Quantum spectral curve at work: from small spin to strong coupling in $\mathcal{N} = 4$ SYM

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ABSTRACT: We apply the recently proposed quantum spectral curve technique to the study of twist operators in planar $\mathcal{N} = 4$ SYM theory. We focus on the small spin expansion of anomalous dimensions in the sl(2) sector and compute its first two orders exactly for any value of the 't Hooft coupling. At leading order in the spin S we reproduced Basso's slope function. The next term of order S^2 structurally resembles the Beisert-Eden-Staudacher dressing phase and takes into account wrapping contributions. This expansion contains rich information about the spectrum of local operators at strong coupling. In particular, we found a new coefficient in the strong coupling expansion of the Konishi operator dimension and confirmed several previously known terms. We also obtained several new orders of the strong coupling expansion of the BFKL pomeron intercept. As a by-product we formulated a prescription for the correct analytical continuation in S which opens a way for deriving the BFKL regime of twist two anomalous dimensions from AdS/CFT integrability.

Contents

1   Introduction

Exploration of the holographic duality between planar 4D $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) and string theory on $AdS_5 \times \mathrm{S}^5$ has led to numerous remarkable results due to integrability discovered on both sides of the duality [1]. Integrability has been particularly successful in application to the problem of computing the planar spectrum of single trace operator anomalous dimensions/string state energies. In the asymptotically large volume limit the spectrum was found to be captured by a system of nested asymptotic Bethe ansatz (ABA) equations [2]. Finite-size corrections [3, 4] were later accounted for via the Thermodynamic Bethe Ansatz (TBA)/Y-system technique [5–10]. This approach led to the formulation of an infinite set of integral equations, which are expected to describe the exact spectrum of the theory at any value of the 't Hooft coupling λ. The main problem of this approach is that the explicit form of the equations requires case-by-case study and is not known in general except for a few explicit examples such as Konishi [11, 12]. They, however, allowed for a detailed numerical study of these simplest operators [11, 13–15] and led to a prediction for string theory which was confirmed in [16–18].

Very recently a new set of equations called the quantum spectral curve or the $\mathbf{P}\mu$-system was proposed [19, 20] which generalizes the original TBA equations to all sectors of the theory and reveals a strikingly simple and concise underlying structure of the spectral problem. It allows one to describe all states of the theory on equal footing.[1] The proposal has the form of a nonlinear Riemann-Hilbert problem for a set of a few functions.

Due to its remarkably transparent structure, the $\mathbf{P}\mu$-system should be suitable to attack a variety of open problems including such a longstanding problem of AdS/CFT integrability as the description of the BFKL scaling regime. Despite its novelty the $\mathbf{P}\mu$-system was already used in various different situations. One application which provided nontrivial tests of the proposal is the exact computation of the Bremsstrahlung function [19, 21]. The new formulation also allowed to find the 9-loop Konishi anomalous dimension at weak coupling [22]. Below in the text we give a short overview of the construction but we advice the reader to refer to [20] where the quantum spectral curve is described in complete detail.

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[1]This is in contrast to the analytic Y-system approach, which requires additional information about the location of poles and/or zeros. For simple states, like Konishi, these poles u^* are prescribed to satisfy the "exact Bethe ansatz equation" $Y_{1,0}^{\mathrm{physical}}(u^*) = -1$. Already for more complicated states in the sl_2 sector there are additional dynamical singularities in the Y-functions which diverge from those appearing in the asymptotic solution when wrapping effects are taken into account. $\mathbf{P}\mu$-system puts under control all such singularities of Y-functions including those in even more complicated states. In particular the BES equations with all types of the Bethe roots are a consequence of the $\mathbf{P}\mu$-system (see [20] for details). The only input information it requires are the integer global R-charges and Lorentz spins entering through the asymptotics of $\mathbf{P}$-functions.

In this paper we will apply the $\mathbf{P}\mu$-system to the calculation of twist operator anomalous dimensions in the $sl(2)$ sector of $\mathcal{N} = 4$ SYM. These operators have the form

$$\mathcal{O} = \text{Tr}\left(Z^{J-1}\,\mathcal{D}^S Z\right) + \dots \tag{1.1}$$

where Z denotes one of the scalars of the theory,[2] $\mathcal{D}$ is a lightcone covariant derivative and the dots stand for permutations. The number of derivatives S is called the spin of the operator, while J is called the twist. We will consider a two-cut configuration with a symmetric distribution of Bethe roots, thus for physical states S is even. We will study the small spin limit, in which the scaling dimension of these operators can be written as

$$\Delta = J + S + \gamma(g), \quad g = \sqrt{\lambda}/(4\pi) \tag{1.2}$$

with the anomalous dimension $\gamma(g)$ given as an expansion

$$\gamma(g) = \gamma^{(1)}(g)S + \gamma^{(2)}(g)S^2 + \mathcal{O}(S^3). \tag{1.3}$$

The first term, $\gamma^{(1)}(g)$, is called the slope function. Remarkably, it can be found exactly at any value of the coupling [23]

$$\gamma^{(1)}(g) = \frac{4\pi g I_{J+1}(4\pi g)}{J I_J(4\pi g)} \ . \tag{1.4}$$

This expression was later derived from the ABA equations in two different ways [24, 25] and further studied and extended in [26–30]. This quantity is protected from finite-size wrapping corrections and thus the ABA prediction is exact. It is also not sensitive to the dressing phase of the ABA, which contributes only starting from order S^2.

Our key observation is that in the small S regime the $\mathbf{P}\mu$-system can be solved iteratively order by order in the spin. In this paper we first solve it at leading order and reproduce the slope function (1.4). Then we compute the coefficient of the S^2 term in the expansion, i.e. the function $\gamma^{(2)}(g)$ which we call the *curvature function*. For twist $J = 2, 3, 4$ we obtain closed exact expressions for it in the form of a double integral. Unlike the slope function, $\gamma^{(2)}(g)$ is affected by the dressing phase in the ABA and by wrapping corrections, all of which are incorporated in the exact $\mathbf{P}\mu$-system.

Furthermore, we use the strong coupling expansion of our result to find the value of a new coefficient in the Konishi operator (i.e. $\text{Tr}\left(\mathcal{D}^2 Z^2\right)$) anomalous dimension at strong coupling. Our result for the Konishi dimension reads

$$\Delta_{konishi} = 2\,\lambda^{1/4} + \frac{2}{\lambda^{1/4}} + \frac{-3\,\zeta_3 + \frac{1}{2}}{\lambda^{3/4}} + \frac{\frac{15\,\zeta_5}{2} + 6\,\zeta_3 - \frac{1}{2}}{\lambda^{5/4}} + \dots \ . \tag{1.5}$$

We have also obtained two new terms in the strong coupling expansion of the BFKL pomeron intercept,

$$j_0 = 2 + S(\Delta)|_{\Delta=0} = 2 - \frac{2}{\lambda^{1/2}} - \frac{1}{\lambda} + \frac{1}{4\,\lambda^{3/2}} + (6\,\zeta_3 + 2)\,\frac{1}{\lambda^2} \tag{1.6}$$

$$+ \left(18\,\zeta_3 + \frac{361}{64}\right)\frac{1}{\lambda^{5/2}} + \left(39\,\zeta_3 + \frac{447}{32}\right)\frac{1}{\lambda^3} + \mathcal{O}\left(\frac{1}{\lambda^{7/2}}\right),$$

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[2]Written in terms of two real scalars as $Z = \Phi_1 + i\Phi_2$.

where the new terms are in the second line. In addition we have checked our results against available results in literature at weak and strong coupling, and found full agreement.

The paper is organized as follows. First in section 2 we review the quantum spectral curve construction in a general setting. In section 3 we demonstrate its applicability by rederiving the exact slope function of $\mathcal{N} = 4$ found in [23]. In section 4 we push the calculation further and find the exact expression for the next coefficient in the small spin expansion, i.e. the curvature function. In sections 5 and 6 we discuss the weak and strong coupling expansions of our result. We then use our results to calculate the previously unknown three loop strong coupling coefficient of the Konishi anomalous dimension in subsection 6.3 and two new coefficients for the BFKL intercept at strong coupling in subsection 6.4. We finish with conclusions and appendices, which contain detailed calculations left out of the main text for brevity.

2 $\mathbf{P}\mu$-system — an overview

In this section we review the formulation of the $\mathbf{P}\mu$-system, and also discuss its symmetries which will be useful later. Below, we will restrict the discussion to states in the $sl(2)$ sector as presented in [19]. Remarkably, the general case is not much more complicated and will appear soon in [20].

2.1 Definitions and notation

The $\mathbf{P}\mu$-system is a nonlinear system of functional equations for a four-vector $\mathbf{P}_a(u)$ and a 4×4 antisymmetric matrix $\mu_{ab}(u)$ depending on the spectral parameter u. For full details about the origin of the construction we refer the reader to [20]. As functions of u, both $\mathbf{P}_a$ and μ_{ab} have prescribed analyticity properties which play a key role. First, $\mathbf{P}_a$ must have only a single branch cut in u going between $-2g$ and $2g$, being analytic in the rest of the complex plane. We call this cut the *short* cut, while the cut on the real line connecting the same two points through infinity is called the *long* cut. The functions μ_{ab} have an infinite set of short branch cuts going between $-2g + in$ and $2g + in$ for all $n \in \mathbb{Z}$ (see figure 1). Most importantly, the analytic continuation of $\mathbf{P}_a$ and μ_{ab} through these cuts is again expressed in terms of these functions, according to the following equations:

$$\tilde{\mathbf{P}}_a = -\mu_{ab}\chi^{bc}\mathbf{P}_c, \quad \text{with} \quad \chi^{ab} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{2.1}$$

and

$$\tilde{\mu}_{ab} - \mu_{ab} = \mathbf{P}_a\tilde{\mathbf{P}}_b - \mathbf{P}_b\tilde{\mathbf{P}}_a . \tag{2.2}$$

Here we denote by $\tilde{\mathbf{P}}_a$ and $\tilde{\mu}_{ab}$ the analytic continuation of $\mathbf{P}_a$ and μ_{ab} through the cut on the real axis. In addition, we have a pseudo-periodicity condition

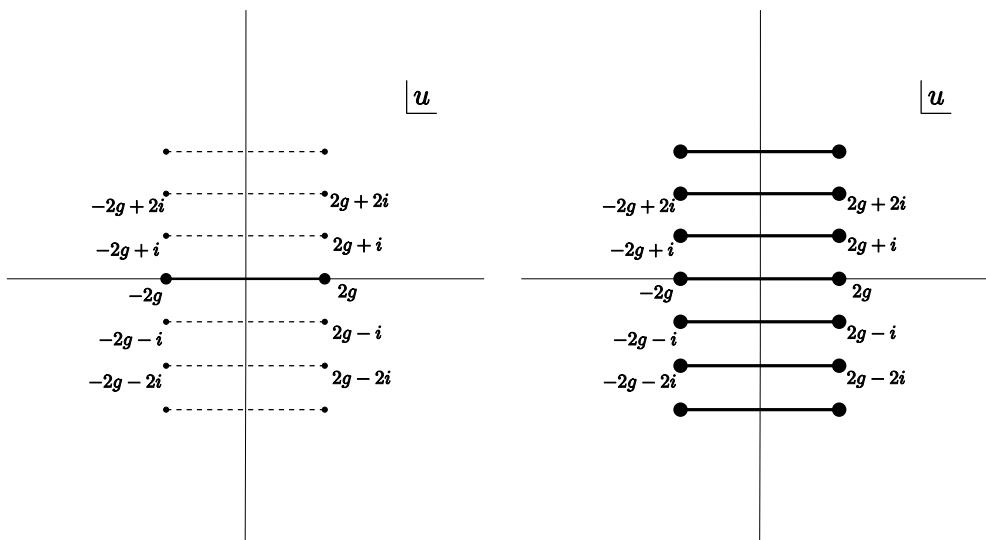$$\tilde{\mu}_{ab}(u) = \mu_{ab}(u + i) \tag{2.3}$$

Figure 1. Cuts in the u plane. We show the location of branch cuts in u for the functions $\mathbf{P}_a(u)$ (left) and $\mu_{ab}(u)$ (right). The infinitely many cuts of $\tilde{\mathbf{P}}_a$ are shown on the left picture by dotted lines.

which, actually, means that $\mu_{ab}(u)$ would be an i-periodic function if defined with long cuts instead of the short cuts.

The functions μ_{ab} are also constrained by the relations

$$\mu_{12}\mu_{34} - \mu_{13}\mu_{24} + \mu_{14}^2 = 1 \ , \tag{2.4}$$

$$\mu_{14} = \mu_{23} \ , \tag{2.5}$$

the first of which states that the Pfaffian of the matrix μ_{ab} is equal to 1. Let us also write the equations (2.1) explicitly:

$$\tilde{\mathbf{P}}_1 = -\mathbf{P}_3\mu_{12} + \mathbf{P}_2\mu_{13} - \mathbf{P}_1\mu_{14} \tag{2.6}$$

$$\tilde{\mathbf{P}}_2 = -\mathbf{P}_4\mu_{12} \qquad\quad + \mathbf{P}_2\mu_{14} - \mathbf{P}_1\mu_{24} \tag{2.7}$$

$$\tilde{\mathbf{P}}_3 = \qquad\quad -\mathbf{P}_4\mu_{13} + \mathbf{P}_3\mu_{14} \qquad\quad - \mathbf{P}_1\mu_{34} \tag{2.8}$$

$$\tilde{\mathbf{P}}_4 = \qquad\qquad\quad -\mathbf{P}_4\mu_{14} + \mathbf{P}_3\mu_{24} - \mathbf{P}_2\mu_{34} \ . \tag{2.9}$$

The above equations ensure that the branch points of $\mathbf{P}_a$ and μ_{ab} are of the square root type, i.e. $\tilde{\tilde{\mathbf{P}}}_a = \mathbf{P}_a$ and $\tilde{\tilde{\mu}}_{ab} = \mu_{ab}$.

Finally, we require that $\mathbf{P}_a$ and μ_{ab} do not have any singularities except these branch points.[3]

2.2 Asymptotics and energy

The quantum numbers and the energy of the state are encoded in the asymptotics of the functions $\mathbf{P}_a$ and μ_{ab} at large real u. The generic case is described in [20], while here we

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[3]For odd values of J the functions $\mathbf{P}_a$ may have an additional branch point at infinity. However, it should cancel in any product of two $\mathbf{P}_a$'s, and therefore it will not appear in any physically relevant quantity (see [19, 20]). We will discuss some explicit examples in the text.

are interested in the states in the $sl(2)$ sector, for which the relations read [19]

$$\mathbf{P}_a \sim (A_1 u^{-J/2}, A_2 u^{-J/2-1}, A_3 u^{J/2}, A_4 u^{J/2-1}) \qquad (2.10)$$

$$(\mu_{12},\ \mu_{13},\ \mu_{14},\ \mu_{24},\ \mu_{34}) \sim \left(u^{\Delta-J},\ u^{\Delta+1},\ u^{\Delta},\ u^{\Delta-1},\ u^{\Delta+J}\right) \qquad (2.11)$$

where J is the twist of the gauge theory operator, and Δ is its conformal dimension. With these asymptotics, the equations (2.1)–(2.5) form a closed system which fixes $\mathbf{P}_a$ and μ_{ab}.

