CHY formulae in 4d

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ABSTRACT: In this paper, we develop a rather general way to reduce integrands with polarization involved in the Cachazo-He-Yuan formulae, such as the reduced Pfaffian, its compactification and its squeezing, as well as the new object for F^3 amplitude. We prove that the reduced Pfaffian vanishes unless evaluated on a certain set of solutions. It leads us to build up the 4d CHY formulae using spinors, which strains off many useless solutions. The supersymmetrization is straightforward and may provide a hint to understand ambitwistor string in 4d.

KEYWORDS: Scattering Amplitudes, String Field Theory

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1 Introduction

A new formulation for S-matrix of massless particles in arbitrary dimensions, dubbed as Cachazo-He-Yuan (CHY) formulation, has been developed for a large variety of theories [1–4]. It expresses tree-level S-matrix as an integral over the moduli space of Riemann spheres, which are localized by a set of constraints, known as scattering equations [1, 5, 6]

$$\mathcal{E}_a := \sum_{b \neq a} \frac{s_{ab}}{\sigma_a - \sigma_b} = 0, \quad \text{for } a = 1, 2, \dots, n, \quad (1.1)$$

where $s_{ab} = (k_a + k_b)^2 = 2k_a \cdot k_b$, σ_a denotes the position of the a^{th} puncture and we denote $\sigma_{ab} := \sigma_a - \sigma_b$. It has been argued that what underpins the formulation is the ambitwistor string theory [7–9].

The formulation has been inspired by Witten’s revolutionary twistor string theory for $\mathcal{N} = 4$ super-Yang-Mills theory (SYM) in four dimensions [10], and in particular the Roiban-Spradlin-Volovich-Witten (RSVW) formulae for all tree amplitudes in the theory [11]. Originally CHY discovered scattering equations in attempts to rewrite the equations in the delta functions of RSVW formulae without using 4d spinor helicity variables [5], thus by construction they reduce to RSVW equations in four dimensions. More precisely, we have $n-3$ different sets of 4d equations, which are *polynomial* equations of degree

$d = 1, 2, \dots, n-3$. The $n-3$ sectors are labeled by $k' = d+1 = 2, \dots, n-2$, which coincide with helicity sectors. A set of equations, which are completely equivalent to RSVW equations, have been proposed in [12] based on ambitwistor string theory in four dimensions. It turns out that they are more convenient for our purposes, and in particular for helicity amplitudes. To write the equations in sector k' , we divide n particles into two sets of k' and $n-k'$ particles denoted as $-'$ and $+'$ respectively:

$$E_b^{\dot{\alpha}} \equiv \tilde{\lambda}_b^{\dot{\alpha}} - t_b \sum_{p \in +'} \frac{t_p \tilde{\lambda}_p^{\dot{\alpha}}}{\sigma_{bp}} = 0 \quad \text{for } b \in -', \quad E_p^{\alpha} \equiv \lambda_p^{\alpha} - t_p \sum_{b \in -'} \frac{t_b \lambda_b^{\alpha}}{\sigma_{pb}} = 0 \quad \text{for } p \in +', \quad (1.2)$$

Here the variables are σ 's and t 's, which can be combined into n variables in $\mathbb{C}^2$, $\sigma_a^{\alpha} = \frac{1}{t_a}(\sigma_a, 1)$. The σ_{bp} is the abbreviation of $\sigma_b - \sigma_p$. The $-'$ and $+'$ are arbitrary two sets of the n external particles, with their length equal to k' and $n-k'$ respectively. Different choices just correspond to different link representation [13, 14], which share the same solution of σ 's. In this paper, we reserve $-$ and $+$ as the negative and positive helicity sets of external particles and k the length of $-$, i.e. the number of external particles of negative helicity. *A priori* there is no relation between solution sector and helicity sector.

We refer the readers to [15] for the direct derivation of (1.2) from (1.1); in the same paper, it has been shown that (1.2) is equivalent to RSVW equations, and one can freely translate between the two forms. Each solution of (1.1) corresponds to a unique solution $\{\sigma_a, t_a\}$ of (1.2) for some k' , with identical cross-ratios of the σ 's. For each k' , (1.2) have an Eulerian number of solutions, $E_{n-3, k'-2}$, and the union of them for all sectors give $(n-3)!$ solutions of (1.1), with $(n-3)! = \sum_{k'=2}^{n-2} E_{n-3, k'-2}$ [5, 16].

It is highly non-trivial to reduce the localized integral measure of CHY formula, with delta functions of (1.1), to that of 4d formula, with (1.2), for some k' sector. The reduction requires a sum over all sectors, and for each of them it results in a complicated conversion factor that depends on k' . In addition, after we plug in spinor-helicity variables for e.g. Yang-Mills amplitudes, the CHY integrand behaves very differently in different helicity and solution sectors. As we will see, the Pfaffian plays the role of “solution-filter”: it is non-vanishing only on the solution sector that coincides with the helicity sector, which is why we have a 4d formula for each helicity sector. What is even more interesting is that in the right sector, the polarization part of the CHY integrand exactly cancels the k' -dependent conversion factor from the measure! Thus two complications cancel out, and for YM we are left with a trivial Parke-Taylor factor in 4d.

Let's make the statement more precisely. For gauge theory and gravity, the most important ingredient is a $2n \times 2n$ skew matrix Ψ_n

$$\Psi_n := \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}; \quad A_{ab} = \begin{cases} \frac{k_a \cdot k_b}{\sigma_{ab}} & a \neq b \\ 0 & a = b \end{cases}, \quad B_{ab} = \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{\sigma_{ab}} & a \neq b \\ 0 & a = b \end{cases}, \quad C_{ab} = \begin{cases} \frac{\epsilon_a \cdot k_b}{\sigma_{ab}} & a \neq b \\ -\sum_{c \neq a} C_{ac} & a = b \end{cases}, \quad (1.3)$$

and we define its reduced Pfaffian $\text{Pf}' \Psi_n := \frac{(-)^{a+b}}{\sigma_{ab}} \text{Pf} |\Psi_n|_{ab}^{ab}$ with $1 \leq a < b \leq n$.

We try to factorize the $\text{Pf}' \Psi_n$ into two parts depending on particles of negative and positive helicity respectively. Then we show in the right sector that is consistent to the

helicity sector, each of the parts combines to a reduced determinant while in other sector one of the part must vanish. That is,

$$\text{Pf}'\Psi_n|_{k'} = \delta_{kk'} \det' \mathbf{h}_k \det' \tilde{\mathbf{h}}_{n-k}, \quad (1.4)$$

Here the two matrices, the $k \times k$ matrix $\mathbf{h}_k$ and $(n-k) \times (n-k)$ one $\tilde{\mathbf{h}}_{n-k}$ essentially introduced in [12] (see also [17, 18]) are given by

$$\begin{aligned} h_{ab} &= \frac{\langle ab \rangle}{\sigma_{ab}} & \text{for } a \neq b, & & h_{aa} &= - \sum_{b \neq a} \frac{t_b}{t_a} h_{ab} & a, b \in -, \\ \tilde{h}_{ab} &= \frac{[ab]}{\sigma_{ab}} & \text{for } a \neq b, & & \tilde{h}_{aa} &= - \sum_{b \neq a} \frac{t_b}{t_a} \tilde{h}_{ab} & a, b \in +, \end{aligned} \quad (1.5)$$

and we define $\det' \mathbf{h}_k = \det |\mathbf{h}_k|_b^a / (t_a t_b)$ (similarly for $\det' \tilde{\mathbf{h}}_{n-k}$) where we use $|\mathbf{h}_k|_b^a$ to denote the minor with any row a and column b deleted.

We rearrange the $\text{Pf}'\Psi_n$ using some fundamental gauge invariant or almost gauge invariant objects. It is either a (modified) trace of linearised field strength ornamented with some σ 's or C_{aa} . We view the 4d scattering equations (1.2) as a change of variables: we refer to $\lambda_{b \in -'}$, $\tilde{\lambda}_{p \in +'}$ and t_a, σ_a as “data” and the 4d scattering equations (1.2) as writing $\tilde{\lambda}_{b \in -'}$ and $\lambda_{p \in +'}$ in terms of the data. After plugging in this change of variables, the C_{aa} in Ψ_n directly reduces to object made up of spinors. What left to do is to deal with all kind of trace. After all, somehow, we find the reduced Pfaffian reduces to the two reduced determinants. This way of reduction is rather general: not only the reduced Pfaffian, but also many other integrands, such as the reduced compactified Pfaffian used in EM, YMS, DBI amplitudes or the new object $\mathcal{P}_n$ used in F^3, R^2, R^3 amplitudes are also related to these two (extended) matrices. It may even be applied at loop level [19].

The paper is organized as follows. In section 2, we introduce the CHY formulae in 4d. In section 3, we study the reduction of Pfaffians to 4d for $k'=k$. First we see how $\text{Pf}\Psi_n$ factorizes in 4d in a manifestly gauge-invariant way, which naturally leads to the 4d matrices $\mathbf{h}_k$ and $\tilde{\mathbf{h}}_{n-k}$. Then we present the beautiful reduction of $\text{Pf}'\Psi_n$, in a similar but more non-trivial way. In section 4, we move to general case with arbitrary k' , which requires generalized version of $\mathbf{h}_k^{k'}$ and $\tilde{\mathbf{h}}_{n-k}^{k'}$ matrices. We show that both $\text{Pf}'\Psi_n$ and $\mathcal{P}_n$ reduce nicely into the generalized $\mathbf{h}_k^{k'}$ and $\tilde{\mathbf{h}}_{n-k}^{k'}$; while $\text{Pf}'\Psi_n$ directly vanishes when $k' \neq k$, $\mathcal{P}_n$ does not and gives interesting formulae in 4d. The reduction of the reduced compactified Pfaffian and squeezed Pfaffian is put in appendix B, C.

2 4d CHY formulae

We start with CHY formula for tree-level S-matrix of n massless particles:

$$M_n = \frac{1}{\text{vol SL}(2, \mathbb{C})} \int \prod_{a=1}^n d\sigma_a \prod_{a=1}^n{}' \delta(\mathcal{E}_a) \mathcal{I}_n(\{\sigma, k, \dots\}) = \sum_{\text{solutions}} \frac{\mathcal{I}_n(\{\sigma, k, \dots\})}{\det' \Phi_n}, \quad (2.1)$$

where the precise definition of the integral measure including delta functions can be found in [1], and $\mathcal{I}_n$ is the CHY integrand defines the theory. In the second equality one sums over

$(n-3)!$ solutions of (1.1), evaluated on the integrand and the Jacobian, which is defined as a reduced determinant:

$$\det' \Phi_n := \frac{\det |\Phi_n|_{abc}^{pqr}}{|pqr||abc|} \quad \text{with} \quad \Phi_{ab} = \frac{s_{ab}}{\sigma_{ab}^2}, \quad \text{for } a \neq b, \quad \Phi_{aa} = -\sum_{b \neq a} \Phi_{ab}, \quad (2.2)$$

where the $n \times n$ matrix (Φ_n) , with entries $\{\Phi_{ab}\} := -\partial\{\mathcal{E}_a\}/\partial\{\sigma_b\}$, is the derivative matrix; the rows p, q, r and columns a, b, c are deleted (corresponding to deleted equations and variables, respectively), and we have two Fadeev-Popov factors, defined as $|abc| := \sigma_{ab}\sigma_{bc}\sigma_{ca}$.

For gauge theory and gravity, the most important ingredients is the reduced Pfaffian $\text{Pf}' \Psi_n$ given in (1.3). Many other integrands can be obtained by doing some operation on it. The CHY integrand for n -point Yang-Mills tree amplitudes reads

$$\mathcal{I}_n^{\text{YM}} = \mathcal{C}_n \text{Pf}' \Psi_n, \quad \mathcal{C}_n = \frac{\text{Tr}(T^{I_1} T^{I_2} \dots T^{I_n})}{\sigma_{12} \sigma_{23} \dots \sigma_{n1}} + \text{permutations}, \quad (2.3)$$

where $\mathcal{C}_n$ is the color-dressed Parke-Taylor factor, with the sum over $(n-1)!$ inequivalent permutations.

The general 4d formula in solution sector k' for n -point amplitudes reads:

$$M_{n,k'} = \frac{1}{\text{vol GL}(2, \mathbb{C})} \int \prod_{a=1}^n d^2 \sigma_a \prod_{\substack{b \in -' \\ p \in +'}} \delta^2(E_b) \delta^2(E_p) I_n(\{\sigma, \lambda, \tilde{\lambda}\}) = \sum_{k' - \text{sec. sol.}} \frac{I_n(\{\sigma, \lambda, \tilde{\lambda}\})}{J_{n,k'}} \quad (2.4)$$

where $d^2 \sigma_a := d\sigma_a \frac{dt_a}{t_a}$, and in addition to 4 deleted variables by $\text{GL}(2)$, 4 redundant equations in (1.2) are deleted which give overall delta functions for momentum conservation. In the second equality, one sums over the Eulerian number, $E_{n-3,k'-2}$, solutions in sector k' . The $J_{n,k'}$ is the Jacobian of the localized $2n-4$ integrals

$$J_{n,k'} = \left(\prod_{a=1}^n t_a \right) \frac{\det \left(\partial \{E_{b \neq c,d}^{\dot{\alpha}}, E_p^{\alpha}\} / \partial \{t_{a \neq m}, \sigma_{a \neq u,v,w}\} \right)}{t_m \sigma_{u,v} \sigma_{v,w} \sigma_{w,u} \langle cd \rangle^2} \quad (2.5)$$

where we have chosen to eliminate $t_m, \sigma_u, \sigma_v, \sigma_w$ and $E_{b=c,d}^{\dot{\alpha}}$, with the FP factor $\langle cd \rangle^2$ (for $E_{p \neq q,r}^{\alpha}$ the FP factor is $[qr]^2$).

