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Black hole microstates from branes at angle

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ABSTRACT: We derive the leading g_s perturbation of the SUGRA fields generated by a supersymmetric configuration of respectively 1, 2 or 4 D3-branes intersecting at an arbitrary angle via the computation of the string theory disk scattering amplitude of one massless NSNS field interacting with open strings stretched between the branes. The configuration with four branes is expected to be relevant for black hole microstate counting in four dimensions.

KEYWORDS: Black Holes in String Theory, D-branes

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1 Introduction and conclusion

In a recent paper was explicitly uncovered an open-closed dictionary between microstates in string theory arising from open string stretched between D-branes and microstate geometries in supergravity for a system of 4 perpendicularly intersecting D3 branes [1]. This configuration of branes is particularly interesting, since it can be seen as the weak g_s coupling picture of a four dimensional black holes with non vanishing area in the strong coupling (SUGRA) picture and computation of its degeneracy should led to the microscopic calculation of the entropy [2, 3].

We stress that for microstate geometries one means SUGRA solutions with the same asymptotic charges as the corresponding black hole, but differing from it near the location of the black hole horizon. The basic idea is that the singular black hole arises as a coarse-grained description of the underlying regular microstates when we trace out the region inside the would-be horizon, that is where the microstates differ from each other. The fact

that these microstates can be found directly in SUGRA is the statement of the fuzzball proposal in a nutshell [4]. More precisely, the strongest form of the conjecture says that every string microstates is associated bijectively to regular and horizon-less microstates geometries. The beauty of this proposal lies in the possibility of solving the information loss paradox with tools already available, namely using mainly supergravity, branes and perturbative string theory. Of course a classical SUGRA solution must be thought as a very special state (coherent state) of the real fully quantum picture, therefore in order to completely describe the features of a quantum black hole, a better knowledge of non perturbative string theory will be required. Nevertheless, we could hope to catch the relevant physics in this approximation.

Much work has been done on 2 and 3 charge black holes, see for instance [5–12], but very little is known for the four charge case, corresponding to black holes in four dimensions (apparently the dimension in which we live). In this paper we generalize the work done in [1] by tilting the branes with some arbitrary angle, but still keeping the system supersymmetric. These configurations are relevant if one wants to study orbifold compactifications rather than the simple toroidal ones, leading to more realistic black hole models.

The strategy of the paper consists in deriving the leading g_s perturbation of the SUGRA fields generated by the configuration of tilted branes via the computation of the string theory disk¹ scattering amplitude of one massless NSNS field interacting with open strings binding the different branes. We begin the stringy computations by evaluating the amplitude for the emission of one closed NSNS field in the presence of one single stack of branes (section 1). To compute the amplitude with two different stacks (section 3) bound by open strings we first construct the open string vertex operators associated to every relevant pair of branes taken into account (section 2), paying attention to be consistent with the supersymmetry preserved by the system. Finally we perform the calculation for the 4 brane system (section 4), that it's the novel result of this letter.

Explicitly, once the amplitude $\mathcal{A}$ has been computed, to extract the leading perturbation is sufficient to use the effective field theory formula

$$\delta\tilde{\Phi}(k) = \left(-\frac{i}{k^2}\right) \frac{\delta\mathcal{A}(k)}{\delta\Phi} \quad (1.1)$$

In [1] explicit agreement was shown between the stringy calculations and a known extremal SUGRA solution, in line with the idea that these classical solutions can account for the microstates of the corresponding black hole. This was achieved by expanding the SUGRA solutions at first order in g_s and matching them with perturbations found from the formula (1.1). On the contrary, the emission found here are completely new and at the moment there is no known SUGRA solution in literature written in terms of arbitrary harmonic functions that can accommodate for the microstates found in string theory. We conjecture that such SUGRA solutions should exist and we give some arguments (section 5) supporting this thesis. It would be relevant to write explicitly such solution and we plan to address this issue in a future work.

¹The disk is the worldsheet surface associated to the leading g_s contribution of a mixed open-closed amplitudes.

Two important points were left open in [1] and are not closed here. First of all, since the number of disk with different boundaries grows with the product of the charges like $Q_1 Q_2 Q_3 Q_4$ and the SUGRA solution is dual to this configuration, we expect that there should be some way to compute the entropy directly from gravity, on the same line of previous successes for the D1D5 case [13, 14], without dealing with the complicated $\mathcal{N} = 4$ quantum mechanics on the branes worldvolume.² Second, to obtain the fuzzball of a black hole it is necessary to show that regular, horizonless and asymptotically flat solutions exist, and this means finding the explicit form of the harmonic function, similarly to what has been done in [16] for $AdS_2 \times S_2$ asymptotics. We plan to also come back on this issues in a future work.

2 One boundary amplitude

In this section we compute the disk amplitude for the emission of one closed NSNS field from a single stack of Dp-branes at angles,³ in the weak coupling regime. By using (1.1) we will be able to extract the corresponding gravitational background generated in the SUGRA limit.

This calculation has already been performed in [1] for perpendicular intersecting branes. The procedure is identical for tilted branes, the only difference is the expression of the reflection matrix R ; therefore in this section we simply state the result of the amplitude and setup notation for the next sections. The relevant vertex operators are

$$\begin{aligned} W_{NSNS}(z, \bar{z}) &= (ER)_{MN} e^{-\varphi} \psi^M e^{ikX}(z) e^{-\varphi} \psi^N e^{ikRX}(\bar{z}) \\ V_{\xi(\phi)}(x_a) &= \sum_{n=0}^{\infty} \xi_{i_1 \dots i_n} \partial X^{i_1}(x_1) \prod_{a=2}^n \int_{-\infty}^{\infty} \frac{dx_a}{2\pi} \partial X^{i_a}(x_a) \end{aligned} \quad (2.1)$$

where $E = h + b$ is the polarization, that encompasses both the graviton and the B-field, R is the reflection matrix encoding the boundary conditions of the open strings, φ is a scalar field that bosonize the superghost, and ψ and X are the worldsheet fermions and bosons respectively. While the first vertex describe closed strings, the second vertex stands for an arbitrary number of untwisted (with endpoints on the same stack of branes) open string scalar fields that can be inserted on the boundary, with some polarization $\xi_{i_1 \dots i_n}$. Remarkably, the insertion of these fields translates into the full multipole expansion of a generic harmonic function appearing in the SUGRA fields.

The amplitude to evaluate is:

$$\mathcal{A}_{NS-NS, \xi(\phi)} = \langle c(z) c(\bar{z}) c(z_1) \rangle \langle W_{NS-NS}(z, \bar{z}) V_{\xi(\phi)} \rangle \quad (2.2)$$

Left and right string sectors are related by:

$$X_{\text{right}}^M = R_N^M X_{\text{left}}^N \quad \psi_{\text{right}}^M = R_N^M \psi_{\text{left}}^N \quad \varphi_{\text{right}} = \varphi_{\text{left}} \quad (2.3)$$

²See for instance [15] for BPS state counting in a pure D-brane configuration, even though the calculation is performed in the limit of small charges, that is the opposite of the SUGRA limit.

³Since there is only one stack, the angle is between the branes and a fixed reference frame. This angle is actually meaningless in an uncompactified geometry, where it can be reabsorbed by a change of coordinates, while is meaningful and not completely arbitrary in a compact space.

The ten dimensional reflection matrix enforce the boundary conditions for branes at angle and therefore contains a non trivial internal part in T^6 :

$$R^M_N = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -\mathbb{1}_{3 \times 3} & 0 & 0 & 0 \\ 0 & 0 & R(\theta_1) & 0 & 0 \\ 0 & 0 & 0 & R(\theta_2) & 0 \\ 0 & 0 & 0 & 0 & R(\theta_3) \end{pmatrix} \quad (2.4)$$

where we have introduced the two dimensional matrices derived in the appendix:

$$R(\theta_i) = \begin{pmatrix} \cos(2\theta_i) & \sin(2\theta_i) \\ \sin(2\theta_i) & -\cos(2\theta_i) \end{pmatrix} \quad (2.5)$$

The final result is:

$$\mathcal{A}_{NSNS, \xi(\phi)} = (ER) \xi(k) \quad (2.6)$$

where $ER = E_{MN} R^N_P \eta^{PM}$ and $\xi(k) = \sum_{n=0}^{\infty} \xi_{i_1 \dots i_n} k^{i_1} \dots k^{i_n}$.

The emitted field is the graviton, therefore using (1.1) and Fourier transforming (setting here and in the following $\xi(k)$ to a constant for simplicity) with

$$\int \frac{d^3 k}{(2\pi)^3} \frac{1}{k^2} e^{ikx} = \frac{1}{4\pi} \frac{1}{r} \quad (2.7)$$

one finds that the deviation from flat space is given by the schematic form:

$$h_{MN} = \frac{(\eta R)_{MN}}{r} \quad (2.8)$$

where all the inessential constants have been absorbed into $\xi(k)$. Notice that the result is proportional to g_s , indeed the factor g_s^2 coming from the massless closed string propagator in (1.1) partially cancels with the g_s^{-1} associated to the disk (open strings don't carry additional g_s factors).

