

## Heat kernel for Newton-Cartan trace anomalies

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**ABSTRACT:** We compute the leading part of the trace anomaly for a free non-relativistic scalar in  $2+1$  dimensions coupled to a background Newton-Cartan metric. The anomaly is proportional to  $1/m$ , where  $m$  is the mass of the scalar. We comment on the implications of a conjectured  $a$ -theorem for non-relativistic theories with boost invariance.

**KEYWORDS:** Anomalies in Field and String Theories, Renormalization Group, Space-Time Symmetries

ARXIV EPRINT: [1605.08684](https://arxiv.org/abs/1605.08684)

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## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Preliminaries</b>	<b>3</b>
2.1	Newton-Cartan gravity	3
2.2	Anomalies nearby a flat background	5
2.3	Non-relativistic scalar	7
<b>3</b>	<b>The heat kernel</b>	<b>8</b>
3.1	The flat space case	8
3.2	The curved-space heat kernel	9
3.3	The metric perturbation	10
3.4	The perturbative calculation	11
<b>4</b>	<b>Conclusions</b>	<b>12</b>
<b>A</b>	<b>Heat kernel at first order</b>	<b>13</b>
<b>B</b>	<b>Heat kernel at second order</b>	<b>14</b>

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## 1 Introduction

Conformal anomalies have a long and glorious history in quantum field theory, see e.g. [1, 2]. For relativistic quantum field theories in even dimension, they are a very useful tool to characterize the irreversibility properties of the Renormalization Group (RG). In 2 dimensions this is established by Zamolodchikov's  $c$ -theorem [3]: in this case it is possible to build a monotonically decreasing quantity defined also outside the fixed points, which coincides with the conformal anomaly at the endpoints of the RG flow. In 4 dimensions a similar property (known as  $a$ -theorem) was conjectured by Cardy [4] and later established nearby weakly coupled fixed points in [5–7]. Dispersion relations of dilaton scattering amplitudes [8, 9] were used for a non-perturbative proof.

Scale-invariant fixed points are common not only in high energy physics, but also in non-relativistic condensed matter systems. It would be of general interest to establish some non-relativistic version of the  $a$ -theorem, which might be used to constraint RG flows in non-relativistic strongly coupled systems, such as fermions at unitarity, and to classify the landscape of possible fixed point realized in many-body physics.

Scale anomalies are natural candidates for such monotonically decreasing functions along the RG flow. In the non-relativistic case, scale invariance is characterized by a different scaling of the time and space coordinates. Such a different scaling can be parameterized

by the dynamical exponent  $z$ :

$$x^i \rightarrow e^\sigma x^i, \quad t \rightarrow e^{z\sigma} t. \quad (1.1)$$

In all these situations, we may expect a quantum violation of scale invariance due to coupling to background curved spacetime:

$$T_i^i - z\epsilon^0 = \mathcal{A}, \quad (1.2)$$

where  $T_{ij}$  and  $\epsilon^0$  are the spatial stress-tensor components and the energy density, and  $\mathcal{A}$  a function of the background curvatures and gauge fields.

In order to study these issues, one needs to couple the field theory to a background curved spacetime. The kind of background depends on the symmetries of the theory. In particular, a very different anomaly structure is found depending if we require or not non-relativistic boost invariance.

In the case without boosts, studied in [10–14], several anomalies are indeed possible at the scale-invariant fixed points. Unfortunately, in all the cases that have been studied, these anomalies have vanishing Weyl variation (type B anomalies [15]). Consequently, an analysis based on the Wess-Zumino consistency conditions as the one in [5–7] would not give any constraint on the RG flow of the anomaly coefficients.

The case with boost invariance instead looks much more promising. In this case it is natural to couple the non-relativistic theory to a Newton-Cartan (NC) gravity background. The study of the anomaly in two spatial dimensions and for  $z = 2$  was initiated in [16]. The outcome was that in this case, by dimensional analysis, an infinite number of terms is in principle possible in the anomaly. Moreover, there is a selection rule which splits these terms into distinct sectors, each with a finite numbers of terms and decoupled Wess-Zumino consistency conditions. In the simplest sector, it turns out that the anomaly structure is exactly the same as for a relativistic theory in four spacetime dimensions.

In order for a NC background to be consistent with causality, the Frobenius condition  $n \wedge dn = 0$  should be satisfied,  $n_\mu$  being a nowhere-vanishing 1-form identifying the local time direction. It turns out that the structure of the anomaly critically depends whether causal backgrounds are or not allowed. If backgrounds which do not satisfy the Frobenius condition are discarded, the structure of the anomaly becomes much simpler; in particular there is just a finite number of terms in the anomaly [17]. Moreover, in a rather subtle way, the type A anomaly disappears [14, 17] and just a type B one survives.

One may worry about the possibility that coupling the theory to a non-causal background could be logically inconsistent. On the other hand, technically the NC gravity background is introduced just as a source for the components of the non-relativistic energy-momentum (EM) tensor. When we make a functional derivative with respect to the background fields, indeed we do not restrict just to casual backgrounds, otherwise we would not be able to get all the independent components of the EM tensor. In other words, functional sources in the path integral are not restricted to any physical condition. Consequently, an unconstrained not necessarily causal background should be used for the purpose of studying anomalies.

In this paper, we compute the NC trace anomalies with the heat kernel method in the case of a Schrödinger-invariant free non-relativistic scalar in 2 spatial dimensions. We find that the anomaly is given by:

$$\mathcal{A} = (-a E_4 + c W^2 + b R^2 + d D^2 R) + \dots, \tag{1.3}$$

where

$$a = \frac{1}{8m\pi^2} \frac{1}{360}, \quad c = \frac{1}{8m\pi^2} \frac{3}{360}, \quad b = \frac{1}{8m\pi^2} \frac{1}{2} \left( \xi - \frac{1}{6} \right)^2, \quad d = \frac{1}{8m\pi^2} \frac{1-5\xi}{30}. \tag{1.4}$$

It is important to stress that the quantities  $(E_4, W^2, R^2, D^2 R)$  in eq. (1.3) are completely determined in terms of the 3-dimensional Newton-Cartan gravity fields and do not have anything of 4-dimensional. Technically,  $R, E_4, W^2, D_A$  are defined as the scalar curvature, the Euler density, the Weyl tensor squared and the covariant derivative of the extra-dimensional null reduction in eq. (2.5), but this is just a trick to build quantities which are automatically invariant under the galilean boost symmetry. The dots in eq. (1.3) correspond to possible terms with a higher number of derivatives. The parameter  $\xi$  is the coupling of the scalar to the null reduction curvature  $R\phi\phi^\dagger$  (conformal coupling is achieved for  $\xi = 1/6$ ). The coefficient  $d$  is a scheme-dependent quantity [18], while  $b$  vanishes for the conformal coupling. The coefficients  $a$  and  $c$  correspond to genuine scheme-independent anomalies. In particular,  $a$  is the coefficient of a type A anomaly and then a good candidate for a quantity which decreases along the RG flow. As far as we know, this is the first explicit calculation of the trace anomaly in the Schrödinger case. Our calculation is genuinely 2+1 dimensional and does not make use of the extra-dimensional null reduction.