Lastly, the spin S of the operator is related [19] to the leading coefficients A_a of the $\mathbf{P}_a$ functions (see (2.10)):

$$A_1 A_4 = \frac{\left((J+S-2)^2 - \Delta^2\right)\left((J-S)^2 - \Delta^2\right)}{16iJ(J-1)} \qquad (2.12)$$

$$A_2 A_3 = \frac{\left((J-S+2)^2 - \Delta^2\right)\left((J+S)^2 - \Delta^2\right)}{16iJ(J+1)} \ . \qquad (2.13)$$

2.3 Symmetries

The $\mathbf{P}\mu$-system enjoys a symmetry preserving all of its essential features. It has the form of a linear transformation of $\mathbf{P}_a$ and μ_{ab} which leaves the system (2.1)–(2.5) and the asymptotics (2.10), (2.11) invariant. Indeed, consider a general linear transformation $\mathbf{P}'_a = R_a{}^b \mathbf{P}_b$ with a non-degenerate constant matrix R. In order to preserve the system (2.1), μ should at the same time be transformed as

$$\mu' = -R\mu\chi R^{-1}\chi. \qquad (2.14)$$

Such a transformation also preserves the form of (2.2) if

$$R^T \chi R \chi = -1 \ , \qquad (2.15)$$

which also automatically ensures antisymmetry of μ_{ab} and (2.4), (2.5). In general, this transformation will spoil the asymptotics of $\mathbf{P}_a$. These asymptotics are ordered as $|\mathbf{P}_2| < |\mathbf{P}_1| < |\mathbf{P}_4| < |\mathbf{P}_3|$, which implies that the matrix R must have the following structure[4]

$$R = \begin{pmatrix} * & * & 0 & 0 \\ 0 & * & 0 & 0 \\ * & * & * & * \\ * & * & 0 & * \end{pmatrix}. \qquad (2.16)$$

The general form of R which satisfies (2.15) and does not spoil the asymptotics generates a 6-parametric transformation, which we will call a γ-transformation. The simplest γ-transformation is the following rescaling:

$$\mathbf{P}_1 \to \alpha \mathbf{P}_1 \ , \qquad \mathbf{P}_2 \to \beta \mathbf{P}_2 \ , \qquad \mathbf{P}_3 \to 1/\beta \mathbf{P}_3 \ , \qquad \mathbf{P}_4 \to 1/\alpha \mathbf{P}_4 \ , \qquad (2.17)$$

$$\mu_{12} \to \alpha\beta\mu_{12} \ , \quad \mu_{13} \to \frac{\alpha}{\beta}\mu_{13} \ , \quad \mu_{14} \to \mu_{14} \ , \qquad \mu_{24} \to \frac{\beta}{\alpha}\mu_{24} \ , \qquad \mu_{34} \to \frac{1}{\alpha\beta}\mu_{34} \ , \quad (2.18)$$

with α, β being constants.

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[4]This matrix would of course be lower triangular if we ordered $\mathbf{P}_a$ by their asymptotics.

In all the solutions we consider in this paper all functions $\mathbf{P}_a$ turn out to be functions of definite parity, so it makes sense to consider γ-transformations which preserve parity. $\mathbf{P}_1$ and $\mathbf{P}_2$ always have opposite parity (as one can see from from (2.10)) and thus should not mix under such transformations; the same is true about $\mathbf{P}_3$ and $\mathbf{P}_4$. Thus, depending on parity of J the parity-preserving γ-transformations are either

$$\mathbf{P}_3 \to \mathbf{P}_3 + \gamma_3 \mathbf{P}_2, \qquad \mathbf{P}_4 \to \mathbf{P}_4 + \gamma_2 \mathbf{P}_1, \tag{2.19}$$

$$\mu_{13} \to \mu_{13} + \gamma_3 \mu_{12}, \qquad \mu_{24} \to \mu_{24} - \gamma_2 \mu_{12}, \qquad \mu_{34} \to \mu_{34} + \gamma_3 \mu_{24} - \gamma_2 \mu_{13} - \gamma_2 \gamma_3 \mu_{12}$$

for odd J or

$$\mathbf{P}_3 \to \mathbf{P}_3 + \gamma_1 \mathbf{P}_1, \qquad \mathbf{P}_4 \to \mathbf{P}_4 - \gamma_1 \mathbf{P}_2, \tag{2.20}$$

$$\mu_{14} \to \mu_{14} - \gamma_1 \mu_{12}, \qquad \mu_{34} \to \mu_{34} + 2\gamma_1 \mu_{14} - \gamma_1^2 \mu_{12} ,$$

for even J.

3 Exact slope function from the $\mathbf{P}\mu$-system

In this section we will find the solution of the $\mathbf{P}\mu$-system (2.1)–(2.5) corresponding to the $sl(2)$ sector operators at leading order in small S. Based on this solution we will compute the slope function $\gamma^{(1)}(g)$ for any value of the coupling.

3.1 Solving the $\mathbf{P}\mu$-system in LO

The solution of the $\mathbf{P}\mu$-system is a little simpler for even J, because for odd J extra branch points at infinity will appear in $\mathbf{P}_a$ due to the asymptotics (2.10). Let us first consider the even J case.

The description of the $\mathbf{P}\mu$-system in the previous section was done for physical operators. Our goal is to take some peculiar limit when the (integer) number of covariant derivatives S goes to zero. As we will see this requires some extension of the asymptotic requirement for μ functions. In this section we will be guided by principles of naturalness and simplicity to deduce these modifications which we will summarize in section 3.2. There we also give a concrete prescription for analytical continuation in S, which we then use to derive the curvature function.

We will start by finding μ_{ab}. Recalling that $\Delta = J + \mathcal{O}(S)$, from (2.12), (2.13) we see that $A_1 A_4$ and $A_2 A_3$ are of order S for small S, so we can take the functions $\mathbf{P}_a$ to be of order $\sqrt{S}$. This is a key simplification, because now (2.2) indicates that the discontinuities of μ_{ab} on the cut are small when S goes to zero. Thus at leading order in S all μ_{ab} are just periodic entire functions without cuts. For power-like asymptotics of μ_{ab} like in (2.11) the only possibility is that they are all constants. However, we found that in this case there is only a trivial solution, i.e. $\mathbf{P}_a$ can only be zero. The reason for this is that for physical states S must be integer and thus cannot be arbitrarily small, nevertheless, it is a sensible question how to define an analytical continuation from integer values of S.[5]

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[5]Restricting the large positive S behavior one can achieve uniqueness of the continuation.

Thus we have to relax the requirement of power-like behavior at infinity. The first possibility is to allow for $e^{2\pi u}$ asymptotics at $u \to +\infty$. We should, however, remember about the constraints (2.4) and (2.5) which restrict our choice and the fact that we can also use γ-symmetry. Let us show that by allowing μ_{24} to have exponential behavior and setting it to $\mu_{24} = C \sinh(2\pi u)$, with other μ_{ab} being constant, we arrive to the correct result. This choice is dictated by our assumptions concerning the analytic continuation of μ_{ab} to non-integer values of S, and this point is discussed in detail in section 3.2. We will also see in that section that by using the γ-transformation (described in section 2.3) and the constraint (2.4) we can set the constant C to 1 and also $\mu_{12} = 1$, $\mu_{13} = 0$, $\mu_{14} = -1$, $\mu_{34} = 0$ (see (3.46)).

Having fixed all μ's at leading order we get the following system of equations[6] for $\mathbf{P}_a$:

$$\tilde{\mathbf{P}}_1 = -\mathbf{P}_3 + \mathbf{P}_1, \tag{3.1}$$

$$\tilde{\mathbf{P}}_2 = -\mathbf{P}_4 - \mathbf{P}_2 - \mathbf{P}_1 \sinh(2\pi u), \tag{3.2}$$

$$\tilde{\mathbf{P}}_3 = \quad\;\; -\mathbf{P}_3, \tag{3.3}$$

$$\tilde{\mathbf{P}}_4 = \quad\;\; +\mathbf{P}_4 + \mathbf{P}_3 \sinh(2\pi u). \tag{3.4}$$

Recalling that the functions $\mathbf{P}_a$ only have a single short cut, we see from these equations that $\tilde{\mathbf{P}}_a$ also have only this cut! This means that we can take all $\mathbf{P}_a$ to be infinite Laurent series in the Zhukovsky variable $x(u)$, which rationalizes the Riemann surface with two sheets and one cut. It is defined as

$$x + \frac{1}{x} = \frac{u}{g} \tag{3.5}$$

where we pick the solution with a short cut, i.e.

$$x(u) = \frac{1}{2}\left(\frac{u}{g} + \sqrt{\frac{u}{g} - 2}\,\sqrt{\frac{u}{g} + 2}\right) \; . \tag{3.6}$$

Solving the equation (3.3) with the asymptotics (2.10) we find

$$\mathbf{P}_3 = \epsilon\left(x^{-J/2} - x^{+J/2}\right) + \sum_{k=1}^{J/2-1} c_k \left(x^{-k} - x^k\right) \tag{3.7}$$

where ϵ and c_k are constants. Now it is useful to rewrite the equation for $\mathbf{P}_1$ (i.e. (3.1)) in the form $\tilde{\mathbf{P}}_1 - \mathbf{P}_1 = -\mathbf{P}_3$, and we see that due to asymptotics of $\mathbf{P}_1$ both sides of this equation must have a gap in the powers of x from $x^{-J/2+1}$ to $x^{J/2-1}$. This means that all coefficients c_k in (3.7) must vanish and we find

$$\mathbf{P}_1 = \epsilon x^{-J/2} \,, \tag{3.8}$$

so we are left with one unfixed constant ϵ (we expect it to be proportional to $\sqrt{S}$).

Thus the equations (3.2) and (3.4) become

$$\tilde{\mathbf{P}}_2 + \mathbf{P}_2 = -\mathbf{P}_4 - \epsilon x^{-J/2} \sinh(2\pi u) \,, \tag{3.9}$$

$$\tilde{\mathbf{P}}_4 - \mathbf{P}_4 = \epsilon(x^{-J/2} - x^{+J/2})\sinh(2\pi u) \,. \tag{3.10}$$

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[6] In this section we only consider the leading order of $\mathbf{P}$'s at small S, so the equations involving them are understood to hold at leading order in S. In section 4 we will study the next-to-leading order and elaborate the notation for contributions of different orders.

We will first solve the second equation. It is useful to introduce operations $[f(x)]_+$ and $[f(x)]_-$, which take parts of Laurent series with positive and negative powers of x respectively. Taking into account that

$$\sinh(2\pi u) = \sum_{n=-\infty}^{\infty} I_{2n+1} x^{2n+1}, \tag{3.11}$$

where $I_k \equiv I_k(4\pi g)$ is the modified Bessel function of the first kind, we can write $\sinh(2\pi u)$ as

$$\sinh(2\pi u) = \sinh_+ + \sinh_-, \tag{3.12}$$

where explicitly

$$\sinh_+ = [\sinh(2\pi u)]_+ = \sum_{n=1}^{\infty} I_{2n-1} x^{2n-1} \tag{3.13}$$

$$\sinh_- = [\sinh(2\pi u)]_- = \sum_{n=1}^{\infty} I_{2n-1} x^{-2n+1} . \tag{3.14}$$

In this notation the general solution of eq. (3.10) with asymptotics at infinity $\mathbf{P}_4 \sim u^{J/2-1}$ can be written as

$$\mathbf{P}_4 = \epsilon(x^{J/2} - x^{-J/2}) \sinh_- + Q_{J/2-1}(u), \tag{3.15}$$

where $Q_{J/2-1}$ is a polynomial of degree $J/2 - 1$ in u. The polynomial $Q_{J/2-1}$ can be fixed from the equation (3.9) for $\mathbf{P}_2$. Indeed, from the asymptotics of $\mathbf{P}_2$ we see that the l.h.s. of (3.9) does not have powers of x from $-J/2 + 1$ to $J/2 - 1$. This fixes

$$Q_{J/2-1}(x) = -\epsilon \sum_{k=1}^{J/2} I_{2k-1} \left(x^{\frac{J}{2}-2k+1} + x^{-\frac{J}{2}+2k-1} \right). \tag{3.16}$$

Once $Q_{J/2-1}$ is found, we set $\mathbf{P}_2$ to be the part of the right hand side of (3.9) with powers of x less than $-J/2$, which gives

$$\mathbf{P}_2 = -\epsilon x^{+J/2} \sum_{n=\frac{J}{2}+1}^{\infty} I_{2n-1} x^{1-2n}. \tag{3.17}$$

Thus (for even J) we have uniquely fixed all $\mathbf{P}_a$ with the only unknown parameter being ϵ. We summarize the solution below:

$$\mu_{12} = 1, \ \mu_{13} = 0, \ \mu_{14} = -1, \ \mu_{24} = \sinh(2\pi u), \ \mu_{34} = 0, \tag{3.18}$$

$$\mathbf{P}_1 = \epsilon x^{-J/2} \tag{3.19}$$

$$\mathbf{P}_2 = -\epsilon x^{+J/2} \sum_{n=J/2+1}^{\infty} I_{2n-1} x^{1-2n} \tag{3.20}$$

$$\mathbf{P}_3 = \epsilon \left(x^{-J/2} - x^{+J/2} \right) \tag{3.21}$$

$$\mathbf{P}_4 = \epsilon \left(x^{J/2} - x^{-J/2} \right) \sinh_- -\epsilon \sum_{n=1}^{J/2} I_{2n-1} \left(x^{\frac{J}{2}-2n+1} + x^{-\frac{J}{2}+2n-1} \right) . \tag{3.22}$$

In the next section we fix the remaining parameter ϵ of the solution in terms of S and find the energy, but now let us briefly discuss the solution for odd J. As we mentioned above the main difference is that the functions $\mathbf{P}_a$ now have a branch point at $u = \infty$, which is dictated by the asymptotics (2.10). In addition, the parity of μ_{ab} is different according to the asymptotics of these functions (2.11). The solution is still very similar to the even J case, and we discuss it in detail in appendix B. Let us present the result here:

$$\mu_{12} = 1, \quad \mu_{13} = 0, \quad \mu_{14} = 0, \quad \mu_{24} = \cosh(2\pi u), \quad \mu_{34} = 1 \tag{3.23}$$

$$\mathbf{P}_1 = \epsilon x^{-J/2}, \tag{3.24}$$

$$\mathbf{P}_2 = -\epsilon x^{J/2} \sum_{k=-\infty}^{-\frac{J+1}{2}} I_{2k} x^{2k}, \tag{3.25}$$

$$\mathbf{P}_3 = -\epsilon x^{J/2}, \tag{3.26}$$

$$\mathbf{P}_4 = \epsilon x^{-J/2} \cosh_- -\epsilon x^{-J/2} \sum_{k=1}^{\frac{J-1}{2}} I_{2k} x^{2k} - \epsilon I_0 x^{-J/2}. \tag{3.27}$$

Note that now $\mathbf{P}_a$ include half-integer powers of x.

Fixing the global charges of the solution. Finally, to fix our solution completely we have to find the value of ϵ and find the energy in terms of the spin using (2.12) and (2.13). For this we first extract the coefficients A_a of the leading terms for all $\mathbf{P}_a$ (see the asymptotics (2.10)). From (3.19)–(3.22) or (3.24)–(3.27) we get

$$A_1 = g^{J/2}\epsilon, \tag{3.28}$$

$$A_2 = -g^{J/2+1}\epsilon I_{J+1}, \tag{3.29}$$

$$A_3 = -g^{-J/2}\epsilon, \tag{3.30}$$

$$A_4 = -g^{-J/2+1}\epsilon I_{J-1}. \tag{3.31}$$

Expanding (2.12), (2.13) at small S with $\Delta = J + S + \gamma$, where $\gamma = \mathcal{O}(S)$, we find at linear order

$$\gamma = i(A_1 A_4 - A_2 A_3) \tag{3.32}$$

$$S = i(A_1 A_4 + A_2 A_3). \tag{3.33}$$

Plugging in the coefficients (3.28)–(3.31) we find that

$$\epsilon = \sqrt{\frac{2\pi i S}{J I_J(\sqrt{\lambda})}} \tag{3.34}$$

and we obtain the anomalous dimension at leading order,

$$\gamma = \frac{\sqrt{\lambda} I_{J+1}(\sqrt{\lambda})}{J I_J(\sqrt{\lambda})} S + \mathcal{O}(S^2), \tag{3.35}$$

which is precisely the slope function of Basso [23].