This result can be directly compared to the SUGRA metric

$$ds^2 = L^{-\frac{1}{2}} \left(-dt^2 + \sum_{I=1}^3 d\tilde{y}_I^2 \right) + L^{\frac{1}{2}} \left(\sum_{I=1}^3 dy_I^2 + \sum_{i=1}^3 dx_i^2 \right) \quad (2.9)$$

of D3 brane with tilted internal axes. Indeed (focusing on the first torus) performing the rotation in the internal tori as

$$\begin{aligned} \tilde{y}'_1 &= \cos(\theta) \tilde{y}_1 - \sin(\theta) y_1 \\ y'_1 &= \sin(\theta) \tilde{y}_1 + \cos(\theta) y_1 \end{aligned} \quad (2.10)$$

and expanding at the leading order in g_s with the ansatz $L = 1 + \frac{g_s}{r}$ one finds:

$$h_{MN} dx^M dx^N = \frac{g_s}{2} \left(dt^2 + \sum_{i=1}^3 dx_i^2 + \cos(2\theta) dy_1^2 + 2 \sin(2\theta) dy_1 d\tilde{y}_1 - \cos(2\theta) d\tilde{y}_1^2 \right) \quad (2.11)$$

having omitted a similar contribution from the other two internal torii. This expression matches what appears in (2.8) after reabsorbing the overall constants.

3 Vertex operators for branes at angle

Boundary conditions imposed by intersecting branes at angle force the open strings stretched between the branes to have non integer mode expansions, therefore leading to vertex operators involving angle-dependent bosonic and fermionic twist field. The T-dual picture is given by open strings ending on D-branes with magnetic fields switched on along the world-volume. In the following we will construct a supersymmetric system of 4 D3 branes at angles and then we will identify the vertex operators corresponding to each pair of branes.

Every Dp-brane imposes some restrictions on the spinors $\epsilon_{L,R}$ parametrizing the SUSY transformations generated by $\epsilon_L Q_L + \epsilon_R Q_R$, where $Q_{L,R}$ are the supercharges (with the same 10d chirality) of *IIB* superstring theory. Solutions to these constraints count the number of unbroken supercharges in the system and when angles are taken into account one finds that if the branes are related by $SU(3)$ rotations some supersymmetry is preserved [18]. In particular, we require that at least $\mathcal{N} = 1$ is preserved between every couples of branes, leading to a condition on the (relative) angles:

$$\theta_1 + \theta_2 + \theta_3 = 0 \quad \text{mod } 2\pi \quad (3.1)$$

where θ_I is the angle between two branes in the $\{y_I, \tilde{y}_I\}$ torus. To preserve $\mathcal{N} = 2$, (3.1) must be true and an angle must be zero, and to have $\mathcal{N} = 4$ all the angles need to be zero.

One can verify that the system of figure 1 satisfies (3.1) for all the six possible pairs of branes. The convention is that a positive angle is taken counterclockwise from the horizontal y direction, so for instance a right arrow stands for $\theta = 0$, a left arrow for $\theta = \pi$ and up arrow for $\theta = \pi/2$.

To be concrete, we construct the vertex operators for the fermionic open string stretched between the (ordered) pair $D3_0 D3_3$ (for a more general discussion on vertex operators at angle see [19]). In the canonical $-\frac{1}{2}$ superghost picture the vertex operator is:⁴

$$V_{\mu}^{\{03\}}(z_n) = e^{-\frac{\varphi}{2}} \mu_{\{03\}}^{\beta} S_{\beta}(\sigma_{1/2-\theta}^{(1)\dagger} e^{i\theta\varphi_1})(\sigma_{1/2-\theta}^{(2)} e^{-i\theta\varphi_2})(e^{-i\varphi_3/2}) e^{ik_n X} \quad (3.2)$$

where S_{β} is an $SO(1,3)$ spin field (see the appendix for notation and conventions) and μ^{β} is the fermionic polarization with Chan Paton indices not shown explicitly. In particular every polarization μ^{β} is actually a Chan-Paton matrix, for instance $(\mu_{\{03\}}^{ab})^{\beta}$ refers to the open string stretched between the brane a in the stack 0 and the brane b in the stack 3 (the order is important).

A bosonic twist $\sigma_{\xi}^{(I)}$ field must be inserted for every I -torus ($I = 1, 2, 3$) in which the branes are not parallel. The twist field is angle dependent and the angle ξ is computed by rotating the first brane, the $D3_0$, to the second brane, with a dagger if the rotation is clockwise and without dagger if it is counterclockwise. Of course $\sigma_{\theta}^{\dagger} = \sigma_{1-\theta}$. Similarly, one must add a fermionic twist $e^{i(\xi-\frac{1}{2})\varphi_I}$ for every torus. As a check one can verify that the conformal dimension⁵ of the vertex operator is equal to 1 and that the vertex is consistent

⁴The subscript μ in V_{μ} is a symbol for the polarization, not an index!

⁵The conformal dimension of a bosonic twist field is given by $h(\sigma_{\theta}) = \frac{1}{2}\theta(1-\theta)$. Other useful formulas are $h(e^{q\varphi}) = -\frac{1}{2}q^2 - q$ and $h(e^{i\lambda\varphi}) = \frac{1}{2}\lambda^2$.

Figure 1. D3-brane configuration in the internal tori.

with the SUSY condition (3.1). Indeed the action of the *IIB* positive chirality supercharge $Q^{+++++} = \oint dz e^{+\frac{i}{2}(\varphi_4+\varphi_5+\varphi_1+\varphi_2+\varphi_3)} e^{-\varphi/2}$ must be well defined, implying that no non-integer powers of z can appear in the commutator $[\epsilon Q, V_\mu^{(-1/2)}]$, which gives the susy-related bosonic vertex operator.

The opposite chirality fermion, the one stretched from $D3_3$ to $D3_0$ is recovered by doing the conjugation of the previous vertex:

$$V_{\bar{\mu}}^{\{30\}}(z_n) = e^{-\frac{\varphi}{2}} \bar{\mu}_{\{30\}}^{\dot{\beta}} C_{\dot{\beta}}(\sigma_{1/2-\theta}^{(1)} e^{-i\theta\varphi_1})(\sigma_{1/2-\theta}^{(2)\dagger} e^{i\theta\varphi_2})(e^{i\varphi_3/2}) e^{ik_n X} \quad (3.3)$$

4 Two boundary amplitude

In this section we evaluate the disk amplitude for the scattering of one closed NSNS fields with two open strings stretched between two stacks of intersecting tilted D3 branes. This case already illustrates the crucial features that will be present in the more physically interesting four boundary computation. Similar calculation without non trivial angles have already been performed for $D1D5$ and $D3D3'$ perpendicularly intersecting system in [1, 20, 21]. The system considered is a non threshold bound state of D-branes, meaning that the solution is not in a naive BPS superposition and individual branes cannot be freely separated (Higgs branch). From the microscopical point of view, this is signaled by the presence of a non zero open string condensate, or in other words of a non zero v.e.v. for the fields associated to open strings stretched between different branes.

The string amplitude to be computed is

$$\mathcal{A}_{\mu\bar{\mu}NSNS} = \int \frac{d^4 z}{V_{CKV}} \langle V_\mu^{(-1/2)}(z_1) V_{\bar{\mu}}^{(-1/2)}(z_2) W_{NSNS}^{(0,-1)}(z_3, z_4) \rangle \quad (4.1)$$

Brane	t	x_1	x_2	x_3	y_1	$\tilde{y}_1$	y_2	$\tilde{y}_2$	y_3	$\tilde{y}_3$
$D3$	—	.	.	.	.	—	.	—	.	—
$D3'$	—	.	.	.	θ		$\pi - \theta$		.	—

Table 1. D3-brane configuration: Neumann and Dirichlet directions are lines and dots respectively. The angle is taken with respect to the non tilded y coordinate.

in which V_{CKV} is the conformal Killing volume, z_1 and z_2 are real, while $z_3 = \bar{z}_4$ is complex.

The brane configuration can be taken to be the $D3_0 D3_3$ as seen before. Here we rewrite the vertex operators needed for the calculation, with a superscript indicating the superghost charge:

$$V_{\mu}^{(-1/2)}(z_1) = \mu_{\alpha} S^{\alpha} e^{-\varphi/2} (\sigma_{1/2-\theta}^{(1)\dagger} e^{i\theta\varphi_1}) (\sigma_{1/2-\theta}^{(2)} e^{-i\theta\varphi_2}) (e^{-i\varphi_3/2}) e^{ik_1 X} \quad (4.2)$$

$$V_{\bar{\mu}}^{(-1/2)}(z_2) = \bar{\mu}_{\dot{\alpha}} C^{\dot{\alpha}} e^{-\varphi/2} (\sigma_{1/2-\theta}^{(1)} e^{-i\theta\varphi_1}) (\sigma_{1/2-\theta}^{(2)\dagger} e^{i\theta\varphi_2}) (e^{i\varphi_3/2}) e^{ik_2 X} \quad (4.3)$$

$$W_{NSNS}^{(0,-1)}(z_3, z_4) = (ER)_{MN} (\partial X^M - ik\psi\psi^M) e^{ikX}(z_3) e^{-\varphi} \psi^N e^{ikRX}(z_4) \quad (4.4)$$

The reflection matrix can be choosen to be the one associated to the $D3_0$ or the one associated to the $D3_3$, the final result will be independent of this choice. The spinorial structure of the condensate v is:

$$\mu^{\alpha} \bar{\mu}^{\dot{\beta}} = (\bar{\sigma}^{\mu})^{\dot{\beta}\alpha} v_{\mu} \quad (4.5)$$

The open string condensate microscopically encodes the different microstates of the black hole geometries. Notice that the condensate v can be linked to the profile function $f(v)$ appearing as a displacement of the center of the $D1D5$ fuzzball harmonic functions or alternatively as the profile of the oscillation of the string in the U-dual F1-P (fundamental string with momentum) system [22–26]. More precisely the condensate is linked to the derivative of $f(v)$ [20], therefore in some sense it gives the dynamic part of the displacement, whether the fixed part can be encoded in the untwisted scalars, as we have already said. While here the condensate is an arbitrary real number, in a proper treatment based on the microscopic theory on the world-volume of the D-branes it should be possible to quantize it, matching all the possible open string configuration to the respective condensate.

