry of the asymptotically flat extreme Reissner-Nordstrom black hole; see e.g. [6]. This situates $AdS_2 \times S^2$ in a privileged position for extending results produced in AdS/CFT investigations to the realm of four-dimensional classical and quantum gravity in asymptotically flat spacetimes. However, any such attempt requires that one first solves the *connection problem*: one needs to know how to extend near-horizon anti-de Sitter solutions to the far asymptotically flat region.

In this paper, we solve the connection problem for coupled gravitational and electromagnetic perturbations of $AdS_2 \times S^2$ in Einstein-Maxwell theory. We show how the $AdS_2 \times S^2$ perturbation equations arise from an appropriate near-horizon approximation of the corresponding equations for the full extreme Reissner-Nordstrom (ERN). We obtain analytically the exact solution to the $AdS_2 \times S^2$ perturbation equations. We then find exact analytic solutions for two more approximations of the full ERN perturbation equations:

an intermediate-region approximation whose solution is given by static perturbations and a far-region approximation whose solution gives the asymptotic behavior of the full ERN perturbations near infinity. Using the method of matched asymptotic expansions, we then glue the three solutions together and obtain the desired answer to our connection problem.

A reduction of the Reissner-Nordstrom perturbation equations to a set of two second-order radial differential equations may be found in Chapter 5 of the monograph [7] and references therein; for a generalization to higher dimensions see [8]. In general, these radial equations can only be solved numerically. The method of matched asymptotic expansions has been used in the context of wave equations of (near-)extreme rotating black holes as early as [9–11]; for an application to the geodesic equations see [12]. Previous work indicating that ERN perturbations are at least partially amenable to analytic treatment includes [13, 14].

This paper is organized as follows. Section 2 contains a telegraphic review of Reissner-Nordstrom, its extreme near-horizon limit, the decoupled $AdS_2 \times S^2$ geometry, and the Regge-Wheeler-Zerilli gauge for the perturbation equations. In section 3 we present a reduction of the ERN perturbation equations to a single fourth-order radial differential equation. This reduction is different from the standard one which produces two second-order equations, but it contains the same information. In section 4 we begin by setting up the matched asymptotic expansions for the full ERN radial equation. We then derive the corresponding $AdS_2 \times S^2$ equation as a near-horizon approximation of the ERN equation and solve it analytically. Finally, we solve the connection problem by finding analytic solutions to intermediate and far region approximations of the full ERN equation and matching. Section 5 specializes the connection formulas to the basis of solutions which are purely ingoing at the horizon in $AdS_2 \times S^2$ and purely outgoing near infinity in ERN.

The case of near-extreme Reissner-Nordstrom and near- $AdS_2 \times S^2$ is treated, to leading order in the deviation from extremality, in the companion paper [15].

2 Review of ERN, $AdS_2 \times S^2$, and Regge-Wheeler-Zerilli gauge

In this paper, we are concerned with the Einstein-Maxwell equations in four dimensions:¹

$$R_{\mu\nu} = 8\pi T_{\mu\nu}, \quad \nabla^\mu F_{\mu\nu} = 0, \quad (2.1)$$

with $4\pi T_{\mu\nu} = F_{\mu\rho}F_{\nu}{}^\rho - \frac{1}{4}g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Linearizing the equations around any solution we get

$$\begin{aligned} & \nabla^\sigma \nabla_\mu h_{\sigma\nu} + \nabla^\sigma \nabla_\nu h_{\sigma\mu} - \square h_{\mu\nu} - \nabla_\mu \nabla_\nu h \\ & = 2g_{\mu\nu} F_{\rho\lambda} F_\sigma{}^\lambda h^{\rho\sigma} - 4F_{\mu\rho} F_{\nu\sigma} h^{\rho\sigma} - F_{\rho\sigma} F^{\rho\sigma} h_{\mu\nu} + 4F_\mu{}^\lambda f_{\nu\lambda} + 4F_\nu{}^\lambda f_{\mu\lambda} - 2g_{\mu\nu} F^{\rho\sigma} f_{\rho\sigma}, \end{aligned} \quad (2.2)$$

$$\nabla^\mu f_{\mu\nu} - h^{\mu\sigma} \nabla_\sigma F_{\mu\nu} - \nabla_\mu h^{\mu\sigma} F_{\sigma\nu} - F^{\mu\sigma} \nabla_\mu h_{\sigma\nu} + \frac{1}{2}(\nabla^\mu h) F_{\mu\nu} = 0, \quad (2.3)$$

where $h_{\mu\nu} \equiv \delta g_{\mu\nu}$, $a_\mu \equiv \delta A_\mu$, $f_{\mu\nu} \equiv \partial_\mu a_\nu - \partial_\nu a_\mu = \delta F_{\mu\nu}$ and $h \equiv g^{\mu\nu} h_{\mu\nu}$.