While the above discussion concerned the ground state, i.e. the $sl(2)$ sector operator with the lowest anomalous dimension at given twist J, it can be generalized for higher mode numbers. In the asymptotic Bethe ansatz for such operators we have two symmetric cuts formed by Bethe roots, with corresponding mode numbers being $\pm n$ (for the ground state $n = 1$). To describe these operators within the $\mathbf{P}\mu$-system we found that we should take $\mu_{24} = C \sinh(2\pi nu)$ instead of $\mu_{24} = C \sinh(2\pi u)$ (and for odd J we similarly use $\mu_{24} = C \cosh(2\pi nu)$ instead of $\mu_{24} = C \cosh(2\pi u)$). Then the solution is very similar to the one above, and we find

$$\gamma = \frac{n\sqrt{\lambda}I_{J+1}(n\sqrt{\lambda})}{JI_J(n\sqrt{\lambda})}S \; , \tag{3.36}$$

which reproduced the result of [23] for non-trivial mode number n. In appendix E.1 we also show how using the $\mathbf{P}\mu$-system one can reproduce the slope function for a configuration of Bethe roots with arbitrary mode numbers and filling fractions.

In summary, we have shown how the $\mathbf{P}\mu$-system correctly computes the energy at linear order in S. In section 4 we will compute the next, S^2 term in the anomalous dimension.

3.2   Prescription for analytical continuation

To deduce the general prescription for the asymptotics of μ_{ab} for non-integer S from our analysis, we first study the possible asymptotics of μ_{ab} for given $\mathbf{P}_a$ in more detail. For that we combine (2.3) with (2.2) and (2.1) to write a finite difference equation on μ_{ab}:

$$\mu_{ab}(u+i) = \mu_{ab}(u) - \mu_{bc}(u)\chi^{cd}\mathbf{P}_d\mathbf{P}_a + \mu_{ac}(u)\chi^{cd}\mathbf{P}_d\mathbf{P}_b. \tag{3.37}$$

As there are 5 linear independent components of μ_{ab} this is a 5th order finite-difference equation which has 5 independent solutions which we denote $\mu_{ab,A}$, $A = 1, \ldots, 5$. Given the asymptotics of $\mathbf{P}_a$ (2.10) and (2.12), (2.13) there are exactly 5 different asymptotics a solution of (3.37) could have as discussed in [19]. We denote these 5 independent solutions of (3.37) as $\mu_{12,A}$ where $A = 1, \ldots, 5$ and summarize their leading asymptotics at large $u > 0$ in the table below

$$
\begin{array}{c||c|c|c|c|c}
A = & 1 & 2 & 3 & 4 & 5 \\
\hline
\mu_{12,A} \sim & u^{\Delta-J} & C_{1,2}u^{-S+1-J} & C_{1,3}u^{-J} & C_{1,4}u^{S-1-J} & C_{1,5}u^{-\Delta-J} \\
\mu_{13,A} \sim & C_{2,1}u^{\Delta+1} & C_{2,2}u^{-S+2} & C_{2,3}u^{+1} & u^S & C_{2,5}u^{-\Delta+1} \\
\mu_{14,A} \sim & C_{3,1}u^{\Delta} & C_{3,2}u^{-S+1} & 1 & C_{3,4}u^{S-1} & C_{3,5}u^{-\Delta} \\
\mu_{24,A} \sim & C_{4,1}u^{\Delta-1} & u^{-S} & C_{4,3}u^{-1} & C_{4,4}u^{S-2} & C_{4,5}u^{-\Delta-1} \\
\mu_{34,A} \sim & C_{5,1}u^{\Delta+J} & C_{5,2}u^{-S+1+J} & C_{5,3}u^{+J} & C_{5,4}u^{S-1+J} & u^{-\Delta+J}
\end{array}
\tag{3.38}
$$

where we fix the normalization of our solutions so that some coefficients are set to 1.[7] As it was pointed out in [19] the asymptotics for different $A's$ are obtained by replacing Δ in (2.11) by $\pm\Delta, \pm(S-1)$ and 0. We label these solutions so that in the small S regime these asymptotics are ordered $\Delta > 1 - S > 0 > S - 1 > -\Delta$.

---

[7]The coefficients $C_{a,A}$ are some rational functions of S, Δ, J and A_1, A_2. In the small S limit all $C_{a,A} \to 0$ in our normalization.

Of course any solution of (3.37) multiplied by an i-periodic function[8] will still remain a solution of (3.37). The true μ_{ab} is thus a linear combination of the partial solutions $\mu_{ab,A}$ with some constant or periodic coefficients. This particular combination should in addition satisfy the analyticity condition (2.3) which is not guaranteed by (3.37).

The prescription for analytical continuation in S which we propose here is based on the large u asymptotics of these periodic coefficients. As we discussed in the previous section the assumption that all these coefficients are asymptotically constant is too constraining already at the leading order in S, and we must assume that at least some of these coefficients grow exponentially as $e^{2\pi u}$. To get some extra insight into the asymptotic behavior of these coefficients it is very instructive to go to the weak coupling regime.

It is known that at one loop the equation (3.37) reduces to a second order equation. When written as a finite difference equation for μ_{12} it coincides exactly with the Baxter equation for the non-compact $sl(2)$ spin chain. For $J = 2$ it reads

$$\left(2u^2 - S^2 - S - \frac{1}{2}\right) Q(u) = \left(u + \frac{i}{2}\right)^2 Q(u+i) + \left(u - \frac{i}{2}\right)^2 Q(u-i) \qquad (3.39)$$

where $Q(u) = \mu_{12}(u + i/2)$. This equation is already very well studied and all its solutions are known explicitly [75] — in particular it is easy to see that one of the solutions must have u^S asymptotics at infinity, while the other behaves as $1/u^{S+1}$. It is also known that at one loop and for any integer S (3.39) has a polynomial solution which gives the energy as $\Delta = J + S + 2ig^2 \partial_u \log \frac{Q(u-i/2)}{Q(u-i/2)}\Big|_{u=0} = S + J + 8g^2 H_S$. At the same time, for non-integer S there are of course no polynomial solutions, and according to [31] and [32] the solution which produces the energy $S + J + 8g^2 H_S$ cannot even have power-like asymptotics, instead the correct large u behavior must be:

$$Q(u) \sim \left(u^S + \ldots\right) + (A + Be^{2\pi u}) \left(\frac{1}{u^{S+1}} + \ldots\right) \ , \ \ u \to +\infty \ . \qquad (3.40)$$

Furthermore, there is a unique entire Q function with the above asymptotics. For $S > -1/2$ we can reformulate the prescription by saying that the correct solution has power-like asymptotics, containing all possible solutions, plus a small solution reinforced with an exponent.

In this form we can try to translate this result to our case. We notice that for $g \to 0$ we have $\mu_{12,1} \sim u^S$ and $\mu_{12,2} \sim u^{-S-1}$, which tells us that at least the second solution must be allowed to have a non-constant periodic coefficient in the asymptotics. We also assume that the coefficient in front of $\mu_{ab,3}$ tends to a constant.[9] This extra condition does not follow from the one loop analysis we deduced from our solution. We will show how this prescription produces the correct known result for the leading order in S. From our analysis it is hard to make a definite statement about the behavior of the periodic

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[8]It could be a periodic function with short cuts. In general the set of these coefficients is denoted in [20] by ω_{ab} whereas $\mu_{ab,A}$ is denoted in [20] as $\mathcal{Q}_{ab,cd}$.

[9]It could be hard or even impossible to separate $\mu_{ab,3}$ from $\mu_{ab,2}$ in a well defined way. In these cases $\mu_{ab,2}$ is defined modulo $\mu_{ab,3}$ and other subleading solutions. Our prescription then means that the exponential part of the coefficient in front of $\mu_{ab,3}$ is proportional to that of in front of $\mu_{ab,2}$.

coefficients in front of $\mu_{12,4}$ and $\mu_{12,5}$, but due to the expected $\Delta \to -\Delta$ symmetry, which interchanges $\mu_{12,5}$ and $\mu_{12,1}$, one may expect that the coefficient of $\mu_{12,5}$ should also go to a constant. To summarize we should have

$$\mu_{ab}(u) = \sum_{A=1}^{5} c_A \mu_{ab,A}(u) + \sum_{A=2,4,5} p_A(u) \mu_{ab,A}(u) \tag{3.41}$$

where c_A are constants whereas $p_A(u)$ are some linear combinations of $e^{\pm 2\pi u}$.[10]

Prescription at small S. In the small S limit, $\mathbf{P}_a \to 0$ and the equation (3.37) simply tells us that $\mu_{ab}(u+i) = \mu_{ab}(u)$ which implies that our 5 independent solutions are just constants at the leading order in S. We begin by noticing that in this limit μ_{12} must be entirely coming from $\mu_{12,1}$ as all the other solutions could only produce negative powers and thus cannot contribute at the leading order. So we start from $\mu_{ab} = C_{ab} + D_{ab} \sinh(2\pi u) + E_{ab} \cosh(2\pi u)$ for some constants C_{ab}, D_{ab}, E_{ab} such that $D_{12} = E_{12} = 0$. Thus we have 5 different C's, 4 different D's and 4 different E's. We notice that this general form of μ_{ab} can be significantly simplified. First, using the Pfaffian constraint (2.4) and the γ-transformation (2.14) any generic μ_{ab} of this form can be reduced to one belonging to the following two-parametric family inside the original 13-parametric space:

$$\mu_{12} = 1, \qquad\qquad \mu_{14} = a^2 \sinh 2\pi u + \frac{a}{2} \cosh 2\pi u , \tag{3.42}$$

$$\mu_{24} = b \sinh 2\pi u + \sinh 2\pi u , \qquad \mu_{34} = \frac{a^2}{4} \frac{(1-2ab)^2}{b^2 - 1} + 1 , \tag{3.43}$$

where μ_{13} is found from the Pfaffian constraint. Second, recall that according to our prescription the 1st and 3rd solutions (columns in the table (3.38)) cannot contain exponential terms. Consider μ_{14} and μ_{24}, we again see that the 4th and 5th solutions could only contain negative powers of u and thus only the 2nd solution can contribute to the parts of μ_{14} and μ_{24} that are non-decaying at infinity. This means that these components can be represented in the following form

$$\mu_{14} = (a_1 \sinh 2\pi u + a_2 \cosh 2\pi u) \mu_{14,2}(u) + \mathcal{O}\left(e^{2\pi u}/u\right) , \tag{3.44}$$

$$\mu_{24} = (a_1 \sinh 2\pi u + a_2 \cosh 2\pi u) \mu_{24,2}(u) + \mathcal{O}\left(e^{2\pi u}/u\right) , \tag{3.45}$$

for $u \to +\infty$. The $\mathcal{O}\left(e^{2\pi u}/u\right)$ terms contain contributions from all of the solutions except for the 2nd. One can see that (3.43) can be of this form only in two cases: if $a = 0$ or if $a = \frac{1}{2b}$. Both of these cases can be brought to the form

$$\mu_{12} = 1, \ \mu_{13} = 0, \ \mu_{14} = 0, \ \mu_{24} = d_1 \sinh 2\pi u + d_2 \cosh 2\pi u, \ \mu_{34} = 1 \tag{3.46}$$

by a suitable γ-transformation (3.43). However, we found that there is an additional constraint which follows from compatibility of μ_{ab} with the decaying asymptotics of $\mathbf{P}_2$. As we show in appendix E.1 for even J one must set $d_2 = 0$. For odd J we must set $d_1 = 0$ as a compatibility requirement. This justifies the choice of μ_{ab} used in the previous section. In the next section we will show how the same prescription can be applied at the next order in S and leads to nontrivial results which we subjected to intensive tests later in the text.

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[10]It could be that some of the coefficients of p_A should be zero due to the constraint (2.4).

4 Exact curvature function

In this section we use the $\mathbf{P}\mu$-system to compute the S^2 correction to the anomalous dimension, which we call the curvature function $\gamma^{(2)}(g)$. First we will discuss the case $J = 2$ in detail and then describe the modifications of the solution for the cases $J = 3$ and $J = 4$, more details on which can be found in appendix C.

4.1 Iterative procedure for the small S expansion of the $\mathbf{P}\mu$-system

For convenience let us repeat the leading order solution of the $\mathbf{P}\mu$-system for $J = 2$ (see (3.18)–(3.22))

$$\mathbf{P}_1^{(0)} = \epsilon \frac{1}{x}, \qquad\qquad \mathbf{P}_2^{(0)} = +\epsilon I_1 - \epsilon x[\sinh(2\pi u)]_- , \qquad (4.1)$$

$$\mathbf{P}_3^{(0)} = \epsilon\left(\frac{1}{x} - x\right), \qquad \mathbf{P}_4^{(0)} = -2\epsilon I_1 - \epsilon\left(\frac{1}{x} - x\right)[\sinh(2\pi u)]_-. \qquad (4.2)$$

Here ϵ is a small parameter, proportional to $\sqrt{S}$ (see (3.34)), and by $\mathbf{P}_a^{(0)}$ we denote the $\mathbf{P}_a$ functions at leading order in ϵ.

The key observation is that the $\mathbf{P}\mu$-system can be solved iteratively order by order in ϵ. Let us write $\mathbf{P}_a$ and μ_{ab} as an expansion in this small parameter:

$$\mathbf{P}_a = \mathbf{P}_a^{(0)} + \mathbf{P}_a^{(1)} + \mathbf{P}_a^{(2)} + \dots \qquad (4.3)$$

$$\mu_{ab} = \mu_{ab}^{(0)} + \mu_{ab}^{(1)} + \mu_{ab}^{(2)} + \dots . \qquad (4.4)$$

where $\mathbf{P}_a^{(0)} = \mathcal{O}(\epsilon)$, $\mathbf{P}_a^{(1)} = \mathcal{O}(\epsilon^3)$, $\mathbf{P}_a^{(2)} = \mathcal{O}(\epsilon^5)$, ..., and $\mu_{ab}^{(0)} = \mathcal{O}(\epsilon^0)$, $\mu_{ab}^{(1)} = \mathcal{O}(\epsilon^2)$, $\mu_{ab}^{(2)} = \mathcal{O}(\epsilon^4)$, etc. This structure of the expansion is dictated by the equations (2.1), (2.2) of the $\mathbf{P}\mu$-system (as we will soon see explicitly). Since the leading order $\mathbf{P}_a$ are of order ϵ, equation (2.2) implies that the discontinuity of μ_{ab} on the cut is of order ϵ^2. Thus to find μ_{ab} in the next to leading order (NLO) we only need the functions $\mathbf{P}_a$ at leading order. After this, we can find the NLO correction to $\mathbf{P}_a$ from equations (2.2). This will be done below, and having thus the full solution of the $\mathbf{P}\mu$-system at NLO we will find the energy at order S^2.

