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$\mathcal{N} = 2$ supersymmetric field theories on 3-manifolds with A-type boundaries

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Abstract: General half-BPS A-type boundary conditions are formulated for $\mathcal{N} = 2$ supersymmetric field theories on compact 3-manifolds with boundary. We observe that under suitable conditions manifolds of the real A-type admitting two complex supersymmetries (related by charge conjugation) possess, besides a contact structure, a natural integrable toric foliation. A boundary, or a general co-dimension-1 defect, can be inserted along any leaf of this preferred foliation to produce manifolds with boundary that have the topology of a solid torus. We show that supersymmetric field theories on such manifolds can be endowed with half-BPS A-type boundary conditions. We specify the natural curved space generalization of the A-type projection of bulk supersymmetries and analyze the resulting A-type boundary conditions in generic $3d$ non-linear sigma models and YM/CS-matter theories.

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1 Introduction

The study of supersymmetric quantum field theories on rigid curved backgrounds in diverse spacetime dimensions has been a powerful source of new non-perturbative results in recent years. So far, a rather complete and systematic understanding of such results has been obtained for supersymmetric field theories on closed manifolds. Most notably, these theories can be engineered by taking appropriate rigid limits of certain supergravity theories. This framework constrains the background geometry and determines the couplings of the field theory to the curvature and the auxiliary background fields in the supergravity multiplet [1–3]. Partition functions and other supersymmetric observables can then be evaluated exactly with the powerful technique of supersymmetric localization providing a new window into non-perturbative physics in quantum field theory. Some of the original work in this direction in two, three, four, and five spacetime dimensions includes [4–11].

Analogous situations on manifolds with boundary, or more generally, on spaces with co-dimension-1 defects, are comparatively much less elaborated upon. There are two key aspects of this story one would like to develop systematically. The first aspect is related to the geometric properties of boundaries. Given a fixed bulk supergravity background that supports supersymmetric field theories, what restrictions should be imposed on the geometry of a co-dimension-1 surface to preserve a subset of the bulk supersymmetry? The second aspect is related more directly to the specific dynamic properties of the field theory in question, in particular, the boundary conditions that can be imposed on the defect.

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Regarding the first point, it is immediately clear that since the commutator of supersymmetries squares to isometries on the compact manifold, the boundary should be oriented along directions parallel to these isometries, in order to preserve the corresponding supersymmetries. Moreover, one can ask if supersymmetry puts any constraints on co-dimension-1 foliations of a compact manifold. A foliation preferred by supersymmetry could be used to decompose closed manifolds into a union of manifolds with boundary. Indeed, we will show that such a foliation exists in a general class of 3-manifolds.

As far as the second point is concerned, it is well known that the invariance of generic observables under bulk symmetries (including supersymmetries) is spoiled, in general, by boundary effects. A symmetry can be restored by cancelling these boundary effects. This can be achieved with the introduction of suitable boundary conditions and/or the introduction of appropriate boundary degrees of freedom.

In the present work we concentrate on three dimensions and develop a systematic treatment of half-BPS boundaries in $\mathcal{N} = 2$ supersymmetric field theories on compact 3-manifolds. We discuss general aspects of the interplay between supersymmetry and the geometry of manifolds with boundary, and analyze a wide class of related half-BPS boundary conditions. We concentrate on the classical aspects of the problem. The main contributions of this work can be summarized as follows.

Summary of main results. We begin in section 2 with a concise collection of useful results on rigid supersymmetry in curved three-dimensional backgrounds. We follow closely the conventions of ref. [3], where it was recognized that the existence of a supersymmetry implies a tranversely holomorphic foliation. Subsequently, we focus on a more specific class of curved 3-manifolds dubbed A-type backgrounds. These backgrounds are introduced in section 3. By definition, they admit two complex Killing spinors related by charge conjugation, [12]. We show, using supersymmetry, that they also admit a contact structure whose Reeb vector is a Killing vector, and, under suitable conditions, a preferred co-dimension-1 foliation whose distribution is defined only in terms of Killing spinors bilinears. The Reeb vector belongs to the foliation, and the algebra of supersymmetry is preserved on the leaves. Geometrically, global properties of the Reeb vector are classified as regular, quasi-regular, and irregular, as reviewed in [13]. Manifolds covered by this analysis include well-known examples of Seifert manifolds, like for instance the round and squashed 3-spheres, and geometries of the $\mathbb{S}^2 \times \mathbb{S}^1$ type.

A boundary can be introduced along a generic leaf of the co-dimension-1 foliation. Technically, our construction of the foliation in terms of vector fields does not require the use of coordinates, which may be problematic if the coordinates are not globally well defined. We argue that the topology of the leaves is that of a torus. Hence, the manifold decomposition, that follows from supersymmetry considerations, selects $3d$ manifolds with boundary in which the boundary is a torus. The main goal of the paper is to formulate $\mathcal{N} = 2$ supersymmetric field theories with half-BPS boundary conditions on such spaces.

In section 4 we show that the geometry of the A-type backgrounds admits a natural half-BPS projection on the bulk supersymmetries that generalizes in curved space the well-known A-type projection familiar from studies of $2d$ $\mathcal{N} = (2, 2)$, [14], and $3d$ $\mathcal{N} = 2$

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theories in flat space, [15]. Unlike the case of flat space where one projects constant spinors, in curved spaces one has to project spinors that are in general non-trivial functions of the spacetime coordinates. We propose a 'canonical' way to implement a generalized A-type projection in curved space, that reduces to the familiar A-type projections in flat space. To the best of our knowledge this formulation is new. A similar generic formulation for B-type projections in curved space is left to future work.

The generalized A-type projection can be employed to formulate corresponding A-type boundary conditions in $\mathcal{N} = 2$ supersymmetric field theories, that preserve half of the bulk supersymmetry. In sections 5–8 we present these boundary conditions for arbitrary non-linear sigma models and YM/CS-matter theories. In both cases, we relate the boundary conditions to the geometry of certain 2-forms defined on the space of field configurations at the boundary. These 2-forms are also relevant in the analysis of the on-shell boundary value problem, that we review in section 6.

Section 7 studies the instructive case of non-linear sigma models with generic Kähler potential and superpotential. The boundary conditions describe Lagrangian submanifolds of the Kähler form in target space. The effect of the curvature and the presence of couplings to the background fields, generalize the more familiar analysis in flat space [14, 16].

The case of general (non-abelian) YM/CS-matter theories is discussed in section 8. We find boundary conditions that include the curved space generalization of holomorphic Neumann boundary conditions for Yang-Mills gauge fields and matter fields, and holomorphic Dirichlet boundary conditions for the gauge fields in CS theories.

A summary of useful formulae, and an exposition of technical details for results used in the main text are relegated to two appendices at the end of the paper.

Prospects. We conclude this short introduction with a few remarks on some of the interesting open questions raised in this work and the prospects of further related developments.

Our main motivation for the study of the classical problem in this paper is the eventual formulation of general half-BPS co-dimension-1 defects in $3d$ $\mathcal{N} = 2$ supersymmetric quantum field theories on curved spaces, and the non-perturbative computation of observables associated with these defects.

The observables we are interested in include the partition function of $\mathcal{N} = 2$ supersymmetric gauge theories on curved backgrounds with boundary. With A-type boundary conditions these partition functions are computing a class of supersymmetric wavefunctions. It would be interesting to explore the dependence of these observables on the moduli of the defects, i.e. the moduli of the boundary conditions we formulate, generalizing the bulk analysis of ref. [17]. A preliminary computation of partition functions on manifolds with boundary in three dimensions using localization techniques has been performed in special cases in [18, 19]. The results in the present paper can be used to extend known results in this direction.

Moreover, one can also attempt to use the information of supersymmetric wavefunctions to study the structure of observables on closed manifolds that do not involve co-dimension-1 defects. Hints of such a possibility come from a variety of previous results: the holomorphic block decomposition of $3d$ partition functions [19, 20], and the analogous phenomenon in different dimensions [7, 21, 22], the recent progress in computing D-brane

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amplitudes in $2d$ $\mathcal{N}=(2,2)$ theories [23–25], and $tt^\star$ arguments in flat toroidal backgrounds in three, and four spacetime dimensions [26].

Boundary conditions also introduce another tool to probe dualities between quantum field theories. If two theories are dual at the quantum level, we expect corresponding boundary conditions on each side to be mapped to each other in a non-trivial way. For instance, in the case of mirror symmetry, the duality between boundary conditions can be understood in the mathematical framework of *symplectic duality* [27]. $3d$ Seiberg duality also acts non-trivially on boundary conditions. We refer the reader to ref. [28] for a recent discussion of the relation between $3d$ Seiberg dualities and $2d$ level-rank dualities in this context. Similar problems with Wilson loops were investigated in [29, 30]. Finally, an intriguing interpretation of co-dimension-1 defects relates the expectation value of these operators to the entanglement structure of the field theory [31].

Another arena of potential applications of such computations is M-theory. The study of boundary conditions in the ABJM theory [32], which is an $\mathcal{N}=6$ Chern-Simons-matter theory, is expected to yield information about the physics of M2, M5-branes and their interactions. For instance, it is anticipated that the low-energy theory at the orthogonal intersection of M2 and M5-branes in $\mathbb{C}^4/\mathbb{Z}_k$ is a $2d$ theory with $\mathcal{N}=(4,2)$ (or in special cases $\mathcal{N}=(4,4)$) supersymmetry. The non-abelian quantum properties of this theory are still illusive. A recent bare Lagrangian formulation of this theory in terms of boundary degrees of freedom motivated by D-brane physics in type IIB Hanany-Witten setups was proposed recently in [33]. For a study of half-BPS boundary conditions in ABJM theory see [34, 35].

Finally, there are several aspects of the general theory of supersymmetric boundaries in three dimensions that are not discussed in this paper. One of these aspects is the general curved space analog of B-type boundary conditions in $2d$ $\mathcal{N}=(2,2)$ theories. Another aspect that is worth exploring further is the formulation of half-BPS boundaries using explicit boundary degrees of freedom and boundary actions [36]. The analysis of supersymmetric boundaries in $2d$ $\mathcal{N}=(2,2)$ theories in [23] was performed in this manner.

2 Review of rigid supersymmetry on curved 3-manifolds

In the modern approach to rigid supersymmetry on curved spaces, the metric tensor $g_{\mu\nu}$ (or any other background field) is embedded into a certain supergravity multiplet, and the field theory is obtained by taking the rigid limit of Festuccia-Seiberg [1] (FS). With a $\mathrm{U}(1)_R$ symmetry, the supergravity of interest in $4d$ is the "new minimal supergravity" of [37], and the supergravity multiplet contains an R-symmetry gauge field $A_\mu^{(R)}$, a conserved vector V^μ and the two gravitini $\Psi_{\mu\alpha}$, $\widetilde{\Psi}_{\mu\dot{\alpha}}$. Following FS, the rigid field theory of chiral and vector superfields on the curved space, is obtained from the action of off-shell supergravity coupled to chiral and vector fields, by freezing the bosonic components of the supergravity multiplet to a configuration in which $\delta\Psi_\mu=\delta\widetilde{\Psi}_\mu=0$. The advantage of this formulation is that the whole procedure can be carried out off-shell, without the need of an explicit solution to the equations $\delta\Psi_\mu=\delta\widetilde{\Psi}_\mu=0$.

In $3d$ it is possible to perform a twisted dimensional reduction of the $4d$ rigid theories to infer a consistent new minimal $3d$ algebra [3]. At the end of this process, the background

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fields are, the metric $g_{\mu\nu}$, an R-symmetry gauge field $A_\mu^{(R)}$, a conserved vector V^μ (as in $4d$), and an extra scalar field H. The conditions $\delta\Psi_\mu = \delta\widetilde{\Psi}_\mu = 0$ reduce to the following two Killing spinor equations

$$\begin{aligned}(\nabla_\mu - iA_\mu^{(R)})\zeta &= -\frac{1}{2}H\gamma_\mu\zeta + \frac{i}{2}V_\mu\zeta - \frac{1}{2}\varepsilon_{\mu\nu\rho}V^\nu\gamma^\rho\zeta \\ &= -\frac{1}{2}\gamma_\mu(H\zeta - iV_\nu\gamma^\nu\zeta)\,, \qquad (2.1)\\ (\nabla_\mu + iA_\mu^{(R)})\tilde{\zeta} &= -\frac{1}{2}H\gamma_\mu\tilde{\zeta} - \frac{i}{2}V_\mu\tilde{\zeta} + \frac{1}{2}\varepsilon_{\mu\nu\rho}V^\nu\gamma^\rho\tilde{\zeta} \\ &= -\frac{1}{2}\gamma_\mu(H\tilde{\zeta} + iV_\nu\gamma^\nu\tilde{\zeta})\ . \qquad (2.2)\end{aligned}$$

The two Weyl spinors ζ and $\tilde{\zeta}$ have R-charges $+1$ and -1 respectively.

In practice, given a choice of the background metric, the other background fields can be adjusted to obtain at least one solution of the Killing spinor equations. On the other hand, assuming that at least one Killing spinor exists as a solution of the equations (2.1), (2.2), it is possible to deduce what geometric structure the manifold needs to possess. In $3d$, this analysis was first carried out in [3, 17, 38]. In section 2.1 we will review in some detail the relevant geometry since it will play an important role in our problem. In fact, in order to set up supersymmetric boundary conditions, it will be useful to improve slightly the way in which the relevant geometric structure is characterized. The new material is presented in section 3. Experts familiar with rigid supersymmetry on spaces without boundary (e.g. the work in [3, 17, 38]) may skip to section 3. We follow closely the notation of ref. [3].

In our presentation, it will be convenient to make an explicit distinction between commuting and anti-commuting Killing spinors. In particular, we will denote the commuting spinors with ζ and $\tilde{\zeta}$, and the anti-commuting spinors with ϵ and $\tilde{\epsilon}$. Both sets of Killing spinors satisfy the same equations. The anti-commuting spinors ϵ and $\tilde{\epsilon}$, will provide the parameters of the supersymmetry transformations of the field theory. The commuting spinors, ζ and $\tilde{\zeta}$, will be used to explore the geometry of the manifold.

2.1 Geometry of $\mathcal{M}_3$

The existence of Killing spinor solutions, ζ and $\tilde{\zeta}$, strongly constrains the geometric structure of the background fields. We will not repeat the general analysis here, but we recall two important results of [3], which will be useful for later purposes. The first states that a solution of (2.1), or (2.2), when it exists, is nowhere vanishing.[1] The second result states that given *one* Killing spinor, say ζ for concreteness, it is possible to cover the manifold with a *transversely holomorphic foliation* (THF), and write the metric in the following form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \eta^2 + c(\tau, z, \bar{z})^2 dz d\bar{z}, \qquad \eta = d\tau + (h(\tau, z, \bar{z})dz + c.c.)\ . \qquad (2.3)$$

By definition of the THF, the adapted coordinate τ is real, whereas $\{z, \bar{z}\}$ are complex. The leaves of the co-dimension-2 foliation are the submanifolds $z = const.$, and two patches

[1] This property will be crucial for the consistency of the *canonical formalism* that we set up in section 4.1. In general situations, depending on the specifics of the Killing spinor equations, a non-trivial solution may or may not admit zeros [2].

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are related by transitions functions, f and h, such that $z' = f(z)$ with f holomorphic, and $\tau' = h(\tau, z, \bar{z})$ with h real. In particular, h can be put in the form $h(\tau, z, \bar{z}) = \tau + t(z, \bar{z})$.

The origin of the transversely holomorphic foliation is an integrability constraint. The one-form $\eta = \eta_\mu dx^\mu$ can be represented as the spinor bilinear

$$\eta_\mu = \frac{1}{|\zeta|^2}\,\zeta^c\gamma_\mu\zeta, \qquad |\zeta|^2 = \zeta^c\zeta\,, \tag{2.4}$$

and the following fields can be defined,[2]

$$\xi^\mu = g^{\mu\nu}\eta_\mu\,, \qquad J^\mu_{\ \nu} = \varepsilon^\mu_{\ \nu\rho}\xi^\rho\ . \tag{2.5}$$

The spinor ζ^c is the charge conjugate to ζ. Notice that from the properties of ξ^μ and $J^\mu_{\ \nu}$ it also follows that the Killing spinor equation of ζ, (2.1), is invariant under the shift symmetry,

$$\begin{aligned} V^\mu \to &\quad V^\mu + X^\mu + k\,\xi^\mu\,, \\ H \to &\quad H + ik\,, \end{aligned} \tag{2.6}$$

where the scalar k and the vector field X^μ are such that $J^\mu_{\ \nu}X^\nu = iX^\mu$ and $\nabla_\mu(X^\mu + k\xi^\mu) = 0$. After gauge fixing the shift invariance, the Killing spinor equation (2.1), implies the constraint

$$J^\mu_{\ \nu}\left(\mathcal{L}_\xi J\right)^\nu_{\ \rho} = 0\ . \tag{2.7}$$

Given the condition (2.7), the authors of ref. [3] showed that it is possible to find the adapted coordinates $\{\tau, z, \bar{z}\}$ introduced in (2.3). This is the THF associated to ζ. On the other hand, if $\tilde{\zeta}$ is the only non trivial solution to the Killing spinor equations, the corresponding THF is defined as in (2.4) with the substitution $\zeta \to \tilde{\zeta}$, i.e. $\tilde{\eta}_\mu = (\tilde{\zeta}^c\gamma_\mu\tilde{\zeta})\,|\tilde{\zeta}|^{-2}$. The Killing spinor equation of $\tilde{\zeta}$ is invariant under a shift similar to (2.6).

Manifolds that admit *two* complex supercharges of opposite R-charge have additional properties compared to the individual THFs given by ζ and $\tilde{\zeta}$. They have a nowhere vanishing Killing vector K^μ, and a *contact structure*. The Killing vector is represented as

$$K^\mu = \tilde{\zeta}\gamma^\mu\zeta\ . \tag{2.8}$$

It solves the equation

$$\nabla_\mu K_\nu = iH\,\varepsilon_{\mu\nu\rho}K^\rho + \varepsilon_{\mu\nu\rho}V^\rho\,\tilde{\zeta}\zeta\,, \tag{2.9}$$

from which $\nabla_{\{\mu}K_{\nu\}} = 0$ follows. The norm of K^μ is $K^\mu K_\mu = (\tilde{\zeta}\zeta)^2 \equiv \Omega^2$, and the function Ω is such that

$$K^\mu\partial_\mu(\tilde{\zeta}\zeta) = -K^\mu\varepsilon_{\mu\alpha\beta}V^\alpha K^\beta = 0\ . \tag{2.10}$$

[2]The triple $(\eta_\mu, \xi^\mu, J^\mu_{\ \nu})$, with η_μ, ξ^μ, and $J^\mu_{\ \nu}$ such that $\eta_\mu\xi^\mu = 1$ and $J^2 = -1 + \xi\,\eta$, is called an *almost contact structure* (ACS). This definition only requires that η_μ, ξ^μ, and $J^\mu_{\ \nu}$, satisfy algebraic constraints. It does not require the manifold to have a metric. For Riemannian manifolds, a metric $g_{\mu\nu}$ is said to be *compatible* with the ACS if $\xi^\mu = g^{\mu\nu}\eta_\mu$. The ACS is then promoted to an *almost contact metric structure* (ACMS). Similarly to the definition of a complex structure, the difference between an *almost* and a *contact structure*, is a differential constraint. However, this constraint is not (2.7) but: $d\eta\,(\xi, \cdot) = 0$ for the *contact structure*, and $d\eta\,(\cdot, \cdot) = g\,(J\cdot, \cdot)$ for the *contact metric structure* [39]. It is perhaps useful to mention that the condition for a contact metric structure resembles the one for Kähler manifolds in even dimensions [40].

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Notice that the Killing spinor equations are linear, therefore ζ and $\lambda\zeta$, with λ an arbitrary complex number, are both solutions. Similarly for $\tilde{\zeta}$. However, the relation $\tilde{\zeta}\zeta = \Omega$ breaks the arbitrariness in the normalization of ζ and $\tilde{\zeta}$, and only the symmetry $\zeta \to \lambda\zeta$ with $\tilde{\zeta} \to \lambda^{-1}\tilde{\zeta}$ remains. Eq. (2.8) is also invariant under this scaling.

When the Killing vector is real, the manifold is a Seifert manifold and the geometry can be further characterized by the orbits of K^μ. Two cases can be distinguished: either the orbits of K^μ are periodic, or they do not close. The first case consists of manifolds with the topology of an $\mathbb{S}^1$-bundle over a $2d$ Riemann surface. In the second case, it can be proved that there exists another independent Killing vector, transverse to K^μ, and that the isometry group of $\mathcal{M}_3$ is at least $\mathrm{U}(1)\times\mathrm{U}(1)$ [41]. Seifert manifolds are also singled out as BRST-preserving backgrounds in $3d$ topological gravity. The relation between $3d$ topological gravity and rigid supersymmetry has been pointed out, and further studied, in [42].

The contact structure $(\hat{\eta}_\mu, \hat{\xi}^\mu, \hat{J}^\mu{}_\nu)$, is defined by the fields

$$\hat{\eta}_\mu = \frac{1}{\Omega^2}K_\mu, \qquad \hat{\xi}^\mu = K^\mu, \qquad \hat{J}^\mu{}_\nu = \frac{1}{\Omega}\varepsilon^\mu{}_{\nu\rho}K^\rho \ , \tag{2.11}$$

subject to the relations: $\hat{\eta}_\mu\hat{\xi}^\mu = 1$, $(d\hat{\eta})_{\mu\nu}\hat{\xi}^\mu = 0$. The latter condition can be checked by means of the Killing spinor equations (2.1) and (2.2). In particular, $(d\hat{\eta})_{\mu\nu}\hat{\xi}^\mu = 0$ implies through Darboux's theorem [40] the existence of local coordinates (ψ, x_1, y_1) such that

$$\hat{\eta} = \frac{K_\mu}{\Omega^2} = d\psi + x_1 dy_1 \,, \qquad \hat{\xi}^\mu\partial_\mu = \partial_\psi \ . \tag{2.12}$$

As a result, the Killing vector K^μ is aligned along ∂_ψ. $\mathcal{M}_3$ endowed with such contact structure is a *contact manifold*, and the vector $\xi^\mu = K^\mu$ is called *Reeb vector*. An equivalent characterization of a contact manifold is the condition that $\hat{\eta}\wedge d\hat{\eta} \neq 0$. The coordinates (ψ, x_1, y_1) are called *canonical* since the condition $\hat{\eta}\wedge d\hat{\eta} \neq 0$ becomes trivial. The contact structure defined in (2.11), shares the same algebraic properties of the triple $(\eta_\mu, \xi^\mu, J^\mu{}_\nu)$ defined in (2.7). These are $\eta_\mu\xi^\mu = 1$ and $J^2 = -\mathbb{I} + \xi\,\eta$, but in addition, an explicit calculation shows that the tensor $\hat{J}^\mu{}_\nu$ satisfies a stronger (integrability) constraint, $\mathcal{L}_{\hat{\xi}}\hat{J} = 0$.

In section 3 we will supplement the above results on 3-manifold geometry with a further new refinement that facilitates the introduction of boundaries preserving a subset of the bulk supersymmetries.

2.2 Supersymmetric multiplets and transformations

Rigid supersymmetric field theories exist on any curved background $\mathcal{M}_3$, equipped with the two Killing spinors ϵ and $\tilde{\epsilon}$. Their Lagrangians are obtained by exploiting the *multiplet calculus* of $4d$ new minimal supergravity [37] and its $3d$ version (see appendix of [3]).

By multiplet calculus we mean the collection of all the supersymmetry transformations of the components of a generic multiplet $\mathcal{S}$. The total number of independent degrees of freedom in $\mathcal{S}$ is 16 bosonic plus 16 fermionic. They are organized as follows:

$$\mathcal{S} = \{C, \chi_a, \tilde{\chi}_\alpha, M, \tilde{M}, a_\mu, \sigma, \lambda_\alpha, \tilde{\lambda}_\alpha, D\} \ . \tag{2.13}$$

The R-charges are $(0, -1, +1, -2, +2, 0, 0, +1, -1, 0)$ relative to the bottom component C. The supersymmetry transformation rules $\delta_\epsilon\mathcal{S} + \tilde{\delta}_{\tilde{\epsilon}}\mathcal{S}$ are summarized in appendix A. The set

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of all these transformations realize an algebra on the space of fields. Denoting with $\varphi_{(r,z)}$ a field of arbitrary spin, R-charge r, and central charge z, the supersymmetric algebra is represented by

$$[\delta_\epsilon, \tilde{\delta}_{\tilde{\epsilon}}]\, \varphi_{(r,z)} = -2i\, (\mathcal{L}_K + \epsilon\tilde{\epsilon}\, (z - rH))\, \varphi_{(r,z)}, \qquad [\delta, \delta] = 0, \qquad [\tilde{\delta}, \tilde{\delta}] = 0\,, \tag{2.14}$$

The symbol $\mathcal{L}_K$ is defined in [3] as a modified Lie derivative along K

$$\mathcal{L}_K \varphi_{(r,z)} = \left[\mathrm{Lie}_K - irK^\mu \left(A_\mu - \frac{1}{2}V_\mu\right) - iz\, K^\mu \mathcal{C}_\mu\right] \varphi_{(r,z)}\ . \tag{2.15}$$

The covariant derivative associated to $\mathcal{L}_K$ will be denoted as

$$\mathcal{D}_\mu \varphi_{(r,z)} = \left[\nabla_\mu - ir\left(A_\mu - \frac{1}{2}V_\mu\right) - iz\, \mathcal{C}_\mu\right] \varphi_{(r,z)}\ . \tag{2.16}$$

Here the background gauge field $\mathcal{C}_\mu$ is related to the background conserved vector V^μ by the relation $V_\mu = -i\varepsilon^{\mu\nu\rho}\partial_\nu \mathcal{C}_\rho$. The gauge field A_μ is not $A_\mu^{(R)}$, but the two are related by a redefinition

$$A_\mu^{(R)} \equiv A_\mu - \frac{3}{2}V_\mu\ . \tag{2.17}$$

The combination $A_\mu - \frac{1}{2}V_\mu$ is not invariant under the shift symmetry (2.6), but $A_\mu^{(R)}$ is. Accordingly, it is convenient to express $(\mathcal{L}_K + \epsilon\tilde{\epsilon}(z - rH))$ as

$$\mathrm{Lie}_K - irK^\mu A_\mu^{(R)} - ir(K^\mu V_\mu - \epsilon\tilde{\epsilon}\, iH) - iz\, K^\mu \mathcal{C}_\mu + \epsilon\tilde{\epsilon}\, z\ . \tag{2.18}$$

In what follows, we will mostly use $\mathcal{D}_\mu$, as defined above, since we adopt the notation of ref. [3]. Sometimes, however, it will be convenient to consider $A^{(R)}$ in the covariant derivative. When this happens we will be very explicit.

For the benefit of the reader we list here two standard short multiplets $\mathcal{S}$ that will play a dominant role in the main discussion. The shortening of the multiplets is obtained by imposing restrictions on its components.

2.2.1 Chiral and the anti-chiral multiplets

Chiral (anti-chiral) multiplets are obtained by imposing the conditions $\tilde{\chi}_\alpha = 0$ ($\chi_\alpha = 0$). This implies that not all components of the generic multiplet are independent. A chiral multiplet Φ, with independent components $\{\phi, \psi_\alpha, F\}$ is organized as follows,

$$\begin{aligned} \mathcal{S}\Big|_{\tilde{\chi}=0} \equiv \Phi = \{&\phi, -i\sqrt{2}\psi_\alpha, 0, -i2F, 0, \\ &-i\mathcal{D}_\mu\phi, (z - rH)\phi, 0, 0, \tfrac{r}{4}(R - 2V^2 - 2H^2)\phi - zH\phi\}. \end{aligned} \tag{2.19}$$

In the above formula, $R[\phi] = r$ is the R-charge of ϕ, and z is the central charge. The transformation rules of $\{\phi, \psi, F\}$ are

$$\begin{aligned} \delta\phi &= \sqrt{2}\,\epsilon\psi\,, \\ \delta\psi &= \sqrt{2}\epsilon F - i\sqrt{2}\,(z - rH)\,\tilde{\epsilon}\,\phi - i\sqrt{2}\gamma^\mu\tilde{\epsilon}\,\mathcal{D}_\mu\phi\,, \\ \delta F &= i\sqrt{2}\,(z - (r-2)H)\,\tilde{\epsilon}\psi - i\sqrt{2}\,\mathcal{D}_\mu(\tilde{\epsilon}\gamma^\mu\psi)\ . \end{aligned} \tag{2.20}$$

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The shorthand notation for Φ will be $\Phi = \{\phi, \psi, F\}$. The case of the anti-chiral multiplet $\tilde{\Phi}$ is analogous. The independent components are $\{\widetilde{\phi}, \widetilde{\psi}_\alpha, \widetilde{F}\}$ and the supersymmetric transformation rules are

$$\begin{aligned}
\delta\widetilde{\phi} &= -\sqrt{2}\,\tilde{\epsilon}\widetilde{\psi}\,, \\
\delta\widetilde{\psi} &= \sqrt{2}\tilde{\epsilon}\widetilde{F} + i\sqrt{2}\,(\tilde{z} - \tilde{r}H)\,\epsilon\,\widetilde{\phi} + i\sqrt{2}\gamma^\mu\epsilon\,\mathcal{D}_\mu\widetilde{\phi}\,, \\
\delta\widetilde{F} &= i\sqrt{2}\,(\tilde{z} - (\tilde{r}-2)H)\,\epsilon\,\widetilde{\psi} - i\sqrt{2}\,\mathcal{D}_\mu(\epsilon\,\gamma^\mu\widetilde{\psi})\,,
\end{aligned} \tag{2.21}$$

where the R-charge of $\widetilde{\phi}$ is $R[\widetilde{\phi}] = -\tilde{r}$ and its central charge is $-\tilde{z}$.

2.2.2 Real and gauge multiplets

A real multiplet Σ arises by imposing on $\mathcal{S}$ the conditions $M = \tilde{M} = 0$, and $r = z = 0$. The subset of independent components can be defined by $\{C^{(\Sigma)}, \chi_\alpha^{(\Sigma)}, \tilde{\chi}_\alpha^{(\Sigma)}, j_\mu, \sigma^{(\Sigma)}\}$, and Σ is organized as follows

$$\begin{aligned}
\Sigma = \{&C^{(\Sigma)}, \chi_\alpha^{(\Sigma)}, \tilde{\chi}_\alpha^{(\Sigma)}, 0, 0, \\
&- j_\mu - V_\mu C^{(\Sigma)}, -\sigma^{(\Sigma)}, \\
&- \tfrac{i}{2}H\tilde{\chi}_\alpha^{(\Sigma)} + i\gamma^\mu{}_\alpha{}^\beta(\nabla_\mu + iV_\mu)\tilde{\chi}_\beta^{(\Sigma)}\,, + \tfrac{i}{2}H\chi_\alpha^{(\Sigma)} - i\gamma^\mu{}_\alpha{}^\beta(\nabla_\mu - iV_\mu)\chi_\beta^{(\Sigma)}\,, \\
&- V^\mu j_\mu - H\sigma^{(\Sigma)} - (\nabla^2 + V^2)C^{(\Sigma)}\}\,.
\end{aligned} \tag{2.22}$$

The vector field j_μ is a conserved current, $\nabla_\mu j^\mu = 0$. The supersymmetric transformations rules are

$$\begin{aligned}
\delta C^{(\Sigma)} &= i\epsilon\chi^{(\Sigma)} + i\tilde{\epsilon}\tilde{\chi}^{(\Sigma)}\,, \\
\delta\chi^{(\Sigma)} &= \tilde{\epsilon}\,\sigma^{(\Sigma)} + i\gamma^\mu\,\tilde{\epsilon}\,(j_\mu + i\partial_\mu C^{(\Sigma)} + V_\mu C^{(\Sigma)})\,, \\
\delta\tilde{\chi}^{(\Sigma)} &= \epsilon\,\sigma^{(\Sigma)} - i\gamma^\mu\,\epsilon\,(j_\mu - i\partial_\mu C^{(\Sigma)} + V_\mu C^{(\Sigma)})\,, \\
\delta j_\mu &= i\varepsilon_{\mu\nu\rho}\nabla^\nu(\epsilon\gamma^\rho\chi^{(\Sigma)} - \tilde{\epsilon}\gamma^\rho\tilde{\chi}^{(\Sigma)})\,, \\
\delta\sigma^{(\Sigma)} &= -i\nabla_\mu(\epsilon\gamma^\rho\chi^{(\Sigma)} + \tilde{\epsilon}\gamma^\rho\tilde{\chi}^{(\Sigma)}) + 2iH(\epsilon\chi^{(\Sigma)} + \tilde{\epsilon}\tilde{\chi}^{(\Sigma)}) - V_\mu(\epsilon\gamma^\rho\chi^{(\Sigma)} - \tilde{\epsilon}\gamma^\rho\tilde{\chi}^{(\Sigma)})\,.
\end{aligned} \tag{2.23}$$

An abelian gauge multiplet $\mathcal{V}$ is a generic multiplet $\mathcal{S}$ subject to the gauge freedom $\delta\mathcal{V} = \Lambda + \tilde{\Lambda}$, where Λ is a chiral multiplet. After the standard procedure of Wess-Zumino gauge fixing the independent fields reduce to $\{\mathcal{A}_\mu, \sigma, \lambda_\alpha, \tilde{\lambda}_\alpha, D\}$. Notice that an abelian gauge multiplet becomes a real multiplet under the identification:

$$\begin{aligned}
&C^{(\Sigma)} = \sigma\,, && \chi_\alpha^{(\Sigma)} = i\tilde{\lambda}_\alpha\,, && \tilde{\chi}_\alpha^{(\Sigma)} = -i\lambda_a\,, \\
&j_\mu = -\tfrac{i}{2}\varepsilon_{\mu\nu\rho}f^{\nu\rho}\,, && \sigma^{(\Sigma)} = D + \sigma H\,,
\end{aligned} \tag{2.24}$$

where $f^{\nu\rho}$ is the field strength of $\mathcal{A}_\mu$. This parametrization will be particularly useful in later sections.

In the case of non-abelian gauge multiplets the supersymmetry transformation rules have extra terms compared to (2.23). The complete set of transformation rules in the

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non-abelian case is

$$
\begin{aligned}
\delta\sigma &= -\epsilon\tilde{\lambda} + \tilde{\epsilon}\lambda\,, \\
\delta\lambda &= +i\epsilon(D+\sigma H) - \tfrac{i}{2}\varepsilon^{\mu\nu\rho}\gamma_\rho\epsilon\,\mathcal{F}_{\mu\nu} - \gamma^\mu\epsilon\,(iD_\mu\sigma - V_\mu\sigma)\ , \\
\delta\tilde{\lambda} &= -i\tilde{\epsilon}(D+\sigma H) - \tfrac{i}{2}\varepsilon^{\mu\nu\rho}\gamma_\rho\tilde{\epsilon}\,\mathcal{F}_{\mu\nu} + \gamma^\mu\tilde{\epsilon}\,(iD_\mu\sigma + V_\mu\sigma)\ , \\
\delta\mathcal{A}_\mu &= -i(\epsilon\gamma_\mu\tilde{\lambda} + \tilde{\epsilon}\gamma_\mu\lambda)\,, \\
\delta D &= D_\mu(\epsilon\gamma^\mu\tilde{\lambda} - \tilde{\epsilon}\gamma^\mu\lambda) - iV_\mu(\epsilon\gamma^\mu\tilde{\lambda} + \tilde{\epsilon}\gamma^\mu\lambda) - H(\epsilon\tilde{\lambda} - \tilde{\epsilon}\lambda) + [\tilde{\lambda}\epsilon + \tilde{\epsilon}\lambda, \sigma]\ .
\end{aligned}
\tag{2.25}
$$

$\mathcal{F}_{\mu\nu}$ is the field strength of $\mathcal{A}_\mu$, and D_μ is the non-abelian gauge covariant derivative (5.28).