¹ $G = c = 1$.

The Reissner-Nordstrom solution of the Einstein-Maxwell equations (2.1) reads:

$$ds^2 = - \left(1 - \frac{2M}{\hat{r}} + \frac{Q^2}{\hat{r}^2}\right) d\hat{t}^2 + \left(1 - \frac{2M}{\hat{r}} + \frac{Q^2}{\hat{r}^2}\right)^{-1} d\hat{r}^2 + \hat{r}^2 d\Omega^2, \quad \hat{A}_{\hat{t}} = -\frac{Q}{\hat{r}}. \quad (2.4)$$

Extreme Reissner-Nordstrom (ERN) is given by $Q = M$:

$$ds^2 = - \left(1 - \frac{M}{\hat{r}}\right)^2 d\hat{t}^2 + \left(1 - \frac{M}{\hat{r}}\right)^{-2} d\hat{r}^2 + \hat{r}^2 d\Omega^2, \quad \hat{A}_{\hat{t}} = -\frac{M}{\hat{r}}. \quad (2.5)$$

The Bertotti-Robinson universe, which arises naturally as the near-horizon limit of ERN, may be obtained as follows. Make the coordinate and gauge transformation,

$$r = \frac{\hat{r} - M}{M}, \quad t = \frac{\hat{t}}{M}, \quad A = \hat{A} + d\hat{t}, \quad (2.6)$$

to obtain

$$\frac{1}{M^2} ds^2 = - \left(\frac{r}{1+r}\right)^2 dt^2 + \left(\frac{r}{1+r}\right)^{-2} dr^2 + (1+r)^2 d\Omega^2, \quad A_t = M \frac{r}{1+r}, \quad (2.7)$$

and then take the limit $r \ll 1$ to get:

$$\frac{1}{M^2} ds^2 = -r^2 dt^2 + \frac{dr^2}{r^2} + d\Omega^2, \quad A_t = Mr. \quad (2.8)$$

This solves the Einstein-Maxwell equations (2.1) on its own. The metric in (2.8) is $AdS_2 \times S^2$. From now on we set $M = 1$.

Spherical symmetry allows one to separate the angular dependence in the linearized Einstein-Maxwell equations (2.2)–(2.3) using scalar, vector, and tensor spherical harmonics labeled by (l, m) , belonging to two sectors of opposite parity: “even” parity $(-1)^l$ and “odd” parity $(-1)^{l+1}$. In this paper, we will consider the even parity sector only because we are primarily interested in perturbations to the AdS_2 part of the near-horizon metric and the odd sector does not allow any such perturbations.² Due to the spherical symmetry one may also set $m = 0$ by an appropriate rotation. Time translation symmetry allows one to separate the time dependence as well using waves of definite energy ω . Finally, we move to the Regge-Wheeler-Zerilli gauge thereby putting the following ansatz for the metric and gauge field perturbations [16, 17]:

$$h_{\mu\nu} = \begin{pmatrix} -Y(r) & X(r) & 0 & 0 \\ & V(r) & 0 & 0 \\ & & K(r) & 0 \\ & & & K(r) \sin^2 \theta \end{pmatrix} e^{i\omega t} Y_{l,0}(\theta, \phi), \quad (2.9)$$

$$a_\mu = \begin{pmatrix} \chi(r) & \psi(r) & 0 & 0 \end{pmatrix} e^{i\omega t} Y_{l,0}(\theta, \phi). \quad (2.10)$$

Note that modes with $l \geq 2$ may be treated in a unified manner while the cases $l = 0$ and $l = 1$ are special and need to be treated separately.³ In this paper, we consider the $l \geq 2$ modes.

²Perturbations h_{tt}, h_{tr}, h_{rr} transform as scalars on the sphere and therefore have the parity $(-1)^l$ of the standard (scalar) spherical harmonic functions $Y_{l,m}$.

³Birkhoff’s theorem implies that the only $l = 0$ solutions are those corresponding to perturbations towards other Reissner-Nordstrom black holes parameterized by infinitesimal changes to the mass and charge parameters $\delta M, \delta Q$.

