

Covariant action for type IIB supergravity

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ABSTRACT: Taking clues from the recent construction of the covariant action for type II and heterotic string field theories, we construct a manifestly Lorentz covariant action for type IIB supergravity, and discuss its gauge fixing maintaining manifest Lorentz invariance. The action contains a (non-gravitating) free 4-form field besides the usual fields of type IIB supergravity. This free field, being completely decoupled from the interacting sector, has no physical consequence.

KEYWORDS: String Field Theory, Superstrings and Heterotic Strings

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1 Introduction and summary

Type IIB string theory in 9+1 dimensions has a 4-form gauge potential whose 5-form field strength obeys a self-duality constraint. As a result the theory is formulated using its equation of motion [1–4] — there is no simple Lorentz invariant action from which the equations of motion can be derived. Alternatively one can write down an action and supplement it with the constraint of self-duality of the 5-form field strength. This constraint needs to be imposed *after deriving the equations of motion from the action*.

The absence of a simple action for type IIB supergravity served as a sort of no go theorem for formulation of a field theory for superstrings. If a manifestly Lorentz invariant superstring field theory could be formulated then by taking its low energy limit one would arrive at an action for low energy supergravity including type IIB supergravity. Therefore, absence of the latter would imply absence of the former.

Recently this difficulty was circumvented and a manifestly Lorentz invariant superstring field theory was formulated [5]. This theory works not only at the classical level but at the full quantum level. The extra ingredient used in this construction was that the theory, besides containing the usual degrees of freedom of string theory, contains a set of free fields that completely decouple from the interacting sector, not only at the classical level but also at the full quantum level. Given this construction one would expect that the low energy limit of this theory should lead to a manifestly Lorentz invariant action for supergravity theories, including type IIB supergravity, at the cost of adding additional fields to the theory representing free decoupled degrees of freedom.

The purpose of this paper is to describe such a construction. In this we shall not try to determine the low energy limit of the string field theory of [5] directly, but use the insights and general structure of this string field theory to guess the form of the action that

describes type IIB supergravity. Some progress towards the study of low energy limit of the string field theory has been achieved in [6]. Our final result will be in the form of an action with no additional constraints. We shall show that under suitable identification of the field variables appearing in the new action with the field variables in the original form of type IIB supergravity, the equations of motion derived from the new action reproduce both the equations of motion and the self-duality constraint on the 5-form field strength present in type IIB supergravity. However as expected, the new formulation has some additional degrees of freedom representing free fields that decouple from the interacting part of the theory.

Different forms of the action for type IIB supergravity have been written down before. These formulations either break manifest Lorentz invariance [7–10], or have infinite number of auxiliary fields [11–19], or have a finite number of auxiliary fields with non-polynomial action [20–24], or requires going to one higher dimension [25, 26].¹ The action we construct in this paper is 9+1 dimensional, preserves manifest Lorentz invariance, has only a finite number of fields and is polynomial in the fields in the absence of gravity.² However the general coordinate transformation acts in an unusual fashion. This is to be expected for two reasons. First of all in string field theory the gauge transformations look different from the standard general coordinate transformations beyond linearized level. Therefore there is no reason to expect that by taking its low energy limit we shall arrive at a theory with standard general coordinate transformation rules. Second, in the standard general coordinate invariant coupling of the metric to other fields, in which we replace the ordinary derivatives by covariant derivatives, there are no free fields since everything gravitates. Therefore if we are to have a field theory in which one set of fields remain free, then the general coordinate transformation laws cannot be standard.

One way to write down a theory of 4-form fields with self-dual field strength will be to begin with a theory of unconstrained 4-form field but arrange the interactions so that only the self-dual part interacts with the rest of the system [25]. In this case the anti-self-dual part would describe a decoupled free field. It may be possible to implement this in the full type IIB supergravity, but one has to take into account the additional subtleties that arise from the fact that the 5-form that obey's self-duality constraint itself depends on the interaction terms. To the best of our knowledge this has not been carried out explicitly maintaining manifest Lorentz invariance. Here we would only like to point out that the procedure we follow, motivated by string field theory, is different from the one described above, In our case the extra free field that decouples also has self-dual field strength. Furthermore it has the wrong sign kinetic term. This will be fatal in an interacting theory, but since these extra modes describe free fields, their presence does not affect the quantization of the interacting part of the theory.

Since the analysis of the paper is somewhat technical, let us summarize the main results. In the usual formulation type IIB theory contains a four form gauge potential

¹Other attempts in this direction can be found in [27, 28].

²Once gravity is turned on the action becomes non-polynomial in the fields since general relativity is intrinsically non-polynomial. Like general relativity, our action is non-polynomial in the metric fluctuations but is polynomial in the derivatives of all fields.

$C^{(4)}$. The action of the theory can be written as $S_1 + S_2$ where S_2 is independent of $C^{(4)}$, and S_1 has the form given in (4.4) with the various quantities appearing in this action defined in (4.1), (4.2). After deriving the equations of motion using this action we are required to impose the self-duality constraint (4.5) on the gauge invariant 5-form field strength. In our formulation we replace the 4-form field $C^{(4)}$ by a 4-form field $P^{(4)}$ and an *independent* self-dual 5-form field $Q^{(5)}$. The action is taken to be $S'_1 + S_2$ where S_2 is the same action as before, and S'_1 is given in (4.47) with the various quantities appearing in this expression defined in (4.2), (4.15), (4.17), (4.37). We find that the equations of motion derived from $S'_1 + S_2$ are equivalent to the ones derived from $S_1 + S_2$ and the self-duality condition (4.5) provided we relate the field $Q^{(5)}$ in the new formalism with the field $C^{(4)}$ in the original formulation via eqs.(4.1), (4.9), (4.32). The degrees of freedom associated with the field $P^{(4)}$ in the new formalism describe (non-gravitating) free fields and decouple from the interacting part of the theory. This is already apparent from the fact that $P^{(4)}$ appears in the action (4.47) only in the linear and quadratic terms, but is clearer in the gauge fixed kinetic term given in (7.3) where the field $\bar{P}^{(4)}$ just has a quadratic action and does not appear anywhere else in the action.

The rest of the paper is organized as follows. In section 2 we use the form of the string field theory action described in [5] to guess the general structure of the action for type IIB supergravity. In section 3 we consider type IIB supergravity with the metric fluctuations and fermion fields set to zero, and show how in this simpler setting one can construct an action whose equations of motion reproduce the equations of motion of type IIB supergravity. In section 4 we include the effect of metric fluctuations as well as the fermion fields and write down the general action whose equations of motion reproduce the full set of equations of motion and self-duality constraint of type IIB supergravity. In section 5 we describe the general coordinate transformation laws of various fields which take a somewhat unusual form in our description. In section 6 we describe how supersymmetry of the original type IIB supergravity can be described as a symmetry of the new action we have constructed in section 4. In section 7 we briefly discuss the Feynman rules derived from this action in a Lorentz covariant gauge.

We expect that the formalism developed in this paper can be generalized to find actions for other chiral theories. It will be interesting to explore if similar techniques can be used to construct an action for the Vasiliev higher spin theories [29–31]. If there is any limit in which the classical Vasiliev theory emerges from classical string field theory, then the existence of an action for the latter implies that the former must also have an action.

2 Expectation from string field theory

In this section we shall review the structure of the action expected from string field theory and describe how we shall implement it in the context of type IIB supergravity.

We begin by recalling some pertinent facts about the action for superstring field theory constructed in [5]. The theory has two sets of fields, which we collectively denote by ψ and $\tilde{\psi}$. The action takes the form

$$-\frac{1}{2}(\tilde{\psi}, \mathcal{Q} \mathcal{X} \tilde{\psi}) + (\tilde{\psi}, \mathcal{Q} \psi) + f(\psi), \tag{2.1}$$

where $(,)$ denotes an inner product and \mathcal{Q} and \mathcal{X} are hermitian, mutually commuting, linear operators made of BRST charge and picture changing operators respectively. The details of these operators will not be important for us. $f(\psi)$ is a non-linear function of the fields ψ only, representing interaction terms. The equations of motion for $\tilde{\psi}$, ψ derived from this action takes the form:

$$\mathcal{Q}\mathcal{X}\tilde{\psi} - \mathcal{Q}\psi = 0, \tag{2.2}$$

and

$$\mathcal{Q}\tilde{\psi} + f'(\psi) = 0, \tag{2.3}$$

where $f'(\psi)$ denotes the derivative of $f(\psi)$ with respect to various components of ψ . Applying the operator \mathcal{X} on (2.3), subtracting it from (2.2) and using the fact that \mathcal{Q} and \mathcal{X} commute we get

$$\mathcal{Q}\psi + \mathcal{X}f'(\psi) = 0. \tag{2.4}$$

This can be identified as the physical equations of motion with ψ containing all the physical fields. On the other hand (2.3) can now be regarded as an equation that determines $\tilde{\psi}$ in terms of ψ . The solution is not unique, but if $\tilde{\psi}$ and $\tilde{\psi} + \Delta\tilde{\psi}$ represent two solutions to this equation for a given ψ then we have

$$\mathcal{Q}\Delta\tilde{\psi} = 0 \tag{2.5}$$

This is a linear equation and hence represent free field degrees of freedom. Furthermore, since these free field modes do not affect the equation for ψ , they decouple from the interacting sector described by the field ψ .

The gauge symmetries of the action (2.1) are generated by two sets of parameters collectively denoted as λ and $\tilde{\lambda}$. The infinitesimal transformation laws take the form

$$\delta\psi = \bar{\mathcal{Q}}\lambda + \mathcal{X}h(\psi)\lambda, \quad \delta\tilde{\psi} = \bar{\mathcal{Q}}\tilde{\lambda} + h(\psi)\lambda, \tag{2.6}$$

where $\bar{\mathcal{Q}}$ is a field independent linear operator and $h(\psi)$ is a linear operator acting on λ , but is a non-linear function of ψ .

In what follows we shall use this insight to construct an action for type IIB supergravity. However we shall use a truncated version of this mechanism in which we introduce the analog of the fields $\tilde{\psi}$ only for the 4-form field of type IIB supergravity. If we try to directly construct the massless field content from the action (2.1) of type IIB string theory, we expect to get a doubling for every field and there will also be additional auxiliary fields / gauge transformations etc. [6].

We proceed as follows. The role of $\tilde{\psi}$ will be played by an unconstrained 4-form field $P^{(4)}$, while the role of ψ will be played by a self-dual 5-form field $Q^{(5)}$ and all the usual fields of type IIB supergravity except the 4-form field. We shall denote these fields collectively by M . The self-duality constraint on $Q^{(5)}$ takes the form

$$*Q^{(5)} = Q^{(5)}, \tag{2.7}$$

where $*$ denotes Hodge dual *with respect to the flat metric*. Note that since $Q^{(5)}$ is an independent field, this is a purely algebraic constraint. (This will be automatic if we express $Q^{(5)}$ as a bispinor field as in type IIB string theory.) The action will be taken to be of the form³

$$S' = \frac{1}{2} \int dP^{(4)} \wedge *dP^{(4)} - \int dP^{(4)} \wedge Q^{(5)} + \widehat{S}(Q^{(5)}, M), \quad (2.8)$$

where $\widehat{S}(Q^{(5)}, M)$ will be determined by demanding that the equations of motion derived from this action agree with those of type IIB supergravity after we make suitable identification of the fields $(P^{(4)}, Q^{(5)})$ with the 4-form field of type IIB supergravity in the usual formulation. We see that as in (2.1), $P^{(4)}$ appears only in the kinetic term, while $Q^{(5)}$ appears in the kinetic term only linearly, but enters the interaction terms. The action has gauge invariance generated by a 3-form valued parameter $\Xi^{(3)}$

$$\delta_g P^{(4)} = d\Xi^{(3)}, \quad (2.9)$$

with all other fields remaining unchanged. $\Xi^{(3)}$ represents a gauge transformation parameter coming from $\tilde{\lambda}$. There are also other gauge transformations originating from λ . They will be discussed later when we consider the explicit form of \widehat{S} .

