$\mathcal{W}$ algebras are L_∞ algebras

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ABSTRACT: It is shown that the closure of the infinitesimal symmetry transformations underlying classical $\mathcal{W}$ algebras give rise to L_∞ algebras with in general field dependent gauge parameters. Therefore, the class of well understood $\mathcal{W}$ algebras provides highly non-trivial examples of such strong homotopy Lie algebras. We develop the general formalism for this correspondence and apply it explicitly to the classical $\mathcal{W}_3$ algebra.

KEYWORDS: Conformal and W Symmetry, Conformal Field Models in String Theory

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1 Introduction

Motivated by bosonic closed string field theory [1], the structure of L_∞ algebras was introduced. In the more mathematical context, these algebras are also called strong homotopy Lie algebras [2]. These are generalizations of Lie algebras and are expected to be closely related to the symmetries of classical field theories. The novel feature is that these algebras do contain higher order products as well as generalized Jacobi-identities among them. Such a structure also appeared in the early work on higher spin theories in [3] and also made its appearance in more recent studies of the Courant bracket [4] and the C-bracket [5] appearing in generalized geometry and double field theory, respectively.

Motivated by the study of field theory truncations of string field theory [6], the authors of [7] argued that the symmetry and the action of any consistent perturbative gauge symmetry is controlled by an L_∞ algebra. For Chern-Simons and Yang-Mills gauge theories as well as for double field theory the symmetries and equations of motion could be expressed in terms of an L_∞ structure.

While reading this paper and the early work on higher spin algebras [3] (see [8] for a more mathematical exposition), as physicists we wondered whether there exist non-trivial examples that really admit field dependent gauge parameters. Taking into account that the higher spin algebras in three dimensions are holographically dual to two-dimensional conformal field theories with extended symmetry [9–11], it is natural to contemplate the idea that $\mathcal{W}$ algebras might be somehow related to L_∞ algebras. These $\mathcal{W}$ algebras have

the feature that the commutators of the generators do not simply close linearly among themselves but involve also (normal ordered) products of the elementary fields.

Taking into account that the general expectation from [1] is, that only classical symmetries are directly related to L_∞ algebras, in this letter we intend to convey the observation that indeed classical $\mathcal{W}$ algebras provide a large class of highly non-trivial L_∞ algebras (see e.g. [12] for some other concrete examples). We can give a very generic definition of the higher products between gauge parameters and fields that are concentrated in the first two graded vector spaces X_0 and X_{-1} . In this CFT context, we recover the general result of [3, 7, 8, 13], that the higher order L_∞ relations are satisfied if and only if the gauge algebra closes and the Jacobi-identity of for three gauge variations vanishes. With this insight, one can turn the logic around and *bootstrap* the form of the classical $\mathcal{W}$ algebras by implementing the L_∞ relations.

First we review some of the for this letter relevant aspects of L_∞ algebras and $\mathcal{W}$ algebras. Then we provide the general prescription of how a classical $\mathcal{W}$ algebra induces higher products and their higher order relations. Some of the ingredients can be given in quite general terms and depend only on the conformal dimensions of the fields involved. What is left open are some algebra dependent structure constants. For the concrete example of the classical $\mathcal{W}_3$ algebra, we evaluate explicitly the L_∞ relations, which indeed uniquely fix the remaining structure constants to precisely the values expected from the closure of the $\mathcal{W}_3$ algebra.

2 Preliminaries

In this section we briefly review the basic structures of an L_∞ algebra and of $\mathcal{W}$ algebras. To keep the presentation as focused as possible, we will only introduce those ingredients that are needed for this letter.

2.1 Basics of L_∞ algebras

Let us first recall the definition of an L_∞ algebra in the so-called ℓ -picture. One has a graded vector space $X = \bigoplus_n X_n$, where X_n is said to have degree n . In addition there are multi-linear products $\ell_n(x_1, \dots, x_n)$ that have degree $\deg(\ell_n) = n - 2$ so that

$$\deg(\ell_n(x_1, \dots, x_n)) = n - 2 + \sum_{i=1}^n \deg(x_i). \quad (2.1)$$

Note that ℓ_2 does not change the degree, while ℓ_1 decreases it by one. The products are graded commutative, i.e.

$$\ell_n(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = (-1)^\sigma \epsilon(\sigma; x) \ell_n(x_1, \dots, x_n), \quad (2.2)$$

where $(-1)^\sigma$ is just the sign of the permutation σ and the Koszul sign $\epsilon(\sigma; x)$ is defined by considering a graded commutative algebra $\Lambda(x_1, x_2, \dots)$ with $x_i \wedge x_j = (-1)^{x_i x_j} x_j \wedge x_i$ and reading the sign from

$$x_1 \wedge \dots \wedge x_k = \epsilon(\sigma; x) x_{\sigma(1)} \wedge \dots \wedge x_{\sigma(n)}. \quad (2.3)$$