3 Reduction of the perturbation equations

Putting the ansatz (2.9)–(2.10) into the linearized Einstein-Maxwell equations (2.2)–(2.3), on the background of the ERN (2.7), we may reduce them to a single fourth-order differential equation for K ,

$$a_4(r)K'''' + a_3(r)K''' + a_2(r)K'' + a_1(r)K' + a_0(r)K = 0. \quad (3.1)$$

The expressions for the coefficients $a_i(r)$ may be found in appendix A. The solution to this differential equation fixes the remaining components of the gravitational and electromagnetic perturbation according to:

$$Y = -\frac{1}{(1+r)^6(l^2(l+1)^2 + 4(1+r)^2\omega^2)} \times [l(l+1)(1+r)^2r^4K'' - 2(1+r)r^2(l(l+1)r^2 - 2(1+r)^4\omega^2)K' + (2l(l+1)r^4 - (1+r)^5(l(l+1)(1+r) + 4r)\omega^2)K], \quad (3.2)$$

$$V = -\frac{(1+r)^4}{r^4}Y, \quad (3.3)$$

$$X = \frac{2i\omega((1+r)rK' - (1+r)K - r^3V)}{l(l+1)(1+r)r}, \quad (3.4)$$

$$\chi = \frac{r^3(1+r)K' - 2r^3K + r(1+r)^5Y' - (1+r)^5Y - r^4(1+r)V + i\omega r(1+r)^5X}{4r(1+r)^3}, \quad (3.5)$$

$$\psi = \frac{(1+r)(i\omega(1+r)K + i\omega r^2(1+r)V - 2rX - r^2(1+r)X')}{4r^2}. \quad (3.6)$$

The reduction described above is not the standard Regge-Wheeler-Zerilli one, who reduce to two second-order equations, but it contains the same information. The reader may wonder what is the reduction path that leads to the above results. If we denote by $\mathcal{E}_{\mu\nu}$ and $\mathcal{M}_\mu$ the linearized Einstein (2.2) and Maxwell (2.3) equations, respectively, then the reduction path may be described as follows. Begin by solving the equations $\mathcal{E}_{tr}, \mathcal{E}_{t\theta}, \mathcal{E}_{r\theta}, \mathcal{E}_{\phi\phi} - \sin^2\theta \mathcal{E}_{\theta\theta}$ which are algebraic equations for X, ψ, χ, V , respectively. This gives rise to (3.3)–(3.6). Then, using (3.3)–(3.6), equation $\mathcal{E}_{tt}$ becomes algebraic for Y and yields (3.2). At this stage, given (3.2)–(3.6), one is left with equations $\mathcal{E}_{\theta\theta}, \mathcal{M}_t, \mathcal{M}_r, \mathcal{M}_\theta$ for K (the remaining equations vanish identically). However, one finds that $\mathcal{M}_r, \mathcal{M}_\theta$ are trivial multiples of $\mathcal{E}_{\theta\theta}$ while $\mathcal{M}_t$ is a linear combination of $\mathcal{E}_{\theta\theta}$ and its derivative. This leaves only equation $\mathcal{E}_{\theta\theta}$ for K , which is equation (3.1).

4 $AdS_2 \times S^2$ answers and the connection problem solved

Consider the scattering of gravitational and electromagnetic waves in the full ERN space-time. Generic modes do not survive in the near-horizon limit. The modes which do survive and solve the $AdS_2 \times S^2$ perturbation equations are low energy modes:

$$\omega \ll 1. \quad (4.1)$$

In this regime, to leading order in ω , we may solve the scattering problem in the full ERN analytically using the method of matched asymptotic expansions as follows. Change variables in equation (3.1) according to

$$K(r) = \left(\frac{r}{1+r} \right)^2 H(r), \quad (4.2)$$

and solve the resulting equation for H ,

$$b_4(r)H'''' + b_3(r)H''' + b_2(r)H'' + b_1(r)H' + b_0(r)H = 0, \quad (4.3)$$

by dividing the spacetime into three regions:

$$\text{Near:} \quad r \ll 1 \quad (4.4)$$

$$\text{Static:} \quad \omega \ll r \ll 1/\omega \quad (4.5)$$

$$\text{Far:} \quad 1 \ll r \quad (4.6)$$

The intermediate Static region is overlapping with both the Near and the Far regions. Therefore, the Static solution may be used to connect the Near and Far solutions.

In the following, we will denote by $b_i^n(r)$, $b_i^s(r)$, and $b_i^f(r)$ the coefficients of eq. (4.3), in the Near, Static, and Far regions, respectively.