2.2.3 Curved D- and F-terms

So far we have not specified whether $\mathcal{S}$ is an elementary or a composite multiplet. The supersymmetric transformations are, of course, valid regardless of this distinction. Once elementary multiplets are defined, any composite multiplet $\mathcal{K}$ of the form $\mathcal{K} = (K, \chi^{(K)}, \tilde{\chi}^{(K)}, M^{(K)}, \ldots)$ is generated by the multiplet calculus. In practice, given the definition of the bottom component K, as a function of the elementary fields C^I, the other components in the multiplet are obtained in a step-by-step procedure: varying $K(C^I)$ with the use of δC^I one reads off the definitions of $\chi^{(K)}$ and $\tilde{\chi}^{(K)}$, and so on. From the composite multiplets it is then possible to construct kinetic terms for the elementary fields and thus generic supersymmetric Lagrangians whose variation is a total derivative.

Such Lagrangians can be understood as follows. Given a generic multiplet $\mathcal{S}$ with $r = 0$ and $z = 0$, its D component almost transforms as a total derivative. Terms that are not total derivatives are proportional to background fields, and the flat space result is recovered when these vanish. In curved space the correct combination transforming into a total derivative is [3]

$$
\text{curved D}-\text{term}: \qquad
\begin{aligned}
\mathscr{L}_D &= -\tfrac{1}{2}(D - a_\mu V^\mu - \sigma H)\,, \\
\delta\mathscr{L}_D &= -\tfrac{1}{2}\nabla_\mu(\epsilon\gamma^\mu\tilde{\lambda} - \tilde{\epsilon}\gamma^\mu\lambda - V^\mu\epsilon\chi + V^\mu\tilde{\epsilon}\tilde{\chi})\ .
\end{aligned}
\tag{2.26}
$$

The result for the F (or $\tilde{F}$) component of a chiral Φ (or anti-chiral $\tilde{\Phi}$) multiplet of R-charge $r = 2$ (or $r = -2$) and central charge $z = 0$ is the same as that in flat space. The F-term is

$$
\text{curved F}-\text{term}: \qquad \mathscr{L}_F = F + \tilde{F}\,, \qquad \delta\mathscr{L}_F = -2i\,\nabla_\mu(\tilde{\epsilon}\gamma^\mu\psi + \epsilon\gamma^\mu\tilde{\psi})\ . \tag{2.27}
$$

3 Manifold decomposition for curved A-type backgrounds

In this paper we focus on a class of background geometries introduced in [12], that we call "A-type".[3] By definition, these backgrounds admit two supercharges related by charge conjugation. The charge conjugate spinors, $\zeta^c \equiv +i\gamma^2\zeta^\star$ and $\tilde{\zeta}^c \equiv +i\gamma^2\tilde{\zeta}^\star$,[4] solve the

[3] In [12] the partition function of $\mathcal{N} = 2$ Chern-Simons theories on generic A-type backgrounds was computed explicitly using supersymmetric localization techniques similar to [5].

[4] Our γ matrix conventions are summarized in appendix A. In deriving the formulae (3.1) and (3.2) we made use of the relation $\gamma^{\mu\star} = -\gamma^2\gamma^\mu\gamma^2$. * denotes the standard complex conjugation.

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equations

$$\big(\nabla_\mu + iA_\mu^{(R)}\big)\zeta^c = +\frac{1}{2}\gamma_\mu(H^\star\zeta^c - iV_\nu^\star\gamma^\nu\zeta^c)\,, \tag{3.1}$$

$$\big(\nabla_\mu - iA_\mu^{(R)}\big)\tilde{\zeta}^c = +\frac{1}{2}\gamma_\mu(H^\star\tilde{\zeta}^c + iV_\nu^\star\gamma^\nu\tilde{\zeta}^c)\ . \tag{3.2}$$

In general, given a Killing spinor, say ζ, its complex conjugate ζ^c is an independent spinor that does not solve any of the Killing spinor equations (2.1) and (2.2). However, if the background fields $A_\mu^{(R)}$ and V_μ are real, and H is purely imaginary, then ζ^c solves the same Killing spinor equation as $\tilde{\zeta}$. Therefore, for an A-type background, ζ and ζ^c are the two Killing spinors of opposite R-charge.

We are going to show that it is possible to understand any A-type background in terms of a *supersymmetric* foliation in which a generic leaf has the topology of a torus. As a mathematical statement about irreducible orientable closed 3-manifolds, it is certainly well known in the literature that such a toric foliation exists, however we will use supersymmetry and the Killing spinors ζ and $\tilde{\zeta}$ to re-derive this result. Very explicitly, the geometry of the foliation will be characterized by a distribution of orthogonal vector fields built out of the Killing spinors. One of these vectors will be the Killing vector K^μ, and we will construct another vector N^μ that: 1) is orthogonal to K^μ, and 2) can be used to define a proper orthogonal submanifold.

The use of vector fields, instead of the adapted coordinates of the THF, will be essential in the formulation of boundary conditions preserving a subset of the bulk supersymmetry. With such a foliation in place, we will be able to decompose the compact manifolds by placing a boundary (or a co-dimension-1 defect) along any leaf of the foliation. Our main purpose will be to formulate rigid supersymmetric fields theories on the resulting spaces with boundary. Since the metric is part of a supergravity multiplet, the decomposition of the manifolds should be combined with certain extra conditions on the remaining background fields. We will discuss concretely how the manifold decomposition is carried out in the rest of this section. In the final subsection 3.5, we revisit some of the well-known examples of compact $3d$ manifolds, and re-discuss them from the perspective of this decomposition.

3.1 Supersymmetric foliation

Normal vector

Let us consider how the existence of the Killing spinors ζ and ζ^c determines the geometry of A-type manifolds. By fixing the normalization of $\tilde{\zeta}$ to be $\tilde{\zeta} = \zeta^c$, we show that supersymmetry provides a "refinement" of the THF in which a special orthogonal direction to K^μ is selected out.

The starting point of our treatment is based on the use of a Fierz identity for commuting spinors that allows us to show that the real vector N^μ, defined as

$$N^\mu = (\zeta^\star\gamma^\mu\zeta) - (\tilde{\zeta}^\star\gamma^\mu\tilde{\zeta}) = (\zeta^\star\gamma^\mu\zeta) + c.c.\,, \tag{3.3}$$

is orthogonal to K^μ, i.e. $K_\mu N^\mu = 0$. The same result about N_μ can be obtained by noticing that

$$\zeta^\star\gamma_\mu\zeta = +i\tilde{\zeta}\gamma_2\gamma_\mu\zeta = +ig_{\nu\mu}e_2^\nu(\tilde{\zeta}\zeta) - \varepsilon_{\nu\mu\rho}K^\rho e_2^\nu\,, \tag{3.4}$$

where e_2^μ is an unspecified vielbein. Hence, the real part of (3.4) gives $N^\mu = 2\varepsilon^{\mu\nu\rho} e_\nu^2 K_\rho$, which is manifestly orthogonal to K^μ. The tangent space $T\mathcal{M}_3$ can then be spanned by the following orthogonal vectors: K^μ, N^μ, and $\tilde{K}^\mu \equiv \varepsilon^{\mu\nu\rho} N_\nu K_\rho$. By construction, we also have

$$N^\mu e_\mu^2 = 0\,, \qquad \tilde{K}^\mu = 2\varepsilon^{\mu\nu\rho}\varepsilon_{\nu\alpha\beta} e_2^\alpha K^\beta K_\rho = -2e_2^\mu ||K||^2 + 2K^\mu \left(K\cdot e^2\right)\ . \tag{3.5}$$

It is, therefore, convenient to choose a reference frame, $\{e_1, e_2, e_3\}$, such that $K^\mu = e_3^\mu$ and $\left(K\cdot e^2\right) = 0$. For such a frame we deduce from (3.4) that $e_2^\mu \propto \varepsilon^\mu{}_{\nu\rho} N^\nu e_3^\rho$, and $N^\nu \propto e_1^\nu$. By consistency, we have to prove that the inverse metric $g^{\mu\nu}$ can be written in terms of the bilinears $K^\mu K^\nu$, $N^\mu N^\nu$ and $\tilde{K}^\mu \tilde{K}^\nu$. Indeed, from the Fierz identity applied to $K^\mu K^\nu$, and from the very definition of $\tilde{K}^\mu \tilde{K}^\nu$, we obtain the relation

$$g^{\mu\nu} = \frac{1}{||K||^2} K^\mu K^\nu + \frac{1}{||N||^2} N^\mu N^\nu + \frac{1}{||N||^2 ||K||^2} \tilde{K}^\mu \tilde{K}^\nu\ . \tag{3.6}$$

Our adapted dreibein fields are

$$e_1^\mu = \frac{N^\mu}{||N||} \equiv n^\mu, \quad e_2^\mu = \frac{\tilde{K}^\mu}{||\tilde{K}||} \equiv \tilde{k}^\mu, \quad e_3^\mu = \frac{K^\mu}{||K||} \equiv k^\mu\,, \tag{3.7}$$

with $||\tilde{K}|| = ||N||\,||K||$. The norms of K^μ and N^μ are

$$K^\mu K_\mu = \Omega^2, \qquad N^\mu N_\mu = 4\left(\zeta^\star \zeta\right)^2 + 4\Omega^2\ . \tag{3.8}$$

The most general metric on $\mathcal{M}_3$ compatible with a THF was given in (2.3). Beginning with manifolds that admit two generic Killing spinors of opposite R-charges, we refine the adapted coordinates of the THF so to describe a local parametrization of the metric (3.6). Recalling the form of (2.3),

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \eta^2 + c(\tau, z, \bar{z})^2 dz d\bar{z}, \qquad \eta = d\tau + (h(\tau, z, \bar{z}) dz + c.c.)\ . \tag{3.9}$$

we seek a metric compatible with a contact structure whose Reeb vector is also a Killing vector. Firstly, let us notice that $K_\mu = \Omega^2\,\hat{\eta}$ and $e^3 = \Omega\,(d\psi + x_1 dy_1)$. Then, instead of $\{x_1, y_1\}$, we can make use of the coordinates $\{z, \bar{z}\}$ by implementing the contact structure condition on the function h. As a result, h is ψ-independent, and since $\partial_\psi \Omega = K^\mu \partial_\mu \Omega = 0$, the function Ω is also ψ-independent. Finally, upon imposing that K^μ is Killing, the metric (3.9) takes the final form [12]

$$ds^2 = \Omega\,(z, \bar{z})^2 \left(d\psi + h(z, \bar{z}) dz + c.c.\right)^2 + c(z, \bar{z})^2 dz d\bar{z}\ . \tag{3.10}$$

For an A-type manifold, $\tilde{\zeta} = \zeta^c$, and the two THFs, the one associated to ζ, and the one associated to $\tilde{\zeta}$, are identified, i.e $\eta = \tilde{\eta} = \Omega\,\hat{\eta} = e^3$ in (3.9). In particular, because of (3.6), we also know that the plane $dz d\bar{z}$ is parametrized by the real vectors N^μ and $\tilde{K}^\mu$. From the point of view of the contact structure, the vectors N^μ and $\tilde{K}^\mu$ span the distribution of the contact plane $\mathcal{H} = \ker \hat{\eta}$.

If follows from Frobenius' theorem, and from the defining property of a contact manifold, namely $\hat{\eta} \wedge d\hat{\eta} \neq 0$, that the distribution $\mathcal{H} = \ker \hat{\eta}$ is not integrable. Instead, we will now study under what conditions the distribution generated by K^μ and $\tilde{K}^\mu$ is integrable. This will provide a regular foliation of the A-type manifold.

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Integrability condition

Frobenius' theorem guarantees that the distribution $\mathcal{E}$, generated by K and $\tilde{K}$, is integrable if the commutator $[K,\tilde{K}]$ belongs to $\mathcal{E}$ [43].[5] This commutator is equal to

$$\begin{aligned}[K,\tilde{K}]^\mu &\equiv K^\alpha\nabla_\alpha\tilde{K}^\mu - \tilde{K}^\alpha\nabla_\alpha K^\mu \\ &= K^\alpha\varepsilon^{\mu\nu\rho}\left(\nabla_\alpha N_\nu\right)K_\rho + K^\alpha\varepsilon^{\mu\nu\rho}N_\nu\left(\nabla_\alpha K_\rho\right) - \varepsilon^{\alpha\nu\rho}\,N_\nu K_\rho\nabla_\alpha K^\mu\,,\end{aligned} \tag{3.11}$$

and we shall consider the equation $N_\mu[K,\tilde{K}]^\mu = 0$.

The second and third terms in (3.11) can be manipulated by using the equation of K_ν, given in (2.9). We obtain the expression

$$+\,K^\alpha\varepsilon^{\mu\nu\rho}N_\nu\left(\nabla_\alpha K_\rho\right) - \varepsilon^{\alpha\nu\rho}\,N_\nu K_\rho\nabla_\alpha K^\mu = -\left(K\cdot V\right)N^\mu\,\tilde{\zeta}\zeta - iH\,||K||^2N^\mu\,. \tag{3.12}$$

A small complication arises in the calculation of $\nabla_\alpha N_\nu$. By definition $\nabla_\alpha N_\nu = D_\alpha\zeta^\star\gamma_\nu\zeta + \zeta^\star\gamma_\nu D_\alpha\zeta + c.c.$, but $D_\alpha\zeta^\star$ is not just $-i\gamma^2 D_\alpha\tilde{\zeta}$. It is given by the more involved expression

$$D_\alpha\zeta^\star = -i\gamma^2 D_\alpha\tilde{\zeta} - \frac{1}{4}\,\varepsilon^{abc}\omega_{\alpha\,ab}\gamma^2\left(\gamma_c^\star + \gamma_c\right)\tilde{\zeta}\,. \tag{3.13}$$

Substituting (3.13) into $\nabla_\alpha N_\nu$, we get several contributions

$$\nabla_\alpha N_\nu = +i\left(D_\alpha\tilde{\zeta}\right)\gamma^2\gamma_\nu\zeta + i\tilde{\zeta}\,\gamma^2\gamma_\nu D_\alpha\zeta + \frac{1}{4}\omega_{\alpha\,ab}\varepsilon^{abc}\tilde{\zeta}\,\left(\gamma^2\gamma_c\gamma_\nu - \gamma_c\gamma^2\gamma_\nu\right)\zeta\;+\;cc. \tag{3.14}$$

In the adapted frame (3.7), and after some algebra, we can show that

$$\varepsilon^{\mu\nu\rho}K_\rho\,\left(K^\alpha\nabla_\alpha N_\nu\right) = -\frac{1}{2}\left(K^\alpha\omega_{\alpha\,ab}\varepsilon^{abc}K_c\right)N^\mu\,. \tag{3.15}$$

Then, the commutator becomes,

$$[K,\tilde{K}]^\mu = \left[-\left(K\cdot V\right)\,\tilde{\zeta}\zeta - iH\,||K||^2 - \frac{1}{2}\left(K^\alpha\omega_{\alpha\,ab}\varepsilon^{abc}K_c\right)\right]N^\mu\,. \tag{3.16}$$

The vector N^μ does not belong to the distribution $\mathcal{E}$, thus the distribution is integrable iff $N_\mu[K,\tilde{K}]^\mu = 0$. This equation determines the component $K^\alpha\omega_{\alpha\,12}$ of the spin connection in terms of the background fields. Since the spin connection enter explicitly the Killing spinor equation, knowing $K^\alpha\omega_{\alpha\,12}$ shall become very useful when we discuss more precise properties of the Killing spinor solutions, in section 3.4.

Once the condition $N_\mu[K,\tilde{K}]^\mu = 0$ is satisfied, Frobenius' theorem [43] implies the integrability of the distribution $\mathcal{E}$ and the existence of the foliation. Considering the general metric (3.10), any A-type metric can always be put locally in the form,

$$ds^2 = \Omega^2\left(\theta,\varphi\right)\left(d\psi + F_\theta d\theta + F_\varphi d\varphi\right)^2 + g_{\theta\theta}^2 d\theta^2 + g_{\varphi\varphi}^2 d\varphi^2\,, \tag{3.17}$$

where F_θ, F_φ, $g_{\theta\theta}$, $g_{\varphi\varphi}$ are ψ-independent, and $N_\mu dx^\mu = d\theta$. The metric (3.17) should be understood as a local parametrization of (3.6), where the normal vector N^μ is obtained

[5] The normalizations of K and $\tilde{K}$ are not important in the argument. Even though they contribute to the commutator, through the terms $\tilde{K}^\mu(K^\alpha\partial_\alpha\frac{1}{||\tilde{K}||})$ and $K^\mu(\tilde{K}^\alpha\partial_\alpha\frac{1}{||K||})$, these contributions belong to $\mathcal{E}$. Thus the statement of Frobenius' theorem remains unchanged.

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directly from the knowledge of the Killing spinors ζ and $\tilde{\zeta} = \zeta^c$.[6] The $2d$ submanifolds $\theta = const.$ define the foliation generated by $\mathcal{E}$.

The foliation generated by K and $\tilde{K}$ will be denoted by $\mathcal{F}$, and a generic leaf in $\mathcal{F}$ will be denoted by $\mathcal{M}_2'$.[7] We refer to $\mathcal{M}_2'$ as a *supersymmetric* leaf of $\mathcal{M}_3$. This terminology follows from the observation that the algebra of supersymmetry

$$[\delta_\epsilon, \tilde{\delta}_{\tilde{\epsilon}}]\, \varphi_{(r,z)} = -2i\, (\mathcal{L}_K + \epsilon\tilde{\epsilon}\, (z - rH))\, \varphi_{(r,z)} \tag{3.18}$$

involves the Killing vector K^μ in the Lie derivative $\mathcal{L}_K$. In the simplest case, since the commutator of two transformations δ_ϵ and $\delta_{\tilde{\epsilon}}$ squares to a translation along the orbit of the Killing vector K^μ, $\mathcal{M}_2'$ preserves supersymmetry because K^μ belongs to $T\mathcal{M}_2'$. In particular, let us notice that $\mathcal{F}$ includes the co-dimendion-2 THF generated by K, since $K \in \mathcal{E}$.

3.2 Topology and manifold decomposition

We can show with a simple argument that the topology of $\mathcal{M}_2'$ cannot be genus zero, i.e. the leaves of the supersymmetric foliations are not spheres. The reasoning goes as follows. $\mathcal{M}_2'$ contains the orbits of the Killing vector K^μ, and K^μ is nowhere vanishing because, as we mentioned in section 2.1, the Killing spinors are nowhere vanishing. If $\mathcal{M}_2'$ was a sphere, K^μ would correspond to the U(1) isometry of the sphere, which is unique. However, this cannot be the case since the U(1) isometry of the sphere vanishes at the north and south poles.

The topology of $\mathcal{M}_2'$ is a torus. We showed that it cannot be genus zero, but also it cannot be a higher genus surface either, because a $2d$ Riemann surface of genus $g > 1$ would not have a Killing vector. Thus locally, an A-type background is topologically a torus fibered over a closed interval. Seifert manifolds have indeed this structure (see for example the review in ref. [44]). The case of the round three-sphere is very instructive: $\mathbb{S}^3$ does admit a genus zero topological (Heegard) decomposition as the union of two 3-balls [45], however the supersymmetry that we are considering rules out this possibility and allows only for $g = 1$ decompositions. A similar phenomenon has been noticed in $4d$: ref. [2] showed that a $4d$ supersymmetric manifold for which $[K, \bar{K}] = 0$, is topologically $\Sigma \times \mathbb{T}^2$, where Σ is a $2d$ Riemann surface. Also in this case, the metric and the complex structures can be written only in terms of spinor bilinears [46].

The results we have obtained so far can be summarized by the statement that any supersymmetric compact space $\mathcal{M}_3$ of A-type admits a toric foliation. We now pick *one* leaf $\mathcal{M}_2'$ of the toric foliation, and slice $\mathcal{M}_3$ along its volume. In this way, we obtain two manifolds $\mathcal{T}_1$ and $\mathcal{T}_2$, which share a common boundary, the leaf $\mathcal{M}_2'$, and such that $\mathcal{T}_1 \# \mathcal{T}_2 \cong \mathcal{M}_3$. Borrowing the terminology from surgery theory, we will refer to $\mathcal{T}$ as a solid torus. For the three-sphere, the solid torus is the analog of the hemisphere in $2d$, and the tip of the hemisphere corresponds here to the shrinking of one of the two boundary cycles. Following the analogy with the lower dimensional case, another interesting $3d$ manifold is represented by the "cylinder", which topologically would be a torus fibered on the interval

[6] We shall remark that the possibility of writing N^μ as spinor bilinear is the difference between an A-type manifold and a manifold with two Killing spinors of opposite R-charge endowed with metric (3.10).

[7] To be pedantic we should also specify a reference point $\theta_0 \in \mathcal{M}_3$ for any leaf. This is usually implied.

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with both a left and a right boundary. We note the obvious fact that when a boundary is inserted the homotopy properties of the manifold change.

Since the metric belongs to the supergravity multiplet, whose components include the R-symmetry gauge field $A_\mu^{(R)}$, the vector field V_μ, and the scalar H, any manifold decomposition should be consistent with the profile of these background fields. Being a scalar field, H is not constrained by the manifold decomposition. However, a condition on V^μ follows from the fact that V^μ is a conserved vector, and therefore we should require $n_\mu V^\mu \equiv V^\perp = 0$ at the boundary. As a further simplifying, but not necessary, assumption in some of the examples that will be analysed below we will also consider $n^\mu A_\mu^{(R)} = 0$.

3.3 Clifford algebra and bilinears at the boundary

The frame fields k^μ, $\tilde{k}^\mu$ and n^μ, split the algebra of the γ matrices into a $2d$ "parallel" Clifford algebra, which lives on $\mathcal{M}_2'$, and an orthogonal matrix $\gamma^\perp = n_\mu \gamma^\mu$. As a consequence, all possible spinor bilinears obtained from ζ and $\tilde{\zeta}$ are classified in terms of scalars and tensors on $T\mathcal{M}_2'$. One obvious example is $K^\mu = \tilde{\zeta}\gamma^\mu\zeta$, which is a vector on $T\mathcal{M}_2'$, and has no scalar component because $n_\mu K^\mu = 0$.

It will be useful for later purposes to have the explicit decomposition for all spinor bilinears. Since we have an expression for n_μ in terms of the Killing spinors, we can use Fierz identities to bring the bilinears in a simple form. It is enough to consider a generic bilinear with at most two γ matrices; higher order bilinears would not be independent, because of the identity $\gamma^\mu\gamma^\nu = g^{\mu\nu} + i\,\varepsilon^{\mu\nu\rho}\gamma_\rho$. For notational convenience we use the indices $\nu_\parallel$ for directions parallel to $\mathcal{M}_2'$.

The bilinears of interest are[8]

$$\begin{aligned}
\zeta\gamma^\perp\zeta &= +\tfrac{2\,\Omega}{||N||}(\tilde{\zeta}\zeta^\star)^\star, & \tilde{\zeta}\gamma^\perp\tilde{\zeta} &= -\tfrac{2\,\Omega}{||N||}(\tilde{\zeta}\zeta^\star),\\
\zeta\gamma^\perp\gamma^{\nu_\parallel}\zeta &= +\tfrac{2}{||N||}(\tilde{\zeta}\zeta^\star)^\star K^{\nu_\parallel}, & \tilde{\zeta}\gamma^\perp\gamma^{\nu_\parallel}\tilde{\zeta} &= +\tfrac{2}{||N||}(\tilde{\zeta}\zeta^\star)K^{\nu_\parallel},\\
\tilde{\zeta}\gamma^\perp\gamma^{\nu_\parallel}\zeta &= -\tfrac{i}{||N||}\tilde{K}^{\nu_\parallel}, & \zeta\gamma^\perp\gamma^{\nu_\parallel}\tilde{\zeta} &= -\tfrac{i}{||N||}\tilde{K}^{\nu_\parallel}.
\end{aligned} \tag{3.19}$$

As a technical remark, we observe that the right column of (3.19) can be obtained by complex conjugation of the left column using $\tilde{\zeta} = +i\gamma^2\zeta^\star$, and $\gamma^{\mu\star} = -\gamma^2\gamma^\mu\gamma^2$. The norm of the normal vector $N^\mu = (\zeta^\star\gamma^\mu\zeta) + c.c.$ was given in (3.8). However, by using the symmetries of the commuting spinors, we can also write $N^\mu N_\mu = N^\mu[(\zeta\gamma_\mu\zeta^\star) + c.c.]$, and from the Fierz identity we obtain

$$\left(\zeta^\star\tilde{\zeta}\right)\left(\zeta\tilde{\zeta}^\star\right) = \frac{1}{4}||N||^2\ . \tag{3.20}$$

Therefore,

$$\left(\zeta\gamma^\perp\zeta\right)\left(\tilde{\zeta}\gamma^\perp\tilde{\zeta}\right) = -\frac{4\Omega^2}{||N||^2}\left(\tilde{\zeta}\zeta^\star\right)^\star\left(\tilde{\zeta}\zeta^\star\right) = -\Omega^2\ . \tag{3.21}$$

[8]For generic commuting spinors ψ and χ, we have $\psi\gamma^\mu\chi = \chi\gamma^\mu\psi$, thus $\psi\gamma^\mu\psi \neq 0$, and $\psi\chi = -\chi\psi$.

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We conclude that the only new geometric information needed, in order to parametrize the bilinears (3.19), is a phase

$$\begin{aligned} \zeta\gamma^{\perp}\zeta &\equiv \Omega\, e^{i\varpi}\,, & \tilde{\zeta}\gamma^{\perp}\tilde{\zeta} &= -\Omega\, e^{-i\varpi}\,, \\ \zeta\gamma^{\perp}\gamma^{\nu_{\parallel}}\zeta &= \Omega\, e^{i\varpi}\, k^{\nu_{\parallel}}\,, & \tilde{\zeta}\gamma^{\perp}\gamma^{\nu_{\parallel}}\tilde{\zeta} &= \Omega\, e^{-i\varpi}\, k^{\nu_{\parallel}}\,, \\ \tilde{\zeta}\gamma^{\perp}\gamma^{\nu_{\parallel}}\zeta &= -i\Omega\, \tilde{k}^{\nu_{\parallel}}\,, & \zeta\gamma^{\perp}\gamma^{\nu_{\parallel}}\tilde{\zeta} &= -i\Omega\, \tilde{k}^{\nu_{\parallel}}\,. \end{aligned} \tag{3.22}$$

The phase ϖ can be calculated explicitly, given the Killing spinor ζ and the norm of the Killing vector. In general, we expect ϖ to be coordinate dependent: $\varpi = \varpi(\psi, z, \bar{z})$, where $\{\psi, z, \bar{z}\}$ are the coordinates adapted to the THF. We will present examples in section 3.5.

Finally, we can ask how the bilinear $\zeta\gamma_\mu\zeta$ decomposes in the basis $\{k_\mu, n_\mu, \tilde{k}_\mu\}$. The answer is again obtained by using Fierz identities and reads

$$U_\mu \equiv \frac{\zeta\gamma_\mu\zeta}{\Omega} = e^{i\varpi}(n_\mu - i\tilde{k}_\mu)\,. \tag{3.23}$$

Consequently, we also find that the metric can be written equivalently as

$$ds^2 = k_\mu k_\nu - U_\mu \widetilde{U}_\nu\,, \qquad \widetilde{U}_\mu \equiv \frac{\tilde{\zeta}\gamma_\mu\tilde{\zeta}}{\Omega} = -U^\star\,. \tag{3.24}$$

3.4 Twisting and phases

So far we have discussed several of the characteristic properties of A-type backgrounds. The appearance of the phase ϖ is one of the properties that will play an important role in the subsequent analysis and as such it deserves some further elaboration.

A constant shift of ϖ can be understood as part of the U(1) invariance that is built into the relations $\Omega = \zeta\tilde{\zeta}$ and $\tilde{\zeta} = \zeta^c$, as we discussed in section 2.1. The coordinate dependent part of $\varpi(\psi, z, \bar{z})$ is due to the non-trivial profile of the background fields and is closely related to the explicit solution of the Killing spinor equations. The choice of the frame fields, and therefore the definition of the curved γ matrices becomes important when we discuss the Killing spinor equation. We fix possible ambiguities in the choice of vielbein by working in the preferred frame $\{n_\mu, k_\mu, \tilde{k}_\mu\}$. The relation between the coordinate dependent phase, $\varpi(\psi, z, \bar{z})$, and the Killing spinor, that we discuss here is made in this frame.[9]

We make the following observation. Given a metric $g_{\mu\nu}$ with corresponding background fields and a generic non-trivial $\varpi(\psi, z, \bar{z})$ we can consider a $\mathrm{U}(1)_R$ gauge transformation that sets it everywhere to zero. As a result of this operation, the new background R-symmetry, in which the phase is constant, is

$$A^{(R)}_{\text{new}} = A^{(R)}_{old} + d\Lambda \tag{3.25}$$

where $d\Lambda$ is a flat connection (in the simplest case a non-zero constant).

[9] The reader familiar with the $\mathbb{S}^3$ geometry may notice that by using the Maurer-Cartan forms two out of four Killing spinors of the $\mathbb{S}^3$ are constant (see for example [47]). As we emphasize in the next section, U^μ is always well defined and so is ϖ. It can be explicitly checked that the phase $\varpi = \psi/2$ will show up in U^μ, even in the Maurer-Cartan formulation. In the frame $\{n_\mu, k_\mu, \tilde{k}_\mu\}$ the phase will appear in the Killing spinors.

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Globally, the addition of a non-trivial flat connection can lead to interesting phenomena. Even though we expect the details of the manifold to become important at this point, we know for sure that the leaves of the supersymmetric foliations are tori, and therefore we can make the following general comments:

- When $\pi_1(\mathcal{M}_3)$ is trivial, the two cycles of a generic leaf $\mathcal{M}_2'$ will shrink in the bulk, identifying the location of the north and south pole. Then, if $A^{(R)}_{old}$ was topologically trivial, $A^{(R)}_{\text{new}}$ is inserting a singularity, effectively changing the topology. For example, it inserts punctures at the north/south pole.

- When $\pi_1(\mathcal{M}_3)$ is non-trivial, e.g. $\pi_1(\mathcal{M}_3) = \mathbb{Z}$, the new flat connection will generically decompose into a combination of an holonomy and a singularity (if both are non vanishing).

- When the manifold has a *toric contact structure*, the Killing vector $K = \partial_\psi$ is a combination of the vectors ∂_{ϕ_1} and ∂_{ϕ_2}, where ϕ_1 and ϕ_2 are 2π-periodic coordinates on the leaves. The effect of a constant $A^{(R)}_{\psi\, new}$, over a topologically trivial connection, will result to the insertion of a vortex loop at the north and south pole, together with an holonomy along the corresponding non-shrinking cycles.

From the point of view of the Killing spinor equations, the addition of a flat connection, from $A^{(R)}_{old}$ to $A^{(R)}_{\text{new}}$, is twisting the original solution.[10] Indeed, assume that in the old background the spinor is of the type $e^{i\varpi}\eta_0$, with η_0 a constant spinor and $\varpi = \varpi(\psi, z, \bar{z})$. Then, in the new background η_0 is the spinor and the Killing spinor equation becomes $\partial_\mu \eta_0 = 0$.

Returning to the coordinate system (3.17) we further observe that the ψ dependence of the Killing spinor is always constrained to be a phase. This is due to the fact that $k = \partial_\psi$, and the fact $k^\mu \partial_\mu(\tilde{\zeta}\zeta) = 0$ that follows from the Killing spinor equations. The generic ansatz for a solution of the Killing spinor equations is then

$$\zeta = e^{if(\psi)}\zeta_0(\theta, \varphi)\,, \qquad \tilde{\zeta} = e^{-if(\psi)}\tilde{\zeta}_0(\theta, \varphi)\ .$$

According to this ansatz, for generic f neither ζ nor $\tilde{\zeta}$ are scalars under translations along the Killing vector, however, the ψ dependence can always be solved by considering a gauge transformation of $A^{(R)}_{old}$ such that $k^\mu \partial_\mu \zeta = k^\mu \partial_\mu \tilde{\zeta} = 0$. By using the integrability condition (3.16) we can prove that

$$k^\mu A^{(R)}_{\mu\,\text{new}} = -iH - k^\mu V_\mu\ . \tag{3.26}$$

To prove this equation contract the Killing spinor equation of ζ with k^μ and $\tilde{\zeta}$. The same result follows by considering the Killing spinor equation for $\tilde{\zeta}$. We shall come back to this relation in sections 7 and 8, where it will be used as an input to solve for boundary conditions preserving a subset of the bulk supersymmetry.

[10] Sometimes, even $A^{(R)}_{old}$ can be interpreted as a twisting of a theory with no $A^{(R)}$ [25]. Here we are saying something slightly different, in particular we identify $A^{(R)}_{\text{new}} - A^{(R)}_{old}$ as a gauge transformation.

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3.5 Examples: spheres and their squashings

Important examples of A-type backgrounds include: the round three-sphere $\mathbb{S}^3$, the ellipsoid $\mathbb{S}^3_b$, the $\mathrm{SU}(2)\times\mathrm{U}(1)$ squashed spheres of [48], and geometries of the type $\mathbb{S}^2\times\mathbb{S}^1$. Round and squashed spheres were the first manifolds on which the use of supersymmetric localization made possible the exact computation of the partition function of $\mathcal{N}=2$ theories [5, 12, 49, 50]. Our main interest here will be to calculate the triple of vectors $\{n_\mu, k_\mu, \tilde{k}_\mu\}$ for the round sphere and its deformations. We will also mention the case of $\mathbb{S}^2\times\mathbb{S}^1$ which admits both an A-type and a different "non-real" structure. In the context of squashed spheres, the distinction between these two structures has been also emphasized in [51].

3.5.1 Ellipsoid

Our first example is $\mathbb{S}^3_b$, defined as the set of points $(z,w)\in\mathbb{C}^2$, with the property $\frac{|z|^2}{\tilde{\ell}^2}+\frac{|w|^2}{\ell^2}=1$. The squashing parameter b is usually defined as the ratio $b^2=\tilde{\ell}/\ell$. The parametrization $z=\tilde{\ell}\sin\theta\, e^{i\phi_1}$, $w=\ell\cos\theta\, e^{i\phi_2}$, gives the metric

$$ds^2_{\mathbb{S}^3_b}=dzd\bar{z}+dwd\bar{w}=f(\theta)^2d\theta^2+\tilde{\ell}^2\sin^2\theta\, d\phi_1^2+\ell^2\cos^2\theta\, d\phi_2^2\,, \tag{3.27}$$

where $f(\theta)^2=\ell^2\sin^2\theta+\tilde{\ell}^2\cos\theta^2$. The coordinates take values in the range $\theta\in[0,\pi/2]$ and $\phi_i\in[0,2\pi]$ for $i=1,2$. They are *toric*, and make manifest the $\mathrm{U}(1)\times\mathrm{U}(1)$ symmetry of the geometry. The north pole at $\theta=0$, and south pole at $\theta=\pi/2$, are conventionally defined by the shrinking of the corresponding $\mathbb{S}^1$ cycles. The precise form of $f(\theta)$ is not important and all of the following calculations will be valid for a generic regular function $g_{\theta\theta}(\theta)$.[11] The background fields can be taken to be (in a gauge where $V_\mu=0$) as

$$H=\pm\frac{i}{g_{\theta\theta}}\,,\qquad A^{(R)}_{\pm}=-\frac{1}{2}\left(1-\frac{\tilde{\ell}}{g_{\theta\theta}}\right)d\phi_1\mp\frac{1}{2}\left(1-\frac{\ell}{g_{\theta\theta}}\right)d\phi_2\,. \tag{3.28}$$

Notice that $A^{(R)}_{\pm}$ is topologically trivial since $A^{(R)}_{\pm\,\phi_1}\to 0$ at the north pole and $A^{(R)}_{\pm\,\phi_2}\to 0$ at the south pole.[12]

There are solutions to the Killing spinor equations with both $+$ and $-$ signs. It is then convenient to distinguish between *positive* and *negative* Killing spinors, respectively.