Here x_i in the exponent means $\deg(x_i)$. Thus, e.g. one has

$$\ell_2(x_1, x_2) = (-1)^{1+x_1x_2} \ell_2(x_2, x_1). \quad (2.4)$$

The defining relations of L_∞ , labelled by n , are then

$$\begin{aligned} \mathcal{J}_n(x_1, \dots, x_n) &:= \sum_{i+j=n+1} (-1)^{i(j-1)} \sum_{\sigma} (-1)^{\sigma} \epsilon(\sigma; x) \\ &\quad \ell_j(\ell_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}) = 0. \end{aligned} \quad (2.5)$$

Here the permutations are restricted to the ones with

$$\sigma(1) < \dots < \sigma(i), \quad \sigma(i+1) < \dots < \sigma(n). \quad (2.6)$$

The L_∞ algebras of interest in this letter are concentrated in the first two vector spaces X_0 and X_{-1} , where X_0 will contain gauge parameters ε and X_{-1} the basic fields W . In this case, only the n -products

$$\ell_n(\varepsilon, W^{n-1}), \quad \ell_n(\varepsilon_1, \varepsilon_2, W^{n-2}) \quad (2.7)$$

can be non-trivial.

Moreover, the order n L_∞ relation with m gauge parameters and $n - m$ fields will be of degree $m - 3$. Therefore, only those relations with $m = 2, 3$ gauge parameters are non-trivial. In particular $\mathcal{J}_1 = \ell_1 \ell_1 = 0$ is trivially satisfied. For this letter all n -products for $n \geq 4$ vanish so that the non-trivial relations are $\mathcal{J}_2(\varepsilon_1, \varepsilon_2) = 0$, $\mathcal{J}_3(\varepsilon_1, \varepsilon_2, x) = 0$, $\mathcal{J}_4(\varepsilon_1, \varepsilon_2, W, x) = 0$, $\mathcal{J}_5(\varepsilon_1, \varepsilon_2, W, W, x) = 0$ with $x \in \{\varepsilon_3, W\}$. Their schematic form is

$$\begin{aligned} \mathcal{J}_2 &= \ell_1 \ell_2 - \ell_2 \ell_1, & \mathcal{L}_3 &= \ell_1 \ell_3 + \ell_2 \ell_2 + \ell_3 \ell_1, \\ \mathcal{J}_4 &= \ell_1 \ell_4 - \ell_2 \ell_3 + \ell_3 \ell_2 - \ell_4 \ell_1, \\ \mathcal{J}_5 &= \ell_1 \ell_5 + \ell_2 \ell_4 + \ell_3 \ell_3 + \ell_4 \ell_2 + \ell_5 \ell_1, \end{aligned} \quad (2.8)$$

where the sign reflects the factor $(-1)^{i(j-1)}$ in (2.5). In our case we have $\ell_4 = 0$, hence the explicit form of these relations reads

$$\ell_1(\ell_2(\varepsilon_1, \varepsilon_2)) = \ell_2(\ell_1(\varepsilon_1), \varepsilon_2) + \ell_2(\varepsilon_1, \ell_1(\varepsilon_2)), \quad (2.9)$$

meaning that ℓ_1 acts as a derivative satisfying the Leibniz rule, and

$$\begin{aligned} 0 &= \ell_1(\ell_3(\varepsilon_1, \varepsilon_2, x)) \\ &\quad + \ell_2(\ell_2(\varepsilon_1, \varepsilon_2), x) + \ell_2(\ell_2(\varepsilon_2, x), \varepsilon_1) + \ell_2(\ell_2(x, \varepsilon_1), \varepsilon_2) \\ &\quad + \ell_3(\ell_1(\varepsilon_1), \varepsilon_2, x) + \ell_3(\varepsilon_1, \ell_1(\varepsilon_2), x) + \ell_3(\varepsilon_1, \varepsilon_2, \ell_1(x)), \\ 0 &= -\ell_2(\ell_3(\varepsilon_1, \varepsilon_2, W), x) + (-1)^x \ell_2(\ell_3(\varepsilon_1, \varepsilon_2, x), W) \\ &\quad + \ell_2(\varepsilon_2, \ell_3(\varepsilon_1, W, x)) - \ell_2(\varepsilon_1, \ell_3(\varepsilon_2, W, x)) \\ &\quad + \ell_3(\ell_2(\varepsilon_1, \varepsilon_2), W, x) - \ell_3(\ell_2(\varepsilon_1, W), \varepsilon_2, x) + (-1)^x \ell_3(\ell_2(\varepsilon_1, x), \varepsilon_2, W) \\ &\quad + \ell_3(\ell_2(\varepsilon_2, W), \varepsilon_1, x) - (-1)^x \ell_3(\ell_2(\varepsilon_2, x), \varepsilon_1, W) + \ell_3(\ell_2(W, x), \varepsilon_1, \varepsilon_2), \end{aligned} \quad (2.10)$$

showing that the usual Jacobi-identity for ℓ_2 does receive some correction terms that are ℓ_1 -derivatives. Here we did not spell out the $\mathcal{J}_5$ relation, since it will turn out to be trivially fulfilled in our case.