4.1 Near region

In the Near region, which corresponds to the $AdS_2 \times S^2$ throat, eq. (4.3) reduces to

$$\begin{aligned} b_4^n(r) &= r^8 \\ b_3^n(r) &= 16r^7 \\ b_2^n(r) &= -2(l^2 + l - 36)r^6 + 2r^4\omega^2 \\ b_1^n(r) &= -12(l^2 + l - 8)r^5 + 8r^3\omega^2 \\ b_0^n(r) &= (l-3)(l-1)(l+2)(l+4)r^4 - 2(l^2 + l - 2)r^2\omega^2 + \omega^4 \end{aligned}$$

and the solution is given by

$$H^n(r) = C_1^n r^{-3} h_{-l-2}^{(1)}(\omega/r) + C_2^n r^{-3} h_{-l-2}^{(2)}(\omega/r) + C_3^n r^{-3} h_{-l}^{(1)}(\omega/r) + C_4^n r^{-3} h_{-l}^{(2)}(\omega/r). \quad (4.7)$$

Here $h_m^{(1)}$, $h_m^{(2)}$ are the spherical Hankel functions of the first and second kind, respectively.⁴

⁴Recall that for $m = 0, 1, 2, \dots$ we have:

$$\begin{aligned} h_m^{(1)}(z) &= i^{-m-1} \frac{e^{iz}}{z} \sum_{k=0}^m (-1)^k \frac{(m+k)!}{k!(m-k)!} \frac{1}{(2iz)^k}, \\ h_m^{(2)}(z) &= i^{m+1} \frac{e^{-iz}}{z} \sum_{k=0}^m (-1)^k \frac{(m+k)!}{k!(m-k)!} \frac{1}{(2iz)^k}, \end{aligned}$$

while $h_{-m-1}^{(1)}(z) = i(-1)^m h_m^{(1)}(z)$ and $h_{-m-1}^{(2)}(z) = -i(-1)^m h_m^{(2)}(z)$.

This Near solution is an exact solution with respect to the Bertotti-Robinson universe. That is to say, using the perturbation ansatz (2.9)–(2.10) on the background of the Bertotti-Robinson universe (2.8), eq. (4.7) gives the exact solution of the linearized Einstein-Maxwell equations (2.2)–(2.3), provided that eqs. (3.2)–(3.6) are replaced by

$$\begin{aligned} Y &= -\frac{r^4 K'' - \omega^2 K}{l(l+1)}, & V &= -\frac{1}{r^4} Y, & X &= -\frac{2i\omega (K - rK')}{l(l+1)r}, \\ \chi &= -\frac{-r^3 K' + Y - rY' + r^4 V - i\omega r X}{4r}, & \psi &= \frac{i\omega K + i\omega r^2 V - 2rX - r^2 X'}{4r^2}, \end{aligned}$$

and eq. (4.2) is replaced by $K(r) = r^2 H(r)$.

4.2 Static region

In the Static region, which corresponds to setting $\omega = 0$, eq. (4.3) reduces to

$$\begin{aligned} b_4^s(r) &= r^4(1+r)^4 \\ b_3^s(r) &= 2r^3(1+r)^3(8+r) \\ b_2^s(r) &= -2r^2(1+r)^2(l(l+1)(1+r)^2 - 36 + 10r) \\ b_1^s(r) &= -2r(1+r)(l(l+1)(1+r)^2(6-r) - 48 + 94r - 20r^2) \\ b_0^s(r) &= l(l+1)(1+r)^2(l(l+1)(1+r)^2 - 2(1-r)(7-r)) + 8(3 - 26r + 29r^2 - 5r^3) \end{aligned}$$

and the solution is given by

$$H^s(r) = (1+r)^3 \left[C_1^s r^{l-1} + C_2^s r^{l-3}(l-r+2lr) + C_3^s r^{-l-2} + C_4^s r^{-l-4}(l+1+3r+2lr) \right]. \quad (4.8)$$