In section 2 we introduce the notation, the coupling of the scalar to the background geometry, and the form of the anomaly nearby flat spacetime. In section 3 we compute the anomaly using the heat kernel method. We conclude in section 4.

## 2 Preliminaries

### 2.1 Newton-Cartan gravity

A NC geometry in  $d+1$  spacetime dimensions is defined by a 1-form  $n_\mu$  (which corresponds to the local time direction), by a positive-definite symmetric tensor  $h^{\mu\nu}$  with rank  $d$  for which  $n_\mu$  is a zero eigenvector

$$n_\mu h^{\mu\alpha} = 0, \tag{2.1}$$

and by a background gauge field  $A_\mu$  for the particle number symmetry. A vector field  $v^\mu$ , whose projection onto  $n_\mu$  is one

$$n_\mu v^\mu = 1, \tag{2.2}$$

is also introduced; once  $v^\mu$  is fixed, it is possible to uniquely define a degenerate rank  $d$  symmetric tensor  $h_{\mu\nu}$ , which corresponds to the metric along the spatial directions, which satisfies:

$$h^{\mu\alpha} h_{\alpha\nu} = \delta_\nu^\mu - v^\mu n_\nu = P_\nu^\mu, \quad h_{\mu\alpha} v^\alpha = 0, \tag{2.3}$$

where  $P_\nu^\mu$  is the projector onto the spatial directions. The NC geometry was first introduced as a tool to write newtonian gravity in a diffeomorphism-invariant fashion; for a review see [19]. Recently it was realized in [20–23] that it is a very useful tool for condensate-matter physics, because it is a very convenient way to parameterize the sources of the non-relativistic energy-momentum tensor. Other uses and applications have been recently discussed in several papers, e.g. [24–30].

Among the symmetries of the NC theory, besides diffeomorphisms and local U(1) gauge symmetry, there is also a local version of the galilean boost symmetry, which is called Milne boost. If we denote by  $\psi_\mu$  the local boost parameter, the geometry fields transform in the following way:

$$\begin{aligned} v'^\mu &= v^\mu + h^{\mu\nu}\psi_\nu \\ h'_{\mu\nu} &= h_{\mu\nu} - (n_\mu P_\nu^\rho + n_\nu P_\mu^\rho)\psi_\rho + n_\mu n_\nu h^{\rho\sigma}\psi_\rho\psi_\sigma, \\ A'_\mu &= A_\mu + P_\mu^\rho\psi_\rho - \frac{1}{2}n_\mu h^{\alpha\beta}\psi_\alpha\psi_\beta, \end{aligned} \tag{2.4}$$

while  $n_\mu$  and  $h^{\mu\nu}$  are invariant.

These non-trivial transformation properties render the classification of local invariant quantities complicated. For this reason it is convenient to use an extra-dimensional null reduction  $(x^-, x^\mu)$  from a relativistic parent space [31]:

$$\begin{aligned} G_{MN} &= \begin{pmatrix} 0 & n_\mu \\ n_\nu & n_\mu A_\nu + n_\nu A_\mu + h_{\mu\nu} \end{pmatrix} = \begin{pmatrix} 0 & n_\mu \\ n_\nu & (h_A)_{\mu\nu} \end{pmatrix}, \\ G^{MN} &= \begin{pmatrix} A^2 - 2v \cdot A & v^\mu - h^{\mu\sigma} A_\sigma \\ v^\nu - h^{\nu\sigma} A_\sigma & h^{\mu\nu} \end{pmatrix} = \begin{pmatrix} \phi_A & v_A^\mu \\ v_A^\mu & h^{\mu\nu} \end{pmatrix}, \end{aligned} \tag{2.5}$$

where the quantities  $h_A$ ,  $\phi_A$  and  $v_A$  (which are Milne boost invariants) are introduced. Diffeomorphism-invariant quantities in  $d + 2$  dimensions are automatically Milne boost-invariant in the non-relativistic  $d + 1$  dimensional theory. We will sometimes refer to this null reduction as DLCQ, Discrete Light-Cone Quantization. We denote by  $D_A$  the covariant derivative defined by the Levi-Civita connection from the metric in eq. (2.5). It is important to stress that, even if we are often using this extra-dimensional trick, we will compute the anomaly of the non-relativistic theory in  $d + 1$  spacetime dimensions, and not of the  $d + 2$  dimensional relativistic parent theory.

It is useful to introduce the DLCQ vector

$$n^M = (1, 0, \dots), \quad n_M = (0, n_\mu) \tag{2.6}$$

which is a null killing vector of the metric (2.5).