2.2 Basics of $\mathcal{W}$ algebras

Let us review just a few facts about $\mathcal{W}$ algebras, focusing on the first non-trivial example of the $\mathcal{W}_3$ algebra. For more details on $\mathcal{W}$ algebras, we refer the reader to the literature as e.g. the collection of early papers [14]. $\mathcal{W}$ algebras arise as extended symmetry algebras of two-dimensional conformal field theories. The minimal symmetry algebra is the Virasoro algebra containing just the generators of infinitesimal conformal transformations.

It was shown in [15] that one can extend the Virasoro algebra by a primary field $W(z)$ of conformal dimension three and still get an algebra that closes and satisfies the Jacobi-identity for the bracket. The new aspect is that the algebra does not close in the set of generators themselves but also involves (normal ordered) products of the latter. In this letter, for concreteness we will focus on the $\mathcal{W}_3$ algebra which has two generators $\mathbf{W} = \{T, W\}$ of conformal dimension 2 and 3. Expanding

$$T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}, \quad W(z) = \sum_{n \in \mathbb{Z}} W_n z^{-n-3} \quad (2.11)$$

defines the modes of the two fields. Their quantum commutator algebra reads

$$\begin{aligned} [L_m, L_n] &= \frac{c}{2} \binom{m+1}{3} \delta_{m+n,0} + (m-n)L_{m+n}, \\ [L_m, W_n] &= (2m-n)W_{m+n}, \\ [W_m, W_n] &= \frac{c}{3} \binom{m+2}{5} \delta_{m+n,0} + \frac{\alpha}{60} (m-n)(2m^2 + 2n^2 - mn - 8)L_{m+n} \\ &\quad + \frac{\beta^{\text{qu}}}{2} (m-n)\Lambda_{m+n}^{\text{qu}} \end{aligned} \quad (2.12)$$

with the normal ordered product $\Lambda^{\text{qu}} = N(TT) - \frac{3}{10}\partial^2 T$. The structure constants α and β appear in the three-point functions $\langle WWT \rangle$ and $\langle WW\Lambda^{\text{qu}} \rangle$ and can be fixed either directly from that or via the Jacobi-identities to the values $\alpha = 2$ and $\beta^{\text{qu}} = 32/(5c + 22)$. In the classical $\hbar \rightarrow 0$ limit [16] the algebra becomes

$$\begin{aligned} [L_m, L_n] &= \frac{c}{2} \binom{m+1}{3} \delta_{m+n,0} + (m-n)L_{m+n}, \\ [L_m, W_n] &= (2m-n)W_{m+n}, \\ [W_m, W_n] &= \frac{c}{3} \binom{m+2}{5} \delta_{m+n,0} + \frac{\alpha}{60} (m-n)(2m^2 + 2n^2 - mn - 8)L_{m+n} \\ &\quad + \frac{\beta^{\text{cl}}}{2} (m-n)\Lambda_{m+n}^{\text{cl}}, \end{aligned} \quad (2.13)$$

where the commutator is meant to be a Poisson-bracket, i.e. $[\cdot, \cdot] := i\{\cdot, \cdot\}_{\text{PB}}$. In this limit the structure constants become $\alpha = 2$, $\beta^{\text{cl}} = 32/(5c)$ and the normal ordered product Λ^{qu} simplifies to $\Lambda^{\text{cl}}(z) = T(z) \cdot T(z)$, involving just the usual product of functions. It is this classical $\mathcal{W}$ algebra that will be related to an L_∞ structure in the next section.

Let us close this brief section with a couple of remarks. This classical $\mathcal{W}_3$ algebra is just the first in the series of so-called $\mathcal{W}_N$ algebras which contain $N - 1$ generators of conformal

dimensions $\{2, 3, \dots, N\}$. In the context of the higher spin $\text{AdS}_3\text{-CFT}_2$ duality [9–11], the classical $\mathcal{W}_\infty[\mu]$ algebra played an important role. This extremely huge non-linear algebra can be considered as a continuous extrapolation of the set of $\mathcal{W}_N$ algebra, in the sense that it truncates as $\mathcal{W}_\infty[N] = \mathcal{W}_N$. It is the asymptotic symmetry algebra of Vasiliev’s $\text{hs}[\mu]$ higher spin theory.

3 $\mathcal{W}$ algebras and L_∞ algebra

In order to relate the classical $\mathcal{W}$ algebra to an L_∞ algebra, not only do we have to consider the fields $T(z), W(z)$ but also how they transform under infinitesimal symmetry transformations. In this section, we first outline the general procedure of how a classical $\mathcal{W}$ algebra gives rise to an L_∞ structure. Second, we provide the classical $\mathcal{W}_3$ algebra as a concrete example.