4.3 Far region

In the Far region, which includes asymptotically flat infinity, eq. (4.3) reduces to

$$\begin{aligned} b_4^f(r) &= r^4(l^2(l+1)^2 + 4r^2\omega^2)^2 \\ b_3^f(r) &= 2r^3(l^4(l+1)^4 - 16r^4\omega^4) \\ b_2^f(r) &= -2r^2[l^5(l+1)^5 - l^2(l+1)^2(l^2(l+1)^2 - 4l(l+1) - 4)r^2\omega^2 \\ &\quad - 8(l^2(l+1)^2 + 6)r^4\omega^4 - 16r^6\omega^6] \\ b_1^f(r) &= 2r[l^5(l+1)^5 + l^2(l+1)^2(l^2(l+1)^2 + 20l(l+1) + 8)r^2\omega^2 - 96r^4\omega^4 - 16r^6\omega^6] \\ b_0^f(r) &= l^6(l+1)^6 - 2l^5(l+1)^5 - 2l^2(l+1)^2(l^3(l+1)^3 - 2l^2(l+1)^2 + 20l(l+1) + 8)r^2\omega^2 \\ &\quad + (l^4(l+1)^4 - 8l^3(l+1)^3 + 56l^2(l+1)^2 + 192)r^4\omega^4 \\ &\quad + 8(l^2(l+1)^2 + 12)r^6\omega^6 + 16r^8\omega^8 \end{aligned}$$

and the solution is given by

$$\begin{aligned}
 H^f(r) = & C_1^f r\omega \left((l+1)(l+2)r\omega h_{l+2}^{(1)}(r\omega) - ((l+1)(l+2)(2l+3) - 2r^2\omega^2) h_{l+1}^{(1)}(r\omega) \right) \\
 & + C_2^f r\omega \left((l+1)(l+2)r\omega h_{l+2}^{(2)}(r\omega) - ((l+1)(l+2)(2l+3) - 2r^2\omega^2) h_{l+1}^{(2)}(r\omega) \right) \\
 & + C_3^f \left(r\omega (l(l+1)^3(l+2) - 2(l^2+l+2)r^2\omega^2) h_{l+2}^{(1)}(r\omega) \right. \\
 & \quad \left. - (l+2)(l(l+1)^3(2l+3) - (l+3)(l^2+l+2)r^2\omega^2) h_{l+1}^{(1)}(r\omega) \right) \\
 & + C_4^f \left(r\omega (l(l+1)^3(l+2) - 2(l^2+l+2)r^2\omega^2) h_{l+2}^{(2)}(r\omega) \right. \\
 & \quad \left. - (l+2)(l(l+1)^3(2l+3) - (l+3)(l^2+l+2)r^2\omega^2) h_{l+1}^{(2)}(r\omega) \right). \quad (4.9)
 \end{aligned}$$

4.4 Overlap regions, matching, and the solution to the connection problem

The Static region overlaps with the Near region in

$$\text{Near-Static overlap:} \quad \omega \ll r \ll 1 \quad (4.10)$$

where eq. (4.3) reduces to

$$r^4 H'''' + 16r^3 H''' - 2(l^2 + l - 36)r^2 H'' - 12(l^2 + l - 8)rH' + (l-3)(l-1)(l+2)(l+4)H = 0$$

and the solution is

$$H^{ns}(r) = C_1^{ns} r^{l-1} + C_2^{ns} r^{l-3} + C_3^{ns} r^{-l-2} + C_4^{ns} r^{-l-4}. \quad (4.11)$$

Expanding the Near solution (4.7) around infinity and the Static solution (4.8) around the origin, one may obtain linear relations between the C_i^n 's and C_i^s 's by matching the coefficients of the two expansions with the Near-Static solution (4.11).

The Static region overlaps with the Far region in

$$\text{Far-Static overlap:} \quad 1 \ll r \ll 1/\omega \quad (4.12)$$

where eq. (4.3) reduces to

$$r^4 H'''' + 2r^3 H''' - 2l(l+1)r^2 H'' + 2l(l+1)rH' + (l-1)l(l+1)(l+2)H = 0$$

and the solution is

$$H^{fs}(r) = C_1^{fs} r^{l+2} + C_2^{fs} r^{l+1} + C_3^{fs} r^{-l+1} + C_4^{fs} r^{-l}. \quad (4.13)$$

Expanding the Far solution (4.9) around the origin and the Static solution (4.8) around infinity, one may obtain linear relations between the C_i^f 's and C_i^s 's by matching the coefficients of the two expansions with the Far-Static solution (4.13).