Let $R^{(5)}$ denote the anti-self-dual 5-form constructed from M and $Q^{(5)}$ that enters the variation of \widehat{S} under a general variation of the fields via the relation

$$\delta \widehat{S} = -\frac{1}{2} \int R^{(5)} \wedge \delta Q^{(5)} + \delta_M \widehat{S}, \quad (2.10)$$

where δ_M denotes variation with respect to all other fields labelled by M . The anti-self-duality of $R^{(5)}$ is due to the fact that $\delta Q^{(5)}$ is self-dual and the wedge product of two self-dual 5-forms vanishes in 9+1 dimensions. Then the equations of motion for $P^{(4)}$, $Q^{(5)}$ and other fields derived from the action (2.8) take respectively the form:

$$d(*dP^{(4)} - Q^{(5)}) = 0, \quad (2.11)$$

$$dP^{(4)} - *dP^{(4)} + R^{(5)} = 0, \quad (2.12)$$

$$\delta_M \widehat{S} = 0. \quad (2.13)$$

Note that in writing the equation of motion (2.12) of $Q^{(5)}$ we have used the fact that $Q^{(5)}$ is self-dual and that the wedge product of any self-dual tensor with another self-dual tensor vanishes identically. Using (2.12) to eliminate $*dP^{(4)}$ from the first equation we get

$$d(Q^{(5)} - R^{(5)}) = 0. \quad (2.14)$$

This is the analog of (2.4).

We shall identify (2.14) and (2.13) as the physical equations of motion of the theory that should reproduce the equations of motion of type IIB supergravity once we make the

³Our convention for the wedge product and $*$ is such that $\int dP^{(4)} \wedge *dP^{(4)} = \frac{1}{5!} \int (dP^{(4)})_{\mu_1 \dots \mu_5} (dP^{(4)})^{\mu_1 \dots \mu_5}$. Therefore the kinetic term for $P^{(4)}$ has the wrong sign. It will not affect us since fluctuations of $P^{(4)}$ will describe a free field.

correct identification of $Q^{(5)}$ with some combination of fields of type IIB supergravity. The remaining equation (2.12) can be regarded as the equation for $P^{(4)}$. We see from this that different solutions to (2.12) for given $Q^{(5)}$, M differ from each other by free field equations of motion

$$d(\Delta P^{(4)}) - *d(\Delta P^{(4)}) = 0. \tag{2.15}$$

Furthermore which solution to this equation we pick does not affect the physical equations encoded in (2.13), (2.14). Therefore the degrees of freedom associated with $P^{(4)}$ decouple from the theory. This is also apparent from the structure of the action — since the interaction term does not depend on $P^{(4)}$, the Feynman diagrams contributing to amplitudes with external states associated with $Q^{(5)}$ and M never have $P^{(4)}$ propagator as internal lines. $P^{(4)}$ only plays a role in determining the $Q^{(5)}$ - $Q^{(5)}$ propagator by inverting the off-diagonal kinetic term in the $P^{(4)}$, $Q^{(5)}$ space after suitable gauge fixing of the gauge symmetry (2.9). This has been described explicitly in section 7.

3 Type IIB supergravity without gravity and fermions

We begin by considering a simpler version of type IIB supergravity action where we freeze the metric to the Minkowski metric $\eta_{\mu\nu}$ and set all the fermion fields to zero. Even though this is not the full action of type IIB supergravity, this example will illustrate how by adding free fields, we can write down manifestly Lorentz covariant form for the action of interacting chiral p -form fields. In the next section we shall include the effect of gravity and fermion fields.

In absence of gravity and fermions the relevant fields of type IIB supergravity are the dilaton ϕ and the 2-form field $B^{(2)}$ from the NSNS sector and the 0-form field $C^{(0)}$, 2-form field $C^{(2)}$ and the 4-form field $C^{(4)}$ in the RR sector. Let us define

$$H^{(3)} \equiv dB^{(2)}, \quad F^{(3)} = dC^{(2)}, \tag{3.1}$$

and⁴

$$F^{(5)} \equiv dC^{(4)}, \quad \widehat{F}^{(5)} \equiv F^{(5)} + B^{(2)} \wedge F^{(3)}. \tag{3.2}$$

Then the type IIB supergravity action is usually written as

$$S = S_1 + S_2, \tag{3.3}$$

⁴We have chosen to work in a formalism in which $C^{(4)}$, $F^{(5)}$ and $P^{(4)}$, $Q^{(5)}$ are invariant under the gauge transformation associated with RR 2-form but not under the gauge transformation associated with the NSNS 2-form (see (3.6), (3.12)). As a result we do not have manifest symmetry under the $SL(2, R)$ duality transformation that mixes the RR and NSNS 2-forms. We can restore this by replacing $B^{(2)} \wedge F^{(3)}$ by $(B^{(2)} \wedge F^{(3)} - C^{(2)} \wedge H^{(3)})/2$ in all expressions in this and the next section. Consequently in the gauge transformation laws of $C^{(4)}$, $P^{(4)}$, $F^{(5)}$ and $Q^{(5)}$ the factors of $\lambda^{(1)} \wedge F^{(3)}$ and $d\lambda^{(1)} \wedge F^{(3)}$ will have to be replaced respectively by $(\lambda^{(1)} \wedge F^{(3)} - \Lambda^{(1)} \wedge H^{(3)})/2$ and $(d\lambda^{(1)} \wedge F^{(3)} - d\Lambda^{(1)} \wedge H^{(3)})/2$. The resulting formalism will have manifest $SL(2, R)$ duality symmetry with $C^{(4)}$, $F^{(5)}$ and $P^{(4)}$, $Q^{(5)}$ remaining invariant under the duality rotation but now they will transform under the gauge transformations associated with both the 2-form potentials. The two formalisms are related by a field redefinition of $C^{(4)}$, $F^{(5)}$, $P^{(4)}$ and $Q^{(5)}$.

where S_2 is a functional of all fields other than the 4-form potential $C^{(4)}$ and

$$S_1 \equiv -\frac{1}{2} \int \widehat{F}^{(5)} \wedge * \widehat{F}^{(5)} + \int F^{(5)} \wedge B^{(2)} \wedge F^{(3)} \quad (3.4)$$

where $*$ denotes the Hodge dual operation. The equations of motion derived from this action have to be supplemented by the self-duality constraint

$$* \widehat{F}^{(5)} = \widehat{F}^{(5)}. \quad (3.5)$$

S_1 and S_2 are individually invariant under the gauge transformation

$$\delta_g B^{(2)} = d\lambda^{(1)}, \quad \delta_g C^{(2)} = d\Lambda^{(1)}, \quad \delta_g C^{(4)} = d\Lambda^{(3)} - \lambda^{(1)} \wedge F^{(3)}, \quad (3.6)$$

where the subscript ‘ g ’ stands for gauge transformation. In particular $\widehat{F}^{(5)}$ remains invariant under these gauge transformations.

The equations of motion of $C^{(4)}$ derived from the action (3.4) takes the form

$$d(*\widehat{F}^{(5)} - B^{(2)} \wedge F^{(3)}) = 0. \quad (3.7)$$

This will be satisfied automatically if we use the self-duality condition (3.5) and the definition of $\widehat{F}^{(5)}$ given in (3.2). Therefore the net field equation for $C^{(4)}$ can be summarized in the self-duality constraint (3.5) and the definition (3.2) of $\widehat{F}^{(5)}$. Alternatively we can treat $F^{(5)}$ or $\widehat{F}^{(5)} = F^{(5)} + B^{(2)} \wedge F^{(3)}$ as the independent variable and use the self-duality constraint (3.5) and the Bianchi identity (3.7) as independent equations of motion.

The equations of motion of the rest of the fields can be expressed as

$$\delta_M S_1 + \delta_M S_2 = 0, \quad (3.8)$$

where δ_M denotes variation with respect to all other fields collectively denoted by M at fixed $F^{(5)}$. For our analysis we only need to note that

$$\delta_M S_1 = \int \left(*\widehat{F}^{(5)} + F^{(5)} \right) \wedge \delta(B^{(2)} \wedge F^{(3)}) = \int \left(2\widehat{F}^{(5)} - B^{(2)} \wedge F^{(3)} \right) \wedge \delta(B^{(2)} \wedge F^{(3)}), \quad (3.9)$$

where in the second step we have used the self-duality constraint (3.5) and the relationship between $F^{(5)}$ and $\widehat{F}^{(5)}$ given in (3.2).

Let us now consider a different theory in which we trade in the field $C^{(4)}$ for a pair of fields — a 4-form field $P^{(4)}$ and an *independent* 5-form field $Q^{(5)}$ satisfying the self-duality constraint (2.7). We now consider the action

$$S = S'_1 + S_2, \quad (3.10)$$

where S_2 is the same action as what appears in (3.3) and

$$S'_1 = \frac{1}{2} \int dP^{(4)} \wedge *dP^{(4)} - \int dP^{(4)} \wedge Q^{(5)} - \int B^{(2)} \wedge F^{(3)} \wedge Q^{(5)} + \frac{1}{2} \int * \left(B^{(2)} \wedge F^{(3)} \right) \wedge \left(B^{(2)} \wedge F^{(3)} \right). \quad (3.11)$$

This action is invariant under the gauge transformations:

$$\begin{aligned}\delta_g B^{(2)} &= d\lambda^{(1)}, & \delta_g C^{(2)} &= d\Lambda^{(1)}, & \delta_g P^{(4)} &= d\Xi^{(3)} - \lambda^{(1)} \wedge F^{(3)}, \\ \delta_g Q^{(5)} &= -d\lambda^{(1)} \wedge F^{(3)} - * \left(d\lambda^{(1)} \wedge F^{(3)} \right).\end{aligned}\tag{3.12}$$

Note that we have used the same symbols $\lambda^{(1)}$ and $\Lambda^{(1)}$ as in the case of the previous action to indicate that these gauge transformations will turn out to be the same as those appearing in (3.6) once we make the correct identification of the fields. On the other hand, the gauge transformation parameter $\Xi^{(3)}$ is *a priori* unrelated to $\Lambda^{(3)}$ appearing in (3.6).