4.2 Correcting μ_{ab}...

In this subsection we find the NLO corrections $\mu_{ab}^{(1)}$ to μ_{ab}. As follows from (2.2) and (2.3), they should satisfy the equation

$$\mu_{ab}^{(1)}(u+i) - \mu_{ab}^{(1)}(u) = \mathbf{P}_a^{(0)}\tilde{\mathbf{P}}_b^{(0)} - \mathbf{P}_b^{(0)}\tilde{\mathbf{P}}_a^{(0)}, \qquad (4.5)$$

in which the right hand is known explicitly. For that reason let us define an apparatus for solving equations of this type, i.e.

$$f(u+i) - f(u) = h(u). \qquad (4.6)$$

More precisely, we consider functions $f(u)$ and $h(u)$ with one cut in u between $-2g$ and $2g$, and no poles. Such functions can be represented as infinite Laurent series in the

Zhukovsky variable $x(u)$, and we additionally restrict ourselves to the case where for $h(u)$ this expansion does not have a constant term.[11]

One can see that the general solution of (4.6) has a form of a particular solution plus an arbitrary i-periodic function, which we also call a zero mode. First we will describe the construction of the particular solution and later deal with zero modes. The linear operator which gives the particular solution of (4.6) described below will be denoted as Σ.

Notice that given the explicit form (4.2) of $\mathbf{P}_a^{(0)}$, the right hand side of (4.5) can be represented in a form

$$\alpha(x)\sinh(2\pi u) + \beta(x), \tag{4.7}$$

where $\alpha(x), \beta(x)$ are power series in x growing at infinity not faster than polynomially. Thus for such α and β we define

$$\Sigma \cdot [\alpha(x)\sinh(2\pi u) + \beta(x)] \equiv \sinh(2\pi u)\Sigma \cdot \alpha(x) + \Sigma \cdot \beta(x). \tag{4.8}$$

We also define $\Sigma \cdot x^{-n} = \Gamma' \cdot x^{-n}$ for $n > 0$, where the integral operator Γ' defined as

$$\left(\Gamma' \cdot h\right)(u) \equiv \oint_{-2g}^{2g} \frac{dv}{4\pi i} \partial_u \log \frac{\Gamma[i(u-v)+1]}{\Gamma[-i(u-v)]} h(v). \tag{4.9}$$

This requirement is consistent because of the following relation[12]

$$\left(\Gamma' \cdot h\right)(u+i) - \left(\Gamma' \cdot h\right)(u) = -\frac{1}{2\pi i}\oint_{-2g}^{2g} \frac{h(v)}{u-v} dv = h_-(u) - \widetilde{h_+}(u). \tag{4.10}$$

What is left is to define Σ on positive powers of x. We do it by requiring

$$\Sigma \cdot [x^a + 1/x^a] \equiv p_a'(u) \tag{4.11}$$

where $p_a'(u)$ is a polynomial in u of degree $a+1$, which is a solution of

$$p_a'(u+i) - p_a'(u) = \frac{1}{2}\left(x^a + 1/x^a\right) \tag{4.12}$$

and satisfies the following additional properties: $p_a'(0) = 0$ for odd a and $p_a'(i/2) = 0$ for even a. One can check that this definition is consistent and defines of $p_a'(u)$ uniquely. Explicit form of the first few $p_a'(u)$, which we call periodized Chebyshev polynomials, can be found in appendix A.

From this definition of Σ one can see that the result of its action on expressions of the form (4.7) can again be represented in this form - what is important for us is that no exponential functions other than $\sinh(2\pi u)$ appear in the result.

---

[11]The r.h.s. of (4.5) has the form $F(u) - \tilde{F}(u)$ and therefore indeed does not have a constant term in its expansion, as the constant in F would cancel in the difference $F(u) - \tilde{F}(u)$.

[12]We remind that f_+ and f_- stand for the part of the Laurent expansion with, respectively, positive and negative powers of x, while $\tilde{f}$ is the analytic continuation around the branch point at $u = 2g$ (which amounts to replacing $x \to \frac{1}{x}$).

A good illustration of how the definitions above work would be the following two simple examples. Suppose one wants to calculate $\Sigma \cdot \left(x - \frac{1}{x}\right)$, then it is convenient to split the argument of Σ in the following way:

$$\Sigma \cdot \left(x - \frac{1}{x}\right) = \Sigma \cdot \left(x + \frac{1}{x}\right) - 2\Sigma \cdot \frac{1}{x}. \tag{4.13}$$

In the first term we recognize $p_1'(u) = \frac{iu(u-i)}{2g}$, whereas in the second the argument of Σ is decaying at infinity, thus Σ is equivalent to Γ' in this context. Notice also that $\Gamma' \cdot \frac{1}{x} = -\Gamma' \cdot x$. All together, we get

$$\Sigma \cdot \left(x - \frac{1}{x}\right) = \Sigma \cdot \left(x + \frac{1}{x}\right) - 2\Sigma \cdot \frac{1}{x} = 2p_1'(u) + 2\Gamma' \cdot x \tag{4.14}$$

In a similar way, in order to calculate $\Sigma \cdot \frac{\sinh_- - \sinh_+}{2}$, one can write $\frac{\sinh_- - \sinh_+}{2} = \sinh_- - \frac{1}{2}\sinh(2\pi u)$. Notice that since $\sinh_-$ decays at infinity,

$$\Sigma \cdot \sinh_- = \Gamma' \cdot \sinh_- . \tag{4.15}$$

Also, since i-periodic functions can be factored out of Σ,

$$\Sigma \cdot \sinh(2\pi u) = \sinh(2\pi u)\Sigma \cdot 1 = \sinh(2\pi u)p_0'(u)/2. \tag{4.16}$$

Finally,

$$\Sigma \cdot \frac{\sinh_- - \sinh_+}{2} = \Gamma' \cdot (\sinh_-) - \frac{1}{2}\sinh(2\pi u)p_0'(u). \tag{4.17}$$

As an example we present the particular solution for two components of μ_{ab} (below we will argue that π_{12} and π_{13} can be chosen to be zero, see (4.26))

$$\mu_{13}^{(1)} - \pi_{13} = \Sigma \cdot \left(\mathbf{P}_1\tilde{\mathbf{P}}_3 - \mathbf{P}_3\tilde{\mathbf{P}}_1\right) = \epsilon^2 \Sigma \cdot \left(x^2 - \frac{1}{x^2}\right) = \epsilon^2 \ \left(\Gamma' \cdot x^2 + p_2'(u)\right), \tag{4.18}$$

$$\mu_{12}^{(1)} - \pi_{12} = \Sigma \cdot \left(\mathbf{P}_1\tilde{\mathbf{P}}_2 - \mathbf{P}_2\tilde{\mathbf{P}}_1\right) =$$
$$= -\epsilon^2 \left[2I_1\Gamma' \cdot x - \sinh(2\pi u)\,\Gamma' \cdot x^2 - \Gamma' \cdot \left(\sinh_- \left(x^2 + \frac{1}{x^2}\right)\right)\right]. \tag{4.19}$$

Now let us apply Σ defined above to (4.5), writing that its general solution is

$$\mu_{ab}^{(1)} = \Sigma \cdot (\mathbf{P}_a^{(0)}\tilde{\mathbf{P}}_b^{(0)} - \mathbf{P}_b^{(0)}\tilde{\mathbf{P}}_a^{(0)}) + \pi_{ab}, \tag{4.20}$$

where the zero mode π_{ab} is an arbitrary i-periodic entire function, which can be written similarly to the leading order as $c_{1,ab}\cosh 2\pi u + c_{2,ab}\sinh 2\pi u + c_{3,ab}$. Again, many of the coefficients $c_{i,ab}$ can be set to zero. First, the prescription from section 3.2 implies that non-vanishing at infinity part of coefficients of $\sinh(2\pi u)$ and $\cosh(2\pi u)$ in μ_{12} is zero. As one can see from the explicit form (4.19) of the particular solution which we choose for μ_{12}, it does not contain $\cosh(2\pi u)$ and the coefficient of $\sinh(2\pi u)$ is decaying at infinity. So in order to satisfy the prescription, we have to set $c_{2,12}$ and $c_{2,12}$ to zero. Second, since the coefficients $c_{n,ab}$ are of order S, we can remove some of them by making an infinitesimal

γ-transformation, i.e. with $R = 1 + \mathcal{O}(S)$ (see section 2.3 and eq. (2.14)). Further, the Pfaffian constraint (2.4) imposes 5 equations on the remaining coefficients, which leaves the following 2-parametric family of zero modes

$$\pi_{12} = 0, \quad \pi_{13} = 0, \quad \pi_{14} = \frac{1}{2}c_{1,34}\cosh 2\pi u, \tag{4.21}$$

$$\pi_{24} = c_{1,24}\cosh 2\pi u, \quad \pi_{34} = c_{1,34}\cosh 2\pi u. \tag{4.22}$$

Let us now look closer at the exponential part of μ_{14} and μ_{24}. Combining the leading order (3.18) and the perturbation (4.20) and taking into account the fact that operator Σ does not produce terms proportional to $\cosh 2\pi u$, we obtain

$$\mu_{14} = \frac{1}{2}c_{1,34}\cosh 2\pi u + \mathcal{O}(\epsilon)\sinh 2\pi u + \mathcal{O}(\epsilon^2) + \ldots, \tag{4.23}$$

$$\mu_{24} = \frac{1}{2}c_{1,24}\cosh 2\pi u + (1 + \mathcal{O}(\epsilon))\sinh 2\pi u + \mathcal{O}(\epsilon^2) + \ldots, \tag{4.24}$$

where dots stand for powers-like terms or exponential terms suppressed by powers of u.

As we remember from section 3.2, only the 2nd solution of the 5th order Baxter equation (3.37) can contribute to the exponential part of μ_{14} and μ_{24}, which means that μ_{14} and μ_{24} are proportional to the same linear combination of $\sinh 2\pi u$ and $\cosh 2\pi u$. From the second equation one can see that this linear combination can be normalized to be $\frac{1}{2}c_{1,24}\cosh 2\pi u + (1+\mathcal{O}(\epsilon))\sinh 2\pi u$. Then $\mu_{14} = C\left(\frac{1}{2}c_{1,24}\cosh 2\pi u + (1 + \mathcal{O}(\epsilon))\sinh 2\pi u\right)$, where C is some constant, which is of order $\mathcal{O}(\epsilon)$, because the coefficient of $\sinh 2\pi u$ in the first equation is $\mathcal{O}(\epsilon)$. Taking into account that $c_{1,24}$ is $\mathcal{O}(\epsilon)$ itself, we find that $c_{1,34} = \mathcal{O}(\epsilon^2)$, i.e. it does not contribute at the order which we are considering. So the final form of the zero mode in (4.20) is

$$\pi_{12} = 0, \quad \pi_{13} = 0, \quad \pi_{14} = 0, \tag{4.25}$$

$$\pi_{24} = c_{1,24}\cosh 2\pi u, \quad \pi_{34} = 0. \tag{4.26}$$

In this way, using the particular solution given by Σ and the form of zero modes (4.26) we have computed all the functions $\mu_{ab}^{(1)}$. The details and the results of the calculation can be found in appendix C.1.

4.3 Correcting $\mathbf{P}_a$...

In the previous section we found the NLO part of μ_{ab}. Now, according to the iterative procedure described in section 4.1, we can use it to write a closed system of equations for $\mathbf{P}_a^{(1)}$. Indeed, expanding the system (2.9) to NLO we get

$$\tilde{\mathbf{P}}_1^{(1)} - \mathbf{P}_1^{(1)} = -\mathbf{P}_3^{(1)} + r_1, \tag{4.27}$$

$$\tilde{\mathbf{P}}_2^{(1)} + \mathbf{P}_2^{(1)} = -\mathbf{P}_4^{(1)} - \mathbf{P}_1^{(1)}\sinh(2\pi u) + r_2, \tag{4.28}$$

$$\tilde{\mathbf{P}}_3^{(1)} + \mathbf{P}_3^{(1)} = r_3, \tag{4.29}$$

$$\tilde{\mathbf{P}}_4^{(1)} - \mathbf{P}_4^{(1)} = \mathbf{P}_3^{(1)}\sinh(2\pi u) + r_4, \tag{4.30}$$

where the free terms are given by

$$r_a = -\mu_{ab}^{(1)}\chi^{bc}\mathbf{P}_c^{(0)}. \tag{4.31}$$

Notice that r_a does not change if we add a matrix proportional to $\mathbf{P}_a^{(0)}\tilde{\mathbf{P}}_b^{(0)} - \mathbf{P}_b^{(0)}\tilde{\mathbf{P}}_a^{(0)}$ to $\mu_{ab}^{(1)}$, due to the relations

$$\mathbf{P}_a\chi^{ab}\mathbf{P}_b = 0, \ \mathbf{P}_a\chi^{ab}\tilde{\mathbf{P}}_b = 0, \tag{4.32}$$

which follow from the $\mathbf{P}\mu$-system equations. In particular we can use this property to replace $\mu_{ab}^{(1)}$ in (4.31) by $\mu_{ab}^{(1)} + \frac{1}{2}\left(\mathbf{P}_a^{(0)}\tilde{\mathbf{P}}_b^{(0)} - \mathbf{P}_b^{(0)}\tilde{\mathbf{P}}_a^{(0)}\right)$. This will be convenient for us, since in expressions for $\mu_{ab}^{(1)}$ in terms of p_a and Γ (see (4.18), (4.19) and appendix C.1) this change amounts to simply replacing Γ' by a convolution with a more symmetric kernel:

$$\Gamma' \to \Gamma, \tag{4.33}$$

$$(\Gamma \cdot h)(u) \equiv \oint_{-2g}^{2g} \frac{dv}{4\pi i}\partial_u \log \frac{\Gamma[i(u-v)+1]}{\Gamma[-i(u-v)+1]}h(v), \tag{4.34}$$

while at the same time replacing

$$p'_a(u) \to p_a(u), \tag{4.35}$$

$$p_a(u) = p'_a(u) + \frac{1}{2}\left(x^a(u) + x^{-a}(u)\right). \tag{4.36}$$

Having made this comment, we will now develop tools for solving the equations (4.27)–(4.30). Notice first that if we solve them in the order (4.29), (4.27), (4.30), (4.28), substituting into each subsequent equation the solution of all the previous, then at each step the problem we have to solve has a form

$$\tilde{f} + f = h \ \text{ or } \ \tilde{f} - f = h \ , \tag{4.37}$$

where h is known, f is unknown and both the right hand side and the left hand side are power series in x. It is obvious that equations (4.37) have solutions only for h such that $h = \tilde{h}$ and $h = -\tilde{h}$ respectively. On the class of such h a particular solution for f can be written as

$$f = [h]_- + [h]_0/2 \equiv H \cdot h \ \Rightarrow \ \tilde{f} + f = h \tag{4.38}$$

and

$$f = [h]_- \equiv K \cdot h \ \Rightarrow \ \tilde{f} - f = h, \tag{4.39}$$

where $[h]_0$ is the constant part of Laurent expansion of h (it does not appear in the second equation, because h such that $h = -\tilde{h}$ does not have a constant part). The operators K and H introduced here can be also defined by their integral kernels

$$H(u,v) = -\frac{1}{4\pi i}\frac{\sqrt{u-2g}\sqrt{u+2g}}{\sqrt{v-2g}\sqrt{v+2g}}\frac{1}{u-v}dv, \tag{4.40}$$

$$K(u,v) = +\frac{1}{4\pi i}\frac{1}{u-v}dv. \tag{4.41}$$

which are equivalent to (4.38), (4.39) of the classes of h such that $h = \tilde{h}$ and $h = -\tilde{h}$ respectively.[13] The particular solution $f = K \cdot h$ of the equation $\tilde{f} + f = h$ is unique in the

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[13]We denote e.g. $K \cdot h = \oint_{-2g}^{2g} K(u,v)h(v)dv$ where the integral is around the branch cut between $-2g$ and $2g$.

class of functions f decaying at infinity, and the solution $f = H \cdot h$ of $\tilde{f} - f = h$ is unique for non-growing f. In all other cases the general solution will include zero modes, which, in our case are fixed by asymptotics of $\mathbf{P}_a$.