Eliminating the C_i^s 's from the matching equations outlined above, we obtain the following linear relation between the C_i^f 's and C_i^n 's:

$$\begin{aligned}
 C_{12}^{m+} &= \frac{(-1)^{l+1}}{2^{2l+2}\pi} \frac{(l+1)(l+2)}{2l-1} \Gamma\left(-l-\frac{1}{2}\right)^2 \omega^{2l+3} \left((25l-8)\omega C_{12}^{f+} - 3l(l+1)^2(3l-1) C_{34}^{f+} \right) \\
 C_{12}^{m-} &= \frac{(-1)^l 2^{2l}}{\pi} (l-1)l(l+1)(2l+1) \Gamma\left(l+\frac{1}{2}\right)^2 \omega^{-2l-1} \left(3\omega C_{12}^{f-} + l^2(l+1) C_{34}^{f-} \right) \\
 C_{34}^{m+} &= \frac{(-1)^{l+1}}{2^{2l+2}\pi} l(l+1)(l+2)(2l+1) \Gamma\left(-l-\frac{1}{2}\right)^2 \omega^{2l+1} \left(3\omega C_{12}^{f+} - l(l+1)^2 C_{34}^{f+} \right) \\
 C_{34}^{m-} &= \frac{(-1)^l 2^{2l}}{\pi} \frac{(l-1)l}{2l+3} \Gamma\left(l+\frac{1}{2}\right)^2 \omega^{-2l+1} \left((25l+33)\omega C_{12}^{f-} + 3l^2(l+1)(3l+4) C_{34}^{f-} \right)
 \end{aligned} \tag{4.14}$$

where we have defined

$$\begin{aligned}
 C_{12}^{n\pm} &= C_1^n \pm C_2^n, & C_{34}^{n\pm} &= C_3^n \pm C_4^n, \\
 C_{12}^{f\pm} &= C_1^f \pm C_2^f, & C_{34}^{f\pm} &= C_3^f \pm C_4^f.
 \end{aligned} \tag{4.15}$$

This is the main result of the paper and completes the solution to our connection problem. The connection formulas (4.14) may also be thought of as computing the classical scattering matrix for gravitational and electromagnetic waves which probe the near-horizon region of extreme Reissner-Nordstrom.

5 A useful basis of solutions

A useful basis of solutions for the full ERN equation (4.3) consists of two solutions H_1, H_2 which are purely ingoing near the horizon and two solutions H_3, H_4 which are purely outgoing near infinity. Given that for $z \rightarrow \infty$ we have $h_m^{(1)}(z) \sim i^{-m-1} z^{-1} e^{iz}$ and $h_m^{(2)}(z) \sim i^{m+1} z^{-1} e^{-iz}$ this basis is defined by:

$$H_1 \quad \text{is the solution with:} \quad C_2^n = 1, C_1^n = C_3^n = C_4^n = 0, \tag{5.1}$$

$$H_2 \quad \text{is the solution with:} \quad C_4^n = 1, C_1^n = C_3^n = C_2^n = 0, \tag{5.2}$$

$$H_3 \quad \text{is the solution with:} \quad C_2^f = 1, C_1^f = C_3^f = C_4^f = 0, \tag{5.3}$$

$$H_4 \quad \text{is the solution with:} \quad C_4^f = 1, C_1^f = C_3^f = C_2^f = 0. \tag{5.4}$$

From the connection formulas (4.14) we find that:

H_1 has the Far amplitudes

$$\begin{aligned}
 C_1^f &= C_2^f = \frac{(-1)^l}{\pi} \frac{2^{2l+1} \Gamma\left(l+\frac{3}{2}\right)^2}{(l+1)(l+2)\omega^{2l+4}}, \\
 C_3^f &= C_4^f = \frac{(-1)^l}{\pi} \frac{3 \cdot 2^{2l+1} \Gamma\left(l+\frac{3}{2}\right)^2}{l(l+1)^3(l+2)\omega^{2l+3}},
 \end{aligned} \tag{5.5}$$

H_2 has the Far amplitudes

$$\begin{aligned}
 C_1^f &= C_2^f = \frac{(-1)^{l+1}}{\pi} \frac{3 \cdot 2^{2l+1} (3l-1) \Gamma\left(l+\frac{3}{2}\right)^2}{l(l+1)(l+2)(2l-1)(2l+1)\omega^{2l+2}}, \\
 C_3^f &= C_4^f = \frac{(-1)^{l+1}}{\pi} \frac{2^{2l+1} (25l-8) \Gamma\left(l+\frac{3}{2}\right)^2}{l^2(l+1)^3(l+2)(2l-1)(2l+1)\omega^{2l+1}},
 \end{aligned} \tag{5.6}$$