In the following we focus on constants open string fields, therefore we can consistently take all the open strings momenta to be vanishing. As we will see, care is needed in taking the zero momentum limit, indeed the amplitude is irreducible only if the disk diagram cannot factorize into two or more disk diagrams connected by propagating massless open fields. If the diagram factorize, the amplitude has massless open string poles at zero momentum exchanged; the associated worldsheet integral is divergent and there is no corresponding SUGRA emission. The closed string momentum satisfies $k^2 = 0$ and $k^M E_{MN} = 0$, moreover we concentrate on a purely spatial momentum $k^M = \{k^0, k^i, k^{\text{int}}\} = \{0, k^i, 0\}$, analytically continuing the momentum to complex values in order to be consistent with the mass-shell condition.

Here we report directly the result of the computation, the reader interested in the details can found the explicit derivation in the appendix. The non compact part and the

$(M, N) = (\bar{3}, 3)$ component have a finite amplitude given by:

$$\mathcal{A}_{\mu\bar{\mu}NSNS}^{NC} = \frac{\pi}{\sqrt{2}}(ER)(kv) \quad (4.6)$$

$$\mathcal{A}_{\mu\bar{\mu}NSNS}^{3\bar{3}} = \frac{-\pi}{\sqrt{2}}(ER)_{[\bar{3}\bar{3}]}(kv) \quad (4.7)$$

with $vk = v^i k^i$, and $(\bar{3}, 3)$ being the complex directions defined from the real ones $(\bar{3}, 3)$. On the contrary the tilted directions give rise to divergent or vanishing amplitudes, depending on the symmetry property of the polarizations. The divergent part require some clarification: it implies that there is no associated SUGRA emission for that choice of the condensate. What's happening can be understood clearly from the $SO(1, 5)$ picture of the directions $\{t, x_1, x_2, x_3, y_3, \tilde{y}_3\}$: the fermionic condensate $\mu^A \bar{\mu}^B$, where $A, B = 1, \dots, 4$, factorizes as $4 \times 4 = \mathbf{6} + \mathbf{10}$. The $\mathbf{6}$, the anti-symmetric part of the product, is simply the vectorial representation, corresponding to the $\Gamma_{[AB]}^{\hat{M}}$ ($\hat{M} = 1, \dots, 6$) appearing in the fermionic kinetic term in the effective lagrangian, therefore giving a factorization channel for the $NSNS$ -fermion-fermion diagram into two diagrams linked by a massless scalar field $\phi_{[AB]}$. To get a non factorizable diagram, one must restrict to the symmetric part of the product for which no fields with suitable indices are present.

A simple way to choose the symmetric part is to restrict to $S^A = S^B = e^{\frac{i}{2}(-\varphi_3 - \varphi_4 - \varphi_5)}$. The vertices to employ are then:

$$V_{\mu}^{(-1/2)}(z_1) = \mu_A S^A e^{-\varphi/2} (\sigma_{1/2-\theta}^{(1)\dagger} e^{i\theta\varphi_1}) (\sigma_{1/2-\theta}^{(2)} e^{-i\theta\varphi_2}) e^{ik_1 X} \quad (4.8)$$

$$V_{\bar{\mu}}^{(-1/2)}(z_2) = \bar{\mu}_B S^B e^{-\varphi/2} (\sigma_{1/2-\theta}^{(1)} e^{-i\theta\varphi_1}) (\sigma_{1/2-\theta}^{(2)\dagger} e^{i\theta\varphi_2}) e^{ik_2 X} \quad (4.9)$$

resulting in the $SO(1, 5)$ amplitude:

$$\mathcal{A}_{\mu\bar{\mu}NSNS}^{SO(1,5)} = \frac{1}{3!} (ER)_{\hat{M}\hat{N}} k_{\hat{P}} v^{\hat{M}\hat{N}\hat{P}} \quad (4.10)$$

having defined a three index condensate contracting $\Gamma_{(AB)}^{\hat{M}\hat{N}\hat{P}}$ with $\mu^{(A} \bar{\mu}^{B)}$ and having used hatted $SO(1, 5)$ indices. Notice that in the final results the angles disappeared altogether. In order to see a non trivial angular dependence one must break the $SO(1, 5)$ symmetry by tilting even the third torus, therefore dealing with three point correlation functions of twist fields. Such correlators will be present in the next section, resulting in an angular dependence of the amplitudes.

5 Four boundary amplitude

Finally we consider the 1/8 BPS configuration of the 4 stack of branes oriented like in figure 1. The string amplitudes correspond to a disk with four different boundaries and polarizations μ chosen in a cyclic order starting and ending on $D3_0$. The associated condensate is complex, meaning that one can run the cycle in two different directions using μ or $\bar{\mu}$ respectively.

The amplitude to be computed is:

$$\mathcal{A}_{\mu^4 NSNS} = \int \frac{d^4 z}{V_{CKV}} \langle V_{\mu}^{(-1/2)}(z_1) V_{\mu}^{(-1/2)}(z_2) V_{\mu}^{(-1/2)}(z_3) V_{\mu}^{(-1/2)}(z_4) W_{NSNS}^{(0,0)}(z_5, z_6) \rangle \quad (5.1)$$

Brane	t	x_1	x_2	x_3	y_1	$\tilde{y}_1$	y_2	$\tilde{y}_2$	y_3	$\tilde{y}_3$
$D3_0$	—	.	.	.	.	—	.	—	.	—
$D3_1$	—	.	.	.	.	—	$\rightarrow$	.	$\leftarrow$	.
$D3_2$	—	.	.	.	$\leftarrow$	.	.	—	$\rightarrow$	.
$D3_3$	—	.	.	.	θ		$\pi - \theta$		.	—

Table 2. D3-brane configuration: Neumann and Dirichlet directions are lines and dots respectively. The arrow distinguishes between the $\theta = 0$ (right arrow) and $\theta = \pi$ (left arrow) case. The angle is taken with respect to the non tilded y coordinate.

with vertex operators constructed as previously explained:

$$V_\mu^{\{01\}}(z_1) = e^{-\frac{\varphi}{2}} \mu_{\{01\}}^\alpha S_\alpha(e^{-i\varphi_1/2})(\sigma_{1/2}^{(2)\dagger})(\sigma_{1/2}^{(3)})e^{ik_1X} \quad (5.2)$$

$$V_\mu^{\{12\}}(z_2) = e^{-\frac{\varphi}{2}} \mu_{\{12\}}^\beta S_\beta(\sigma_{1/2}^{(1)})(\sigma_{1/2}^{(2)})(e^{-i\varphi_3/2})e^{ik_2X} \quad (5.3)$$

$$V_\mu^{\{23\}}(z_3) = e^{-\frac{\varphi}{2}} \mu_{\{23\}}^{\dot{\alpha}} C_{\dot{\alpha}}(\sigma_\theta^{(1)}e^{i\varphi_1(\theta+1/2)})(\sigma_{1/2-\theta}^{(2)}e^{-i\theta\varphi_2})(\sigma_{1/2}^{(3)})e^{ik_3X} \quad (5.4)$$

$$V_\mu^{\{30\}}(z_4) = e^{-\frac{\varphi}{2}} \mu_{\{30\}}^{\dot{\beta}} C_{\dot{\beta}}(\sigma_{1/2-\theta}^{(1)}e^{-i\theta\varphi_1})(\sigma_{1/2-\theta}^{(2)\dagger}e^{i\theta\varphi_2})(e^{i\varphi_3/2})e^{ik_4X} \quad (5.5)$$

$$W^{(0,0)}(z_5, z_6) = c_{\text{NS}}(ER)_{MN}(\partial X^M - ik\psi\psi^M)e^{ikX}(z_5)(\partial X^N - ikR\psi\psi^N)e^{ikRX}(z_6) \quad (5.6)$$

It's convenient to take the choice $\mu_{\{01\}}^{(\alpha)} \mu_{\{12\}}^{(\beta)} \bar{\mu}_{\{23\}}^{(\dot{\alpha})} \bar{\mu}_{\{30\}}^{(\dot{\beta})}$ for the condensate. This configuration lives in the $(\mathbf{3}_L, \mathbf{3}_R)$, that can be seen as a symmetric traceless tensor v^{ij} , and leads to non factorizable diagrams.