Now it is easy to write the explicit solution of the equations (4.27)–(4.30):

$$\mathbf{P}_3^{(1)} = H \cdot r_3, \tag{4.42}$$

$$\mathbf{P}_1^{(1)} = \frac{1}{2}\mathbf{P}_3^{(1)} + K \cdot \left(r_1 - \frac{1}{2}r_3\right), \tag{4.43}$$

$$\mathbf{P}_4^{(1)} = K \cdot \left(-\frac{1}{2}\left(\tilde{\mathbf{P}}_3^{(1)} - \mathbf{P}_3^{(1)}\right)\sinh(2\pi u) + \frac{2r_4 + r_3\sinh(2\pi u)}{2}\right) - 2\delta, \tag{4.44}$$

$$\mathbf{P}_2^{(1)} = H \cdot \left(-\frac{1}{2}\left(\mathbf{P}_4^{(1)} + \sinh(2\pi u)\mathbf{P}_1^{(1)} + \tilde{\mathbf{P}}_4^{(1)} + \sinh(2\pi u)\tilde{\mathbf{P}}_1^{(1)}\right) + \tag{4.45}$$

$$+ \frac{r_4 + \sinh(2\pi u)r_1 + 2r_2}{2}\right) + \delta,$$

where δ is a constant fixed uniquely by requiring $\mathcal{O}(1/u^2)$ asymptotics for $\mathbf{P}_2$. This asymptotic also sets the last coefficient $c_{1,24}$ left in π_{12} to zero. Thus in the class of functions with asymptotics (2.10) the solution for μ_{ab} and $\mathbf{P}_a$ is unique up to a γ-transformation.

4.4 Result for $J = 2$

In order to obtain the result for the anomalous dimension, we again use the formulas (2.12), (2.13) which connect the leading coefficients of $\mathbf{P}_a$ with Δ, J and S. After plugging in A_i which we find from our solution, we obtain the result for the S^2 correction to the anomalous dimension:

$$\gamma_{J=2}^{(2)} = \frac{\pi}{g^2(I_1 - I_3)^3}\oint\frac{du_x}{2\pi i}\oint\frac{du_y}{2\pi i}\left[\frac{8I_1^2(I_1 + I_3)\left(x^3 - \left(x^2 + 1\right)y\right)}{(x^3 - x)y^2}\right. \tag{4.46}$$

$$+ \frac{8\mathrm{sh}_-^x\mathrm{sh}_-^y\left(x^2y^2 - 1\right)\left(I_1(x^4y^2 + 1) - I_3x^2(y^2 + 1)\right)}{x^2\left(x^2 - 1\right)y^2}$$

$$- \frac{4(\mathrm{sh}_-^y)^2x^2\left(y^4 - 1\right)\left(I_1(2x^2 - 1) - I_3\right)}{(x^2 - 1)y^2}$$

$$+ \frac{8I_1^2\mathrm{sh}_-^y x\left(2\left(x^3 - x\right)\left(y^3 + y\right) - 2x^2\left(y^4 + y^2 + 1\right) + y^4 + 4y^2 + 1\right)}{(x^2 - 1)y^2}$$

$$- \frac{8(I_1 - I_3)I_1\mathrm{sh}_-^y x(x - y)(xy - 1)}{(x^2 - 1)y}$$

$$\left. - \frac{4(I_1 - I_3)(\mathrm{sh}_-^x)^2\left(x^2 + 1\right)y^2}{(x^2 - 1)}\right]\frac{1}{4\pi i}\partial_u\log\frac{\Gamma(iu_x - iu_y + 1)}{\Gamma(1 - iu_x + iu_y)}\,.$$

Here the integration contour goes around the branch cut at $(-2g, 2g)$. We also denote $\mathrm{sh}_-^x = \sinh_-(x)$, $\mathrm{sh}_-^y = \sinh_-(y)$ (recall that $\sinh_-$ was defined in (3.14)). This is our final result for the curvature function at any coupling.

It is interesting to note that our result contains the combination $\log\frac{\Gamma(iu_x - iu_y + 1)}{\Gamma(1 - iu_x + iu_y)}$ which plays an essential role in the construction of the BES dressing phase. We will use this identification in section 6.3 to compute the integral in (4.46) numerically with high precision.

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In the next subsections we will describe generalizations of the $J = 2$ result to operators with $J = 3$ and $J = 4$.

4.5 Results for higher J

Solving the $\mathbf{P}\mu$-system for $J = 3$ is similar to the $J = 2$ case described above, except for several technical complications, which we will describe here, leaving the details for the appendix C.2. As in the previous section, the starting point is the LO solution of the $\mathbf{P}\mu$ system, which for $J = 3$ reads

$$\mathbf{P}_1 = \epsilon x^{-3/2}, \quad \mathbf{P}_3 = -\epsilon x^{3/2}, \tag{4.47}$$

$$\mathbf{P}_2 = -\epsilon x^{3/2}\cosh_- + \epsilon x^{-1/2}I_2, \tag{4.48}$$

$$\mathbf{P}_4 = -\epsilon x^{1/2}I_2 - \epsilon x^{-3/2}I_0 - \epsilon x^{-3/2}\cosh_-, \tag{4.49}$$

$$\mu_{12} = 1, \quad \mu_{13} = 0, \quad \mu_{14} = 0, \quad \mu_{24} = \cosh(2\pi u), \quad \mu_{34} = 1 . \tag{4.50}$$

The first step is to construct $\mu_{ab}^{(1)}$ from its discontinuity given by the equation (4.5). The full solution consists of a particular solution and a general solution of the corresponding homogeneous equation, i.e. zero mode π_{ab}. In our case the zero mode can be an i-periodic function, i.e. a linear combination of $\sinh(2\pi u)$, $\cosh(2\pi u)$ and constants. As in the case of $J = 2$, we use a combination of the Pfaffian constraint, prescription from section 3.2 and a γ-transformation to reduce all the parameters of the zero mode to just one, sitting in μ_{24}:

$$\pi_{12} = 0, \ \pi_{13} = 0, \ \pi_{14} = 0, \ \pi_{24} = c_{24,2}\sinh(2\pi u), \ \pi_{34} = 0. \tag{4.51}$$

As in the previous section, the next step is to find $\mathbf{P}_a^{(1)}$ from the $P\mu$ system expanded to the first order, namely from

$$\tilde{\mathbf{P}}_1^{(1)} + \mathbf{P}_3^{(1)} = r_1, \tag{4.52}$$

$$\tilde{\mathbf{P}}_2^{(1)} + \mathbf{P}_4^{(1)} + \mathbf{P}_1^{(1)}\cosh(2\pi u) = r_2, \tag{4.53}$$

$$\tilde{\mathbf{P}}_3^{(1)} + \mathbf{P}_1^{(1)} = r_3, \tag{4.54}$$

$$\tilde{\mathbf{P}}_4^{(1)} + \mathbf{P}_2^{(1)} - \mathbf{P}_3^{(1)}\cosh(2\pi u) = r_4, \tag{4.55}$$

where r_a are defined by (4.31) and for $J = 3$ are given explicitly in appendix C.2. In attempt to solve this system, however, we encounter another technical complication. As one can see from (4.47)–(4.49), the LO solution contains half-integer powers of J, meaning that the $\mathbf{P}_a$ now have an extra branch point at infinity. However, the operations H and K defined by (4.41) work only for functions which have Laurent expansion in integer powers of x. In order to solve equations of the type (4.5) on the class of functions which allow Laurent-like expansion in x with only half-integer powers x, we introduce operations H^*, K^*:

$$H^* \cdot f \equiv \frac{x+1}{\sqrt{x}}H \cdot \frac{\sqrt{x}}{x+1}f, \tag{4.56}$$

$$K^* \cdot f \equiv \frac{x+1}{\sqrt{x}}K \cdot \frac{\sqrt{x}}{x+1}f. \tag{4.57}$$

In terms of these operations the solution of the system (4.52)–(4.55) is

$$\mathbf{P}_1^{(1)} = \frac{1}{2}\left(H^*(r_1 + r_3) + K^*(r_1 - r_3)\right) + \mathbf{P}_1^{\text{zm}}, \tag{4.58}$$

$$\mathbf{P}_3^{(1)} = \frac{1}{2}\left(H^*(r_1 + r_3) - K^*(r_1 - r_3)\right) + \mathbf{P}_2^{\text{zm}}, \tag{4.59}$$

$$\mathbf{P}_2^{(1)} = \frac{1}{2}\left(H^*(r_2 + r_4) + K^*(r_2 - r_4) - \right.$$
$$\left. - H^*\left(\cosh(2\pi u)K^*(r_1 - r_3)\right) - K^*\left(\cosh(2\pi u)H^*(r_1 + r_3)\right)\right) + \mathbf{P}_3^{\text{zm}}, \tag{4.60}$$

$$\mathbf{P}_4^{(1)} = \frac{1}{2}\left(H^*(r_2 + r_4) - K^*(r_2 - r_4) - \right.$$
$$\left. - H^*\left(\cosh(2\pi u)K^*(r_1 - r_3)\right) + K^*\left(\cosh(2\pi u)H^*(r_1 + r_3)\right)\right) + \mathbf{P}_4^{\text{zm}}, \tag{4.61}$$

where $\mathbf{P}_a^{\text{zm}}$ is a solution of the system (4.52)–(4.55) with right hand side set to zero, whose explicit form $\mathbf{P}_a^{\text{zm}}$ is given in appendix C.2 (see (C.15)–(C.16)) and which is parametrized by four constants L_1, L_2, L_3, L_4, e.g.

$$\mathbf{P}_1^{\text{zm}} = L_1 x^{-1/2} + L_3 x^{1/2}. \tag{4.62}$$

These constants are fixed by requiring correct asymptotics of $\mathbf{P}_a$, which also fixes the parameter $c_{24,2}$ in the zero mode (4.51) of μ_{ab}.[14] Indeed, a priori $\mathbf{P}_2$ and $\mathbf{P}_1$ have wrong asymptotics. Imposing a constraint that $\mathbf{P}_2$ decays as $u^{-5/2}$ and $\mathbf{P}_1$ decays as $u^{-3/2}$ produces five equations, which fix all the parameters uniquely.

Skipping the details of the intermediate calculations, we present the final result for the anomalous dimension:

$$\gamma_{J=3}^{(2)} = \oint \frac{du_x}{2\pi i} \oint \frac{du_y}{2\pi i} i \frac{1}{g^2 (I_2 - I_4)^3} \left[\frac{2\left(x^6 - 1\right) y (\text{ch}_-^y)^2 (I_2 - I_4)}{x^3 \left(y^2 - 1\right)} - \right. \tag{4.63}$$
$$- \frac{4\text{ch}_-^x \text{ch}_-^y \left(x^3 y^3 - 1\right)\left(I_2 x^5 y^3 + I_2 - I_4 x^2 \left(x y^3 + 1\right)\right)}{x^3 \left(x^2 - 1\right) y^3} +$$
$$+ \frac{(y^2 - 1)(\text{ch}_-^y)^2 I_2 \left((x^8 + 1)\left(2y^4 + 3y^2 + 2\right) - (x^6 + x^2)\left(y^2 + 1\right)^2\right)}{x^3 \left(x^2 - 1\right) y^3} -$$
$$- \frac{(y^2 - 1)(\text{ch}_-^y)^2 I_4 \left((x^8 + 1)y^2 + (x^6 + x^2)\left(y^4 + 1\right)\right)}{x^3 \left(x^2 - 1\right) y^3} -$$
$$- \frac{4I_2 \text{ch}_-^y (x - y)(xy - 1)\left(I_2 \left((x^6 + 1)\left(y^3 + y\right) + (x^5 + x)\left(y^4 + y^2 + 1\right) - x^3\left(y^4 + 1\right)\right) + I_4 x^3 y^2\right)}{x^3 \left(x^2 - 1\right) y^3} -$$
$$- \frac{I_2^2 (y^2 - 1)(x - y)(xy - 1)\left(I_2 \left((x^6 + x^4 + x^2 + 1\right) y + 2x^3\left(y^2 + 1\right)\right) + I_4 \left(x^5 + x\right)\left(y^2 + 1\right)\right)}{x^3 \left(x^2 - 1\right) y^3} \right]$$
$$\frac{1}{4\pi i} \partial_u \log \frac{\Gamma(iu_x - iu_y + 1)}{\Gamma(1 - iu_x + iu_y)}.$$

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[14]Actually in this way $c_{24,2}$ is fixed to be zero.

We defined $\mathrm{ch}^x_- = \cosh_-(x)$ and $\mathrm{ch}^y_- = \cosh_-(y)$, where $\cosh_-(x)$ is the part of the Laurent expansion of $\cosh\left(g(x+1/x)\right)$ vanishing at infinity, i.e.

$$\cosh_-(x) = \sum_{k=1}^{\infty} I_{2k} x^{-2k}. \tag{4.64}$$

The result for $J = 4$ is given in appendix C.4.

5 Weak coupling tests and predictions

Our results for the curvature function $\gamma^{(2)}(g)$ at $J = 2, 3, 4$ (eqs. (4.46), (4.63), (C.27)) are straightforward to expand at weak coupling. We give expansions to 10 loops in appendix D. Let us start with the $J = 2$ case, for which we found

$$\gamma^{(2)}_{J=2} = -8g^2\zeta_3 + g^4\left(140\zeta_5 - \frac{32\pi^2\zeta_3}{3}\right) + g^6\left(200\pi^2\zeta_5 - 2016\zeta_7\right) \tag{5.1}$$
$$+ g^8\left(-\frac{16\pi^6\zeta_3}{45} - \frac{88\pi^4\zeta_5}{9} - \frac{9296\pi^2\zeta_7}{3} + 27720\zeta_9\right)$$
$$+ g^{10}\left(\frac{208\pi^8\zeta_3}{405} + \frac{160\pi^6\zeta_5}{27} + 144\pi^4\zeta_7 + 45440\pi^2\zeta_9 - 377520\zeta_{11}\right) + \dots$$

Remarkably, at each loop order all contributions have the same transcendentality, and only simple zeta values (i.e. ζ_n) appear. This is also true for the $J = 3$ and $J = 4$ cases.

We can check this expansion against known results, as the anomalous dimensions of twist two operators have been computed up to five loops for arbitrary spin [33–40] (see also [41] and the review [42]). To three loops they can be found solely from the ABA equations, while at four and five loops wrapping corrections need to be taken into account which was done in [39, 40] by utilizing generalized Luscher formulas. All these results are given by linear combinations of harmonic sums

$$S_a(N) = \sum_{n=1}^{N} \frac{(\mathrm{sign}(a))^n}{n^{|a|}}, \quad S_{a_1,a_2,a_3,\dots}(N) = \sum_{n=1}^{N} \frac{(\mathrm{sign}(a_1))^n}{n^{|a_1|}} S_{a_2,a_3,\dots}(n) \tag{5.2}$$

with argument equal to the spin S. To make a comparison with our results we expanded these predictions in the $S \to 0$ limit. For this lengthy computation, as well as to simplify the final expressions, we used the `Mathematica` packages HPL [43, 44], the package [45] provided with the paper [46], and the HarmonicSums package [47–53].