The equations of motion for $P^{(4)}$ and $Q^{(5)}$ derived from the action (3.10), (3.11) take the form

$$d(*dP^{(4)} - Q^{(5)}) = 0,\tag{3.13}$$

$$dP^{(4)} + B^{(2)} \wedge F^{(3)} - * \left(dP^{(4)} + B^{(2)} \wedge F^{(3)} \right) = 0,\tag{3.14}$$

respectively. Using (3.14) to eliminate $d*P^{(4)}$ term in (3.13), we get

$$dQ^{(5)} = d(B^{(2)} \wedge F^{(3)}) - d* \left(B^{(2)} \wedge F^{(3)} \right).\tag{3.15}$$

We now claim that the theory described by the action (3.10), (3.11) is equivalent to that described by the action (3.3) together with a free 4-form field with self-dual 5-form field strength, under the identification

$$\widehat{F}^{(5)} = \frac{1}{2} \left[Q^{(5)} + B^{(2)} \wedge F^{(3)} + * \left(B^{(2)} \wedge F^{(3)} \right) \right].\tag{3.16}$$

For this claim to be valid the following must hold:

1. $\widehat{F}^{(5)}$ defined in (3.16) should satisfy the self-duality constraint (3.5) and the Bianchi identity (3.7) as a consequence of (3.15).
2. Once we make the identification (3.16), we must have

$$\delta_M S'_1 = \delta_M S_1,\tag{3.17}$$

so that the equations of motion for all other fields derived from the action $S_1 + S_2$ agree with those derived from the action $S'_1 + S_2$. $\delta_M S'_1$ has to be calculated at fixed $P^{(4)}$ and $Q^{(5)}$.

3. Given a solution to the equations of motion derived from the action $S_1 + S_2$, the identification (3.16) should produce a set of solutions to the equations of motion derived from $S'_1 + S_2$ which differ from each other by addition of plane wave solutions. The latter correspond to free fields and do not affect the interacting part of the theory.

We begin by proving the first proposition. $\widehat{F}^{(5)}$ defined in (3.16) clearly satisfies the self-duality constraint (3.5) since $Q^{(5)}$ is self-dual. Furthermore using (3.15) and (3.16) we get

$$d\widehat{F}^{(5)} = d(B^{(2)} \wedge F^{(3)}) = H^{(3)} \wedge F^{(3)}.\tag{3.18}$$

This agrees with (3.7). This establishes the first proposition.

Let us now verify the second proposition given in (3.17). $\delta_M S_1$ is already computed in (3.9), so for verifying (3.17) we need to compute $\delta_M S'_1$. Since $P^{(4)}$ and $Q^{(5)}$ are held fixed while computing $\delta_M S'_1$, we get from (3.11):

$$\delta_M S'_1 = - \int \delta(B^{(2)} \wedge F^{(3)}) \wedge Q^{(5)} + \int * (B^{(2)} \wedge F^{(3)}) \wedge \delta(B^{(2)} \wedge F^{(3)}) . \quad (3.19)$$

Using the antisymmetry of the wedge product, and (3.16), we can express this as

$$\delta_M S'_1 = 2 \int \widehat{F}^{(5)} \wedge \delta(B^{(2)} \wedge F^{(3)}) - \int (B^{(2)} \wedge F^{(3)}) \wedge \delta(B^{(2)} \wedge F^{(3)}) . \quad (3.20)$$

This agrees with $\delta_M S_1$ computed in (3.9), thereby establishing (3.17).

Finally we turn to the third proposition. Given a solution $\widehat{F}^{(5)}$ to eqs.(3.5) and (3.7), eq.(3.16) gives us a value of $Q^{(5)}$ that solves the equations of motion (3.15). But this still leaves open the possibility of getting different $P^{(4)}$ satisfying (3.13), (3.14). A particular solution to these equations is provided by setting

$$P^{(4)} = C^{(4)} , \quad (3.21)$$

where $C^{(4)}$ is related to $\widetilde{F}^{(5)}$ via (3.2). To see this, we note that the solution (3.21) satisfies (3.14) as a consequence of (3.2) and the self-duality condition (3.5). Once (3.14) and (3.15) are satisfied, (3.13) follows automatically. Now suppose a general solution to (3.13), (3.14) for $P^{(4)}$ for given $B^{(2)}$, $C^{(2)}$, $Q^{(5)}$ has the form

$$P^{(4)} = C^{(4)} + \widetilde{P}^{(4)} . \quad (3.22)$$

Then using (3.13), (3.14) we get

$$d * d\widetilde{P}^{(4)} = 0, \quad d\widetilde{P}^{(4)} - *d\widetilde{P}^{(4)} = 0 . \quad (3.23)$$

Furthermore the gauge transformation generated by $\Xi^{(3)}$ acts as

$$\delta_g \widetilde{P}^{(4)} = d\Xi^{(3)} . \quad (3.24)$$

Eqs. (3.23) and the gauge transformation (3.24) are precisely those of a free 4-form gauge field with a self-duality constraint on its field strength. Furthermore which solution of (3.23) we pick does not affect the solutions for the other fields $B^{(2)}$, $C^{(2)}$, $Q^{(5)}$ etc. which are determined completely in terms of the solution to the equations of motion derived from $S_1 + S_2$ via the identification (3.16). This establishes the third proposition.

It is also easy to verify that the gauge transformations generated by $\lambda^{(1)}$ and $\Lambda^{(1)}$ in (3.6) agree with those given in (3.12) under the identification (3.16). Therefore the theory described by the action $S'_1 + S_2$ is equivalent to the one described by the action $S_1 + S_2$ and the self-duality constraint (3.5) up to addition of free fields.

Finally, note that the action (3.11) has a finite number of fields and is polynomial in these fields. Non-polynomiality will arise when we couple this theory to gravity, but this is an inevitable consequence of the fact that gravity is non-polynomial.

4 Inclusion of gravity and fermions

We now consider the effect of inclusion of gravity and fermions. In this case $H^{(3)}$ and $F^{(3)}$ are defined as in (3.1) but the definition of $\widehat{F}^{(5)}$ is modified to

$$F^{(5)} \equiv dC^{(4)}, \quad \widehat{F}^{(5)} = F^{(5)} + Y, \quad (4.1)$$

$$Y \equiv B^{(2)} \wedge F^{(3)} + \text{fermionic terms} \quad (4.2)$$

where ‘fermionic terms’ in the definition of Y describe 5-forms constructed from the fermion bilinear. As in (3.3), the total action is still written as

$$S = S_1 + S_2, \quad (4.3)$$

but the action S_1 given in (3.4) is replaced by

$$S_1 \equiv -\frac{1}{2} \int \widehat{F}^{(5)} \wedge \star_g \widehat{F}^{(5)} + \int F^{(5)} \wedge Y, \quad (4.4)$$

where \star_g denotes the Hodge dual operation with respect to the dynamical metric $g_{\mu\nu}$. Similarly S_2 is covariantized with respect to the general coordinate transformation and includes the Einstein-Hilbert term and fermionic contribution, but continues to be independent of $C^{(4)}$. The self-duality constraint (3.5) is generalized to

$$\star_g \widehat{F}^{(5)} = \widehat{F}^{(5)}. \quad (4.5)$$

In order to check the internal consistency of this procedure we examine the equations of motion for $C^{(4)}$. This takes the form

$$d(\star_g \widehat{F}^{(5)} - Y) = 0. \quad (4.6)$$

Using (4.5) this reduces to

$$d(\widehat{F}^{(5)} - Y) = 0, \quad (4.7)$$

which holds identically as a consequence of (4.1). Therefore once we impose the self-duality condition (4.5) and the definition (4.1) of $\widehat{F}^{(5)}$, the equation of motion for $C^{(4)}$ holds identically.

We introduce vielbein fields $\hat{e}_\mu{}^a$ and its inverse $\hat{E}_a{}^\mu$ via

$$\hat{e}_\mu{}^a \hat{e}_\nu{}^b \eta_{ab} = g_{\mu\nu}, \quad \hat{E}_a{}^\mu \hat{E}_b{}^\nu \eta^{ab} = g^{\mu\nu}, \quad \hat{e}_{\mu a} = \hat{e}_\mu{}^b \eta_{ba}, \quad \hat{E}^{a\mu} = \eta^{ab} \hat{E}_b{}^\mu, \quad (4.8)$$

and define⁵

$$\widetilde{F}_{a_1 \dots a_5}^{(5)} = \hat{E}_{a_1}{}^{\mu_1} \dots \hat{E}_{a_5}{}^{\mu_5} \widehat{F}_{\mu_1 \dots \mu_5}^{(5)}. \quad (4.9)$$

⁵One cautionary comment is in order here. Often one uses the same symbol to denote tensors under general coordinate transformation and tensors under local Lorentz transformation which are related to each other by contraction with vielbeins, e.g. $A_a = \hat{E}_a{}^\mu A_\mu$. We shall not use this convention and make all factors of vielbeins explicit. For example we have used a different symbol \widetilde{F} to denote the transform of \widehat{F} to a tensor under local Lorentz transformation. \widehat{F} will always denote the quantity whose components are given by the components of the right hand side of (4.1) describing a tensor under general coordinate transformation.

Then the self-duality condition (4.5) on the 5-form field strength can be reexpressed as

$$* \tilde{F}^{(5)} = \tilde{F}^{(5)}, \tag{4.10}$$

where, as in section 3, $*$ now denotes the Hodge dual with respect to flat Minkowski metric.

In the following we shall gauge fix the local Lorentz transformation by choosing $\hat{E}^{a\mu}$ and $\hat{e}_{\mu a}$ to be symmetric matrices. The insight for this again comes from string field theory whose gauge symmetries do not include local Lorentz transformation. To facilitate this choice of gauge, let us express the first equation of (4.8) in the matrix form as

$$\hat{e}\eta\hat{e}^T = g, \tag{4.11}$$

where \hat{e} denotes the matrix with components $\hat{e}_{\mu a}$. When the metric $g_{\mu\nu}$ is close to $\eta_{\mu\nu}$ a solution to (4.11) for which $\hat{e}_{\mu a} = \hat{e}_{a\mu}$ may be expressed as

$$\hat{e}\eta = (g\eta)^{1/2}, \tag{4.12}$$

where in defining the square root we take the matrix that has all positive eigenvalues. Writing $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, (4.12) can be written as

$$\hat{e}\eta = (1 + h\eta)^{1/2} = \left(1 + \frac{1}{2}h\eta - \frac{1}{8}h\eta h\eta + \dots\right), \tag{4.13}$$

so that

$$\hat{e} = \left(\eta + \frac{1}{2}h - \frac{1}{8}h\eta h + \dots\right) \tag{4.14}$$

is symmetric. In component this corresponds to

$$\hat{e}_{ad} = \eta_{ad} + \frac{1}{2}h_{ad} - \frac{1}{8}h_{ab}\eta^{bc}h_{cd} + \dots \tag{4.15}$$

Note that in this gauge we no longer have the distinction between the coordinate indices μ, ν, \dots and the tangent space indices a, b, \dots . We shall raise and lower all indices with the flat metric η . There is a rigid Lorentz transformation that preserves this gauge: under this e_{ab} transforms as a covariant rank two tensor. A general coordinate transformation must be accompanied by a compensating local Lorentz transformation in order to preserve this gauge.