In particular let's fix:

$$S_\alpha = S_\beta = e^{\frac{i}{2}(\varphi_4 + \varphi_5)} \quad C_{\dot{\alpha}} = C_{\dot{\beta}} = e^{-\frac{i}{2}(\varphi_4 - \varphi_5)} \quad (5.7)$$

therefore the only way to saturate the +2 superghost charge in the $\{5, \bar{5}\}$ torus is taking the term with four fermions in the closed string vertex. We split the calculation in non-compact and compact directions. We start by computing the correlators common to both cases:

$$\begin{aligned} \langle c(z_1) c(z_2) c(z_4) \rangle &= z_{12} z_{24} z_{41} \\ \left\langle \prod_{j=1}^4 e^{-\varphi/2}(z_j) \right\rangle &= \prod_{i < j} z_{ij}^{-1/4} \end{aligned} \quad (5.8)$$

The twist fields correlator in the third torus is a two point function, while in the other torus we have a three and a four point function. The latter correlators in principle receive a classical contribution proportional to the area of the polygon formed by the intersecting branes, but it can be neglected in the SUGRA limit ([19, 27]). The explicit expressions for the quantum part are:

$$\begin{aligned} \langle \sigma_{1/2}^{(3)}(z_1) \sigma_{1/2}^{(3)}(z_3) \rangle &= z_{13}^{-\frac{1}{4}} \\ \langle \sigma_{1/2}^{(1)}(z_2) \sigma_\theta^{(1)}(z_3) \sigma_{1/2-\theta}^{(1)}(z_4) \rangle &= (4\pi\Gamma_{\{\frac{1}{2}, \theta, \frac{1}{2}-\theta\}})^{\frac{1}{4}} z_{23}^{-\frac{\theta}{2}} z_{24}^{+\frac{\theta}{2}-\frac{1}{4}} z_{34}^{-\frac{\theta}{2}+\theta^2} \\ \langle \sigma_{1/2}^{(2)}(z_1) \sigma_{1/2}^{(2)}(z_2) \sigma_{1/2-\theta}^{(2)}(z_3) \sigma_{1/2+\theta}^{(2)}(z_4) \rangle &= z_{12}^{-\frac{1}{4}} z_{34}^{-\frac{1}{4}+\theta^2} \left(\frac{z_{13}z_{24}}{z_{14}z_{23}} \right)^{\frac{1}{4}} \mathcal{I}^{-\frac{1}{2}}[w] \end{aligned} \quad (5.9)$$

where we have used

$$\langle \sigma_\alpha(z_1) \sigma_\beta(z_2) \sigma_\gamma(z_3) \rangle = (4\pi \Gamma_{\{\alpha, \beta, \gamma\}})^{\frac{1}{4}} z_{12}^{-\alpha\beta} z_{13}^{-\alpha\gamma} z_{23}^{-\beta\gamma} \quad (5.10)$$

$$\Gamma_{\alpha, \beta, \gamma} = \frac{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(1-\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)} \quad (5.11)$$

$$\langle \sigma_{1-a}(z_1) \sigma_a(z_2) \sigma_{1-b}(z_3) \sigma_b(z_4) \rangle = z_{12}^{-a(1-a)} z_{34}^{-b(1-b)} \left(\frac{z_{13}z_{24}}{z_{14}z_{23}} \right)^{\frac{1}{2}(a+b)-ab} \mathcal{I}^{-\frac{1}{2}}(w) \quad (5.12)$$

$$\mathcal{I}[w] = \frac{1}{2\pi} (B_1(a, b) G_2(w) H_1(1-w) + B_2(a, b) G_1(w) H_2(1-w)) \quad (5.13)$$

$$B_1(a, b) = \frac{\Gamma(a)\Gamma(1-b)}{\Gamma(1+a-b)} \quad B_2(a, b) = \frac{\Gamma(b)\Gamma(1-a)}{\Gamma(1+b-a)} \quad (5.14)$$

$$G_1(w) = {}_2F_1[a, 1-b; 1; w] \quad G_2(w) = {}_2F_1[1-a, b; 1; w] \quad (5.15)$$

$$H_1(w) = {}_2F_1[a, 1-b; 1+a-b; w] \quad H_2(w) = {}_2F_1[1-a, b; 1-a+b; w] \quad (5.16)$$

$$w = \frac{z_{12}z_{34}}{z_{13}z_{24}} = 1-x \quad (5.17)$$

where the last equality is valid after the gauge fixing

$$z_1 = -\infty \quad z_2 = 0 \quad z_3 = x \quad z_4 = 1 \quad z_5 = z \quad z_6 = \bar{z} \quad (5.18)$$

5.1 Fermions in extended space-time dimensions

Now we specialize to the amplitude with the choice $(M, N) = (4, \bar{4})$: all the worldsheet fermions ψ are in the extended space-time directions.

$$\begin{aligned} \left\langle e^{-i\varphi_1/2}(z_1) e^{i\varphi_1(\theta+1/2)}(z_3) e^{-i\theta\varphi_1}(z_4) \right\rangle &= z_{13}^{-\frac{\theta}{2}-\frac{1}{4}} z_{14}^{\frac{\theta}{2}} z_{34}^{-\theta^2-\frac{\theta}{2}} \\ \left\langle e^{-i\theta\varphi_2}(z_3) e^{i\theta\varphi_2}(z_4) \right\rangle &= z_{34}^{-\theta^2} \\ \left\langle e^{-i\varphi_3/2}(z_2) e^{i\varphi_3/2}(z_4) \right\rangle &= z_{24}^{-\frac{1}{4}} \end{aligned} \quad (5.19)$$

The correlator of the four spin fields and four fermions fields after bosonization is:

$$\left\langle e_{(z_1)}^{\frac{i}{2}(\varphi_4+\varphi_5)} e_{(z_2)}^{\frac{i}{2}(\varphi_4+\varphi_5)} e_{(z_3)}^{-\frac{i}{2}(\varphi_4-\varphi_5)} e_{(z_4)}^{-\frac{i}{2}(\varphi_4-\varphi_5)} e_{(z_5)}^{i(\varphi_4-\varphi_5)} e_{(z_6)}^{i(-\varphi_4-\varphi_5)} \right\rangle = \frac{1}{4} z_{12}^{\frac{1}{2}} z_{16}^{-1} z_{26}^{-1} z_{34}^{\frac{1}{2}} z_{35}^{-1} z_{45}^{-1} \quad (5.20)$$

Using the gauge fixing (5.18), the remaining worldsheet triple integral is:⁶

$$I_4 = \int_0^1 dx (-x)^{-\frac{1}{2}-\frac{\theta}{2}} (x-1)^{-\theta} \mathcal{I}^{-\frac{1}{2}}[1-x] \int_{\mathbb{C}^+} \frac{d^2 z}{\bar{z}(1-x)(x-z)} \quad (5.21)$$

that can be reduced to a single integration by computing the integral in the upper half of the complex plane (for instance using $z = re^{i\alpha}$ and performing the r integration first):

$$\int_{\mathbb{C}^+} \frac{d^2 z}{\bar{z}(1-x)(x-z)} = -\frac{\pi \log(x)}{(x-1)} \quad (5.22)$$

⁶Notice that by putting $\theta = 0$ in one recover the case of perpendicular branes, involving the correlator of 4 $\mathbb{Z}_2$ twist fields [1]. In particular $\mathcal{I}^{-\frac{1}{2}}[1-x]_{\theta=0} = ({}_2F_1[\frac{1}{2}, \frac{1}{2}; 1; x] {}_2F_1[\frac{1}{2}, \frac{1}{2}; 1; 1-x])^{-1/2}$.

The remaining integral must be performed numerically and is a finite overall constant that can be reabsorbed into the open string condensate. In the particular case of $\theta = 0$ the integral is purely imaginary, but for a generic angle the result has a real part too.

Recombining all the results obtained so far one obtains the amplitude:

$$\mathcal{A}_{\mu^4 NSNS}^{NC} = (ER)_{4\bar{4}} k_i k_j v^{ij}(\theta) \quad (5.23)$$

where all the constants, group theory factors and the Chan-Paton trace over the polarizations have been reabsorbed into the symmetric traceless condensate $v^{ij}(\theta)$. The real and imaginary components of the amplitude contribute respectively to the symmetric and anti-symmetric parts of $(ER)_{M\bar{M}}$, therefore for a generic angle and generic complex condensate the system will emit both the graviton and the B-field.⁷ The associated harmonic is given by $\frac{3x_j x_i - \delta_{ij} r^2}{r^5}$, having used

$$\int \frac{d^3 k}{(2\pi)^3} \frac{-ik_j k_i}{|k|^2} e^{ikx} = \frac{i}{4\pi} \frac{3x_j x_i - \delta_{ij} |x|^2}{|x|^5}. \quad (5.24)$$

5.2 Fermions in compact dimensions

Now we focus on the choice $(M, N) = (J, \bar{J})$ with $J = 1, 2, 3$. We have already computed the correlators (5.8) and (5.9). The correlator of spin fields and fermions is the same for all J :

$$\left\langle S_{(\alpha}(z_1) S_{\beta)}(z_2) C_{(\dot{\alpha}}(z_3) C_{\dot{\beta})}(z_4) \psi^\mu(z_5) \psi^\nu(z_6) \right\rangle = \frac{(z_{12} z_{34})^{1/2} z_{56}}{2 \prod_{i=1}^4 (z_{i5} z_{i6})^{1/2}} \sigma_{\alpha\dot{\alpha}}^{(\mu} \sigma_{\beta\dot{\beta}}^{\nu)} \quad (5.25)$$

In the following we list the remaining correlators and the final gauge fixed integrals.