A Weyl transformation on the NC background is equivalent to a Weyl transformation on the DLCQ background which is independent from the null direction  $x^-$ :

$$n^A D_A \sigma = 0. \tag{2.7}$$

The Weyl transformation parameter  $\sigma$  is an arbitrary function of  $x^\mu$ ; the transformation laws of the basic metric objects is as follows:

$$G_{MN} \rightarrow e^{2\sigma} G_{MN}, \quad n_\mu \rightarrow e^{2\sigma} n_\mu, \quad h_{\mu\nu} \rightarrow e^{2\sigma} h_{\mu\nu}. \quad (2.8)$$

In order to define a spacetime volume element we introduce:

$$\sqrt{g} = \sqrt{\det(n_\mu n_\nu + h_{\mu\nu})} = \sqrt{-\det G_{AB}}. \quad (2.9)$$

## 2.2 Anomalies nearby a flat background

Arbitrary variations on background fields are not in general allowed, because one must satisfy both eq. (2.1) and eq. (2.2). The most general perturbations can be parameterized in terms of an arbitrary  $\delta n_\mu$ , a transverse perturbation  $\delta u^\mu$  with  $\delta u^\mu n_\mu = 0$  and a transverse metric perturbation  $\delta \tilde{h}^{\alpha\beta} n_\beta = 0$ . The variation of the metric fields are then:

$$\delta n_\mu, \quad \delta v^\mu = -v^\mu v^\alpha \delta n_\alpha + \delta u^\mu, \quad \delta h^{\mu\nu} = -v^\mu \delta n^\nu - \delta n^\mu v^\nu - \delta \tilde{h}^{\mu\nu}. \quad (2.10)$$

Specializing eq. (2.10) nearby the flat limit gives:

$$\begin{aligned} n_\mu &= (1 + \delta n_0, \delta n_i), & v^\mu &= (1 - \delta n_0, \delta u_i), & \delta \tilde{h}^{0i} &= 0, \\ h_{\mu\nu} &= \begin{pmatrix} 0 & -\delta u_i \\ -\delta u_i & \delta_{ij} + \delta \tilde{h}_{ij} \end{pmatrix}, & h^{\mu\nu} &= \begin{pmatrix} 0 & -\delta n_i \\ -\delta n_i & \delta_{ij} - \delta \tilde{h}_{ij} \end{pmatrix}. \end{aligned} \quad (2.11)$$

In terms of DLCQ extra-dimensional fields, this corresponds to:

$$\begin{aligned} G_{AB} &= \begin{pmatrix} 0 & 1 + \delta n_0 & \delta n_i \\ 1 + \delta n_0 & 2\delta A_0 & \delta A_i - \delta u_i \\ \delta n_i & \delta A_i - \delta u_i & \delta_{ij} + \delta \tilde{h}_{ij} \end{pmatrix}, \\ G^{AB} &= \begin{pmatrix} -2A_0 & 1 - \delta n_0 & -\delta A_i + \delta u_i \\ 1 - \delta n_0 & 0 & -\delta n_i \\ -\delta A_i + \delta u_i & -\delta n_i & \delta_{ij} + \delta \tilde{h}_{ij} \end{pmatrix}. \end{aligned} \quad (2.12)$$

We can use these sources to define conserved currents. Let us consider the vacuum functional  $W[g_{\mu\nu}]$ :

$$e^{iW[g_{\mu\nu}]} = \int \mathcal{D}\phi e^{iS[\phi, g_{\mu\nu}]} \quad (2.13)$$

where  $\phi$  runs over the dynamical fields of the theory. We can define the expectation values of the energy-momentum tensor multiplet through:

$$\delta W = \int d^d x \sqrt{-g} \left( \frac{1}{2} T_{ij} \delta \tilde{h}_{ij} + j^\mu \delta A_\mu - \epsilon^\mu \delta n_\mu - p_i \delta u_i \right). \quad (2.14)$$

Here  $p_i$  is the momentum density,  $T_{ij}$  is the spatial stress tensor,  $j^\mu = (j^0, j^i)$  contains the number density and current and  $\epsilon^\mu = (\epsilon^0, \epsilon^i)$  the energy density and current. Number current is proportional to the momentum density (this is direct consequence of eq. (2.12), because only the combination  $\delta A_i - \delta u_i$  enters inside the DLCQ metric).

Conservations laws in the flat limit give:

$$\partial_\mu j^\mu = 0, \quad \partial_\mu \epsilon^\mu = 0, \quad \partial_t p^j + \partial_i T^{ij} = 0. \quad (2.15)$$

If one makes a Weyl variation  $\Delta$  of the vacuum function, nearby flat spacetime at the first order finds:

$$\Delta W = \sigma G_{AB} \frac{\delta W}{\delta G_{AB}} = \sigma \left( \delta^{ij} \frac{\delta W}{\delta(\tilde{h}_{ij})} + 2 \frac{\delta W}{\delta(\delta n_0)} \right) = \sigma(T_i^i - 2\epsilon^0). \quad (2.16)$$

The non relativistic trace anomaly in  $d = 2$  and for  $z = 2$  can be conveniently written [16] in this way:

$$\Delta W = \int \sqrt{g} d^3x \sigma \left( -aE_4 + cW^2 + bR^2 + dD_A D^A R + eR^{AB}_{CD} R_{ABEF} \frac{\epsilon^{CDEF}}{\sqrt{g}} \right) + \dots \quad (2.17)$$

In this equation the tensors  $R_{ABCD}$ ,  $E_4$ ,  $W^2$  are the Riemann curvature, the Euler density and the Weyl tensor squared of the DLCQ metric eq. (2.5). As in the relativistic case in 4 dimensions, in this equation  $a$ ,  $c$  and  $e$  correspond to anomaly coefficients, while  $b = 0$  from the Wess-Zumino consistency conditions [32] and  $d$  can be removed by local counterterms. The dots in eq. (2.17) correspond to an infinite number of possible terms with a higher number of derivatives, which however belong to separated Weyl sectors. These terms are obtained contracting the DLCQ vector  $n_A$  with combinations of curvatures. By dimensional analysis, for each  $n_A$  one can add one extra DLCQ derivatives  $D_A$  (being a DLCQ curvature a commutator of two covariant derivatives, two  $n_A$  are needed in order to buy a curvature). Examples of possible terms with the right dimension in order to enter the anomaly are:

$$n^A D_A R_{BC} R^{BC}, \quad R_{ABCP} R^{ABCQ} R^P_{MQN} n^M n^N. \quad (2.18)$$

The number of  $n_A$  vectors, which we denote by  $N_n$ , is unchanged by a Weyl transformation, so the Wess-Zumino consistency conditions can be solved independently in each sector with different  $N_n$ . Eq. (2.17) refers to  $N_n = 0$ , while an analysis of the sectors of the anomaly with  $N_n > 0$  is left as a topic for future investigation.

We can now expand the anomaly around the flat background. We set  $\delta u_i = 0$ , because nearby flat space it is equivalent to  $-\delta A_i$ .