In this way we have confirmed the coefficients in (5.1) to four loops. Let us note that expansion of harmonic sums leads to multiple zeta values (MZVs), which however cancel in the final result leaving only ζ_n.

Importantly, the part of the four-loop coefficient which comes from the wrapping correction is essential for matching with our result. This is a strong confirmation that our calculation based on the $\mathbf{P}\mu$-system is valid beyond the ABA level. Additional evidence that our result incorporates all finite-size effects is found at strong coupling (see section 6).

For operators with $J = 3$, our prediction at weak coupling is

$$\gamma^{(2)}_{J=3} = -2g^2\zeta_3 + g^4\left(12\zeta_5 - \frac{4\pi^2\zeta_3}{3}\right) + g^6\left(\frac{2\pi^4\zeta_3}{45} + 8\pi^2\zeta_5 - 28\zeta_7\right) \tag{5.3}$$
$$+ g^8\left(-\frac{4\pi^6\zeta_3}{45} - \frac{4\pi^4\zeta_5}{15} - 528\zeta_9\right) + \dots$$

The known results for any spin in this case are available at up to six loops, including the wrapping correction which first appears at five loops [54–56]. Expanding them at $S \to 0$ we have checked our calculation to four loops.[15]

For future reference, in appendix D we present an expansion of known results for $J = 2, 3$ up to order S^3 at first several loop orders. In particular, we found that multiple zeta values appear in this expansion, which did not happen at lower orders in S.

Let us now discuss the $J = 4$ case. The expansion of our result reads:

$$\gamma^{(2)}_{J=4} = g^2\left(-\frac{14\zeta_3}{5} + \frac{48\zeta_5}{\pi^2} - \frac{252\zeta_7}{\pi^4}\right) \tag{5.4}$$
$$+ g^4\left(-\frac{22\pi^2\zeta_3}{25} + \frac{474\zeta_5}{5} - \frac{8568\zeta_7}{5\pi^2} + \frac{8316\zeta_9}{\pi^4}\right)$$
$$+ g^6\left(\frac{32\pi^4\zeta_3}{875} + \frac{3656\pi^2\zeta_5}{175} - \frac{56568\zeta_7}{25} + \frac{196128\zeta_9}{5\pi^2} - \frac{185328\zeta_{11}}{\pi^4}\right)$$
$$+ g^8\left(-\frac{4\pi^6\zeta_3}{175} - \frac{68\pi^4\zeta_5}{75} - \frac{55312\pi^2\zeta_7}{125} + \frac{1113396\zeta_9}{25} - \frac{3763188\zeta_{11}}{5\pi^2}\right.$$
$$\left. + \frac{3513510\zeta_{13}}{\pi^4}\right) + \dots$$

Unlike for the $J = 2$ and $J = 3$ cases, we could not find a closed expression for the energy at any spin S in literature even at one loop, however there is another way to check our result. One can expand the asymptotic Bethe ansatz equations at large J for fixed values of $S = 2, 4, 6, \dots$ and then extract the coefficients in the expansion which are polynomial in S. This was done in [26] (see appendix C there) where at one loop the expansion was found up to order $1/J^6$:

$$\gamma(S, J) = g^2\left(\frac{S}{2\,J^2} - \left(\frac{S^2}{4} + \frac{S}{2}\right)\frac{1}{J^3} + \left[\frac{3S^3}{16} + \left(\frac{1}{8} - \frac{\pi^2}{12}\right)S^2 + \frac{S}{2}\right]\frac{1}{J^4} + \dots\right) + \mathcal{O}(g^4) \tag{5.5}$$

Now taking the part proportional to S^2 and substituting $J = 4$ one may expect to get a numerical approximation to the 1-loop coefficient in our result (5.4), i.e. $-\frac{14\zeta_3}{5} + \frac{48\zeta_5}{\pi^2} - \frac{252\zeta_7}{\pi^4}$. To increase the precision we extended the expansion in (5.5) to order $1/J^8$. Remarkably, in this way we confirmed the 1-loop part of the $\mathbf{P}\mu$ prediction (5.4) with about 1% accuracy! In figure 2 one can also see that the ABA result converges to our prediction when the order of expansion in $1/J$ is being increased.

Also, in contrast to $J = 2$ and $J = 3$ cases we see that negative powers of π appear in (5.4) (although still all the contributions at a given loop order have the same transcendentality). It would be interesting to understand why this happens from the gauge theory

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[15]As a further check it would be interesting to expand to order S^2 the known results for twist 2 operators at five loops, and for twist 3 operators at five and six loops — all of which are given by huge expressions.
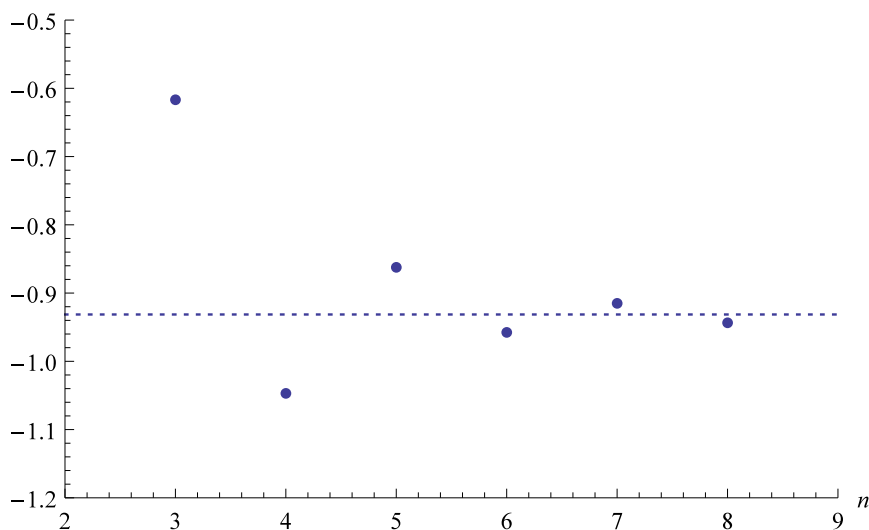
Figure 2. One-loop energy at $J = 4$ from the Bethe ansatz. The dashed line shows the result from the $\mathbf{P}\mu$-system for the coefficient of S^2 in the 1-loop energy at $J = 4$, i.e. $-\frac{14\zeta_3}{5} + \frac{48\zeta_5}{\pi^2} - \frac{252\zeta_7}{\pi^4} \approx -0.931$ (see (5.4)). The dots show the Bethe ansatz prediction (5.5) expanded to orders $1/J^3, 1/J^4, \ldots, 1/J^8$ (the order of expansion n corresponds to the horizontal axis), and it appears to converge to the $\mathbf{P}\mu$-system result.

perspective, especially since expansion of the leading S term (1.4) has the same structure for all J,

$$\gamma_J^{(1)} = \frac{8\pi^2 g^2}{J(J+1)} - \frac{32\pi^4 g^4}{J(J+1)^2(J+2)} + \frac{256\pi^6 g^6}{J(J+1)^3(J+2)(J+3)} + \ldots \qquad (5.6)$$

The change of structure at $J = 4$ might be related to the fact that for $J \geq 4$ the ground state anomalous dimension even at one loop is expected to be an irrational number for integer $S > 0$ (see [57, 58]), and thus cannot be written as a linear combination of harmonic sums with integer coefficients.

In the next section we will discuss tests and applications of our results at strong coupling.

6   Strong coupling tests and predictions

In this section we will present the strong coupling expansion of our results for the curvature function, and link these results to anomalous dimensions of short operators at strong coupling. We will also obtain new predictions for the BFKL pomeron intercept.

6.1   Expansion of the curvature function for $J = 2, 3, 4$

To obtain the strong coupling expansion of our exact results for the curvature function, we evaluated it numerically with high precision for a range of values of g and then made a fit to find the expansion coefficients. It would also be interesting to carry out the expansion analytically, and we leave this for the future.

For numerical study it is convenient to write our exact expressions (4.46), (4.63), (C.27) for $\gamma^{(2)}(g)$, which have the form

$$\gamma^{(2)}(g) = \oint du_x \oint du_y f(x,y) \partial_{u_x} \log \frac{\Gamma(iu_x - iu_y + 1)}{\Gamma(1 - iu_x + iu_y)} \tag{6.1}$$

where the integration goes around the branch cut between $-2g$ and $2g$, in a slightly different way (we remind that we use notation $x + \frac{1}{x} = \frac{u_x}{g}$ and $y + \frac{1}{y} = \frac{u_y}{g}$). Namely, by changing the variables of integration to x, y and integrating by parts one can write the result as

$$\gamma^{(2)}(g) = \oint dx \oint dy F(x,y) \log \frac{\Gamma(iu_x - iu_y + 1)}{\Gamma(iu_y - iu_x + 1)} \tag{6.2}$$

where $F(x,y)$ is some polynomial in the following variables: x, $1/x$, y, $1/y$, sh^x_- and sh^y_- (for $J = 3$ it includes ch^x_-, ch^y_- instead of the sh_- functions). The integral in (6.2) is over the unit circle. The advantage of this representation is that plugging in sh^x_-, sh^y_- as series expansions (truncated to some large order), we see that it only remains to compute integrals of the kind

$$C_{r,s} = \frac{1}{i} \oint \frac{dx}{2\pi} \oint \frac{dy}{2\pi} x^r y^s \log \frac{\Gamma(iu_x - iu_y + 1)}{\Gamma(iu_y - iu_x + 1)} \tag{6.3}$$

These are nothing but the coefficients of the BES dressing phase [59–62]. They can be conveniently computed using the strong coupling expansion [59]

$$C_{r,s} = \sum_{n=0}^{\infty} \left[-\frac{2^{-n-1}(-\pi)^{-n} g^{1-n} \zeta_n \left(1 - (-1)^{r+s+4}\right) \Gamma\left(\frac{1}{2}(n-r+s-1)\right) \Gamma\left(\frac{1}{2}(n+r+s+1)\right)}{\Gamma(n-1) \Gamma\left(\frac{1}{2}(-n-r+s+3)\right) \Gamma\left(\frac{1}{2}(-n+r+s+5)\right)} \right] \tag{6.4}$$

However this expansion is only asymptotic and does not converge. For fixed g the terms will start growing with n when n is greater than some value N, and we only summed the terms up to $n = N$ which gives the value of $C_{r,s}$ with very good precision for large enough g.

Using this approach we computed the curvature function for a range of values of g (typically we took $7 \leq g \leq 30$) and then fitted the result as an expansion in $1/g$. This gave us only numerical values of the expansion coefficients, but in fact we found that with very high precision the coefficients are as follows. For $J = 2$

$$\gamma^{(2)}_{J=2} = -\pi^2 g^2 + \frac{\pi g}{4} + \frac{1}{8} - \frac{1}{\pi g}\left(\frac{3\zeta_3}{16} + \frac{3}{512}\right) - \frac{1}{\pi^2 g^2}\left(\frac{9\zeta_3}{128} + \frac{21}{512}\right) \tag{6.5}$$
$$+ \frac{1}{\pi^3 g^3}\left(\frac{3\zeta_3}{2048} + \frac{15\zeta_5}{512} - \frac{3957}{131072}\right) + \dots,$$

then for $J = 3$

$$\gamma^{(2)}_{J=3} = -\frac{8\pi^2 g^2}{27} + \frac{2\pi g}{27} + \frac{1}{12} - \frac{1}{\pi g}\left(\frac{1}{216} + \frac{\zeta_3}{8}\right) - \frac{1}{\pi^2 g^2}\left(\frac{3\zeta_3}{64} + \frac{743}{13824}\right) \tag{6.6}$$
$$+ \frac{1}{\pi^3 g^3}\left(\frac{41\zeta_3}{1024} + \frac{35\zeta_5}{512} - \frac{5519}{147456}\right) + \dots,$$

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and finally for $J = 4$

$$\gamma_{J=4}^{(2)} = -\frac{\pi^2 g^2}{8} + \frac{\pi g}{32} + \frac{1}{16} - \frac{1}{\pi g}\left(\frac{3\zeta_3}{32} + \frac{15}{4096}\right) - \frac{0.01114622551913}{g^2} \tag{6.7}$$
$$+ \frac{0.004697583899}{g^3} + \dots .$$

To fix coefficients for the first four terms in the expansion we were guided by known analytic predictions which will be discussed below, and found that our numerical result matches these predictions with high precision. Then for $J = 2$ and $J = 3$ we extracted the numerical values obtained from the fit for the coefficients of $1/g^2$ and $1/g^3$, and plugging them into the online calculator EZFace [63] we obtained a prediction for their exact values as combinations of ζ_3 and ζ_5. Fitting again our numerical results with these exact values fixed, we found that the precision of the fit at the previous orders in $1/g$ increased. This is a highly nontrivial test for the proposed exact values of $1/g^2$ and $1/g^3$ terms. For $J = 2$ we confirmed the coefficients of these terms with absolute precision 10^{-17} and 10^{-15} at $1/g^2$ and $1/g^3$ respectively (at previous orders of the expansion the precision is even higher). For $J = 3$ the precision was correspondingly 10^{-15} and 10^{-13}.

For $J = 4$ we were not able to get a stable fit for the $1/g^2$ and $1/g^3$ coefficients from EZFace, so above we gave their numerical values (with uncertainty in the last digit). However below we will see that based on $J = 2$ and $J = 3$ results one can make a prediction for these coefficients, which we again confirmed by checking that precision of the fit at the previous orders in $1/g$ increases. The precision of the final fit at orders $1/g^2$ and $1/g^3$ is 10^{-16} and 10^{-14} respectively.