The relation between the two jacobians is simple. Viewing (1.2) as a change of variables and plugging in it, we find

$$\det' \Phi_n(\{s_{ab}, \sigma_a\})|_{k'} = J_{n,k'} \det' \mathbf{h}_{k'} \det' \tilde{\mathbf{h}}_{n-k'}. \quad (2.6)$$

Here we don't need to plug in any solutions, but simply make a change of variables, so this is really an equality between rational functions of the data, i.e. λ_b 's, $\tilde{\lambda}_p$'s, and σ_a, t_a 's. The two reduced determinants $\det' \mathbf{h}_{k'}$ and $\det' \tilde{\mathbf{h}}_{n-k'}$ can be thought as two resultants and are divided by $\det' \Phi_n$ as discussed in [20]. We find that the quotient is just $J_{n,k'}$. A conjecture about the closed form of $J_{n,k'}$ is put in appendix A.

Thanks to (1.4), then for gluon amplitudes, the integrand is nothing but the (color-dressed) Parke-Taylor factor $I_n^{\text{YM}} = \mathcal{C}_n$. Different from (2.1), any $\frac{\sigma_{bc}}{t_b t_c} \frac{\sigma_{pq}}{t_p t_q}$ or $\frac{\sigma_{bp}}{t_b t_p}$ with $b, c \in -'$ and $p, q \in +'$ is $\text{GL}(2, \mathbb{C})$ invariant and any known 4d integrand added with these objects could be a new 4d integrand, for example we add some $\frac{\sigma_{bp}}{t_b t_p}$ to the I_n^{YM} and we get those for QCD in [21].

In this paper, we explicitly demonstrate the first identity (1.4). Compared to this identity, the second one (2.6) is a more boring one, as there is no polarization involved and just kinematics reducing to 4 dimensions. One can check as many points as we want, without any difficulties (we have checked up to 50 points with all solution sectors numerically). A proof based on direct inspection should be straightforward.

3 Reduced Pfaffian in 4d for the $k' = k$ sector

In this section, we show in a constructive way how the reduced Pfaffian factorizes in four dimensions for the solution sector that coincides with its helicity sector, $k' = k$. We will proceed in two steps: as a warm up, we show how it works for the vanishing Pfaffian $\text{Pf} \Psi_n$, which factorizes into two vanishing determinants in 4d; then we apply it to the more non-trivial case of the reduced Pfaffian and show $\text{Pf}' \Psi_n = \det' \mathbf{h} \det' \tilde{\mathbf{h}}$. The reason for doing so is that both $\text{Pf} \Psi_n$ and $\text{Pf}' \Psi_n$ have similar expansions, as first studied in [22], and we review them here.

From the definition of Pfaffian and thanks to the special structure of $2n \times 2n$ matrix Ψ_n , we can expand $\text{Pf} \Psi_n$ as a sum over $n!$ permutations of labels $1, 2, \dots, n$, denoted as $p \in S_n$

$$\text{Pf} \Psi_n = \sum_{p \in S_n} \text{sgn}(p) \Psi_p = \sum_{p \in S_n} \text{sgn}(p) \Psi_I \Psi_J \cdots \Psi_K, \quad (3.1)$$

where $\text{sgn}(p)$ denotes the signature of the permutation p and in the second equality, we use the unique decomposition of any permutation p into disjoint cycles $I, J, \dots, K$ given by

$$I = (a_1 a_2 \cdots a_i), \quad J = (b_1 b_2 \cdots b_j), \dots, \quad K = (c_1 c_2 \cdots c_k); \quad (3.2)$$

each Ψ_p is the product of its “cycle factors” $\Psi_I \Psi_J \cdots \Psi_K$, which we define now. When the length of a cycle equals one, its cycle factor $\Psi_{(a)}$ is given by the diagonal of C -matrix:

$$\Psi_{(a)} := C_{aa} = - \sum_{b \neq a} \frac{\epsilon_a \cdot k_b}{\sigma_{ab}}, \quad (3.3)$$

and when the length exceeds one e.g. $i > 1$, the cycle factor is given by

$$\Psi_I = \Psi_{(a_1 a_2 \cdots a_i)} := \frac{\frac{1}{2} \text{tr}(f_{a_1} f_{a_2} \cdots f_{a_i})}{\sigma_{(a_1 a_2 a_3 \cdots a_i)}} \quad \text{with} \quad f_a^{\mu\nu} = k_a^\mu \epsilon_a^\nu - \epsilon_a^\mu k_a^\nu. \quad (3.4)$$

Here $\sigma_{(a_1 a_2 a_3 \cdots a_i)} = \sigma_{a_1 a_2} \sigma_{a_2 a_3} \cdots \sigma_{a_i a_1}$. The trace is over Lorentz indices and $f^{\mu\nu}$ is the linearized field strengths of gluons. Note that the decomposition is manifestly gauge invariant: for cycle factors with length more than 1 (3.4), the trace of $f^{\mu\nu}$ is gauge invariant, while for 1-cycles, (3.3), the factor is gauge invariant on the support of scattering equations (1.1).

The reduced Pfaffian $\text{Pf}'\Psi_n$, as discussed in [22], is different from $\text{Pf}\Psi_n$. Because the $1^{\text{th}}, n^{\text{th}}$ columns and rows have been deleted, the numerator of the cycle containing 1 and n becomes $\frac{1}{2}\epsilon_1 \cdot (f_{a_2} f_{a_3} \cdots f_{a_{i-1}}) \cdot \epsilon_n$ instead of a trace. Then

$$\text{Pf}'\Psi_n = \sum_{p \in S_n}' \text{sgn}(p) W_I \Psi_J \cdots \Psi_K, \quad (3.5)$$

with

$$W_I = W_{[1a_2 \cdots a_{i-1}n]} = \frac{\frac{1}{2}\epsilon_1 \cdot (f_{a_2} f_{a_3} \cdots f_{a_{i-1}}) \cdot \epsilon_n}{\sigma_{(1a_2 a_3 \cdots a_{i-1}n)}}. \quad (3.6)$$

Here $I, J, \cdots K$ are the cycles of permutation p . The prime on the summation sign indicates that the sum is taken over all $p \in S_n$ such that 1 is changed into n . There are $(n-1)!$ such permutations in S_n so the sum consists of $(n-1)!$ terms.

The key observation in [22] allows us to expand the reduced Pfaffian in terms of building blocks, each of which is either the product of various closed cycles or an open cycle involving the two deleted labels. Closed cycles have a very good property that they will vanish unless all of their elements belong to same helicity. While the open cycle is much tougher, as it's not gauge invariant individually (dependent on the gauge of the two deleted particles) and won't vanish when their elements come from different helicity sets. As a warm up, we show in the first subsection the Pfaffian, which is the product of only closed cycles [22], factorizes. Though the Pfaffian equals zero, it very non-trivially factorizes into determinants of two matrices. Also it is the naturel way to introduce the two matrices $\mathbf{h}_k$ and $\tilde{\mathbf{h}}_{n-k}$ (1.5). In the next subsection we carefully deal with the open cycle and finally factorize the reduced Pfaffian to two reduced determinants.

3.1 The Pfaffian in 4d

Let's start with the Pfaffian, $\text{Pf}\Psi_n$. In 4 dimension, $f^{\mu\nu}$ reduces to a self-dual part and an anti-self-dual part: $f^{\mu\nu} \rightarrow \epsilon^{\alpha\beta} f^{\dot{\alpha}\dot{\beta}} + \epsilon^{\dot{\alpha}\dot{\beta}} f^{\alpha\beta}$. We denote these two parts as f^- and f^+ respectively. An important property is that any two adjoint linearised strength fields $f_b^- f_p^+$ in the trace can exchange their place if the helicity of b, p are different, i.e.

$$\cdots f_b^- f_p^+ \cdots = \cdots f_p^+ f_b^- \cdots. \quad (3.7)$$

So we can always reduce those traces where particles of negative or positive helicity are mixed each other to split ones which have a simple reduction in 4d. Then

$$\text{tr} (f_{a_1} f_{a_2} \cdots f_{a_i}) = \begin{cases} 2 \langle a_1 a_2 \rangle \langle a_2 a_3 \rangle \cdots \langle a_i a_1 \rangle, & \{a_1, a_2, \cdots, a_i\} \subset - \\ 2 [a_i a_{i-1}] [a_{i-1} a_{i-2}] \cdots [a_1 a_i], & \{a_1, a_2, \cdots, a_i\} \subset + \\ \langle b_1 b_2 \rangle \cdots \langle b_x b_1 \rangle [p_y p_{y-1}] \cdots [p_1 p_y], & \text{otherwise} \end{cases}, \quad (3.8)$$

Here $b_1, b_2, \cdots, b_x$ are all the particles of negative helicity from $a_1, a_2, \cdots, a_i$ with their ordering unchanged and similarly $p_1, p_2, \cdots, p_y$ are all the particles of positive helicity from $a_1, a_2, \cdots, a_i$ with their ordering unchanged. Note that $\text{tr} (f_{a_1} f_{a_2} \cdots f_{a_i})$ directly vanishes if there is only one particle of negative helicity or only one particle of positive

helicity in $a_1, a_2, \dots, a_i$. However we see that the remaining case still effectively vanish as we always add up all permutations (see (3.1)) while

$$\sum_{\{\alpha\} \in \text{OP}(\{b_1, b_2, \dots, b_x\}, \{p_1, p_2, \dots, p_y\})} \frac{1}{\sigma(\{\alpha\})} = 0. \quad (3.9)$$

Here the sum is over ordered permutations “OP”, namely permutations of the labels in the joined set $\{b_1, b_2, \dots, b_x\}, \{p_1, p_2, \dots, p_y\}$ such that the ordering within $\{b_1, b_2, \dots, b_x\}$ and $\{p_1, p_2, \dots, p_y\}$ is preserved. Therefore, in the sum of (3.1), we can effectively write $\text{tr}(f_{a_1} f_{a_2} \dots f_{a_i})$ in 4d in a remarkably simple way:

$$\frac{1}{2} \text{tr}(f_{a_1} f_{a_2} \dots f_{a_i}) \rightarrow \begin{cases} \langle a_1 a_2 \rangle \langle a_2 a_3 \rangle \dots \langle a_i a_1 \rangle, & \{a_1, a_2, \dots, a_i\} \subset - \\ [a_1 a_2] [a_2 a_3] \dots [a_i a_1], & \{a_1, a_2, \dots, a_i\} \subset + \\ 0, & \text{otherwise} \end{cases} \quad (3.10)$$

Motivated by (3.10), we recall the off-diagonal elements of the $k \times k$ matrix $\mathbf{h}_k$ and $(n-k) \times (n-k)$ one $\tilde{\mathbf{h}}_{n-k}$ essentially introduced in [12] (see also [17, 18]):

$$h_{ab} = \frac{\langle ab \rangle}{\sigma_{ab}} \quad a \neq b, \quad a, b \in -, \quad \tilde{h}_{ab} = \frac{[ab]}{\sigma_{ab}} \quad a \neq b, \quad a, b \in +. \quad (3.11)$$

It is clear that when we have any cycle factor with length at least 2, it must be given by the chain product of such off-diagonal elements

$$\Psi_{(a_1 a_2 \dots a_i)} \rightarrow \begin{cases} h_{a_1 a_2} h_{a_2 a_3} \dots h_{a_i a_1} & \{a_1, a_2, \dots, a_i\} \subset - \\ \tilde{h}_{a_1 a_2} \tilde{h}_{a_2 a_3} \dots \tilde{h}_{a_i a_1} & \{a_1, a_2, \dots, a_i\} \subset + \\ 0 & \text{otherwise} \end{cases} \quad (3.12)$$

To this point we have not used scattering equations and solution sectors in four dimensions. The non-trivial part of the reduction concerns 1-cycle, or the diagonal entries of C -matrix. Note that $\Psi_{(a)} = C_{aa}$ is only gauge invariant on the support of scattering equations, so it is not surprising that to reduce it nicely one needs to use scattering equations in four dimensions. Now we derive the explicit expression of C_{aa} . When $a \in -$ and $a \in -'$, we have

$$C_{aa}^- = - \sum_{b \in -', b \neq a} \frac{\langle ab \rangle [b\mu]}{[a\mu] \sigma_{ab}} - \sum_{p \in +'} \frac{\langle ap \rangle [p\mu]}{[a\mu] \sigma_{ap}}. \quad (3.13)$$

Note that C_{aa} depends on σ and because of the 4d scattering equations (1.2), we can make the change of variables

$$\tilde{\lambda}_b^\alpha = t_b \sum_{p \in +'} \frac{t_p \tilde{\lambda}_p^\alpha}{\sigma_{bp}} \quad \text{for } b \in -', \quad \lambda_p^\alpha = t_p \sum_{b \in -'} \frac{t_b \lambda_b^\alpha}{\sigma_{pb}} \quad \text{for } p \in +', \quad (3.14)$$

Such that C_{aa}^- reduces to:

$$\begin{aligned}
 C_{aa}^- &= -\frac{1}{[a\mu]} \sum_{b \neq a; p} \left(\frac{\langle ab \rangle t_b t_p [p\mu]}{\sigma_{bp} \sigma_{ab}} + \frac{\langle ab \rangle t_b t_p [p\mu]}{\sigma_{pb} \sigma_{ap}} \right) \\
 &= -\frac{1}{[a\mu]} \sum_{b \neq a; p} \frac{\langle ab \rangle t_b t_p [p\mu]}{\sigma_{bp}} \left(\frac{1}{\sigma_{ab}} - \frac{1}{\sigma_{ap}} \right) \\
 &= -\frac{1}{[a\mu]} \sum_{b \neq a; p} \frac{\langle ab \rangle t_b t_p [p\mu]}{\sigma_{ab} \sigma_{ap}}. \tag{3.15}
 \end{aligned}$$

In the last equality, we have collected the denominators together such that σ_{bp} is canceled. Now C_{aa}^- factorizes into two factors

$$C_{aa}^- = - \left(\sum_{b \neq a} \frac{t_b \langle ab \rangle}{t_a \sigma_{ab}} \right) \left(\sum_p \frac{t_a t_p [p\mu]}{\sigma_{ap} [a\mu]} \right) = - \sum_{b \in -', b \neq a} \frac{t_b \langle ab \rangle}{t_a \sigma_{ab}}. \tag{3.16}$$

All gauge dependence is in the latter factor and it can be eliminated by scattering equations as $t_a \sum_{p \in +'} \frac{t_p \tilde{\lambda}_p^\alpha}{\sigma_{ap}} = \tilde{\lambda}_a^\alpha$ (3.14).