In the following eqs. (2.19)–(2.21) we drop the  $\delta$ 's in front of the perturbations of  $n_\mu$ ,  $A_\mu$ , and  $\tilde{h}_{ij}$ . The DLCQ scalar curvature is:

$$R = -\partial_k^2 \tilde{h}_{ii} + \partial_{ij} \tilde{h}_{ij} - 2\partial_k^2 n_0 + 2\partial_0(\partial_i n_i), \quad (2.19)$$

where  $\partial_{ij} = \partial_i \partial_j$  and  $\partial_k^2 = \partial_k \partial_k$  is the spatial flat laplacian. The Euler density  $E_4$  and the

Weyl tensor squared  $W^2$  read:

$$E_4 = 2(\partial_k(\partial_k n_0 + \partial_0 n_k))^2 - 2(\partial_i(\partial_j n_0 + \partial_0 n_j))^2 - 2(\partial_0(\epsilon_{ij}\partial_i n_j))^2 + 4\partial_0(\partial_0 n_i - \partial_i n_0)\partial_k(\partial_k n_i - \partial_i n_k), \quad (2.20)$$

$$W^2 = \frac{1}{3} \left( -\partial_k^2 n_0 + \partial_0(\partial_i n_i) + \partial_k^2 \tilde{h}_{ii} - \partial_{ij} \tilde{h}_{ij} \right)^2 - 3(\partial_0(\epsilon_{ij}\partial_i n_j))^2 + 2\partial_k(\epsilon_{ij}\partial_i n_j)[\partial_k(\epsilon_{lm}\partial_l A_m) + \epsilon_{kl}\partial_0(\partial_0 n_l - \partial_l n_0) - \epsilon_{lm}\partial_0 \tilde{h}_{km}]. \quad (2.21)$$

### 2.3 Non-relativistic scalar

The action for a non-relativistic scalar in a generic NC background is:

$$\int d^3x \sqrt{g} \left\{ imv^\mu \left( \phi^\dagger D_\mu \phi - D_\mu \phi^\dagger \phi \right) - h^{\mu\nu} D_\mu \phi^\dagger D_\nu \phi - \xi R \phi^\dagger \phi \right\}. \quad (2.22)$$

Here the covariant derivative include just the gauge part:

$$D_\mu \phi = \partial_\mu \phi - imA_\mu \phi. \quad (2.23)$$

One can obtain the action (2.22) from DLCQ reduction on a circle  $x^-$  with radius  $4\pi$  of a relativistic scalar:

$$S = \frac{1}{4\pi} \int d^4x \sqrt{-\det G_{AB}} \left( -G^{MN} \partial_M \Phi^\dagger \partial_N \Phi - \xi R \Phi^\dagger \Phi \right), \quad (2.24)$$

using

$$\Phi(x^-, x^\mu) = \phi(x^\mu) e^{imx^-}. \quad (2.25)$$

Let us specialize to the case with  $A_\mu = 0$ . We choose the positive branch  $i\partial_t = \sqrt{-\partial_t^2}$  and we perform the Euclidean rotation:  $t \rightarrow -it_E$ ,  $m \rightarrow im_E$ ; we will omit the subscript  $E$  in what follows. In curved space, this can be realized by

$$v^\mu \rightarrow iv^\mu, \quad m \rightarrow im, \quad n_\mu \rightarrow -in_\mu, \quad \sqrt{g} \rightarrow i\sqrt{g}. \quad (2.26)$$

Introducing the spatial laplacian:

$$\mathcal{D}^2 \phi = \frac{\partial_\mu (\sqrt{g} h^{\mu\nu} \partial_\nu \phi)}{\sqrt{g}}, \quad (2.27)$$

we can write the euclidean action as:

$$S_E = - \int d^3x \sqrt{g} \phi^\dagger \hat{\Delta} \phi = \int d^3x \sqrt{g} \phi^\dagger \left\{ mv^\mu \left( \sqrt{-\partial_\mu^2} \phi \right) + m \frac{\sqrt{-\partial_\mu^2} (\sqrt{g} v^\mu \phi)}{\sqrt{g}} - \mathcal{D}^2 \phi + \xi R \phi \right\}. \quad (2.28)$$

We will consider perturbations around the flat NC spacetime. It is convenient to split the Schrödinger operator in a flat part plus a perturbation:

$$\Delta = -2m\sqrt{-\partial_t^2} + \partial_t^2, \quad \hat{\Delta} = \Delta + \delta\Delta. \quad (2.29)$$



### 3 The heat kernel

#### 3.1 The flat space case

The relativistic conformal anomaly for a scalar was computed by numerous authors, see e.g. [33–36]. A convenient way to compute it is by the heat kernel formalism; see [37–39] for reviews. In flat space, for a euclidean relativistic scalar in  $d_R$  dimensions, the heat kernel is as follows:

$$K_{\partial_i^2}(s; x, y) = \langle y | e^{s\partial_i^2} | x \rangle = \frac{1}{(4\pi s)^{d_R/2}} \exp\left(-\frac{(x-y)^2}{4s}\right) \quad (3.1)$$

where  $d_R$  is the total number of dimensions. We will denote by  $\tilde{K}_{\mathcal{O}}$  the restriction to  $x = y$  of the heat kernel  $K_{\mathcal{O}}$  of the operator  $\mathcal{O}$ , e.g.:

$$\tilde{K}_{\partial_i^2}(s) = \frac{1}{(4\pi s)^{d_R/2}}. \quad (3.2)$$

We can use the results in [40] for the heat kernel in the non-relativistic case. The following Euclidean negative-definite operator is introduced:

$$\Delta = \Delta_t + \partial_i^2, \quad \Delta_t = -2m\sqrt{-\partial_t^2}, \quad (3.3)$$

One can use the parameterization in [40]:

$$e^{-2ms\sqrt{-\partial_t^2}} = \int_0^\infty d\sigma \frac{ms}{\sqrt{\pi}} \frac{1}{\sigma^{3/2}} e^{-\frac{s^2 m^2}{\sigma}} e^{-\sigma(-\partial_t^2)}, \quad (3.4)$$

to rewrite the time-dependent part of the heat kernel as

$$K_{\Delta_t} = \langle t | e^{-2ms\sqrt{-\partial_t^2}} | t' \rangle = \int_0^\infty d\sigma \frac{m}{2\pi} \frac{s}{\sigma^2} e^{-\frac{4s^2 m^2 + (t-t')^2}{4\sigma}} = \frac{ms}{2\pi} \frac{1}{m^2 s^2 + \frac{(t-t')^2}{4}}. \quad (3.5)$$