H_3 has the Near amplitudes

$$\begin{aligned} C_1^m = -C_2^m &= \frac{(-1)^{l+1}}{\pi} \frac{3 \cdot 2^{2l-1} (l-1) l (l+1) (2l+1) \Gamma(l + \frac{1}{2})^2}{\omega^{2l}}, \\ C_3^m = -C_4^m &= \frac{(-1)^{l+1}}{\pi} \frac{2^{2l-1} (l-1) l (25l+33) \Gamma(l + \frac{1}{2})^2}{(2l+3) \omega^{2l-2}}, \end{aligned} \quad (5.7)$$

H_4 has the Near amplitudes

$$\begin{aligned} C_1^m = -C_2^m &= \frac{(-1)^{l+1}}{\pi} \frac{2^{2l-1} (l-1) l^3 (l+1)^2 (2l+1) \Gamma(l + \frac{1}{2})^2}{\omega^{2l+1}}, \\ C_3^m = -C_4^m &= \frac{(-1)^{l+1}}{\pi} \frac{3 \cdot 2^{2l-1} (l-1) l^3 (l+1) (3l+4) \Gamma(l + \frac{1}{2})^2}{(2l+3) \omega^{2l-1}}. \end{aligned} \quad (5.8)$$

A word of caution is in order here. H_1, H_2 and H_3, H_4 differ from the standard “in” and “up” solutions⁵ one would define for ERN by infinitely oscillating phases at the horizon and infinity, respectively. For example, an “in” solution in ERN is defined as one that behaves like $e^{i\omega r_*}$ near the horizon, where $r_* = -1/r + 2 \ln r + r$ is the ERN tortoise coordinate. On the other hand, the $AdS_2 \times S^2$ tortoise coordinate is $r_* = -1/r$ so that the behavior of H_1, H_2 near the $AdS_2 \times S^2$ horizon agrees, for sufficiently small ω , with that of an “in” ERN solution everywhere near the horizon except precisely at the horizon where the two differ by the infinitely oscillating phase $e^{2i\omega \ln r}$. Similar comments apply to the behavior at infinity.

6 Conclusion

In this paper we have analytically solved the coupled gravitational and electromagnetic perturbation equations of the Bertotti-Robinson universe. Moreover, using matched asymptotic expansions, we have shown how to analytically extend these solutions to the asymptotically flat region of extreme Reissner-Nordstrom. We expect this extension to prove useful for transferring results from studies of AdS_2 holography to the realm of four-dimensional classical and quantum gravity in asymptotically flat spacetimes.

It has been argued that AdS_2 has no finite energy excitations [6]. The reason is that, with anti-de Sitter boundary conditions, backreaction in AdS_2 is so strong that it destroys its asymptotics. The reader might therefore wonder how do the results of this paper fit in that picture. We note that it is possible to study the backreaction problem using the linearized perturbations of $AdS_2 \times S^2$ that we have explicitly found in section 4.1. The technique uses the so-called “linearization stability constraints” and we expect that, with anti-de Sitter boundary conditions, one would confirm that the solutions of section 4.1 should be discarded. Indeed, we expect that the only solution which obeys asymptotically $AdS_2 \times S^2$ boundary conditions is diffeomorphic to $AdS_2 \times S^2$ itself; see e.g. [21]. However, the main focus of this paper is the connection with the asymptotically flat extreme Reissner-Nordstrom and in this context “leaky boundary conditions” at the location of the would-be

⁵The labels “in” and “up” for solutions purely ingoing at the horizon and purely outgoing at infinity, respectively, were introduced in [18] and have become standard in black hole perturbation theory [19, 20].

$AdS_2 \times S^2$ boundary are both allowed and desired. That is to say, energy is allowed to flow in and out of the anti-de Sitter boundary as needed in accordance with the boundary conditions imposed at asymptotically flat infinity.⁶ Consider for example the scattering problem set up by sending gravitational and electromagnetic waves from the asymptotically flat region in extreme Reissner-Nordstrom. The boundary conditions for such a problem are those of the solutions H_1 and H_2 in section 5. In this scattering problem, gravitational and electromagnetic energy flows in from outside the $AdS_2 \times S^2$ boundary, scatters off the horizon, and then flows out of the $AdS_2 \times S^2$ boundary, with a ratio that is dictated by the amplitudes in eqs. (5.5) and (5.6).