Similarly we can work out the case of $a \in +$ and $a \in +'$

$$C_{aa}^+ = - \sum_{b \neq a; b \in +'} \frac{t_b [ab]}{t_a \sigma_{ab}}. \tag{3.17}$$

We first discuss the $k' = k$ case and without loss of generality let's consider $-' = -$, which makes our discussion simpler. Then the above two cases are already enough here, postponing other two cases in the following sections. Miraculously, C_{aa} reduces to diagonal entries of $\mathbf{h}_k$ or $\tilde{\mathbf{h}}_{n-k}$ [12] depending on the helicity:

$$h_{aa} = C_{aa}^- = - \sum_{\substack{b \neq a \\ b \in -}} \frac{t_b \langle ab \rangle}{t_a \sigma_{ab}} \quad a \in -, \quad \tilde{h}_{aa} = C_{aa}^+ = - \sum_{\substack{b \neq a \\ b \in +}} \frac{t_b [ab]}{t_a \sigma_{ab}} \quad a \in +. \tag{3.18}$$

The important thing is that the diagonal entry is a linear combination of off-diagonal entries in that row/column. With these diagonal entries of $\mathbf{h}_k$ or $\tilde{\mathbf{h}}_{n-k}$, the reduction for $\Psi_{(a_1 a_2 \dots a_i)}$ with $i > 1$ or $i = 1$ (for $k' = k$) are both spelled out in one nice formula, (3.12).

We find $h_{a_1 a_2} h_{a_2 a_3} \dots h_{a_i a_1}$ in (3.12) is just the ingredient of $\det \mathbf{h}_k$,

$$\det \mathbf{h}_k = \sum_{q \in S_k} \text{sgn}(q) h_{I_1} h_{I_2} \dots h_{I_s}, \quad \text{with } h_I = h_{(a_1 a_2 \dots a_i)} = h_{a_1 a_2} h_{a_2 a_3} \dots h_{a_i a_1}, \tag{3.19}$$

where the sum is over all permutations of particles of negative helicity, i.e. $q \in S_k$ and $I_1, I_2, \dots, I_s$ are the cycles of the permutation q . Similarly works for $\tilde{h}_{a_1 a_2} \tilde{h}_{a_2 a_3} \dots \tilde{h}_{a_i a_1}$.

Then, we see that $\text{Pf} \Psi_n$ factorizes to two parts depending on particles of negative or positive helicity respectively, with most of the terms vanishing and the surviving terms combining to $\det \mathbf{h}_k$ or $\det \tilde{\mathbf{h}}_{n-k}$,

$$\text{Pf} \Psi_n \big|_{k'=k} = \det \mathbf{h}_k \det \tilde{\mathbf{h}}_{n-k}. \tag{3.20}$$

Obviously both $\det \mathbf{h}_k$ and $\det \tilde{\mathbf{h}}_{n-k}$ vanish since they both have a null vector; this is consistent with the fact that $\text{Pf} \Psi_n$ vanishes due to the two null vectors.

3.2 Reduced Pfaffian in 4 dimensions

Now we turn to $\text{Pf}'\Psi_n$. Now we need to deal with the open cycle. Similarly, we can always reduce these mixed open brackets into split one as any two adjoint linearised strength fields $f_b^- f_p^+$ in the kinematic numerator of open brackets $\epsilon_1 \cdots f_b^- f_p^+ \cdots \epsilon_n$ can exchange their place if the helicity of b, p are different, i.e.

$$\epsilon_1 \cdots f_b^- f_p^+ \cdots \epsilon_n = \epsilon_1 \cdots f_p^+ f_b^- \cdots \epsilon_n. \quad (3.21)$$

Note that this equality is true no matter what the helicity of 1 and n are. In the following demonstration we need to delete two columns and rows from negative and positive helicity set respectively, so we assign 1^- and n^+ . Using this property, we can always rearrange the kinematic numerator in a split form with the ordering of particles of negative helicity and the ordering of particles of positive helicity unchanged respectively. For example, with $n > 6$,

$$\begin{aligned} \epsilon_1^- \cdot f_5^+ f_2^- f_3^- f_6^+ f_4^- \cdot \epsilon_n^+ &= \epsilon_1^- \cdot f_2^- f_5^+ f_3^- f_6^+ f_4^- \cdot \epsilon_n^+ = \cdots = \epsilon_1^- \cdot f_2^- f_3^- f_4^- f_5^+ f_6^+ \cdot \epsilon_n^+ \\ &= \frac{2 \langle 12 \rangle \langle 23 \rangle \langle 34 \rangle \langle 4\mu \rangle [\mu 5] [56] [6n]}{[1\mu] \langle n\mu \rangle}. \end{aligned} \quad (3.22)$$

All $\binom{3}{5} = 10$ such kinematic numerators of open cycles whose ordering of negative and positive particles between 1 and n are 2, 3, 4 and 5, 6 respectively equal to $\epsilon_1^- \cdot f_2^- f_3^- f_4^- f_5^+ f_6^+ \cdot \epsilon_n^+$. Further on, all such kinematic numerator can reduce to a product of some simple angle brackets and square brackets as shown in the last equality. Here $|\mu\rangle, |\mu\rangle$ are the reference of 1, n respectively.

For the general case with x particles of negative and y particles of positive helicity between 1 and n , there are $\binom{x}{x+y}$ cycles whose kinematic numerators are equal to those of a certain split open cycles and they all reduce to a product of some simple angle brackets and square brackets,

$$\frac{1}{2} \epsilon_1^- \cdot f_{b_1}^- f_{b_2}^- \cdots f_{b_x}^- f_{p_y}^+ f_{p_{y-1}}^+ \cdots f_{p_1}^+ \cdot \epsilon_n^+ = \frac{\langle 1b_1 \rangle \langle b_1 b_2 \rangle \cdots \langle b_{x-1} b_x \rangle \langle b_x \mu \rangle [\mu p_y] [p_y p_{y-1}] \cdots [p_2 p_1] [p_1 n]}{[1\mu] \langle n\mu \rangle}. \quad (3.23)$$

Here $|\mu\rangle, |\mu\rangle$ are the reference of 1, n respectively, i.e. $\epsilon_1^- = \frac{|1\rangle[\mu]}{[1\mu]}, \epsilon_n^+ = \frac{|n\rangle\langle\mu|}{\langle n\mu \rangle}$ and we have used the reversed ordering $p_y, p_{y-1}, \cdots, p_1$ for later convenience.

Since $\binom{x}{x+y}$ such open brackets share same kinematic numerator, we try to combine their denominators. They happen to be combined to the partial fraction identity (analogous to Kleiss-Kuijff relations of amplitudes),

$$(-)^{|\rho|} \sum_{\{\alpha\} \in \text{OP}(\{\beta\}, \{\rho^T\})} \frac{1}{\sigma_{(1, \{\alpha\}, n)}} = \frac{1}{\sigma_{(1, \{\beta\}, n, \{\rho\})}}, \quad (3.24)$$

here $\{\alpha\}$ means $a_2, a_3, \cdots, a_{i-1}$ and $\{\beta\}, \{\rho\}$ means $b_1, b_2, \cdots, b_x$, and $p_1, p_2, \cdots, p_y$ respectively. $\{\rho^T\}$ denotes the reverse ordering of the labels $\{\rho\}$.

Then $(-)^{|\rho|} \sum_{\{\alpha\} \in \text{OP}(\{\beta\}, \{\rho^T\})} \Psi_{[1a_2 \dots a_{i-1}n]}$ combines to

$$\begin{aligned} \frac{\frac{1}{2} \epsilon_1^- \cdot f_{b_1}^- f_{b_2}^- \dots f_{b_x}^- f_{p_y}^+ f_{p_{y-1}}^+ \dots f_{p_1}^+ \cdot \epsilon_n^+}{\sigma_{(1b_1 \dots b_x n p_1 p_2 \dots p_y)}} &= \left(h_{1b_1} h_{b_1 b_2} \dots h_{b_{x-1} b_x} \frac{\langle b_x \mu \rangle}{\langle n \mu \rangle \sigma_{b_x n}} \right) \left(\tilde{h}_{np_1} \tilde{h}_{p_1 p_2} \dots \tilde{h}_{p_{y-1} p_y} \frac{[\mu p_y]}{[1\mu] \sigma_{1p_y}} \right) \\ &:= h_{[1b_1 b_2 \dots b_x]} \tilde{h}_{[np_1 p_2 \dots p_y]} \end{aligned} \quad (3.25)$$

In the first equality, we have plugged in (3.23). In the second equality, we have defined $h_{[1b_1 b_2 \dots b_x]}$ as $h_{1b_1} h_{b_1 b_2} \dots h_{b_{x-1} b_x} \frac{\langle b_x \mu \rangle}{\langle n \mu \rangle \sigma_{b_x n}}$ and $\tilde{h}_{[np_1 p_2 \dots p_y]}$ as $\tilde{h}_{np_1} \tilde{h}_{p_1 p_2} \dots \tilde{h}_{p_{y-1} p_y} \frac{[\mu p_y]}{[1\mu] \sigma_{1p_y}}$. Here we can treat 1 as b_0 and if there is no particles of negative helicity between 1 and n , $\frac{\langle b_x \mu \rangle}{\langle n \mu \rangle \sigma_{b_x n}}$ reduces to $\frac{\langle 1 \mu \rangle}{\langle n \mu \rangle \sigma_{1n}}$; similarly we can treat n as p_0 and if there is no particles of positive helicity between 1 and n , $\frac{[\mu p_y]}{[1\mu] \sigma_{1p_y}}$ reduces to $\frac{[\mu n]}{[1\mu] \sigma_{1n}}$. Note that these prefactors $\frac{\langle b_x \mu \rangle}{\langle n \mu \rangle \sigma_{b_x n}}$, $\frac{[\mu p_y]}{[1\mu] \sigma_{1p_y}}$ only depend on b_x or c_y respectively.

Though $\Psi_{[1a_2 \dots a_{i-1}n]}$ has particles with mixed helicity, $h_{[1b_1 b_2 \dots b_x]}$ and $\tilde{h}_{[np_1 p_2 \dots p_y]}$ do have only particles of negative or positive helicity respectively. Adding that closed cycles vanish unless all of their elements have same helicity, $\text{Pf}' \Psi_n$ decouples to two parts which are dependent on particles of negative and positive helicity respectively,

$$\text{Pf}' \Psi_n = \left(\text{sgn}(r) \sum_{\beta} h_{[1b_1 \dots b_x]} \sum_{I \dots J} h_I \dots h_J \right) \left(\text{sgn}(\tilde{r}) \sum_{\rho} \tilde{h}_{[1p_1 \dots p_y]} \sum_{K \dots L} \tilde{h}_K \dots \tilde{h}_L \right). \quad (3.26)$$

Here we have explicitly written out the open cycles to emphasis them. $\beta, I, \dots, J$ are the cycles of permutations r of negative helicity particles except 1 and $\rho, K, \dots, L$ are the cycles of permutations $\tilde{r}$ of positive helicity except n .