The total heat kernel reads (here  $d$  is the number of spatial dimensions):

$$K_{\Delta}(s) = \langle xt | e^{s\Delta} | yt' \rangle = \frac{1}{2\pi} \frac{ms}{m^2 s^2 + \frac{(t-t')^2}{4}} \frac{1}{(4\pi s)^{d/2}} \exp\left(-\frac{(x-y)^2}{4s}\right). \quad (3.6)$$

The  $(x, t) = (y, t')$  restriction of the unperturbed heat kernel is then:

$$\tilde{K}_{\Delta}(s) = \langle xt | e^{s\Delta} | xt \rangle = \frac{2}{m(4\pi s)^{1+d/2}}. \quad (3.7)$$

A comparison between eq. (3.2) and eq. (3.7) shows that the Schrödinger operator in  $d+1$  spacetime dimensions feels the same spectral dimension

$$d_{\mathcal{O}} = -2 \frac{\partial \log \tilde{K}_{\mathcal{O}}(s)}{\partial \log s}, \quad (3.8)$$

as a relativistic laplacian in  $d+2$  dimensions. For this reason, the non-relativistic trace anomaly appears in odd spacetime dimensions and not in even ones as in the relativistic case.

### 3.2 The curved-space heat kernel

Let us consider the curved space correction for the heat kernel of the operator  $\hat{\Delta}$  defined in eq. (2.28). One can decompose the total heat kernel as the sum of the flat space contribution generated by  $\Delta$  and a correction due to  $\delta\Delta$ , see eq. (2.29). The operator  $\hat{\Delta}$  is usually defined as a differential operator in the functional space with scalar product:

$$\langle xt|x't'\rangle_g = \frac{\delta(x-x')\delta(t-t')}{\sqrt{g}}. \quad (3.9)$$

One can define the heat kernel as

$$\hat{K}_{\hat{\Delta}}(s) = \exp(s\hat{\Delta}). \quad (3.10)$$

The  $(x, t) = (y, t')$  restriction of the heat kernel of the operator  $\hat{\Delta}$  can be expanded in powers of  $s$ :

$$\tilde{K}_{\hat{\Delta}}(s) = \langle xt|e^{s\hat{\Delta}}|xt\rangle_g = \frac{1}{s^{d/2+1}} \left( a_0(\hat{\Delta}) + a_2(\hat{\Delta})s + a_4(\hat{\Delta})s^2 + \dots \right). \quad (3.11)$$

We shall be interested in particular to the coefficient  $a_4$ , which will give us the trace anomaly of the  $d = 2$  theory.

Let us sketch the derivation, taken from [38]. A convenient expression for the renormalized vacuum functional can be given in terms of the  $\zeta$  function

$$W^{\text{ren}} = -\frac{1}{2}\zeta'(0, \hat{\Delta}) - \frac{1}{2}\log \mu^2 \zeta(0, \hat{\Delta}), \quad (3.12)$$

where  $\mu$  is a renormalization scale and the zeta function of the operator  $\hat{\Delta}$  is defined as:

$$\zeta(s, \hat{\Delta}) = \text{Tr}(\hat{\Delta}^{-s}). \quad (3.13)$$

Under a variation of the operator  $\hat{\Delta}$ , the  $\zeta$  function transforms as:

$$\delta\zeta(s, \hat{\Delta}) = -s\text{Tr}((\delta\hat{\Delta})\hat{\Delta}^{-s-1}). \quad (3.14)$$

Specializing to a Weyl transformation, the heat kernel generator transforms as:

$$\delta\hat{\Delta} = -2\sigma\hat{\Delta}, \quad \delta\zeta = 2\sigma s\zeta. \quad (3.15)$$

Moreover, the  $\zeta$  function is regular at  $s = 0$ . Consequently, the variation of the vacuum functional is:

$$\delta W^{\text{ren}} = -\zeta(0, \hat{\Delta}) = -a_4(\hat{\Delta}). \quad (3.16)$$

The second equality in eq. (3.16) follows from the relation:

$$\tilde{K}_{\hat{\Delta}}(t) = \frac{1}{2\pi i} \oint ds t^{-s} \Gamma(s) \zeta(s, \hat{\Delta}) \quad (3.17)$$

and by taking the residue at the pole at  $s = 0$ . This shows that for  $d = 2$ :

$$T_i^i - 2\epsilon^0 = a_4(x, \hat{\Delta}). \quad (3.18)$$

It is useful to introduce an operator  $\hat{M}$  for which

$$\langle xt|\hat{\Delta}|x't'\rangle_g = \langle xt|\hat{M}|x't'\rangle, \quad \text{where} \quad \langle xt|x't'\rangle = \delta(x-x')\delta(t-t'). \quad (3.19)$$

One can decompose the heat kernel generator  $\hat{M}$  as the sum of the flat one plus a perturbation  $\hat{V}$  that encodes all the gravitational effects:

$$\langle xt|\hat{M}|x't'\rangle = g^{1/4}(\Delta + \delta\Delta)[g^{-1/4}\delta(x-x')\delta(t-t')], \quad \hat{M} = \Delta + \hat{V}, \quad (3.20)$$

where  $\delta\Delta$  is the difference between the curved and the flat space Schrödinger operator, see eq. (2.29). The trace of the operator  $\hat{K}_{\hat{M}}$  can be expanded in powers of  $s$ :

$$\tilde{K}_{\hat{M}}(s) = \langle xt|e^{s\hat{M}}|xt\rangle = \frac{1}{s^{d/2+1}} \left( a_0(\hat{M}) + a_2(\hat{M})s + a_4(\hat{M})s^2 + \dots \right). \quad (3.21)$$

With our choice of conventions, we have that  $\tilde{K}_{\hat{M}}(s) = \sqrt{g}\tilde{K}_{\hat{\Delta}}(s)$ .