We now introduce the following notation for operators acting on 5-forms. We use the indices A, B, \dots to denote the index $(a_1 \dots a_5), (b_1 \dots b_5), \dots$ of 5-forms. Therefore A, B, \dots each takes $\binom{10}{5}$ independent values. However in defining sum over one of these indices — say A — we shall find it more convenient to define it as a sum over all values of a_1, \dots, a_5 , i.e.

$$\sum_A \equiv \sum_{a_1} \sum_{a_2} \dots \sum_{a_5}. \tag{4.16}$$

In this notation the 5-form $\tilde{F}_{a_1 \dots a_5}^{(5)}$ will be denoted as $\tilde{F}_A^{(5)}$. We also introduce the following matrices in this space:

$$\begin{aligned} \zeta^{AB} &= \eta^{a_1 b_1} \dots \eta^{a_5 b_5}, & \zeta_{AB} &= \eta_{a_1 b_1} \dots \eta_{a_5 b_5}, & E^{AB} &= \hat{E}^{a_1 b_1} \dots \hat{E}^{a_5 b_5}, \\ e_{AB} &= \hat{e}_{a_1 b_1} \dots \hat{e}_{a_5 b_5}, & \varepsilon^{AB} &= \frac{1}{5!} \epsilon^{a_1 \dots a_5 b_1 \dots b_5}, \end{aligned} \tag{4.17}$$

where $\epsilon^{a_0 \dots a_9}$ is totally anti-symmetric in all the indices and $\epsilon^{01 \dots 9} = 1$. Note that we have used the same symbol ζ for labelling a matrix with both upper index and both lower index, but which one to use should be clear from the expression in which it appears and the rule that an upper index can only contract with a lower index and vice versa. For example in ζe we have to use ζ with upper indices while in ζE we shall use ζ with lower index. These matrices satisfy the identities:

$$\zeta^T = \zeta, \quad e^T = e, \quad E^T = E, \quad \varepsilon^T = -\varepsilon, \quad \varepsilon \zeta \varepsilon = \zeta, \quad \zeta \zeta = I, \quad e \varepsilon e = (-\det \hat{e}) \zeta \varepsilon \zeta, \tag{4.18}$$

etc. while acting on 5-forms. Here I denotes identity matrix and $\det \hat{e}$ is the determinant of the 10×10 matrix $\hat{e}_{\mu a}$. Since all of the quantities appearing in (4.17) transform covariantly under rigid Lorentz transformation, an action built out of these ingredients will have manifest Lorentz invariance.

The self-duality condition (4.10) on $\tilde{F}^{(5)}$ can be expressed as

$$\zeta \varepsilon \tilde{F}^{(5)} = \tilde{F}^{(5)}. \tag{4.19}$$

Also in this notation (4.9) can be written as

$$\tilde{F}^{(5)} = \zeta E \hat{F}^{(5)} = \zeta E (dC^{(4)} + Y). \tag{4.20}$$

Using the fact that $e\zeta$ is the inverse matrix of ζE , we get

$$\hat{F}^{(5)} \equiv dC^{(4)} + Y = e \zeta \tilde{F}^{(5)}. \tag{4.21}$$

This gives

$$d(e\zeta \tilde{F}^{(5)} - Y) = 0. \tag{4.22}$$

We can regard $F^{(5)} = e\zeta \tilde{F}^{(5)} - Y$ as independent variable instead of $C^{(4)}$, and eqs. (4.19) and (4.22) as the independent equations that determine $F^{(5)}$.

We shall now attempt to replace the action S_1 by an action S'_1 :

$$S'_1 = \frac{1}{2} \int dP^{(4)} \wedge *dP^{(4)} - \int dP^{(4)} \wedge Q^{(5)} + \hat{S}_1(Q^{(5)}, M), \tag{4.23}$$

and write the total action as

$$S' = S'_1 + S_2, \tag{4.24}$$

in the spirit of (3.10), (3.11). Here, as in (2.8), $P^{(4)}$ is an unconstrained 4-form field, $Q^{(5)}$ is a 5-form field satisfying $Q^{(5)} = *Q^{(5)}$ and \hat{S}_1 is a functional of $Q^{(5)}$ and all the usual fields of type IIB supergravity other than the 4-form field $C^{(4)}$, collectively called M . S_2 is the same action as what appears in (4.3). Our goal will be to determine \hat{S}_1 by demanding that $S'_1 + S_2$ gives the same equations of motion as $S_1 + S_2$ and the self-duality constraint (4.19), as long as we make proper identification of fields between the two formalisms. While doing so, we shall maintain manifest Lorentz covariance at all stages, but invariance under general coordinate transformation will not be manifest.

Since $S'_1 + S_2$ has the same structure as the action (2.8) with all the $Q^{(5)}$ and $P^{(4)}$ dependence coming through S'_1 we have the analogs of (2.11), (2.12) and (2.14) as equations of motion of $P^{(4)}$, $Q^{(5)}$:

$$d(*dP^{(4)} - Q^{(5)}) = 0, \tag{4.25}$$

$$dP^{(4)} - *dP^{(4)} + R^{(5)} = 0, \tag{4.26}$$

and

$$d(Q^{(5)} - R^{(5)}) = 0, \tag{4.27}$$

where $R^{(5)}$ is an anti-self-dual 5-form, defined via the equation

$$\delta\widehat{S}_1 = -\frac{1}{2} \int R^{(5)} \wedge \delta Q^{(5)} + \delta_M \widehat{S}_1, \tag{4.28}$$

and δ_M denotes variation with respect to all other fields labelled by M at fixed $P^{(4)}$, $Q^{(5)}$. Comparing (4.27) with (4.22) we arrive at the identification

$$Q^{(5)} - R^{(5)} = 2(e\zeta\widetilde{F}^{(5)} - Y), \tag{4.29}$$

where the normalization factor of 2 on the right hand side has been chosen to ensure compatibility between the normalization of the action (4.4) and (4.23) (see e.g. (3.16)). Comparing the self-dual and anti-self-dual parts on the two sides and using the fact that the Hodge star operation corresponds to matrix multiplication by $\zeta\varepsilon$ from the left, we get

$$Q^{(5)} = (1 + \zeta\varepsilon)e\zeta\widetilde{F}^{(5)} - (1 + \zeta\varepsilon)Y, \tag{4.30}$$

and

$$-R^{(5)} = (1 - \zeta\varepsilon)e\zeta\widetilde{F}^{(5)} - (1 - \zeta\varepsilon)Y. \tag{4.31}$$

Our goal will be to eliminate $\widetilde{F}^{(5)}$ from these equations to express $R^{(5)}$ as a function of $Q^{(5)}$ and the fields M appearing in (4.23), and then solve (4.28) to determine the form of \widehat{S}_1 . For this we shall determine $\widetilde{F}^{(5)}$ in terms of $Q^{(5)}$ using (4.30) and then substitute in (4.31). Using the self-duality condition $\zeta\varepsilon\widetilde{F}^{(5)} = \widetilde{F}^{(5)}$, we can solve (4.30) as

$$\widetilde{F}^{(5)} = \left\{ 1 + \frac{1}{2}(1 + \zeta\varepsilon)(e\zeta - 1) \right\}^{-1} \left(\frac{1}{2}Q^{(5)} + \frac{1}{2}(1 + \zeta\varepsilon)Y \right). \tag{4.32}$$

Substituting this into (4.31) we get

$$-R^{(5)} = \frac{1}{2}(1 - \zeta\varepsilon)e\zeta \left\{ 1 + \frac{1}{2}(1 + \zeta\varepsilon)(e\zeta - 1) \right\}^{-1} \left(Q^{(5)} + (1 + \zeta\varepsilon)Y \right) - (1 - \zeta\varepsilon)Y. \tag{4.33}$$

Using the fact that $Q^{(5)}$ and $(1 + \zeta\varepsilon)$ are annihilated by $(1 - \zeta\varepsilon)$ from the left, we can subtract terms proportional to $(1 - \zeta\varepsilon)Q^{(5)}$ and $(1 - \zeta\varepsilon)(1 + \zeta\varepsilon)$ from this expression. This leads to

$$-R^{(5)} = \frac{1}{2}(1 - \zeta\varepsilon)(e\zeta - 1) \left\{ 1 + \frac{1}{2}(1 + \zeta\varepsilon)(e\zeta - 1) \right\}^{-1} \left(Q^{(5)} + (1 + \zeta\varepsilon)Y \right) - (1 - \zeta\varepsilon)Y. \tag{4.34}$$

We now note that

$$\int P \wedge Q = \frac{1}{5!} \int \varepsilon^{AB} P_A Q_B = -\frac{1}{5!} \int \varepsilon^{AB} Q_A P_B = - \int Q^T \varepsilon P, \quad (4.35)$$

where Q^T denotes the transpose of Q multiplied by a factor of $1/5!$. Using this and (4.34), (4.28) gives

$$\begin{aligned} \delta \widehat{S}_1 = & -\frac{1}{2} \int \delta Q^{(5)T} \varepsilon (1 - \zeta \varepsilon) \left[\frac{1}{2} (e\zeta - 1) \left\{ 1 + \frac{1}{2} (1 + \zeta \varepsilon) (e\zeta - 1) \right\}^{-1} Q^{(5)} \right. \\ & \left. + \frac{1}{2} (e\zeta - 1) \left\{ 1 + \frac{1}{2} (1 + \zeta \varepsilon) (e\zeta - 1) \right\}^{-1} (1 + \zeta \varepsilon) Y - Y \right] + \delta_M \widehat{S}_1. \end{aligned} \quad (4.36)$$

Our goal will be to see if we can integrate this to get \widehat{S}_1 . To this end we define:

$$\mathcal{M} \equiv (\zeta - \varepsilon) \left\{ (e\zeta - 1) \left(1 + \frac{1}{2} (1 + \zeta \varepsilon) (e\zeta - 1) \right)^{-1} \zeta \right\} (\varepsilon + \zeta). \quad (4.37)$$

It is now easy to see using (4.18) and the relation $Q^{(5)} = \zeta \varepsilon Q^{(5)} = \frac{1}{2} (1 + \zeta \varepsilon) Q^{(5)}$ that we can express (4.36) as

$$\delta \widehat{S}_1 = \frac{1}{8} \int \delta Q^{(5)T} \mathcal{M} Q^{(5)} + \frac{1}{2} \int \delta Q^{(5)T} \left[\frac{1}{2} \mathcal{M} Y - (\zeta - \varepsilon) Y \right] + \delta_M \widehat{S}_1. \quad (4.38)$$

We can evaluate \mathcal{M} by first expanding the terms inside the curly bracket on the right hand side of (4.37) in a Taylor series in $e\zeta - 1$, and then expanding each term in this series in binomial expansion. In any given term in this expansion containing products of e , ζ and ε we can now try to reduce the number of terms in the product using (4.18) and the fact that $\varepsilon\zeta$ acting on $(\varepsilon + \zeta)$ from the left gives $(\varepsilon + \zeta)$ and $\zeta\varepsilon$ acting on $(\zeta - \varepsilon)$ from the right gives $-(\zeta - \varepsilon)$. Using this it is easy to check that each term in the expansion can be brought to $(\zeta - \varepsilon)(e\zeta)^n \zeta(\zeta + \varepsilon)$ possibly multiplied by a power of $\det \hat{e}$. Since each of these represent a symmetric matrix, we conclude that \mathcal{M} is a symmetric matrix. Therefore the following action satisfies (4.38)

$$\widehat{S}_1 = \frac{1}{16} \int Q^{(5)T} \mathcal{M} Q^{(5)} + \frac{1}{2} \int Q^{(5)T} \left[\frac{1}{2} \mathcal{M} Y - (\zeta - \varepsilon) Y \right] + \widetilde{S}_1(M), \quad (4.39)$$

where \widetilde{S}_1 is independent of $Q^{(5)}$ and $P^{(4)}$ but could depend on the other fields of the theory.