6.2 Generalization to any J

Here we will find an analytic expression for the strong coupling expansion of the curvature function which generalizes the formulas (6.5) and (6.6) to any J. To this end it will be beneficial to consider the structure of classical expansions of the scaling dimension. A good entry point is considering the inverse relation $S(\Delta)$, frequently encountered in the context of BFKL. It satisfies a few basic properties, namely the curve $S(\Delta)$ goes through the points $(\pm J, 0)$ at any coupling, because at $S = 0$ the operator is BPS. At the same time for non-BPS states one should have $\Delta(\lambda) \propto \lambda^{1/4} \to \infty$ [76] which indicates that if Δ is fixed, S should go to zero, thus combining this with the knowledge of fixed points $(\pm J, 0)$ we conclude that at infinite coupling $S(\Delta)$ is simply the line $S = 0$. As the coupling becomes finite $S(\Delta)$ starts bending from the $S = 0$ line and starts looking like a parabola going through the points $\pm J$, see figure 3. Based on this qualitative picture and the scaling $\Delta(\lambda) \propto \lambda^{1/4}$ at $\lambda \to \infty$ and fixed J and S, one can write down the following ansatz,

$$S(\Delta) = \left(\Delta^2 - J^2\right)\left(\alpha_1 \frac{1}{\lambda^{1/2}} + \alpha_2 \frac{1}{\lambda} + (\alpha_3 + \beta_3\Delta^2)\frac{1}{\lambda^{3/2}} + (\alpha_4 + \beta_4\Delta^2)\frac{1}{\lambda^2} \tag{6.8}$$
$$+ (\alpha_5 + \beta_5\Delta^2 + \gamma_5\Delta^4)\frac{1}{\lambda^{5/2}} + (\alpha_6 + \beta_6\Delta^2 + \gamma_6\Delta^4)\frac{1}{\lambda^3} + \dots\right).$$

We omit odd powers of the scaling dimension from the ansatz, as only the square of Δ enters the $\mathbf{P}\mu$-system. We can now invert the relation and express Δ in terms of S at

strong coupling, which gives

$$\Delta^2 = J^2 + S\left(A_1\sqrt{\lambda} + A_2 + \ldots\right) + S^2\left(B_1 + \frac{B_2}{\sqrt{\lambda}} + \ldots\right) + S^3\left(\frac{C_1}{\lambda^{1/2}} + \frac{C_2}{\lambda} + \ldots\right) + \mathcal{O}(S^4)\,,$$

(6.9)

where the coefficients A_i, B_i, C_i are some functions of J. There exists a one-to-one mapping between the coefficients α_i, β_i, etc. and A_i, B_i etc, which is rather complicated but easy to find. We note that this structure of Δ^2 coincides with Basso's conjecture in [23] for mode number $n = 1$.[16] The pattern in (6.9) continues to higher orders in S with further coefficients D_i, E_i, etc. and powers of λ suppressed incrementally. This structure is a nontrivial constraint on Δ itself as one easily finds from (6.9) that

$$\Delta = J + \frac{S}{2J}\left(A_1\sqrt{\lambda} + A_2 + \frac{A_3}{\sqrt{\lambda}} + \ldots\right)$$

(6.10)

$$+ S^2\left(-\frac{A_1^2}{8J^3}\,\lambda - \frac{A_1 A_2}{4J^3}\,\sqrt{\lambda} + \left[\frac{B_1}{2J} - \frac{A_2^2 + 2A_1 A_3}{8J^3}\right] + \left[\frac{B_2}{2J} - \frac{A_2 A_3 + A_1 A_4}{4J^3}\right]\frac{1}{\sqrt{\lambda}} + \ldots\right).$$

By definition the coefficients of S and S^2 are the slope and curvature functions respectively, so now we have their expansions at strong coupling in terms of A_i, B_i, C_i, etc. Since the S coefficient only contains the constants A_i, we can find all of their values by simply expanding the slope function (3.35) at strong coupling. We get

$$A_1 = 2\,, \quad A_2 = -1\,, \quad A_3 = J^2 - \frac{1}{4}\,, \quad A_4 = J^2 - \frac{1}{4}\ldots\,.$$

(6.11)

Note that in this series the power of J increases by two at every other member, which is a direct consequence of omitting odd powers of Δ from (6.8). We also expect the same pattern to hold for the coefficients B_i, C_i, etc.

The curvature function written in terms of A_i, B_i, etc. is given by

$$\gamma_J^{(2)}(g) = -\frac{2\pi^2 g^2 A_1^2}{J^3} - \frac{\pi g A_1 A_2}{J^3} - \frac{A_2^2 + 2A_1 A_3 - 4B_1 J^2}{8J^3} - \frac{A_2 A_3 + A_1 A_4 - 2B_2 J^2}{16\pi g J^3}$$

(6.12)

$$-\frac{A_3^2 + 2A_2 A_4 + 2A_1 A_5 - 4B_3 J^2}{128\pi^2 g^2 J^3} - \frac{A_3 A_4 + A_2 A_5 + A_1 A_6 - 2B_4 J^2}{256\pi^3 g^3 J^3} + \mathcal{O}\left(\frac{1}{g^4}\right).$$

The remaining unknowns here (up to order $1/g^4$) are B_1, B_2, which we expect to be constant due to the power pattern noticed above and B_3, B_4, which we expect to have the form $aJ^2 + b$ with a and b constant. These unknowns are immediately fixed by comparing the general curvature expansion (6.12) to the two explicit cases that we know for $J = 2$ and $J = 3$. We find

$$B_1 = 3/2\,, \quad B_2 = -3\,\zeta_3 + \frac{3}{8},$$

(6.13)

and

$$B_3 = -\frac{J^2}{2} - \frac{9\,\zeta_3}{2} + \frac{5}{16}\,, \quad B_4 = \frac{3}{16}J^2(16\,\zeta_3 + 20\,\zeta_5 - 9) - \frac{15\,\zeta_5}{2} - \frac{93\,\zeta_3}{8} - \frac{3}{16}.$$

(6.14)

---

[16]The generalization of (6.9) for $n > 1$ is not fully clear, as noted in [64], and this case will be discussed in appendix E.2.

Having fixed all the unknowns we can write the strong coupling expansion of the curvature function for arbitrary values of J as

$$\gamma_J^{(2)}(g) = -\frac{8\pi^2 g^2}{J^3} + \frac{2\pi g}{J^3} + \frac{1}{4J} + \frac{1 - J^2(24\,\zeta_3 + 1)}{64\pi g J^3} - \frac{8J^4 + J^2(72\,\zeta_3 + 11) - 4}{512 g^2 \left(\pi^2 J^3\right)}$$
$$+ \frac{3\left(8J^4(16\,\zeta_3 + 20\,\zeta_5 - 7) - 16J^2(31\,\zeta_3 + 20\,\zeta_5 + 7) + 25\right)}{16384\pi^3 g^3 J^3} + \mathcal{O}\left(\frac{1}{g^4}\right). \quad (6.15)$$

Expanding $\gamma_{J=4}^{(2)}$ defined in (C.27) at strong coupling numerically we were able to confirm the above result with high precision.

6.3 Anomalous dimension of short operators

In this section we will use the knowledge of slope functions $\gamma_J^{(n)}$ at strong coupling to find the strong coupling expansions of scaling dimensions of operators with finite S and J, in particular we will find the three loop coefficient of the Konishi operator by utilizing the techniques of [23, 64]. What follows is a quick recap of the main ideas in these papers.

We are interested in the coefficients of the strong coupling expansion of Δ, namely

$$\Delta = \Delta^{(0)}\lambda^{\frac{1}{4}} + \Delta^{(1)}\lambda^{-\frac{1}{4}} + \Delta^{(2)}\lambda^{-\frac{3}{4}} + \Delta^{(3)}\lambda^{-\frac{5}{4}} + \dots \quad (6.16)$$

First, we use Basso's conjecture (6.9) and by fixing S and J we re-expand the square root of Δ^2 at strong coupling to find

$$\Delta = \sqrt{A_1 S}\,\sqrt[4]{\lambda} + \frac{\sqrt{A_1}\left(J^2 + A_2 S + B_1 S^2\right)}{2A_1\sqrt{S}}\,\frac{1}{\sqrt[4]{\lambda}} + \mathcal{O}\left(\frac{1}{\lambda^{\frac{3}{4}}}\right). \quad (6.17)$$

Thus we reformulate the problem entirely in terms of the coefficients A_i, B_i, C_i, etc. For example, the next coefficient in the series, namely the two-loop term is given by

$$\Delta^{(2)} = -\frac{\left(2A_2 + 4B_1 + J^2\right)^2 - 16A_1(A_3 + 2B_2 + 4C_1)}{16\sqrt{2}A_2^{3/2}}. \quad (6.18)$$

Further coefficients become more and more complicated, however a very clear pattern can be noticed after looking at these expressions: we see that the term $\Delta^{(n)}$ only contains coefficients with indices up to $n + 1$, e.g. the tree level term $\Delta^{(0)}$ only depends on A_1, the one-loop term depends on A_1, A_2, B_1, etc. Thus we can associate the index of these coefficients with the loop level. Conversely, from the last section we learned that the letter of A_i, B_i, etc. can be associated with the order in S, i.e. the slope function fixed all A_i coefficients and the curvature function in principle fixes all B_i coefficients.

6.3.1 Matching with classical and semiclassical results

Looking at (6.17) we see that knowing A_i and B_i only takes us to one loop, in order to proceed we need to know some coefficients in the C_i and D_i series. This is where the next ingredient in this construction comes in, which is the knowledge of the classical energy and its semiclassical correction in the Frolov-Tseytlin limit, i.e. when $\mathcal{S} \equiv S/\sqrt{\lambda}$ and

$\mathcal{J} \equiv J/\sqrt{\lambda}$ remain fixed, while S, J, $\lambda \to \infty$. Additionally we will also be taking the limit $\mathcal{S} \to 0$ in all of the expressions that follow. In particular, the square of the classical energy has a very nice form in these limits and is given by [14, 64]

$$\mathcal{D}^2_{\text{classical}} = \mathcal{J}^2 + 2\,\mathcal{S}\,\sqrt{\mathcal{J}^2 + 1} + \mathcal{S}^2\,\frac{2\mathcal{J}^2 + 3}{2\mathcal{J}^2 + 2} - \mathcal{S}^3\,\frac{\mathcal{J}^2 + 3}{8\,(\mathcal{J}^2 + 1)^{5/2}} + \mathcal{O}\left(\mathcal{S}^4\right)\;,\quad(6.19)$$

where $\mathcal{D}_{\text{classical}} \equiv \Delta_{\text{classical}}/\sqrt{\lambda}$. The 1-loop correction to the classical energy is given by

$$\Delta_{sc} \simeq \frac{-\mathcal{S}}{2\,(\mathcal{J}^3 + \mathcal{J})} + \mathcal{S}^2\left[\frac{3\mathcal{J}^4 + 11\mathcal{J}^2 + 17}{16\mathcal{J}^3\,(\mathcal{J}^2 + 1)^{5/2}} - \sum_{\substack{m>0 \\ m \neq n}} \frac{n^3 m^2\,\left(2m^2 + n^2\mathcal{J}^2 - n^2\right)}{\mathcal{J}^3\,(m^2 - n^2)^2\,(m^2 + n^2\mathcal{J}^2)^{3/2}}\right]\quad(6.20)$$

If the parameters $\mathcal{S}$ and $\mathcal{J}$ are fixed to some values then the sum can be evaluated explicitly in terms of zeta functions. We now add up the classical and the 1-loop contributions,[17] take S and J fixed at strong coupling and compare the result to (6.9). By requiring consistency we are able to extract the following coefficients,

$$
\begin{array}{llll}
A_1 = & 2, & A_2 = & -1 \\
B_1 = & 3/2, & B_2 = & -3\zeta_3 + \frac{3}{8} \\
C_1 = & -3/8, & C_2 = & \frac{1}{16}\left(60\,\zeta_3 + 60\,\zeta_5 - 17\right) \\
D_1 = & 31/64, & D_2 = & \frac{1}{512}(-5520\,\zeta_3 - 5120\,\zeta_5 - 3640\,\zeta_7 + 901)
\end{array}
$$

As discussed in the previous section, we can in principle extract all coefficients with indices 1 and 2. In order to find e.g. B_3 we would need to extend the quantization of the classical solution to the next order. Note that the coefficients A_1, A_2 and B_1, B_2 have the same exact values that we extracted from the slope and curvature functions.

6.3.2 Result for the anomalous dimensions at strong coupling

The key observation in [64] was that once written in terms of the coefficients A_i, B_i, C_i, the two-loop term $\Delta^{(2)}$ only depends on $A_{1,2,3}$, $B_{1,2}$, C_1 as can be seen in (6.18). As discussed in the last section, the one-loop result fixes all of these constants except A_3, which in principle is a contribution from a true two-loop calculation. However we already fixed it from the slope function and thus we are able to find

$$\Delta^{(2)} = \frac{-21\,S^4 + (24 - 96\,\zeta_3)S^3 + 4\left(5J^2 - 3\right)S^2 + 8J^2 S - 4J^4}{64\sqrt{2}\,S^{3/2}}.\quad(6.21)$$

Now that we know the strong coupling expansion of the curvature function and thus all the coefficients B_i, we can do the same trick and find the three loop strong coupling scaling dimension coefficient $\Delta^{(3)}$, which now depends on $A_{1;2;3;4}$, $B_{1,2,3}$, $C_{1,2}$, D_1. We find it to be

$$\Delta^{(3)} = \frac{187\,S^6 + 2\left(624\,\zeta_3 + 480\,\zeta_5 - 193\right)S^5 + \left(-146\,J^2 - 4\left(336\,\zeta_3 - 41\right)\right)S^4}{512\sqrt{2}\,S^{5/2}} +$$

$$+ \frac{\left(32\left(6\,\zeta_3 + 7\right)J^2 - 88\right)S^3 + \left(-28\,J^4 + 40\,J^2\right)S^2 - 24\,J^4 S + 8\,J^6}{512\sqrt{2}\,S^{5/2}},\quad(6.22)$$

---

[17]Note that they mix various orders of the coupling.

(S, J)	$\lambda^{-5/4}$ prediction		$\lambda^{-5/4}$ fit	error	fit order
$(2, 2)$	$\frac{15\,\zeta_5}{2} + 6\,\zeta_3 - \frac{1}{2}$	$= 14.48929958$	14.12099034	2.61%	6
$(2, 3)$	$\frac{15\,\zeta_5}{2} + \frac{63\,\zeta_3}{8} - \frac{1131}{512}$	$= 15.03417190$	14.88260078	1.02%	5
$(2, 4)$	$\frac{21\,\zeta_3}{2} + \frac{15\,\zeta_5}{2} - \frac{25}{8}$	$= 17.27355565$	16.46106336	4.94%	7

Table 1. Comparisons of strong coupling expansion coefficients for $\lambda^{-5/4}$ obtained from fits to TBA data versus our predictions for various operators. The fit order is the order of polynomials used for the rational fit function (see [64] for details).

for $S = 2$ it simplifies to

$$\Delta_{S=2}^{(3)} = \frac{1}{512}\left(J^6 - 20J^4 + 48J^2(4\zeta_3 - 1) + 192(12\,\zeta_3 + 20\,\zeta_5 + 1)\right) \qquad (6.23)$$

and finally for the Konishi operator, which has $S = 2$ and $J = 2$ we get[18]

$$\Delta_{S=2,J=2}^{(3)} = \frac{15\,\zeta_5}{2} + 6\,\zeta_3 - \frac{1}{2}. \qquad (6.24)$$

In order to compare our predictions with data available from TBA calculations [13], we employed Padé type fits as explained in [64]. The fit results are shown in table 1, we see that our predictions are within 5% error bounds, which is a rather good agreement. However we must be honest that for the $J = 3$ and especially $J = 4$ states we did not have as many data points as for the $J = 2$ state and the fit is somewhat shaky.

6.4 BFKL pomeron intercept

The gauge theory operators that we consider in this paper are of high importance in high energy scattering amplitude calculations, especially in the Regge limit of high energy and fixed momentum transfer [65, 66]. In this limit one can approximate the scattering amplitude as an exchange of effective particles, the so-called reggeized gluons, compound states of which are frequently called *pomerons*. When momentum transfer is large, perturbative computations are possible and the so-called 'hard pomeron' appears, the BFKL pomeron [67–69]. The BFKL pomeron leads to a power law behaviour of scattering amplitudes $s^{j(\Delta)}$, where $j(\Delta)$ is called the Reggeon spin and s is the energy transfer of the process. The remarkable connection between the pomeron and the operators we consider can be symbolically stated as

$$\text{pomeron} = \text{Tr}\left(Z\,\mathcal{D}_+^S\,Z\right) + \ldots \qquad (6.25)$$

where we are now considering twist two operators ($J = 2$) and the spin S can take on complex values by analytic continuation. The Reggeon spin $j(\Delta)$ (also refered to as a Regge trajectory) is a function of the anomalous dimension of the operator and is related to spin S as $j(\Delta) = S(\Delta) + 2$. Some of these trajectories are shown in figure 3. A very important quantity in this story is the BFKL intercept $j(0)$, which we consider next.