Our immediate task is to obtain the Killing spinors, $\zeta_\pm$, $\tilde{\zeta}_\pm$, and calculate the vector fields K^μ, N^μ and $\tilde{K}^\mu$. Notice that (θ,ϕ_1,ϕ_2) are not the adapted coordinates introduced in the previous section, but since we have coordinate-independent expressions for K^μ, N^μ and $\tilde{K}^\mu$, the choice of coordinates is not an issue. In the frame

$$E^1=\ell\cos\theta\, d\phi_2\,,\qquad E^2=\tilde{\ell}\sin\theta\, d\phi_1\,,\qquad E^3=g_{\theta\theta}\, d\theta\,, \tag{3.29}$$

the explicit expression of the Killing spinors is

$$\zeta_\pm=\mathfrak{M}_{[\,\pm\,\theta,\,(\phi_1\pm\phi_2)]}\,\eta\,,\qquad \eta=\tfrac{1}{\sqrt{2}}\begin{pmatrix}+1\\-1\end{pmatrix}\,, \tag{3.30}$$

[11] Regularity means any function that asymptotes to $\tilde{\ell}$, ℓ at $\theta=0$ and $\theta=\pi/2$, respectively.

[12] The background $A^{(R)}$ of [48] is recovered by the substitution $\phi_i\to-\phi_i$. The difference in the sign is due to our choice of γ matrices that differs from the one in [48].

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$$\tilde{\zeta}_\pm = \mathfrak{M}_{[\pm\theta,(\phi_1\pm\phi_2)]}\,\bar{\eta}\,, \qquad \bar{\eta} = \tfrac{1}{\sqrt{2}}\begin{pmatrix}+1\\+1\end{pmatrix}\,, \tag{3.31}$$

with the matrix $\mathfrak{M}$ given by

$$\mathfrak{M}_{[\theta,\varpi]} = \exp\left(-i\frac{\theta}{2}\gamma^3\right)\exp\left(-i\frac{\varpi}{2}\gamma^1\right) = \begin{pmatrix} e^{-i\theta}\cos\frac{\varpi}{2} & ie^{-i\theta}\sin\frac{\varpi}{2} \\ ie^{+i\theta}\sin\frac{\varpi}{2} & e^{+i\theta}\cos\frac{\varpi}{2}\end{pmatrix}\,. \tag{3.32}$$

In (3.31) we have chosen a normalization such that $\tilde{\zeta}_\pm = \zeta^c_\pm$. In fact, since the curved background is real, we are guaranteed that ζ^c solves the equation of $\tilde{\zeta}$. The Killing vector associated to $\zeta_\pm$ is

$$K^\mu_\pm\,\partial_\mu = \tilde{\zeta}_\pm\gamma^\mu\zeta_\pm\,\partial_\mu = \pm\tilde{\ell}^{-1}\partial_{\phi_1} + \ell^{-1}\partial_{\phi_2}\,, \tag{3.33}$$

and the novel vectors, $n^\mu \equiv N^\mu/||N||$, and $\tilde{k}^\mu = \tilde{K}^\mu/||\tilde{K}||$ are

$$n^\mu_\pm\,\partial_\mu = -\frac{1}{g_{\theta\theta}}\partial_\theta\,, \tag{3.34}$$

$$\tilde{k}^\mu_\pm\,\partial_\mu = -\tilde{\ell}^{-1}\cot\theta\,\partial_{\phi_1} \pm \ell^{-1}\tan\theta\,\partial_{\phi_2}\,. \tag{3.35}$$

It is interesting to write the metric in the adapted frame $\{k_\mu, n_\mu, \tilde{k}_\mu\}$. In the case of positive Killing spinors, the metric takes the form (3.17)

$$ds^2_{\mathbb{S}^3_b} = \frac{1}{4}\left(d\psi + \cos\theta_H d\varphi\right)^2 + \frac{1}{4}g^2_{\theta\theta}\,d\theta^2_H + \frac{1}{4}\sin^2\theta_H d\varphi^2\,, \tag{3.36}$$

where

$$d\psi = \ell d\phi_2 + \tilde{\ell}d\phi_1\,, \qquad d\varphi = \ell d\phi_2 - \tilde{\ell}d\phi_1\,, \qquad d\theta_H = 2d\theta\,. \tag{3.37}$$

For the round three-sphere $\ell = \tilde{\ell}$ and we recover well known results. The coordinates $\{\psi, \theta_H, \varphi\}$ coincide with the familiar Hopf coordinates, for which $\mathbb{S}^3$ is seen as a U(1) fibration over the two-sphere $d\theta^2_H + \sin^2\theta_H d\varphi^2$.[13] The interpretation of the Killing spinors is manifest, $K_+ = \partial_\psi$ and sits along the Hopf fiber, whereas $K_- = \partial_\varphi$ generates the U(1) isometry of $\mathbb{S}^2$. Furthermore, in the example of the round three-sphere written in Hopf coordinates, we can write the $\mathbb{S}^2$ at the base of the fibration as $\mathbb{CP}^1$, and exhibit the THF of $\mathbb{S}^3$

$$ds^2_{\mathbb{S}^3} = \left(d\tau + \frac{i\,\bar{z}dz}{2(1+|z|^2)} + c.c.\right) + \frac{dzd\bar{z}}{1+|z|^2}, \tag{3.38}$$

where $\tau = (\psi+\varphi)/2$ and $z = \tan(\theta/2)\,e^{i\varphi}$. It is worth emphasizing that the Killing spinors ζ_- and $\tilde{\zeta}_-$, which generates $K_- = \partial_\varphi$, become the standard spinors of the $\mathbb{S}^2$, after a change of frame. In the next section we will make this statement more precise.

[13] Notice that when $\ell \neq \tilde{\ell}$ the periodicities of φ and ψ are different from those of the round three-sphere. In the coordinates $\psi_H \equiv \phi_1 + \phi_2$, $\varphi_H \equiv \phi_2 - \phi_1$ the metric of $\mathbb{S}^2_b$ is [52]

$$ds^2 = \frac{R^2}{4}\left[(1+\mathfrak{b}\cos\theta)\,d\theta^2_H + \frac{1-\mathfrak{b}^2}{1-\mathfrak{b}\cos\theta_H}\sin^2\theta\,d\varphi^2_H\right] + \frac{R^2}{4}\left(1-\mathfrak{b}\cos\theta_H\right)\left(d\psi_H + \frac{\cos\theta_H - \mathfrak{b}}{1-\mathfrak{b}\cos\theta_H}d\varphi_H\right)^2,$$

where $2R^2 = \ell^2 + \tilde{\ell}^2$ and $\mathfrak{b} = (\tilde{\ell}^2 - \ell^2)/(\tilde{\ell}^2 + \ell^2)$.

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The Killing spinors $\zeta_\pm$ in (3.30) have non-trivial dependence on θ and ϖ. As can be seen from evaluating $\mathfrak{M}_{[\theta,\varpi]}\eta$, the ϖ dependence reduces to a phase. Following the discussion in the previous section, we can twist away the phase by performing a gauge transformation on the background R-symmetry connection. To see how this works in practice, let us observe that we can indeed decompose $A^{(R)}_\pm$ as

$$A^{(R)}_\pm = +\frac{1}{2}(d\phi_1 \pm d\phi_2) + \frac{1}{2g_{\theta\theta}}(\tilde{\ell} d\phi_1 \pm \ell d\phi_2) \ . \tag{3.39}$$

The background $A^{(R)}_{\pm\, new}$ in which the spinors $\zeta_\pm$ are constant along the direction of the Killing vectors, can be obtained either by an explicit computation, or by solving the general relation $k^\mu A^{(R)}_{\mu\,\mathrm{new}} = -iH - k^\mu V_\mu$ from the knowledge of k^μ and $A^{(R)}_{\mu\, old}$ above. The latter strategy implies that

$$A^{(R)}_{\pm\, new} = A^{(R)}_\pm - \frac{1}{2}(d\phi_1 \pm d\phi_2) = \frac{1}{2g_{\theta\theta}}(\tilde{\ell} d\phi_1 \pm \ell d\phi_2) \ . \tag{3.40}$$

As we expect $A^{(R)}_{\pm\, new}$ becomes well defined on the ellipsoid $\mathbb{S}^3_b$ with punctures at the north and south poles.

3.5.2 On U(1) fibrations and non A-type geometries

Another class of interesting real curved spaces are the $\mathrm{SU}(2)\times\mathrm{U}(1)$ squashings of the round three-sphere of [48]. We will consider a slightly more general class of backgrounds, whose metric is given by

$$ds^2 = \frac{\tilde{\ell}^2}{4}\left(d\psi + u(\theta)d\varphi\right)^2 + \frac{\ell^2}{4}\left(g^2_{\theta\theta}d\theta^2 + \sin^2\theta d\varphi^2\right) \ . \tag{3.41}$$

When $u(\theta) = \cos\theta$, $g_{\theta\theta} = 1$, and $\ell = \tilde{\ell}$ we recover the Hopf fibration of the $\mathbb{S}^3$. When $\ell \neq \tilde{\ell}$, the U(1) fiber of the round sphere gets squashed, and the metric only preserves the $\mathrm{SU}(2)\times\mathrm{U}(1)$ subgroup of the original SO(4) isometry group. We may also take $u(\theta) = u_0$ constant, and for the particular value $u_0 = 0$ we recover the metric of $\mathbb{S}^2\times\mathbb{S}^1$.

It will be useful to define the parameter $\beta = \tilde{\ell}/\ell$. It measures the squashing for geometries that are deformations of $\mathbb{S}^3$, hereafter $\mathbb{S}^3_\beta$. Also, it measures the inverse temperature for geometries of the type $\mathbb{S}^2\times\mathbb{S}^1$. By a global rescaling we can set $\ell = 1$. We will work with the dreibeins

$$E^1 = \frac{1}{2}g_{\theta\theta}d\theta\,, \qquad E^2 = \frac{1}{2}\sin\theta d\varphi\,, \qquad E^3 = \frac{\beta}{2}(d\psi + u(\theta)d\varphi) \ . \tag{3.42}$$

The background scalar field H is taken to be purely imaginary, and we turn on

$$V_3 = -iH + \frac{\beta}{g_{\theta\theta}}\frac{u'(\theta)}{\sin\theta}\,, \tag{3.43}$$

$$A^{(R)}_\varphi = -\frac{1}{2}\frac{\cos\theta}{g_{\theta\theta}} - \frac{\beta^2}{2}\frac{u(\theta)}{g_{\theta\theta}}\frac{u'(\theta)}{\sin\theta}\,, \qquad A^{(R)}_\psi = -\frac{1}{2} - \frac{\beta^2}{2g_{\theta\theta}}\frac{u'(\theta)}{\sin\theta} \ . \tag{3.44}$$

In this setup, the metrics (3.41) admit two Killing spinors of opposite charge

$$\zeta = e^{-i\psi/2}\begin{pmatrix}1\\0\end{pmatrix}, \qquad \zeta^c = \tilde{\zeta} = e^{+i\psi/2}\begin{pmatrix}0\\1\end{pmatrix} \ . \tag{3.45}$$

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From these Killing spinors we calculate the frame $\{n_\mu, \tilde{k}_\mu, k_\mu\}$, and find

$$n^\mu \partial_\mu = \frac{2}{g_{\theta\theta}} \partial_\theta \,, \qquad k^\mu \partial_\mu = \frac{2}{\beta} \partial_\psi \,, \qquad \tilde{k}^\mu \partial_\mu = \frac{2}{\sin\theta} \left(u(\theta) \partial_\psi - \partial_\varphi \right) \,. \tag{3.46}$$

We also recognize that the dreibeins $\{E^1, E^2, E^3\}$ correspond to the triple $\{n_\mu, -\tilde{k}_\mu, k_\mu\}$. The phases $\pm i\psi$ of the spinors ζ and $\tilde{\zeta}$ in (3.45) can be re-absorbed by twisting $A^{(R)}$. The corresponding gauge transformation leaves $A^{(R)}_\varphi$ invariant and changes $A^{(R)}_\psi$ as follows

$$A^{(R)}_\psi \quad \to \quad A^{(R)}_{\psi\, new} = -\frac{\beta^2}{2 g_{\theta\theta}} \frac{u'(\theta)}{\sin\theta} \,. \tag{3.47}$$

Observe that for $\mathbb{S}^2 \times \mathbb{S}^1$ geometries, the function $u(\theta)$ is trivial, and therefore

$$A^{(R)}_{\psi\, new} = 0 \,, \qquad A^{(R)}_\varphi = -\frac{1}{2} \frac{\cos\theta}{g_{\theta\theta}} \,. \tag{3.48}$$

By taking $H = 0$ this $\mathbb{S}^2 \times \mathbb{S}^1$ background becomes the topologically twisted background of [53].

The reasoning that led to the background fields (3.43) and (3.44) is based on simple observations, which we now elucidate. First of all, $A^{(R)}$ and V are real when H is imaginary, hence the family of backgrounds is of the A-type. For example, considering the round three-sphere, $\beta = 1$, $u = \cos\theta$, we find $A^{(R)}_\mu = 0$, and $V_3 = -iH - 1$, thus in the gauge $H = +i$, the spinors ζ and $\tilde{\zeta}$ correspond to the positive Killing spinors (3.30) and (3.31) calculated in the new frame (3.42). The more general background fields, (3.43) and (3.44), are obtained by solving the Killing spinor equation for $A^{(R)}$ and V, upon insisting that ζ in (3.45) is a solution. By writing the Killing spinor equation in the following form

$$(\partial_\mu - i A^{(R)}_\mu)\zeta = -\frac{1}{4} \omega^{ab}_\mu \gamma_{ab} \zeta - \frac{1}{2} \gamma_\mu (H - i V_\nu \gamma^\nu) \zeta \,, \tag{3.49}$$

we get V_3 from the θ component, $A^{(R)}_\varphi$, and $A^{(R)}_\psi$ from the other two equations.

Some of the details of this calculation can be seen explicitly in the cases of $\mathbb{S}^3_\beta$ and $\mathbb{S}^2 \times \mathbb{S}^1$ geometries. The equations (3.49) become

$$\partial_\theta \zeta - i A_\theta \zeta = +\frac{i}{4} \left(\mathfrak{p}\, \beta + iH - V_3 \gamma_3 \right) \gamma^1 \zeta \,, \tag{3.50}$$

$$\partial_\psi \zeta - i A_\psi \zeta = -\frac{i}{4} \beta \left(\mathfrak{p}\, \beta - iH - V_3 \gamma_3 \right) \gamma^3 \zeta \,, \tag{3.51}$$

$$\begin{aligned} \partial_\varphi \zeta - i A_\varphi \zeta = +\frac{i}{4} \Big[& ((2 - \mathfrak{p}\, \beta^2) + iH + V_3 \gamma_3) \gamma^3 \cos\theta \\ & + (\mathfrak{p}\, \beta + iH + V_3 \gamma_3) \gamma^2 \sin\theta \Big] \zeta \,, \end{aligned} \tag{3.52}$$

where $\mathfrak{p} = 1$ for $\mathbb{S}^3_\beta$, and $\mathfrak{p} = 0$ for $\mathbb{S}^2 \times \mathbb{S}^1$. For the round three sphere $\beta = 1$, $H = +i$, $V_3 = 0$ and the r.h.s. of (3.50) and (3.52) vanish identically. The use of the frame fields (3.42), compared to the toric frame of the previous section, makes the computation of the positive Killing spinors particularly simple: two out of three equations can be trivially satisfied,

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and the remaining one, $\partial_\psi \zeta = -\frac{i}{2}\gamma_3\zeta$, is solved by (3.45). For the SU(2) × U(1) squashing $\mathbb{S}^3_\beta$, the background field V_3 is tuned in such a way that the r.h.s. of (3.49) becomes a projector, as one can check from (3.50). Then, the positive Killing spinor of the round sphere is promoted to a Killing spinor of the squashed sphere.[14] In this case, the R-symmetry background is proportional to $A_3^{(R)}$, and it is aligned with V_3.

The analysis of $\mathbb{S}^2 \times \mathbb{S}^1$ geometries follows the same logic.

Before we move on, let us notice that when we consider the negative Killing spinors of the round three-sphere, a different simplification takes place in the equations (3.50)–(3.52): the trivial equation becomes $\partial_\psi \zeta = 0$, whereas the equations along θ, and φ become effectively those of the $\mathbb{S}^2$ in its standard parametrization [7],

$$\nabla_\theta \zeta = +\frac{i}{2}\gamma_\theta \zeta\,, \qquad \nabla_\varphi \zeta = +\frac{i}{2}\gamma_\varphi \zeta\,. \tag{3.53}$$

whose solutions are[15]

$$\zeta = C_1\, e^{+i\frac{\varphi}{2}} \begin{pmatrix} \cos\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} \end{pmatrix} + C_2\, e^{-i\frac{\varphi}{2}} \begin{pmatrix} \sin\frac{\theta}{2} \\ +i\cos\frac{\theta}{2} \end{pmatrix}\,. \tag{3.54}$$

$\mathbb{S}^2 \times \mathbb{S}^1$ and non A-type geometry. Metrics of the type $\mathbb{S}^2 \times \mathbb{S}^1$ are interesting for a second reason: they are perhaps the simplest $3d$ example admitting both a real and a non-real structure. The non-real structure is obtained by considering the following background fields

$$H = 0\,, \qquad V = -\frac{2i}{g_{\theta\theta}} E^3\,, \qquad A^{(R)} = +\frac{i}{g_{\theta\theta}} E^3\,, \tag{3.55}$$

with $E^3 = \frac{\beta}{2}(d\psi + u_0 d\varphi)$. The Killing spinor equation (3.51) becomes trivial: $\partial_\psi \zeta = 0$. After $A^{(R)}$ and V have been subtracted, the equations on the $\mathbb{S}^2$ base, (3.50), and (3.52), become

$$\nabla_\mu \zeta = +\frac{1}{2g_{\theta\theta}} \gamma_\mu \gamma^3 \zeta\,. \tag{3.56}$$

The Killing spinor equations for $\tilde{\zeta}$ are not obtained from (3.56) by charge conjugation. Indeed, the background is not real. Instead, from the original Killing spinor equation (2.2) we find,

$$\partial_\psi \tilde{\zeta} = 0 \qquad \nabla_\theta \tilde{\zeta} = -\frac{1}{2g_{\theta\theta}} \gamma_\theta \gamma^3 \tilde{\zeta}\,, \qquad \nabla_\varphi \tilde{\zeta} = -\frac{1}{2g_{\theta\theta}} \gamma_\varphi \gamma^3 \tilde{\zeta}\,. \tag{3.57}$$

The equations (3.57) appear in the same form in [8, 19] for $g_{\theta\theta} = 1$. The explicit solutions are proportional to the following four spinors

$$\zeta_1 = e^{-\frac{i}{2}\varphi} \begin{pmatrix} \sin\frac{\theta}{2} \\ +\cos\frac{\theta}{2} \end{pmatrix}\,, \qquad \zeta_2 = e^{+\frac{i}{2}\varphi} \begin{pmatrix} -\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}\,, \tag{3.58}$$

$$\tilde{\zeta}_1 = e^{-\frac{i}{2}\varphi} \begin{pmatrix} \sin\frac{\theta}{2} \\ -\cos\frac{\theta}{2} \end{pmatrix}\,, \qquad \tilde{\zeta}_2 = e^{+\frac{i}{2}\varphi} \begin{pmatrix} +\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}\,. \tag{3.59}$$

[14] Squashings whose Killing spinors reduce to the negative Killing spinors of the round sphere, have been studied in [50]. In this case, the ansatz for Killing spinors need to be slightly modified.

[15] These $\mathbb{S}^2$ Killing spinors can be uplifted to $\mathbb{S}^3$, as explained in [54]. In two dimensions, γ_3 anti-commutes with $\gamma_\mu^{(2d)}$. Therefore, the positive Killing spinors of the $\mathbb{S}^2$ are proportional to $\gamma_3\zeta$, with ζ given in (3.53).

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If the background fields are all purely imaginary (as in (3.55)), it follows from (3.1) that the charge conjugate spinor ζ^c is independent of ζ and solves the same equation. For example, $\zeta_2 = \zeta_1^c$ in (3.58). The same statement applies to $\tilde{\zeta}$ and $\tilde{\zeta}^c$. We conclude that if a background admits two Killing spinors of opposite R-charge, and all the background fields are purely imaginary, by construction it supports $\mathcal{N} = 4$ supersymmetry.

4 Boundary effects in theories with rigid supersymmetry

Given a supersymmetric field theory on a compact manifold $\mathcal{M}_3$, defined by an action

$$\mathcal{S} = \int_{\mathcal{M}_3} \mathscr{L} \,, \tag{4.1}$$

it is not guaranteed that the action will remain supersymmetric when we insert a boundary along $\mathcal{M}_2'$, and restrict the fields to the manifolds, $\mathcal{T}_1$ or $\mathcal{T}_2$, obtained from $\mathcal{T}_1 \# \mathcal{T}_2 \cong \mathcal{M}_3$. In fact, for any symmetry δ acting on the fields, the Lagrangian is locally invariant up to a total derivative, $\delta\mathscr{L} = \nabla_\mu \mathscr{V}^\mu$, hence the action restricted on $\mathcal{T}$ will be invariant under the symmetry δ iff

$$\delta\mathcal{S} = \int_{\mathcal{T}} \nabla_\mu \mathscr{V}^\mu = \oint_{\mathcal{M}_2'} n_\mu \mathscr{V}^\mu = 0 \,, \tag{4.2}$$

where in the last step we used the divergence theorem. Typically, the condition (4.2) is solved by imposing appropriate boundary conditions such that $n_\mu \mathscr{V}^\mu = 0$, or by adding appropriate degrees of freedom on the boundary. In this paper we consider only the first possibility. In the case of supersymmetry $\mathscr{V}^\mu$ is both a function of the anticommuting Killing spinors, ϵ and $\tilde{\epsilon}$, and the fields of the theory. Therefore, in order to solve (4.2), one generally synchronizes the boundary conditions on the fields with certain conditions on the spinors. For example, if we assume that a certain projection on the spinors realizes a specific sub-algebra of the bulk supersymmetry, we can insert this knowledge into $n_\mu \mathscr{V}^\mu$ to simplify the problem and deduce definite boundary conditions for the fields of the theory.

For example, in the case of boundary conditions in two-dimensional $\mathcal{N} = (2,2)$ theories on a strip [14, 55], one can consider two different types of $\frac{1}{2}$-BPS boundary conditions, called A- and B-type. They are characterized by the spinor projections

- $\bar{\epsilon}_+ = +e^{i\alpha}\epsilon_-$ for A-type ,
- $\epsilon_+ = -e^{i\alpha}\epsilon_-$ for B-type .

$\epsilon_\pm$ and $\bar{\epsilon}_\pm$ are the complex components of the $2d$ Weyl spinors ϵ and $\bar{\epsilon}$, and $\bar{\epsilon}$ is the complex conjugate of ϵ. The phase α is an arbitrary constant and the minus sign is a convention. An $\mathcal{N} = (2,2)$ theory has 4 real supercharges and the 1/2-BPS projections preserve $(1,1)$ or $(2,0)$ supersymmetry, for A-type or B-type, respectively. Such conditions play an important role in D-brane physics described by setups with $\mathcal{N} = (2,2)$ worldsheet supersymmetry. In $3d$ theories with $\mathcal{N} = 2$ supersymmetry similar projections (and corresponding boundary conditions) have been formulated in flat space in [15].

When one attempts to apply this standard logic to a theory on a curved background, as in this paper, one encounters inevitably some obvious difficulties. Most notably, on curved

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backgrounds many of the simplifications of constant flat space spinors are absent. The Killing spinors ϵ, $\tilde{\epsilon}$ are, in general, non-trivial functions of the coordinates and an A- or a B-type projection cannot be imposed in the simple standard flat space form written above.

In what follows we will describe how to impose a direct generalization of the A-type condition on the anticommuting spinors ϵ and $\tilde{\epsilon}$ in a generic three-dimensional A-type background. We will do so by introducing a "canonical" formalism that builds on the observations of the previous two sections. We anticipate that a similar generic formulation exists also for B-type projections. However, in this paper we will focus exclusively on A-type boundary conditions leaving B-type projections and B-type boundary conditions to a separate treatment in future work.

4.1 Generalized A-type projections on supersymmetry

Out of the commuting spinors ζ and $\tilde{\zeta}$ we construct two natural projectors, $\mathscr{P}$ and $\widetilde{\mathscr{P}}$

$$\mathscr{P}\psi = \frac{1}{\Omega}(\tilde{\zeta}\psi)\zeta, \qquad \widetilde{\mathscr{P}}\psi = \frac{1}{\Omega}(\psi\zeta)\tilde{\zeta}, \qquad \forall\,\psi. \tag{4.3}$$

It is simple to check that $\mathscr{P}^2 = \mathscr{P}$, $\widetilde{\mathscr{P}}^2 = \widetilde{\mathscr{P}}$ and $\mathscr{P} + \widetilde{\mathscr{P}} = \mathbb{I}$. Since the Killing spinors $\zeta, \tilde{\zeta}$ are nowhere vanishing these projectors are everywhere well defined. Moreover, both $\mathscr{P}$ and $\widetilde{\mathscr{P}}$ are invariant under the symmetry $\zeta \to \lambda\zeta$, $\tilde{\zeta} \to \lambda^{-1}\tilde{\zeta}$, with $\lambda \in \mathbb{C}$.

By acting with $\mathscr{P}$ and $\widetilde{\mathscr{P}}$ on both ϵ and $\tilde{\epsilon}$ we formulate the generalized A-type conditions

$$\widetilde{\mathscr{P}}\epsilon = 0\,, \qquad \mathscr{P}\tilde{\epsilon} = 0\,, \tag{4.4}$$

$$(\mathscr{P}\epsilon\,\tilde{\zeta}) = (\zeta\,\widetilde{\mathscr{P}}\tilde{\epsilon})\,, \qquad \tilde{\zeta} = \zeta^c\,. \tag{4.5}$$

Defining the parameters

$$\vartheta \equiv \frac{1}{\Omega}(\tilde{\zeta}\epsilon)\,, \qquad \widetilde{\vartheta} \equiv \frac{1}{\Omega}\,(\tilde{\epsilon}\,\zeta)\ . \tag{4.6}$$

the above relations become

$$\epsilon = \vartheta\zeta\,, \qquad \tilde{\epsilon} = \widetilde{\vartheta}\tilde{\zeta}\,, \tag{4.7}$$

$$\vartheta = \widetilde{\vartheta}\,. \tag{4.8}$$

The restriction $\tilde{\zeta} = \zeta^c$ (which is possible in A-type backgrounds) is imposed here because the scalar product $(\tilde{\epsilon}\,\zeta) = (\tilde{\zeta}\epsilon)$ alone does not enforce a relation between ϵ and $\tilde{\epsilon}$. Indeed, by rescaling $\tilde{\epsilon} \to \alpha\tilde{\epsilon}$ and $\epsilon \to \beta\epsilon$, with arbitrary $\alpha, \beta \in \mathbb{C}$, it is always possible to find two representatives of the commuting spinors, $\lambda\zeta$ and $\lambda^{-1}\tilde{\zeta}$, for which the relation $(\tilde{\epsilon}\,\zeta) = (\tilde{\zeta}\epsilon)$ is satisfied. The condition $\tilde{\zeta} = \zeta^c$ is needed to break the invariance under the rescalings by $\lambda \in \mathbb{C}$ to a residual U(1).

As a simple check that exhibits why this is the natural curved space generalization of the A-type projection we notice that for constant spinors in flat space the relation (4.8) reduces to the familiar A-type condition $\tilde{\epsilon}_+ = +e^{i\alpha}\epsilon_-$. Indeed, in flat space, we may set $\zeta = (1,0)$, $\tilde{\zeta} = (0,1)$, $\Omega = 1$, and then the relation $(\tilde{\epsilon}\,\zeta) = (\tilde{\epsilon}\,\zeta)$ becomes the expected

$\tilde{\epsilon}_+ = \epsilon_-$. The residual U(1) transformation gives the most general boundary condition, which is precisely $\tilde{\epsilon}_+ = e^{i\alpha}\epsilon_-$.

We emphasize that the curved space version of the above A-type condition is, by construction, compatible only with A-type curved manifolds, for which $\tilde{\zeta} = \zeta^c$. The projections (4.7), (4.8) reduce the amount of supersymmetry by one half.

In sections 6–8, we will demostrate how the input of the projections (4.7), (4.8) affects the (in)variance of a generic $\mathcal{N} = 2$ field theory under supersymmetry, and we will study corresponding general A-type boundary conditions on $\mathcal{N} = 2$ supersymmetric gauge theories that preserve half of the bulk supersymmetry at the boundary.

4.2 Bulk A-type supersymmetries and BPS equations

Having understood how to project the anticommuting Killing spinors of a generic A-type background, we now go back to the supersymmetry transformations of chiral and vector superfields, and reformulate them accordingly. First we spell out the supersymmetry transformations with generic ϑ and $\widetilde{\vartheta}$, and then we study what happens upon enforcing the projection $\vartheta = \widetilde{\vartheta}$.

Before entering the details we point out that we can decompose any spinor ψ as

$$\psi = \frac{1}{\Omega}(\tilde{\zeta}\psi)\zeta + \frac{1}{\Omega}(\psi\zeta)\tilde{\zeta} \ . \tag{4.9}$$

Moreover, we notice the useful identities

$$\begin{aligned}
(\gamma^\mu\zeta)_\alpha &= \tfrac{\tilde{\zeta}\gamma^\mu\zeta}{\Omega}\,\zeta_\alpha - \tfrac{\zeta\gamma^\mu\zeta}{\Omega}\,\tilde{\zeta}_\alpha = k^\mu\,\zeta_\alpha - U^\mu\,\tilde{\zeta}_\alpha\,, \\
(\gamma^\mu\tilde{\zeta})_\alpha &= \tfrac{\tilde{\zeta}\gamma^\mu\tilde{\zeta}}{\Omega}\,\zeta_\alpha - \tfrac{\tilde{\zeta}\gamma^\mu\zeta}{\Omega}\,\tilde{\zeta}_\alpha = \tilde{U}^\mu\,\zeta_\alpha - k^\mu\,\tilde{\zeta}_\alpha\ .
\end{aligned} \tag{4.10}$$

A similar decomposition holds in $4d$ for manifolds which are a torus fibration [46].

Chiral and anti-chiral multiplets. In (2.20) and (2.21) we wrote down the supersymmetric transformation rules for chiral and anti-chiral multiplets for generic Killing spinors. When we further specialize the supersymmetry to an A-type background we obtain the following expressions.

- For a chiral multiplet:

$$\begin{aligned}
\tfrac{1}{\sqrt{2}}\ \delta\phi &= +\vartheta\,\zeta\psi\,, \\
\tfrac{1}{\sqrt{2}}\ \delta\psi_\alpha &= +\vartheta\ F\,\zeta_\alpha + i\,\widetilde{\vartheta}\,\Big[\Big(k^\mu\mathcal{D}_\mu\phi - ir(iH)\phi - (z-q\sigma)\phi\Big)\tilde{\zeta}_\alpha - (\widetilde{U}^\mu\mathcal{D}_\mu\phi)\zeta_\alpha\Big]\,, \\
\tfrac{1}{\sqrt{2}}\ \delta F &= +i\,\widetilde{\vartheta}\,\Big[\Big(k^\mu(\mathcal{D}_\mu - \tfrac{i}{2}V_\mu)\psi - i(r-\tfrac{1}{2})(iH)\psi - (z-q\sigma)\psi\Big)\tilde{\zeta} \\
&\qquad\qquad + \widetilde{U}^\mu\,\zeta(\mathcal{D}_\mu - \tfrac{i}{2}V_\mu)\psi + \sqrt{2}q\,\tilde{\zeta}\tilde{\lambda}\,\phi\Big]\ .
\end{aligned} \tag{4.11}$$

- For an anti-chiral multiplet:

$$
\begin{aligned}
\tfrac{1}{\sqrt{2}}\ \delta\widetilde{\phi} &= -\,\widetilde{\vartheta}\,\tilde{\zeta}\psi\,,\\
\tfrac{1}{\sqrt{2}}\ \delta\widetilde{\psi}_\alpha &= +\,\widetilde{\vartheta}\ \widetilde{F}\,\tilde{\zeta}_\alpha + i\,\vartheta\ \Big[\Big(k^\mu\mathcal{D}_\mu\widetilde{\phi} + ir(iH)\widetilde{\phi} + (z-q\sigma)\widetilde{\phi}\Big)\zeta_\alpha - (U^\mu\mathcal{D}_\mu\widetilde{\phi})\tilde{\zeta}_\alpha\Big]\,,\\
\tfrac{1}{\sqrt{2}}\ \delta\widetilde{F} &= -i\,\vartheta\ \Big[\ \Big(k^\mu(\mathcal{D}_\mu + \tfrac{i}{2}V_\mu)\widetilde{\psi} + i(r-\tfrac{1}{2})(iH)\widetilde{\psi} + (z-q\sigma)\widetilde{\psi}\Big)\zeta\, +\\
&\qquad\qquad + U^\mu\,\tilde{\zeta}(\mathcal{D}_\mu + \tfrac{i}{2}V_\mu)\widetilde{\psi} - \sqrt{2}q\,\zeta\lambda\,\widetilde{\phi}\Big]\ .
\end{aligned}
\tag{4.12}
$$

It is clear, in particular, that the fixed point (BPS) equations, in which the fermions are set to zero and the bosons satisfy $\delta f = 0$ for any fermion f of the multiplet, depend on the assumption we make about ϑ and $\widetilde{\vartheta}$. For the A-type projection, $\vartheta = \widetilde{\vartheta}$, we obtain

$$
k^\mu\mathcal{D}_\mu\phi - ir(iH)\phi - (z-q\sigma)\phi = 0\,, \qquad i\widetilde{U}^\mu\mathcal{D}_\mu\phi - F = 0\,, \tag{4.13}
$$

$$
k^\mu\mathcal{D}_\mu\widetilde{\phi} + ir(iH)\widetilde{\phi} + (z-q\sigma)\widetilde{\phi} = 0\,, \qquad iU^\mu\mathcal{D}_\mu\widetilde{\phi} - \widetilde{F} = 0\ . \tag{4.14}
$$

Further assuming the reality conditions $\widetilde{\phi} = \phi^\star$ and $\widetilde{F} = F^\star$, these equations reduce to

$$
k^\mu\mathcal{D}_\mu\phi - ir(iH)\phi = 0 \qquad \& \qquad (z-q\sigma)\phi = 0 \qquad \& \qquad iU^\mu\mathcal{D}_\mu\phi - F = 0\ . \tag{4.15}
$$

We obtained the last equation using the property $\widetilde{U} = -U^\star$. In the case of arbitrary ϑ and $\widetilde{\vartheta}$ we would have instead $F = \widetilde{F} = 0$ and $U^\mu\mathcal{D}_\mu\phi = 0$ independently. In the presence of a superpotential, we should integrate out F^a in favor of $g^{a\bar{c}}\partial_{\bar{c}}\widetilde{W}$. Recalling that $U^\mu = e^{i\varpi}(n^\mu - i\tilde{k}^\mu)$, we see that the equation $iU^\mu\mathcal{D}_\mu\phi^a - F^a = 0$ becomes the natural $3d$ generalization of the domain wall equations in two dimensional $(2,2)$ theories.

Real and gauge multiplets. The supersymmetric transformation rules for the gauge field were discussed in subsection (2.2.2). There we made a connection between the real multiplet and the gauge multiplet:

$$
j_\mu = -\frac{i}{2}\varepsilon_{\mu\nu\rho}\mathcal{F}^{\nu\rho}, \qquad a_\mu = -j_\mu - \sigma V_\mu\,, \qquad \psi_\Sigma = i\tilde{\lambda}\,, \qquad \widetilde{\psi}_\Sigma = -i\lambda\ . \tag{4.16}
$$

Here we use the real multiplet parametrization for the fermions, and write the field strength $\mathcal{F}$ in terms of the vector a_μ. The supersymmetry transformations on an A-type background then takes the following form.