For example, with $1^- 2^- 3^+ 4^+$,

$$\text{Pf}' \Psi_4 = \left(h_{[1]} h_{(2)} + h_{[12]} \right) \left(\tilde{h}_{[4]} \tilde{h}_{(3)} + \tilde{h}_{[43]} \right), \quad (3.27)$$

with $1^- 2^- 3^- 4^+ 5^+$,

$$\begin{aligned} \text{Pf}' \Psi_5 &= \left(h_{[1]} h_{(2)} h_{(3)} + h_{[1]} h_{(23)} + h_{[12]} h_{(3)} + h_{[132]} + h_{[13]} h_{(2)} + h_{[123]} \right) \\ &\quad \times \left(\tilde{h}_{[5]} \tilde{h}_{(4)} + \tilde{h}_{[54]} \right). \end{aligned} \quad (3.28)$$

Without the loss of generality, we let $- = \{1, 2, \dots, k\}$ and $+ = \{k+1, k+2, \dots, n\}$. We try to prove the two parts in (3.26) combine to two reduced determinants of matrices $\mathbf{h}_k$, $\tilde{\mathbf{h}}_{n-k}$ respectively, defined as $\det' \mathbf{h}_k = \frac{\det |\mathbf{h}_k|_{t_b}^{t_c}}{t_b t_c}$, $\det' \tilde{\mathbf{h}}_{n-k} = \frac{\det |\tilde{\mathbf{h}}_{n-k}|_{t_p}^{t_q}}{t_p t_q}$ with $b, c \in -, p, q \in +$ as $\mathbf{h}_k$ has a null vector $(t_1, t_2, \dots, t_k)$ and $\tilde{\mathbf{h}}_{n-k}$ has a null vector $(t_{k+1}, t_{k+2}, \dots, t_n)$.

Note that

$$\det |\mathbf{h}_k|_1^{b_x} = (-)^x \left(\det |\mathbf{h}_k|_1^1 \right) \Big|_{h_{b_x c} \rightarrow h_{1c}} = (-)^x \sum_{r \in S_{k-1}} \text{sgn}(r) h_{(b_x \dots)} \Big|_{h_{b_x c} \rightarrow h_{1c}} h_I \dots h_J. \quad (3.29)$$

Here r is any permutation of particles of negative helicity except 1, and $(b_x \dots), I, \dots, J$ are the cycles of r . c can be anyone of $1, 2, \dots, k$.

Since

$$h_{[1b_1b_2\cdots b_x]} = \frac{\langle b_x \mu \rangle}{\langle n \mu \rangle \sigma_{b_x n}} h_{b_1 b_2 \cdots b_{x-1} b_x} h_{b_x b_1} \Big|_{h_{b_x b_1} \rightarrow h_{1b_1}} = \frac{\langle b_x \mu \rangle}{\langle n \mu \rangle \sigma_{b_x n}} (b_1 b_2 \cdots b_x)_h \Big|_{h_{b_x b_1} \rightarrow h_{1b_1}} \quad (3.30)$$

we write the first part in the r.h.s. of (3.26) as a sum over all possible b_x , i.e. $b_x = 1, 2, \dots, k$. This equality can also be seen by collecting terms with the same prefactor $\frac{\langle b_x \mu \rangle}{\langle n \mu \rangle \sigma_{b_x n}}$,

$$\text{sgn}(r) \sum_{\beta} h_{[1b_1 \cdots b_x]} \sum_{I \cdots J} h_I \cdots h_J = \sum_{b_x=1}^k \left(\text{sgn}(r) \sum_{\beta'} h_{[1 \underbrace{b_1 \cdots b_{x-1}}_{\beta'} b_x]} \sum_{I \cdots J} h_I \cdots h_J \right), \quad (3.31)$$

Here $\beta' = \{b_1, \dots, b_{x-1}\}$. $\beta', I, \dots, J$ are the cycles of permutations of particles of negative helicity except 1 and b_x . Then each term of the summation in r.h.s. of the above equation equals $\det |\mathbf{h}_k|_1^{b_x}$ up to a prefactor. Summing over all possible b_x , i.e. $b_x = 1, 2, \dots, k$, gives the left parenthesis of r.h.s. in (3.26). Similar derivations leads to the right parenthesis. Then

$$\text{Pf}' \Psi_n = \left(\sum_{b_x=1}^k \frac{\langle b_x \mu \rangle}{\langle n \mu \rangle \sigma_{b_x n}} \det |\mathbf{h}_k|_1^{b_x} \right) \left(\sum_{p_y=k+1}^n \frac{[\mu p_y]}{[1\mu] \sigma_{1p_y}} \det |\tilde{\mathbf{h}}_{n-k}|_n^{p_y} \right). \quad (3.32)$$

We insert $\frac{t_n t_{b_x}}{t_1 t_{b_x}}$ in every term of the first sum of above equation and $\frac{t_1 t_{p_y}}{t_n t_{p_y}}$ in every term of the second sum, which doesn't change the value of $\text{Pf}' \Psi_n$. Then

$$\text{Pf}' \Psi_n = \left(\sum_{b_x=1}^k \frac{\langle b_x \mu \rangle t_{b_x} t_n}{\langle n \mu \rangle \sigma_{b_x n}} \frac{\det |\mathbf{h}_k|_1^{b_x}}{t_1 t_{b_x}} \right) \left(\sum_{p_y=k+1}^n \frac{[\mu p_y] t_1 t_{p_y}}{[1\mu] \sigma_{1p_y}} \frac{\det |\tilde{\mathbf{h}}_{n-k}|_n^{p_y}}{t_n t_{p_y}} \right). \quad (3.33)$$

While all $\frac{\det |\mathbf{h}_k|_1^{b_x}}{t_1 t_{b_x}}$ with $b_x = 1, 2, \dots, k$ reduce to $\det' \mathbf{h}_k$, all $\frac{\det |\tilde{\mathbf{h}}_{n-k}|_n^{p_y}}{t_n t_{p_y}}$ with $p_y = k+1, k+2, \dots, n$ reduce to $\det' \tilde{\mathbf{h}}_{n-k}$. Then

$$\text{Pf}' \Psi_n = \frac{\sum_{b_x=1}^k \frac{t_{b_x} t_n \langle b_x \mu \rangle}{\sigma_{b_x n}}}{\langle n \mu \rangle} \frac{\sum_{p_y=k+1}^n \frac{t_1 t_{p_y} [\mu p_y]}{\sigma_{1p_y}}}{[1\mu]} \det' \mathbf{h}_k \det' \tilde{\mathbf{h}}_{n-k}. \quad (3.34)$$

All gauge dependence of particle 1 and n combine to one factor respectively and on the support of 4d scattering equation (1.2),

$$t_n \sum_{b \in -'} \frac{t_b \lambda_b^\alpha}{\sigma_{nb}} = \lambda_n^\alpha, \quad t_1 \sum_{p \in +'} \frac{t_p \tilde{\lambda}_p^\alpha}{\sigma_{1p}} = \tilde{\lambda}_1^\alpha, \quad (3.35)$$

the two prefactors before the determinants in (3.34) reduce to 1 respectively. Then we get

$$\text{Pf}' \Psi_n \Big|_{k'=k} = \det' \mathbf{h}_k \det' \tilde{\mathbf{h}}_{n-k}. \quad (3.36)$$

For example, with $1^-2^-3^-4^+5^+$,

$$\begin{aligned}
 \text{Pf}'\Psi_5 &= \left(\left(h_{(23)} + h_{(2)}h_{(3)} \right) \frac{\langle 1\mu \rangle}{\langle 5\mu \rangle \sigma_{15}} + \left(h_{(2)} \Big|_{h_{22} \rightarrow h_{12}} h_{(3)} + h_{(32)} \Big|_{h_{23} \rightarrow h_{13}} \right) \frac{\langle 2\mu \rangle}{\langle 5\mu \rangle \sigma_{25}} \right. \\
 &\quad \left. + \left(h_{(2)}h_{(3)} \Big|_{h_{33} \rightarrow h_{13}} + h_{(23)} \Big|_{h_{32} \rightarrow h_{12}} \right) \frac{\langle 3\mu \rangle}{\langle 5\mu \rangle \sigma_{35}} \right) \\
 &\quad \times \left(\frac{[\mu 5]}{[1\mu] \sigma_{15}} \tilde{h}_{(4)} + \frac{[\mu 4]}{[1\mu] \sigma_{14}} \tilde{h}_{(4)} \Big|_{\tilde{h}_{44} \rightarrow \tilde{h}_{54}} \right) \\
 &= \sum_{a_x=1}^3 \frac{\langle b_x \mu \rangle}{\langle 5\mu \rangle \sigma_{b_x 5}} \det |\mathbf{h}_3|_1^{a_x} \sum_{b_y=4}^5 \frac{[\mu p_y]}{[1\mu] \sigma_{1p_y}} \det |\tilde{\mathbf{h}}_2|_5^{b_y} \\
 &= \det' \mathbf{h}_3 \det' \tilde{\mathbf{h}}_2
 \end{aligned} \tag{3.37}$$

4 Extension to all solution sectors

We have arrived at (3.32) without using the explicitly form of 1-length cycle, i.e. C_{aa} . When extended to all solution sectors, those cycles whose length are longer than 1 don't change, while the 1-length cycles change to C_{aa} with the solutions of k' sectors plugged in. That is, we need to enhance the origin two matrices to $\mathbf{h}_k^{k'}$ and $\tilde{\mathbf{h}}_{n-k}^{k'}$ with their diagonal entries depending on the solution sector k' while the off-diagonal entries unchanged. The expression of C_{aa} with $a \in -$ and $a \in -'$ has been given in (3.16). Note that this expression is true even when $k' \neq k$,

$$h_{aa} = - \sum_{\substack{b \neq a \\ b \in -'}} \frac{t_b}{t_a} \frac{\langle ab \rangle}{\sigma_{ab}} \quad a \in - \text{ and } a \in -'. \tag{4.1}$$

Now we derive the expression of C_{aa} with a not consistent in helicity sector and solution sector. When $a \in -$ but $a \notin -'$, we have

$$C_{aa} = - \sum_{p \in +', p \neq a} \frac{\langle ap \rangle [p\mu]}{[a\mu] \sigma_{ap}} - \sum_{b \in -'} \frac{\langle ab \rangle [b\mu]}{[a\mu] \sigma_{ab}} \tag{4.2}$$

After we plug in the changes of variables (3.14), unlike (3.15), terms with $a \in +'$ and $p = a$ both contribute.

$$C_{aa} = - \frac{1}{[a\mu]} \sum_{\substack{p \in +', p \neq a \\ b \in -'}} \left(\frac{\langle ab \rangle t_b t_p [p\mu]}{\sigma_{pb} \sigma_{ap}} + \frac{\langle ab \rangle t_b t_p [p\mu]}{\sigma_{bp} \sigma_{ab}} \right) + \sum_{b \in -'} \frac{\langle ab \rangle t_b t_a}{\sigma_{ba}^2} \tag{4.3}$$

The first term on the r.h.s. also factorizes into two parts following the trick used in (3.15), (3.16),

$$- \frac{1}{[a\mu]} \sum_{\substack{p \in +', p \neq a \\ b \in -'}} \frac{\langle ab \rangle t_b t_p [p\mu]}{\sigma_{ab} \sigma_{ap}} = - \left(\sum_{b \in -'} \frac{t_a t_b \langle ab \rangle}{\sigma_{ab}} \right) \left(\sum_{p \in +', p \neq a} \frac{t_p [p\mu]}{t_a \sigma_{ap} [a\mu]} \right) = 0, \tag{4.4}$$

while it vanishes as shown in the last equality because the part in the first parenthesis vanishes on the support of 4d scattering equation (1.2), (note that $a \in +'$)

$$t_a \sum_{b \in -'} \frac{t_b \lambda_b^\alpha}{\sigma_{ab}} = \lambda_a^\alpha \quad (4.5)$$

then we see that C_{aa} only has contribution from the term of $p = a$, and we obtain

$$C_{aa} = - \sum_{\substack{b < c \\ b, c \in -'}} \frac{\langle bc \rangle t_b t_c t_a^2 \sigma_{bc}}{\sigma_{ba}^2 \sigma_{ca}^2} \quad \text{when } a \in - \text{ but } a \notin -' . \quad (4.6)$$

Consequently, we have

$$h_{aa} = -t_a^2 \sum_{b < c; b, c \in -'} \frac{t_b t_c \sigma_{bc} \langle bc \rangle}{\sigma_{ab}^2 \sigma_{ac}^2} \quad a \in - \text{ but } a \notin -' , \quad (4.7)$$

By a parity transformation, we can directly obtain $\tilde{h}_{aa}$

$$\begin{aligned} \tilde{h}_{aa} &= - \sum_{\substack{b \neq a \\ b \in +'}} \frac{t_b}{t_a} \frac{[ab]}{\sigma_{ab}} & a \in + \text{ and } a \in +' \\ \tilde{h}_{aa} &= -t_a^2 \sum_{b < c; b, c \in +'} \frac{t_b t_c \sigma_{bc} [bc]}{\sigma_{ab}^2 \sigma_{ac}^2} & a \in + \text{ but } a \notin +' . \end{aligned} \quad (4.8)$$

When $k' = k$, these extended matrices come back to their original ones. When $k' \neq k$, one of $\det \mathbf{h}_k^k$, $\det \tilde{\mathbf{h}}_{n-k}^k$ must vanish. Further more, when $k' < k$, after deleting appropriate row and column of $\mathbf{h}_k^k$, the determinant of the remaining matrix still vanishes, so does the $\tilde{\mathbf{h}}_{n-k}^k$ when $k' > k$, which results in the vanishing of $\text{Pf}' \Psi_n$ in $k' \neq k$ sectors. We will discuss this in section 4.1. Some integrands receive the contribution from the $k' \neq k$ sectors, such as $\mathcal{P}_n$, which will be discussed in section 4.2.