### 3.3 The metric perturbation

To compute the curved space heat kernel, we specialize to a simple perturbation in which just the background fields  $(n_0, v^0)$  are perturbed in a time-independent way, i.e. we take

$$h_{ij} = \delta_{ij}, \quad n_i = v^i = 0, \quad A_\mu = 0. \quad (3.22)$$

We will use the following parameterization:

$$n_0 = \frac{1}{1-\eta(x)}, \quad v^0 = 1-\eta(x), \quad g^{1/2} = \frac{1}{1-\eta}, \quad (3.23)$$

where  $\eta(x)$  is a function of space. We need  $R$  at next-to-leading order:

$$R \approx -2\partial^2\eta - 2\eta\partial^2\eta - \frac{7}{2}\partial_i\eta\partial_i\eta + \dots. \quad (3.24)$$

The dimension 4 curvature invariants are:

$$\begin{aligned} R^2 &\approx 4(\partial^2\eta)^2, & W^2 &\approx \frac{1}{3}(\partial^2\eta)^2, & E_4 &\approx 2(\partial^2\eta)^2 - 2\partial_{ij}\eta\partial_{ij}\eta, \\ D_A D^A R &\approx -2\partial^2\partial^2\eta - 2(\partial^2\eta)^2 - 2\eta\partial^2\partial^2\eta - 13\partial_k\eta\partial_k\partial^2\eta - 7\partial_{ij}\eta\partial_{ij}\eta. \end{aligned} \quad (3.25)$$

The covariant spatial laplacian, as defined in eq. (2.27), is:

$$\mathcal{D}^2\phi = g^{-1/2}\partial_i(g^{1/2}\partial_i\phi) \approx \partial^2\phi + (\partial_i\eta + \eta\partial_i\eta)\partial_i\phi. \quad (3.26)$$

One should also take into account the normalization factors of the  $\delta$  function in the heat kernel, which at the second order in  $\eta$  reads:

$$-g^{1/4}\mathcal{D}^2(g^{-1/4}\delta(x)) \approx -\partial^2\delta(x) + \delta(x) \left( \frac{\partial^2\eta}{2} + \frac{1}{2}\eta\partial^2\eta + \frac{3}{4}\partial_i\eta\partial_i\eta \right). \quad (3.27)$$

In the Euclidean, the heat kernel generator  $\hat{M}$  is:

$$\langle xt|\hat{M}|x't'\rangle = \langle xt| \left( \Delta + S(x)\sqrt{-\partial_0^2}\delta(x-x')\delta(t-t') + P(x)\delta(x-x')\delta(t-t') \right) |x't'\rangle, \quad (3.28)$$

where

$$S = 2m\eta, \quad P = - \left( \frac{\partial^2\eta}{2} + \frac{1}{2}\eta\partial^2\eta + \frac{3}{4}\partial_i\eta\partial_i\eta \right) + \xi \left( 2\partial^2\eta + 2\eta\partial^2\eta + \frac{7}{2}(\partial_i\eta)^2 \right). \quad (3.29)$$

### 3.4 The perturbative calculation

One can use the perturbative approach explained for example in the textbook [39] and in the paper [41]:

$$K_{\hat{M}}(s) = \exp(s(\Delta + \hat{V})) = \sum_{n=0}^{\infty} K_n(s), \quad (3.30)$$

where

$$K_n(s) = \int_0^s ds_n \int_0^{s_n} ds_{n-1} \dots \int_0^{s_2} ds_1 e^{(s-s_n)\Delta} \hat{V} e^{(s_n-s_{n-1})\Delta} \hat{V} \dots e^{(s_2-s_1)\Delta} \hat{V} e^{s_1\Delta}. \quad (3.31)$$

The operator  $\hat{V}$  is given by:

$$\langle xt | \hat{V} | x't' \rangle = \langle xt | \left( S(x) \sqrt{-\partial_0^2} \delta(x-x') \delta(t-t') + P(x) \delta(x-x') \delta(t-t') \right) | x't' \rangle. \quad (3.32)$$

To determine  $a_4$  at the lowest order in  $\eta$ , we need to compute the  $K_1$  contribution using the functions  $P, S$  in eq. (3.29) at the second order in  $\eta$  and the  $K_2$  contribution using  $P, S$  at the first order in  $\eta$ .

The contribution from  $K_1$  splits in a part due to  $P$  and a part due to  $S$ :

$$\begin{aligned} \tilde{K}_{1P} &= \frac{2}{m(4\pi s)^{d/2+1}} \left( sP + \frac{1}{6} s^2 \partial_x^2 P + \dots \right) \\ \tilde{K}_{1S} &= \frac{1}{m^2} \frac{1}{(4\pi s)^{d/2+1}} \left( S + \frac{s}{6} \partial_x^2 S + \frac{s^2}{60} \partial^2 \partial^2 S + \dots \right). \end{aligned} \quad (3.33)$$

The contribution due to  $K_2$  splits in four pieces:

$$\begin{aligned} \tilde{K}_{2SS} &= \frac{1}{2m^3(4\pi s)^{d/2+1}} \left( S^2 + \frac{s}{3} S \partial_k^2 S + \frac{s}{6} \partial_k S \partial_k S \right. \\ &\quad \left. + \frac{s^2}{30} S \partial^2 \partial^2 S + \frac{s^2}{36} \partial^2 S \partial^2 S + \frac{s^2}{15} \partial_i \partial^2 S \partial_i S + \frac{s^2}{45} \partial_i \partial_j S \partial_i \partial_j S + \dots \right) \\ \tilde{K}_{2PP} &= \frac{2}{m(4\pi s)^{d/2+1}} \left( \frac{s^2}{2} P(x)^2 + \dots \right) \\ \tilde{K}_{2PS} &= K_{2SP} = \frac{1}{m^2(4\pi s)^{d/2+1}} \left( \frac{s}{2} SP + \frac{s^2}{12} (\partial^2 SP + S \partial^2 P + \partial_i S \partial_i P) + \dots \right). \end{aligned} \quad (3.34)$$

The calculations are sketched in appendices A and B.

We can then re-express the  $a_4$  coefficient in terms of the curvature invariants, see eq. (3.25). There is a degeneracy between  $W^2$  and  $R^2$ , due to the fact that in the simple background that we have chosen they are proportional to each other. In order to fix these coefficients, we can use the fact that for the conformal coupling  $\xi = 1/6$ , the coefficient of the  $R^2$  term must vanish due to the Wess-Zumino consistency conditions. Up to quadratic order in  $\eta$ , the result is:

$$a_4(\hat{M}) = \sqrt{g} (-aE_4 + cW^2 + bR^2 + dD_A D^A R), \quad (3.35)$$

where the coefficients are given in eq. (1.4).