- For the ϵ variation of the bosons

$$
\begin{aligned}
\delta_\epsilon\sigma &= +i\vartheta\,(\zeta\psi_\Sigma)\,,\\
\delta_\epsilon\mathcal{A}_\mu &= -\vartheta\,\big[+e^{i\varpi}n_\mu\ (\tilde{\zeta}\psi_\Sigma) - ie^{i\varpi}\tilde{k}_\mu(\tilde{\zeta}\psi_\Sigma) - k_\mu\ (\zeta\psi)\big]\,,\\
\delta_\epsilon D &= +i\vartheta\ \Big[\zeta\big(k^\mu(\mathcal{D}_\mu - \tfrac{i}{2}V_\mu) - \tfrac{1}{2}H\big)\psi_\Sigma - \tilde{\zeta}\,U^\mu(\mathcal{D}_\mu - \tfrac{i}{2}V_\mu)\psi_\Sigma\Big]\ .
\end{aligned}
\tag{4.17}
$$

- For the $\tilde{\epsilon}$ variation of the bosons

$$
\begin{aligned}
\delta_{\tilde{\epsilon}}\sigma &= +i\widetilde{\vartheta}\ (\tilde{\zeta}\widetilde{\psi}_\Sigma)\,,\\
\delta_{\tilde{\epsilon}}\mathcal{A}_\mu &= -\widetilde{\vartheta}\big[-e^{-i\varpi}n_\mu\ (\zeta\widetilde{\psi}_\Sigma) - ie^{-i\varpi}\tilde{k}_\mu\ (\zeta\widetilde{\psi}_\Sigma) - k_\mu\ (\tilde{\zeta}\widetilde{\psi}_\Sigma)\big]\,,\\
\delta_{\tilde{\epsilon}}D &= -i\widetilde{\vartheta}\ \Big[\tilde{\zeta}\big(k^\mu(\mathcal{D}_\mu + \tfrac{i}{2}V_\mu) + \tfrac{1}{2}H\big)\widetilde{\psi}_\Sigma - \zeta\,\widetilde{U}^\mu(\mathcal{D}_\mu + \tfrac{i}{2}V_\mu)\widetilde{\psi}_\Sigma\Big]\ .
\end{aligned}
\tag{4.18}
$$

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- For the fermionic fields

$$\begin{aligned}\delta\psi_\Sigma &= \widetilde{\vartheta}\Big[\big[D - i\sigma(iH + k^\mu V_\mu) - ik^\mu(j_\mu + i\partial_\mu\sigma)\big]\tilde{\zeta} - i\widetilde{U}^\mu(a_\mu - i\partial_\mu\sigma)\zeta\Big]\,,\\ \delta\widetilde{\psi}_\Sigma &= \vartheta\Big[\big[D - i\sigma(iH + k^\mu V_\mu) - ik^\mu(j_\mu - i\partial_\mu\sigma)\big]\tilde{\zeta} - iU^\mu(a_\mu + i\partial_\mu\sigma)\zeta\Big]\,.\end{aligned} \tag{4.19}$$

The projection $\widetilde{\vartheta} = \vartheta$ becomes relevant when we consider the full variation $\delta = \delta_\epsilon + \delta_{\tilde{\epsilon}}$ of the bosonic fields. The variations of the fermions is by construction chiral, and therefore $\delta\psi_\Sigma$ and $\delta\widetilde{\psi}_\Sigma$ is not modified when we impose $\widetilde{\vartheta} = \vartheta$. The fixed point equations $\psi_\Sigma = \widetilde{\psi}_\Sigma = 0$ and $\delta\psi_\Sigma = \delta\widetilde{\psi}_\Sigma = 0$ are

$$D - i\sigma(iH + k^\mu V_\mu) - ik^\mu(j_\mu + i\partial_\mu\sigma) = (n^\mu + i\tilde{k}^\mu)(a_\mu - i\partial_\mu\sigma) = 0\,, \tag{4.20}$$

$$D - i\sigma(iH + k^\mu V_\mu) - ik^\mu(j_\mu - i\partial_\mu\sigma) = (n^\mu - i\tilde{k}^\mu)(a_\mu + i\partial_\mu\sigma) = 0\,. \tag{4.21}$$

There are two well known solutions to these BPS equations. In both cases, we shall assume σ and a_μ to be real. The first one is,

$$D - i\sigma(iH + k^\mu V_\mu) = 0\,, \qquad \partial_\mu\sigma = a_\mu = 0\,, \tag{4.22}$$

where the combination $iH + k^\mu V_\mu$ is correctly invariant under the shift symmetry (2.6). If matter fields are set to zero, this solution represents a generalized 'Coulomb branch' solution. In the gauge $k^\mu V_\mu = 0$ the solution (4.22) takes a more familiar form [48]. The second solution to the BPS equations is:

$$n^\mu a_\mu + \tilde{k}^\mu\partial_\mu\sigma = 0\,, \qquad k^\mu\partial_\mu\sigma = 0\,, \tag{4.23}$$

$$\tilde{k}^\mu a_\mu - n^\mu\partial_\mu\sigma = 0\,, \qquad D - i\sigma(iH + k^\mu V_\mu) = ik^\mu j_\mu\,. \tag{4.24}$$

The equations (4.23) and (4.24) generalize to arbitrary A-type backgrounds those of [52, 53].

5 $\mathcal{N} = 2$ Lagrangians

With all the geometric prerequisites in place we need one more element before we can start discussing concretely how to treat $\mathcal{N} = 2$ supersymmetric field theories on A-type curved backgrounds with boundaries. We need to collect all the surface terms that arise in the supersymmetric variation of explicit Lagrangians. This is the main purpose of this section.

5.1 $\mathcal{N} = 2$ non-linear sigma models

In this subsection we study first the most general (classical) $\mathcal{N} = 2$ theory of chiral superfields on A-type curved manifolds. In flat space such theories are characterized in standard fashion by an action governed by a Kähler potential K and a superpotential W. The curved space generalization of this action is straightforward. We spell out the details for a non-linear sigma model (NLσ) of $\mathfrak{s}$ *elementary* chiral superfields $\{\phi^a, \psi^a_\alpha, F^a\}$, and their conjugate $\{\widetilde{\phi}^{\bar{c}}, \widetilde{\psi}^{\bar{c}}_\alpha, \tilde{F}^{\bar{c}}\}$, with a generic superpotential. As far as we know, some of the following calculations are not listed in the literature.

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5.1.1 General Kähler interactions

In flat space, supersymmetry turns a generic target space into a Kähler manifold. This continues to be true in curved space. In addition, the Lagrangian contains a set of new couplings between the dynamical fields and the background fields H and V^μ. By following the strategy outlined in the review section 2, the Lagrangian of the curved non-linear sigma model is obtained from the curved D-term combination (2.26) evaluated on the composite multiplet

$$\mathcal{K} = \{K, \chi^{(K)}, \tilde{\chi}^{(K)}, M^{(K)}, \tilde{M}^{(K)}, a_\mu^{(K)}, \sigma^{(K)}, \lambda^{(K)}, \tilde{\lambda}^{(K)}, D^{(K)}\}\,, \tag{5.1}$$

whose bottom component is the generic real function $K = K(\phi^a, \widetilde{\phi}^{\bar{c}})$. Derivatives of K w.r.t. the fields will be indicated by $K_{I_1 I_2,\ldots I_n}$, where I can be either an unbarred or a barred index. For $n > 1$ the tensor $K_{I_1 I_2,\ldots I_n}$ is totally symmetric. The assignment of R- and central charges is

$$R[\phi^a] = r^a, \quad R[\widetilde{\phi}^{\bar{c}}] = -r^{\bar{c}}, \quad Z[\phi^a] = z^a, \quad Z[\widetilde{\phi}^{\bar{c}}] = -z^{\bar{c}}, \tag{5.2}$$

and the Lagrangian takes the form

$$\begin{aligned}\mathscr{L}_{NL\sigma} &= -\tfrac{1}{2}\left(D^{(K)} - a_\mu^{(K)} V^\mu - \sigma^{(K)} H\right) \\ &= \mathscr{L}^{\text{flat}} - \tfrac{\mathfrak{R}}{8}\left(r^a K_a \phi^a + r^{\bar{c}} K_{\bar{c}} \widetilde{\phi}^{\bar{c}}\right) + \mathscr{L}_H^{\text{bos}} + \mathscr{L}_H^{\text{ferm}} + \mathscr{L}_V^{\text{bos}} + \mathscr{L}_V^{\text{ferm}},\end{aligned} \tag{5.3}$$

where $\mathfrak{R}$ is the curvature of the background manifold and we have defined:

$$\begin{aligned}\mathscr{L}^{\text{flat}} &= +g^{\mu\nu} D_\mu \phi^a K_{a\bar{c}} D_\nu \widetilde{\phi}^{\bar{c}} - \frac{i}{2} K_{a\bar{c}}\, \widetilde{\psi}^{\bar{c}} \gamma^\mu (\mathbb{D}_\mu \psi^a) + \frac{i}{2} (\mathbb{D}_\mu \widetilde{\psi}^{\bar{c}}) \gamma^\mu \psi^a K_{a\bar{c}} \\ &\quad - F^a \widetilde{F}^{\bar{c}} K_{a\bar{c}} - \frac{1}{2} F^a K_{\bar{c}\bar{n}a} (\widetilde{\psi}^{\bar{c}} \widetilde{\psi}^{\bar{n}}) + \frac{1}{2} \widetilde{F}^{\bar{c}} K_{\bar{c}am} (\psi^m \psi^a) + \frac{1}{4} K_{\bar{c}\bar{n}am} \widetilde{\psi}^{\bar{c}} \widetilde{\psi}^{\bar{n}} \psi^a \psi^m\,, \end{aligned} \tag{5.4a}$$

$$\begin{aligned}\mathscr{L}_H^{\text{bos}} &= +\left(H r^{\bar{c}} - z^{\bar{c}}\right)\left(H r^a - z^a\right) \widetilde{\phi}^{\bar{c}} K_{\bar{c}a} \phi^a \\ &\quad - \frac{H}{4}\Big[\left(H r^a - z^a\right) K_a \phi^a + \left(H r^{\bar{c}} - z^{\bar{c}}\right) K_{\bar{c}} \widetilde{\phi}^{\bar{c}}\Big] + \frac{3H}{4}\left(z^a K_a \phi^a + z^{\bar{c}} K_{\bar{c}} \widetilde{\phi}^{\bar{c}}\right),\end{aligned} \tag{5.4b}$$

$$\begin{aligned}\mathscr{L}_H^{\text{ferm}} &= -\frac{i}{2}\Bigg[\left(H\left(r^{\bar{c}} - \frac{1}{2}\right) - z^{\bar{c}}\right) K_{a\bar{c}} + \left(H r^m - z^m\right) \phi^m K_{\bar{c}ma}\Bigg] \psi^a \widetilde{\psi}^{\bar{c}} \\ &\quad - \frac{i}{2}\Bigg[\left(H\left(r^a - \frac{1}{2}\right) - z^{\bar{c}}\right) K_{a\bar{c}} + \left(H r^{\bar{n}} - z^{\bar{n}}\right) \widetilde{\phi}^{\bar{n}} K_{a\bar{n}\bar{c}}\Bigg] \psi^a \widetilde{\psi}^{\bar{c}}\,,\end{aligned} \tag{5.4c}$$

$$\begin{aligned}\mathscr{L}_V^{\text{bos}} &= +iV_\mu\Bigg[\left(r^{\bar{c}} \widetilde{\phi}^{\bar{c}} K_{\bar{c}a} - \frac{1}{2} K_a\right) D^\mu \phi^a - \left(r^a \phi^a K_{a\bar{c}} - \frac{1}{2} K_{\bar{c}}\right) D^\mu \widetilde{\phi}^{\bar{c}} \\ &\qquad + \frac{i}{4} V^\mu \left(r^a K_a \phi^a + r^{\bar{c}} K_{\bar{c}} \widetilde{\phi}^{\bar{c}} - 4 r^a \phi^a r^{\bar{c}} \widetilde{\phi}^{\bar{c}}\right)\Bigg],\end{aligned} \tag{5.4d}$$

$$\begin{aligned}\mathscr{L}_V^{\text{ferm}} &= +\frac{i}{2} V_\mu \Bigg[i \widetilde{\psi}^{\bar{c}} \gamma^\mu \left(\left(r^a - \frac{1}{2}\right) \psi^a + \Gamma^a_{mn} r^m \phi^m \psi^n\right) \\ &\qquad + i\left(\left(r^{\bar{c}} - \frac{1}{2}\right) \psi^{\bar{c}} + \Gamma^{\bar{c}}_{\bar{m}\bar{n}} r^{\bar{m}} \widetilde{\phi}^{\bar{m}} \psi^{\bar{n}}\right) \gamma^\mu \psi^a\Bigg] K_{a\bar{c}}\,.\end{aligned} \tag{5.4e}$$

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In (5.3) we are using the covariant derivatives

$$\begin{aligned} D_\mu \varphi_{(r,z)} &= \mathcal{D}_\mu \varphi_{(r,z)} + irV_\mu \varphi_{(r,z)} \\ &= \nabla_\mu \varphi_{(r,z)} - irA_\mu^{(R)} \varphi_{(r,z)}, \\ \mathbb{D}_\mu \psi^a &= D_\mu \psi^a + K^{a\bar{c}} K_{\bar{c}mn} D_\mu \phi^m \psi^n, \\ \mathbb{D}_\mu \widetilde{\psi}^{\bar{c}} &= D_\mu \widetilde{\psi}^{\bar{c}} + K^{\bar{c}a} K_{a\bar{m}\bar{n}} D_\mu \bar{\phi}^{\bar{m}} \psi^{\bar{n}}. \end{aligned} \tag{5.5}$$

The background connection appearing in D_μ is $A^{(R)} = A - \frac{3}{2}V$. Let us also mention that the R-charges of the derivatives of the composite fields are

$$R[K_a] = -r_a, \qquad R[K_{\bar{c}}] = r^{\bar{c}}, \qquad R[K_{a\bar{c}}] = -r^a + r^{\bar{c}} \ . \tag{5.6}$$

As in flat space, the function K defines a Kähler potential for the metric $G_{a\bar{c}} \equiv K_{a\bar{c}}$. Consistency of the supersymmetric transformation rules requires K to be a quasi-homogeneous function of vanishing R- and central charge.[16] Collecting the fields ϕ^a and $\widetilde{\phi}^{\bar{c}}$ under the variable $C^I = (\phi^a, \widetilde{\phi}^{\bar{c}})$, the two conditions on K are

$$\begin{aligned} &\textstyle\sum_I r^I C^I K_I = 0, \qquad r^I = (r^a, -r^{\bar{c}}), \\ &\textstyle\sum_I z^I C^I K_I = 0, \qquad z^I = (z^a, -z^{\bar{c}}) \ . \end{aligned} \tag{5.8}$$

These extra conditions on the Kähler potential arise from coupling the theory to the background field H.

5.1.2 Superpotential interactions

Superpotential interactions are introduced as F-terms for a chiral multiplet $\Omega_W = (W, \psi^{(W)}, F^{(W)})$, where W is a holomorphic function of the chiral fields ϕ^a. The resulting Lagrangian in components is

$$\mathscr{L}_W = F^m \partial_m W - \frac{1}{2}\psi^i \psi^j \partial_i \partial_j W + \widetilde{F}^{\bar{n}} \partial_{\bar{n}} \widetilde{W} + \frac{1}{2}\widetilde{\psi}^{\bar{n}} \widetilde{\psi}^{\bar{m}} \partial_{\bar{n}} \partial_{\bar{m}} \widetilde{W} \ . \tag{5.9}$$

Invariance under supersymmetry requires W to be a quasi-homogeneus function of the ϕ^a of degree 2

$$-2W + \sum_i r^a \phi^a \partial_a W = 0 \ . \tag{5.10}$$

In a similar way $\widetilde{W}$ is quasi-homogeneous of degree -2. The R-charges of $\partial_a W$ and $\partial_c \widetilde{W}$ are

$$R[\partial_a W] = 2 - r^a, \qquad R[\partial_{\bar{c}} W] = r^{\bar{c}} - 2 \ . \tag{5.11}$$

The most general Lagrangian for a set of chiral superfields is then specified by the two functions K and W, and by the assignment of charges. Schematically, from (5.3) and (5.9) we find

$$\mathscr{L}_{NL\sigma} = \mathscr{L}_K + \mathscr{L}_W \ . \tag{5.12}$$

[16] This restriction can be understood from the computation of $\sigma^{(K)}$. Extracting $\sigma^{(K)}$ from $\delta\chi^{(K)}$ and $\delta\tilde{\chi}^{(K)}$ leads to two different expressions:

$$\sigma^{(K)} = -2\left(r^a H - z^a\right) K_a \phi^a + iK_{a\bar{c}}\, \psi^a \widetilde{\psi}^{\bar{c}} = -2\left(r^{\bar{c}} H - z^{\bar{c}}\right) K_{\bar{c}} \widetilde{\phi}^{\bar{c}} + iK_{a\bar{c}}\, \psi^a \widetilde{\psi}^{\bar{c}} \ . \tag{5.7}$$

In order for $\sigma^{(K)}$ to be well defined, K has to be quasi-homogeneous of the type (5.8).

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5.1.3 Variation under supersymmetry

Given $\mathscr{L}_{NL\sigma}$, the object of interest for us is the total derivative that arises in a supersymmetric variation

$$\delta\mathscr{L}_{NL\sigma} + \tilde{\delta}\mathscr{L}_{NL\sigma} = \nabla_\mu(\mathscr{V}^\mu_{NL\sigma}). \tag{5.13}$$

The supervariation can be obtained either by varying the action explicitly or by evaluating (2.26) and (2.27) for the multiplets $\mathcal{K}$, Ω_W and $\widetilde{\Omega}_{\widetilde{W}}$. The result in both cases is

$$\begin{aligned}\sqrt{2}\,\mathscr{V}^\mu_{NL\sigma} = &+\epsilon\Big[\gamma^\mu\gamma^\nu\psi^a\,K_{a\bar{c}}\mathcal{D}_\nu\widetilde{\phi}^{\bar{c}} - (r^{\bar{c}}H - z^{\bar{c}})\,\gamma^\mu\psi^a\,K_{a\bar{c}}\widetilde{\phi}^{\bar{c}} - iV^\mu\psi^a K_a - 2i\gamma^\mu\widetilde{\psi}^{\bar{c}}\partial_{\bar{c}}\widetilde{W}\Big]\\ &-\tilde{\epsilon}\Big[\gamma^\mu\gamma^\nu\widetilde{\psi}^{\bar{c}}\,K_{\bar{c}a}\mathcal{D}_\nu\phi^a - (r^aH - z^a)\,\gamma^\mu\widetilde{\psi}^{\bar{c}}\,K_{\bar{c}a}\phi^a + iV^\mu\widetilde{\psi}^{\bar{c}}K_{\bar{c}} + 2i\gamma^\mu\psi^a\partial_a W\Big]\\ &+i\epsilon\gamma^\mu\widetilde{\psi}^{\bar{c}}\Big(F^aK_{\bar{c}a} - \frac{1}{2}K_{am\bar{c}}(\psi^a\psi^m)\Big) + i\tilde{\epsilon}\gamma^\mu\psi^a\Big(\widetilde{F}^{\bar{c}}K_{\bar{c}a} + \frac{1}{2}K_{\bar{c}\bar{n}a}(\widetilde{\psi}^{\bar{c}}\widetilde{\psi}^{\bar{n}})\Big)\,.\end{aligned} \tag{5.14}$$

The equations of motion of the auxiliary fields F^a and $\bar{F}^{\bar{c}}$ are

$$\begin{aligned}K_{a\bar{c}}\widetilde{F}^{\bar{c}} + \tfrac{1}{2}K_{a\bar{c}\bar{n}}(\widetilde{\psi}^{\bar{c}}\widetilde{\psi}^{\bar{n}}) &= \partial_a W\,,\\ F^aK_{a\bar{c}} - \tfrac{1}{2}K_{\bar{c}am}(\psi^a\psi^m) &= \partial_{\bar{c}}\widetilde{W}\,.\end{aligned} \tag{5.15}$$

Integrating out F^a and $\bar{F}^{\bar{c}}$ we obtain the final expression

$$\begin{aligned}\sqrt{2}\,\mathscr{V}^\mu_{NL\sigma} = &+\epsilon\Big[\gamma^\mu\gamma^\nu\psi^a\,\mathcal{D}_\nu\widetilde{\phi}^{\bar{c}} - (r^{\bar{c}}H - z^{\bar{c}})\,\gamma^\mu\psi^a\,\widetilde{\phi}^{\bar{c}} - iV^\mu\psi^aK^{\bar{c}} - i\gamma^\mu\widetilde{\psi}^{\bar{c}}W^a\Big]K_{a\bar{c}}\\ &-\tilde{\epsilon}\Big[\gamma^\mu\gamma^\nu\widetilde{\psi}^{\bar{c}}\,\mathcal{D}_\nu\phi^a - (r^aH - z^a)\,\gamma^\mu\widetilde{\psi}^{\bar{c}}\,\phi^a + iV^\mu\widetilde{\psi}^{\bar{c}}K^a + i\gamma^\mu\psi^a\widetilde{W}^{\bar{c}}\Big]K_{a\bar{c}}\,,\end{aligned} \tag{5.16}$$

where we have defined the vectors

$$W^a \equiv K^{a\bar{c}}\partial_{\bar{c}}\widetilde{W}, \qquad \widetilde{W}^{\bar{c}} \equiv K^{\bar{c}a}\partial_a W, \qquad K^{\bar{c}} \equiv K^{\bar{c}a}K_a, \qquad K^a \equiv K^{a\bar{c}}K_{\bar{c}}\,. \tag{5.17}$$

The R-charges of these vectors can be deduced from (5.6) and (5.11): $R[W^a] = r^a - 2$, $R[\widetilde{W}^{\bar{c}}] = 2 - r^{\bar{c}}$, and so on. Observe that the bilinears appearing in $\mathscr{V}_{NL\sigma}$, are the most general bilinears of vanishing R-charge with the correct index structure built out of ϵ and $\tilde{\epsilon}$, ψ and $\widetilde{\psi}$, and the corresponding bosonic fields. For example, it is obvious that derivatives of the superpotential W^a only couple to $\epsilon\gamma^\mu\widetilde{\psi}$, not to $\epsilon\gamma^\mu\psi$.

5.1.4 Digression on target space geometry

In differential geometry, a Kähler manifold is defined as a symplectic (real) manifold $(\mathcal{N}, \omega)$, equipped with a complex structure J such that $G(\cdot,\cdot) \equiv \omega(\cdot, J\,\cdot)$ is a Riemaniann metric on $\mathcal{TN}$. The last condition is called *ω-compatibility* [40]. In a local description with coordinates $(\phi^a, \bar{\phi}^{\bar{c}})$ the metric is represented as

$$ds^2_{\mathcal{N}} = G_{a\bar{c}}d\phi^a d\bar{\phi}^{\bar{c}} + G_{\bar{c}a}d\phi^a d\bar{\phi}^{\bar{c}} = 2G_{a\bar{c}}d\phi^a d\bar{\phi}^{\bar{c}}\,, \qquad G_{a\bar{c}} = G^\star_{\bar{a}c}\,, \tag{5.18}$$

and the two-form ω is represented as $\omega_{\alpha\bar{c}} \propto G_{a\bar{c}}d\phi^a \wedge d\phi^{\bar{c}}$. The target space of the non-linear sigma model, listed above, is such a Kähler manifold.

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For many of the explicit computations in the following sections, a different parametrization will turn out to be especially useful. This involves the change of variables

$$\phi^a = \Phi^a + i\Phi^{a+\mathfrak{s}}, \qquad \bar{\phi}^{\bar{c}} = \Phi^{\bar{c}} - i\Phi^{\bar{c}+\mathfrak{s}}, \tag{5.19}$$

where $a, \bar{c} = 1, \ldots, \mathfrak{s}$. When the reality condition $\bar{\phi}^{\bar{c}} = \phi^{c\star}$ holds, the fields Φ^I are real. However, in general, we may consider ϕ and $\bar{\phi}$ as two independent complex variables. Then the fields Φ^I are also complex and (5.19) is a standard change of variables in $\mathrm{GL}(2\mathfrak{s}, \mathbb{C})$.

Collecting the labels of the type $(a, a+\mathfrak{s})$ into one index $I = 1, \ldots, 2\mathfrak{s}$, the matrix that represents the change of variable is

$$\begin{pmatrix} \phi^a \\ \bar{\phi}^{\bar{c}} \end{pmatrix} = \mathfrak{M} \begin{pmatrix} \Phi^i \\ \Phi^{i+\mathfrak{s}} \end{pmatrix}, \qquad \mathfrak{M} = \begin{pmatrix} \delta^a_i & +i\delta^a_{i+\mathfrak{s}} \\ \delta^{\bar{c}}_i & -i\delta^{\bar{c}}_{i+\mathfrak{s}} \end{pmatrix}, \qquad \mathfrak{M}^{-1} = \tfrac{1}{2}\,\mathfrak{M}^\dagger, \tag{5.20}$$

where the symbol $\delta^a_{i+\mathfrak{s}}$ stands for a diagonal matrix in the off-diagonal blocks of $\mathfrak{M}$, and is defined to be $\delta^a_{i+\mathfrak{s}} = 1$ (or 0) if $a = i$ (or $a \neq i$), as is clear from (5.19). The metric changes accordingly

$$ds^2_{\mathcal{N}} = G_{IJ}\, d\Phi^I d\Phi^J, \qquad G_{IJ} = \begin{pmatrix} \delta^{\{a}_i \delta^{\bar{c}\}}_j\, G_{a\bar{c}} & -i\delta^{[a}_i \delta^{\bar{c}]}_{j+\mathfrak{s}}\, G_{a\bar{c}} \\ +i\delta^{[a}_{i+\mathfrak{s}} \delta^{\bar{c}]}_j\, G_{a\bar{c}} & \delta^{\{a}_{i+\mathfrak{s}} \delta^{\bar{c}\}}_{j+\mathfrak{s}}\, G_{a\bar{c}} \end{pmatrix}. \tag{5.21}$$

The matrix G_{IJ} is real and symmetric, $G = G^T$. On the other hand, the complex structure and the two-form are given by

$$J^M_{\ N} = \begin{pmatrix} 0 & \delta^m_{n+\mathfrak{s}} \\ -\delta^m_{n+\mathfrak{s}} & 0 \end{pmatrix}, \qquad \omega_{MN} = -G_{MI} J^I_{\ N}\,. \tag{5.22}$$

Important relations are $J = -J^T$, $J^2 = -\mathbb{I}$, and $G = JGJ^T$.[17] The second one in (5.22) is precisely the condition of ω-compatibility, which is part of the definition of $\mathcal{N}$.

By construction, two types of "products" exist on a Kähler manifold, one is the symplectic product defined from the tensor ω_{IJ} and the other one is the metric. In components, we find

$$\left(v^a \bar{w}^{\bar{c}} + w^a \bar{v}^{\bar{c}}\right) G_{a\bar{c}} = V^I G_{IJ} W^J\,, \tag{5.23}$$

$$\left(v^a \bar{w}^{\bar{c}} - w^a \bar{v}^{\bar{c}}\right) G_{a\bar{c}} = i V^I \omega_{IJ} W^J\,, \tag{5.24}$$

for any pair of vectors V^I, W^J. The formulae (5.23) and (5.24) will be useful in several occasions. Here we mention one simple application regarding the kinetic energy, which in the new variables Φ^I is the sum of both the metric and the symplectic product. Because of the following identity

$$\begin{aligned} &G_{a\bar{c}}(\partial_\mu \phi^a - i r^a a_\mu \phi^a)(\partial^\mu \bar{\phi}^{\bar{c}} + i r^{\bar{c}} a^\mu \bar{\phi}^{\bar{c}}) = \\ &\quad = \frac{1}{2} G_{MN} \left(\partial_\mu \Phi^M + a_\mu \sum_I J^M_{\ I} r^I \Phi^I\right) \left(\partial^\mu \Phi^N + a^\mu \sum_K J^N_{\ K} r^K \Phi^K\right) \end{aligned} \tag{5.25}$$

[17] When we write matrix products we always understand row by column multiplication, from right to left.

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valid for any connection a_μ, it is possible to introduce the analog of the covariant derivatives $(\mathcal{D}_\mu\phi^a, \mathcal{D}_\mu\bar{\phi}^{\bar{c}})$ acting on Φ^I. In particular, we define

$$(\mathcal{D}_\mu\phi^a, \mathcal{D}_\mu\phi^{\bar{c}}) \;\to\; \partial_\mu\mathbf{\Phi} + \left(A_\mu - \frac{1}{2}V_\mu\right) J\,\mathcal{R}\,\mathbf{\Phi} + J\,\mathcal{Z}\,\mathbf{\Phi} \;, \tag{5.26}$$

where the bold symbol $\mathbf{\Phi}$ represents the vector Φ^I and the matrices $\mathcal{R}$ and $\mathcal{Z}$ are given by

$$\mathcal{R}_{IJ} = \begin{pmatrix} r^a\,\delta^a_i\delta^a_j & 0 \\ 0 & r^{\bar{c}}\,\delta^{\bar{c}}_i\delta^{\bar{c}}_j \end{pmatrix}, \qquad \mathcal{Z}_{IJ} = \begin{pmatrix} z^a\,\delta^a_i\delta^a_j & 0 \\ 0 & z^{\bar{c}}\,\delta^{\bar{c}}_i\delta^{\bar{c}}_j \end{pmatrix} \;. \tag{5.27}$$

Notice the absence of negative signs in the right bottom corner of $\mathcal{R}$ ($\mathcal{Z}$), corresponding to $r^{\bar{c}}$ ($z^{\bar{c}}$). The bold symbols $\mathbf{\Psi}$, $\mathbf{W}$, and $\mathbf{K}$, will be used to describe the vectors corresponding to Ψ^I, W^I and K^I, that indeed appear in the supervariation (5.16).

5.2 $\mathcal{N} = 2$ gauge theories coupled to matter

5.2.1 YM and CS theories

Next, consider a vector multiplet $\mathcal{V} = \{\mathcal{A}_\mu, \lambda, \tilde{\lambda}, \sigma, D\}$ valued in the Lie algebra $\mathfrak{g}$ of a gauge group $\mathfrak{S}$, possibly non-abelian. The field strength $\mathcal{F}_{\mu\nu}$ of the gauge field and the covariant derivatives of the various fields in the vector multiplet are

$$\begin{aligned} \mathcal{F}_{\mu\nu} &= \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - i[\mathcal{A}_\mu, \mathcal{A}_\nu]\,, \\ D_\mu\lambda &= \mathcal{D}_\mu\lambda + iV_\mu\lambda - i[\mathcal{A}_\mu, \lambda]\,, \\ D_\mu\tilde{\lambda} &= \mathcal{D}_\mu\tilde{\lambda} - iV_\mu\lambda + i[\tilde{\lambda}, \mathcal{A}_\mu]\,, \\ D_\mu\sigma &= \partial_\mu\sigma - i[\mathcal{A}_\mu, \sigma]\;. \end{aligned} \tag{5.28}$$

In three dimensions a gauge field admits both Yang-Mills (YM) kinetic terms and Chern-Simons (CS) kinetic terms. For abelian theories the supersymmetric Lagrangian is obtained as the curved D-term of the composite multiplet $-\frac{1}{\mathbf{e}^2}\Sigma^2$, where Σ is the real multiplet associated to $\mathcal{V}$, and $\mathbf{e}$ is the coupling constant. The non-abelian Lagrangian is the standard generalization of this construction, and the result in components is

$$\begin{aligned} \mathbf{e}^2\,\mathscr{L}_{YM} = \mathrm{Tr}\Big\{ &\frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + \frac{1}{2}D^\mu\sigma D_\mu\sigma - \frac{i}{2}\tilde{\lambda}(\gamma^\mu D_\mu\lambda) + \frac{i}{2}(D_\mu\tilde{\lambda}\,\gamma^\mu)\lambda \\ &+ i\tilde{\lambda}[\sigma,\lambda] - \frac{1}{2}\,(D+\sigma H)^2 + \frac{i}{2}H\tilde{\lambda}\lambda + \mathscr{L}^V_{YM}\Big\}\,, \end{aligned} \tag{5.29}$$

$$\mathbf{e}^2\,\mathscr{L}^V_{YM} = +\frac{i}{2}V_\mu\Big\{\sigma\varepsilon^{\mu\nu\rho}\mathcal{F}_{\nu\rho} - \frac{1}{2}V^\mu\sigma^2 + \frac{i}{2}\tilde{\lambda}\gamma^\mu\lambda\Big\}\;. \tag{5.30}$$

For CS theories the supersymmetric Lagrangian is

$$\mathscr{L}_{CS} = \frac{k}{4\pi}\mathrm{Tr}\left\{i\varepsilon^{\mu\nu\rho}\left(\mathcal{A}_\mu\partial_\nu\mathcal{A}_\rho + \frac{2}{3}\mathcal{A}_\mu\mathcal{A}_\nu\mathcal{A}_\rho\right) - 2D\sigma + 2i\tilde{\lambda}\lambda\right\}\;. \tag{5.31}$$

Finally, if the gauge group contains a (product of) U(1) factors we can add for each abelian factor the corresponding FI term

$$\mathscr{L}_{FI} = +\frac{1}{2}\xi(D - \mathcal{A}_\mu V^\mu - \sigma H)\;. \tag{5.32}$$

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5.2.2 Matter couplings

Matter can be added both to CS and YM theories by coupling the vector multiplet to chiral and anti-chiral superfields in arbitrary representations of the gauge group $\mathfrak{S}$. We consider matter superfields $\Phi^{\mathbf{a}}$ and $\widetilde{\Phi}^{\bar{\mathbf{c}}}$ labelled by a bold index which collectively indicates both the color index a and flavor index m, i.e. $\mathbf{a} = (a, m)$. The color indices are contracted in scalar products defined in the appropriate representation of the chiral and anti-chiral fields. Similarly, the components of the gauge multiplets act on the matter fields according to their representation, and the covariant derivatives contain both the background and the gauge fields, $D_\mu \varphi_{(r,z)} = \mathcal{D}_\mu \varphi_{(r,z)} + irV_\mu \varphi_{(r,z)} - i\mathcal{A}_\mu \varphi_{(r,z)}$ for any field $\varphi_{(r,z)}$.