4.1 The vanishing of reduced Pfaffian in other sectors

We start from the equation (3.32). Note that we have got this by deleting the 1th and n^{th} rows and columns of Ψ_n . We can also delete other rows and columns to get a similar expression. What's more, along the demonstration of (3.5) to (3.32), we don't use the scattering equations (1.2), in other words (3.32) is true for any solutions. After (3.32), the scattering equation is used and we demonstrate (3.36). Now we move to other solution sectors. Without loss of generality, let's consider $- = \{1, 2, \dots, k\}$ while $- ' = \{1, 2, \dots, k'\}$, which makes our discussion simpler. Then when $k' < k$, further on, we can also and always delete $(k' + 1)^{\text{th}}$ and n^{th} column and row instead of the of 1th and n^{th} ones, then the reduced Pfaffian becomes

$$\text{Pf}' \Psi_n = \left(\sum_{b_x=1}^k \frac{\langle b_x \mu \rangle}{\langle n \mu \rangle \sigma_{b_x n}} \det |\mathbf{h}_k^{k'}|_{k'+1}^{b_x} \right) \left(\sum_{p_y=k+1}^n \frac{[\mu p_y]}{[k'+1, \mu] \sigma_{k'+1, p_y}} \det |\tilde{\mathbf{h}}_{n-k}^{n-k'}|_n^{p_y} \right) \quad (4.9)$$

Notice that we still have to calculate the determinants of series of matrices. Instead of both summations in r.h.s. of (4.9) being combined to simple factors as shown in (3.34), we show that the determinants of matrices $|\mathbf{h}_k^{k'}|_{k'+1}^{b_x}$ with $b_x = 1, 2, \dots, k$ in the first summation vanish identically.

These matrices all come from the original matrix $\mathbf{h}_k^{k'}$ with the $(k'+1)^{\text{th}}$ column deleted and the $1^{\text{th}}, 2^{\text{th}}, \dots, k^{\text{th}}$ row deleted respectively. An important observation is that the first k' columns of these matrices are linearly dependent as

$$t_1 \boldsymbol{\eta}_1 + t_2 \boldsymbol{\eta}_2 + \dots + t_{k'} \boldsymbol{\eta}_{k'} = \mathbf{0} \quad (4.10)$$

here $\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \dots, \boldsymbol{\eta}_{k'}$ are the $1^{\text{th}}, 2^{\text{th}}, \dots, k'^{\text{th}}$ columns of anyone of matrices $|\mathbf{h}_k^{k'}|_{k'+1}^{b_x}$ with $b_x = 1, 2, \dots, k$. This is equivalent to say that

$$\sum_{b=1}^{k'} t_b h_{ab} = 0, \quad \text{for } a = 1, 2, \dots, k \quad (4.11)$$

These equations come from two totally different origins as $a \leq k'$ or $a > k'$.

For the case of $a \in -'$ i.e. $a = 1, 2, \dots, k'$, the establishment of (4.11) come from the fact that the diagonal elements h_{aa} are a linear combination of some off-diagonal entries as shown in (4.1) with some appropriate coefficients.

While for the cases of $a > k'$, note that a now belongs to the set $+$, and the validity of (4.11) come from the change of variables (3.14). What we need here is the cases of $a = k' + 1, k' + 2, \dots, k$,

$$t_a \sum_{b \in -'} \frac{t_b \lambda_b^\alpha}{\sigma_{ab}} = \lambda_a^\alpha \quad \text{for } a = k' + 1, k' + 2, \dots, k \quad (4.12)$$

Obviously after we act λ_a on both sides of above equation, both sides vanish, that is

$$\sum_{b=1}^{k'} t_b h_{ab} = \sum_{b=1}^{k'} t_b \frac{\langle ab \rangle}{\sigma_{ab}} = 0 \quad \text{for } a = k' + 1, k' + 2, \dots, k \quad (4.13)$$

After understanding (4.10), now it is easy to understand the vanishing of all matrices $|\mathbf{h}_k^{k'}|_{k'+1}^{b_x}$ with $b_x = 1, 2, \dots, k$. We take a multiple of the a^{th} row of these matrix by t_a for $a = 1, 2, \dots, k'$, and then add $2^{\text{th}}, 3^{\text{th}}, \dots, k'^{\text{th}}$ column to the 1^{th} column; in this way we obtain a new 1^{th} column whose entries all equal to zero because of (4.10). Since we just do some fundamental operation on these matrices and we obtain a column with all entries equal to zero, all determinants of these matrices vanish.

When $k' < k$, the determinants of matrices in the first summation in (4.9) vanish; when $k' > k$, the determinants of matrices in the second summation in (4.9) vanish. $\text{Pf}' \boldsymbol{\Psi}_n$ only receives the contribution of $k' = k$ sector, then we proved the identity (1.4) given in the introduction.

In the reduction of $\text{Pf}' \boldsymbol{\Psi}_n$, we reorganize the Pfaffian using some fundamental (almost) gauge invariant objects and then deal with these objects, finally we reconstruct the reduced Pfaffian using $\det \mathbf{h}_k$ and $\det \tilde{\mathbf{h}}_{n-k}$. This procedure is quite general. The reduced

Pfaffian can be thought as putting two kinematic of the deleted particles in higher dimensions and the Lorentz contraction of them and anything else vanish unless contraction of them each other equal to 1.¹ Similarly, we can put polarizations of m pairs of particles in higher dimension and then we get the reduced compactified Pfaffian, which is the integrand of Einstein-Maxwell, Einstein-Maxwell-scalar, Yang-Mills-scalar, Born-Infeld amplitudes. This can be viewed as the “fancy reduced Pfaffian” as we may meet several open brackets in each term of the expansion. Besides, the valid solution sector for the reduced compactified Pfaffian is shifted to $k' = k + m$ sector with $2m$ the number of the particles whose polarization are set in higher dimension. These are presented in the Appendix B. Furthermore, the reduced squeezed Pfaffian, which is the integrand of Einstein-Yang-Mills, can be obtained by some combination of the reduced compactified Pfaffian. So all these methods can be applied in the reduced squeezed Pfaffian, seen in Appendix C.

There are some integrands that can't be organized as a matrix, let alone its Pfaffian, such as the $\mathcal{P}_n$ used in F^3, R^2, R^3 amplitudes. We can still reduce it into objects related to $\det \mathbf{h}_k$ and $\det \tilde{\mathbf{h}}_{n-k}$ and make some properties manifest. Besides, it receives the contribution from several sectors and one need to add up all of these to get the corresponding amplitudes, as discussed in the following subsection.

4.2 The new object $\mathcal{P}_n$ for higher dimension operator

As shown in [21], as a generalization of the reduced Pfaffian in Yang-Mills theory, $\mathcal{P}_n$ is a new, gauge-invariant object that leads to gluon amplitudes with a single insertion of F^3 , and gravity amplitudes by Kawai-Lewellen-Tye relations. When reduced to four dimensions for given helicities, this new object vanishes for any solution of scattering equations on which the reduced Pfaffian is non-vanishing. This intriguing behavior in four dimensions explains the vanishing of graviton helicity amplitudes produced by the Gauss-Bonnet R^2 term, and provides a scattering-equation origin of the decomposition into self-dual and anti-self-dual parts for F^3 and R^3 amplitudes.

No matter what $\mathcal{P}_n$ is, it must be gauge invariant. It's most natural to start from the expansion of Pfaffian in a manifest gauge invariant way (3.1). Reorganize these gauge invariant objects according to their length and we define the minimal gauge invariant and gauge invariant objects P as

$$P_{i_1 i_2 \dots i_r} := \sum_{|I_1|=i_1, |I_2|=i_2, \dots, |I_r|=i_r} \Psi_{I_1} \Psi_{I_2} \dots \Psi_{I_r} , \quad (4.14)$$

with $i_1 + i_2 + \dots + i_r = n$ and the convention $i_1 \leq i_2 \dots \leq i_r$.

Then Pfaffian can be written as

$$\text{Pf} \Psi_n = \sum_i (-)^{n-m} P_{i_1 i_2 \dots i_m} = 0 , \quad (4.15)$$

¹Here is some subtlety. We have to complex k_1, k_n set in higher dimensions such that they dotting anything else equal to 0, while they dotting each other equal to 1. It is just a mathematics trick after all nothing about k_1, k_n changes beyond the reduced Pfaffian.

Decorated with some appropriate coefficient, P can be used to build up some unknown CHY integrands, such as the new integrand is defined as

$$\mathcal{P}_n = \sum_{\substack{1 \leq i_1 \leq i_2 \leq \dots \leq i_m \leq n \\ i_1 + i_2 + \dots + i_m = n}} (-)^{n-m} (N_{i>1} + c) P_{i_1 i_2 \dots i_m}, \quad (4.16)$$

where $N_{i>1}$ denotes the number of indices in $i_1, i_2, \dots, i_m$ which are larger than 1, or the number of cycles with length at least 2; c is just any constant because we can add any multiplet of (4.15) without changing the answer.

In 4d, $\text{Pf}\Psi_n$ reduce to the determinants of $\mathbf{h}_k^{k'}$ and $\tilde{\mathbf{h}}_{n-k}^{k'}$. Similarly we define

$$H_{i_1 i_2 \dots i_\ell} = \sum_{|I_1|=i_1, |I_2|=i_2, \dots, |I_\ell|=i_\ell} h_{I_1} h_{I_2} \dots h_{I_\ell}, \quad (4.17)$$

with $i_1 + i_2 + \dots + i_\ell = k$ and the convention $i_1 \leq i_2 \leq \dots \leq i_\ell$. Then $\det \mathbf{h}_k^{k'}$ can be rewritten as a sum of H and similarly works $\det \tilde{\mathbf{h}}_{n-k}^{k'}$,

$$\det \mathbf{h}_k^{k'} = \sum_{\{i\}_k^\ell} (-)^{k-\ell} H_{i_1 i_2 \dots i_\ell}. \quad \det \tilde{\mathbf{h}}_{n-k}^{k'} = \sum_{\{\tilde{i}\}_{n-k}^\ell} (-)^{n-k-\ell} \tilde{H}_{\tilde{i}_1 \tilde{i}_2 \dots \tilde{i}_\ell}, \quad (4.18)$$

where we have introduced shorthand notation for the summation range, $\{i\}_k^\ell$ means $i_1 + i_2 + \dots + i_\ell = k$ and $i_1 \leq i_2 \leq \dots \leq i_\ell$ and similarly for $\{\tilde{i}\}_{n-k}^\ell$.

Further, similar to the definition of (4.16), we define two auxiliary objects $\mathcal{H}_k^{k'}$ and $\tilde{\mathcal{H}}_{n-k}^{k'}$,

$$\mathcal{H}_k^{k'} = \sum_{\{i\}_k^\ell} (-)^{k-\ell} N_{i>1} H_{i_1 i_2 \dots i_\ell}, \quad \tilde{\mathcal{H}}_{n-k}^{k'} = \sum_{\{\tilde{i}\}_{n-k}^\ell} (-)^{n-k-\ell} N_{\tilde{i}>1} \tilde{H}_{\tilde{i}_1 \tilde{i}_2 \dots \tilde{i}_\ell}. \quad (4.19)$$

Thanks to (3.12), P reduces to several products H and $\tilde{H}$ in 4d,

$$P_{i_1 i_2 \dots i_m} = \sum_{j, \tilde{j}} H_{j_1 j_2 \dots j_\ell} \tilde{H}_{\tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_{\tilde{\ell}}}, \quad (4.20)$$

Here the sum is over all distinct partition of $i_1 i_2 \dots i_m$ into two parts $j_1 j_2 \dots j_\ell$ and $\tilde{j}_1 \tilde{j}_2 \dots \tilde{j}_{\tilde{\ell}}$, with $j_1 + j_2 + \dots + j_\ell = k$ and $\tilde{j}_1 + \tilde{j}_2 + \dots + \tilde{j}_{\tilde{\ell}} = n - k$. Dividing $N_{i>1}$ in (4.16) into two parts $N_{j>1}$ and $N_{\tilde{j}>1}$ (set $c = 0$) which depend on $-$ and $+$ sets respectively, $\mathcal{P}_n$ reduces to:

$$\begin{aligned} \mathcal{P}_n = & \left(\sum_{\{i\}_k^\ell} (-)^{k-\ell} N_{i>1} H_{i_1 i_2 \dots i_\ell} \right) \left(\sum_{\{\tilde{i}\}_{n-k}^\ell} (-)^{n-k-\ell} \tilde{H}_{\tilde{i}_1 \tilde{i}_2 \dots \tilde{i}_{\tilde{\ell}}} \right) \\ & + \left(\sum_{\{i\}_k^\ell} (-)^{k-\ell} H_{i_1 i_2 \dots i_\ell} \right) \left(\sum_{\{\tilde{i}\}_{n-k}^\ell} (-)^{n-k-\ell} N_{\tilde{i}>1} \tilde{H}_{\tilde{i}_1 \tilde{i}_2 \dots \tilde{i}_{\tilde{\ell}}} \right) \end{aligned} \quad (4.21)$$

where each mix summation over i and $\tilde{i}$ decouples to two independent summation over i and over $\tilde{i}$ respectively. Then

$$\mathcal{P}_n|_{k'} = \mathcal{H}_k^{k'} \det \tilde{\mathbf{h}}_{n-k}^{k'} + \det \mathbf{h}_k^{k'} \tilde{\mathcal{H}}_{n-k}^{k'}. \quad (4.22)$$

When $k' = k$, both terms in the r.h.s. of (4.22) vanish, which is orthogonal to $\text{Pf}'\Psi_n$ and answers the vanishing of R^2 theory which is a Gauss-Bonnet term in 4 dimensions. When $k' < k$, the second term vanishes and $\mathcal{P}_n$ reduces to $\mathcal{H}_k \det \tilde{\mathbf{h}}_{n-k}^{k'}$, which gives the self-dual amplitude of F^3, R^3 theory etc. When $k' > k$, the first term vanishes and $\mathcal{P}_n$ reduces to $\det \mathbf{h}_k^{k'} \tilde{\mathcal{H}}_{n-k}$, which gives the anti-self-dual amplitude of those theory.