## 4 Conclusions

It is natural to conjecture that an analogous of the  $a$  theorem may hold for the  $E_4$  anomaly coefficient in  $d = 2$  Schrödinger-invariant theories. If we consider two fixed points in the UV and in the IR with matter content given just by free scalars, it would mean that the following quantity should decrease from UV to IR:

$$a_{\text{UV}} \propto \sum_k^{\text{UV}} \frac{1}{m_k} \geq \sum_k^{\text{IR}} \frac{1}{m_k} \propto a_{\text{IR}}, \quad (4.1)$$

where the sum over  $k$  is over the number of scalar species. In this class of theories indeed the mass is conserved, and so the mass of bound states is the sum of the elementary constituents, with no bound-state deficit mass. The statement in eq. (4.1) may give a quantitative formulation to the physical intuition that bound states should form in the IR: in the process of adding energy to a system, bound states are broken instead of formed.

Up to an overall  $1/m$  factor, the anomaly coefficients in eq. (1.4) are numerically identical to the corresponding ones in the relativistic case in 4 dimensions, see e.g. [33–36]. It would be interesting to check if this numerical coincidence is valid in more general cases (e.g. for fermions) and if it has a physical explanation.

Several aspects deserve consideration for further investigations, for instance the trace anomaly for a free fermion. The Chern-Simons term is also very interesting: it describes anyons, and in three spacetime dimension this gives a continuous interpolation between the bosonic and the fermionic case. These calculations will be useful to check the conjectured  $a$ -theorem in practical condensed-matter examples. Possible techniques which might be used for a proof are the local renormalization group [5–7] and the dispersion relations method [8, 9].

We considered only the simplest sector  $N_n = 0$  in the anomaly, while an infinite number of higher derivatives terms is present in the other sectors, e.g. eq. (2.18). The general structure of the anomaly terms with an arbitrary number of  $n_A$  is not known and in particular it is not known if additional type A anomalies are present.

In the supersymmetric case, it is possible that exact expressions for  $a$  might be found also in the interacting case, by coupling the theory to a supergravity background as in [42–44]. Newton-Cartan supergravity was recently studied in [45, 46]. Another interesting direction is holography [47–52].

## Acknowledgments

We are grateful to Carlos Hoyos for useful comments.

## A Heat kernel at first order

Let us first consider a multiplicative perturbation  $P(x)$ :

$$\begin{aligned}
 K_{1P}(s) &= \int_0^s ds' \int d^d \tilde{x} \int d\tilde{t} \langle xt | e^{-(s-s')\Delta} | \tilde{x}\tilde{t} \rangle P(\tilde{x}) \langle \tilde{x}\tilde{t} | e^{-s'\Delta} | yt' \rangle \\
 &= \frac{1}{(2\pi)^2} \int_0^s ds' \frac{1}{(4\pi(s-s'))^{d/2}} \frac{1}{(4\pi s')^{d/2}} \int d\tilde{t} \frac{m(s-s')}{m^2(s-s')^2 + \frac{(t-\tilde{t})^2}{4}} \frac{ms'}{m^2 s'^2 + \frac{(\tilde{t}-t')^2}{4}} \\
 &\quad \int d^d \tilde{x} P(\tilde{x}) \exp\left(-\frac{(x-\tilde{x})^2}{4(s-s')} - \frac{(\tilde{x}-y)^2}{4s'}\right). \tag{A.1}
 \end{aligned}$$

The  $d\tilde{t}$  integral can be computed explicitly; moreover we can Fourier transform  $P$ :

$$P(\tilde{x}) = \int \frac{d^d k}{(2\pi)^{d/2}} e^{ik\tilde{x}} P(k). \tag{A.2}$$

We get the following expression:

$$\begin{aligned}
 K_{1P}(s) &= \frac{1}{(2\pi)^2} \int_0^s ds' \frac{1}{(4\pi(s-s'))^{d/2}} \frac{1}{(4\pi s')^{d/2}} \frac{8m\pi s}{4m^2 s^2 + (t-t')^2} \\
 &\quad \int d^d \tilde{x} \int \frac{d^d k}{(2\pi)^{d/2}} P(k) \exp\left(-\frac{(x-\tilde{x})^2}{4(s-s')} - \frac{(\tilde{x}-y)^2}{4s'} + ik\tilde{x}\right). \tag{A.3}
 \end{aligned}$$

Doing the gaussian integral, we get:

$$\begin{aligned}
 K_{1P}(s) &= \frac{1}{(2\pi)^2} \int_0^s ds' \frac{1}{(4\pi s)^{d/2}} \frac{8m\pi s}{4m^2 s^2 + (t-t')^2} \\
 &\quad \int \frac{d^d k}{2\pi^{d/2}} \exp\left(-\frac{(x-y)^2}{4s} + ik \cdot \left(x \frac{s'}{s} + y \frac{s-s'}{s}\right) - k^2 \frac{s'}{s} (s-s')\right) P(k). \tag{A.4}
 \end{aligned}$$

Setting  $t = t'$  and  $x = y$  and expanding we recover the first of eq. (3.33).

The single insertion of  $S$  can be reduces to a derivative acting on the single insertion of a  $P$ :

$$\begin{aligned}
 K_{1S}(s) &= \int_0^s ds' \int d^d \tilde{x} \int d\tilde{t} \langle xt | e^{-(s-s')\Delta} | \tilde{x}\tilde{t} \rangle S(\tilde{x}) \sqrt{-\partial_{\tilde{t}}^2} \langle \tilde{x}\tilde{t} | e^{-s'\Delta} | yt' \rangle \\
 &= \sqrt{-\partial_{t'}^2} \left( \int_0^s ds' \int d^d \tilde{x} \int d\tilde{t} \langle xt | e^{-(s-s')\Delta} | \tilde{x}\tilde{t} \rangle S(\tilde{x}) \langle \tilde{x}\tilde{t} | e^{-s'\Delta} | yt' \rangle \right). \tag{A.5}
 \end{aligned}$$

We can now use eq. (A.4). In order to perform the  $\sqrt{-\partial_{t'}^2}$  operator, we can use the following formula:

$$\sqrt{-\partial_t^2} \left( \frac{1}{1 + \frac{t^2}{A^2}} \right) = \frac{A(A^2 - t^2)}{(A^2 + t^2)^2}, \tag{A.6}$$

which can be derived directly using Fourier transform:

$$\mathcal{F} \left( \frac{1}{1 + \frac{t^2}{A^2}} \right) = \sqrt{\frac{\pi}{2}} A e^{-A|\omega|}. \tag{A.7}$$