3.1 The general picture

As mentioned we have to define $\delta_{\varepsilon_i} W_j$, i.e. the infinitesimal transformation of the fields $\{W_2, W_3, \dots\}$ under the symmetry generated by W_i . Here we have written $W_2 = T$. The L_∞ algebra consists of two graded vector-spaces X_0 and X_{-1} . Here, the field space X_{-1} is a direct sum of the W_i , thus $X_{-1} = \oplus W_i$ and the space of gauge parameters is a direct sum of transformations with generators W_i , thus $X_0 = \oplus \varepsilon_i$.

The infinitesimal variation of the chiral fields under the symmetries generated by W_i can be determined using

$$\delta_{\varepsilon_i} W_j(z) = \frac{1}{2\pi i} \oint dw \varepsilon_i(w) [W_i(w), W_j(z)], \quad (3.1)$$

where $[\cdot, \cdot] := i\{\cdot, \cdot\}_{\text{PB}}$. The right hand side can be generically evaluated using the form of the OPE between quasi-primary fields [17] (see also [18] for a more pedagogical exposition)

$$W_i(w) W_j(z) = \sum_k C_{ij}{}^k \frac{a_{ijk}^n}{n!} \frac{\partial^n \phi_k(z)}{(w-z)^{h_i+h_j-h_k-n}} \quad (3.2)$$

with

$$a_{ijk}^n = \binom{2h_k + n - 1}{n}^{-1} \binom{h_k + h_i - h_j + n - 1}{n}. \quad (3.3)$$

In a $\mathcal{W}$ algebra the fields $\phi_k(z)$ can themselves be products of the generators $W_i(z)$. Note that the coefficients a_{ijk}^n only depend on the conformal dimensions of the fields involved and are the same in the quantum and classical case. Only the structure constants $C_{ij}{}^k$ and the concrete form of the product of fields receive quantum corrections. Thus, for the variation of the generators we obtain

$$\delta_{\varepsilon_i} W_j(z) = \sum_{\substack{m, n \in \mathbb{Z}_0^+ \\ m+n = h_{ijk} - 1}} C_{ij}{}^k \frac{a_{ijk}^n}{m! n!} \partial^m \varepsilon_i(z) \partial^n \phi_k(z) \quad (3.4)$$

with $h_{ijk} = h_i + h_j - h_k$.

To define the higher products (2.7) of an L_∞ algebra, we follow [7] and use that the closure requirement

$$[\delta_{\varepsilon_i}, \delta_{\varepsilon_j}]W_k = \delta_{-\mathbf{C}(\varepsilon_i, \varepsilon_j, \mathbf{W})}W_k \quad (3.5)$$

is equivalent to the L_∞ relations with two gauge parameters once one identifies

$$\delta_{\varepsilon_i}W_j = \sum_{n \geq 0} \frac{1}{n!} (-1)^{\frac{n(n-1)}{2}} \ell_{n+1}^{W_j}(\varepsilon_i, \mathbf{W}^n) \quad (3.6)$$

and

$$\mathbf{C}(\varepsilon_i, \varepsilon_j, \mathbf{W}) = \sum_l \sum_{n \geq 0} \frac{1}{n!} (-1)^{\frac{n(n-1)}{2}} \ell_{n+2}^{\varepsilon_l}(\varepsilon_i, \varepsilon_j, \mathbf{W}^n). \quad (3.7)$$

We introduced an upper index to the L_∞ products which indicates in which W_i or ε_i direction the image of the higher products is located. Moreover, $\mathbf{C}$ and $\mathbf{W}$ denote elements in these vector spaces, where the short-hand notation $\mathbf{W}^n$ means that the higher product depends on n generators.

The combinatorial prefactors in (3.6) and (3.7) have been determined in [7] and guarantee that the so defined higher order products satisfy the L_∞ relations with two gauge parameters. Furthermore, it is shown in [7] that the L_∞ relations with three gauge parameters are equivalent to a vanishing Jacobi-identity for three gauge transformations

$$\sum_{\text{cycl}} [\delta_{\varepsilon_i}, [\delta_{\varepsilon_j}, \delta_{\varepsilon_k}]] = 0. \quad (3.8)$$

Together with (3.5) this relation ensures that the gauge transformations δ_{ε_i} form a well-defined associative Lie algebra.