A model of two-dimensional anti-de Sitter gravity that takes into account backreaction is the Jackiw-Teitelboim theory [28–30] and it has been recently shown that the effective action which describes all gravitational dynamics in this theory is given by the Schwarzian derivative of an AdS_2 boundary reparametrization $t \rightarrow f(t)$ [31]. The Jackiw-Teitelboim theory describes the s-wave sector of higher dimensional theories with an AdS_2 throat geometry. In the context of this paper, this means that the Schwarzian action for a linearized reparametrization $t \rightarrow t + \epsilon(t)$ should capture the gravitational dynamics of the $l = 0$ solutions of the linearized Einstein-Maxwell equations in extreme Reissner-Nordstrom. However, Birkhoff’s theorem in four-dimensions implies that there are no nontrivial $l = 0$ solutions other than linear perturbations towards other Reissner-Nordstrom black holes parametrized by small changes $\delta M, \delta Q$ of the mass and charge of the black hole. Indeed, it is not difficult to verify (see e.g. [32]) that, up to the $SL(2, R)$ isometries of AdS_2 , the only solution to the Schwarzian action expanded to second order in ϵ is given by $\epsilon(t) = \alpha t^3$ with $\alpha \propto \delta M - \delta Q$.

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A The master radial equation for ERN perturbations

In section 3 we presented a reduction of the ERN perturbation equations to the fourth-order differential equation (3.1). The expressions for the coefficients $a_i(r)$ appearing in eq. (3.1) are given by:

$$\begin{aligned} a_4(r) &= r^8(1+r)^4 \left(l^2(1+l)^2 + 4(1+r)^2\omega^2 \right)^2 \\ a_3(r) &= 2r^7(1+r)^3 \left(l^4(1+l)^4(4+r) + 32l^2(1+l)^2(1+r)^2\omega^2 + 16(4-r)(1+r)^4\omega^4 \right) \end{aligned}$$

⁶For readers who are familiar with the NHEK geometry [22] in the extreme Kerr throat, we note that the situation here is exactly analogous to the one in Kerr/CFT [23]. Linearization stability constraints imply that, up to diffeomorphisms and boundary gravitons, asymptotically NHEK boundary conditions are consistent only with trivial solutions $h_{\mu\nu} = 0$ [24, 25]. On the other hand, with astrophysically meaningful boundary conditions imposed at future null infinity of extreme Kerr, refs. [26, 27] analytically computed the gravitational wave signals from extreme-mass-ratio-inspirals in NHEK.

$$\begin{aligned}
a_2(r) &= 2r^4(1+r)^2 \left[-l^4(1+l)^4 r^2 (l(1+l)(1+r)^2 + 4r - 6) \right. \\
&\quad + l^2(1+l)^2(1+r)^2 (l^2(1+l)^2(1+r)^4 - 4l(1+l)r^2(1+r)^2 + 4r^2(12-16r-r^2)) \omega^2 \\
&\quad \left. + 8(1+r)^4 (l^2(1+l)^2(1+r)^4 + 6r^2(2-4r+r^2)) \omega^4 + 16(1+r)^{10} \omega^6 \right] \\
a_1(r) &= 2r^4(1+r) \left[-l^4(1+l)^4 r (l(1+l)(2+3r-r^3) + 4r(3-2r)) \right. \\
&\quad + l^2(1+l)^2(1+r)^2 (l^2(1+l)^2(1+r)^4 - 4l(1+l)r(1+r)^2(2-5r) - 8r^2(12-16r-r^2)) \omega^2 \\
&\quad \left. - 96r^2(1+r)^4 (2-4r+r^2) \omega^4 - 16(1+r)^{10} \omega^6 \right] \\
a_0(r) &= l^4(1+l)^4 r^4 (l^2(1+l)^2(1+r)^4 - 2l(1+l)(1-r)^2(1+r)^2 + 8r^2(3-2r)) \\
&\quad - 2l^2(1+l)^2 r^2(1+r)^2 (l^3(1+l)^3(1+r)^6 + 2l^2(1+l)^2(1-r)r(1+r)^4 \\
&\quad + 4l(1+l)r^2(1+r)^2 (4-r+5r^2) - 8r^4(12-16r-r^2)) \omega^2 \\
&\quad + (1+r)^4 (l^4(1+l)^4(1+r)^8 - 8l^3(1+l)^3 r^2(1+r)^6 + 56l^2(1+l)^2 r^4(1+r)^4 \\
&\quad - 96l(1+l)(1-r)r^4(1+r)^2 + 192r^6(2-4r+r^2)) \omega^4 \\
&\quad + 8(1+r)^{10} (l^2(1+l)^2(1+r)^4 + 4r^3(2+3r)) \omega^6 + 16(1+r)^{16} \omega^8
\end{aligned}$$

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