In order to determine \widetilde{S}_1 we have to compare $\delta_M S_1$ with $\delta_M S'_1$. Recall that in computing $\delta_M S'_1$ we keep fixed $P^{(4)}$ and $Q^{(5)}$ while in computing $\delta_M S_1$ we keep fixed $C^{(4)}$ or equivalently $F^{(5)}$. Now we get from (4.23) and (4.39):

$$\delta_M S'_1 = \delta_M \widehat{S}_1 = \frac{1}{2} \int \delta Y^T \left[\frac{1}{2} \mathcal{M} - (\varepsilon + \zeta) \right] Q^{(5)} + \delta_M \widetilde{S}_1 + \mathcal{O}(\delta \hat{e}) \quad (4.40)$$

where $\mathcal{O}(\delta \hat{e})$ denote terms proportional to $\delta \hat{e}$ — these would come from variation of \mathcal{M} . Note that we have transposed the matrix sandwiched between $Q^{(5)T}$ and Y using the symmetry of \mathcal{M} . Using (4.37), and after some algebra, this can be expressed as

$$\delta_M S'_1 = - \int \delta Y^T \varepsilon e\zeta \left(1 + \frac{1}{2} (1 + \zeta \varepsilon) (e\zeta - 1) \right)^{-1} Q^{(5)} + \delta_M \widetilde{S}_1 + \mathcal{O}(\delta \hat{e}). \quad (4.41)$$

On the other hand $\delta_M S_1$ can be computed from (4.4), (4.1):

$$\delta_M S_1 = - \int \delta Y \wedge (\star_g \widehat{F}^{(5)} + F^{(5)}) + \mathcal{O}(\delta \hat{e}) = - \int \delta Y \wedge (2\widehat{F}^{(5)} - Y) + \mathcal{O}(\delta \hat{e}), \quad (4.42)$$

where in the second step we have used the relation $\star_g \widehat{F}^{(5)} = \widehat{F}^{(5)}$. In the matrix notation this equation takes the form

$$\delta_M S_1 = - \int \delta Y^T \varepsilon (2\widehat{F}^{(5)} - Y) + \mathcal{O}(\delta \hat{e}). \quad (4.43)$$

Using (4.21), (4.32), and some algebra, we arrive at

$$\delta_M S_1 = - \int \delta Y^T \varepsilon e \zeta \left(1 + \frac{1}{2}(1 + \zeta \varepsilon)(e \zeta - 1) \right)^{-1} Q^{(5)} - \int \delta Y^T \zeta Y + \frac{1}{2} \int \delta Y^T \mathcal{M} Y + \mathcal{O}(\delta \hat{e}). \quad (4.44)$$

Comparing (4.41) and (4.44) we get

$$\delta_M \widetilde{S}_1 = - \int \delta Y^T \zeta Y + \frac{1}{2} \int \delta Y^T \mathcal{M} Y + \mathcal{O}(\delta \hat{e}), \quad (4.45)$$

and hence

$$\widetilde{S}_1 = - \frac{1}{2} \int Y^T \zeta Y + \frac{1}{4} \int Y^T \mathcal{M} Y + \dots, \quad (4.46)$$

where \dots now denotes some functional of $\hat{e}_{a\mu}$ only. However such a term, under a variation of $\hat{e}_{a\mu}$, will give a non-vanishing contribution to $\delta_M S'_1$ even when $Q^{(5)}$ and Y vanish. It is easy to see from (4.4) that $\delta_M S_1$ does not have such terms. Therefore if we want the equality of $\delta_M S'_1$ and $\delta_M S_1$ to hold even for variation with respect to $\hat{e}_{a\mu}$, then the \dots terms in (4.46) must vanish. Therefore we get from (4.23), (4.39), (4.46)

$$\begin{aligned} S'_1 &= \frac{1}{2} \int dP^{(4)} \wedge *dP^{(4)} - \int dP^{(4)} \wedge Q^{(5)} + \frac{1}{16} \int Q^{(5)T} \mathcal{M} Q^{(5)} \\ &\quad + \frac{1}{2} \int Q^{(5)T} \left[\frac{1}{2} \mathcal{M} Y - (\zeta - \varepsilon) Y \right] - \frac{1}{2} \int Y^T \zeta Y + \frac{1}{4} \int Y^T \mathcal{M} Y. \end{aligned} \quad (4.47)$$

This is what should replace the action (4.4) in this formulation. Note that unlike in the case of the action (4.4), where a self-duality constraint has to be imposed after deriving the equations of motion, there is no such additional constraint for the action (4.47).

In order to verify that the classical equations of motion derived from (4.47) are equivalent to the usual equations of motion of type IIB string theory, we also need to check that the variation of S'_1 with respect to \hat{e}_{ab} at fixed $P^{(4)}$, $Q^{(5)}$ agrees with the variation of S_1 with respect to \hat{e}_{ab} at fixed $F^{(5)}$. In making this comparison we can ignore possible dependence on \hat{e}_{ab} entering through Y since we have already ensured that the terms involving δY agree between $\delta_M S_1$ and $\delta_M S'_1$. Let us denote by δ_e the variation with respect to \hat{e}_{ab} at fixed Y , $P^{(4)}$, $Q^{(5)}$ while acting on S'_1 and fixed Y , $F^{(5)}$ while acting on S_1 . We need to show the equality of $\delta_e S_1$ and $\delta_e S'_1$. Now from (4.47) we have

$$\delta_e S'_1 = \frac{1}{16} \int Q^{(5)T} \delta \mathcal{M} Q^{(5)} + \frac{1}{4} \int Q^{(5)T} \delta \mathcal{M} Y + \frac{1}{4} \int Y^T \delta \mathcal{M} Y. \quad (4.48)$$

Using (4.32) and the results $(1 - \varepsilon\zeta)\delta\mathcal{M} = 2\delta\mathcal{M} = \delta\mathcal{M}(1 + \zeta\varepsilon)$ that follows from (4.37), we can express this as

$$\delta_e S'_1 = \frac{1}{4} \int \tilde{F}^{(5)T} \left\{ 1 + \frac{1}{2}(1 + \zeta\varepsilon)(e\zeta - 1) \right\}^T \delta\mathcal{M} \left\{ 1 + \frac{1}{2}(1 + \zeta\varepsilon)(e\zeta - 1) \right\} \tilde{F}^{(5)}. \quad (4.49)$$

From (4.37) we get

$$\delta\mathcal{M} = (\zeta - \varepsilon) \left(1 + \frac{1}{2}(e\zeta - 1)(1 + \zeta\varepsilon) \right)^{-1} \delta e \zeta \left(1 + \frac{1}{2}(1 + \zeta\varepsilon)(e\zeta - 1) \right)^{-1} (1 + \zeta\varepsilon). \quad (4.50)$$

Using this, and the result $(1 + \zeta\varepsilon)\tilde{F}^{(5)} = 2\tilde{F}^{(5)}$, we get

$$\delta_e S'_1 = \frac{1}{2} \int \tilde{F}^{(5)T} \left\{ 1 + \frac{1}{2}(1 + \zeta\varepsilon)(e\zeta - 1) \right\}^T (\zeta - \varepsilon) \left(1 + \frac{1}{2}(e\zeta - 1)(1 + \zeta\varepsilon) \right)^{-1} \delta e \zeta \tilde{F}^{(5)}. \quad (4.51)$$

Let us now turn to the computation of $\delta_e S_1$. To make the metric dependence of (4.4) manifest we introduce the matrix notation:

$$G^{AB} = g^{a_1 b_1} \dots g^{a_5 b_5} \quad \text{for } A = (a_1, \dots, a_5), \quad B = (b_1, \dots, b_5), \quad (4.52)$$

and express (4.4) as

$$S_1 = -\frac{1}{2 \times 5!} \int \sqrt{-\det g} \hat{F}_A^{(5)} G^{AB} \hat{F}_B^{(5)} + \frac{1}{5!} \int F_A^{(5)} \varepsilon^{AB} Y_B. \quad (4.53)$$

This gives

$$\delta_e S_1 = -\frac{1}{2 \times 5!} \int (\delta \sqrt{-\det g}) \hat{F}_A^{(5)} G^{AB} \hat{F}_B^{(5)} - \frac{1}{2 \times 5!} \int \sqrt{-\det g} \hat{F}_A^{(5)} \delta G^{AB} \hat{F}_B^{(5)}. \quad (4.54)$$

The first term vanishes due to the self-duality constraint (4.5) on $\hat{F}^{(5)}$. The second term can be simplified using (4.21) and

$$G = E\zeta E, \quad \sqrt{-\det g} = -\det \hat{e}. \quad (4.55)$$

This gives, recalling that the definition of $\tilde{F}^{(5)T}$ includes a transpose and a multiplicative factor of $1/5!$:

$$\delta_e S_1 = -\frac{1}{2} \int (-\det \hat{e}) \tilde{F}^{(5)T} \zeta e (\delta E \zeta E + E \zeta \delta E) e \zeta \tilde{F}^{(5)}. \quad (4.56)$$

Since $E = e^{-1}$ we have $\delta E = -e^{-1} \delta e e^{-1}$. Using this we can simplify this equation as

$$\delta_e S_1 = \frac{1}{2} \int (-\det \hat{e}) \tilde{F}^{(5)T} (\zeta \delta e e^{-1} + e^{-1} \delta e \zeta) \tilde{F}^{(5)} = \int (-\det \hat{e}) \tilde{F}^{(5)T} e^{-1} \delta e \zeta \tilde{F}^{(5)}, \quad (4.57)$$

where in the second step we have replaced the first term in the middle expression by its transpose. Using (4.18) and (4.19) we can write

$$\tilde{F}^{(5)T} = -\tilde{F}^{(5)T} \varepsilon \zeta = -\tilde{F}^{(5)T} \zeta \zeta \varepsilon \zeta = -(-\det \hat{e})^{-1} \tilde{F}^{(5)T} \zeta e \varepsilon e. \quad (4.58)$$

Substituting this into (4.57) we get

$$\delta_e S_1 = - \int \tilde{F}^{(5)T} \zeta e \varepsilon \delta e \zeta \tilde{F}^{(5)}. \quad (4.59)$$

From (4.51) and (4.59) we now get

$$\begin{aligned} & \delta_e S'_1 - \delta_e S_1 \\ &= \frac{1}{2} \int \tilde{F}^{(5)T} \left[\left\{ 1 + \frac{1}{2}(1+\zeta\varepsilon)(e\zeta-1) \right\}^T (\zeta - \varepsilon) + 2\zeta e \varepsilon \left(1 + \frac{1}{2}(e\zeta - 1)(1+\zeta\varepsilon) \right) \right] \\ & \quad \left(1 + \frac{1}{2}(e\zeta - 1)(1 + \zeta\varepsilon) \right)^{-1} \delta e \zeta \tilde{F}^{(5)}. \end{aligned} \quad (4.60)$$

Straightforward manipulation of the expression inside the square bracket using the self-duality of $\tilde{F}^{(5)}$ gives

$$\delta_e S'_1 - \delta_e S_1 = 0. \quad (4.61)$$

This shows complete equivalence between the equations of motion derived from $S_1 + S_2$ and $S'_1 + S_2$.