For example, the integrand for F^3 theory reads $\mathcal{I}_n^{F^3} = \mathcal{C}_n \mathcal{P}_n$. In 4 dimension, when $k' < k$, $I_n^{F^3}$ reads

$$I_n^{F^3} = \frac{\mathcal{C}_n \mathcal{H}_k^{k'} \det \tilde{\mathbf{h}}_{n-k}^{k'}}{\det' \mathbf{h}_{k'} \det' \tilde{\mathbf{h}}_{n-k'}} \quad (4.23)$$

Here $\det' \mathbf{h}_{k'} \det' \tilde{\mathbf{h}}_{n-k'}$ comes from the transition of two forms of scattering equations as shown in (2.6).

5 Discussion

In CHY representation, the fundamental gauge invariant objects are quite common, either C_{aa} or the trace of some linearised field strength together with some σ 's. In this paper, we find a rather general way to reduce this gauge invariant objects into that made up of spinors using 4d scattering equations. Particularly, we show how the reduced Pfaffian reduces to some determinants and why it vanishes on the support of most solutions. This explains why only some particular solutions contribute to the YM or GR amplitudes according to their helicity structure in 4d and provides a basis for dividing the solutions of scattering equations into MHV, NMHV, $\dots$, $\overline{\text{MHV}}$ sectors which contributes to corresponding YM or GR amplitudes respectively, also seen in [23].

We extend this methods to reduced compactified Pfaffian where some polarizations are set in higher dimensions, which is building block for EM, EMS, YMS or DBI amplitudes. We give the explicit reduction results of the compactified Pfaffian up to 3 pairs of particles whose polarizations are set in higher dimension and provides the general way to get the reduction with arbitrary pairs. Another interesting integrand with polarization involved is the reduced squeezed Pfaffian, which is the building block of EYM theory [3, 4, 24, 25]. As it can be expressed by some combination of the reduced compactified Pfaffian, its reduction in 4d is directly obtained from that of the reduced compactified Pfaffian. The results of one and two gluon color traces are explicitly presented in the appendix C.

Even some integrands which can't be organized as a matrix, let alone its Pfaffian, such as the new integrand $\mathcal{P}_n$ used in F^3, R^2, R^3 theory, also can be enclosed in this procedure. We decompose $\mathcal{P}_n$ to some fundamental gauge invariant objects, reduce these fundamental gauge invariant objects first and then organize them into a compacted form, which apparently shows most information of the $\mathcal{P}_n$, and explains the orthogonality of F^3 and YM amplitudes, the vanishing of Gauss-Bonnet term R^2 and the self-dual and anti-self-dual amplitudes of F^3 or R^3 amplitudes in 4d. In fact, we use these properties to guess what the compacted form in 4d of $\mathcal{P}_n$ should be, then fix the coefficient of the fundamental gauge invariant objects and finally get the $\mathcal{P}_n$. This is quite general to find the CHY integrand of an unknown theory. Even when the scattering equations or σ dependence of

the entries in the matrix Ψ_n has been changed, their CHY integrand are very likely to be decomposed into some C_{aa} or trace like fundamental gauge invariant objects. And we can reduce these objects first, organize them into a form manifest showing some properties the theory requires and finally confirm their CHY integrand. This can even be applied at loop level, as shown in [19, 26].

Instead of reducing all kinds of integrands in 4d, we now turn to the general 4d CHY formulae. After overcoming the obstacles to reduce integrands with polarization involved, the calculation of CHY formulae becomes much simpler. We develop the 4d CHY formulae to directly calculate the amplitude of the some theory. The reduced Pfaffian behaves like a solution filter, making the building of 4d CHY formulae natural. As if the general CHY formulae has been reduced to 4d CHY formulae and the number of solutions decrease from $(n-3)!$ to $E_{n-3,k'-2}$.

We have discussed the reduction of the reduced compactified Pfaffian and squeezed Pfaffian in appendix B, C and discussed how the valid solution sector shift from the helicity sector. The more polarizations there are, the more efficient our procedure is. Even when there is no polarization involved, and the reduced compactified Pfaffian totally reduce to $\text{Pf}'\mathbf{A}_n$ times something, our reduction procedure still holds and it tells us only the $k' = n/2$ solution sector contributes. This means CHY formulae of some effective field theory such as Born-Infeld, Dirac-Born-Infeld, Non-Linear Sigma Model, Special Galileon theories with $\text{Pf}'\mathbf{A}_n$ acting as CHY integrand also reduce to a set of 4d CHY formulae. Even for some theories that receive the contribution of several solution sectors such as those with $\mathcal{P}_n$ acting as CHY integrand, the physical meaning of 4d CHY formulae is also apparent: the contribution from the $k' < k$ sectors gives self-dual amplitude and that of $k' > k$ gives anti-self-dual amplitudes.

Many good properties shared by CHY formulae are still inherited by the 4d CHY formulae. The soft limits has been discussed in [15, 27–30]. Factorisation should also be easy to study. Not only the CHY formulae have a simple representation in 4d, 4d CHY formulae can also help us understand the CHY formulae in general dimension. Besides, the supersymmetrization of the 4d CHY formulae is directly and we just need to replace the $\tilde{\lambda}_a^\alpha$ with $\{\tilde{\lambda}_a^\alpha|\eta_a^A\}$ in the scattering equation in (2.4) as shown in [15]. This way we can involve fermions in CHY formulae, for example we use SYM amplitudes to build up QCD amplitudes as shown [31]. In the same paper, we use two set of spinors to describe the massive Higgs, which has been generalized to calculate form factors [32, 33].

We tentatively study whether the solutions divide beyond in 4d. Especially we hope something interesting come out in 6 dimensions where we also have a good spinor representation [34–36] and some nice result of CHY formulae in 6d comes out. We can treat a massive particle in 4d as a massless particles in 6d, especial the massive loop particles in 4d. Up to now, our result is negative and we didn't find the solutions of scattering equation divide again in other dimension.

CHY formulae has been extended to 1-loop level, as discussed in [37, 38]. It has been known that what underpins the CHY formulae is ambitwistor string. And ambitwistor string theory has been extended to 1-loop level, as shown in [7, 39, 40]. However we find the solutions at 1-loop level don't divide into several sectors again in SYM or SUGRA

theory. CHY formulae has singular solutions at 1-loop, how about 4d CHY formulae and how does it contribute to the divergence bubble or tadpole? Also it is interesting to check whether the integrand still factorizes to two objects that depend on particles of negative helicity or positive helicity respectively. Also it will help us to build the general 1-loop CHY integrand.

As discussed in [41], also it is very useful to study the positivity of the jacobian or integrand in 4d CHY formulae. (2.6) is a useful identity as it link several objects. As shown in [41], $\det' \Phi_n(\{s_{ab}, \sigma_a\})$ is positive at the positive region. We know that $\det' \mathbf{h}_{k'} \det' \tilde{\mathbf{h}}_{n-k'}$ is exactly the result of $\text{Pf}' \Psi_n$ with k' external particles of negative helicity. If the 4d jacobian $J_{n,k'}$ is also positive at the positive region, it strongly supports that the YM amplitude is also positive in some regions.

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A A conjecture on $J_{n,k'}$

In this section, without making confusion, we denote $\frac{\sigma_a - \sigma_b}{t_a t_b} =: (ab)$. We have strong evidence to support the following conjecture,

$$J_{n,k'} = \frac{1}{\prod_{a=1}^n t_a^2 \prod_{b \in -', p \in +'} (bp)^2} \sum_{b < c, p < q} \left(\prod_{\substack{1 \leq x \leq n-k'-2 \\ 1 \leq y \leq k'-2}} (b_x c_x) \langle b_x c_x \rangle (p_y q_y) [p_y q_y] (d_{xy} r_{yx})^2 \right) \quad (\text{A.1})$$

Here the sum is over all $b_1 < c_1, b_2 < c_2, \dots, b_{n-k'-2} < c_{n-k'-2}$; $b_1, c_1, b_2, c_2, \dots, b_{n-k'-2}, c_{n-k'-2} \in -'$ and $p_1 < q_1, p_2 < q_2, \dots, p_{k'-2} < q_{k'-2}$; $p_1, q_1, p_2, q_2, \dots, p_{k'-2}, q_{k'-2} \in +'$. The x^{th} row of matrix d is $-'\setminus\{b_x, c_x\}$ for $1 \leq x \leq n-k'-2$ and the y^{th} row of matrix r is $+' \setminus \{p_y, q_y\}$ for $1 \leq y \leq k'-2$. When $k'=2$ or $n-2$, $(d_{xy} r_{yx})^2$ reduces to 1. We have numerically checked this formula up to 9 points in all solution sectors, and 15 points in $k'=3$ sector.

Here are some examples.

For the MHV solution sector, it can be analytically proved that

$$J_{n,2} = \frac{(bc)^{n-4} \langle bc \rangle^{n-4}}{\prod_{a=1}^n t_a^2 \prod_{p \in +'} (bp)^2 (cp)^2} \quad (\text{A.2})$$

Here $\{b, c\} = -'$. After deleting the 4 columns and rows about b, c of the matrix in (2.5) with a compensate $\frac{\prod_{a=1}^n t_a}{t_b^3 t_c^3 (bc)^2 \langle bc \rangle^2}$, the minor becomes a “diagonal” matrix whose diagonal entries are 2×2 matrices and their determinants can be easily calculated out as $\frac{(bc) \langle bc \rangle}{(pb)^2 (pc)^2 t_p^3}$ for $p = +'$, then we get (A.2).

For the NMHV solution sector,

$$J_{6,3} = \frac{1}{\prod_{a=1}^n t_a^2 \prod_{b \in -', p \in +'} (bp)^2} \sum_{b < c, p < q} (bc) \langle bc \rangle (pq) [pq] (dr)^2 \quad (\text{A.3})$$

Here d, r is a particle label as the breviate of d_{11}, r_{11} and $\{b, c, d\} = -', \{p, q, r\} = +'$.

$$J_{7,3} = \frac{1}{\prod_{a=1}^n t_a^2 \prod_{b \in -', p \in +'} (bp)^2} \sum_{b_1 < c_1, b_2 < c_2, p < q} (b_1 c_1) \langle b_1 c_1 \rangle (b_2 c_2) \langle b_2 c_2 \rangle (pq) [pq] (d_1 r_1)^2 (d_2 r_2)^2 \quad (\text{A.4})$$

Here d_1, d_2 is a particle label as the breviate of d_{11}, d_{21} and $\{b_1, c_1, d_{11}\} = -', \{b_2, c_2, d_{21}\} = -'$. r_1, r_2 is a particle label as the breviate of r_{11}, r_{12} and $\{p, q, r_{11}, r_{12}\} = +'$. Note that this restrain doesn't fix r_{11}, r_{12} totally as r_{11}, r_{12} can exchange their value. However it doesn't affect the value of $J_{7,3}$ as we always sum over all $b_1 < c_1, b_2 < c_2$.

$$J_{n,3} = \frac{1}{\prod_{a=1}^n t_a^2 \prod_{b \in -', p \in +'} (bp)^2} \sum_{b < c, p < q} \left(\prod_{1 \leq x \leq n-5} (b_x c_x) \langle b_x c_x \rangle (pq) [pq] (d_x r_x)^2 \right) \quad (\text{A.5})$$

Here d_x is a particle label as the breviate of d_{x1} and $\{b_x, c_x, d_x\} = -'$. r_x is a particle label as the breviate of r_{1x} and $\{r_{11}, r_{12}, \dots, r_{1, n-5}\} = +' \setminus \{p, q\}$.

For NNMHV solution sector,

$$J_{8,4} = \frac{1}{\prod_{a=1}^n t_a^2 \prod_{b \in -', p \in +'} (bp)^2} \sum_{\substack{b_1 < c_1, b_2 < c_2 \\ p_1 < q_1, p_2 < q_2}} (b_1 c_1) \langle b_1 c_1 \rangle (b_2 c_2) \langle b_2 c_2 \rangle (p_1 q_1) [p_1 q_1] (p_2 q_2) [p_2 q_2] \\ \times (d_{11} r_{11})^2 (d_{21} r_{12})^2 (d_{21} r_{12})^2 (d_{22} r_{22})^2. \quad (\text{A.6})$$

Here $\{b_1, c_1, d_{11}, d_{12}\} = -', \{b_2, c_2, d_{21}, d_{22}\} = -', \{p_1, q_1, r_{11}, r_{12}\} = +', \{p_2, q_2, r_{21}, r_{22}\} = +'$. Note that this restrain doesn't fix d_{11}, d_{12} totally as d_{11}, d_{12} can exchange their value neither does r_{11}, r_{12} . In most cases, it doesn't affect the value of $J_{8,4}$ however a few cases do rely on particular ordering of d_{11}, d_{12} or r_{11}, r_{12} , leaving a further study to fix the final expression of $J_{n,k'}$.