The gauge invariant interactions among different flavors are fixed by a choice of Kähler potential and superpotential. For the simplicity of the presentation, we will consider a canonical Kähler potential. Each flavor may also have different background R-charge r^m and central charge z^m. Assuming that chiral and anti-chiral superfields have opposite charges, it is convenient to define the diagonal matrices of $\mathcal{R}$- and $\mathcal{Z}$-charges. The Lagrangian is

$$\mathscr{L}_{\text{matter}} = \mathscr{L}_K + \mathscr{L}_W \,, \tag{5.33}$$

where $\mathscr{L}_W$ contains a gauge invariant superpotential, (5.9), and $\mathscr{L}_K$ is given by

$$\mathscr{L}_K = \mathscr{L}^{\text{flat}} - \tfrac{\mathfrak{R}}{4}\, \widetilde{\phi}\mathcal{R}\phi + \mathscr{L}_H^{\text{bos}} + \mathscr{L}_H^{\text{ferm}} + \mathscr{L}_V^{\text{bos}} + \mathscr{L}_V^{\text{ferm}} \,. \tag{5.34}$$

In this formula $\mathfrak{R}$ is the curvature of the background manifold and

$$\mathscr{L}^{\text{flat}} = g^{\mu\nu} D_\mu \phi D_\nu \widetilde{\phi} - \frac{i}{2}\, \widetilde{\psi}\gamma^\mu (D_\mu \psi) + \frac{i}{2}(D_\mu \widetilde{\psi})\gamma^\mu \psi - F\widetilde{F} + \sqrt{2}i \left(\widetilde{\phi}\lambda\psi + \widetilde{\psi}\widetilde{\lambda}\phi\right) + \widetilde{\phi} D\phi \,, \tag{5.35a}$$

$$\mathscr{L}_H^{\text{bos}} = \widetilde{\phi}\left(H^2 \mathcal{R}\left(\mathcal{R} - \frac{1}{2}\right) + (\mathcal{Z} + \sigma)^2 - 2H(\mathcal{R} - 1)(\mathcal{Z} - \sigma)\right)\phi + H\widetilde{\phi}\sigma\phi \,, \tag{5.35b}$$

$$\mathscr{L}_H^{\text{ferm}} = -i\psi\left(H\left(\mathcal{R} - \frac{1}{2}\right) - (\mathcal{Z} - \sigma)\right)\widetilde{\psi} \,, \tag{5.35c}$$

$$\mathscr{L}_V^{\text{bos}} = iV_\mu \left(\widetilde{\phi}\left(\mathcal{R} - \frac{1}{2}\right) D^\mu \phi - \phi\left(\mathcal{R} - \frac{1}{2}\right) D^\mu \widetilde{\phi} - \frac{i}{2}\, V^\mu \widetilde{\phi}\mathcal{R}\left(\mathcal{R} - \frac{1}{2}\right)\phi\right) \,, \tag{5.35d}$$

$$\mathscr{L}_V^{\text{ferm}} = -V_\mu \widetilde{\psi}\left(\mathcal{R} - \frac{1}{2}\right)\gamma^\mu \psi \,. \tag{5.35e}$$

Equipped with the precise form of $\mathscr{L}_{YM}$, $\mathscr{L}_{CS}$ and $\mathscr{L}_{\text{matter}}$ it is possible to write down the most generic quiver gauge theory. In this case, the gauge fields will be also labelled by a bold index of the type, $\mathbf{m} = (a, m)$, where m labels the nodes of the quiver theory, and a labels the generators of the gauge group $\mathfrak{S}_m$ at the node m. Considering normalized generators for the gauge groups, the CS coupling κ is promoted to a matrix of the form $\kappa_{\mathbf{mn}} = \delta_{ac} \otimes \kappa_{mn}$, with κ_{mn} a symmetric tensor.

5.2.3 Variation under supersymmetry

The supersymmetric variation of the actions $\mathscr{L}_{YM}$, $\mathscr{L}_{CS}$ and $\mathscr{L}_{\text{matter}}$ has the following properties. Let us begin with the non-abelian YM theory. The change in the action under

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a supersymmetric transformation is given by the total derivative of

$$\begin{aligned}\mathbf{e}^2\,\mathscr{V}^\mu_{YM} = \mathrm{Tr}\Big[&+\frac{1}{4}\epsilon\,\gamma^\mu\gamma^\rho\tilde{\lambda}\,(\hat{\mathcal{F}}_\rho + 2i\sigma V_\rho) - \frac{1}{2}\epsilon\,\gamma^\mu\gamma^\rho\tilde{\lambda}\,\partial_\rho\sigma \\ &+\frac{1}{4}\tilde{\epsilon}\,\gamma^\mu\gamma^\rho\lambda\,(\hat{\mathcal{F}}_\rho + 2i\sigma V_\rho) + \frac{1}{2}\tilde{\epsilon}\,\gamma^\mu\gamma^\rho\lambda\,\partial_\rho\sigma \\ &+\frac{1}{2}\epsilon\,\gamma^\mu\psi_\Sigma\,(iD + \sigma(iH)) + \frac{1}{2}\tilde{\epsilon}\,\gamma^\mu\widetilde{\psi}_\Sigma\,(iD + \sigma(iH))\Big]\,, \end{aligned} \tag{5.36}$$

where $\hat{\mathcal{F}}_\rho = \varepsilon_{\mu\nu\rho}\mathcal{F}^{\mu\nu}$. In the real multiplet parametrization,

$$j_\rho = -\frac{i}{2}\hat{\mathcal{F}}_\rho\,, \qquad a_\rho = -j_\rho - \sigma V_\rho\,, \qquad \lambda = i\widetilde{\psi}_\Sigma\,, \qquad \tilde{\lambda} = -i\psi_\Sigma\,, \tag{5.37}$$

we can rewrite $\mathscr{V}^\mu_{YM}$ in a more compact form as follows

$$\begin{aligned}\mathbf{e}^2\,\mathscr{V}^\mu_{YM} = \mathrm{Tr}\Big[&-\frac{1}{2}\Big(\epsilon\gamma^\mu\gamma^\rho\psi_\Sigma\,(a_\rho - i\partial_\rho\sigma) - \epsilon\gamma^\mu\psi_\Sigma\,\big(iD + (iH)\sigma\big)\Big) \\ &+\frac{1}{2}\Big(\tilde{\epsilon}\gamma^\mu\gamma^\rho\widetilde{\psi}_\Sigma\,(a_\rho + i\partial_\rho\sigma) + \tilde{\epsilon}\gamma^\mu\widetilde{\psi}_\Sigma\,\big(iD + (iH)\sigma\big)\Big)\Big]\,. \end{aligned} \tag{5.38}$$

For the CS action (5.31) the variation under supersymmetry gives

$$\mathscr{V}^\mu_{CS} = +\frac{i}{4\pi}\kappa_{\mathbf{ac}}\left[\varepsilon^{\mu\nu\rho}\,(\epsilon\gamma_\rho\psi^{\mathbf{a}}_\Sigma - \tilde{\epsilon}\gamma_\rho\widetilde{\psi}^{\mathbf{a}}_\Sigma)\mathcal{A}^{\mathbf{c}}_\nu + 2(\epsilon\gamma^\mu\psi^{\mathbf{a}}_\Sigma + \tilde{\epsilon}\gamma^\mu\widetilde{\psi}^{\mathbf{a}}_\Sigma)\sigma^{\mathbf{c}}\right]\,. \tag{5.39}$$

The case of the FI Lagrangian (5.32) is straightforward, and we obtain

$$\mathscr{V}^\mu_{FI} = +\frac{1}{2}\,\xi\,(\epsilon\gamma^\mu\tilde{\lambda} - \tilde{\epsilon}\gamma^\mu\lambda) = -\frac{i}{2}\xi\left[\epsilon\gamma^\mu\widetilde{\psi}_\Sigma + \tilde{\epsilon}\gamma^\mu\psi_\Sigma\right]\,. \tag{5.40}$$

Finally, the variation of the matter action generates

$$\begin{aligned}\sqrt{2}\,\mathscr{V}^\mu_{\mathrm{matter}} = &+\epsilon\Big[\gamma^\mu\gamma^\nu\psi^{\mathbf{a}}\,\mathcal{D}_\nu\widetilde{\phi}^{\bar{\mathbf{c}}} - (r^{\bar{\mathbf{c}}}H - z^{\bar{\mathbf{c}}})\,\gamma^\mu\psi^{\mathbf{a}}\,\widetilde{\phi}^{\bar{\mathbf{c}}} - iV^\mu\psi^{\mathbf{a}}\widetilde{\phi}^{\bar{\mathbf{c}}} + i\,\gamma^\mu\widetilde{\psi}^{\bar{\mathbf{c}}}F^{\mathbf{a}}\Big]G_{\mathbf{a}\bar{\mathbf{c}}} \\ &-\tilde{\epsilon}\Big[\gamma^\mu\gamma^\nu\widetilde{\psi}^{\bar{\mathbf{c}}}\,\mathcal{D}_\nu\phi^{\mathbf{a}} - (r^{\mathbf{a}}H - z^{\mathbf{a}})\,\gamma^\mu\widetilde{\psi}^{\bar{c}}\,\phi^{\mathbf{a}} + iV^\mu\widetilde{\psi}^{\bar{c}}\phi^{\mathbf{a}} - i\,\gamma^\mu\psi^{\mathbf{a}}\widetilde{F}^{\bar{\mathbf{c}}}\Big]G_{\mathbf{a}\bar{\mathbf{c}}} \\ &-\epsilon\,\gamma^\mu\psi^{\mathbf{a}}(\sigma\widetilde{\phi})^{\bar{\mathbf{c}}}G_{\mathbf{a}\bar{\mathbf{c}}} + \tilde{\epsilon}\,\gamma^\mu\widetilde{\psi}^{\bar{\mathbf{c}}}(\sigma\phi)^{\mathbf{a}}G_{\mathbf{a}\bar{\mathbf{c}}} - i\sqrt{2}\Big[\epsilon\,\gamma^\mu\widetilde{\phi}^{\bar{\mathbf{c}}}\,(\psi_\Sigma\,\phi)^{\mathbf{a}} + \tilde{\epsilon}\,\gamma^\mu\widetilde{\phi}^{\bar{\mathbf{c}}}\,(\widetilde{\psi}_\Sigma\,\phi)^{\mathbf{a}}\Big]G_{\mathbf{a}\bar{\mathbf{c}}}\,. \end{aligned} \tag{5.41}$$

The contraction of the color and flavor indices is packaged into $G_{\mathbf{a}\bar{\mathbf{c}}}$. Notice that in the last line σ, ψ_Σ and $\widetilde{\psi}_\Sigma$ act appropriately on color indices.

6 Boundary conditions: a preview

In the previous sections we made precise two key elements of our initial discussion: we decomposed any compact A-type background $\mathcal{M}_3$ into the union of submanifolds with boundary, called $\mathcal{T}$, and we wrote down supersymmetric field theories for $\mathcal{N} = 2$ chiral and vector superfields on $\mathcal{M}_3$, explicitly calculating the expressions for the supersymmetric variation $\mathscr{V}^\mu$. When these field theories are restricted on $\mathcal{T}$, the action can only be invariant under a subset of the bulk supersymmetries if there are boundary conditions solving the corresponding constraints $\mathscr{V}^\perp = 0$.

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In addition, a well-defined classical problem requires appropriate boundary conditions that annihilate all the surface contributions in the Euler-Lagrange variation of the system. Schematically, given a field Φ, and a bulk action $\mathcal{S} = \int_{\mathcal{M}_3} \mathscr{L}[\Phi]$ the equations of motion of the theory require $\delta\mathcal{S} = 0$, where

$$\delta\mathcal{S} = \int_{\mathcal{M}_3} \delta\Phi \left[\frac{\partial\mathscr{L}}{\partial\Phi} - \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\Phi_\mu}\right)\right] + \int_{\mathcal{M}_3} \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\Phi_\mu}\delta\Phi\right), \qquad \Phi_\mu \equiv \partial_\mu\Phi\,. \tag{6.1}$$

On a space with boundary, one demands simultaneously

$$\mathbb{E}[\Phi] = \frac{\partial\mathscr{L}}{\partial\Phi} - \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial\Phi_\mu}\right) = 0\,, \qquad \mathbb{B}[\Phi,\delta\Phi] = n_\mu \frac{\partial\mathscr{L}}{\partial\Phi_\mu}\delta\Phi\Big|_{\text{bdy}} = 0\,. \tag{6.2}$$

A priori, the boundary equations $\mathbb{B} = 0$ are a set of on-shell equations. In what follows, some of these boundary equations will be required to hold also off-shell and will be used to find solutions of $\mathcal{V}^\perp = 0$, which is our main goal.

6.1 The boundary value problem

6.1.1 Fermions

Let us focus first on the boundary value problem for the fermions in $\mathscr{L}_{YM}$ and $\mathscr{L}_{\text{matter}}$, respectively. In $\mathscr{L}_{CS}$ the fermions do not have a kinetic term and do not contribute boundary terms. The corresponding boundary contributions are

$$-\frac{i}{2}\int_{\mathcal{M}_2'} \text{Tr}\left(\tilde{\lambda}\gamma^\perp\delta\lambda - \delta\tilde{\lambda}\gamma^\perp\lambda\right) \subset \delta\mathcal{S}_{YM}\,, \tag{6.3}$$

$$-\frac{i}{2}\int_{\mathcal{M}_2'} K_{a\tilde{c}}\left(\widetilde{\psi}^{\tilde{c}}\gamma^\perp\delta\psi^a - \delta\widetilde{\psi}^{\tilde{c}}\gamma^\perp\psi^a\right) \subset \delta\mathcal{S}_{\text{matter}}\,. \tag{6.4}$$

It is convenient to rewrite both terms in a uniform way. Defining the doublet $\delta\mathbf{\Psi} = (\delta\psi^a, \delta\widetilde{\psi}^{\tilde{c}})$ and $\mathbf{\Psi} = (\psi^a, \widetilde{\psi}^{\tilde{c}})$, we obtain the expression

$$\mathbb{B}^f[\mathbf{\Psi},\delta\mathbf{\Psi}] = -\frac{i}{2}\mathbf{\Psi}^T \begin{pmatrix} 0 & K_{a\tilde{c}} \\ K_{\tilde{c}a} & 0 \end{pmatrix} \otimes \gamma^\perp\,\delta\mathbf{\Psi}\,. \tag{6.5}$$

The form (6.5) also covers the case of (6.3). It is convenient to use ψ_Σ instead of λ. If the generators of the Lie algrebra $\{\mathbf{t}^a\}$ are normalized so that $\text{Tr}[\mathbf{t}^a\mathbf{t}^c] = \delta^{ac}$, the corresponding metric K is the identity. In the real notation of subsection 5.1.4 both (6.3) and (6.4) can be written in the compact form

$$\mathbb{B}^f[\mathbf{\Psi},\delta\mathbf{\Psi}] = -\frac{i}{2}G_{IJ}\mathbf{\Psi}^I\gamma^\perp\,\delta\mathbf{\Psi}^J\,, \tag{6.6}$$

where G is the appropriate metric. Notice that because of the anti-symmetry of $\mathbf{\Psi}^I\gamma^\perp\,\delta\mathbf{\Psi}^J$, the boundary term $\mathbb{B}[\mathbf{\Psi},\delta\mathbf{\Psi}]$ is a 2-form on the space of fermions, i.e. $\mathbb{B}[\mathbf{\Psi},\delta\mathbf{\Psi}] = -\mathbb{B}[\delta\mathbf{\Psi},\mathbf{\Psi}]$.

As we did before, we decompose

$$\mathbf{\Psi} = \frac{\tilde{\zeta}\mathbf{\Psi}}{\Omega}\zeta + \frac{\mathbf{\Psi}\zeta}{\Omega}\tilde{\zeta}\,, \qquad \delta\mathbf{\Psi} = \frac{\tilde{\zeta}\delta\mathbf{\Psi}}{\Omega}\zeta + \frac{\delta\mathbf{\Psi}\zeta}{\Omega}\tilde{\zeta}\,. \tag{6.7}$$

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Then, the equation $\mathbb{B}[\mathbf{\Psi}, \delta\mathbf{\Psi}] = 0$ becomes

$$G_{IJ}\,\frac{\zeta\gamma^{\perp}\zeta}{\Omega}\,(\tilde{\zeta}\mathbf{\Psi})^I\,(\tilde{\zeta}\delta\mathbf{\Psi})^J + G_{IJ}\,\frac{\tilde{\zeta}\gamma^{\perp}\tilde{\zeta}}{\Omega}\,(\mathbf{\Psi}\zeta)^I\,(\delta\mathbf{\Psi}\zeta)^J = 0\ . \tag{6.8}$$

Recalling (3.22), we solve this equation by requiring the boundary conditions

$$\frac{\zeta\gamma^{\perp}\zeta}{\Omega}\,(\tilde{\zeta}\mathbf{\Psi})^I = M^I{}_K\,(\mathbf{\Psi}\zeta)^K \tag{6.9}$$

with a general (possibly field-dependent) matrix M that has the property

$$M^T G M = G\ . \tag{6.10}$$

The boundary condition (6.9) respects the R-symmetry whatever r-charge is assigned to $\mathbf{\Psi}$.

6.1.2 Vectors

There are two possible actions for a vector field $\mathcal{A}_\mu$ in $3d$: $\mathscr{L}_{CS}$, and $\mathscr{L}_{YM}$. The Euler-Lagrange variation with respect to $\mathcal{A}_\mu$, yields the boundary terms

$$-\frac{i}{4\pi}\int_{\mathcal{M}_2'} \kappa_{mn}\,\mathrm{Tr}\left[\varepsilon^{\perp\nu\rho}\mathcal{A}^m_\nu\delta\mathcal{A}^n_\rho\right] \subset \delta\mathcal{S}_{CS}\,, \tag{6.11}$$

$$\int_{\mathcal{M}_2'}\mathrm{Tr}\Big[\Big(\mathcal{F}^{\perp\nu} + i\,\varepsilon^{\perp\nu\rho}V_\rho\,\sigma\Big)\delta\mathcal{A}_\nu\Big] = \int_{\mathcal{M}_2'}\mathrm{Tr}\Big[+i\varepsilon^{\perp\rho\nu}a_\rho\delta\mathcal{A}_\nu\Big] \subset \delta\mathcal{S}_{YM}\ . \tag{6.12}$$

$\mathcal{F}$ is the full non-abelian field strength and a_ρ is defined in eq. (5.37). For a given set of generators $\{\mathbf{t}^a\}$ of the gauge group, we can write $\mathcal{A} = \mathcal{A}^c\mathbf{t}^c$ and $a = a^c\mathbf{t}^c$. Then, both (6.11) and (6.12) can be expressed in terms of the tensor

$$\mathbb{B}^v[\mathcal{V}, \delta\mathcal{A}] = G_{\mathbf{mn}}\,\varepsilon^{\perp\rho\nu}\,\mathcal{V}^{\mathbf{m}}_\rho\,\delta\mathcal{A}^{\mathbf{n}}_\nu \tag{6.13}$$

with $\mathcal{V}_\rho = \mathcal{A}_\rho$ for CS, and $\mathcal{V}_\rho = a_\rho$ for YM. We introduced bold indices $\mathbf{m}$ and $\mathbf{n}$ to describe general quiver gauge theories. Specifically, $\mathbf{m} = (a, m)$ is a double index where m labels the nodes of the quiver and a labels the generators of the gauge group $\mathfrak{S}_m$, that refers to the node m of the quiver. Considering orthonormal generators, the matrix G is $G_{\mathbf{mn}} = \delta_{ac}\otimes\kappa_{mn}$.

In the orthogonal frame $\{k_\mu, \tilde{k}_\mu\}$ on $T\mathcal{M}_2'$, we can further decompose $\mathcal{V}$ and $\delta\mathcal{A}$ along k and $\tilde{k}$ to obtain

$$\mathbb{B}^v[\mathcal{V}, \delta\mathcal{A}] = -G_{\mathbf{mn}}\begin{pmatrix}\mathcal{V}^{\mathbf{m}}_{\tilde{k}}\\ \mathcal{V}^{\mathbf{m}}_{k}\end{pmatrix}^T\begin{pmatrix}0 & +1\\ -1 & 0\end{pmatrix}\begin{pmatrix}\delta\mathcal{A}^{\mathbf{n}}_{\tilde{k}}\\ \delta\mathcal{A}^{\mathbf{n}}_{k}\end{pmatrix}\ . \tag{6.14}$$

We used $\varepsilon^{\perp\rho\nu}\tilde{k}_\rho k_\nu = -1$.

After tracing over the bold indices, $\mathbb{B}^v[\mathcal{V}, \delta\mathcal{A}]$ becomes a 2-form on the cotangent space of $\mathcal{M}_2'$. Equation $\mathbb{B}^v[\mathcal{V}, \delta\mathcal{A}] = 0$ is solved by finding appropriate Lagrangian submanifolds associated to this 2-form. Concretely, we may pick any $\mathrm{Sp}(2,\mathbb{C})$ matrix with unit determinant, call it M, and impose the boundary conditions

$$(1-M)\delta\mathcal{A} = (1-M)\mathcal{V} = 0 \qquad \forall\, p\in\mathcal{M}_2'\ . \tag{6.15}$$

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When $\mathcal{M}_2'$ is endowed with a complex structure, the action of $\mathrm{Sp}(2,\mathbb{C})$ has a natural interpretation. By construction, these solutions are valid both for CS and YM gauge theories.

We point out that an additional interesting solution of $\mathbb{B}^v[\mathcal{V},\delta\mathcal{A}]=0$ is available in the case of CS theories. In general, the tensor κ_{mn} is symmetric, but need not be positive definite. In that case, it may have isotropic subspaces. On this subspaces $\mathbb{B}^v[\mathcal{V},\delta\mathcal{A}]$ vanishes automatically, independently of the coordinate dependence of $\mathcal{V}$ and $\delta\mathcal{A}$. For example, given an isotropic vector v^m such that $v^m\kappa_{mn}v^n=0$, we may consider boundary conditions $\delta\mathcal{A}=v^m\delta\mathcal{A}^m_\mu dx^\mu$ and $\mathcal{V}=v^m\mathcal{V}^m_\mu dx^\mu$ with arbitrary components $\delta\mathcal{A}_\mu$ and $\mathcal{V}_\mu$ on $\mathcal{M}_2'$. For a general treatment of such boundary conditions in CS theory we refer the reader to [56].

6.1.3 Scalars

In the non-linear sigma model, the variation of $\mathscr{L}_{scalar}$ with respect to ϕ^a, $\widetilde{\phi}^{\bar{c}}$, yields the result[18]

$$\delta\mathcal{S}_{NL\sigma}\supset-\int_{\mathcal{M}_2'}\Big(\mathcal{D}^\perp\widetilde{\phi}^{\bar{c}}\,K_{a\bar{c}}\,\delta\phi^a+\delta\widetilde{\phi}^{\bar{c}}\,K_{a\bar{c}}\,\mathcal{D}^\perp\phi^a\Big)-iV^\perp\Big(\tfrac{1}{2}\,K_a\delta\phi^a-\tfrac{1}{2}\,K_{\bar{c}}\delta\widetilde{\phi}^{\bar{c}}\Big)\,. \tag{6.16}$$

The term proportional to $V^\perp$ does not contribute, because $V^\perp=0$ at the boundary. The first term can be written in compact notation as

$$\delta\mathbf{\Phi}^T G\,\mathcal{D}^\perp\mathbf{\Phi}\,,\qquad G=\begin{pmatrix}0 & K_{a\bar{c}}\\ K_{\bar{c}a} & 0\end{pmatrix} \tag{6.17}$$

where G is the target space metric and $\mathbf{\Phi}$ the vector of scalars, introduced in section 5.1.4. We can set (6.17) to zero by assuming that the two vectors $\delta\mathbf{\Phi}$ and $\mathcal{D}^\perp\mathbf{\Phi}=0$ are orthogonal. The standard way to do this, is to consider Dirichlet, $\delta\phi^a=0$, or Neumann, $\mathcal{D}^\perp\phi^a=0$, boundary conditions (and similarly for the scalars $\widetilde{\phi}$). Notice that in general $\mathcal{D}^\perp$ contains non-vanishing normal components of a gauge connection.

In supersymmetric YM theories, the gauge multiplet contains a real scalar σ in the adjoint representation of the gauge group. The variation $\delta\sigma$ of the action yields the boundary term $\mathrm{Tr}\,(\delta\sigma D^\perp\sigma)$. This term is similar to (6.17), and can be set to zero in the same way.

6.2 Path integral and closure under supersymmetry

We conclude this section with an additional remark. In the ensuing sections 7 and 8 we solve the equations $\mathscr{V}^\perp=0$ to obtain half-BPS boundary conditions for general supersymmetric gauge theories. This is sufficient for the purposes of the classical problem.

In the quantum problem we are integrating over generic field configurations in a path integral. In the presence of a boundary the integration is further restricted to configurations with specific boundary conditions. Consequently, in this context the invariance of the path integral with respect to a given symmetry requires that the boundary conditions are also invariant under the symmetry in question. In general, this is not automatic and it may lead to further restrictions on the boundary conditions.

[18] We remind the reader that in this, and the next two sections, we are referring to a flat target space for which the coefficients $K_{a\bar{c}}$ are constants independent of the field profiles ϕ^a, $\widetilde{\phi}^{\bar{c}}$.

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Although we are mainly interested in the classical problem in this paper, we will partially address the issue of the closure of boundary conditions under supersymmetry in the following sections.

7 Boundary conditions I

In this section we address the precise form of A-type boundary conditions in general three-dimensional non-linear sigma models. A good prototype for this exercise are A-type boundary conditions in $2d$ $\mathcal{N}=(2,2)$ non-linear sigma models on the strip that define D-branes in a Kähler target space $\mathcal{X}$. In that case we know, [14], that the solution of the A-type boundary conditions is describing D-branes wrapping Lagrangian submanifolds in $\mathcal{X}$. We will describe how similar solutions arise in three-dimensional theories. We work out first the case of a flat space background, and then explain how things are modified when the $3d$ theory is placed on a general curved A-type background.

7.1 Non-linear sigma models

7.1.1 General equations

The action $\mathcal{S}$ of a supersymmetric non-linear sigma model is specified by a Kähler potential K, a superpotential W, and finally the R-charges and central charges of the chiral superfields. In this subsection K is generic (a flat Kähler potential for chiral superfields will be considered in the ensuing section 8). We continue to call the target space $\mathcal{X}$.

In section 5 we calculated the variation of $\mathcal{S}$ under supersymmetry, and found a generic expression for $\mathscr{V}^{\mu}_{NL\sigma}$. Here we are interested in solutions of the equations $\mathscr{V}^{\perp}_{NL\sigma}=0$ at $\mathcal{M}_2'$. We have

$$\begin{aligned}\sqrt{2}\,\mathscr{V}^{\perp}_{NL\sigma} = &+\epsilon\Big[\gamma^{\perp}\gamma^{\nu}\psi^{a}\,\mathcal{D}_{\nu}\widetilde{\phi}^{\bar{c}} - (r^{\bar{c}}H - z^{\bar{c}})\,\gamma^{\perp}\psi^{a}\,\widetilde{\phi}^{\bar{c}} - iV^{\perp}\psi^{a}K^{\bar{c}} - i\gamma^{\perp}\widetilde{\psi}^{\bar{c}}W^{a}\Big]K_{a\bar{c}}\\ &-\tilde{\epsilon}\Big[\gamma^{\perp}\gamma^{\nu}\widetilde{\psi}^{\bar{c}}\,\mathcal{D}_{\nu}\phi^{a} - (r^{a}H - z^{a})\,\gamma^{\perp}\widetilde{\psi}^{\bar{c}}\,\phi^{a} + iV^{\perp}\widetilde{\psi}^{\bar{c}}K^{a} + i\gamma^{\perp}\psi^{a}\widetilde{W}^{\bar{c}}\Big]K_{a\bar{c}}\,. \qquad (7.1)\end{aligned}$$

The indices $a,\bar{c}$ run from 1 to $\mathfrak{s}$, where $2\mathfrak{s}$ is the real dimension of the target space $\mathcal{X}$. It is convenient to use the identity $\gamma^{\mu}\gamma^{\nu}=g^{\mu\nu}+\gamma^{\mu\nu}$ and rewrite

$$+\epsilon\gamma^{\perp}\gamma^{\nu}\psi^{a}\,\mathcal{D}_{\nu}\widetilde{\phi}^{\bar{c}} = +\epsilon\psi^{a}\mathcal{D}^{\perp}\widetilde{\phi}^{\bar{c}} + \epsilon\gamma^{\perp\nu}\psi^{a}\,\mathcal{D}_{\nu}\widetilde{\phi}^{\bar{c}}\,, \qquad (7.2)$$

$$-\tilde{\epsilon}\gamma^{\perp}\gamma^{\nu}\widetilde{\psi}^{\bar{c}}\,\mathcal{D}_{\nu}\phi^{a} = -\tilde{\epsilon}\widetilde{\psi}^{\bar{c}}\mathcal{D}^{\perp}\phi^{a} - \tilde{\epsilon}\gamma^{\perp\nu}\widetilde{\psi}^{\bar{c}}\,\mathcal{D}_{\nu}\phi^{a}\,. \qquad (7.3)$$

In equations (7.1)–(7.3) we recognize the combinations

$$V^{\perp}\left[K_{\bar{c}}(\delta\widetilde{\phi}^{\bar{c}})_{\text{susy}} - K_{a}(\delta\phi^{a})_{\text{susy}}\right] \qquad \& \qquad (\delta\phi)_{\text{susy}}\mathcal{D}^{\perp}\widetilde{\phi} + (\delta\widetilde{\phi})_{\text{susy}}\mathcal{D}^{\perp}\phi\,, \qquad (7.4)$$

which appeared in the analysis of $\mathbb{B}^{s}[\Phi,\delta\Phi]$ (6.16). This is expected because on-shell we can always use the Noether current to rewrite $\mathscr{V}^{\perp}$.

Following the discussion in section 6, we require $\mathscr{V}^{\perp}_{NL\sigma}=0$. The analysis of this equation reduces naturally to the study of four types of terms:

$$\mathscr{V}_{1} = +\left[\epsilon\psi^{a}\mathcal{D}^{\perp}\widetilde{\phi}^{\bar{c}} - \tilde{\epsilon}\widetilde{\psi}^{\bar{c}}\mathcal{D}^{\perp}\phi^{a}\right]K_{a\bar{c}}\,, \qquad (7.5)$$

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$$\mathscr{V}_2 = +\left[\epsilon\gamma^{\perp\nu}\psi^a\,\mathcal{D}_\nu\widetilde{\phi}^{\bar{c}} - \tilde{\epsilon}\gamma^{\perp\nu}\widetilde{\psi}^{\bar{c}}\,\mathcal{D}_\nu\phi^a\right]K_{a\bar{c}}\,, \tag{7.6}$$

$$\mathscr{V}_3 = -\left[\epsilon\,\gamma^{\perp}\psi^a\,\widetilde{\phi}^{\bar{c}} - \tilde{\epsilon}\,\gamma^{\perp}\widetilde{\psi}^{\bar{c}}\,\phi^a\right]K_{a\bar{c}}\,, \tag{7.7}$$

$$\mathscr{V}_4 = -\left[\epsilon\gamma^{\perp}\widetilde{\psi}^{\bar{c}}W^a + \tilde{\epsilon}\gamma^{\perp}\psi^a\widetilde{W}^{\bar{c}}\right]K_{a\bar{c}}\,. \tag{7.8}$$

In order to obtain explicit boundary conditions for the fields that appear in these equations we have to disentangle the spinorial and target space structures. The reader can find the details of this computation in appendix B. Here we outline the main steps.

Firstly, the anticommuting spinors are decomposed in components using the projectors $\mathscr{P}$ and $\widetilde{\mathscr{P}}$. As a result, all the geometric information can be packaged into the bilinears (3.22)

$$\begin{aligned}
\zeta\gamma^{\perp}\zeta &\equiv \Omega\,e^{i\varpi}\,, & \tilde{\zeta}\gamma^{\perp}\tilde{\zeta} &= -\Omega\,e^{-i\varpi}\,,\\
\zeta\gamma^{\perp}\gamma^{\nu\parallel}\zeta &= \Omega\,e^{i\varpi}\,k^{\nu\parallel}\,, & \tilde{\zeta}\gamma^{\perp}\gamma^{\nu\parallel}\tilde{\zeta} &= \Omega\,e^{-i\varpi}\,k^{\nu\parallel}\,.\\
\tilde{\zeta}\gamma^{\perp}\gamma^{\nu\parallel}\zeta &= -i\Omega\,\tilde{k}^{\nu\parallel}\,, & \zeta\gamma^{\perp}\gamma^{\nu\parallel}\tilde{\zeta} &= -i\Omega\,\tilde{k}^{\nu\parallel}\,.
\end{aligned} \tag{7.9}$$

Secondly, we impose the A-type projection (4.4)–(4.5) on the spinors ϵ and $\tilde{\epsilon}$,

$$\widetilde{\mathscr{P}}\epsilon = 0\,, \qquad \mathscr{P}\tilde{\epsilon} = 0\,, \qquad \vartheta = \widetilde{\vartheta}\,. \tag{7.10}$$

Finally, we impose the boundary condition (6.9) on the spinors, i.e. $e^{i\varpi}(\tilde{\zeta}\mathbf{\Psi})^I = M^I_K(\mathbf{\Psi}\zeta)^K$. These manipulations introduce the orthogonal matrix M and the phase ϖ in $\mathscr{V}_i$. At the end, the $\mathscr{V}_i$ depend only on $\tilde{\zeta}\epsilon$ and $\mathbf{\Psi}\zeta$. Hence, a bilinear $\epsilon\mathbf{\Psi}$ common in all terms can be factorized out, and the result for $\mathscr{V}^{\perp}_{NL\sigma}$ can be understood as a condition on the bosons. This is nicely expressed in the matrix notation $\mathbf{\Phi}$ and $\mathbf{\Psi}$ of section 5.1.4. As a simple example of these manipulations, we obtain

$$\mathscr{V}_1 = +\epsilon\psi^a K_{a\bar{c}}\mathcal{D}^{\perp}\widetilde{\phi}^{\bar{c}} - \tilde{\epsilon}\widetilde{\psi}^{\bar{c}}K_{a\bar{c}}\mathcal{D}^{\perp}\phi^a = (\epsilon\mathbf{\Psi})^T\left(\frac{1-iJ}{2} + e^{-i\varpi}M^T\frac{1+iJ}{2}\right)GD^{\perp}\mathbf{\Phi}\,.$$

The complete result is

$$\mathscr{V}^{\perp}_{NL\sigma} = +(\epsilon\mathbf{\Psi})^T\,\left[(1-iJ)\right]\,G\,P^{(\varpi,+)}_M\left[n^\mu\mathcal{D}_\mu\mathbf{\Phi} + J\,\tilde{k}^\mu\mathcal{D}_\mu\mathbf{\Phi}\right] \tag{7.11a}$$

$$+(\epsilon\mathbf{\Psi})^T\,\left[e^{-i\varpi}(1+iJ)\right]\,G\,P^{(\varpi,-)}_M\left[k^\mu\mathcal{D}_\mu\mathbf{\Phi} + J\,(iH)\,\mathcal{R}\,\mathbf{\Phi} - iJ\,\mathcal{Z}\,\mathbf{\Phi}\right] \tag{7.11b}$$

$$-(\epsilon\mathbf{\Psi})^T\,\left[e^{-i\varpi}(1-iJ)\right]\,P^{(-\varpi,+)}_{M^T}\,J\left[G\,\mathbf{W}\right], \tag{7.11c}$$

where the matrix $P^{(\varpi,\pm)}_M$ is a *target space projector* defined as

$$P^{(\varpi,\pm)}_M \equiv \frac{1}{2}\Big(1 \pm M[\varpi]\Big)\,, \tag{7.12}$$

$$M[\varpi] \equiv M\mathscr{R}[\varpi] = \mathscr{R}[-\varpi/2]M\mathscr{R}[\varpi/2]\,, \tag{7.13}$$

$$\mathscr{R}[\varpi] \equiv \cos\varpi\,1 + \sin\varpi\,J\,. \tag{7.14}$$

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In deriving (7.11) we imposed $\{M, J\} = 0$ from which (7.13) follows. With this condition, P_M is a projector if $M^2 = 1$. Collecting all requirements, the matrix M is an orthogonal matrix with the properties

$$M^2 = 1\,, \quad \{M, J\} = 0\ . \tag{7.15}$$

The matrix $\mathscr{R}[\varpi]$ is the matrix of local R-symmetry.

7.1.2 Solutions in flat space

Having obtained the general formula (7.11) we are now in position to study solutions to equation $\mathscr{V}^{\perp}_{NL\sigma} = 0$. Flat space is of course a special case of our discussion. It is instructive to exhibit first how Lagrangian 'D-branes' come out of (7.11) for a theory defined on a euclidean $3d$ half-plane. In this case the boundary leaf $\mathcal{M}'_2$ is a 2-plane.