Therefore, the higher order products with one gauge parameter can be read off directly from the expression (3.4). Due to the appearance of products of fields in the $\mathcal{W}$ algebra, in general one also gets products ℓ_n with $n > 2$. Moreover, in (3.5) we allowed the gauge parameter on the right hand side, $\mathbf{C}(\varepsilon_i, \varepsilon_j, \mathbf{W})$, to depend on the fields $\mathbf{W}$. This structure appeared in the context of higher spin theories in the early paper [3]. In order to determine them we proceed as follows. Using (3.1) and the usual Jacobi-identity for the bracket $[\cdot, \cdot]$, one can write

$$[\delta_{\varepsilon_i}, \delta_{\varepsilon_j}]W_l = -\frac{1}{2\pi i} \int \sum_k C_{ij}{}^k \mathcal{P}_{ijk}(\varepsilon_i, \varepsilon_j) [\phi_k, W_l] \quad (3.9)$$

with universal $\mathcal{P}_{ijk}(\varepsilon_i, \varepsilon_j)$ that only depend on the conformal dimensions of the fields involved

$$\mathcal{P}_{ijk}(\varepsilon_i, \varepsilon_j) = \sum_{\substack{r, s \in \mathbb{Z}_0^+ \\ r+s = h_{ijk} - 1}} \kappa_{ijk}^{rs} \partial^r \varepsilon_i \partial^s \varepsilon_j \quad (3.10)$$

with

$$\kappa_{ijk}^{rs} = \frac{(-1)^r (2h_k - 1)!}{r! s! (h_i + h_j + h_k - 2)!} \prod_{t=0}^{s-1} (2h_i - 2 - r - t) \prod_{u=0}^{r-1} (2h_j - 2 - s - u). \quad (3.11)$$

Using the Leibniz-rule for $[\phi_k, W_l]$ in (3.9), one can read off

$$\mathbf{C}(\varepsilon_i, \varepsilon_j, \mathbf{W}) = \sum_l \sum_k C_{ij}{}^k \mathcal{P}_{ijk}(\varepsilon_i, \varepsilon_j) \partial_l \phi_k \in X_0. \quad (3.12)$$

Using (3.7) and expanding the right-hand side, one can read off the higher order L_∞ products with two gauge parameters. If we have fixed the structure constants of the initial $\mathcal{W}$ -algebra such that the Jacobi-identity for the Poisson bracket is satisfied then by construction it is guaranteed that the L_∞ relations are satisfied. However, as we will see, one can initially leave some of the structure constants open and fix them by the L_∞ relations. In the CFT context this is called the bootstrap approach.

3.2 The classical $\mathcal{W}_3$ algebra

Since the general formalism developed in the previous section might appear quite abstract, let us now exemplify all this explicitly for the classical $\mathcal{W}_3$ algebra.

3.2.1 L_∞ products with one gauge parameter

Using the algebra (2.13) and the general relations (3.4), (3.6), one can derive the infinitesimal variations and read off the higher products with one gauge parameter

$$\begin{aligned} \delta_\varepsilon T &= \underbrace{\frac{c}{12} \partial^3 \varepsilon}_{\ell_1^T(\varepsilon)} + \underbrace{(2 \partial \varepsilon T + \varepsilon \partial T)}_{\ell_2^T(\varepsilon, T)}, \\ \delta_\varepsilon W &= \underbrace{(3 \partial \varepsilon W + \varepsilon \partial W)}_{\ell_2^W(\varepsilon, W)} \end{aligned} \quad (3.13)$$

and

$$\begin{aligned} \delta_\eta W &= \underbrace{\frac{c}{360} \partial^5 \eta}_{\ell_1^W(\eta)} + \underbrace{\alpha \left(\frac{1}{6} \partial^3 \eta T + \frac{1}{4} \partial^2 \eta \partial T + \frac{3}{20} \partial \eta \partial^2 T + \frac{1}{30} \eta \partial^3 T \right)}_{\ell_2^W(\eta, T)} \\ &\quad + \underbrace{\beta \left(\partial \eta (TT) + \frac{1}{2} \eta \partial (TT) \right)}_{-\frac{1}{2} \ell_3^W(\eta, T, T)}, \\ \delta_\eta T &= \underbrace{(3 \partial \eta W + 2 \eta \partial W)}_{\ell_2^T(\eta, W)}, \end{aligned} \quad (3.14)$$

where here and in the following we use the shorter notation where an ε denotes an ε_T and an η always denotes an ε_W . In the following we will first leave the two structure constants $\alpha = C_{WW}^T$ and $\beta = C_{WW}^\Lambda$ undetermined. We will see that the L_∞ relations indeed fix them to their expected values $\alpha = 2$ and $\beta = 32/(5c)$. It is amusing that the central terms in the $\mathcal{W}$ -algebra are related to the maps $\ell_1 : X_0 \rightarrow X_{-1}$.