B The reduced compactified Pfaffian in 4d

In the main text, we have discussed the reduced Pfaffian which we delete 1^{th} and n^{th} rows and columns of the matrix Ψ_n . Then we introduce the open cycle to reduce it into two reduced determinants. Also we can effectively think that we set the momenta of the particles $1, n$ in higher dimension and they dotting everything equal to zero unless they dotting themselves equal to 1 to make the complement $\frac{1}{\sigma_{1n}}$. Then we can still decompose the reduced Pfaffian into some (modified) closed cycles. One closed cycle must contain $1, n$ both or vanish if it just contains one of them, and then it reduces to open cycle as k_1, k_n are set in higher dimension. This can be extended to other cases as now we set the

polarisation of some pairs of particles in higher dimension. We call it reduced compactified Pfaffian which is the building block for EM, EMS, YMS amplitudes, as discussed in [4]. Then we can use the similar trick to reduce the reduced compactified Pfaffian in 4d.

We donate the set of particles whose polarisation are set in higher dimension as h . Obviously, the length of set h must be even. Further on, we let the polarisation of particles in h be anyone of an orthogonal bases and they dotting each other equal to 1 or 0, donated as $\delta^{I_a I_b}$. Then the compactified Pfaffian can be think of as being deleted the rows and columns of h in the last n rows and columns of the matrix Ψ_n donated as $\text{Pf}'|\Psi_n|_{\{h\}}^{\{h\}}$ and complement it with a Pfaffian. That is,

$$\text{Pf}'\Psi_{n;m} = \text{Pf}'|\Psi_n|_{\{h\}}^{\{h\}} \text{Pf}[\mathcal{X}]_h \quad (\text{B.1})$$

with

$$\text{Pf}[\mathcal{X}]_h = \sum_{\{a,b\} \in \text{p.m.}(h)} \text{sgn}(\{a,b\}) \frac{\delta^{I_{a_1}, I_{b_1}}}{\sigma_{a_1, b_1}} \frac{\delta^{I_{a_2}, I_{b_2}}}{\sigma_{a_2, b_2}} \cdots \frac{\delta^{I_{a_m}, I_{b_m}}}{\sigma_{a_m, b_m}}. \quad (\text{B.2})$$

Here $2m$ is length of the set h . First we considering the case with $m = 1$, that is only one pair of particles donated as e_1, e_2 that needs dimension reduction. For simplicity, we also delete the rows and columns of e_1, e_2 in the first n rows and columns to satisfy the mass dimension, i.e. we effectively set the momenta of e_1, e_2 in higher dimension. Then in the expansion of the reduced compactified Pfaffian, all cycles that contain e_1, e_2 vanish unless they contain and only contain both e_1, e_2 . Then this cycle, which equals to $\frac{\delta^{I_{e_1} I_{e_2}}}{\sigma_{e_1 e_2}}$, factor out, leaving all other cycles normal as if e_1, e_2 not existed. They factor into two determinants of two matrices in 4d, just like the factorisation of Pfaffian in (3.20), with the diagonal elements equal to C_{aa} plugging a certain solution sector k' , as expressed in (4.1), (4.7), (4.8). One of two determinants of these two matrices will vanish trivially unless $k' = k + 1$. That is, we need to assign e_1, e_2 to into two sets, for example, we let $- = - \cup \{e_1\}$ and $+ = + \cup \{e_2\}$. Then the reduced compactified Pfaffian with only one pair of particles needing dimension reduction reduces to

$$\text{Pf}'\Psi_{n;1}|_{k'} = \delta_{k+1, k'} \frac{1}{\sigma_{e_1 e_2}^2} \det |\mathbf{h}_{k'}|_{e_1}^{e_1} \det |\tilde{\mathbf{h}}_{n-k'}|_{e_2}^{e_2} \delta^{I_{e_1} I_{e_2}}. \quad (\text{B.3})$$

The expression of h_{ab} with $a, b \in -$ and $\tilde{h}_{ab}$ with $a, b \in +$ are given in (3.11), (4.1), (4.8). Here we can extend the definition domain from $-$ to $- \cup \{e_1\}$ and from $+$ to $+ \cup \{e_2\}$, to enclose $h_{e_1 b}$ or $\tilde{h}_{e_2 b}$, though it is not important as such entries will always been deleted from the matrices $\mathbf{h}_{k'}$ and $\tilde{\mathbf{h}}_{n-k'}$ in the above equation. In this case, the exchange of $e_1 \leftrightarrow e_2$ will affect the expression of the diagonal elements of $\mathbf{h}_{k'}$ and $\tilde{\mathbf{h}}_{n-k'}$ but it won't affect the final result. For later convenience, we write the above equation in a slightly different way,

$$\text{Pf}'\Psi_{n;1}|_{k'} = \delta_{k+1, k'} \frac{1}{\sigma_{e_1 e_2}} \det |\mathbf{h}_{k'}|_{e_1}^{e_1} \det |\tilde{\mathbf{h}}_{n-k'}|_{e_2}^{e_2} \text{Pf}[\mathcal{X}]_h. \quad (\text{B.4})$$

Now we consider the case with $m = 2$, i.e. 4 particles donated as e_1, e_2, e_3, e_4 need dimension reduction. There are be 3 perfect matching to make pairs in the expansion of $\text{Pf}[\mathcal{X}]_h$, as shown in (B.2). We can take these perfect individually and at last add them

up. For example, we consider a perfect matching that e_1, e_2 a pair and e_3, e_4 a pair. Still we effectively set the momenta of e_1, e_2 in higher dimension and then $\frac{\delta^{I_{e_1} I_{e_2}}}{\sigma_{e_1 e_2}}$ factors out. The left pair e_3, e_4 must be adjoint and enclosed in one cycle and it reduces to an open cycle similar to (3.6) with the polarisations on the ends replaced by kinematics as

$$\text{tr} \left(f_{e_4}^h f_{a_3}^\pm f_{a_4}^\pm \cdots f_{a_i}^\pm f_{e_3}^h \right) = k_{e_4} \cdot (f_{a_3}^\pm f_{a_4}^\pm \cdots f_{a_i}^\pm) \cdot k_{e_3} \delta^{I_{e_3} I_{e_4}} \quad (\text{B.5})$$

Here f_e^h just means the polarisation of particle e is set in higher dimension and $f_a^\pm$ means when reduced to 4 dimension, the helicity of particle a can be negative or positive. Also any two adjoint linearised strength fields $f_b^- f_p^+$ in the trace can exchange their place if the helicity of b, p are different. So we use this property to split the kinematic numerator of this open cycle in respect of negative and positive helicity first, then use the partial fraction identity to spilt the denominator, and finally cut the open cycle into two closed cycles,

$$\begin{aligned} & (-)^{|\rho|} \sum_{\{\alpha\} \in \text{OP}(\{\beta\}, \{\rho^T\})} \frac{\text{tr} \left(f_{e_4}^h f_{a_3}^\pm f_{a_4}^\pm \cdots f_{a_i}^\pm f_{e_3}^h \right)}{\sigma_{e_4 a_3} \sigma_{a_3 a_4} \cdots \sigma_{a_{i-1} a_i} \sigma_{a_i e_3} \sigma_{e_3 e_4}} \\ &= h_{e_4 b_1} h_{b_1 b_2} \cdots h_{b_{x-1} b_x} h_{b_x e_3} \tilde{h}_{e_4 p_y} \tilde{h}_{p_y p_{y-1}} \cdots \tilde{h}_{p_2 p_1} \tilde{h}_{p_1 e_3} \\ &= h_{(b_1 b_2 \cdots b_x)} \Big|_{h_{b_x b_1} \rightarrow h_{e_4 b_1}} h_{b_x e_3} \tilde{h}_{(p_1 p_2 \cdots p_y)} \Big|_{\tilde{h}_{p_y p_1} \rightarrow \tilde{h}_{e_3 p_1}} \tilde{h}_{p_y e_4} \end{aligned} \quad (\text{B.6})$$

Here $h_{eb} = \frac{\langle eb \rangle}{\sigma_{eb}}$, $\tilde{h}_{ep} = \frac{[ep]}{\sigma_{ep}}$. Compared to (3.25), we have replaced the $\epsilon_1^- = \frac{[1]\mu}{[1\mu]}$ or $\epsilon_n^+ = \frac{[n]\langle\mu\rangle}{\langle n\mu\rangle}$ by $k_e = |e\rangle[e]$, and then the prefactor $\frac{\langle b_x \mu \rangle}{\langle n \mu \rangle \sigma_{b_x n}}$ or $\frac{[\mu p_y]}{[1\mu] \sigma_{1 p_y}}$ are replaced by $\frac{\langle b_x e_3 \rangle}{\sigma_{b_x e_3}} = h_{b_x e_3}$, $\frac{[e_4 p_y]}{\sigma_{e_4 p_y}} = \tilde{h}_{e_4 p_y}$.

Then followed by other closed cycles, similar to (3.32), the perfect matching with $\{e_1, e_2\}$ a pair and $\{e_3, e_4\}$ a pair gives

$$\delta_{k+2, k'} \frac{1}{\sigma_{e_1 e_2}^2} \left(\sum_{b_x \in \{e_3\} \cup -} h_{b_x e_4} \det |\mathbf{h}_{k'}|_{e_1 b_x}^{e_1 b_x} \right) \left(\sum_{p_y \in \{e_4\} \cup +} \tilde{h}_{p_y e_3} \det |\tilde{\mathbf{h}}_{n-k'}|_{e_2 p_y}^{e_2 p_y} \right) \delta^{I_{e_1} I_{e_2}} \delta^{I_{e_3} I_{e_4}}. \quad (\text{B.7})$$

The factor $\delta_{k+2, k'}$ come from the fact that we must assign e_1, e_2 in different sets and e_3, e_4 in different sets, for example $- = - \cup \{e_1, e_3\}$ and $+ = + \cup \{e_2, e_4\}$. In this case, the exchange of $e_1 \leftrightarrow e_2$ or $e_3 \leftrightarrow e_4$ doesn't affect the final result. The exchange of $\{e_1, e_2\} \leftrightarrow \{e_3, e_4\}$ doesn't affect the final result. The other two perfect matching can be think of as the exchange of $\{e_1, e_2, e_3, e_4\} \leftrightarrow \{e_1, e_3, e_2, e_4\}$ and $\{e_1, e_2, e_3, e_4\} \leftrightarrow \{e_1, e_4, e_2, e_3\}$. However we have a cleverer choice that each perfect matching in (B.2) must share the same coefficient $\text{Pf}'|\Psi_{n;2}|_{\{h\}}^{\{h\}}$. We can read this coefficient $\text{Pf}'|\Psi_{n;2}|_{\{h\}}^{\{h\}}$ from (B.7) and then the reduced compactified Pfaffian with two pairs of particles needing dimension reduction reduces to

$$\text{Pf}'\Psi_{n;2}|_{k'} = \delta_{k+2, k'} \left(\sum_{b_x \in \{e_3\} \cup -} h_{b_x e_4} \det |\mathbf{h}_{k'}|_{e_1 b_x}^{e_1 b_x} \right) \left(\sum_{p_y \in \{e_4\} \cup +} \tilde{h}_{p_y e_3} \det |\tilde{\mathbf{h}}_{n-k'}|_{e_2 p_y}^{e_2 p_y} \right) \frac{\sigma_{e_3 e_4}}{\sigma_{e_1 e_2}} \text{Pf}[\mathcal{X}]_h. \quad (\text{B.8})$$

One can change the $\{e_1, e_2, e_3, e_4\}$ by any other permutation and it won't change the final result. They just different representation of $\text{Pf}'|\Psi_{n;2}|_{\{h\}}^{\{h\}}$ as the choice of prefactor in (B.6) is rather arbitrary: one choose $h_{b_x e_3}$ as prefactor as well as $h_{e_4 b_1}$, so does $\tilde{h}_{p_y e_4}$ and $\tilde{h}_{p_1 e_3}$.