In flat space the profile of the background fields is trivial, and the covariant derivative $\mathcal{D}_\mu$ reduces to the standard partial derivative ∂_μ. In what follows we will also set, for convenience, $\mathcal{Z} = 0$ for the central charges. The role of $\mathcal{Z}$ in (7.11b) is the same as that of a real mass obtained by giving a vev to the bottom component of a real multiplet coupled to $\mathbf{\Phi}$. We will consider such masses in relation to YM and CS theories in section 8.

Before going into the details of the solution, it is worth emphasizing two simplifying special properties of flat space:

1) There is always a choice of coordinates, say $\{\theta, x, \tilde{x}\}$, such that the frame $\{n_\mu, k_\mu, \tilde{k}_\mu\}$ is precisely $\{\partial_\theta, \partial_x, \partial_{\tilde{x}}\}$. The boundary is placed at a fixed value of θ.

2) The phase ϖ appearing in $M[\varpi]$ is a constant.

Both of these features are generically absent in curved space because of the background curvature.

Focusing on the vanishing of the components (7.11a)–(7.11b), we obtain the conditions

$$\partial_\theta \mathbf{\Phi} \in \mathrm{Ker}(1 + M[\varpi]) \tag{7.16}$$

$$\partial_x \mathbf{\Phi} \in \mathrm{Ker}(1 - M[\varpi]) \qquad \& \qquad \partial_{\tilde{x}} \mathbf{\Phi} \in \mathrm{Ker}(1 - M[\varpi])\ . \tag{7.17}$$

Since $M^2 = 1$, the eigenvalues of the matrix M are ± 1. Moreover, since $\{M, J\} = 0$, the complex structure of the target space is a bijection between $\mathrm{Ker}(1 - M)$ and $\mathrm{Ker}(1 + M)$. As a result, $\mathrm{Ker}(1 \pm M)$ is middle dimensional in the target space, and the direct sum $\mathrm{Ker}(1 - M) \oplus \mathrm{Ker}(1 + M)$ is a basis for $T\mathcal{X}$. The submanifold corresponding to the distribution $\mathrm{Ker}(1 - M)$ is a *Lagrangian* submanifold $\mathcal{L}$.[19] The effect of the matrix $\mathscr{R}[\varpi]$ is to change the orientation of the Lagrangian submanifold by a constant angle ϖ.

[19] For the convenience of the reader we remind that a *Lagrangian* submanifold $\mathcal{L}$ (defined on a symplectic manifold $(\mathcal{N}, \omega)$, where ω is the symplectic form) is characterized by the two conditions:

$$\omega\big|_{\mathcal{TL}} = 0, \qquad \dim \mathcal{L} = \frac{1}{2} \dim \mathcal{N}\ . \tag{7.18}$$

When the symplectic manifold $\mathcal{N}$ is Kähler, the Riemaniann metric G_{IJ} can be used to characterize $\mathcal{L}$, and the definition just given is equivalent to the condition

$$\mathcal{TL}^{\perp} = J\,\mathcal{TL}, \qquad \mathcal{TL}^{\perp} = \{\vec{v} \in \mathcal{TN} \,|\, v^I G_{IJ} w^J = 0\ \forall\, \vec{w} \in \mathcal{TL}\}\ . \tag{7.19}$$

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The Lagrangian submanifold just described contains $\mathbf{\Phi}(\mathcal{M}_2')$, the image of $\mathcal{M}_2'$ under the maps $\mathbf{\Phi}$. Both M and the derivatives of $\mathbf{\Phi}$ are objects in $T\mathcal{X}$. The solutions (7.16)–(7.17) transform correctly under a change of coordinates in the target space. Locally, we may take a chart such that the Lagrangian submanifold is described by mixed Dirichlet and Neumann boundary conditions. We impose Neumann boundary conditions along the directions parallel to the submanifold, and Dirichlet conditions along the directions transverse to the submanifold.

In the simplest situation, in which $\mathcal{X}$ is an affine vector space and the Kähler potential is canonical, the Neumann and Dirichlet boundary conditions can be seen explicitly by solving (7.16)–(7.17). This is done by considering a basis $\{v_i\}_{i=1}^{\mathfrak{s}}$ of $\mathrm{Ker}(1+M)$, and writing $\mathbf{\Phi} = \sum_{i=1}^{\mathfrak{s}} \left[f_i v^i + g_i J v^i\right]$ with f_i and g_i functions of the coordinates. The solution to (7.16) is $\partial_\theta g_i = 0$ at the boundary, i.e. Neumann boundary conditions along the direction of the submanifold. The generic solution to (7.17) is $f_i = f_i(\theta)$, and therefore $f = const$ at the boundary, i.e. Dirichlet boundary conditions in the direction transverse to the submanifold. The worldvolume of $\mathcal{L}$ is along the span of $\{Jv_i\}_{i=1}^{\mathfrak{s}}$. The case with $\varpi \neq 0$, is solved by rotating the fields accordingly with the projector. The latter can be written as

$$P_M^{(\varpi,\pm)} = \mathscr{R}[-\varpi/2]\frac{(1 \pm M)}{2}\mathscr{R}[\varpi/2]\,, \tag{7.20}$$

and the solution is $\mathbf{\Phi} \equiv \mathscr{R}[-\varpi/2]\mathbf{\Phi}'$, where $\mathbf{\Phi}'$ satisfies the Neumann/Dirichlet boundary conditions that depend on M.

Along similar lines consider the boundary conditions derived from the superpotential term, namely the equation that arises by requiring the last term (7.11c) to vanish,

$$P_{M^T}^{(-\varpi,+)}\, J\, G\, \mathbf{W} = \mathscr{R}[\varpi/2](1 + M^T)\mathscr{R}[-\varpi/2]\, J\, G\, \mathbf{W} = 0\ . \tag{7.21}$$

In this case the projector depends on $M^T\mathscr{R}[-\varpi]$, in agreement with R-symmetry considerations. The vector $\mathbf{W}$ was defined in section 5.1.4, and in the complex basis it has components $W^a = K^{a\bar{c}}\partial_{\bar{c}}\widetilde{W}$, $\widetilde{W}^{\bar{c}} = K^{\bar{c}a}\partial_a W$. Since $W = \mathrm{Re}\, W + i\mathrm{Im}\, W$ is a holomorphic function of the fields, the Cauchy-Riemann equations imply the relations

$$\begin{bmatrix} \partial_m \mathrm{Re}\, W \\ \partial_{m+\mathfrak{s}} \mathrm{Re}\, W \end{bmatrix} = J \begin{bmatrix} \partial_m \mathrm{Im}\, W \\ \partial_{m+\mathfrak{s}} \mathrm{Im}\, W \end{bmatrix}, \qquad \frac{\partial}{\partial\phi^m} W = \frac{\partial}{\partial\Phi^m}\mathrm{Re}\, W - i\frac{\partial}{\partial\Phi^{m+\mathfrak{s}}}\mathrm{Re}\, W\ . \tag{7.22}$$

The quantity $G\,\mathbf{W}$ is

$$G\,\mathbf{W} = \begin{bmatrix} \partial_m \mathrm{Re}\, W(\mathbf{\Phi}) \\ \partial_{m+\mathfrak{s}} \mathrm{Re}\, W(\mathbf{\Phi}) \end{bmatrix}, \tag{7.23}$$

where ∂_i is shorthand notation for $\partial_i = \partial/\partial\Phi^i$. Implementing the rotation $\mathbf{\Phi} = \mathscr{R}[-\varpi/2]\mathbf{\Phi}'$, we obtain from (7.21) the projection equations

$$(1 + M^T) \begin{bmatrix} \partial'_m \mathrm{Im}\, W(\mathbf{\Phi}') \\ \partial'_{m+\mathfrak{s}} \mathrm{Im}\, W(\mathbf{\Phi}') \end{bmatrix} = 0, \tag{7.24}$$

where $\partial' = \partial/\partial\Phi'$. Because $\mathrm{Ker}(1 \pm M)$ span the tangent space $T\mathcal{M}$, and J is a bijection between these two kernels, we can understand the boundary condition (7.24) by considering

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the action of $v^T(1+M^T)$ and $(Jv)^T(1+M^T)$ on $\partial'\mathrm{Im}\,W(\mathbf{\Phi}')$, for any $v \in \mathrm{Ker}(1+M)$. By definition $v^T(1+M^T)=0$, thus only $(Jv)^T(1+M^T)$ is non-trivial. The latter can be calculated explicitly $(Jv)^T(1+M^T)=2(Jv)^T$, and from (7.24) we obtain the boundary condition

$$(Jv)^I\partial'_I\mathrm{Im}\,W(\mathbf{\Phi}')=0\,, \tag{7.25}$$

which translates into the statement that $\partial'_I\mathrm{Im}W(\mathbf{\Phi}')$ has no component along the span of $\{Jv_i\}_{i=1}^{\mathfrak{s}}$ and therefore $\mathrm{Im}W(\mathbf{\Phi}')$ is constant along the wordvolume of the submanifold $\mathcal{L}$.

7.1.3 Solutions in curved space

In the previous section, we solved the equations $\mathscr{V}^{\perp}_{NL\sigma}=0$ relying on two special features of flat space: the fact that the phase ϖ is constant, and the fact that there is a coordinate-adapted orthogonal basis in $T\mathcal{M}_3$. In curved space we do not expect in general these two features to hold.

For example, in the case of the ellipsoid in toric coordinates with background fields (3.28)

$$H=\pm\frac{i}{g_{\theta\theta}},\qquad A^{(R)}_{\pm}=-\frac{1}{2}\left(1-\frac{\tilde{\ell}}{g_{\theta\theta}}\right)d\phi_1\mp\frac{1}{2}\left(1-\frac{\ell}{g_{\theta\theta}}\right)d\phi_2\,, \tag{7.26}$$

we find $\varpi_{\pm}=\psi$ with frame vectors

$$n^{\mu}_{\pm}\,\partial_\mu=-\frac{1}{g_{\theta\theta}}\partial_\theta\,, \tag{7.27}$$

$$k^{\mu}_{\pm}\,\partial_\mu=\pm\tilde{\ell}^{-1}\partial_{\phi_1}+\ell^{-1}\partial_{\phi_2}\,, \tag{7.28}$$

$$\tilde{k}^{\mu}_{\pm}\partial_\mu=\mp\cot(2\theta)\,k^{\mu}_{\pm}\partial_\mu\pm\frac{1}{\sin(2\theta)}\left(\frac{1}{\ell}\partial_{\phi_2}\mp\frac{1}{\tilde{\ell}}\partial_{\phi_1}\right)\,. \tag{7.29}$$

Consider now a more general manifold $\mathcal{M}_3$ in toric coordinates (θ,ϕ_1,ϕ_2), in similar notation to the one above for the ellipsoid. By definition, the Killing vector $k=\frac{1}{\Omega}\partial_\psi$ is expressed as a combination of ∂_{ϕ_1} and ∂_{ϕ_2}, and ϖ is only a function of ψ. The triple of vectors $(k^\mu,n^\mu,\tilde{k}^\mu)$ takes the form

$$k=\frac{1}{\Omega}\partial_\psi\,,\quad n=f_n\,\partial_\theta\,,\quad \tilde{k}=\tilde{f}\,\partial_\psi+v^\mu\partial_\mu\,. \tag{7.30}$$

The functions $\tilde{f}$, f_n and v^μ depend on the details of the background, however, the integrability condition implies $[v^\mu\partial_\mu,\partial_\psi]=0$. $\mathcal{M}_3$ is decomposed as before, $\mathcal{M}_3\cong\mathcal{T}_1\#\mathcal{T}_2$, and the fields are restricted on one of the solid tori, call it $\mathcal{T}$ for simplicity.

In this case, the general solution of $\mathscr{V}^{\perp}=0$ has (see (7.11a), (7.11b))

$$\partial_\theta\mathbf{\Phi}\in\mathrm{Ker}(1+M[\varpi]) \tag{7.31}$$

$$\mathcal{D}_\psi\mathbf{\Phi}+J\,(iH)\,\mathcal{R}\,\mathbf{\Phi}\in\mathrm{Ker}(1-M[\varpi])\,,\qquad \tilde{k}^\mu\mathcal{D}_\mu\mathbf{\Phi}\in\mathrm{Ker}(1-M[\varpi])\,. \tag{7.32}$$

In the first line we used, for illustration purposes, the simplifying assumption $A^{(R)}_{\perp}=0$, which clearly holds for the example of the ellipsoid (7.26). The covariant derivatives in (7.32) are

$$\tilde{k}^\mu\mathcal{D}_\mu\mathbf{\Phi}=\tilde{f}\,\partial_\psi\mathbf{\Phi}+v^\mu\partial_\mu\mathbf{\Phi}+\tilde{k}^\mu A^{(R)}_\mu\,J\mathcal{R}\,\mathbf{\Phi}\,, \tag{7.33}$$

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$$k^\mu \mathcal{D}_\mu \mathbf{\Phi} + J(iH)\mathcal{R}\mathbf{\Phi} = k^\mu \partial_\mu \mathbf{\Phi} + \left[k^\mu A^{(R)}_\mu + (iH + k^\mu V_\mu)\right] J\mathcal{R}\,\mathbf{\Phi}\ . \tag{7.34}$$

We can always solve (7.31) with Neumann boundary conditions because regardless of whether the phase ϖ is constant or coordinate dependent, $\partial_\theta \mathscr{R}[\varpi] = 0$. The solution of the other two equations instead depends on ϖ.

Consider first the case of background fields where ϖ is constant. As we explained in section 3.4, this can be achieved from a general background with a gauge transformation of the original $A^{(R)}_\mu$ to a new R-symmetry background $A^{(R)}_{\text{new}}$. In that case, the term that appears inside the parenthesis on the r.h.s. of equation (7.34), with the substitution $A^{(R)} \to A^{(R)}_{\text{new}}$, vanishes because of the condition we found in (3.26). Consequently, we obtain as in flat space

$$k^\mu \mathcal{D}_\mu \mathbf{\Phi} + J(iH)\mathcal{R}\mathbf{\Phi}\Big|_{\text{twisted}} = k^\mu \partial_\mu \mathbf{\Phi} \in \text{Ker}(1 - M[\varpi])\ . \tag{7.35}$$

The analysis of $\tilde{k}^\mu \mathcal{D}_\mu \mathbf{\Phi}$ requires more detailed knowledge of $\tilde{f}$ and $\tilde{k}^\mu A^{(R)}_{\mu\, new}$. To be concrete, in the case of the geometries introduced in section 3.5 we obtain the following expressions:

- For the ellipsoid, $A^{(R)}_{\mu\,\text{new}}$ and its scalar product with $\tilde{k}^\mu$, given in (7.29), are

$$A^{(R)}_{\pm\, new} = A^{(R)}_{\pm} - \frac{1}{2}(d\phi_1 \pm d\phi_2) = \frac{1}{2g_{\theta\theta}}(\tilde{\ell} d\phi_1 \pm \ell d\phi_2), \tag{7.36}$$

$$\tilde{k}^\mu A^{(R)}_{\mu\,\text{new}}\Big|_{\pm} = \pm \cot(2\theta)\, iH_\pm\ . \tag{7.37}$$

  The function $\tilde{f}$ is also proportional to $\cot(2\theta)$.

- For the circle bundles of section 3.5.2, we find

$$A^{(R)}_{\mu\,\text{new}} dx^\mu = -\left(\frac{\cos\theta}{2g_{\theta\theta}} + \frac{\beta^2}{2}\frac{u(\theta)}{g_{\theta\theta}}\frac{u'(\theta)}{\sin\theta}\right) d\varphi - \frac{\beta^2}{2g_{\theta\theta}}\frac{u'(\theta)}{\sin\theta}\, d\psi\,, \tag{7.38}$$

$$\tilde{k} = \frac{2}{\sin\theta}\left(u(\theta)\partial_\psi - \partial_\varphi\right)\ , \tag{7.39}$$

$$\tilde{k}^\mu A^{(R)}_{\mu\,\text{new}} = \frac{\cot\theta}{g_{\theta\theta}}\ . \tag{7.40}$$

  The function $\tilde{f}$ is proportional to $u(\theta)/\sin\theta$.

We notice that the boundary condition from $\tilde{k}^\mu \mathcal{D}_\mu \mathbf{\Phi}$ simplifies when the boundary is placed at the *equator* of the corresponding geometries, because at that point $\tilde{k}^\mu A^{(R)}_{\mu\,\text{new}} = 0$. When this happens, the covariant derivative $\tilde{k}^\mu \mathcal{D}_\mu \mathbf{\Phi}$ becomes a combination of partial derivatives, and again we can solve the boundary conditions as in flat space. Namely, we impose Neumann boundary conditions along the directions parallel to the submanifold, and Dirichlet for the directions transverse to the submanifold. The value of $\tilde{f}$ at the equator is not important in this statement. When the boundary is placed away from the equator a more complicated boundary condition (7.32) has to be imposed.

In more general setups, a background $\mathcal{M}_3$ exhibits a coordinate-dependent phase ϖ. In that case the boundary equations (7.31)–(7.32)

$$\partial_\theta \mathbf{\Phi} \in \text{Ker}(1 + M[\varpi])\,, \tag{7.41}$$

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$$\mathcal{D}_\psi \mathbf{\Phi} + J\,(iH)\,\mathcal{R}\,\mathbf{\Phi} \in \mathrm{Ker}(1 - M[\varpi])\,, \qquad \tilde{k}^\mu \mathcal{D}_\mu \mathbf{\Phi} \in \mathrm{Ker}(1 - M[\varpi]) \tag{7.42}$$

do not exhibit any simplification in the covariant derivatives, and the boundary conditions are functionals of both the derivatives and the values of the fields. As a result, the direct geometric meaning of the boundary conditions in target space, that was present in backgrounds with constant ϖ, is now less manifest. Nevertheless, one can still solve the boundary conditions by diagonalizing $M[\varpi]$, for a given choice of M, and arranging the combinations (7.31)–(7.32) to belong to $\mathrm{Ker}(1 \pm M[\varpi])$. Since $[M, J] \neq 0$, the eigenvectors of $M[\varpi]$ are not the ones of $M = M[\varpi = 0]$. Consequently, as we move along the orbit of the Killing vector, or more generically, along the boundary $\mathcal{M}_2'$, these eigenvectors change according to their ϖ dependence.

7.2 Real multiplets

Before we tackle general gauge theories, there is another comparatively simple example we would like to discuss. It is well-known that in $3d$ flat space there is a simple duality between a chiral superfield and an abelian gauge field.[20] We expect the corresponding boundary conditions to be mapped trivially under this duality. With this in mind, in this subsection we present A-type boundary conditions for $\mathcal{N} = 2$ theories of $\mathfrak{s}$ abelian vector superfields interacting via a constant target space metric G.

The supersymmetric variation $\mathscr{V}^\mu$ is expressed most conveniently in terms of the real parametrization of the abelian vector superfields in (5.38):

$$\begin{aligned}\mathscr{V}^\mu_{\text{real}} = &-\frac{1}{2}\Big(\epsilon\gamma^\mu\gamma^\rho\psi^a_\Sigma\,(a_\rho - i\partial_\rho\sigma)^c - \epsilon\gamma^\mu\psi^a_\Sigma\,\big(iD + (iH)\sigma\big)^c\Big)G_{ac} \\ &+ \frac{1}{2}\Big(\tilde{\epsilon}\gamma^\mu\gamma^\rho\widetilde{\psi}^a_\Sigma\,(a_\rho + i\partial_\rho\sigma)^c + \tilde{\epsilon}\gamma^\mu\widetilde{\psi}^a_\Sigma\,\big(iD + (iH)\sigma\big)^c\Big)G_{ac}\ .\end{aligned} \tag{7.43}$$

We can further rearrange $\mathscr{V}^\perp_{\text{real}}$ by borrowing results from the study of $\mathscr{V}^\perp_{\widetilde{N}L\sigma}$ in the previous section. In particular, let us define the two complex combinations: $\partial_\rho\phi_\Sigma \equiv a_\rho - i\partial_\rho\sigma$ and $\mathrm{Im}\,\varphi_\Sigma \equiv (D + (H)\sigma)$, $\mathrm{Re}\,\varphi_\Sigma \equiv 0$. Then, we can rewrite $\mathscr{V}^\perp_{\text{real}}$ as

$$\mathscr{V}^\mu_{\text{real}} = -\frac{1}{2}\Big(\epsilon\gamma^\mu\gamma^\rho\psi^a_\Sigma\,\partial_\rho\widetilde{\phi}^c_\Sigma + \epsilon\gamma^\mu\psi^a_\Sigma\,\widetilde{\varphi}^c_\Sigma\Big)G_{ac} + \frac{1}{2}\Big(\tilde{\epsilon}\gamma^\mu\gamma^\rho\widetilde{\psi}^a_\Sigma\,\partial_\rho\phi^c_\Sigma + \tilde{\epsilon}\gamma^\mu\widetilde{\psi}^a_\Sigma\,\varphi^c_\Sigma\Big)G_{ac}\ ,$$

and with an obvious change of variables, it is clear that we have obtained an expression that is essentially the sum of $\mathscr{V}_1$, $\mathscr{V}_2$ and $\mathscr{V}_3$, given in (7.5), (7.6), (7.7), respectively.

Consequently, the surface term $\mathscr{V}^\perp_{\text{real}}$ takes the suggestive form

$$\begin{aligned}\mathscr{V}^\perp_{\text{real}} = &-(\epsilon\mathbf{\Psi}_\Sigma)^T\ [(1 - iJ)G]\,P^{(\varpi,+)}_M\left(n^\mu\begin{bmatrix}a_\mu\\ \partial_\mu\sigma\end{bmatrix} + J\,\tilde{k}^\mu\begin{bmatrix}a_\mu\\ \partial_\mu\sigma\end{bmatrix}\right) \\ &-(\epsilon\mathbf{\Psi}_\Sigma)^T\ \left[e^{-i\varpi}(1 + iJ)G\right]P^{(\varpi,-)}_M\left(k^\mu\begin{bmatrix}a_\mu\\ \partial_\mu\sigma\end{bmatrix} - \begin{bmatrix}iD + (iH)\sigma\\ 0\end{bmatrix}\right)\end{aligned} \tag{7.44}$$

[20] In the simplest case, the duality is obtained by considering $\int d^4\theta(\Sigma^2 - \Sigma(\Phi + \mathbf{\Phi}))$, where Φ is a chiral superfield and Σ a generic superfield. Integrating out Φ constrains Σ to be a real superfield and produces a U(1) gauge theory. Alternatively, integrating out Σ gives the action of a chiral superfield. It is interesting to reconsider this exercise in curved spaces.

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where the matrix M fixes the spinor boundary conditions $e^{i\varpi}\tilde{\zeta}\mathbf{\Psi}_\Sigma = M\mathbf{\Psi}_\Sigma\zeta$. From the definition of $a_\mu = -j_\mu - \sigma V_\mu$, and the fact that $V^\perp = 0$, we finally obtain the boundary conditions

$$
\begin{aligned}
& n^\mu \begin{bmatrix} j_\mu \\ \partial_\mu\sigma \end{bmatrix} \in \mathrm{Ker}(1+M[\varpi])\,, \quad \left\{ \tilde{k}^\mu \begin{bmatrix} a_\mu \\ \partial_\mu\sigma \end{bmatrix}\,,\; k^\mu \begin{bmatrix} j_\mu \\ \partial_\mu\sigma \end{bmatrix} \right\} \in \mathrm{Ker}(1-M[\varpi])\,, \\
& D - i\sigma((iH) + k^\mu V_\mu) = 0\ . \qquad\qquad (7.45)
\end{aligned}
$$

The last condition is correctly invariant under the shift symmetry (2.6). Assuming $\tilde{k}^\mu V_\mu = 0$, the boundary conditions for j_μ and $\partial_\mu\sigma$ are arranged as those of a neutral chiral multiplet.

7.3 Closure under supersymmetry

As we noted in subsection 6.2 the boundary conditions may transform non-trivially under supersymmetry. We would like to know if the boundary conditions that were formulated above are invariant under the A-type supersymmetries, and if not, whether invariance can be restored by imposing further constraints. Since the boundary conditions on the fermions are algebraic, it is immediately possible to examine how things work in some generality. In particular, when the matrix M is field independent we find that supersymmetry invariance of the fermion boundary conditions does not impose any new constraints.

In contrast, the analysis of the transformation of the boson boundary conditions is more involved and case-dependent. Since the boson boundary conditions involve derivatives of the bosons, their transformation leads to expressions that involve derivatives of the corresponding fermions. The details of the resulting expressions depend on the specifics of the differential operators and, in general, have to be analyzed case by case. For that reason, and in order to keep the discussion as generic as possible, in what follows we will concentrate mostly on the transformation properties of the fermion boundary conditions.

Chiral and anti-chiral multiplets. The supersymmetry transformations of the fermions $(\psi, \widetilde{\psi})$ in a chiral multiplet are

$$
\begin{aligned}
\delta\psi_\alpha &= +\vartheta\ F\,\zeta_\alpha + i\,\widetilde{\vartheta}\ \Big[\Big(k^\mu\mathcal{D}_\mu\phi - ir(iH)\phi\Big)\tilde{\zeta}_\alpha - (\widetilde{U}^\mu\mathcal{D}_\mu\phi)\zeta_\alpha\Big]\,, \\
\delta\widetilde{\psi}_\alpha &= +\widetilde{\vartheta}\ \widetilde{F}\,\tilde{\zeta}_\alpha + i\,\vartheta\ \Big[\Big(k^\mu\mathcal{D}_\mu\widetilde{\phi} + ir(iH)\widetilde{\phi}\Big)\zeta_\alpha - (U^\mu\mathcal{D}_\mu\widetilde{\phi})\tilde{\zeta}_\alpha\Big]\ .
\end{aligned} \qquad (7.46)
$$

For A-type supersymmetries, we set $\theta = \widetilde{\theta}$. We want to examine how A-type supersymmetry transforms the boundary conditions $e^{i\varpi}\tilde{\zeta}\mathbf{\Psi} = M\,\mathbf{\Psi}\zeta$. Assuming for simplicity that the matrix M is invariant we only need to consider the bilinears $\delta\mathbf{\Psi}\zeta$, $\tilde{\zeta}\delta\mathbf{\Psi}$. Straightforward manipulations yield the scalar products

$$
\delta\mathbf{\Psi}\zeta = +i\frac{1+iJ}{2}\Big(k^\mu\mathcal{D}_\mu + r(iH)J\Big)\mathbf{\Phi} + \frac{1-iJ}{2}\mathbf{F} - i\frac{1-iJ}{2}e^{+i\varpi}(n^\mu - ik^\mu)\mathcal{D}_\mu\mathbf{\Phi}\,, \qquad (7.47)
$$

$$
\tilde{\zeta}\delta\mathbf{\Psi} = +i\frac{1-iJ}{2}\Big(k^\mu\mathcal{D}_\mu + r(iH)J\Big)\mathbf{\Phi} + \frac{1+iJ}{2}\mathbf{F} + i\frac{1+iJ}{2}e^{-i\varpi}(n^\mu + ik^\mu)\mathcal{D}_\mu\mathbf{\Phi}\ . \qquad (7.48)
$$

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Consequently, the condition $e^{i\varpi}\tilde{\zeta}\delta\mathbf{\Psi} = M\,\delta\mathbf{\Psi}\zeta$ holds if the following equations are satisfied

$$\begin{aligned}(1+iJ)(1-M\mathscr{R}[\varpi])\Big(k^\mu\mathcal{D}_\mu + r(iH)J\Big)\mathbf{\Phi} &= 0\,,\\ (1-iJ)(1+M\mathscr{R}[\varpi])(n^\mu\mathcal{D}_\mu\mathbf{\Phi} - J\tilde{k}^\mu\mathcal{D}_\mu\mathbf{\Phi}) &= 0\,,\\ (1-iJ)(1-M\mathscr{R}[-\varpi])\mathbf{F} &= 0\ .\end{aligned} \tag{7.49}$$

In these formulae we recognize the boundary conditions that we derived previously for the bosons. As a minor difference comparing (7.49) to the original boundary condition (7.11a), we notice a sign change in front of the term $J\tilde{k}^\mu\mathcal{D}_\mu\mathbf{\Phi}$. This sign difference, however, is irrelevant in the final boundary conditions, since the two terms in the second equation in (7.49) have to vanish independently. We note that both $n^\mu\mathcal{D}_\mu\mathbf{\Phi}$ and $J\tilde{k}^\mu\mathcal{D}_\mu\mathbf{\Phi}$ belong in $\mathrm{Ker}(1+M[\varpi])$ and their relative normalization is not fixed by the boundary conditions.

We conclude that the supersymmetry invariance of the fermion boundary conditions does not impose any new constraints when M is separately invariant. In the more general case of a field dependent M one needs to consider extra contributions from the supersymmetric variation of M.

Finally, regarding the variation of the bosons at the boundary, it is possible to prove in complete generality the orthogonality condition $\delta\mathbf{\Phi}\,G\,\mathcal{D}^\perp\mathbf{\Phi} = 0$. From the A-type supersymmetry, the definition of $\delta\phi$ and $\delta\widetilde{\phi}$, and the boundary condition on the spinors $e^{i\varpi}\tilde{\zeta}\mathbf{\Psi} = \mathbf{\Psi}\zeta$, we deduce the boundary variation

$$\delta\mathbf{\Phi} = \frac{1}{2}\Big((1+iJ) + (1-iJ)e^{-i\varpi}M\Big)\epsilon\mathbf{\Psi} = P_M^{(\varpi,+)}\frac{1+iJ}{2}\epsilon\mathbf{\Psi}\ . \tag{7.50}$$

Therefore, $P_M^{(\varpi,-)}\delta\mathbf{\Phi} = 0$ and $\delta\mathbf{\Phi}$ belongs to $\mathrm{Ker}(1-M[\varpi])$. From the orthogonality of the two kernels $\mathrm{Ker}(1\pm M[\varpi])$, the condition $\delta\mathbf{\Phi}\,G\,\mathcal{D}^\perp\mathbf{\Phi} = 0$ follows. Let us notice that on-shell this orthogonality condition corresponds to $\mathbb{B}^s[\delta\mathbf{\Phi},\mathbf{\Phi}] = 0$ (see (6.16)).

Real multiplets. The supersymmetry transformations $\delta\psi_\Sigma$ and $\delta\widetilde{\psi}_\Sigma$ in a real multiplet are very similar to the ones of the chiral fermions $\delta\psi$ and $\delta\widetilde{\psi}$. The only difference is the contribution of the D-term

$$\begin{aligned}\delta\psi_\Sigma &= \vartheta\left[\big[D - i\sigma(iH + k^\mu V_\mu) - ik^\mu(j_\mu + i\partial_\mu\sigma)\big]\tilde{\zeta} + ie^{-i\varpi}(n^\mu + i\tilde{k}^\mu)(a_\mu - i\partial_\mu\sigma)\zeta\right],\\ \delta\widetilde{\psi}_\Sigma &= \vartheta\left[\big[D - i\sigma(iH + k^\mu V_\mu) - ik^\mu(j_\mu - i\partial_\mu\sigma)\big]\zeta - ie^{+i\varpi}(n^\mu - i\tilde{k}^\mu)(a_\mu + i\partial_\mu\sigma)\tilde{\zeta}\right]\ .\end{aligned}$$

Repeating the evaluation of $e^{i\varpi}\tilde{\zeta}\delta\mathbf{\Psi}_\Sigma = M\,\delta\mathbf{\Psi}_\Sigma\zeta$ we obtain results similar to the chiral multiplet case. Also in this case supersymmetry invariance of the fermion boundary conditions does not impose any further constraints.

8 Boundary conditions II

In this section we study A-type boundary conditions in general (non-abelian) $\mathcal{N} = 2$ supersymmetric CS/YM-matter theories. The corresponding actions on curved backgrounds

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and their supersymmetric variations $\mathscr{V}^\mu$ were obtained in section 5.2. Our analysis recovers previously known results in special cases, e.g. flat space, and extends them to general A-type backgrounds $\mathcal{T}$, in particular backgrounds with a solid torus topology.

We discuss first the conditions arising from the supersymmetric variation in the gauge sector. The corresponding conditions in the matter sector are presented separately.

8.1 Gauge sector

8.1.1 Description and summary of results

From the supersymmetric variation of the Yang-Mills and Chern-Simons actions, respectively, we obtain the boundary terms

$$\begin{aligned}\mathbf{e}^2\,\mathscr{V}^\perp_{YM} = \mathrm{Tr}\Big[&-\frac{i}{4}\epsilon\,\gamma^\perp\gamma^\rho\psi_\Sigma\,(\hat{\mathcal{F}}_\rho + 2i\sigma V_\rho) + \frac{i}{2}\epsilon\,\gamma^\perp\gamma^\rho\psi_\Sigma\,\partial_\rho\sigma \\ &+ \frac{i}{4}\tilde{\epsilon}\,\gamma^\perp\gamma^\rho\widetilde{\psi}_\Sigma\,(\hat{\mathcal{F}}_\rho + 2i\sigma V_\rho) + \frac{i}{2}\tilde{\epsilon}\,\gamma^\perp\gamma^\rho\widetilde{\psi}_\Sigma\,\partial_\rho\sigma \\ &+ \frac{1}{2}\epsilon\,\gamma^\perp\psi_\Sigma\,(iD + \sigma(iH)) + \frac{1}{2}\tilde{\epsilon}\,\gamma^\perp\widetilde{\psi}_\Sigma\,(iD + \sigma(iH))\Big]\,,\end{aligned} \tag{8.1}$$

$$\mathscr{V}^\perp_{CS} = +\frac{i}{4\pi}\kappa_{\mathbf{ac}}\left[\varepsilon^{\perp\nu\rho}\,(\epsilon\gamma_\rho\psi^{\mathbf{a}}_\Sigma - \tilde{\epsilon}\gamma_\rho\widetilde{\psi}^{\mathbf{a}}_\Sigma)\mathcal{A}^{\mathbf{c}}_\nu + 2(\epsilon\gamma^\perp\psi^{\mathbf{a}}_\Sigma + \tilde{\epsilon}\gamma^\perp\widetilde{\psi}^{\mathbf{a}}_\Sigma)\sigma^{\mathbf{c}}\right]\,. \tag{8.2}$$

In the gauge sector analysis we will also include a term coming from the vector-matter couplings

$$\mathscr{V}^\perp_{\mathrm{matter}} \supset -i\langle\widetilde{\phi}\,\mathbf{t}^a\phi\rangle\left[\epsilon\gamma^\perp\psi^a_\Sigma + \tilde{\epsilon}\gamma^\perp\widetilde{\psi}^a_\Sigma\right]\,. \tag{8.3}$$

$\{\mathbf{t}^a\}$ is a basis for the generators of the gauge groups in play, and $\langle\phi\,\mathbf{t}^a\widetilde{\phi}\rangle$ denotes the action of the adjoint fermions $\lambda = \lambda^a\mathbf{t}^a$ and $\tilde{\lambda} = \tilde{\lambda}^a\mathbf{t}^a$ in the representation of each of the matter fields ϕ and $\widetilde{\phi}$. Recall that we use bold indices $\mathbf{a}$ to describe general quiver gauge theories. In the multi-index $\mathbf{a} = (a, m)$ m labels the nodes of the quiver theory, and a the generators of the gauge group $\mathfrak{S}_m$ at the node m. For any set $\{\mathbf{t}^a\}$ of generators we set $\mathrm{Tr}[\mathbf{t}^a\mathbf{t}^c] = G_{ac}$, and $\kappa_{\mathbf{ac}} = G_{ac}\otimes\kappa_{mn}$. For canonically normalized generators $G_{ac} = \delta_{ac}$.

The expressions in (8.1)–(8.3) are a collection of all the terms in $\mathscr{V}^\perp_{YM+matter}$ and $\mathscr{V}^\perp_{CS+matter}$ that are functions of the spinors ψ_Σ and $\widetilde{\psi}_\Sigma$ of the $\mathcal{N} = 2$ vector multiplets. In what follows, we refer to the sum of (8.1) and (8.3) as $\mathscr{V}^\perp_{gs\,YM}$, and the sum of (8.2) and (8.3) as $\mathscr{V}^\perp_{gs\,CS}$ (gs stands for gauge sector).