## B Heat kernel at second order

The four contributions  $K_{2X_1X_2}(s)$ , where

$$X_1 = \{P(x_1), S(x_1)\}, \quad X_2 = \{P(x_2), S(x_2)\}, \quad (\text{B.1})$$

have a very similar structure:

$$K_{2X_1X_2}(s) = \int_0^s ds_2 \int_0^{s_2} ds_1 \langle x't'|e^{-(s-s_2)\Delta}|x_2t_2\rangle \hat{X}_2 \langle x_2t_2|e^{-(s_2-s_1)\Delta}|x_1t_1\rangle \hat{X}_1 \langle x_1t_1|e^{-s_1\Delta}|xt\rangle, \quad (\text{B.2})$$

where

$$\hat{X}_1 = \left\{ P(x_1), S(x_1) \sqrt{-\partial_{t_1}^2} \right\}, \quad \hat{X}_2 = \left\{ P(x_2), S(x_2) \sqrt{-\partial_{t_2}^2} \right\}. \quad (\text{B.3})$$

We can split it as follows:

$$K_{2X_1X_2}(s) = \int_0^s ds_2 \int_0^{s_2} ds_1 \frac{1}{(4\pi(s-s_2))^{d/2}} \frac{1}{(4\pi(s_2-s_1))^{d/2}} \frac{1}{(4\pi s_1)^{d/2}} \Xi^{X_1X_2} \Psi^{X_1X_2}, \quad (\text{B.4})$$

where  $\Xi^{X_1X_2}$  and  $\Psi^{X_1X_2}$  correspond to the space and time part of the integrals. The space part is:

$$\begin{aligned} \Xi^{X_1X_2} &= \int dx_1 \int dx_2 \exp\left( ik_1x_1 + ik_2x_2 - \frac{(x'-x_2)^2}{4(s-s_2)} - \frac{(x_2-x_1)^2}{4(s_2-s_1)} - \frac{(x_1-x)^2}{4s_1} \right) X_1(k_1) X_2(k_2) \\ &= (4\pi)^d \left( \frac{s_1(s-s_2)(s_2-s_1)}{s} \right)^{d/2} \exp\left( \frac{ik_1s_1x'}{s} + \frac{ik_2s_2x'}{s} - \frac{ik_1s_1x}{s} - \frac{ik_2s_2x}{s} + \frac{k_1^2s_1^2}{s} + \frac{k_2^2s_2^2}{s} \right. \\ &\quad \left. - k_1^2s_1 - 2k_1k_2s_1 - k_2^2s_2 + \frac{2k_1k_2s_1s_2}{s} + ik_1x + ik_2x - \frac{x^2}{4s} + \frac{xx'}{2s} - \frac{(x')^2}{4s} \right) X_1 X_2, \end{aligned}$$

which, specializing for  $x = x'$ , reads:

$$\begin{aligned} \Xi^{X_1X_2}|_{x=x'} &= \exp\left( -\left( \frac{s_1^2}{s} - s_1 \right) \partial_{x_1}^2 - \left( \frac{s_2^2}{s} - s_2 \right) \partial_{x_2}^2 - 2\left( \frac{s_1s_2}{s} - s_1 \right) \partial_{x_1} \cdot \partial_{x_2} \right) X_1(x_1) X_2(x_2) \\ &\quad (4\pi)^d \left( \frac{s_1(s-s_2)(s_2-s_1)}{s} \right)^{d/2}. \end{aligned} \quad (\text{B.5})$$

The time part is:

$$\begin{aligned} \Psi^{PP} &= \frac{1}{(2\pi)^3} \int dt_1 \int dt_2 \frac{m(s-s_2)}{m^2(s-s_2)^2 + \frac{(t_2-t')^2}{4}} \frac{m(s_2-s_1)}{m^2(s_2-s_1)^2 + \frac{(t_2-t_1)^2}{4}} \frac{ms_1}{m^2s_1^2 + \frac{(t_1-t)^2}{4}}, \\ \Psi^{SP} &= \frac{1}{4\pi^3} \int dt_1 \int dt_2 \frac{m(s-s_2)}{m^2(s-s_2)^2 + \frac{(t_2-t')^2}{4}} \frac{m(s_2-s_1)}{m^2(s_2-s_1)^2 + \frac{(t_2-t_1)^2}{4}} \frac{4m^2s_1^2 - (t_1-t)^2}{(4m^2s_1^2 + (t_1-t)^2)^2}, \\ \Psi^{PS} &= \frac{1}{4\pi^3} \int dt_1 \int dt_2 \frac{m(s-s_2)}{m^2(s-s_2)^2 + \frac{(t_2-t')^2}{4}} \frac{4m^2(s_2-s_1)^2 - (t_2-t_1)^2}{(4m^2(s_2-s_1)^2 + (t_2-t_1)^2)^2} \frac{ms_1}{m^2s_1^2 + \frac{(t_1-t)^2}{4}}, \\ \Psi^{SS} &= \frac{1}{2\pi^3} \int dt_1 \int dt_2 \frac{m(s-s_2)}{m^2(s-s_2)^2 + \frac{(t_2-t')^2}{4}} \frac{4m^2(s_2-s_1)^2 - (t_2-t_1)^2}{(4m^2(s_2-s_1)^2 + (t_2-t_1)^2)^2} \frac{4m^2s_1^2 - (t_1-t)^2}{(4m^2s_1^2 + (t_1-t)^2)^2}. \end{aligned}$$

The result of the integration is:

$$\begin{aligned}\Psi^{PP} &= \frac{1}{\pi} \frac{2ms}{4m^2s^2 + (t-t')^2}, & \Psi^{SS} &= \frac{1}{\pi} \frac{4ms(4m^2s^2 - 3(t-t')^2)}{(4m^2s^2 + (t-t')^2)^3}, \\ \Psi^{PS} = \Psi^{SP} &= \frac{1}{\pi} \frac{(4m^2s^2 - (t-t')^2)}{(4m^2s^2 + (t-t')^2)^2}.\end{aligned}\tag{B.6}$$

Putting all together and specializing to  $t = t'$  and  $x = x'$ , we find the expressions in eq. (3.34).

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