Now we consider the case with $m = 3$, i.e. 6 particles donated as $e_1, e_2, e_3, e_4, e_5, e_6$ need dimension reduction. Still we consider the perfect matching with $\{e_1, e_2\}$ a pair, $\{e_3, e_4\}$ a pair and $\{e_5, e_6\}$ a pair first. Still we effectively set the kinematics of e_1, e_2 in higher dimension and then $\frac{\delta^{I_{e_1} I_{e_2}}}{\sigma_{e_1 e_2}}$ factors out. The left two pairs $\{e_3, e_4\}$ and $\{e_5, e_6\}$ must be adjoint in the trace respectively, or vanish. However these two pairs can be enclosed in two different cycles or in a common cycle. The former case is simpler. We split the open cycles, cut it into modified open cycles and rearrange them together with other closed cycles into determinants. That is, those of the expansion of $\text{Pf}' \Psi_{n;2}|_{k'}$ that contain such cycles $\frac{\text{tr}(f_{e_3}^h \dots f_{e_4}^h)}{\sigma_{(e_3 \dots e_4)}}$, $\frac{\text{tr}(f_{e_5}^h \dots f_{e_6}^h)}{\sigma_{(e_5 \dots e_6)}}$ reduce to

$$\delta_{k+3,k'} \frac{1}{\sigma_{e_1 e_2}^2} \left(\sum_{\substack{b_x \in \{e_3\} \cup - \\ c_z \in \{e_5\} \cup - \\ b_x \neq c_z}} h_{b_x e_4} h_{c_z e_6} \det |h_{k'}|_{e_1 e_3 e_5}^{e_1 b_x c_z} \right) \times \left(\sum_{\substack{p_y \in \{e_4\} \cup + \\ q_w \in \{e_6\} \cup + \\ p_y \neq q_w}} \tilde{h}_{p_y e_3} \tilde{h}_{q_w e_5} \det |\tilde{h}_{n-k'}|_{e_2 e_4 e_6}^{e_2 p_y q_w} \right) \delta^{I_{e_1} I_{e_2}} \delta^{I_{e_3} I_{e_4}} \delta^{I_{e_5} I_{e_6}}. \quad (\text{B.9})$$

The factor $\delta_{k+3,k'}$ come from the fact that we must assign each pair of particles in this perfect matching into different sets, for example $- = - \cup \{e_1, e_3, e_5\}$ and $+ = + \cup \{e_2, e_4, e_6\}$. In this case, the exchange of particles in each pair doesn't affect the final result. The exchange of different pairs also doesn't affect the final result.

However the case where $\{e_3, e_4\}$ and $\{e_5, e_6\}$ are enclosed in a common cycle also contribute and we need to deal with it more carefully. We find that the equation (B.5) can be extended to

$$\begin{aligned} & \text{tr} \left(f_{e_4}^h f_{a_3}^\pm \dots f_{a_i}^\pm f_{e_5}^h f_{e_6}^h f_{a_3}^\pm \dots f_{a_j}^\pm f_{e_3}^h \right) \\ &= k_{e_4} \cdot f_{a_3}^\pm \dots f_{a_i}^\pm \cdot k_{e_5} \delta^{I_{e_5} I_{e_6}} k_{e_6} \cdot f_{a_3}^\pm \dots f_{a_j}^\pm \cdot k_{e_3} \delta^{I_{e_3} I_{e_4}} \end{aligned} \quad (\text{B.10})$$

or even more general form. Still any two adjoint linearised strength fields $f_b^- f_p^+$ in the trace can exchange their place if the helicity of b, p are different. However this time the exchanging is blocked by f_e^h . So we treat the region $f_{e_4}^h f_{a_3}^\pm \dots f_{a_i}^\pm f_{e_5}^h$ and $f_{e_6}^h f_{a_3}^\pm \dots f_{a_j}^\pm f_{e_3}^h$ individually and split the kinematic numerator of these region separately, followed by the splitting of denominators using partial fraction identity. Finally,

$$\begin{aligned} & (-)^{|\rho|} \sum_{\{\alpha\} \in \text{OP}(\{\beta\}, \{\rho^T\})} \frac{\text{tr} \left(\dots f_{e_{\tau_2}}^h f_{a_3}^\pm f_{a_4}^\pm \dots f_{a_i}^\pm f_{e_{\tau_3}}^h \dots \right)}{\dots \sigma_{e_{\tau_2} a_3} \sigma_{a_3 a_4} \dots \sigma_{a_{i-1} a_i} \sigma_{a_i e_{\tau_3}} \sigma_{e_{\tau_3} e_{\tau_4}} \dots} \\ &= \dots h_{e_{\tau_2} b_1} h_{b_1 b_2} \dots h_{b_{x-1} b_x} h_{b_x e_{\tau_3}} \tilde{h}_{e_{\tau_2} p_y} \tilde{h}_{p_y p_{y-1}} \dots \tilde{h}_{p_2 p_1} \tilde{h}_{p_1 e_{\tau_3}} \frac{\sigma_{e_{\tau_3} e_{\tau_2}}}{\sigma_{e_{\tau_3} e_{\tau_4}}} \dots \end{aligned} \quad (\text{B.11})$$

Here we have given the general form with arbitrary pairs of particles in h enclosed in a common trace. $h_{e_\tau b} = \frac{\langle e_\tau b \rangle}{\sigma_{e_\tau b}}$, $\tilde{h}_{e_\tau p} = \frac{[e_\tau p]}{\sigma_{e_\tau p}}$. And $f_{e_\tau 4}^h$ is the linearised strength fields next to $f_{e_\tau 3}^h$. It could just be $f_{e_\tau 2}^h$ and at this case $\frac{\sigma_{e_\tau 3 e_\tau 2}}{\sigma_{e_\tau 3 e_\tau 4}}$ reduces to 1.

Compared to (B.6), here comes out a factor $\frac{\sigma_{e_\tau 3 e_\tau 2}}{\sigma_{e_\tau 3 e_\tau 4}}$. We can think there is also a factor $\frac{\sigma_{e_3 e_4}}{\sigma_{e_3 e_4}} = 1$ in equation (B.6). However there is something different essentially. For example, in the former case, $\frac{\text{tr}(f_{e_3}^h \dots f_{e_4}^h) \text{tr}(f_{e_5}^h \dots f_{e_6}^h)}{\sigma_{(e_3 \dots e_4)} \sigma_{(e_5 \dots e_6)}}$, e_3 groups with e_4 and e_5 groups with e_6 , while in the later case, $\frac{\text{tr}(f_{e_4}^h \dots f_{e_5}^h f_{e_6}^h \dots f_{e_3}^h)}{\sigma_{(e_4 \dots e_5 e_6 \dots e_3)}}$, e_3 groups with e_6 and e_4 groups with e_5 . In the former case, the group pairs are consistent with the perfect matching pairs and the exchanging of particles in the same perfect matching pair is identical, while in the latter cases, the group pairs are not consistent with the perfect matching pairs and the exchanging of particles in the same perfect matching pair is two different contributions. However it is still not tough. The reduction of those that contain such cycles $\frac{\text{tr}(f_{e_4}^h \dots f_{e_5}^h f_{e_6}^h \dots f_{e_3}^h)}{\sigma_{(e_4 \dots e_5 e_6 \dots e_3)}}$ in the expansion of $\text{Pf}' \Psi_{n;2}|_{k'}$ can be got from (B.9) by exchanging $e_5 \leftrightarrow e_3$ if ignoring the factor $\frac{\sigma_{e_\tau 3 e_\tau 2}}{\sigma_{e_\tau 3 e_\tau 4}}$ and the reduction of those that contain such cycles $\frac{\text{tr}(f_{e_4}^h \dots f_{e_6}^h f_{e_5}^h \dots f_{e_3}^h)}{\sigma_{(e_4 \dots e_6 e_5 \dots e_3)}}$ in the expansion of $\text{Pf}' \Psi_{n;2}|_{k'}$ can be got from that of $\frac{\text{tr}(f_{e_4}^h \dots f_{e_5}^h f_{e_6}^h \dots f_{e_3}^h)}{\sigma_{(e_4 \dots e_5 e_6 \dots e_3)}}$ by exchanging $e_5 \leftrightarrow e_6$. Taking all the perfect matching into consideration, then the reduced compactified Pfaffian with three pairs of particles needing dimension reduction reduces to

$$\begin{aligned} \text{Pf}' \Psi_{n;3}|_{k'} = & \delta_{k+3,k'} \left(\frac{\sigma_{e_4 e_5} \sigma_{e_6 e_3}}{\sigma_{e_1 e_2}} \left(\sum_{\substack{b_x \in \{e_3\} \cup - \\ c_z \in \{e_5\} \cup - \\ b_x \neq c_z}} h_{b_x e_4} h_{c_z e_6} \det |h_{k'}|_{e_1 b_x c_z}^{e_1 e_3 e_5} \right) \right. \\ & \times \left(\sum_{\substack{p_y \in \{e_4\} \cup + \\ q_w \in \{e_6\} \cup + \\ p_y \neq q_w}} \tilde{h}_{p_y e_5} \tilde{h}_{q_w e_3} \det |\tilde{h}_{n-k'}|_{e_2 p_y q_w}^{e_2 e_4 e_6} \right) + \left(e_5 \leftrightarrow e_3 \right) - \left(e_5 \leftrightarrow e_6 \right) \Big) \text{Pf}[\mathcal{X}]_h. \end{aligned} \quad (\text{B.12})$$

The minus before the exchanging of $e_5 \leftrightarrow e_6$ comes from $\sigma_{e_6 e_5} = -\sigma_{e_5 e_6}$ and $\sigma_{e_5 e_6}$ is absorbed in $\text{Pf}[\mathcal{X}]_h$. When it comes to the cases with $m \geq 3$, there are no more new objects coming out and just some more calculation and we can always reduce the reduced compactified Pfaffian into some determinants.

C The reduced squeezed Pfaffian

We write the reduced compactified Pfaffian (B.1), (B.2) in a slightly different way,

$$\begin{aligned} \text{Pf}' \Psi_{n;m} = & \sum_{\{a,b\} \in \text{p.m.}(h)} (-1)^m \frac{\text{Tr}(T^{I_{a_1}} T^{I_{b_1}})}{\sigma_{a_1, b_1} \sigma_{b_1, a_1}} \frac{\text{Tr}(T^{I_{a_2}} T^{I_{b_2}})}{\sigma_{a_2, b_2} \sigma_{b_2, a_2}} \dots \frac{\text{Tr}(T^{I_{a_m}} T^{I_{b_m}})}{\sigma_{a_m, b_m} \sigma_{b_m, a_m}} \\ & \times \text{sgn}(\{a, b\}) \sigma_{a_1, b_1} \sigma_{a_2, b_2} \dots \sigma_{a_m, b_m} \text{Pf}[\Psi]_{\text{H}, a_1, b_1, a_2, b_2, \dots, a_{m-1}, b_{m-1}; \text{H}} \end{aligned} \quad (\text{C.1})$$

Here we use $\text{Pf}|\Psi|_{\text{H},a_1,b_1,a_2,b_2,\dots,a_{m-1},b_{m-1};\text{H}}$ to show what column and rows are left in the matrix Ψ_n and we denote the set of particles whose polarisation are not set in higher dimension as H. Here $\frac{\text{Tr}(T^{I_{a_1}} T^{I_{b_1}})}{\sigma_{a_1,b_1}\sigma_{b_1,a_1}}$ can be seen as a two-gluon Parke-Taylor factor. As shown in [4], it can be extended to a Parke-Taylor factor with arbitrary number of gluons,

$$\mathcal{C}_{a_1,a_2,\dots,a_s} = \sum_{\omega \in S_s/\mathbb{Z}_s} \frac{\text{Tr}\left(T^{I_{\omega(a_1)}} T^{I_{\omega(a_2)}} \dots T^{I_{\omega(a_s)}}\right)}{\sigma_{\omega(a_1)\omega(a_2)} \sigma_{\omega(a_2)\omega(a_3)} \dots \sigma_{\omega(a_s)\omega(a_1)}}. \quad (\text{C.2})$$

We denote the set of gluons as g and the subsets sharing in the same color trace as $\text{Tr}_1, \text{Tr}_2, \dots, \text{Tr}_m$. Then the half integrand for EYM are given by

$$\mathcal{C}_{\text{Tr}_1} \dots \mathcal{C}_{\text{Tr}_m} \sum_{\substack{a_1 < b_1 \in \text{Tr}_1 \\ \dots \\ a_{m-1} < b_{m-1} \in \text{Tr}_{m-1}}} \text{sgn}(\{a, b\}) \sigma_{a_1,b_1} \dots \sigma_{a_{m-1},b_{m-1}} \text{Pf}|\Psi|_{\text{H},a_1,b_1,a_2,b_2,\dots,a_{m-1},b_{m-1};\text{H}}. \quad (\text{C.3})$$

The reduction of $\text{Pf}|\Psi|_{\text{H},a_1,b_1,a_2,b_2,\dots,a_{m-1},b_{m-1};\text{H}}$ can be obtained from the above section (the explicit form with $m = 1, 2, 3$ are given in (B.4), (B.8), (B.12) except removing $\text{Pf}[\mathcal{X}]_{\text{h}}$ respectively).

Here we present the reduction of the reduced squeezed Pfaffian for EYM amplitudes with single gluon color trace,

$$\mathcal{C}_g \det[\mathbf{h}]_{\text{H}} - \det[\tilde{\mathbf{h}}]_{\text{H}^+} \quad (\text{C.4})$$

and that of double gluon color traces

$$\mathcal{C}_{\text{Tr}_1} \mathcal{C}_{\text{Tr}_2} \sum_{e_1 < e_2 \in \text{Tr}_1} \left(\sum_{b_x \in \{e_1\} \cup \text{H}^-} h_{b_x e_2} \det[\mathbf{h}]_{\text{H}^-, e_1} \Big|_{e_1}^{b_x} \right) \left(\sum_{p_y \in \{e_2\} \cup \text{H}^+} \tilde{h}_{p_y e_1} \det[\tilde{\mathbf{h}}]_{\text{H}^+, e_2} \Big|_{e_2}^{p_y} \right) \sigma_{e_1 e_2}^2. \quad (\text{C.5})$$

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