3.2.2 L_∞ products with two gauge parameters

By taking (3.12) as an ansatz and comparing it with (3.7), we can read off the L_∞ products with two gauge parameters. We find

$$\mathbf{C}(\varepsilon_1, \varepsilon_2, \mathbf{W}) = \varepsilon_1 \partial \varepsilon_2 - \partial \varepsilon_1 \varepsilon_2 =: \ell_2^\varepsilon(\varepsilon_1, \varepsilon_2), \quad (3.15)$$

$$\mathbf{C}(\varepsilon, \eta, \mathbf{W}) = \varepsilon \partial \eta - 2 \partial \varepsilon \eta =: \ell_2^\eta(\varepsilon, \eta) \quad (3.16)$$

and a non-trivial higher order correction in

$$\mathbf{C}(\eta_1, \eta_2, \mathbf{W}) = \ell_2^\varepsilon(\eta_1, \eta_2) + \ell_3^\varepsilon(\eta_1, \eta_2, T) \quad (3.17)$$

with

$$\ell_2^\varepsilon(\eta_1, \eta_2) = \alpha \left(\frac{1}{30} \eta_1 \partial^3 \eta_2 - \frac{1}{30} \partial^3 \eta_1 \eta_2 + \frac{1}{20} \partial^2 \eta_1 \partial \eta_2 - \frac{1}{20} \partial \eta_1 \partial^2 \eta_2 \right) \quad (3.18)$$

and

$$\ell_3^\varepsilon(\eta_1, \eta_2, T) = \beta T (\eta_1 \partial \eta_2 - \partial \eta_1 \eta_2). \quad (3.19)$$

Please note the highly non-trivial form of (3.18) and the T dependence in (3.19).

3.2.3 L_∞ relations with two gauge parameters and closure

The following section is dedicated to the equivalence of the closure condition

$$[\delta_{\varepsilon_i}, \delta_{\varepsilon_j}] W_k = \delta_{-\mathbf{C}(\varepsilon_i, \varepsilon_j, \mathbf{W})} W_k \quad (3.20)$$

and the L_∞ relations (2.9) and (2.10). For every combination (ij, k) we will explicitly state the corresponding L_∞ relations.

- **(TT,T):** the closure condition (3.20) is equivalent to the L_∞ relations

$$\begin{aligned} \ell_1^T(\ell_2^\varepsilon(\varepsilon_1, \varepsilon_2)) &= \ell_2^T(\ell_1^T(\varepsilon_1), \varepsilon_2) + \ell_2^T(\varepsilon_1, \ell_1^T(\varepsilon_2)), \\ 0 &= \ell_2^T(\ell_2^\varepsilon(\varepsilon_1, \varepsilon_2), T) + \ell_2^T(\ell_2^T(\varepsilon_2, T), \varepsilon_1) + \ell_2^T(\ell_2^T(T, \varepsilon_1), \varepsilon_2). \end{aligned} \quad (3.21)$$

- **(TT,W):** due to $\ell_1^W(\varepsilon) = 0$ the closure condition corresponds to the non-trivial relation

$$0 = \ell_2^W(\ell_2^\varepsilon(\varepsilon_1, \varepsilon_2), W) + \ell_2^W(\ell_2^W(\varepsilon_2, W), \varepsilon_1) + \ell_2^W(\ell_2^W(W, \varepsilon_1), \varepsilon_2). \quad (3.22)$$

- **(TW,T):** one gets the single non-trivial relation

$$0 = \ell_2^T(\ell_2^\eta(\varepsilon, \eta), W) + \ell_2^T(\ell_2^T(\eta, W), \varepsilon) + \ell_2^T(\ell_2^W(W, \varepsilon), \eta). \quad (3.23)$$

- **(TW,W)**: in this case the closure condition also involves the products (TT) and therefore for the first time implies higher order L_∞ relations. One finds

$$\ell_1^W(\ell_2^\eta(\varepsilon, \eta)) = \ell_2^W(\ell_1^T(\varepsilon), \eta) + \ell_2^W(\varepsilon, \ell_1^W(\eta)) \quad (3.24)$$

which holds only for $\alpha = 2$, and

$$\begin{aligned} 0 = & \ell_2^W(\ell_2^\eta(\varepsilon, \eta), T) + \ell_2^W(\ell_2^W(\eta, T), \varepsilon) + \ell_2^W(\ell_2^T(T, \varepsilon), \eta) \\ & + \ell_3^W(\ell_1^T(\varepsilon), \eta, T) \end{aligned} \quad (3.25)$$

that vanishes only for $16\alpha = 5c\beta$. Finally, one also has the non-trivial $\mathcal{J}_4$ relation

$$0 = \ell_3^W(\ell_2^\eta(\varepsilon, \eta), T, T) - 2\ell_3^W(\ell_2^T(\varepsilon, T), \eta, T) - \ell_2^W(\varepsilon, \ell_3^W(\eta, T, T)) \quad (3.26)$$

that is zero for any choice of α and β . Together these are precisely the values familiar from the classical $\mathcal{W}_3$ algebra. Note that here we have only presented those terms from the general L_∞ relation that are non-vanishing. The combinatorial factor of two in (3.26) is due to the exchange of the two T -fields.