Given the action $S'_1 + S_2$, one can formally quantize this using Batalin-Vilkovisky (BV) formalism following the same route as for the string field theory action of [5], since the structure of gauge transformations in the two theories are similar. However this theory will suffer from the usual ultraviolet divergences of superstring theory. Therefore the full quantization of the theory will require using the full string field theory. For this reason, the utility of this action lies not in the fact that we can use it to quantize type IIB supergravity, but in that it is through action of this type that one can make a direct link between superstring field theory — needed for a systematic quantization of superstring theory — and supergravity describing the low energy dynamics of the theory. This construction also throws light on an apparent puzzle — it has been known since early days of string theory that the RR vertex operators in the canonical $(-1/2, -1/2)$ picture couple directly to the RR field strengths instead of RR gauge fields, while supergravity theories, including type IIA supergravity which has an action, are naturally formulated in terms of the gauge fields. The formulation of type IIB supergravity presented here illustrates how supergravity theories can be formulated directly in terms of field strengths. This also tells us that the version of supergravity that will emerge naturally from string field theory should involve a formulation in which the other RR fields are also described in terms of their field strengths. Such a formulation can be easily obtained by generalizing the construction described here, although this is not necessary for being able to write down the action.

5 General coordinate transformation

Even though our action is not manifestly invariant under general coordinate transformation, since the equations of motion are equivalent to the equations of motion of type IIB supergravity, the formalism has hidden general coordinate invariance. In this section we shall determine the general coordinate transformation laws for various fields in our formalism.

First of all, for all the fields other than $P^{(4)}$ and $Q^{(5)}$, collectively called M in (2.8), the general coordinate transformation rules are the same as in the original formulation of type IIB supergravity since these fields are in one to one correspondence in the two formalisms. This includes also the vielbein fields, but we should keep in mind that general coordinate transformations have to be accompanied by a local Lorentz transformation to keep the vielbein symmetric. Since under combined infinitesimal general coordinate and local Lorentz transformations generated by the parameters ξ^a and $\omega_{ab} = -\omega_{ba}$, we have

$$\delta \hat{e}_{ab} = \partial_a \xi^c \hat{e}_{cb} + \xi^c \partial_c \hat{e}_{ab} + \hat{e}_{ac} \eta^{cd} \omega_{db}, \quad (5.1)$$

the requirement that $\delta \hat{e}_{ab}$ remains symmetric determines ω_{ab} in terms of ξ^a via the relations:

$$\hat{e}_{ac} \eta^{cd} \omega_{db} - \hat{e}_{bc} \eta^{cd} \omega_{da} = \partial_b \xi^c \hat{e}_{ca} - \partial_a \xi^c \hat{e}_{cb}. \quad (5.2)$$

This compensating local Lorentz transformation must also act on the fermions under a general coordinate transformation.

For determining the transformation laws of $P^{(4)}$ and $Q^{(5)}$ we again draw our insight from the structure of gauge transformations in superstring field theory given in (2.6). In the truncated version of the theory in which the only field coming from $\tilde{\psi}$ is $P^{(4)}$, the gauge transformation parameters are also truncated with $\tilde{\lambda}$ giving only the 3-form gauge transformation parameter $\Xi^{(3)}$ that generates $P^{(4)} \rightarrow P^{(4)} + d\Xi^{(3)}$ transformation, and λ containing all other gauge transformation parameters including general coordinate transformation. If we denote by δ_ξ the general coordinate transformation with infinitesimal parameter ξ then it follows from (2.6) that $\delta_\xi P^{(4)}$ and $\delta_\xi Q^{(5)}$ will be a function of the fields coming from ψ . This includes $Q^{(5)}$ and other fields collectively denoted by M in (2.8) but does not include $P^{(4)}$. This is clearly an unusual transformation law for $P^{(4)}$ since even the usual term $\xi^a \partial_a P^{(4)}$ will not be present in the transformation of $P^{(4)}$.⁶

We now note the following:

1. We have seen in section 4 that the variation δ_M of S_1 with respect to all fields at fixed $F^{(5)}$ agrees with the variation δ'_M of S'_1 with respect to all fields at fixed $P^{(4)}$ and $Q^{(5)}$. As a special case, this also applies to variations induced by general coordinate transformation.
2. S_1 given in (4.4) is manifestly invariant under general coordinate transformation *if we regard $F^{(5)}$ as an independent 5-form field variable* and $\hat{F}^{(5)}$ to be given by $F^{(5)} + Y$.
3. Therefore if we can ensure that the variation of S'_1 under the general coordinate transformation of $P^{(4)}$ and $Q^{(5)}$ agrees with the variation of S_1 under the general coordinate transformation of $F^{(5)}$ then we would have proved the general coordinate invariance of S'_1 . Denoting by the symbol δ'_ξ the transformation of the action induced by the general coordinate transformation of $F^{(5)}$, $P^{(4)}$ and $Q^{(5)}$ only, the requirement given above translates to

$$\delta'_\xi S'_1 = \delta'_\xi S_1. \quad (5.3)$$

⁶Unusual form of general coordinate transformation laws also appear in double field theories [32–35].

4. Since S_2 is manifestly invariant under general coordinate transformation, this will also prove the general coordinate invariance of $S'_1 + S_2$.

Note the emphasis on the fact that in computing $\delta'_\xi S_1$ we need to regard $F^{(5)}$ as an independent field variable. The reason for this is as follows. As is well known, while checking symmetries of the action under a given transformation we cannot use equations of motion. For S'_1 this translates to the statement that we should not use eqs. (4.25), (4.26) and (4.27). However we are allowed to use the self-duality of $Q^{(5)}$ since this is an algebraic constraint on the field imposed at the beginning. We shall follow this guideline while computing $\delta'_\xi S'_1$. Now for S_1 we normally regard $C^{(4)}$ as independent variable. If we calculate the corresponding change in S_1 under general coordinate transformation of $C^{(4)}$ then in the resulting expression we would have used the Bianchi identity $dF^{(5)} = 0$ already since this is automatic when $F^{(5)}$ is expressed in terms of $C^{(4)}$. Since under the identification (4.29) this translates to the equation of motion (4.27) derived from S'_1 , we can no longer compare $\delta'_\xi S_1$ and $\delta'_\xi S'_1$ off-shell. To avoid this we proceed by noting that S_1 given in (4.4), with the identification $\widehat{F}^{(5)} = F^{(5)} + Y$, is invariant under general coordinate transformation even if we regard $F^{(5)}$ as an independent 5-form field variable. The resulting expression for $\delta'_\xi S_1$ holds even when $F^{(5)}$ does not satisfy Bianchi identity, and furthermore since this is computed for general $F^{(5)}$, the formula for $\delta'_\xi S_1$ will continue to hold when we impose the self-duality constraint (4.5) *after computing the variation*. Therefore if we can ensure that the identity (5.3) holds for this expression for $\delta'_\xi S_1$, this would establish the invariance of S'_1 under general coordinate transformation.

We begin with the computation of $\delta'_\xi S'_1$. It follows from (4.23) and (4.28) that

$$\delta'_\xi S'_1 = \int dP^{(4)} \wedge *d\delta_\xi P^{(4)} - \int dP^{(4)} \wedge \delta_\xi Q^{(5)} - \int d\delta_\xi P^{(4)} \wedge Q^{(5)} - \frac{1}{2} \int R^{(5)} \wedge \delta_\xi Q^{(5)}. \quad (5.4)$$

This will eventually have to be compared with $\delta'_\xi S_1$. Now $\delta'_\xi S_1$ can depend on $Q^{(5)}$ through its dependence on $F^{(5)}$ and the relation between $F^{(5)}$ and $Q^{(5)}$ encoded in (4.32), (4.1), (4.20), but it does not have any dependence on $P^{(4)}$. Therefore the terms involving $P^{(4)}$ in $\delta'_\xi S'_1$ must cancel among themselves. Since we have argued that neither $\delta_\xi P^{(4)}$ nor $\delta_\xi Q^{(5)}$ depend on $P^{(4)}$ we see that only the first two terms on the right hand side of (5.4) have $P^{(4)}$ dependence and hence they must cancel. This can be achieved by setting

$$\delta_\xi Q^{(5)} = d\delta_\xi P^{(4)} + *\delta_\xi P^{(4)}. \quad (5.5)$$

With this, (5.4) reduces to

$$\delta'_\xi S'_1 = - \int d\delta_\xi P^{(4)} \wedge Q^{(5)} - \frac{1}{2} \int R^{(5)} \wedge \delta_\xi Q^{(5)}. \quad (5.6)$$

Using (5.5) again and the anti-self-duality of $R^{(5)}$, we can express this as

$$\delta'_\xi S'_1 = - \int d\delta_\xi P^{(4)} \wedge (Q^{(5)} - R^{(5)}). \quad (5.7)$$

Let us now compute $\delta'_\xi S_1$. Using (4.4) and the fact that in computing $\delta'_\xi S_1$ we only allow $F^{(5)}$ to vary, we get

$$\delta'_\xi S_1 = - \int \delta_\xi F^{(5)} \wedge (\star_g \widehat{F}^{(5)} - Y) = - \int \delta_\xi F^{(5)} \wedge F^{(5)}, \quad (5.8)$$

where in the second step we have used the self-duality relation (4.5) and the relation $\widehat{F}^{(5)} - Y = F^{(5)}$ given in (4.1). Now for any pair of p and $(11-p)$ forms $K^{(p)}$ and $L^{(11-p)}$ in 9+1 dimensions, we have the general relations

$$\delta_\xi K^{(p)} = 1_\xi dK^{(p)} + d(1_\xi K^{(p)}), \quad \int 1_\xi K^{(p)} \wedge L^{(11-p)} = - \int 1_\xi L^{(11-p)} \wedge K^{(p)}, \quad (5.9)$$

where 1_ξ denotes the contraction of ξ with the differential form. Using this we get

$$\delta_\xi F^{(5)} = d 1_\xi F^{(5)} + 1_\xi dF^{(5)}, \quad (5.10)$$

and hence

$$\begin{aligned} \delta'_\xi S_1 &= - \int (d 1_\xi F^{(5)} + 1_\xi dF^{(5)}) \wedge F^{(5)} = - \int (d 1_\xi F^{(5)} \wedge F^{(5)} - 1_\xi F^{(5)} \wedge dF^{(5)}) \\ &= - \int (d 1_\xi F^{(5)} \wedge F^{(5)} + d 1_\xi F^{(5)} \wedge F^{(5)}) = -2 \int d 1_\xi F^{(5)} \wedge F^{(5)}. \end{aligned} \quad (5.11)$$

Comparing the right hand sides of (5.7) and (5.11), and using the identification (4.29), we now see that $\delta'_\xi S_1$ and $\delta'_\xi S'_1$ agree if we set

$$\delta_\xi P^{(4)} = 1_\xi F^{(5)} = 1_\xi (\widehat{F}^{(5)} - Y). \quad (5.12)$$

Using (4.21), (4.32) this can be expressed as

$$\delta_\xi P^{(4)} = 1_\xi \left[e\zeta \left\{ 1 + \frac{1}{2}(1 + \zeta\varepsilon)(e\zeta - 1) \right\}^{-1} \left(\frac{1}{2}Q^{(5)} + \frac{1}{2}(1 + \zeta\varepsilon)Y \right) - Y \right]. \quad (5.13)$$

This, together with (5.5), determines the general coordinate transformation laws of all the fields appearing in the action (4.47).