- $(M, N) = (1, \bar{1})$

$$\begin{aligned} \left\langle e_{(z_1)}^{-i\varphi_1/2} e_{(z_3)}^{i\varphi_1(\theta+1/2)} e_{(z_4)}^{-i\theta\varphi_1} \psi_{(z_5)}^1 \psi_{(z_6)}^{\bar{1}} \right\rangle &= \frac{1}{2} z_{13}^{-\frac{\theta}{2}-\frac{1}{4}} z_{14}^{\frac{\theta}{2}} z_{34}^{-\theta^2-\frac{\theta}{2}} z_{15}^{-\frac{1}{2}} z_{16}^{\frac{1}{2}} z_{35}^{\frac{\theta}{2}+\frac{1}{2}} z_{46}^{-\frac{\theta}{2}-\frac{1}{2}} z_{45}^{-\theta} z_{46}^{\theta} z_{56}^{-1} \\ \left\langle e^{-i\theta\varphi_2}(z_3) e^{i\theta\varphi_2}(z_4) \right\rangle &= z_{34}^{-\theta^2} \\ \left\langle e^{-i\varphi_3/2}(z_2) e^{i\varphi_3/2}(z_4) \right\rangle &= z_{24}^{-\frac{1}{4}} \end{aligned} \quad (5.26)$$

$$I_1 = \int_0^1 dx (-x)^{-\frac{1}{2}-\frac{\theta}{2}} (x-1)^{-\theta} \mathcal{I}^{-\frac{1}{2}} [1-x] \int_{\mathbb{C}^+} \frac{d^2 z (x-z)^\theta (x-\bar{z})^{\theta-1}}{|z|(1-z)^{\theta+1/2} (1-\bar{z})^{-\theta+1/2}} \quad (5.27)$$

- $(M, N) = (2, \bar{2})$

$$\begin{aligned} \left\langle e^{-i\varphi_1/2}(z_1) e^{i\varphi_1(\theta+1/2)}(z_3) e^{-i\theta\varphi_1}(z_4) \right\rangle &= z_{13}^{-\frac{\theta}{2}-\frac{1}{4}} z_{14}^{\frac{\theta}{2}} z_{34}^{-\theta^2-\frac{\theta}{2}} \\ \left\langle e^{-i\theta\varphi_2}(z_3) e^{i\theta\varphi_2}(z_4) \psi^2(z_5) \psi^{\bar{2}}(z_6) \right\rangle &= \frac{1}{2} z_{34}^{-\theta^2} z_{35}^{-\theta} z_{36}^{\theta} z_{45}^{\theta} z_{46}^{-\theta} z_{56}^{-1} \\ \left\langle e^{-i\varphi_3/2}(z_2) e^{i\varphi_3/2}(z_4) \right\rangle &= z_{24}^{-\frac{1}{4}} \end{aligned} \quad (5.28)$$

$$I_2 = \int_0^1 dx (-x)^{-\frac{1}{2}-\frac{\theta}{2}} (x-1)^{-\theta} \mathcal{I}^{-\frac{1}{2}} [1-x] \int_{\mathbb{C}^+} \frac{d^2 z (x-z)^{-\theta-1/2} (x-\bar{z})^{\theta-1/2}}{|z|(1-z)^{-\theta+1/2} (1-\bar{z})^{\theta+1/2}} \quad (5.29)$$

⁷It's nevertheless possible to select the complex open string condensate in such way that only the graviton is emitted.

- $(M, N) = (3, \bar{3})$

$$\begin{aligned}
\left\langle e^{-i\varphi_1/2}(z_1)e^{i\varphi_1(\theta+1/2)}(z_3)e^{-i\theta\varphi_1}(z_4) \right\rangle &= z_{13}^{-\frac{\theta}{2}-\frac{1}{4}} z_{14}^{\frac{\theta}{2}} z_{34}^{-\theta^2-\frac{\theta}{2}} \\
\left\langle e^{-i\theta\varphi_2}(z_3)e^{i\theta\varphi_2}(z_4) \right\rangle &= z_{34}^{-\theta^2} \\
\left\langle e^{-i\varphi_3/2}(z_2)e^{i\varphi_3/2}(z_4)\psi^3(z_5)\psi^{\bar{3}}(z_6) \right\rangle &= \frac{1}{2} z_{24}^{-\frac{1}{4}} z_{25}^{-\frac{1}{2}} z_{26}^{\frac{1}{2}} z_{45}^{\frac{1}{2}} z_{46}^{-\frac{1}{2}} z_{56}^{-1} \quad (5.30)
\end{aligned}$$

$$I_3 = \int_0^1 dx (-x)^{-\frac{1}{2}-\frac{\theta}{2}} (x-1)^{-\theta} \mathcal{I}^{-\frac{1}{2}}[1-x] \int_{\mathbb{C}^+} \frac{d^2 z}{z(1-\bar{z})|x-z|} \quad (5.31)$$

In all these cases for a generic angle and generic complex condensate both the graviton and the B-field are emitted.

6 Comments on the supergravity solution

It's very common in supergravity to find extremal solutions parametrized by arbitrary harmonic functions. The prototypical example is the Reissner-Nordstrom: in the limit in which the horizons coincide, the solution is extremal and becomes part of the larger Majumdar-Papapetrou family of solutions. To our knowledge, even though there is a conspicuous literature on SUGRA bound states of branes at angles (see for instance [28–32]) there is no extremal BPS solution expressed in terms of arbitrary harmonics functions for branes intersecting at angles. The arbitrary harmonics are crucial in order to interpret the solutions as excitations of the ‘naive’ intersection of branes and to match with the stringy worldsheet computations.

To be concrete, let's remember what happen in the case of perpendicular $D3$ branes. The ‘naive’ solution for marginally (zero binding energy) bound states of intersecting susy branes can be obtained by the harmonic superposition rule [33, 34]. For instance, limiting for simplicity to the metric, the system of 4 perpendicular $D3$ brane pictured in figure 1 for $\theta = 0$ is given by:

$$\begin{aligned}
ds^2 &= (H_0 H_1 H_2 H_3)^{-\frac{1}{2}} dt^2 + (H_0 H_1 H_2 H_3)^{\frac{1}{2}} \sum_{i=1}^3 dx_i^2 + \left(\frac{H_2 H_3}{H_0 H_1} \right)^{\frac{1}{2}} dy_1^2 + \left(\frac{H_0 H_1}{H_2 H_3} \right)^{\frac{1}{2}} d\tilde{y}_1^2 \\
&+ \left(\frac{H_0 H_2}{H_1 H_3} \right)^{\frac{1}{2}} dy_2^2 + \left(\frac{H_1 H_3}{H_0 H_2} \right)^{\frac{1}{2}} d\tilde{y}_2^2 + \left(\frac{H_0 H_3}{H_1 H_2} \right)^{\frac{1}{2}} dy_3^2 + \left(\frac{H_1 H_2}{H_0 H_3} \right)^{\frac{1}{2}} d\tilde{y}_3^2 \quad (6.1)
\end{aligned}$$

where $H_A = 1 + \frac{Q_A}{r}$.

The associated metric parametrized in terms of arbitrary harmonic functions has been given has been written in [1, 16]. In particular in [1] it was given in terms of 8 harmonic functions $H_a = \{V, L_I, K_I, M\}$:

$$ds^2 = -e^{2U} (dt + w)^2 + e^{-2U} \sum_{i=1}^3 dx_i^2 + \sum_{I=1}^3 \left[\frac{dy_I^2}{V e^{2U} Z_I} + V e^{2U} Z_I \tilde{e}_I^2 \right] \quad (6.2)$$

with

$$\begin{aligned}
 Z_I &= L_I + \frac{|\epsilon_{IJK}|}{2} \frac{K_J K_K}{V} \\
 \mu &= \frac{M}{2} + \frac{L_I K_I}{2V} + \frac{|\epsilon_{IJK}|}{6} \frac{K_I K_J K_K}{V^2} \\
 e^{-4U} &= Z_1 Z_2 Z_3 V - \mu^2 V^2 \\
 b_I &= \frac{K_I}{V} - \frac{\mu}{Z_I} \\
 \tilde{e}_I &= d\tilde{y}_I - b_I dy_I \\
 *_3^{flat} dw &= V d\mu - \mu dV - V Z_I d\left(\frac{K_I}{V}\right)
 \end{aligned} \tag{6.3}$$

This solution, that besides the metric includes the C_4 Ramond-Ramond field, reduces to the naive one if one chooses $L_I = H_I$, $V = H_0$ and $K_I = M = 0$, but in principle it can account for the hairs of the 4d black holes for a proper choice of the harmonic functions, as suggested by the matching with stringy microstates obtained in [1]. This is precisely what has been done in the case of the D1-D5 fuzzball [22], in which the harmonic functions are such that: a) far away from the fuzzball the metric resembles a black hole with same asymptotic charges, b) the metric has a ‘throat’ region in which it approximates the ‘naive’ solution, c) the fuzzball deviates strongly from the black hole geometry at the location of the would-be horizon and d) the geometry is non singular and horizonless.

In the case of branes at angles the ‘naive’ solution can be written, for instance specializing to the case of two $D3$ branes tilted in the first two internal torii (the $D3_0 D3_3$ pair in figure 1), as:

$$\begin{aligned}
 ds^2 &= e^{2U_\theta} \left(-dt^2 + \sum_{I=1}^2 dy_I^2 + \sum_{I=1}^3 d\tilde{y}_I^2 + \right. \\
 &\quad \left. + \sum_{A=1,2} \frac{Q_A}{r} \left([(R_A)^1_a dy_a]^2 + [(R_A)^2_a dy_a]^2 \right) \right) + e^{-2U_\theta} \left(dy_3^2 + \sum_{i=1}^3 dx_i^2 \right)
 \end{aligned} \tag{6.4}$$

$$\begin{aligned}
 F_5 &= dt \wedge dr \wedge d\tilde{y}_3 \wedge \partial_r \left[e^{4U_\theta} \left(\sum_{A=1,2} \frac{Q_A}{r} [(R_A)^{\tilde{1}}_a dy_a \wedge (R_A)^{\tilde{2}}_a dy_a] - \right. \right. \\
 &\quad \left. \left. - (dy_1 \wedge dy_2 - d\tilde{y}_1 \wedge d\tilde{y}_2) \frac{Q_1 Q_2}{r^2} \sin^2(\theta_1 - \theta_2) \right) \right] + \\
 &\quad + g e^{4U_\theta} d\theta \wedge d\phi \wedge dy_3 \wedge \partial_r \left[\sum_{A=1,2} \frac{Q_A}{r} [(R_A)^1_a dy_a \wedge (R_A)^2_a dy_a] \right]
 \end{aligned} \tag{6.5}$$

with $a = 1, \tilde{1}, 2, \tilde{2}$ running only in the tilted torii (we define $d\tilde{y}_1 \equiv dy_{\tilde{1}}$), $A = 1, 2$ labelling

the two $D3$ branes, g the determinant of the metric and

$$e^{2U_\theta} = \left(1 + \frac{Q_1 + Q_2}{r} + \frac{Q_1 Q_2}{r^2} \sin^2(\theta_1 - \theta_2) \right)^{-1/2} \quad (6.6)$$

$$R_A = \begin{pmatrix} \cos(\theta_A) & -\sin(\theta_A) & 0 & 0 \\ \sin(\theta_A) & \cos(\theta_A) & 0 & 0 \\ 0 & 0 & \cos(\theta_A) & \sin(\theta_A) \\ 0 & 0 & -\sin(\theta_A) & \cos(\theta_A) \end{pmatrix} \quad (6.7)$$

Even though the associated solution with arbitrary harmonics is still missing, we don't find any particular obstruction to its existence. For instance after dimensional reduction on the internal T^6 , focusing on the metric, the tilting of the internal torii translates in the presence of massless scalar fields grouped into the $h_{2,1}$ vector multiplets. Rotating the branes supersymmetrically is a continuous deformation, parametrized by some moduli fields, of a BPS system. In particular these moduli pertain to the deformations of the complex structure and although the attractor mechanism [35–37] fixes many of them in terms of the charges of the solution, in general some fields remains undetermined and they can account for the freedom to tilt the perpendicular brane configuration while still keeping the system supersymmetric.