- **(WW,T)**: here another new aspect appears, namely that the closure condition requires a higher order correction to $\mathbf{C}(\eta_1, \eta_2, \mathbf{W})$ (3.17). The non-trivial parts in the L_∞ relations now read

$$\begin{aligned} \ell_1^T(\ell_2^\varepsilon(\eta_1, \eta_2)) &= \ell_2^T(\ell_1^W(\eta_1), \eta_2) + \ell_2^T(\eta_1, \ell_1^W(\eta_2)), \\ 0 &= \ell_1^T(\ell_3^\varepsilon(\eta_1, \eta_2, T)) + \ell_2^T(\ell_2^\varepsilon(\eta_1, \eta_2), T) \\ &\quad + \ell_2^T(\ell_2^W(\eta_2, T), \eta_1) + \ell_2^T(\ell_2^W(T, \eta_1), \eta_2), \\ 0 &= -2\ell_2^T(\ell_3^\varepsilon(\eta_1, \eta_2, T), T) \\ &\quad + \ell_2^T(\eta_2, \ell_3^W(\eta_1, T, T)) - \ell_2^T(\eta_1, \ell_3^W(\eta_2, T, T)). \end{aligned} \quad (3.27)$$

Again, the $\mathcal{J}_2$ relation is satisfied for $\alpha = 2$ and the $\mathcal{J}_3$ relation requires $16\alpha = 5c\beta$ to hold.

- **(WW,W)**: for this final closure condition, we find the equivalence to the L_∞ relations

$$\begin{aligned} 0 &= \ell_2^W(\ell_2^\varepsilon(\eta_1, \eta_2), W) + \ell_2^W(\ell_2^T(\eta_2, W), \eta_1) + \ell_2^W(\ell_2^T(W, \eta_1), \eta_2), \\ 0 &= \ell_2^W(\ell_3^\varepsilon(\eta_1, \eta_2, T), W) + \ell_3^W(\ell_2^T(\eta_1, W), \eta_2, T) + \ell_3^W(\eta_1, \ell_2^T(\eta_2, W), T) \end{aligned} \quad (3.28)$$

that are both independent of α and β .

One can also show that the above relations cover all non-trivial L_∞ relations with two gauge parameters, so that we have explicitly checked their equivalence to the closure conditions.

3.2.4 L_∞ relations with three gauge parameters and the Jacobi-identity

It only remains to check the L_∞ with three gauge parameters. Recall that they are supposed to be equivalent to the Jacobi-identity (3.8) of three gauge transformations, i.e.

$$\sum_{\text{cycl}} [\delta_{\varepsilon_i}, [\delta_{\varepsilon_j}, \delta_{\varepsilon_k}]] = 0. \quad (3.29)$$

Again we list the L_∞ relations that ensure this relation for a given combination of (ijk) .

- **(TTT):**

$$0 = \ell_2^\varepsilon(\ell_2^\varepsilon(\varepsilon_1, \varepsilon_2), \varepsilon_3) + \ell_2^\varepsilon(\ell_2^\varepsilon(\varepsilon_3, \varepsilon_1), \varepsilon_2) + \ell_2^\varepsilon(\ell_2^\varepsilon(\varepsilon_2, \varepsilon_3), \varepsilon_1)$$

- **(TTW):**

$$0 = \ell_2^\eta(\ell_2^\varepsilon(\varepsilon_1, \varepsilon_2), \eta) + \ell_2^\eta(\ell_2^\eta(\eta, \varepsilon_1), \varepsilon_2) + \ell_2^\eta(\ell_2^\eta(\varepsilon_2, \eta), \varepsilon_1)$$

- **(WWT):**

$$\begin{aligned} 0 &= \ell_2^\varepsilon(\ell_2^\varepsilon(\eta_1, \eta_2), \varepsilon) + \ell_2^\varepsilon(\ell_2^\eta(\varepsilon, \eta_1), \eta_2) + \ell_2^\varepsilon(\ell_2^\eta(\eta_2, \varepsilon), \eta_1) \\ &\quad + \ell_3^\varepsilon(\eta_1, \eta_2, \ell_1^T(\varepsilon)), \\ 0 &= -\ell_2^\varepsilon(\ell_3^\varepsilon(\eta_1, \eta_2, T), \varepsilon) + \ell_3^\varepsilon(\ell_2^\eta(\eta_1, \varepsilon), \eta_2, T), \\ &\quad - \ell_3^\varepsilon(\ell_2^\eta(\eta_2, \varepsilon), \eta_1, T) + \ell_3^\varepsilon(\ell_2^T(T, \varepsilon), \eta_1, \eta_2). \end{aligned}$$

The first $\mathcal{J}_3$ -type relation requires $16\alpha = 5c\beta$ to hold and shows that the two-product ℓ_2 violates the Jacobi-identity.