It is easy to see that when equations of motion are satisfied, the transformation law of $Q^{(5)}$ given in (5.5), (5.13) agrees with the one induced from the transformation law of $F^{(5)}$ via the identification (4.29). To this end note that (5.5) and (5.12) give

$$\delta_\xi Q^{(5)} = d 1_\xi F^{(5)} + *d 1_\xi F^{(5)}. \quad (5.14)$$

On the other hand (4.29) gives

$$Q^{(5)} = F^{(5)} + *F^{(5)}. \quad (5.15)$$

Thus on-shell, when $dF^{(5)} = 0$, the transformation induced on $Q^{(5)}$ from (5.10) is

$$\delta_\xi Q^{(5)} = \delta_\xi F^{(5)} + *\delta_\xi F^{(5)} = d 1_\xi F^{(5)} + *d 1_\xi F^{(5)}. \quad (5.16)$$

We see that (5.16) and (5.14) are in perfect agreement. Eq. (5.15) also explains why the transformation laws of $Q^{(5)}$ are somewhat unusual. Whereas $F^{(5)}$ transforms as a 5-form under general coordinate transformation, the $*$ in the second term represents Hodge dual with respect to Minkowski metric, leading to non-standard transformation laws of this term.

6 Supersymmetry

In this section we shall discuss supersymmetry of the action constructed in section 4. Our goal will be to propose supersymmetry transformation laws δ'_s of the new variables $P^{(4)}$, $Q^{(5)}$ and M that leave the new action $S'_1 + S_2$ given in (4.24) invariant. We propose the following transformations:

$$\delta'_s M = \delta_s M, \quad \delta'_s P^{(4)} = \delta_s C^{(4)}, \quad \delta'_s Q^{(5)} = d\delta_s C^{(4)} + *d\delta_s C^{(4)}, \quad (6.1)$$

where δ_s denotes the usual supersymmetry transformation laws described in [3, 4]. It is understood that on the right hand side of (6.1) all factors of $dC^{(4)}$ have to be replaced in terms of $Q^{(5)}$ using (4.21) and (4.32). To this end it is important that in the expressions for $\delta_s C^{(4)}$ and $\delta_s M$, $C^{(4)}$ always appears in the combination $dC^{(4)}$ [4], since an explicit factor of $C^{(4)}$ without derivative could not have been expressed back in terms of $Q^{(5)}$. Our goal will be to show that $\delta'_s(S'_1 + S_2)$ vanishes. In doing so, we can use the self-duality of $Q^{(5)}$ since this condition is valid off-shell, but not the relation (4.27) since the latter is an equation of motion derived from $S'_1 + S_2$.

For computing $\delta'_s(S'_1 + S_2)$ we shall make use of the known results on the δ_s transformation properties of the original action $S_1 + S_2$. However instead of regarding $C^{(4)}$ as an independent variable, it will be more useful for us to regard $F^{(5)}$ as an independent variable satisfying the self-duality condition $\star_g \widehat{F}^{(5)} = \widehat{F}^{(5)}$. In this case we can no longer use the Bianchi identity $dF^{(5)} = 0$. The expression for $\delta_s(S_1 + S_2)$ under these conditions can be found using explicit computation, but we shall extract the result from known results in the literature as follows:

1. An action for type IIB supergravity was proposed in eq. (4.7) of [23]. This action had, besides the usual fields of type IIB supergravity which we have called $C^{(4)}$ and M , an additional scalar field a . The scalar field enters the action through a combination f_4 which is also proportional to $\widehat{F}^{(5)} - \star_g \widehat{F}^{(5)}$. The f_4 dependent term in the action is quadratic in f_4 . We can identify the action $S_1 + S_2$ appearing in (4.3) as the one given in [23] without the quadratic term in f_4 and without the additional scalar field a . (There are also some obvious changes in the normalizations and notations that can be easily identified but will not be described here.)
2. The action given in [23] was shown to be invariant under supersymmetry transformations that agree with those used in [3, 4] after setting $f_4 = 0$. During this analysis $C^{(4)}$ was taken as the independent variable instead of $F^{(5)}$, and as a consequence the Bianchi identity $dF^{(5)} = 0$ was used.
3. Since the action of [23] depends on f_4 through a term quadratic in f_4 , its first order variation with respect to f_4 vanishes at $f_4 = 0$. Therefore the supersymmetry of the action of [23] guarantees that the action $S_1 + S_2$ is supersymmetric if after taking the supersymmetry variation we set $\widehat{F}^{(5)}$ to be equal to $\star_g \widehat{F}^{(5)}$ since this sets f_4 to 0. However if we do not use the Bianchi identity $dF^{(5)} = 0$ then in general there will be

additional terms proportional to $dF^{(5)}$ in the expression for $\delta_s(S_1 + S_2)$. This allows us to write

$$\delta_s(S_1 + S_2) = \Xi, \tag{6.2}$$

where Ξ denotes some term proportional to $dF^{(5)}$.

4. For computing Ξ we can organize each term in $\delta_s(S_1 + S_2)$ using integration by parts such that the supersymmetry transformation parameter has no derivative acting on it. In this case it is easy to see that since S_2 does not depend on $F^{(5)}$, the entire contribution to Ξ comes from the variation of S_1 . The variation of $C^{(4)}$ generates $\int \delta_s C^{(4)} \wedge dF^{(5)}$. On the other hand, using the result of [3, 4] that $\delta Y = -d\delta C^{(4)} + \dots$, where \dots contain terms without derivatives of the supersymmetry transformation parameters, one finds from (4.42) that the variation of Y generates $-2 \int \delta_s C^{(4)} \wedge dF^{(5)}$. This gives

$$\Xi = - \int \delta_s C^{(4)} \wedge dF^{(5)}. \tag{6.3}$$

Let us now return to our main goal, which is to show that $\delta'_s(S'_1 + S_2)$ vanishes. Since S_2 depends only on the set of variables M , we have, from (6.1), $\delta'_s S_2 = \delta_s S_2$. Therefore using (6.2), we get

$$\delta'_s(S'_1 + S_2) = \delta'_s S'_1 - \delta_s S_1 + \Xi. \tag{6.4}$$

The Ξ term on the right hand side is important since using $dF^{(5)} = 0$ would translate to (4.27) under the identification (4.29), and we are not allowed to use this relation. We now note from (4.23), (4.28) that

$$\delta'_s S'_1 = \int \delta'_s P^{(4)} \wedge d(Q^{(5)} - *dP^{(4)}) + \frac{1}{2} \int \delta'_s Q^{(5)} \wedge (dP^{(4)} - *dP^{(4)} + R^{(5)}) + \tilde{\delta}_s S'_1, \tag{6.5}$$

where $\tilde{\delta}_s$ denotes the variation induced by δ'_s (or equivalently δ_s) variation of M . Using (6.1) this takes the form

$$\delta'_s S'_1 = \int \delta_s C^{(4)} \wedge d(Q^{(5)} - R^{(5)}) + \tilde{\delta}_s S'_1. \tag{6.6}$$

On the other hand we have from (4.4),

$$\begin{aligned} \delta_s S_1 &= - \int \delta_s F^{(5)} \wedge (\star_g \widehat{F}^{(5)} - Y) + \tilde{\delta}_s S_1 = - \int d\delta_s C^{(4)} \wedge (\star_g \widehat{F}^{(5)} - Y) + \tilde{\delta}_s S_1 \\ &= \int \delta_s C^{(4)} \wedge d(\widehat{F}^{(5)} - Y) + \tilde{\delta}_s S_1 = \frac{1}{2} \int \delta_s C^{(4)} \wedge d(Q^{(5)} - R^{(5)}) + \tilde{\delta}_s S_1, \end{aligned} \tag{6.7}$$

where in the second line we have used the self-duality constraint $\star_g \widehat{F}^{(5)} = \widehat{F}^{(5)}$ which we are allowed to use, and the identification (4.21), (4.29). Now since we have shown in section 4 that the variation of S'_1 and S_1 with respect to M are identical, we have $\tilde{\delta}_s S'_1 = \tilde{\delta}_s S_1$. Therefore we get from (6.4), (6.6), (6.7):

$$\delta'_s(S'_1 + S_2) = \frac{1}{2} \int \delta_s C^{(4)} \wedge d(Q^{(5)} - R^{(5)}) + \Xi = 0, \tag{6.8}$$

where in the last step we have used (6.3), (4.21), (4.29). This establishes supersymmetry of the action.

We end the section with two observations:

1. The form of the transformation laws given in (6.1) is consistent with the general form of gauge transformations described in [5] and reviewed in (2.6), according to which the supersymmetry transformation laws of various fields, which is a special case of the gauge transformation generated by λ , should be independent of $P^{(4)}$.
2. It is easy to verify that the supersymmetry transformation laws δ'_s agree with δ_s after using the identification (4.30). For all fields encoded in M this is automatic consequence of (6.1); so we only need to check this for $Q^{(5)}$. We have from (4.21), (4.30)

$$Q^{(5)} = \widehat{F}^{(5)} + * \widehat{F}^{(5)} - Y - *Y = F^{(5)} + *F^{(5)}. \quad (6.9)$$

Therefore

$$\delta_s Q^{(5)} = \delta_s F^{(5)} + * \delta_s F^{(5)} = d \delta_s C^{(4)} + * d \delta_s C^{(4)} = \delta'_s Q^{(5)}, \quad (6.10)$$

where in the last step we have used (6.1). This shows that the transformations δ_s and δ'_s agree.

7 Lorentz covariant gauge fixing and Feynman rules

String field theory action of [5] admits a Lorentz covariant gauge fixing at the full quantum level — the ‘Siegel gauge’. This suggests that the action given in (4.47) (together with S_2) must also admit a Lorentz covariant gauge fixing. In this section we shall describe how this can be done in flat space-time background.