A possibility is that the tilted solution can be obtained by a chain of boost and dualities from the perpendicular solution (6.2), in line with the results of [29].

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A Notations and conventions

We will always take $\alpha' = 2$.

The index structure is the following:

- $M, N, R \dots$ for 10-dimensional vector indices.
- $\hat{M}, \hat{N}, \hat{R} \dots$ for 6-dimensional vector indices.
- $\alpha, \beta \dots$ for left spinorial indices in directions $\{t, x_1, x_2, x_3\}$
- $\dot{\alpha}, \dot{\beta} \dots$ for right spinorial indices in directions $\{t, x_1, x_2, x_3\}$
- $A, B, C \dots$ for $SO(1, 5)$ spinor indices. Upper position means right spinor, lower position left spinor.
- $\mu, \nu, \rho \dots$ for vectorial indices in extended space-time directions $\{t, x_1, x_2, x_3\}$

- $i, j, k \dots$ for vectorial indices in extended spatial directions $\{x_1, x_2, x_3\}$
- $1, \bar{1}, 2, \bar{2} \dots$ for complex vectorial indices. In particular $\{1, \bar{1}\}$ parametrize the first internal torus $\{y_1, \bar{y}_1\}$, whilst $\{4, \bar{4}\} \rightarrow \{t, x_3\}$, $\{5, \bar{5}\} \rightarrow \{x_1, x_2\}$.

The left and right spin fields of directions $\{t, x_1, x_2, x_3\}$ are respectively bosonized by the scalar field φ_I according to the formula:

$$\text{left:} \quad S_\alpha = e^{\frac{i}{2}(\pm\varphi_4 \pm \varphi_5)} \quad \#(-) = \text{even} \quad (\text{A.1})$$

$$\text{right:} \quad C_{\dot{\alpha}} = e^{\frac{i}{2}(\pm\varphi_4 \pm \varphi_5)} \quad \#(-) = \text{odd} \quad (\text{A.2})$$

The complex fermions Ψ are built from the real ones ψ via⁸

$$\Psi^I = \frac{1}{\sqrt{2}}(\psi^{2I-1} + i\psi^{2I}) \quad \Psi^{\bar{I}} = \frac{1}{\sqrt{2}}(\psi^{2I-1} - i\psi^{2I}) \quad (\text{A.3})$$

and are bosonized as:

$$\Psi^I = e^{i\varphi_I} \quad (\text{A.4})$$

In the text we will always use the notation ψ , even when referring to the complex fermion, as it should be clear from the context which of the two is used. Cocycle factors, needed to implement anticommutation, will be also omitted.

We follow [38] for the spinorial indices conventions. In particular we need the following formula:

$$\eta_{\mu\nu} = (-, +, +, +) \quad (\text{A.5})$$

$$\epsilon^{12} = \epsilon_{21} = +1 \quad \epsilon^{21} = \epsilon_{12} = -1 \quad (\text{A.6})$$

$$\epsilon_{0123} = -1 \quad (\text{A.7})$$

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (\text{A.8})$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\text{A.9})$$

$$\sigma^0 = \bar{\sigma}^0, \quad \sigma^1 = -\bar{\sigma}^1, \quad \sigma^2 = -\bar{\sigma}^2, \quad \sigma^3 = -\bar{\sigma}^3 \quad (\text{A.10})$$

$$(\sigma^{[\mu\nu]})_\alpha{}^\beta \equiv \frac{1}{4}((\sigma^\mu)_{\alpha\dot{\gamma}}(\bar{\sigma}^\nu)^{\dot{\gamma}\beta} - (\sigma^\nu)_{\alpha\dot{\gamma}}(\bar{\sigma}^\mu)^{\dot{\gamma}\beta}) \quad (\text{A.11})$$

$$(\bar{\sigma}^{[\mu\nu]})^{\dot{\alpha}}{}_{\dot{\beta}} \equiv \frac{1}{4}((\bar{\sigma}^\mu)^{\dot{\alpha}\gamma}(\sigma^\nu)_{\gamma\dot{\beta}} - (\bar{\sigma}^\nu)^{\dot{\alpha}\gamma}(\sigma^\mu)_{\gamma\dot{\beta}}) \quad (\text{A.12})$$

The four dimensional gamma matrices in the Weyl basis are:

$$\Gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad (\text{A.13})$$

⁸It's convenient to treat the torus $\{4, \bar{4}\} \rightarrow \{t, x_3\}$ in a different way by defining $\Psi^4 = \frac{1}{\sqrt{2}}(\psi^0 + \psi^3)$, $\Psi^{\bar{4}} = \frac{1}{\sqrt{2}}(\psi^0 - \psi^3)$ in order to use $\bar{\sigma}^\mu = \{\bar{\sigma}^0, \bar{\sigma}^1, \bar{\sigma}^2, \bar{\sigma}^3\}$. Otherwise one can use (A.3) and add an i factor in front of $\bar{\sigma}^3$ in the definition of $\bar{\sigma}^\mu$.

with associated charge conjugation matrix:

$$(\Gamma^m)^T = -C(\Gamma^m)C^{-1} \quad (\text{A.14})$$

$$C = \begin{pmatrix} i(\sigma^2 \bar{\sigma}^0)_a{}^b & 0 \\ 0 & i(\bar{\sigma}^2 \sigma^0)^{\dot{c}}{}_{\dot{d}} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} \epsilon_{ab} & 0 \\ 0 & -\epsilon_{\dot{a}\dot{b}} \end{pmatrix} \quad (\text{A.15})$$

B OPEs and correlators

$$\langle \psi^\mu(z) \psi^\nu(w) \rangle = \frac{\eta^{\mu\nu}}{(z-w)} \quad (\text{B.1})$$

$$\langle X^\mu(z) X^\nu(w) \rangle = -\eta^{\mu\nu} \log(z-w) \quad (\text{B.2})$$

$$\langle \partial_z X^\mu(z) e^{ikX}(w) \rangle = -\frac{ik^\mu}{(z-w)} \quad (\text{B.3})$$

$$\langle \prod_{i=1}^N e^{q_i \varphi(z_i)} \rangle = \prod_{i<j}^N (z_i - z_j)^{-q_i \cdot q_j} \quad (\text{superghost}) \quad (\text{B.4})$$

$$\langle \prod_{i=1}^N e^{i\lambda_i \varphi(z_i)} \rangle = \prod_{i<j}^N (z_i - z_j)^{+\lambda_i \cdot \lambda_j} \quad (\text{B.5})$$

We use the following basics OPEs in four dimensions:

$$C^{\dot{\alpha}}(z) S^\beta(w) \sim -\frac{1}{\sqrt{2}} (\bar{\sigma}_\mu)^{\dot{\alpha}\beta} \psi^\mu(w) \quad (\text{B.6})$$

$$\psi^\mu \sim \frac{1}{\sqrt{2}} (\bar{\sigma}^\mu)^{\dot{\beta}\alpha} C_{\dot{\beta}} S_\alpha \quad (\text{B.7})$$

$$S_\alpha(z) S_\beta(w) \sim \frac{\epsilon_{\alpha\beta}}{(z-w)^{1/2}} \quad (\text{B.8})$$

$$C_{\dot{\alpha}}(z) C_{\dot{\beta}}(w) \sim -\frac{\epsilon_{\dot{\alpha}\dot{\beta}}}{(z-w)^{1/2}} \quad (\text{B.9})$$

$$\psi^m(z) C^{\dot{\beta}}(w) \sim \frac{1}{\sqrt{2}} \frac{(\bar{\sigma}^m)^{\dot{\beta}\alpha} S_\alpha(w)}{(z-w)^{1/2}} \quad (\text{B.10})$$

$$\psi^\mu(z) S_\alpha(w) \sim -\frac{1}{\sqrt{2}} \frac{(\sigma^\mu)_{\alpha\dot{\beta}} C^{\dot{\beta}}(w)}{(z-w)^{1/2}} \quad (\text{B.11})$$

$$\psi^\mu \psi^\nu(z) S_\alpha(w) \sim \frac{(\sigma^{\mu\nu})_\alpha{}^\beta S_\beta(w)}{(z-w)} \quad (\text{B.12})$$

$$\psi^\mu \psi^\nu(z) C^{\dot{\alpha}}(w) \sim \frac{(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} C^{\dot{\beta}}(w)}{(z-w)} \quad (\text{B.13})$$