To analyze the supersymmetric variations $\mathscr{V}^\mu_{gs\,YM}$ and $\mathscr{V}^\mu_{gs\,CS}$ we need to disentangle the geometric and spinorial structures. This can be achieved, as before, by using the projectors $\mathscr{P}$, $\widetilde{\mathscr{P}}$, and the A-type projection on ϵ and $\tilde{\epsilon}$. On the anti-commuting spinors $\mathbf{\Psi}_\Sigma = (\psi_\Sigma, \widetilde{\psi}_\Sigma)$ we impose the general boundary condition

$$e^{i\varpi}\tilde{\zeta}\mathbf{\Psi}_\Sigma = M\,\mathbf{\Psi}_\Sigma\zeta\,. \tag{8.4}$$

Supersymmetry will soon fix some of the properties of the matrix M, as we found for the non-linear sigma model in section 7. Nevertheless, the case of the non-linear sigma model and the case of the general gauge theory discussed here exhibit conceptually different properties. Let us highlight the origin of these differences.

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In the boundary condition (8.4), the spinors of the vector multiplets ψ_Σ, $\widetilde{\psi}_\Sigma$ have been arranged conveniently as a doublet $\mathbf{\Psi}_\Sigma$. The same doublet can be formed in non-linear sigma models out of the fermions in the chiral superfields. In that case we can also form naturally a corresponding doublet of bosons $\mathbf{\Phi} = (\phi, \widetilde{\phi})$. This is also possible for abelian real superfields, where the role of $\mathbf{\Phi}$ is played by the complex combination of the dual photon and the real scalar σ. In the case of a non-abelian gauge theory, however, there is no obvious natural bosonic $\mathbf{\Phi}$ that we can associate to $\mathbf{\Psi}_\Sigma$. As a result, we cannot proceed identically to the non-linear sigma model case thinking in terms of a generalized target space structure on the gauge indices.

An alternative approach is suggested by the 2-form

$$\mathbb{B}^v[\mathcal{V}, \delta\mathcal{A}] = G_{\mathbf{mn}}\, \varepsilon^{\perp\rho\nu}\, \mathcal{V}_\rho^{\mathbf{m}}\, \delta\mathcal{A}_\nu^{\mathbf{n}}\,, \tag{8.5}$$

that appears in the on-shell boundary value problem for vectors. In (8.5) both $\mathcal{V}$ and $\delta\mathcal{A}$ are 1-forms on the boundary. For example, $\mathbb{B}^v$ appears in the Euler-Lagrange variation of CS theories, (6.13), as well as in the supersymmetric variation $\mathscr{V}^\perp_{CS}$, (5.39),

$$\mathbb{B}^v[\delta\mathcal{A}, \mathcal{A}] \;\propto\; \kappa_{\mathbf{ac}}\, \varepsilon^{\perp\nu\rho}\, \delta\mathcal{A}_\rho^{\mathbf{a}} \mathcal{A}_\nu^{\mathbf{c}} \;=\; \kappa_{\mathbf{ac}}\, \varepsilon^{\perp\nu\rho}\, (\tilde{\epsilon}\gamma_\rho \widetilde{\psi}_\Sigma^{\mathbf{a}} - \epsilon\gamma_\rho \psi_\Sigma^{\mathbf{a}}) \mathcal{A}_\nu^{\mathbf{c}}\,. \tag{8.6}$$

In this equation $\mathbb{B}^v$ couples the two components of $\mathcal{A}_\nu$ in the boundary directions to a combination of the spinors. It is therefore natural to think in terms of doublets distinguished by the spacetime indices of vectors parallel to the boundary.

Similar manipulations can be employed in $\mathscr{V}^\perp_{YM}$ using the identity $\gamma^\mu\gamma^\nu = g^{\mu\nu} + \gamma^{\mu\nu}$ to rewrite the kinetic terms as follows

$$\mathscr{V}^\perp_{YM} \supset \; +\tfrac{1}{4\mathrm{e}^2}\, G_{ac}\, \varepsilon^{\perp\nu\rho}\, (\epsilon\gamma_\rho \psi_\Sigma^a - \tilde{\epsilon}\gamma_\rho \widetilde{\psi}_\Sigma^a) \hat{\mathcal{F}}_\nu^c \;\; -\tfrac{i}{4\mathrm{e}^2}\, G_{ac}\, (\epsilon\psi_\Sigma^a - \tilde{\epsilon}\widetilde{\psi}_\Sigma^a)\, g^{\perp\nu} \hat{\mathcal{F}}_\nu^c \tag{8.7a}$$

$$-\tfrac{1}{2\mathrm{e}^2}\, G_{ac}\, \varepsilon^{\perp\nu\rho} (\epsilon\gamma_\rho \psi_\Sigma^a + \tilde{\epsilon}\gamma_\rho \widetilde{\psi}_\Sigma^a) \mathcal{D}_\nu \sigma^c \;\; +\tfrac{i}{2\mathrm{e}^2}\, G_{ac}\, (\epsilon\psi_\Sigma^a + \tilde{\epsilon}\widetilde{\psi}_\Sigma^a)\, g^{\perp\nu} \mathcal{D}_\nu \sigma^c\,. \tag{8.7b}$$

The first two terms in (8.7a) and (8.7b) have the same structure as $\mathbb{B}^v$ in (8.5) (up to a difference in $\pm$ signs).

Introducing the notation

$$\mathfrak{I}_\pm(\mathbf{\Psi}, \mathcal{V}) \equiv \frac{1}{2} \varepsilon^{\perp\nu\rho} \big[\epsilon\gamma_\rho \psi^{\mathbf{a}} \pm \tilde{\epsilon}\gamma_\rho \widetilde{\psi}^{\mathbf{a}}\big] G_{\mathbf{ac}} \mathcal{V}_\nu^{\mathbf{c}} \tag{8.8}$$

we show in the next subsection (see eqs. (8.25), (8.28)) that $\mathfrak{I}_\pm(\mathbf{\Psi}_\Sigma, \mathcal{V})$ is closely related to $\mathbb{B}^v[\epsilon\mathbf{\Psi}_\Sigma, P_U\mathcal{V}]$, where P_U is a certain projector depending on a matrix U that has only gauge indices and satisfies $U^T G U = G$ and $U^2 = 1$. The interplay between A-type supersymmetry and the geometry of the form $\mathbb{B}^v$ fixes the relation between U and M by setting

$$M = 1_{2\times 2} \otimes U = \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}, \qquad e^{i\varpi} \tilde{\zeta} \psi_\Sigma^a = U_c^a \psi_\Sigma^c \zeta\,, \qquad e^{i\varpi} \tilde{\zeta} \widetilde{\psi}_\Sigma^a = U_c^a \widetilde{\psi}_\Sigma^c \zeta\,. \tag{8.9}$$

Note that unlike the boundary conditions in the non-linear sigma model case, (6.9), (7.15), in (8.9) the gauge indices of ψ_Σ and $\widetilde{\psi}_\Sigma$ do not mix.

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With these boundary conditions and the standard, by now, manipulations on spinor bilinears we arrive at compact expressions for $\mathscr{V}^{\perp}_{gsCS}$ and $\mathscr{V}^{\perp}_{gsYM}$. In order to keep the notation simple and most transparent, let us quote the pertinent results in the case of a single gauge group. In Chern-Simons-matter theories we obtain

$$\mathscr{V}^{\perp}_{gsCS} = \frac{\kappa}{2\pi}\left[(P_U\mathcal{A})^a\, G_{ac} - i\left(\sigma^a G_{ac} - \frac{2\pi}{\kappa}\langle\phi\,\mathbf{t}^a\widetilde{\phi}\rangle\right) U^a_c\right]\left[\epsilon\psi^c_\Sigma + \tilde{\epsilon}\widetilde{\psi}^c_\Sigma\right], \tag{8.10}$$

where $P_U\mathcal{A} = \tilde{k}^\mu\mathcal{A}_\mu + iUk^\mu\mathcal{A}_\mu$. In the case of Yang-Mills theories

$$\begin{aligned}\mathscr{V}^{\perp}_{gsYM} = &+\frac{i}{2\mathbf{e}^2}\Bigg[-\frac{i}{2}\left(P_U\hat{\mathcal{F}}\right)^a G_{ac} + \mathcal{D}_\perp\sigma^a\, G_{ac} \\ &\qquad - U^a_{\ b}\big(D^b - i\sigma^b(\,iH + k^\mu V_\mu\,)\big)G_{ac} + 2\mathbf{e}^2\, U^a_{\ c}\langle\phi\,\mathbf{t}^a\widetilde{\phi}\rangle\Bigg]\left[\epsilon\psi^c_\Sigma + \tilde{\epsilon}\widetilde{\psi}^c_\Sigma\right] \\ &+\frac{1}{2\mathbf{e}^2}\, G_{ac}\left[j^a_\perp - (P_U\mathcal{D}\sigma)^a\right]\left[\epsilon\psi^c_\Sigma - \tilde{\epsilon}\widetilde{\psi}^c_\Sigma\right],\end{aligned} \tag{8.11}$$

where

$$P_U\hat{\mathcal{F}} = \tilde{k}^\mu(\hat{\mathcal{F}}_\mu + 2i\sigma V_\mu) + iUk^\mu\hat{\mathcal{F}}_\mu\,, \qquad P_U\mathcal{D}\sigma = \tilde{k}^\mu\mathcal{D}_\mu\sigma + iUk^\mu\mathcal{D}_\mu\sigma\,, \tag{8.12}$$

$$j_\perp = -\frac{i}{2}\varepsilon_{\perp\nu\rho}\mathcal{F}^{\nu\rho} = -\frac{i}{2}\hat{\mathcal{F}}_\perp\ . \tag{8.13}$$

When the gauge group has an abelian component, a FI term can also be added to the Lagrangian. Since the variation of this term is of the type

$$\mathscr{V}^{\perp}_{FI} = +\frac{1}{2}\xi(\epsilon\gamma^\perp\tilde{\lambda} - \tilde{\epsilon}\gamma^\perp\lambda) = -\frac{i}{2}\xi\left[\epsilon\gamma^\mu\widetilde{\psi}_\Sigma + \tilde{\epsilon}\gamma^\mu\psi_\Sigma\right], \tag{8.14}$$

we can easily include the FI parameters in (8.11) by considering the shift $D \to D - \xi$.

In summary, without assuming any further constraints on the spinors $\mathbf{\Psi}_\Sigma$ other than (8.9), the most generic boundary conditions on the bosonic fields of the gauge multiplet are

$$\text{CS}-\text{theories}: \quad P_U\mathcal{A} - iU\left(\sigma - \frac{2\pi}{\kappa}\langle\phi\,\mathbf{t}\,\widetilde{\phi}\rangle\right) = 0\,, \tag{8.15}$$

$$\text{YM}-\text{theories}: \quad D_\perp\sigma - \tfrac{i}{2}\,P_U\hat{\mathcal{F}} - U\big(D - i\sigma(\,iH + k^\mu V_\mu\,)\big) + 2\mathbf{e}^2\,U\langle\phi\,\mathbf{t}\,\widetilde{\phi}\rangle = 0 \tag{8.16}$$

$$j_\perp - P_U\mathcal{D}\sigma = 0\ . \tag{8.17}$$

As a special solution, one can further impose $P_U\mathcal{V} = 0$ both in CS and YM theories. In the next subsection we show that this is equivalent to requiring $\mathfrak{I}_\pm = 0$. This projection, which is natural from the point of view of the Euler-Lagrange variations in Chern-Simons theory (6.15), selects a Lagrangian submanifold of $\mathbb{B}^v$, as we explain in the next section. The remaining conditions yield:

- In the case of CS theory, (8.15) reduces to the algebraic equation of motion of the auxiliary field D,

$$\delta\mathscr{L}_{\text{CS}-\text{matter}} \supset \left(-\frac{\kappa}{2\pi}\sigma^a + \langle\phi\,\mathbf{t}^a\widetilde{\phi}\rangle\right)\delta D^a = 0\ . \tag{8.18}$$

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- In the case of YM, the condition $\hat{\mathcal{F}}_\perp = 0$ translates into $\varepsilon_{\perp\mu\nu}\mathcal{F}^{\mu\nu} = 0$, where the free indices are constrained to run over the boundary indices by anti-symmetry. Then, $\hat{\mathcal{F}}_\perp = 0$ is satisfied if the non-abelian connection is flat at the boundary, namely $\mathcal{F} = 0$ at the boundary. In components, the boundary condition on σ becomes

$$\partial_\perp \sigma^a - i[\mathcal{A}_\perp, \sigma]^a = U^a{}_b\big(D^b - i\sigma^b(\,iH + k^\mu V_\mu\,)\big) - 2\mathbf{e}^2\langle \phi\, U^a{}_b \mathbf{t}^b \widetilde{\phi}\rangle\ . \tag{8.19}$$

The contribution from a FI term (5.40) can be incorporated by a shift of D.

8.1.2 Technical details

Let us elaborate further on the details that led to the above boundary conditions. The key quantity is $\mathfrak{I}_\pm(\mathbf{\Psi}, \mathcal{V})$ defined in (8.8). We re-express this quantity using the A-type projection on ϵ and $\tilde{\epsilon}$. Leaving the label Σ of the spinors implicit, the resulting expression is

$$\begin{aligned}&\mathfrak{I}_\pm(\mathbf{\Psi}, \mathcal{V}) = \\ &\quad = +\tfrac{1}{2}\tfrac{\tilde{\zeta}\epsilon}{\Omega}\ \varepsilon^{\perp\nu\rho}\tilde{k}_\rho k_\nu \Big[\big[-e^{i\varpi}\tilde{\zeta}\psi^{\mathbf{m}} \mp e^{-i\varpi}\widetilde{\psi}^{\mathbf{m}}\zeta\big]\,(i\mathcal{V}_k) - \big[+\psi^{\mathbf{m}}\zeta \pm \tilde{\zeta}\widetilde{\psi}^{\mathbf{m}}\big]\,\mathcal{V}^{\mathbf{n}}_{\tilde{k}}\Big] G_{\mathbf{mn}}\,,\end{aligned} \tag{8.20}$$

or equivalently in matrix notation (with $\varepsilon^{\perp\nu\rho}\tilde{k}_\rho k_\nu = +1$)

$$\mathfrak{I}_\pm = +\frac{1}{2}\frac{\tilde{\zeta}\epsilon}{\Omega}\ G_{\mathbf{mn}}\ \begin{pmatrix} -e^{i\varpi}\tilde{\zeta}\psi^{\mathbf{m}} \mp e^{-i\varpi}\widetilde{\psi}^{\mathbf{m}}\zeta \\ +\psi^{\mathbf{m}}\zeta \pm \tilde{\zeta}\widetilde{\psi}^{\mathbf{m}} \end{pmatrix}^T \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{V}^{\mathbf{n}}_{\tilde{k}} \\ (i\mathcal{V}^{\mathbf{n}}_k) \end{pmatrix}\ . \tag{8.21}$$

It is clear that $\mathfrak{I}_\pm$ is similar in form to $\mathbb{B}^v$ evaluated on specific complex combinations of the components of $\mathcal{V}$ and the spinors. We mentioned in section 6.1.2 that the most general solution to the equations $\mathbb{B}^v = 0$, are the Lagrangian submanifolds of the two-form $\mathbb{B}^v$. In special cases the general A-type boundary conditions (8.15)–(8.17) are solved by these Lagrangian submanifolds. We proceed to examine this aspect more closely.

Starting with $\mathfrak{I}_-$, which appears in the CS case, we notice that we can rewrite the fermions in (8.21) as follows

$$\begin{aligned}&\begin{pmatrix} -e^{i\varpi}\tilde{\zeta}\psi^{\mathbf{m}} + e^{-i\varpi}\widetilde{\psi}^{\mathbf{m}}\zeta \\ +\psi^{\mathbf{m}}\zeta - \tilde{\zeta}\widetilde{\psi}^{\mathbf{m}} \end{pmatrix} = \\ &= -\begin{pmatrix} e^{+i\varpi/2} & 0 \\ 0 & e^{-i\varpi/2} \end{pmatrix} (\tilde{\zeta}\mathbf{\Psi})^{\mathbf{m}}\, e^{i\varpi/2} + \begin{pmatrix} e^{-i\varpi/2} & 0 \\ 0 & e^{+i\varpi/2} \end{pmatrix} \sigma_1\,(\mathbf{\Psi}\zeta)^{\mathbf{m}}\, e^{-i\varpi/2}\ .\end{aligned} \tag{8.22}$$

Then, imposing the boundary condition $e^{i\varpi}\tilde{\zeta}\mathbf{\Psi} = M\,\mathbf{\Psi}\zeta$, on the spinors ψ and $\widetilde{\psi}$ we obtain

$$\frac{1}{2}\frac{\tilde{\zeta}\epsilon}{\Omega}\begin{pmatrix} -e^{i\varpi}\tilde{\zeta}\psi^{\mathbf{m}} + e^{-i\varpi}\widetilde{\psi}^{\mathbf{m}}\zeta \\ +\psi^{\mathbf{m}}\zeta - \tilde{\zeta}\widetilde{\psi}^{\mathbf{m}} \end{pmatrix} = -\frac{1}{2}\left(1 - \sigma_1\, e^{+i\sigma_3\frac{\varpi}{2}}\, M^{-1} e^{-i\sigma_3\frac{\varpi}{2}}\right) e^{+i\sigma_3\frac{\varpi}{2}}\,(\epsilon\mathbf{\Psi})\, e^{i\varpi/2}\ .$$

The last expression can be written as a projector P^-_M acting on $\epsilon\mathbf{\Psi}$ with

$$P^\pm_M \equiv +\frac{1}{2}\left(1 \pm \sigma_1\, e^{+i\sigma_3\frac{\varpi}{2}}\, M^{-1} e^{-i\sigma_3\frac{\varpi}{2}}\right)\ . \tag{8.23}$$

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The matrix M, which acts on the doublet $\mathbf{\Psi} = (\psi^{\mathbf{m}}, \widetilde{\psi}^{\mathbf{m}})$, is of the general form $M = R_{(2\times 2)} \otimes U$, where U acts on the gauge indices and R is a 2-by-2 matrix. $P_M^{\pm}$ is a projector only if $R = \pm 1$. Choosing $R = +1$ for concreteness, (the $R = -1$ choice is very similar), P_M becomes

$$P_U^{\pm} = +\frac{1}{2}\left(1 \pm \sigma_1 \otimes U\right) \ , \tag{8.24}$$

and the matrix U is required to be orthogonal with respect to G, and to satisfy $U^2 = 1$. The quantity $\mathfrak{I}_-$ takes the final form

$$\mathfrak{I}_- = +e^{i\varpi/2}\, e^{+i\sigma_3 \frac{\varpi}{2}} \begin{pmatrix} \epsilon\psi^{\mathbf{m}'} \\ \epsilon\widetilde{\psi}^{\mathbf{m}'} \end{pmatrix}^T (i\sigma_2) \begin{pmatrix} 1 & U^{\mathbf{m}}_{\mathbf{m}'} \\ U^{\mathbf{m}}_{\mathbf{m}'} & 1 \end{pmatrix} G_{\mathbf{mn}} \begin{pmatrix} \mathcal{V}^{\mathbf{n}}_{\tilde{k}} \\ i\mathcal{V}^{\mathbf{n}}_{k} \end{pmatrix} \ , \tag{8.25}$$

$$(i\sigma_2) = \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \ . \tag{8.26}$$

Since U is orthogonal with respect to G, the condition $\mathfrak{I}_- = 0$ can be achieved by setting

$$\mathcal{V}^{\mathbf{n}}_{\tilde{k}} + U^{\mathbf{n}}_{\mathbf{c}}\ i\mathcal{V}^{\mathbf{c}}_{k} = 0 \ . \tag{8.27}$$

Notice that the dependence on the phase ϖ has disappeared in the above manipulations.

The calculation of $\mathfrak{I}_+$ proceeds along similar lines. In this case, the relevant combination of spinors in (8.21) can be recast as

$$\frac{1}{2}\frac{\tilde{\zeta}\epsilon}{\Omega} \begin{pmatrix} -e^{i\varpi}\tilde{\zeta}\psi^{\mathbf{m}} - e^{-i\varpi}\widetilde{\psi}^{\mathbf{m}}\zeta \\ +\psi^{\mathbf{m}}\zeta + \tilde{\zeta}\widetilde{\psi}^{\mathbf{m}} \end{pmatrix} = -\sigma_3(1 + \sigma_1\, e^{+i\sigma_3\frac{\varpi}{2}}\, M^{-1} e^{-i\sigma_3\frac{\varpi}{2}})\, e^{+i\sigma_3\frac{\varpi}{2}}\, (\epsilon\mathbf{\Psi})\, e^{i\varpi/2} \ ,$$

yielding the final expression

$$\mathfrak{I}_+ = +e^{i\varpi/2}\, e^{+i\sigma_3\frac{\varpi}{2}} \begin{pmatrix} \epsilon\psi^{\mathbf{m}'} \\ \epsilon\widetilde{\psi}^{\mathbf{m}'} \end{pmatrix}^T \sigma_3\,(i\sigma_2) \begin{pmatrix} 1 & U^{\mathbf{m}}_{\mathbf{m}'} \\ U^{\mathbf{m}}_{\mathbf{m}'} & 1 \end{pmatrix} G_{\mathbf{mn}} \begin{pmatrix} \mathcal{V}^{\mathbf{n}}_{\tilde{k}} \\ i\mathcal{V}^{\mathbf{n}}_{k} \end{pmatrix} \ . \tag{8.28}$$

In conclusion, with $R = +1$, both conditions $\mathfrak{I}_+ = 0$ and $\mathfrak{I}_- = 0$ lead to (8.27). Considering instead the choice $R = -1$ would lead to $\mathcal{V}^{\mathbf{n}}_{\tilde{k}} - U^{\mathbf{n}}_{\mathbf{c}}\ i\mathcal{V}^{\mathbf{c}}_{k} = 0$. Clearly, this choice is equivalent to the substitution $U \to -U$.

We noted in the previous subsection that by setting $\mathfrak{I}_\pm = 0$ in $\mathscr{V}^{\perp}_{CS}$ or $\mathscr{V}^{\perp}_{YM}$ we are led to a special solution of the boundary conditions (8.15)–(8.17) where $P_U\mathcal{V} = 0$:

- In CS theories, where $\mathcal{V} = \mathcal{A}$, this is equivalent to a single boundary condition on the gauge field

$$\mathcal{A}^{\mathbf{n}}_{\tilde{k}} + U^{\mathbf{n}}_{\mathbf{c}}\ i\mathcal{A}^{\mathbf{c}}_{k} = 0 \ . \tag{8.29}$$

- In YM theories, where $\mathcal{V} = \hat{\mathcal{F}}, \mathcal{D}\sigma$, one obtains two separate boundary conditions: one on the non-abelian field strength and another one on $\mathcal{D}_\mu\sigma$

$$\hat{\mathcal{F}}^{\mathbf{n}}_{\tilde{k}} + U^{\mathbf{n}}_{\mathbf{c}}\ i\hat{\mathcal{F}}^{\mathbf{c}}_{k} = \tilde{k}^\mu\mathcal{D}_\mu\sigma^{\mathbf{n}} + U^{\mathbf{n}}_{\mathbf{c}}\ ik^\mu\mathcal{D}_\mu\sigma^{\mathbf{c}} = 0 \ . \tag{8.30}$$

  These equations are natural covariant generalizations of corresponding boundary conditions in flat space that set components parallel to the boundary of the dual field strength $\hat{\mathcal{F}}_\mu$ and $\mathcal{D}_\mu\sigma$ to zero.

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8.2 Matter sector

Next we focus on terms that arise from the supersymmetric variation of the matter sector of the gauge theory. These terms are functions of the spinors ψ and $\widetilde{\psi}$ of the chiral and anti-chiral multiplet. The relevant boundary contributions can be summarized in the expression

$$\sqrt{2}\,\mathscr{V}^{\perp}_{\text{matter}} = +\epsilon\Big[\gamma^{\perp}\gamma^{\nu}\psi^{\mathbf{a}}\,\mathcal{D}_{\nu}\widetilde{\phi}^{\bar{\mathbf{c}}} - (r^{\bar{\mathbf{c}}}H - (z^{\bar{\mathbf{c}}} - q^{\bar{\mathbf{c}}}\sigma))\,\gamma^{\perp}\psi^{\mathbf{a}}\,\widetilde{\phi}^{\bar{\mathbf{c}}} - iV^{\perp}\psi^{\mathbf{a}}\widetilde{\phi}^{\bar{\mathbf{c}}} + i\,\gamma^{\perp}\widetilde{\psi}^{\bar{\mathbf{c}}}F^{\mathbf{a}}\Big]G_{\mathbf{a}\bar{\mathbf{c}}}$$
$$-\,\tilde{\epsilon}\Big[\gamma^{\perp}\gamma^{\nu}\widetilde{\psi}^{\bar{\mathbf{c}}}\,\mathcal{D}_{\nu}\phi^{\mathbf{a}} - (r^{\mathbf{a}}H - (z^{\mathbf{a}} - q^{\mathbf{a}}\sigma))\,\gamma^{\perp}\widetilde{\psi}^{\bar{c}}\,\phi^{\mathbf{a}} + iV^{\perp}\widetilde{\psi}^{\bar{c}}\phi^{\mathbf{a}} - i\,\gamma^{\perp}\psi^{\mathbf{a}}\widetilde{F}^{\bar{\mathbf{c}}}\Big]G_{\mathbf{a}\bar{\mathbf{c}}}\,. \quad (8.31)$$

The effects of a gauge invariant superpotential W can be incorporated, as already done in (5.14), by considering the on-shell relations

$$G_{a\bar{c}}\widetilde{F}^{\bar{c}} = \partial_a W\,, \qquad F^a G_{a\bar{c}} = \partial_{\bar{c}}\widetilde{W}\,. \quad (8.32)$$

In (8.31) the chiral and anti-chiral superfields transform in arbitrary representations of the gauge group. In the bold multi-indices $\mathbf{a} = (a, m)$, a is a color index and m a flavor index. The metric G is the scalar product in the combined flavor/color index space. In non-abelian theories σ acts on ϕ ($\widetilde{\phi}$) and ψ ($\widetilde{\psi}$) according to their representations. We will make a slight abuse of notation where the specifics of this action are suppressed.

By making use of the standard identity $\gamma^{\perp}\gamma^{\nu} = g^{\perp\nu} + i\,\varepsilon^{\perp\nu\rho}\gamma_{\rho}$ and the fact that $V^{\perp} = 0$ at the boundary, $\mathscr{V}^{\perp}_{\text{matter}}$ can be rewritten in the form:

$$\sqrt{2}\,\mathscr{V}^{\perp}_{\text{matter}} = +i\Big[\varepsilon^{\perp\nu\rho}\epsilon\gamma_{\rho}\psi^{\mathbf{a}}\mathcal{D}_{\nu}\widetilde{\phi}^{\bar{\mathbf{c}}} + (iH)\epsilon\gamma^{\perp}\psi^{\mathbf{a}}\,r^{\bar{\mathbf{c}}}\widetilde{\phi}^{\bar{\mathbf{c}}}\Big]G_{\mathbf{a}\bar{\mathbf{c}}}$$
$$+\Big[\epsilon\psi^{\mathbf{a}}\,\mathcal{D}^{\perp}\widetilde{\phi}^{\bar{\mathbf{c}}} + (z^{\bar{\mathbf{c}}} - q^{\bar{\mathbf{c}}}\sigma)\,\epsilon\gamma^{\perp}\psi^{\mathbf{a}}\,\widetilde{\phi}^{\bar{\mathbf{c}}}\Big]G_{\mathbf{a}\bar{\mathbf{c}}} + i\,\epsilon\gamma^{\perp}\widetilde{\psi}^{\bar{\mathbf{c}}}\,G_{\bar{\mathbf{c}}\mathbf{a}}\,F^{\mathbf{a}}$$
$$-i\Big[\varepsilon^{\perp\nu\rho}\tilde{\epsilon}\gamma_{\rho}\widetilde{\psi}^{\bar{\mathbf{c}}}\,\mathcal{D}_{\nu}\phi^{\mathbf{a}} + (iH)\tilde{\epsilon}\gamma^{\perp}\widetilde{\psi}^{\bar{\mathbf{c}}}\,r^{\mathbf{a}}\phi^{\mathbf{a}}\Big]G_{\bar{\mathbf{c}}\mathbf{a}}$$
$$-\Big[\tilde{\epsilon}\widetilde{\psi}^{\bar{\mathbf{c}}}\,\mathcal{D}^{\perp}\phi^{\mathbf{a}} + (z^{\mathbf{a}} - q^{\mathbf{a}}\sigma)\,\tilde{\epsilon}\gamma^{\perp}\widetilde{\psi}^{\bar{c}}\,\phi^{\mathbf{a}}\Big]G_{\bar{\mathbf{c}}\mathbf{a}} + i\,\tilde{\epsilon}\gamma^{\perp}\psi^{\mathbf{a}}\,G_{\mathbf{a}\bar{\mathbf{c}}}\widetilde{F}^{\bar{\mathbf{c}}}\,. \quad (8.33)$$

The analysis of $\mathscr{V}^{\perp}_{CS}$ and $\mathscr{V}^{\perp}_{YM}$ selected boundary conditions in the gauge sector on the basis of the two form $\mathbb{B}^{v}$. Even though $\mathscr{V}^{\perp}_{\text{matter}}$ can still be thought of as $\mathscr{V}^{\perp}_{NLs}$, on the basis of the flavor indices, in this subsection we will not follow the approach of section 7. Instead, we will explore the extension of the manipulations of the previous subsection 8.1 to the matter sector. Accordingly, we assume from the start the following boundary conditions on the matter fermions

$$\frac{\zeta\gamma^{\perp}\zeta}{\Omega}(\tilde{\zeta}\mathbf{\Psi}) = M(\mathbf{\Psi}\zeta)\,, \qquad M = \begin{pmatrix} S & 0 \\ 0 & \widetilde{S} \end{pmatrix}\,. \quad (8.34)$$

The matrices S and $\widetilde{S}$ act on the representation space of the matter. They are required to have the properties $S^2 = \widetilde{S}^2 = 1$, and $S^T G\widetilde{S} = G$. M acts diagonally on the doublet $\mathbf{\Psi} = (\psi^{\mathbf{a}}, \widetilde{\psi}^{\bar{\mathbf{c}}})$, i.e. representations do not mix. This is the same type of ansatz that emerged in the gauge sector. Here two possibly different S and $\widetilde{S}$ are allowed because of the two chiralities.

With standard manipulations of the spinor bilinears, we recast $\mathscr{V}^{\perp}_{\text{matter}}$ in terms of two independent spinor components $\epsilon\psi$ and $\tilde{\epsilon}\widetilde{\psi}$,

$$\sqrt{2}\,\mathscr{V}^{\perp}_{\text{matter}} = +\epsilon\psi^{\mathbf{a}}\,G_{\mathbf{a}\bar{\mathbf{c}}}\,\Big[iP_{\widetilde{S}}\,\mathcal{D}\widetilde{\phi}^{\bar{\mathbf{c}}} + \big(\mathcal{D}^{\perp}\widetilde{\phi}^{\bar{\mathbf{c}}} - \widetilde{S}^{\bar{\mathbf{c}}}_{\bar{\mathbf{n}}}(z^{\bar{\mathbf{n}}} - q^{\bar{\mathbf{n}}}\sigma)\widetilde{\phi}^{\bar{\mathbf{n}}}\big) + ie^{-i\varpi}\widetilde{F}^{\bar{\mathbf{c}}}\Big]$$

$$+\tilde{\epsilon}\widetilde{\psi}^{\bar{\mathbf{c}}}\, G_{\bar{\mathbf{c}}\mathbf{a}} \left[iP_S\, \mathcal{D}\phi^{\mathbf{a}} - \big(\mathcal{D}^{\perp}\phi^{\mathbf{a}} - S^{\mathbf{a}}_{\mathbf{m}}(z^{\mathbf{m}} - q^{\mathbf{m}}\sigma)\phi^{\mathbf{m}}\big) + ie^{i\varpi}F^{\mathbf{a}} \right] . \quad (8.35)$$

We defined

$$P_S\, \mathcal{D}\phi^{\mathbf{a}} \equiv \mathcal{D}_{\tilde{k}}\phi^{\mathbf{a}} + iS^{\mathbf{a}}_{\mathbf{m}}\, (\mathcal{D}_k\phi^{\mathbf{m}} - ir^{\mathbf{m}}(iH)\phi^{\mathbf{m}})\,, \quad (8.36)$$

$$P_{\widetilde{S}}\, \mathcal{D}\widetilde{\phi}^{\bar{\mathbf{c}}} \equiv \mathcal{D}_{\tilde{k}}\widetilde{\phi}^{\bar{\mathbf{c}}} + i\widetilde{S}^{\bar{\mathbf{c}}}_{\bar{\mathbf{n}}}\, (\mathcal{D}_k\widetilde{\phi}^{\bar{\mathbf{n}}} + ir^{\bar{\mathbf{n}}}(iH)\widetilde{\phi}^{\bar{\mathbf{n}}})\ . \quad (8.37)$$

The projectors P_S and $P_{\widetilde{S}}$ are the analog of P_U in the gauge sector. For matter charged under the R-symmetry, we see that the terms in $\mathscr{V}^{\perp}_{\text{matter}}$ proportional to the R-charges, $\pm r(iH)$, correctly combine with the covariant derivatives along the Killing vector.

The expression (8.35) allows us to read off the following general boundary conditions on the matter sector

$$\text{chiral}: \qquad iP_{\widetilde{S}}\, \mathcal{D}\widetilde{\phi}^{\bar{\mathbf{c}}} + \big(\mathcal{D}^{\perp}\widetilde{\phi}^{\bar{\mathbf{c}}} - \widetilde{S}^{\bar{\mathbf{c}}}_{\bar{\mathbf{n}}}(z^{\bar{\mathbf{n}}} - q^{\bar{\mathbf{n}}}\sigma)\widetilde{\phi}^{\bar{\mathbf{n}}}\big) + ie^{-i\varpi}\widetilde{F}^{\bar{\mathbf{c}}} = 0\,, \quad (8.38)$$

$$\text{anti}-\text{chiral}: \qquad iP_S\, \mathcal{D}\phi^{\mathbf{a}} - \big(\mathcal{D}^{\perp}\phi^{\mathbf{a}} - S^{\mathbf{a}}_{\mathbf{m}}(z^{\mathbf{m}} - q^{\mathbf{m}}\sigma)\phi^{\mathbf{m}}\big) + ie^{i\varpi}F^{\mathbf{a}} = 0\ . \quad (8.39)$$

A special solution of these boundary conditions is obtained by imposing the conditions $P_S\mathcal{D}\phi = P_{\widetilde{S}}\mathcal{D}\widetilde{\phi} = 0$. Then, setting to zero the remaining terms in (8.38), (8.39) we obtain

$$\mathcal{D}^{\perp}\widetilde{\phi} - \widetilde{S}(z - q\sigma)\widetilde{\phi} + ie^{-i\varpi}\widetilde{F} = 0\,, \qquad \mathcal{D}^{\perp}\phi - S(z - q\sigma)\phi - ie^{i\varpi}F = 0\ . \quad (8.40)$$

where the term $(z - q\sigma)$ in (8.40) corresponds to the standard real mass.

In euclidean space, we may consider $S = 1$, then $P_S\mathcal{D}\phi = 0$ would become $D_+\phi = 0$, with D_+ a covariantized holomorphic derivative along the coordinates of the boundary, which in this case would be a plane. The same is true for $P_{\widetilde{S}}\mathcal{D}\widetilde{\phi} = 0$ if $\widetilde{\phi}$ is regarded as a field independent of ϕ. Thus, $P_S\mathcal{D}\phi = P_{\widetilde{S}}\mathcal{D}\widetilde{\phi} = 0$ are the natural generalization to curved space of such boundary conditions.