Since gauge transformations of most fields are standard and we can choose the analog of Lorentz / Feynman gauge for them maintaining manifest Lorentz covariance, we shall focus on the $P^{(4)} \rightarrow P^{(4)} + d \Xi^{(3)}$ gauge transformation. We can fix a gauge by adding a gauge fixing term of the form

$$\frac{1}{2} \int * d * P^{(4)} \wedge d * P^{(4)}. \quad (7.1)$$

Since in flat space-time the background value of e is η , $(e\eta - 1)$ and hence \mathcal{M} has its expansion beginning at the first order in the fluctuations. Therefore the only terms quadratic in $P^{(4)}$, $Q^{(5)}$ in the original action are the first two terms on the right hand side of (4.47). After adding (7.1) to the action (4.47) the quadratic term involving $P^{(4)}$ and $Q^{(5)}$ takes the form

$$\frac{1}{2} \int P^{(4)} \wedge *(* d * d + d * d *) P^{(4)} + \int P^{(4)} \wedge d Q^{(5)}. \quad (7.2)$$

In momentum space this corresponds to a term proportional to

$$\begin{aligned} & \frac{1}{2 \times 4!} \int d^{10} k \left[P^{(4)abcd}(-k) k^2 P_{abcd}^{(4)}(k) + 2i P^{(4)abcd}(-k) k^e Q_{eabcd}^{(5)}(k) \right] \\ &= \frac{1}{2 \times 4!} \int d^{10} k \left[(P^{(4)abcd}(-k) \right. \\ & \quad \left. - i(k^2)^{-1} k_f Q^{(5)fabcd}(-k)) k^2 (P_{abcd}^{(4)}(k) + i(k^2)^{-1} k^e Q_{eabcd}^{(5)}(k)) \right. \\ & \quad \left. - Q^{(5)fabcd}(-k) (k^2)^{-1} k_f k^e Q_{eabcd}^{(5)}(k) \right] \\ &= \frac{1}{2 \times 4!} \int d^{10} k \left[\bar{P}^{(4)abcd}(-k) k^2 \bar{P}_{abcd}^{(4)}(k) - Q^{(5)fabcd}(-k) (k^2)^{-1} k_f k^e Q_{eabcd}^{(5)}(k) \right], \end{aligned} \quad (7.3)$$

where

$$\bar{P}_{abcd}^{(4)}(k) \equiv P_{abcd}^{(4)}(k) + i(k^2)^{-1} k^e Q_{eabcd}^{(5)}(k). \tag{7.4}$$

We can now treat $\bar{P}^{(4)}$ as the independent field instead of $P^{(4)}$. Since this does not appear anywhere else in the action, this describes a free field and hence decouples. Therefore the only kinetic operator that is of relevance is that of $Q^{(5)}$. If we define the following operator acting on 5-forms:

$$K_A{}^B(k) = (k^2)^{-1} \left(k_{a_1} k^{b_1} \delta_{a_2}{}^{b_2} \dots \delta_{a_5}{}^{b_5} + \delta_{a_1}{}^{b_1} k_{a_2} k^{b_2} \delta_{a_3}{}^{b_3} \delta_{a_4}{}^{b_4} \delta_{a_5}{}^{b_5} + \dots + \delta_{a_1}{}^{b_1} \dots \delta_{a_4}{}^{b_4} k_{a_5} k^{b_5} \right), \tag{7.5}$$

then the kinetic operator acting on $Q^{(5)}$ may be written as

$$-\frac{1}{4} \zeta (1 - \zeta \varepsilon) K(k) (1 + \zeta \varepsilon), \tag{7.6}$$

up to a constant of proportionality. The operator $(1 + \zeta \varepsilon)/2$ on the right projects onto self-dual 5-forms, whereas the operator $(1 - \zeta \varepsilon)/2$ on the left projects onto anti-self-dual 5-forms reflecting the fact that only anti-self-dual 5-forms have non-zero contraction with self-dual 5-forms. Thus the kinetic operator is a map from the space of self-dual 5-forms to the space of anti-self-dual 5-forms. The propagator, which is i times the inverse of the kinetic term, should be a map from the space of anti-self-dual 5-forms to the space of self-dual 5-forms, also reflecting the fact that the propagator naturally acts on current dual to field which is in this case anti-self-dual 5-form. It is easy to verify that the following operator constitutes the inverse of the kinetic term in this sense:

$$\Delta = -(1 + \zeta \varepsilon) K(k) (1 - \zeta \varepsilon) \zeta. \tag{7.7}$$

This is the gauge invariant propagator of a 5-form field strength given e.g. in [36]. With this propagator $i\Delta$ for the $Q^{(5)}$ field, and the vertices computed in the usual way from the action $S'_1 + S_2$, we can now compute the tree level Green's functions and S-matrix elements of type IIB supergravity in the standard way. Loop corrections will require embedding this theory into the full string field theory described in [5].

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References

- [1] N. Marcus and J.H. Schwarz, *Field theories that have no manifestly Lorentz invariant formulation*, *Phys. Lett. B* **115** (1982) 111 [INSPIRE].
- [2] J.H. Schwarz and P.C. West, *Symmetries and transformations of chiral $N = 2$ $D = 10$ supergravity*, *Phys. Lett. B* **126** (1983) 301 [INSPIRE].
- [3] J.H. Schwarz, *Covariant field equations of chiral $N = 2$ $D = 10$ supergravity*, *Nucl. Phys. B* **226** (1983) 269 [INSPIRE].
- [4] P.S. Howe and P.C. West, *The complete $N = 2$, $D = 10$ supergravity*, *Nucl. Phys. B* **238** (1984) 181 [INSPIRE].
- [5] A. Sen, *BV master action for heterotic and type II string field theories*, *JHEP* **02** (2016) 087 [arXiv:1508.05387] [INSPIRE].
- [6] N. Kiryu and Y. Okawa, work in progress.
- [7] M. Henneaux and C. Teitelboim, *Dynamics of chiral (selfdual) P forms*, *Phys. Lett. B* **206** (1988) 650 [INSPIRE].
- [8] J.H. Schwarz and A. Sen, *Duality symmetric actions*, *Nucl. Phys. B* **411** (1994) 35 [hep-th/9304154] [INSPIRE].
- [9] J.H. Schwarz, *Coupling a selfdual tensor to gravity in six-dimensions*, *Phys. Lett. B* **395** (1997) 191 [hep-th/9701008] [INSPIRE].
- [10] D. Belov and G.W. Moore, *Holographic action for the self-dual field*, hep-th/0605038 [INSPIRE].
- [11] B. McClain, F. Yu and Y.S. Wu, *Covariant quantization of chiral bosons and $OSp(1,1|2)$ symmetry*, *Nucl. Phys. B* **343** (1990) 689 [INSPIRE].
- [12] C. Wotzasek, *The Wess-Zumino term for chiral bosons*, *Phys. Rev. Lett.* **66** (1991) 129 [INSPIRE].
- [13] I. Martin and A. Restuccia, *Duality symmetric actions and canonical quantization*, *Phys. Lett. B* **323** (1994) 311 [INSPIRE].
- [14] F.P. Devecchi and M. Henneaux, *Covariant path integral for chiral p forms*, *Phys. Rev. D* **54** (1996) 1606 [hep-th/9603031] [INSPIRE].
- [15] L.D. Faddeev and S.L. Shatashvili, *Realization of the Schwinger term in the Gauss law and the possibility of correct quantization of a theory with anomalies*, *Phys. Lett. B* **167** (1986) 225 [INSPIRE].
- [16] I. Bengtsson and A. Kleppe, *On chiral p forms*, *Int. J. Mod. Phys. A* **12** (1997) 3397 [hep-th/9609102] [INSPIRE].
- [17] N. Berkovits, *Manifest electromagnetic duality in closed superstring field theory*, *Phys. Lett. B* **388** (1996) 743 [hep-th/9607070] [INSPIRE].
- [18] N. Berkovits, *Local actions with electric and magnetic sources*, *Phys. Lett. B* **395** (1997) 28 [hep-th/9610134] [INSPIRE].
- [19] N. Berkovits, *SuperMaxwell actions with manifest duality*, *Phys. Lett. B* **398** (1997) 79 [hep-th/9610226] [INSPIRE].

- [20] P. Pasti, D.P. Sorokin and M. Tonin, *Space-time symmetries in duality symmetric models*, in *Gauge theories, applied supersymmetry, quantum gravity*, B. de Wit et al. eds, Leuven University Press, Leuven, Belgium (1995), [DFPD-TH-95-46](#) (1995).
- [21] P. Pasti, D.P. Sorokin and M. Tonin, *On Lorentz invariant actions for chiral p forms*, *Phys. Rev. D* **55** (1997) 6292 [[hep-th/9611100](#)] [[INSPIRE](#)].
- [22] G. Dall'Agata, K. Lechner and D.P. Sorokin, *Covariant actions for the bosonic sector of $D = 10$ IIB supergravity*, *Class. Quant. Grav.* **14** (1997) L195 [[hep-th/9707044](#)] [[INSPIRE](#)].
- [23] G. Dall'Agata, K. Lechner and M. Tonin, *$D = 10$, $N = IIB$ supergravity: Lorentz invariant actions and duality*, *JHEP* **07** (1998) 017 [[hep-th/9806140](#)] [[INSPIRE](#)].
- [24] G. Dall'Agata, K. Lechner and M. Tonin, *Action for IIB supergravity in 10-dimensions*, [hep-th/9812170](#) [[INSPIRE](#)].
- [25] E. Witten, *Five-brane effective action in M-theory*, *J. Geom. Phys.* **22** (1997) 103 [[hep-th/9610234](#)] [[INSPIRE](#)].
- [26] E. Witten, *Duality relations among topological effects in string theory*, *JHEP* **05** (2000) 031 [[hep-th/9912086](#)] [[INSPIRE](#)].
- [27] L. Castellani and I. Pesando, *The complete action of chiral $D = 10$, $N = 2$ supergravity*, *Conf. Proc. C* **9115201** (1991) 24 [[INSPIRE](#)].
- [28] L. Castellani and I. Pesando, *The complete superspace action of chiral $D = 10$, $N = 2$ supergravity*, *Int. J. Mod. Phys. A* **8** (1993) 1125 [[INSPIRE](#)].
- [29] M.A. Vasiliev, *Consistent equation for interacting gauge fields of all spins in $(3 + 1)$ -dimensions*, *Phys. Lett. B* **243** (1990) 378 [[INSPIRE](#)].
- [30] M.A. Vasiliev, *Nonlinear equations for symmetric massless higher spin fields in $(A)dS(d)$* , *Phys. Lett. B* **567** (2003) 139 [[hep-th/0304049](#)] [[INSPIRE](#)].
- [31] X. Bekaert, S. Cnockaert, C. Iazeolla and M.A. Vasiliev, *Nonlinear higher spin theories in various dimensions*, [hep-th/0503128](#) [[INSPIRE](#)].
- [32] W. Siegel, *Two vierbein formalism for string inspired axionic gravity*, *Phys. Rev. D* **47** (1993) 5453 [[hep-th/9302036](#)] [[INSPIRE](#)].
- [33] W. Siegel, *Superspace duality in low-energy superstrings*, *Phys. Rev. D* **48** (1993) 2826 [[hep-th/9305073](#)] [[INSPIRE](#)].
- [34] W. Siegel, *Manifest duality in low-energy superstrings*, in the proceedings of *Stringe '93*, May 24–29, Berkeley, U.S.A. (1993), [hep-th/9308133](#) [[INSPIRE](#)].
- [35] C. Hull and B. Zwiebach, *Double field theory*, *JHEP* **09** (2009) 099 [[arXiv:0904.4664](#)] [[INSPIRE](#)].
- [36] L. Álvarez-Gaumé and E. Witten, *Gravitational anomalies*, *Nucl. Phys. B* **234** (1984) 269 [[INSPIRE](#)].