B.1 How to fix the normalization

As an example, let's explicitly fix the normalization of (B.6).

$$S^1 = e^{\frac{i}{2}(\varphi_1 + \varphi_2)} \quad S^2 = e^{\frac{i}{2}(-\varphi_1 - \varphi_2)} \quad (\text{B.14})$$

$$C^{\dot{1}} = e^{\frac{i}{2}(-\varphi_1 + \varphi_2)} \quad C^{\dot{2}} = e^{\frac{i}{2}(+\varphi_1 - \varphi_2)} \quad (\text{B.15})$$

Therefore, choosing $\dot{\alpha} = \dot{1}$, $\beta = 2$ and using (A.3):

$$C^{\dot{1}}(z)S^2(w) \sim e^{-\varphi_1} = \Psi^{\bar{1}} = \frac{1}{\sqrt{2}}(\psi^1 - i\psi^2) = N(\bar{\sigma}_\mu)^{\dot{1}2}\psi^\mu \quad (\text{B.16})$$

So we have:

$$N(\bar{\sigma}_\mu)^{\dot{1}2}\psi^\mu = N(\psi^1(\bar{\sigma}_1)^{\dot{1}2} + \psi^2(\bar{\sigma}_2)^{\dot{1}2}) = N(\psi^1(-1) + \psi^2(i)) = -N(\psi^1 - i\psi^2) \quad (\text{B.17})$$

From the comparison, it follows that $N = -\frac{1}{\sqrt{2}}$.

C Rotation matrix for tilted branes

Consider two parallel branes in the (x, y) plane. At the beginning the first and second brane are fixed in $y = 0$, that is they are Neumann in x and Dirichlet in y , where X, Y are the open string coordinates in target space. This is equivalent to $\partial_z X = \partial_{\bar{z}} X$ and $\partial_z Y = -\partial_{\bar{z}} Y$ in complex coordinates. If we tilt a brane by an angle θ with respect to $y = 0$, we need to impose $\partial_z X' = \partial_{\bar{z}} X'$ and $\partial_z Y' = -\partial_{\bar{z}} Y'$ where:

$$\begin{pmatrix} X' \\ Y' \end{pmatrix} = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} \quad (\text{C.1})$$

After some algebra, for a counterclockwise rotation we find:

$$\partial_z X'^\mu = R^\mu_\nu \partial_{\bar{z}} X^\nu \quad X^1 = X, X^2 = Y \quad (\text{C.2})$$

$$R(\theta) = \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ -\sin(2\theta) & -\cos(2\theta) \end{pmatrix} \quad (\text{C.3})$$

For a clockwise rotation we have:

$$R(\theta) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} \quad (\text{C.4})$$

D Two boundary amplitude

We split the calculation into two sections: the first one dealing with the amplitude with all the fermions in extended space-time directions and the second focusing on compact directions. Here we start by computing the correlators common to both sections:

$$\langle \sigma_{1/2-\theta}^{(1)\dagger}(z_1) \sigma_{1/2-\theta}^{(1)}(z_2) \rangle \langle \sigma_{1/2-\theta}^{(2)}(z_1) \sigma_{1/2-\theta}^{(2)\dagger}(z_2) \rangle = z_{12}^{-\frac{1}{2}+2\theta^2} \quad (\text{D.1})$$

$$\langle e^{-\frac{1}{2}\varphi}(z_1) e^{-\frac{1}{2}\varphi}(z_2) e^{-\varphi}(z_4) \rangle = z_{12}^{-\frac{1}{4}} z_{14}^{-\frac{1}{2}} z_{24}^{-\frac{1}{2}} \quad (\text{D.2})$$

The ∂X piece in (4.4) gives vanishing contribution due to $k^M G_{MN} = 0$.

D.1 Fermions in extended space-time dimensions

The remaining internal part gives:

$$\langle e^{i\theta\varphi_1} e^{-i\theta\varphi_2} e^{-i\varphi_3/2}(z_1) e^{-i\theta\varphi_1} e^{i\theta\varphi_2} e^{i\varphi_3/2}(z_2) \rangle = z_{12}^{-2\theta^2-1/4} \quad (\text{D.3})$$

Taking $M = \mu$ and $N = \nu$, the last correlator, apart from overall constants, is:

$$(k)_\rho (ER)_{\mu\nu} (\bar{\sigma}^\nu)^{\dot{\gamma}\delta} \langle S^\alpha(z_1) C^{\dot{\beta}}(z_2) J^{[\rho\mu]}(z_3) C_{\dot{\gamma}} S_\delta(z_4) \rangle \quad (\text{D.4})$$

in which we have defined the current $J^{[\rho\mu]} =: \psi^\rho \psi^\mu$: and used the $\text{SO}(1,3)$ relation $\psi^\nu \sim \frac{1}{\sqrt{2}} (\bar{\sigma}^\nu)^{\dot{\gamma}\delta} C_{\dot{\gamma}} S_\delta$. Now we split the current in the dual (left) and anti-self dual part (right):⁹

$$\begin{aligned} J^{[\rho\mu]} &= J_L^{[\rho\mu]} + J_R^{[\rho\mu]} \\ J_L^{[\rho\mu]} &= \frac{1}{2} (J^{[\rho\mu]} + \frac{i}{2} \epsilon^{\rho\mu\tau\zeta} J_{[\tau\zeta]}) \\ J_R^{[\rho\mu]} &= \frac{1}{2} (J^{[\rho\mu]} - \frac{i}{2} \epsilon^{\rho\mu\tau\zeta} J_{[\tau\zeta]}) \end{aligned} \quad (\text{D.5})$$

Using the fact that $\text{SO}(1,3)$ Weyl spinors of different chirality have vanishing 2-point correlation function, we find:

$$(k)_\rho (ER)_{\mu\nu} (\bar{\sigma}^\nu)^{\dot{\gamma}\delta} [\langle S_{(z_1)}^\alpha (J_L^{[\rho\mu]})_{(z_3)} (S_\delta)_{(z_4)} \rangle (-\delta_{\dot{\gamma}}^{\dot{\beta}} z_{24}^{-1/2}) + \langle C_{(z_2)}^{\dot{\beta}} ((J_R^{[\rho\mu]})_{(z_3)} (C_{\dot{\gamma}})_{(z_4)}) \rangle (\delta_\delta^\alpha z_{14}^{-1/2})] \quad (\text{D.6})$$

having used $\epsilon^{\alpha\beta} \epsilon_{\beta\delta} = \delta_\delta^\alpha$ and

$$S_\alpha(z) S_\beta(w) \sim \frac{\epsilon_{\alpha\beta}}{(z-w)^{1/2}} \quad C_{\dot{\alpha}}(z) C_{\dot{\beta}}(w) \sim -\frac{\epsilon_{\dot{\alpha}\dot{\beta}}}{(z-w)^{1/2}} \quad (\text{D.7})$$

Conformal symmetry fixes the 3-point correlation functions of quasi-primary fields to be of the standard form

$$\langle \Phi_1(z_1) \Phi_2(z_2) \Phi_3(z_3) \rangle = \frac{\mathcal{N}_{3Pt}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_1+h_3-h_2}} \quad (\text{D.8})$$

Where h is the conformal dimension of the fields and $\mathcal{N}_{3Pt}$ is a normalization $\mathcal{N}_{3Pt}$ constant. Using (A.11), (A.12) and that $h(J) = 1$, $h(S) = h(C) = 1/4$, we get:¹⁰

$$\langle S_{(z_1)}^\alpha (J_L^{[\rho\mu]})_{(z_3)} (S_\delta)_{(z_4)} \rangle = \frac{(\sigma^{[\rho\mu]})_\delta^\alpha}{z_{13} z_{14}^{-1/2} z_{34}} \quad (\text{D.9})$$

$$\langle C_{(z_2)}^{\dot{\beta}} (J_R^{[\rho\mu]})_{(z_3)} (C_{\dot{\gamma}})_{(z_4)} \rangle = -\frac{(\bar{\sigma}^{[\rho\mu]})_{\dot{\gamma}}^{\dot{\beta}}}{z_{23} z_{24}^{-1/2} z_{34}} \quad (\text{D.10})$$

⁹According to the convention $\star_4 A^{\mu\nu} \equiv \frac{i}{2} \epsilon^{\mu\nu\rho\sigma} A_{\rho\sigma}$.