- **(WWW):**

$$\begin{aligned} 0 &= \ell_2^\eta(\ell_2^\varepsilon(\eta_1, \eta_2), \eta_3) + \ell_2^\eta(\ell_2^\varepsilon(\eta_3, \eta_1), \eta_2) + \ell_2^\eta(\ell_2^\varepsilon(\eta_2, \eta_3), \eta_1), \\ 0 &= \ell_2^\eta(\ell_3^\varepsilon(\eta_1, \eta_2, T), \eta_3) + \ell_2^\eta(\ell_3^\varepsilon(\eta_3, \eta_1, T), \eta_2) + \ell_2^\eta(\ell_3^\varepsilon(\eta_2, \eta_3, T), \eta_1), \\ 0 &= \ell_3^\varepsilon(\ell_2^T(\eta_1, W), \eta_2, \eta_3) + \ell_3^\varepsilon(\ell_2^T(\eta_2, W), \eta_3, \eta_1) + \ell_3^\varepsilon(\ell_2^T(\eta_3, W), \eta_1, \eta_2). \end{aligned}$$

Again the listed relations cover all non-trivial L_∞ relations with three gauge parameters. In particular, all relations of type $\mathcal{J}_5 = 0$ are trivially satisfied for the $\mathcal{W}_3$ algebra.

3.2.5 Summary

Looking back one realizes that the constraint $\alpha = 2$ arose solely from the relations $\mathcal{J}_2 = 0$, while $16\alpha = 5c\beta$ was needed for $\mathcal{J}_3 = 0$ to hold. The other non-trivial L_∞ relations of type $\mathcal{J}_4 = 0$ were satisfied regardless of the particular value α and β .

Therefore for the known classical values $\alpha = 2$ and $\beta = 32/(5c)$ all higher order relations of the L_∞ algebra are fulfilled and we have shown that the classical $\mathcal{W}_3$ algebra induces a highly non-trivial L_∞ algebra. Since the concrete form of the $\mathcal{W}_3$ algebra was

known before, it looks like that we have just checked what we argued must be true on general grounds in section 3.1. However, from the L_∞ point of view, the general conformal field theory structure provides a concrete recipe to read off higher order products that satisfy all L_∞ relations.

Moreover, as we actually did, one can turn the logic around and determine the unknown structure constants and relations among them from the L_∞ relations. In other words, with general input from CFT, one can also use the L_∞ structure to *bootstrap* the structure constants in the extended classical conformal symmetry algebras.

4 Conclusions

The purpose of this letter was to point out that classical $\mathcal{W}$ algebras do naturally define an L_∞ algebra whose objects are concentrated in X_0 and X_{-1} . We provided a general prescription of how to read off the higher order L_∞ products from the conformal field theory structure. Moreover, we argued that the Poisson algebra relations for the $\mathcal{W}$ algebra are equivalent to the non-trivial higher order L_∞ relation for two and three gauge parameters. As an application, we explicitly worked out the example of a $\mathcal{W}_3$ algebra. In that case, the highest appearing products are of order three and as a consequence only a finite number of higher order relations are non-trivial. Turning the logic around, the structure constants of the $\mathcal{W}_3$ could also be bootstrapped by the L_∞ relations.

It should be clear how this generalizes to $\mathcal{W}$ algebras with more generators. For instance, the $\mathcal{W}_N$ algebra contains generators of conformal dimension $\{2, 3, \dots, N\}$. The highest order product of fields is (T^{N-1}) and appears in $\delta_{\varepsilon_N} W_N$. Therefore, the expectation is that this also defines an L_∞ algebra with the highest higher order product being $\ell_N^{W_N}(\varepsilon_N, T^{N-1})$. Of course, in this case higher order relations up to $\mathcal{J}_{2N-1}$ can become non-trivial. Therefore, the classical $\mathcal{W}_N$ algebras provide an infinite set of highly non-trivial L_∞ algebras. One could also consider classical super $\mathcal{W}$ algebras, which will correspond to super L_∞ algebras.

Coming back to the $\text{AdS}_3\text{-CFT}_2$ duality, one might suspect that the classical $\mathcal{W}_\infty[\mu]$ algebra is related to an L_∞ algebra with an infinite number of higher products satisfying an infinite number of higher order relations. As mentioned, for positive integer μ one has the truncation $\mathcal{W}_\infty[N] = \mathcal{W}_N$ which will also hold for the L_∞ algebra.

So far we were discussing just the classical $\mathcal{W}$ symmetry and one could wonder whether also the quantum case admits a description in terms of an L_∞ algebra or a quantum version of it. This questions seems to be fairly non-trivial, as the procedure laid out in section 3.1 does not go through straightforwardly. In the quantum case all products of fields need to be normal ordered and it is not clear what a field dependent gauge parameter should mean. We leave this interesting question for future research.

Finally, one might ask whether one can generalize this L_∞ structure beyond

- the two vector spaces X_0 and X_{-1} ,
- the chiral sector of a 2D CFT,
- two-dimensions.

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