The covariant derivatives in P_S and $P_{\widetilde{S}}$ contain both the dynamical gauge fields $\mathcal{A}$ and the R-symmetry connection $A^{(R)}_\mu$. Regarding the dependence on $A^{(R)}_\mu$, we may borrow part of the discussion in section 7.1.3 to understand the precise form of $\mathcal{D}_k$ and $\mathcal{D}_{\tilde{k}}$. As simple illustrating examples, let us consider the case of A-type backgrounds with twisted spinors. The twisted R-symmetry gauge field is such that $\mathcal{D}_k$ becomes

$$\mathcal{D}_k\phi^{\mathbf{m}} - ir^{\mathbf{m}}(iH)\phi^{\mathbf{m}} = k^\mu\partial_\mu\phi^{\mathbf{m}} - ik^\mu(\mathcal{A}_\mu\phi)^{\mathbf{m}}\ . \quad (8.41)$$

For the ellipsoid and the manifolds with SU(2) × U(1) symmetry that we introduced in section 3.5, we may also use $\tilde{k}^\mu A^{(R)}_\mu = 0$ at the equator to obtain $\mathcal{D}_{\tilde{k}}$ in the simplified form

$$\mathcal{D}_{\tilde{k}}\phi^{\mathbf{a}} = f\, k^\mu\partial_\mu\phi^{\mathbf{m}} + v^\mu\partial_\mu\phi^{\mathbf{m}} - i\tilde{k}^\mu(\mathcal{A}_\mu\phi)^{\mathbf{m}}\ . \quad (8.42)$$

In that case the boundary condition $P_S\mathcal{D}\phi = 0$ reads

$$P_S\mathcal{D}\phi = \big[f\, k^\mu\partial_\mu\phi + v^\mu\partial_\mu\phi - i\tilde{k}^\mu(\mathcal{A}_\mu\phi)\big] + iS\big[k^\mu\partial_\mu\phi - ik^\mu(\mathcal{A}_\mu\phi)\big] = 0\ . \quad (8.43)$$

A similar result holds for $P_{\widetilde{S}}\mathcal{D}\widetilde{\phi} = 0$.[21]

[21] The action of S on $(\mathcal{A}_\mu\phi)^{\mathbf{m}}$ and of $\widetilde{S}$ on $(\mathcal{A}_\mu\widetilde{\phi})^{\bar{\mathbf{n}}}$ should not be confused with the separate action of U that was defined in the gauge sector.

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The expressions (8.41)–(8.43) hold under a set of simplifying assumptions for the background fields. Let us also notice that the covariant derivative normal to the boundary, $\mathcal{D}^\perp$, simplifies under the additional assumption $A_\perp^{(R)} = 0$, and reads $\mathcal{D}^\perp = \partial^\perp - iq\mathcal{A}_\perp$. For a generic A-type background, the full convariant derivatives, including the R-symmetry gauge fields, should be considered.

Finally, we should stress that the boundary conditions $P_S\mathcal{D}\phi = 0$ and $P_{\widetilde{S}}\mathcal{D}\widetilde{\phi} = 0$ are genuinely complex. The reality condition on the bosonic fields of the chiral matter, $\widetilde{\phi} = \phi^\star$ and $\widetilde{F} = F^\star$, would impose a restricted set of boundary conditions. Considering $\widetilde{\phi} = \phi^\star$, we find either $\mathcal{D}_k\phi = \mathcal{D}_{\tilde{k}}\phi = 0$ or $S^\star = -\widetilde{S}$, provided the latter is compatible with $S^T G\widetilde{S} = G$. Higgs-like solutions can be defined as the solutions of $(z - q\sigma) = 0$, where the gauge group is broken to $\mathrm{U}(1)^{\mathrm{rank}\,\mathfrak{S}}$. Then, the boundary conditions $\mathcal{D}_k\phi = 0$ and $\mathcal{D}_{\tilde{k}}\phi = 0$ imply that ϕ is constant at the boundary, hence each U(1) can be Higgsed by the vevs of the charged matter fields. Together with $D^\perp\phi - ie^{i\varpi}F = 0$ and its complex conjugate, we recover the ordinary Dirichlet and Neumann[22] boundary conditions for chiral fields.

8.3 Closure under supersymmetry

We conclude the analysis of the above boundary conditions, both in the gauge and the matter sector, with a study of their transformation under supersymmetry. We already looked at this problem when we discussed the boundary conditions of Lagrangian branes, and similar comments continue to apply here. In particular, the variation of the boundary conditions on the fermions are algebraic, and it is immediate to check whether they are closed under supersymmetry or not. For CS theories, the variation of the boundary conditions on $\mathcal{A}_\mu$ and σ are also simple and both turn out to be algebraic. In YM theories, the boundary conditions on the bosons are boundary conditions on the derivatives, hence their analysis requires specific information about the details of the background.

8.3.1 Gauge sector

The boundary conditions on the fermions ψ_Σ and $\widetilde{\psi}_\Sigma$ are

$$e^{i\varpi}\,\tilde{\zeta}\psi_\Sigma^{\mathbf{a}} \;=\; U_{\mathbf{m}}^{\mathbf{a}}\,\psi_\Sigma^{\mathbf{m}}\zeta\,, \qquad e^{i\varpi}\,\tilde{\zeta}\widetilde{\psi}^{\mathbf{c}} \;=\; U_{\mathbf{n}}^{\mathbf{c}}\,\widetilde{\psi}^{\mathbf{n}}\zeta\,. \tag{8.44}$$

The supersymmetric variation of the fermions $\delta\psi_\Sigma$ and $\delta\widetilde{\psi}_\Sigma$ under the A-type supersymmetry, $\theta = \widetilde{\theta}$, is

$$\begin{aligned}
\delta\psi_\Sigma &= \vartheta\left[\big[D - i\sigma(iH + k^\mu V_\mu) - ik^\mu(j_\mu + i\partial_\mu\sigma)\big]\tilde{\zeta} + ie^{-i\varpi}(n^\mu + i\tilde{k}^\mu)(a_\mu - i\mathcal{D}_\mu\sigma)\zeta\right],\\
\delta\widetilde{\psi}_\Sigma &= \vartheta\left[\big[D - i\sigma(iH + k^\mu V_\mu) - ik^\mu(j_\mu - i\partial_\mu\sigma)\big]\zeta - ie^{+i\varpi}(n^\mu - i\tilde{k}^\mu)(a_\mu + i\mathcal{D}_\mu\sigma)\tilde{\zeta}\right],
\end{aligned}$$

and the conditions we would like to check (assuming the matrix U is invariant) are

$$e^{i\varpi}\,\tilde{\zeta}\delta\psi_\Sigma^{\mathbf{a}} \;=\; U_{\mathbf{m}}^{\mathbf{a}}\,\delta\psi_\Sigma^{\mathbf{m}}\zeta\,, \qquad e^{i\varpi}\,\tilde{\zeta}\delta\widetilde{\psi}^{\mathbf{c}} \;=\; U_{\mathbf{n}}^{\mathbf{c}}\,\delta\widetilde{\psi}^{\mathbf{n}}\zeta\,. \tag{8.45}$$

[22] In the presence of a superpotential, the Neumann condition $D^\perp\phi = 0$ (where $F = 0$) is eventually promoted to a 'domain wall' condition $D^\perp\phi - ie^{i\varpi}F = 0$, when F is integrated-out.

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Both conditions are satisfied if

$$
\begin{aligned}
&n^\mu j_\mu = \tilde{k}^\mu \mathcal{D}_\mu \sigma^{\mathbf{a}} + iU^{\mathbf{a}}_{\mathbf{m}}\, k^\mu \mathcal{D}_\mu \sigma^{\mathbf{m}} \ , \\
&n^\mu \mathcal{D}_\mu \sigma^{\mathbf{a}} + \tilde{k}^\mu (j^{\mathbf{a}}_\mu + \sigma^{\mathbf{a}} V_\mu) + iU^{\mathbf{a}}_{\mathbf{m}}\, k^\mu j^{\mathbf{m}}_\mu = U^{\mathbf{a}}_{\mathbf{m}} \big[D - i\sigma(iH + k^\mu V_\mu)\big]^{\mathbf{m}} \ .
\end{aligned}
\tag{8.46}
$$

Closer inspection reveals that in YM theories, (8.45) reduces to a subset of the boundary conditions that we found in section 8.1 and 8.1.2. Since the vector-matter couplings cannot appear in $\delta\psi_\Sigma$ and $\delta\widetilde{\psi}_\Sigma$, the conditions (8.45) cannot lead to the most general boundary conditions (8.16), (8.17). Instead, they lead to the boundary conditions

$$
n^\mu j_\mu = P_U \mathcal{D}\sigma \ , \qquad \mathcal{D}^\perp \sigma + \big(\tilde{k}^\mu (j_\mu + \sigma V_\mu) + iUk^\mu j_\mu\big) = U\big[D - i\sigma(iH + k^\mu V_\mu)\big] \ . \tag{8.47}
$$

In CS theories, consider the boundary condition $\mathcal{A}^{\mathbf{n}}_{\tilde{k}} + U^{\mathbf{n}}_{\mathbf{c}}\ i\mathcal{A}^{\mathbf{c}}_k = 0$. The condition on the fermions ψ_Σ and $\widetilde{\psi}_\Sigma$ is the same as in YM, and therefore (8.45) leads to additional constraints on D, and on the derivatives of $\mathcal{A}$ and σ, which are precisely given by (8.47). On the other hand, the variation of the gauge boson at the boundary is

$$
\delta\mathcal{A}_\mu = -\theta\Big[n_\mu(e^{i\varpi}\tilde{\zeta}\psi_\Sigma + e^{-i\varpi}\widetilde{\psi}_\Sigma\zeta) - i\tilde{k}^\mu(e^{i\varpi}\tilde{\zeta}\psi_\Sigma - e^{-i\varpi}\widetilde{\psi}_\Sigma\zeta) + k_\mu(\psi_\Sigma\zeta - \tilde{\zeta}\widetilde{\psi}_\Sigma)\Big] \ . \tag{8.48}
$$

Consequently, the special boundary condition $\delta\mathcal{A}^{\mathbf{n}}_{\tilde{k}} + U^{\mathbf{n}}_{\mathbf{c}}\ i\delta\mathcal{A}^{\mathbf{c}}_k = 0$ is trivially satisfied:

$$
\begin{aligned}
\delta\mathcal{A}^{\mathbf{n}}_{\tilde{k}} + U^{\mathbf{n}}_{\mathbf{c}}\ i\delta\mathcal{A}^{\mathbf{c}}_k &= -i(e^{i\varpi}\tilde{\zeta}\psi^{\mathbf{n}} - e^{-i\varpi}\widetilde{\psi}^{\mathbf{n}}\zeta) + iU^{\mathbf{n}}_{\mathbf{c}}\ (\psi^{\mathbf{c}}_\Sigma\zeta - \tilde{\zeta}\widetilde{\psi}^{\mathbf{c}}_\Sigma) \\
&= +i(e^{-i\varpi}\widetilde{\psi}^{\mathbf{n}}\zeta - U^{\mathbf{n}}_{\mathbf{c}}\tilde{\zeta}\widetilde{\psi}^{\mathbf{c}}_\Sigma) - i(e^{i\varpi}\tilde{\zeta}\psi^{\mathbf{n}} - U^{\mathbf{n}}_{\mathbf{c}}\ \psi^{\mathbf{c}}_\Sigma\zeta) = 0 \ .
\end{aligned}
\tag{8.49}
$$

Finally, the scalar field σ, which is auxiliary in CS theories, but dynamical in YM theories, exhibits the supersymmetric variation

$$
\delta\sigma = i\theta(\zeta\psi_\Sigma + \tilde{\zeta}\widetilde{\psi}_\Sigma) \ . \tag{8.50}
$$

We notice that the boundary conditions on the spinors relate the ζ and $\tilde{\zeta}$ component of each fermionic field, and thus do not fix (8.50). We could impose $\delta\sigma = 0$ by requiring

$$
\tilde{\zeta}\widetilde{\psi}_\Sigma = \psi_\Sigma\zeta \ . \tag{8.51}
$$

In that case, out of four fermionic variables (two for ψ_Σ and two for $\widetilde{\psi}_\Sigma$), the boundary conditions (8.45) would fix two in terms of the rest, and by imposing (8.51) only one would remain unconstrained.

8.3.2 Matter sector

The supersymmetric variation of the matter fermions $\delta\psi$ and $\delta\widetilde{\psi}$ under the A-type supersymmetry, $\theta = \widetilde{\theta}$, is

$$
\begin{aligned}
\delta\psi_\alpha &= +\vartheta\ F\,\zeta_\alpha + \vartheta\ \Big[i\Big(k^\mu\mathcal{D}_\mu\phi - ir(iH)\phi - (z - q\sigma)\phi\Big)\tilde{\zeta}_\alpha + i\Big(e^{-i\varpi}(n^\mu + i\tilde{k}^\mu)\mathcal{D}_\mu\phi\Big)\zeta_\alpha\Big] \ , \\
\delta\widetilde{\psi}_\alpha &= +\vartheta\ \widetilde{F}\,\tilde{\zeta}_\alpha + \vartheta\ \Big[i\Big(k^\mu\mathcal{D}_\mu\widetilde{\phi} + ir(iH)\widetilde{\phi} + (z - q\sigma)\widetilde{\phi}\Big)\zeta_\alpha - i\Big(e^{+i\varpi}(n^\mu - i\tilde{k}^\mu)\mathcal{D}_\mu\widetilde{\phi}\Big)\tilde{\zeta}_\alpha\Big] \ .
\end{aligned}
$$

The conditions we want to check are in this case

$$e^{i\varpi}\,\tilde{\zeta}\delta\psi^{\mathbf{a}}_{\Sigma} \;=\; S^{\mathbf{a}}_{\mathbf{m}}\,\delta\psi^{\mathbf{m}}_{\Sigma}\zeta\,, \qquad e^{i\varpi}\,\tilde{\zeta}\delta\widetilde{\psi}^{\bar{\mathbf{c}}} \;=\; \widetilde{S}^{\bar{\mathbf{c}}}_{\bar{\mathbf{n}}}\,\delta\widetilde{\psi}^{\mathbf{n}}\zeta\;. \tag{8.52}$$

A short calculation leads to the constraints

$$iP_S\mathcal{D}\phi + \big(\mathcal{D}^{\perp}\phi + S(z-q\sigma)\phi\big) - ie^{+i\varpi}F \;=\; 0\,, \tag{8.53}$$

$$iP_{\widetilde{S}}\mathcal{D}\widetilde{\phi} - \big(\mathcal{D}^{\perp}\widetilde{\phi} - \widetilde{S}(z-q\sigma)\widetilde{\phi}\big) - ie^{-i\varpi}\widetilde{F} \;=\; 0\;. \tag{8.54}$$

Compare these formulae with the boundary conditions (8.38) and (8.39). The two sets of conditions do not coincide, because several signs do not match. They hold simultaneously under the restriction,

$$P_S\mathcal{D}\phi = P_{\widetilde{S}}\mathcal{D}\widetilde{\phi} \;=\; 0\,, \tag{8.55}$$

$$(z-q\sigma)\phi = (z-q\sigma)\widetilde{\phi} \;=\; 0\,, \tag{8.56}$$

$$\mathcal{D}^{\perp}\phi - ie^{i\varpi}F = \mathcal{D}^{\perp}\widetilde{\phi} + ie^{-i\varpi}\widetilde{F} \;=\; 0\;. \tag{8.57}$$

This restricted set of conditions, can be further constrained by imposing the reality condition $\widetilde{\phi} = \phi^{\star}$, as we mentioned at the end of section (8.2).

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A Conventions

Clifford algebra. The flat space γ matrices are

$$\gamma^1 = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^2 = -\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \gamma^3 = +\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{A.1}$$

These γ matrices satisfy the relation $\gamma^a\gamma^b = \delta^{ab} + i\,\varepsilon^{abc}\gamma_c$. In particular, $\gamma^{ab} \equiv \frac{1}{2}[\gamma^a, \gamma^b] = i\,\varepsilon^{abc}\gamma_c$. Spinors χ_a and χ_b are contracted as follows,

$$\chi_a\chi_b \equiv \chi^{\alpha}_a\,C_{\alpha\beta}\chi^{\beta}_b \qquad \text{with} \qquad C = \begin{pmatrix} 0 & -1 \\ +1 & 0 \end{pmatrix}, \tag{A.2}$$

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and also

$$\chi_a \, \gamma^\mu \, \chi_b \equiv \chi_a^\alpha \, C_{\alpha\beta} \, (\gamma^\mu)^\beta_{\ \sigma} \, \chi_b^\sigma \ . \tag{A.3}$$

Note the properties $C_{\alpha\beta} = -C_{\beta\alpha}$ and $(C\gamma)_{\alpha\beta} = (C\gamma)_{\beta\alpha}$. Thus, for anticommuting spinors $\chi\gamma^\mu\zeta = -\zeta\gamma^\mu\chi$ whereas for commuting spinors $\chi\gamma^\mu\zeta = +\zeta\gamma^\mu\chi$.

The Fierz Identity for anticommuting spinors is

$$(\chi_d\gamma^\mu\chi_c)\,(\chi_b\gamma_\mu\chi_a) = -\,(\chi_d\chi_c)\,(\chi_b\chi_a) - 2\,(\chi_d\chi_b)\,(\chi_c\chi_a) \ . \tag{A.4}$$

For commuting spinors we have instead

$$(\chi_d\gamma^\mu\chi_c)\,(\chi_b\gamma_\mu\chi_a) = +\,(\chi_d\chi_c)\,(\chi_b\chi_a) - 2\,(\chi_d\chi_b)\,(\chi_c\chi_a) \ . \tag{A.5}$$

Differential geometry. Given an euclidean metric $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, the frame fields are defined by $ds^2 = e^a_\mu \delta_{ab} e^b_\nu$. The inverse frame fields are $e^\mu_a = g^{\mu\nu}\delta_{ab}e^b_\nu$, with $g^{\mu\nu}$ the inverse metric. The Levi-Civita covariant derivative ∇_μ acting on 1) a spinor χ, 2) a vector field $\mathbf{V}^\nu$, and 3) a 1-form field $\mathbf{A}_\nu$ is

$$\begin{aligned} \nabla_\mu \, \chi \ &\equiv \partial_\mu\chi + \tfrac{1}{4}\omega_{\mu\,ab}\gamma^{ab}\chi \,, \\ \nabla_\mu \mathbf{V}^\nu &\equiv \partial_\mu \mathbf{V}^\nu + \Gamma^\nu_{\mu\alpha}\mathbf{V}^\alpha \,, \\ \nabla_\mu \mathbf{A}_\nu &\equiv \partial_\mu \mathbf{A}_\nu - \Gamma^\alpha_{\mu\nu}\mathbf{A}_\alpha \,, \end{aligned} \tag{A.6}$$

where $\Gamma^\nu_{\mu\alpha}$ is the Levi-Civita connection, and we have defined the spin connection ω_μ out of

$$\widetilde{\nabla}_\mu e^a_\nu \equiv \partial_\mu e^a_\nu + (\omega_\mu)^a_{\ b}\, e^b_\nu - \Gamma^\rho_{\mu\nu} e^a_\rho = \nabla_\mu e^a_\nu + (\omega_\mu)^a_{\ b}\, e^b_\nu = 0 \ . \tag{A.7}$$

Supersymmetry transformations from [3]. We list the transformations rules of the components of the generic multiplet $\mathcal{S}$,

$$\begin{aligned}
\delta C \ &= i\epsilon\chi + i\tilde{\epsilon}\widetilde{\chi} \,, \\
\delta\chi \ &= \epsilon M - \tilde{\epsilon}(\sigma + (z - rH)C) - \gamma^\mu\tilde{\epsilon}\,(\mathcal{D}_\mu C + ia_\mu) \,, \\
\delta\widetilde{\chi} \ &= \tilde{\epsilon}\widetilde{M} - \epsilon(\sigma - (z - rH)C) - \gamma^\mu\epsilon\,(\mathcal{D}_\mu C - ia_\mu) \,, \\
\delta M \ &= -2\tilde{\epsilon}\tilde{\lambda} + 2i(z - (r-2)H)\tilde{\epsilon}\chi - 2i\mathcal{D}_\mu(\tilde{\epsilon}\gamma^\mu\chi) \,, \\
\delta\widetilde{M} \ &= +2\epsilon\lambda - 2i(z - (r+2)H)\epsilon\widetilde{\chi} - 2i\mathcal{D}_\mu(\epsilon\gamma^\mu\widetilde{\chi}) \,, \\
\delta a_\mu \ &= -i(\epsilon\gamma_\mu\tilde{\lambda} + \tilde{\epsilon}\gamma_\mu\lambda) + \mathcal{D}_\mu(\epsilon\chi - \tilde{\epsilon}\widetilde{\chi}) \,, \\
\delta\sigma \ &= -\epsilon\tilde{\lambda} + \tilde{\epsilon}\lambda + i(z - rH)(\epsilon\chi - \tilde{\epsilon}\widetilde{\chi}) \,, \\
\delta\lambda \ &= +i\epsilon(D + \sigma H) - i\varepsilon^{\mu\nu\rho}\gamma_\rho\epsilon\,\mathcal{D}_\mu a_\nu - \gamma^\mu\epsilon((z - rH)a_\mu + i\mathcal{D}_\mu\sigma - V_\mu\sigma) \,, \\
\delta\tilde{\lambda} \ &= -i\tilde{\epsilon}(D + \sigma H) - i\varepsilon^{\mu\nu\rho}\gamma_\rho\tilde{\epsilon}\,\mathcal{D}_\mu a_\nu + \gamma^\mu\tilde{\epsilon}((z - rH)a_\mu + iD_\mu\sigma + V_\mu\sigma) \,, \\
\delta D \ &= \mathcal{D}_\mu(\epsilon\gamma^\mu\tilde{\lambda} - \tilde{\epsilon}\gamma^\mu\lambda) - iV_\mu(\epsilon\gamma^\mu\tilde{\lambda} + \tilde{\epsilon}\gamma^\mu\lambda) - H(\epsilon\tilde{\lambda} - \tilde{\epsilon}\lambda) \\
&\qquad + (z - rH)(\epsilon\tilde{\lambda} + \epsilon\lambda - iH(\epsilon\chi - \tilde{\epsilon}\widetilde{\chi})) + \tfrac{ir}{4}(R - 2V^2 - 6H^2)(\epsilon\chi - \tilde{\epsilon}\widetilde{\chi}) \ .
\end{aligned}$$

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B Factorization of bilinears

In this section we explain the details of the manipulations of $\mathscr{V}^{\perp}_{NL\sigma}$ that were used to obtain the final formula (7.11b) in section 7.1. Let us recall the two basic inputs of this discussion: 1) the main contributions to $\mathscr{V}^{\perp}_{NL\sigma}$ that we want to analyze:

$$\mathscr{V}_1 + \mathscr{V}_2 = +\left[\epsilon\gamma^{\perp}\gamma^{\mu}\psi^a\mathcal{D}_\mu\widetilde{\phi}^{\bar{c}} - \tilde{\epsilon}\gamma^{\perp}\gamma^{\mu}\widetilde{\psi}^{\bar{c}}\mathcal{D}_\mu\phi^a\right]K_{a\bar{c}} \tag{B.1}$$

$$\mathscr{V}_3 = -\left[\epsilon\,\gamma^{\perp}\psi^a\,\widetilde{\phi}^{\bar{c}} - \tilde{\epsilon}\,\gamma^{\perp}\widetilde{\psi}^{\bar{c}}\,\phi^a\right]K_{a\bar{c}} \tag{B.2}$$

$$\mathscr{V}_4 = -\left[\epsilon\gamma^{\perp}\widetilde{\psi}^{\bar{c}}W^a + \tilde{\epsilon}\gamma^{\perp}\psi^a\widetilde{W}^{\bar{c}}\right]K_{a\bar{c}}\,, \tag{B.3}$$

and 2) the decomposition of the spinors with the use of the projectors $\mathscr{P}$ and $\widetilde{\mathscr{P}}$:

$$\begin{aligned} \epsilon &= \tfrac{1}{\Omega}(\tilde{\zeta}\epsilon)\zeta\,, & \tilde{\epsilon} &= \tfrac{1}{\Omega}(\tilde{\epsilon}\,\zeta)\tilde{\zeta}\,, \\ \psi &= \tfrac{1}{\Omega}(\tilde{\zeta}\psi)\zeta + \tfrac{1}{\Omega}(\psi\zeta)\tilde{\zeta}\,, & \widetilde{\psi} &= \tfrac{1}{\Omega}(\tilde{\zeta}\widetilde{\psi})\zeta + \tfrac{1}{\Omega}(\widetilde{\psi}\zeta)\tilde{\zeta}\,. \end{aligned} \tag{B.4}$$

We begin by studying $\mathscr{V}_1 + \mathscr{V}_2$. From (B.4) we get

$$\begin{aligned} \mathscr{V}_1 + \mathscr{V}_2 &= \left[+\,\epsilon\,\gamma^{\perp}\gamma^{\nu}\psi^a\mathcal{D}_\nu\widetilde{\phi}^{\bar{c}} - \tilde{\epsilon}\,\gamma^{\perp}\gamma^{\nu}\widetilde{\psi}^{\bar{c}}\mathcal{D}_\nu\phi^a\right] \\ &= \frac{(\tilde{\zeta}\epsilon)}{\Omega^2}\Big[(\zeta\gamma^{\perp}\gamma^{\nu}\zeta)\,(\tilde{\zeta}\psi^a) + (\zeta\gamma^{\perp}\gamma^{\nu}\tilde{\zeta})\,(\psi^a\zeta)\Big]G_{a\bar{c}}\mathcal{D}_\nu\widetilde{\phi}^{\bar{c}} + \\ &\qquad - \frac{(\tilde{\epsilon}\,\zeta)}{\Omega^2}\Big[(\tilde{\zeta}\gamma^{\perp}\gamma^{\nu}\zeta)\,(\tilde{\zeta}\widetilde{\psi}^{\bar{c}}) + (\tilde{\zeta}\gamma^{\perp}\gamma^{\nu}\tilde{\zeta})\,(\widetilde{\psi}^{\bar{c}}\zeta)\Big]G_{\bar{c}a}\mathcal{D}_\nu\phi^a\,. \end{aligned}$$

By using the knowledge of the bosonic bilinears (3.22), we obtain[23]

$$\mathscr{V}_1 + \mathscr{V}_2 = \frac{(\tilde{\zeta}\epsilon)}{\Omega^2}\,(\zeta\tilde{\zeta})\left[(\psi^a\zeta)\mathcal{D}^{\perp}\widetilde{\phi}^{\bar{c}} + (\tilde{\zeta}\widetilde{\psi}^{\bar{c}})\mathcal{D}^{\perp}\phi^a\right]G_{a\bar{c}} \tag{B.5}$$

$$+\frac{(\tilde{\zeta}\epsilon)}{\Omega^2}\,(\zeta\gamma^{\perp}\gamma^{\nu\parallel}\zeta)\Big[(\tilde{\zeta}\psi^a)\mathcal{D}_{\nu\parallel}\widetilde{\phi}^{\bar{c}} - e^{-2i\varpi}(\widetilde{\psi}^{\bar{c}}\zeta)\mathcal{D}_{\nu\parallel}\phi^a\Big]G_{a\bar{c}} \tag{B.6}$$

$$+\frac{(\tilde{\zeta}\epsilon)}{\Omega^2}\,(\zeta\gamma^{\perp}\gamma^{\nu\parallel}\tilde{\zeta})\Big[(\psi^a\zeta)\mathcal{D}_{\nu\parallel}\widetilde{\phi}^{\bar{c}} - (\tilde{\zeta}\widetilde{\psi}^{\bar{c}})\mathcal{D}_{\nu\parallel}\phi^a\Big]G_{a\bar{c}}\,. \tag{B.7}$$

In order to simplify our formulae, it is now convenient to use the matrix notation where $(\psi^a, \widetilde{\psi}^{\bar{c}}) \to \Psi^I$ and $(\phi^a, \widetilde{\phi}^{\bar{c}}) \to \Phi^I$. Each vector will be denoted by a corresponding bold symbol: $\mathbf{\Phi}$, $\mathbf{\Psi}$, $\mathbf{W}$, and $\mathbf{K}$. The change of variables for $(\psi^a\zeta)$ and $(\widetilde{\psi}^{\bar{c}}\zeta)$ results in

$$\begin{aligned} (\psi^a\zeta)\,G_{a\bar{c}}\,\mathcal{D}_\nu\widetilde{\phi}^{\bar{c}} &= \tfrac{1}{2}(\mathbf{\Psi}\zeta)^T(1-iJ)G\,\mathcal{D}_\nu\mathbf{\Phi}\,, \\ (\widetilde{\psi}^{\bar{c}}\zeta)\,G_{a\bar{c}}\,\mathcal{D}_\nu\phi^a &= \tfrac{1}{2}(\mathbf{\Psi}\zeta)^T(1+iJ)G\,\mathcal{D}_\nu\mathbf{\Phi}\,. \end{aligned} \tag{B.8}$$

For the scalar products involving $\tilde{\zeta}$ we shall use the boundary condition $e^{i\varpi}\tilde{\zeta}\mathbf{\Psi} = M\mathbf{\Psi}\zeta$, and write

$$\begin{aligned} (\tilde{\zeta}\psi^a)\,G_{a\bar{c}}\,\mathcal{D}_\nu\widetilde{\phi}^{\bar{c}} &= \tfrac{1}{2}(\tilde{\zeta}\mathbf{\Psi})^T(1-iJ)G\,\mathcal{D}_\nu\mathbf{\Phi} = \tfrac{1}{2}e^{-i\varpi}(\mathbf{\Psi}\zeta)^T M^T(1-iJ)G\,\mathcal{D}_\nu\mathbf{\Phi}\,, \\ (\tilde{\zeta}\widetilde{\psi}^{\bar{c}})\,G_{a\bar{c}}\,\mathcal{D}_\nu\phi^a &= \tfrac{1}{2}(\tilde{\zeta}\mathbf{\Psi})^T(1+iJ)G\,\mathcal{D}_\nu\mathbf{\Phi} = \tfrac{1}{2}e^{-i\varpi}(\mathbf{\Psi}\zeta)^T M^T(1+iJ)G\,\mathcal{D}_\nu\mathbf{\Phi}\,. \end{aligned} \tag{B.9}$$

[23] In (B.6) and (B.7), the sum over $\nu_\parallel$ is understood to run over the indices of the boundary $\mathcal{M}_2$.

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From (B.8) and (B.9), it is a simple exercise to show that $\mathscr{V}_1+\mathscr{V}_2$ can be put in the following form

$$\begin{aligned}\mathscr{V}_1+\mathscr{V}_2 = &\left[\frac{1}{\Omega}(\tilde{\zeta}\epsilon)\,(\mathbf{\Psi}\zeta)^T\right]\frac{(\zeta\tilde{\zeta})}{\Omega}\left[\frac{1-iJ}{2}+e^{-i\varpi}M^T\frac{1+iJ}{2}\right]G\,(\mathcal{D}^{\perp}\mathbf{\Phi})\\ &+e^{-i\varpi}\left[\frac{1}{\Omega}(\tilde{\zeta}\epsilon)\,(\mathbf{\Psi}\zeta)^T\right]\left[e^{i\varpi}M^T\frac{1-iJ}{2}-\frac{1+iJ}{2}\right]G\,(k^{\mu}\mathcal{D}_{\mu}\mathbf{\Phi})\\ &-i\left[\frac{1}{\Omega}(\tilde{\zeta}\epsilon)\,(\mathbf{\Psi}\zeta)^T\right]\left[\frac{1-iJ}{2}-e^{-i\varpi}M^T\frac{1+iJ}{2}\right]G\,(\tilde{k}^{\mu}\mathcal{D}_{\mu}\mathbf{\Phi})\ .\end{aligned}$$

We can then introduce the projectors $P_M^{(\varpi,\pm)}$, and by using the properties:

$$\begin{aligned}P_{M^T}^{(\varpi,+)}J &= J\,P_{M^T}^{(\varpi,-)}\,, & P_{M^T}^{(\varpi,-)}J &= J\,P_{M^T}^{(\varpi,+)}\,,\\ P_{M^T}^{(\varpi,+)}G &= G\,P_{M}^{(\varpi,+)}\,, & P_{M^T}^{(\varpi,-)}G &= G\,P_{M}^{(\varpi,-)}\,,\end{aligned}\tag{B.10}$$

we arrive at the final expression

$$\mathscr{V}_1+\mathscr{V}_2 = +(\epsilon\Psi)^T\left[(1-iJ)G\,P_M^{(\varpi,+)}\,(\mathcal{D}^{\perp}\Phi)+i(1-iJ)G\,P_M^{(\varpi,-)}\,(\tilde{k}^{\mu}D_{\mu}\Phi)\right]\tag{B.11}$$

$$+(\epsilon\Psi)^T\left[e^{-i\varpi}(1+iJ)G\,P_M^{(\varpi,-)}\,(k^{\mu}D_{\mu}\Phi)\right]\ .\tag{B.12}$$

We rearrange $\mathscr{V}_3$ and $\mathscr{V}_4$ with similar manipulations. In the case of $\mathscr{V}_3$ we find

$$\mathscr{V}_3 = \big[-\epsilon\,\gamma^{\perp}\psi^a\widetilde{\phi}^{\bar{c}}+\tilde{\epsilon}\,\gamma^{\perp}\widetilde{\psi}^{\bar{c}}\phi^a\big]G_{a\bar{c}}\tag{B.13}$$

$$= -\frac{(\tilde{\zeta}\epsilon)}{\Omega^2}(\zeta\gamma^{\perp}\zeta)\left[(\tilde{\zeta}\psi^a)\widetilde{\phi}^{\bar{c}}+e^{-2i\varpi}(\widetilde{\psi}^{\bar{c}}\zeta)\phi^a\right]G_{\bar{c}a}\tag{B.14}$$

$$= \left[-\frac{(\tilde{\zeta}\epsilon)}{\Omega}(\mathbf{\Psi}\zeta)^T\right]\frac{(\zeta\gamma^{\perp}\zeta)}{\Omega}\left[e^{-i\varpi}M^T\frac{1-iJ}{2}+e^{-2i\varpi}\frac{1+iJ}{2}\right]G\mathbf{\Phi}\tag{B.15}$$

$$= +ie^{-i\varpi}(\epsilon\mathbf{\Psi})^T(1+iJ)G\,P_M^{(\varpi,-)}J\,\mathbf{\Phi}\ .\tag{B.16}$$

In the case of $\mathscr{V}_4$ we find

$$\mathscr{V}_4 = \big[-\tilde{\epsilon}\,\gamma^{\perp}\psi^a\widetilde{W}^{\bar{c}}-\epsilon\,\gamma^{\perp}\widetilde{\psi}^{\bar{c}}W^a\big]G_{a\bar{c}}\tag{B.17}$$

$$= -\frac{(\tilde{\zeta}\epsilon)}{\Omega^2}(\zeta\gamma^{\perp}\zeta)\left[-e^{-2i\varpi}(\psi^a\zeta)\widetilde{W}^{\bar{c}}+(\tilde{\zeta}\widetilde{\psi}^{\bar{c}})W^a\right]G_{\bar{c}a}\tag{B.18}$$

$$= \left[-\frac{(\tilde{\zeta}\epsilon)}{\Omega}(\mathbf{\Psi}\zeta)^T\right]\frac{(\zeta\gamma^{\perp}\zeta)}{\Omega}\left[-e^{-2i\varpi}\frac{1-iJ}{2}+e^{-i\varpi}M^T\frac{1+iJ}{2}\right]G\mathbf{W}\tag{B.19}$$

$$= -e^{-i\varpi}(\epsilon\mathbf{\Psi})^T(1-iJ)\,P_{M^T}^{(-\varpi,-)}G\,\mathbf{W}\ .\tag{B.20}$$

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