¹⁰For instance the first correlator, in the group theory language of $so(4) \sim so(2) \oplus so(2)$ is $\mathbf{2}_L \otimes (\mathbf{4} \otimes \mathbf{4})_{asd} \otimes \mathbf{2}_L = \mathbf{2}_L \otimes (\mathbf{3}_L, \mathbf{1}_R) \otimes \mathbf{2}_L = (\mathbf{3}_L, \mathbf{1}_R) \otimes (\mathbf{1}_L + \mathbf{3}_L, \mathbf{1}_R)$ where $(\mathbf{4} \otimes \mathbf{4})_{asd}$ is the self dual antisymmetric part of the product and $\mathbf{4} = (\mathbf{2}_L, \mathbf{2}_R)$. Therefore the correlator has a singlet coming from $(\mathbf{3}_L, \mathbf{1}_R) \otimes (\mathbf{3}_L, \mathbf{1}_R) = \mathbf{1}_L + \mathbf{3}_L + \mathbf{5}_L$. This means that the product is symmetric in the spinor indices (if properly lowered or raised) and antisymmetric in the vectorial indices.

having used the self-duality/anti self-duality properties of the Lorentz generators. Now we employ the relations

$$(\bar{\sigma}^\nu)^{\dot{\beta}\delta}(\sigma^{[\rho\mu]})_\delta{}^\alpha = \frac{1}{2}[-i\epsilon^{\rho\mu\nu\eta}(\bar{\sigma}_\eta)^{\dot{\beta}\alpha} + \eta^{\mu\nu}(\bar{\sigma}^\rho)^{\dot{\beta}\alpha} - \eta^{\rho\nu}(\bar{\sigma}^\mu)^{\dot{\beta}\alpha}] \quad (\text{D.11})$$

$$(\bar{\sigma}^\nu)^{\dot{\gamma}\alpha}(\bar{\sigma}^{[\rho\mu]})_{\dot{\gamma}}{}^\beta = \frac{1}{2}[-i\epsilon^{\rho\mu\nu\eta}(\bar{\sigma}_\eta)^{\dot{\beta}\alpha} - \eta^{\mu\nu}(\bar{\sigma}^\rho)^{\dot{\beta}\alpha} + \eta^{\rho\nu}(\bar{\sigma}^\mu)^{\dot{\beta}\alpha}] \quad (\text{D.12})$$

to obtain the integrand. Indeed putting everything together and gauge fixing to $z_1 = -\infty$, $z_2 = x$, $z_3 = i$, $z_4 = -i$ we get:

$$\langle c(z_1)c(z_3)c(z_4) \rangle = z_{13}z_{14}z_{34} \quad (\text{D.13})$$

$$\begin{aligned} \mathcal{A}_{\mu\bar{\mu}NSNS}^{NC} &= \mu_\alpha \bar{\mu}_{\dot{\beta}} \int_{-\infty}^{+\infty} dx (z_{13}z_{14}z_{34}) z_{12}^{-\frac{1}{2}+2\theta^2} z_{12}^{-\frac{1}{4}} z_{14}^{-\frac{1}{2}} z_{24}^{-\frac{1}{2}} z_{12}^{-2\theta^2-1/4} \\ &\times \left(\frac{i}{2\sqrt{2}} \right) [(-i\epsilon^{\rho\mu\nu\eta}(\bar{\sigma}_\eta)^{\dot{\beta}\alpha}(k)_\rho (ER)_{\mu\nu} + (ER)(k\bar{\sigma})^{\dot{\beta}\alpha} - (\bar{\sigma}ERk)^{\dot{\beta}\alpha}) z_{13}^{-1} z_{34}^{-1} z_{14}^{1/2} z_{24}^{-1/2} + \\ &+ (-i\epsilon^{\rho\mu\nu\eta}(\bar{\sigma}_\eta)^{\dot{\beta}\alpha}(k)_\rho (ER)_{\mu\nu} - (ER)(k\bar{\sigma})^{\dot{\beta}\alpha} + (\bar{\sigma}ERk)^{\dot{\beta}\alpha}) z_{23}^{-1} z_{24}^{1/2} z_{34}^{-1} z_{14}^{-1/2}] \quad (\text{D.14}) \end{aligned}$$

Where $(ER) = (ER)_{\mu\nu}\eta^{\mu\nu}$ and NC stands for non compact. The term $\bar{\sigma}ERk$ can be discarded if we are interested in a purely spatial (non internal) closed string momentum, using the fact that the momentum is transverse and the particular form of the reflection matrix in that directions, that is $kR = -k$. After some manipulations we obtain:

$$\mathcal{A}_{\mu\bar{\mu}NSNS}^{NC} = \left(\frac{i}{2\sqrt{2}} \right) \int_{-\infty}^{+\infty} dx \left[(-i\epsilon^{\rho\mu\nu\eta}v_\eta(k)_\rho (ER)_{\mu\nu}) \frac{2x}{(1+x^2)} + (ER)(kv) \frac{-2i}{(1+x^2)} \right] \quad (\text{D.15})$$

substituting the open string condensate $v^\rho = \mu_\alpha \bar{\mu}_{\dot{\beta}}(\bar{\sigma}^\rho)^{\dot{\beta}\alpha}$ and with $kv = k^i v^i$. The first part integrates to zero, while the second integral is simply:¹¹

$$\int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)} = \pi \quad (\text{D.18})$$

Finally the amplitude of a massless closed string scattering on two tilted $D3$ branes is given by:

$$\mathcal{A}_{\mu\bar{\mu}NSNS}^{NC} = \frac{\pi}{\sqrt{2}}(ER)(kv) \quad (\text{D.19})$$

¹¹In all these calculations we have put the open strings momenta to zero from the beginning. If one wants to be rigorous, he should first do the full computation and then send all the open string momenta to zero. For instance the Koba-Nielsen factor, after using gauge fixing and kinematics of the momenta (and apart from an overall infinite constant due to $z_1 = -\infty$) is:

$$\langle e^{ik_1 X}(z_1) e^{ik_2 X}(z_2) e^{ik X}(z_3) e^{ik R X}(z_4) \rangle = |1+x^2|^{\alpha' k_2 k} |2i|^{\alpha' k R k/2} \quad (\text{D.16})$$

Therefore the correct integral (D.18), using the integral beta function representation is [39]:

$$\lim_{k^2 \rightarrow 0} \int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)} |1+x^2|^{\alpha' k_2 k} = \lim_{k^2 \rightarrow 0} B(-\alpha' k_2 k + \frac{1}{2}, \frac{1}{2}) = \pi. \quad (\text{D.17})$$

Employing (1.1), one finds that the only emitted fields are the diagonal components of the metric and using

$$\int \frac{d^3k}{(2\pi)^3} \frac{-ik_j}{|k|^2} e^{ikx} = \frac{1}{4\pi} \frac{x_j}{|x|^3} \quad (\text{D.20})$$

one finds that the scaling of the harmonic function is given by $\frac{x_j}{r^3}$.

D.2 Fermions in compact dimensions

One of the fermion of the closed string vertex still needs to be in the first four directions, in order to saturate the spin fields charge. The remaining fermions must be of opposite chirality: $(M, N) = (I, \bar{I})$ in (4.1). We start with $(M, N) = (3, \bar{3})$ and the conjugate choice $(M, N) = (\bar{3}, 3)$. The correlators (D.1), (D.2) and (D.13) are unchanged. What is left is:¹²

$$(k)_\rho \langle S_{(z_1)}^\alpha C_{(z_2)}^{\dot{\beta}} \psi_{(z_3)}^\rho \rangle = (k)_\rho \frac{1}{\sqrt{2}} \frac{(\bar{\sigma}^\rho)^{\dot{\beta}\alpha}}{z_{13}^{1/2} z_{23}^{1/2}} \quad (\text{D.21})$$

$$\langle e^{i\theta\varphi_1} e^{-i\theta\varphi_2}(z_1) e^{-i\theta\varphi_1} e^{i\theta\varphi_2}(z_2) \rangle = z_{12}^{-2\theta^2} \quad (\text{D.22})$$

$$\langle e^{-i\varphi_3/2}(z_1) e^{i\varphi_3/2}(z_2) e^{\pm i\varphi_3}(z_3) e^{\mp i\varphi_3}(z_4) \rangle = \frac{1}{2} z_{12}^{-\frac{1}{4}} z_{13}^{\mp\frac{1}{2}} z_{14}^{\pm\frac{1}{2}} z_{23}^{\pm\frac{1}{2}} z_{24}^{\mp\frac{1}{2}} z_{34}^{-1} \quad (\text{D.23})$$

Summing these two contributions and gauge fixing to $z_1 = -\infty$, $z_2 = x$, $z_3 = i$, $z_4 = -i$ we get:

$$\mathcal{A}_{\mu\bar{\mu}NSNS}^{3\bar{3}} = \left(\frac{i}{2\sqrt{2}} \right) \int_{-\infty}^{+\infty} dx(kv) \left(\frac{(ER)_{3\bar{3}}}{x-i} + \frac{(ER)_{\bar{3}3}}{x+i} \right) \quad (\text{D.24})$$

that select the antisymmetric part of $(ER)_{3\bar{3}}$, having used that $\int_{-\infty}^{+\infty} \frac{dx}{x-i} = \int_{-\infty}^{+\infty} \frac{dx(x+i)}{x^2+1} = i\pi$. Since the reflection matrix is diagonal with $(-1, 1)$ eigenvalues in the third torus, the emitted field is the metric.

$$\mathcal{A}_{\mu\bar{\mu}NSNS}^{3\bar{3}} = \frac{-\pi}{\sqrt{2}} (ER)_{[3\bar{3}]}(kv) \quad (\text{D.25})$$

The emitted field is therefore the off-diagonal (real) metric in the third torus and the scaling of the harmonic function is the one found previously for NC components.

If instead we compute the $(M, N) = (1, \bar{1})$ we find a divergent or vanishing integral for every angle. Indeed one finds:

$$\mathcal{A}_{\mu\bar{\mu}NSNS}^{1\bar{1}} \propto \int_{-\infty}^{+\infty} dx (x-i)^{-\theta-\frac{1}{2}} (x+i)^{\theta-\frac{1}{2}} \quad (\text{D.26})$$

that can be shown to be divergent using [40]:

$$\int_{-\infty}^{+\infty} dx (x-i)^a (x+i)^b = -\pi(2i)^{2+a+b} e^{-\pi ia} \frac{\Gamma(-1-a-b)}{\Gamma(-a)\Gamma(-b)} \quad (\text{D.27})$$

The same reasoning applies to $(M, N) = (2, \bar{2})$.

The calculation leading to (4.10) has been done in [1].

¹²Notice that we have simplified the calculation by employing only scalar fields in the non compact space, otherwise we should have computed excited twist fields coming from the correlators of scalars and twist fields.

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