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[18]The ζ_3 and ζ_5 terms are coming from semi-classics and were already known before [27] and match our result.
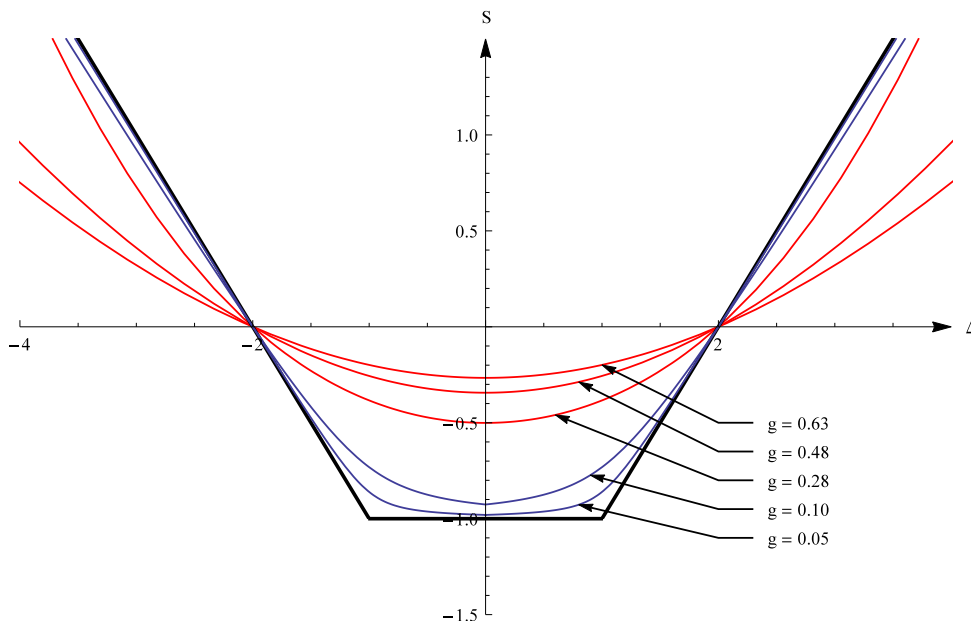
Figure 3. The BFKL trajectories $S(\Delta)$ at various values of the coupling. Blue lines are obtained using the known two loop weak coupling expansion [70, 71] and red lines are obtained using the strong coupling expansion [72–74].

One can also use the same techniques as in the previous section to calculate the strong coupling expansion of the BFKL intercept. As stated before, the intercept of a BFKL trajectory $j(\Delta)$ is simply $j(0)$ and we already wrote down an ansatz for $S(\Delta)$ in (6.8). The coefficients α_i, β_i, etc. are in one-to-one correspondence with the coefficients A_i, B_i etc. from (6.10), values of which we found in the previous sections. Plugging in their values we find

$$\alpha_1 = 1/2, \qquad \alpha_2 = 1/4, \qquad \alpha_3 = -1/16\,, \qquad \alpha_4 = -\frac{3\zeta_3}{2} - \frac{1}{2}, \tag{6.26}$$

$$\alpha_5 = -\frac{9\zeta_3}{2} - \frac{361}{256}, \qquad\qquad \alpha_6 = -\frac{39\zeta_3}{4} - \frac{447}{128} \tag{6.27}$$

$$\beta_3 = -3/16, \quad \beta_4 = \frac{3\zeta_3}{8} - \frac{21}{64}, \quad \beta_5 = \frac{9\zeta_3}{8} - \frac{51}{128}, \quad \beta_6 = \frac{45\zeta_3}{8} + \frac{15\zeta_5}{16} + \frac{13}{512} \tag{6.28}$$

$$\gamma_5 = \frac{21}{128}, \qquad \gamma_6 = -\frac{51\zeta_3}{64} - \frac{15\zeta_5}{64} + \frac{137}{256} \tag{6.29}$$

Furthermore, setting $\Delta = 0$ we find the intercept to be

$$j(0) = 2 + S(0) = 2 - \frac{2}{\lambda^{1/2}} - \frac{1}{\lambda} + \frac{1}{4\,\lambda^{3/2}} + (6\zeta_3 + 2)\,\frac{1}{\lambda^2}$$
$$+ \left(18\,\zeta_3 + \frac{361}{64}\right)\frac{1}{\lambda^{5/2}} + \left(39\,\zeta_3 + \frac{447}{32}\right)\frac{1}{\lambda^3} + \mathcal{O}\left(\frac{1}{\lambda^{7/2}}\right). \tag{6.30}$$

The first four terms successfully reproduce known results [72–74] and the last two terms of the series are a new prediction (their derivation relies on the knowledge of the constants $B_{3,4;J=2}$ found in the last section). On figure 4 we show plots of the intercept at weak and at strong coupling.
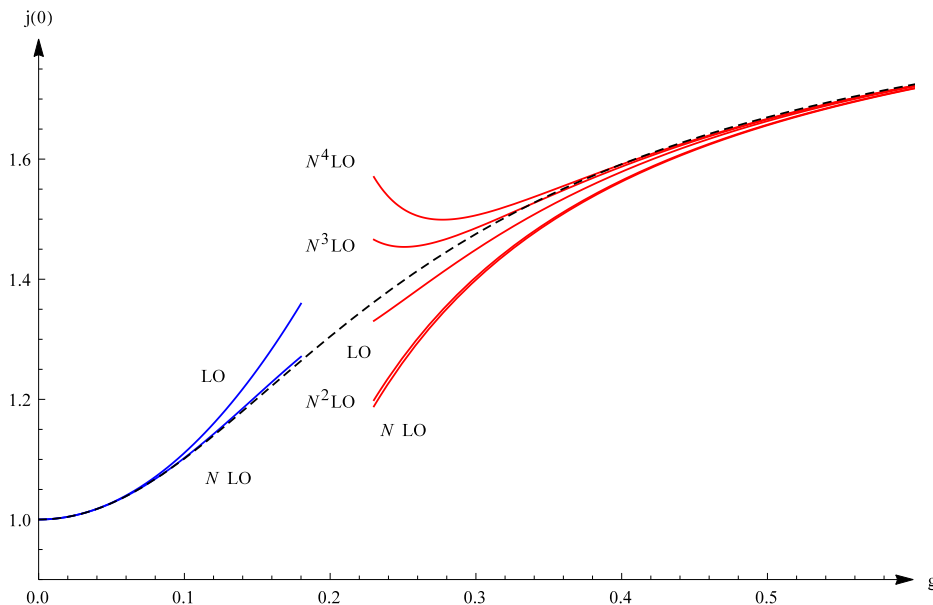
Figure 4. The BFKL intercept $j(0) = 2 + S(0)$ dependence on the coupling constant g at two orders at weak coupling (blue lines), four orders at strong coupling (red lines) and a Padé type interpolating function in between (dashed line).

7 Conclusions

In this paper we applied the recently proposed $\mathbf{P}\mu$-system of Riemann-Hilbert type equations to study anomalous dimensions in the $sl(2)$ sector of planar $\mathcal{N} = 4$ SYM theory. Our main result are the expressions (4.46), (4.63) and (C.27) for the curvature function $\gamma_J^{(2)}(g)$, i.e. the coefficient of the S^2 term in the anomalous dimension at small spin S. These results correspond to operators with twist $J = 2, 3$ and 4. Curiosly, we found that they involve essential parts of the BES dressing phase in the integral representation.

We derived these results by solving the $\mathbf{P}\mu$-system to order S^2 and they are exact at any coupling. While expansion in small S (but at any coupling) seems hardly possible to perform in the TBA approach, here it resembles a perturbative expansion — the $\mathbf{P}\mu$-system is solved order by order in S and the coupling is kept arbitrary.

For $J = 2$ and $J = 3$ our calculation perfectly matches known results to four loops at weak coupling. This includes in particular the leading finite-size correction at $J = 2$. At strong coupling we obtained the expansion of our results numerically, and also found full agreement with known predictions. This provides yet another check that our result incorporates all wrapping corrections. Going to higher orders in this expansion we were able to use the EZFace calculator [63] to fit the coefficients as linear combinations of ζ_3 and ζ_5 (and confirmed the outcome with high precision). By combining these coefficients with the other known results, we obtained the 3-loop coefficient in the Konishi anomalous dimension at strong coupling. This serves as a highly nontrivial prediction for a direct string theory calculation, which hopefully may be done along the lines of [16, 17]. Our results also predict the value of two new coefficients for the pomeron intercept at strong coupling.

For the future analysis it would be interesting to build an integral equation which would generate iteratively small S corrections and to be able to approach finite values of S at any coupling. Furthermore, extension of this approach to the boundary problems, twisted boundary conditions and even q-deformations [77–86] would give a rich set of new analytical results. Finally, applications of our methods to ABJM model [87–89] and comparison with the localization results [90–92] would give the unknown interpolation function $h(\lambda)$, the only ingredient missing in the integrability framework [93–95]. As the **P**μ-system [96] for ABJM model has various peculiar features compared to $\mathcal{N} = 4$ SYM it would be especially interesting to study this case.

Acknowledgments

A Summary of notation and definitions

In this appendix we summarize some notation used throughout the paper.

Laurent expansions in x. We often represent functions of the spectral parameter u as a series in x

$$f(u) = \sum_{n=-\infty}^{\infty} f_n x^n \tag{A.1}$$

with

$$u = g(x + 1/x). \tag{A.2}$$

We denote by $[f]_+$ and $[f]_-$ part of the series with positive and negative powers of x:

$$[f]_+ = \sum_{n=1}^{\infty} f_n x^n, \tag{A.3}$$

$$[f]_- = \sum_{n=1}^{\infty} f_{-n} x^{-n}. \tag{A.4}$$

As a function of u, $x(u)$ has a cut from $-2g$ to $2g$. For any function $f(u)$ with such a cut we denote another branch of $f(u)$ obtained by analytic continuation (from Im $u > 0$) around the branch point $u = 2g$ by $\tilde{f}(u)$. In particular, $\tilde{x} = 1/x$.

Functions sinh$_\pm$ and cosh$_\pm$. We define $I_k = I_k(4\pi g)$, where $I_k(u)$ is the modified Bessel function of the first kind. Then

$$\sinh_+ = [\sinh(2\pi u)]_+ = \sum_{k=1}^{\infty} I_{2k-1} x^{2k-1}, \tag{A.5}$$

$$\sinh_- = [\sinh(2\pi u)]_- = \sum_{k=1}^{\infty} I_{2k-1} x^{-2k+1}, \tag{A.6}$$

$$\cosh_+ = [\cosh(2\pi u)]_+ = \sum_{k=1}^{\infty} I_{2k} x^{2k}, \tag{A.7}$$

$$\cosh_- = [\cosh(2\pi u)]_- = \sum_{k=1}^{\infty} I_{2k} x^{-2k}. \tag{A.8}$$

In some cases we denote for brevity

$$\mathrm{sh}_-^x = \sinh_-(x), \quad \mathrm{ch}_-^x = \cosh_-(x). \tag{A.9}$$

Integral kernels. In order to solve for $\mathbf{P}_a^{(1)}$ in section 4.3 we introduce integral operators H and K with kernels

$$H(u,v) = -\frac{1}{4\pi i} \frac{\sqrt{u-2g}\sqrt{u+2g}}{\sqrt{v-2g}\sqrt{v+2g}} \frac{1}{u-v} dv, \tag{A.10}$$

$$K(u,v) = +\frac{1}{4\pi i} \frac{1}{u-v} dv, \tag{A.11}$$

which satisfy

$$\tilde{f} + f = h\,, \quad f = H \cdot h \quad \text{and} \quad \tilde{f} - f = h\,, \quad f = K \cdot h. \tag{A.12}$$

Since the purpose of H and K is to solve equations of the type A.12, H usually acts on functions h such that $\tilde{h} = h$, whereas K acts on h such that $\tilde{h} = -h$. On the corresponding classes of functions H and K can be represented by kernels which are equal up to a sign

$$H(u,v) = -\left.\frac{1}{2\pi i} \frac{1}{x_u - x_v} dx_v\right|_{\tilde{h}=h}, \tag{A.13}$$

$$K(u,v) = \left.\frac{1}{2\pi i} \frac{1}{x_u - x_v} dx_v\right|_{\tilde{h}=-h}. \tag{A.14}$$

In order to be able to deal with series in half-integer powers of x in section 4.5 we introduce modified kernels:

$$H^* \cdot f \equiv \frac{x+1}{\sqrt{x}} H \cdot \frac{\sqrt{x}}{x+1} f, \tag{A.15}$$

$$K^* \cdot f \equiv \frac{x+1}{\sqrt{x}} K \cdot \frac{\sqrt{x}}{x+1} f. \tag{A.16}$$

Finally, to write the solution to equations of the type (4.5), we introduce the operator Γ' and its more symmetric version Γ

$$\left(\Gamma' \cdot h\right)(u) \equiv \oint_{-2g}^{2g} \frac{dv}{4\pi i} \partial_u \log \frac{\Gamma[i(u-v)+1]}{\Gamma[-i(u-v)]} h(v), \tag{A.17}$$

$$\left(\Gamma \cdot h\right)(u) \equiv \oint_{-2g}^{2g} \frac{dv}{4\pi i} \partial_u \log \frac{\Gamma[i(u-v)+1]}{\Gamma[-i(u-v)+1]} h(v). \tag{A.18}$$

Periodized Chebyshev polynomials. Periodized Chebyshev polynomials appearing in $\mu_{ab}^{(1)}$ are defined as

$$p_a'(u) = \Sigma \cdot [x^a + 1/x^a] = 2\Sigma \cdot \left[T_a\left(\frac{u}{2g}\right)\right], \tag{A.19}$$

$$p_a(u) = p_a'(u) + \frac{1}{2}\left(x^a(u) + x^{-a}(u)\right), \tag{A.20}$$

where $T_a(u)$ are Chebyshev polynomials of the first kind. Here is the explicit form for the first five of them:

$$p_0' = -i(u - i/2), \tag{A.21}$$

$$p_1' = -i\frac{u(u-i)}{4g}, \tag{A.22}$$

$$p_2' = -i\frac{(u-i/2)(-6g^2+u^2-iu)}{6g^2}, \tag{A.23}$$

$$p_3' = -i\frac{u(u-i)\left(-6g^2+u(u-i)\right)}{8g^3}, \tag{A.24}$$

$$p_4' = -i\frac{\left(u-\frac{i}{2}\right)\left(30g^4-20g^2u^2+20ig^2u+3u^4-6iu^3-2u^2-iu\right)}{30g^4}. \tag{A.25}$$

B   The slope function for odd J

Here we give details on solving the $\mathbf{P}\mu$-system for odd J at leading order in the spin. First, the parity of the μ_{ab} functions is different from the even J case, which can be seen from the asymptotics (2.11). Following arguments similar to the discussion for even J in section 3.1, we obtain

$$\mu_{12} = 1, \ \mu_{13} = 0, \ \mu_{14} = 0, \ \mu_{24} = \cosh(2\pi u), \ \mu_{34} = 1. \tag{B.1}$$

Plugging these μ_{ab} into (2.9) we get a system of equations for $\mathbf{P}_a$

$$\tilde{\mathbf{P}}_1 = -\mathbf{P}_3, \tag{B.2}$$

$$\tilde{\mathbf{P}}_2 = -\mathbf{P}_4 - \mathbf{P}_1 \cosh(2\pi u), \tag{B.3}$$

$$\tilde{\mathbf{P}}_3 = -\mathbf{P}_1, \tag{B.4}$$

$$\tilde{\mathbf{P}}_4 = -\mathbf{P}_2 + \mathbf{P}_3 \cosh(2\pi u). \tag{B.5}$$

This system can be solved in a similar way to the even J case. The only important difference is that due to asymptotics (2.10) the $\mathbf{P}_a$ acquire an extra branch point at $u = \infty$.

Let us first rewrite the equations for $\mathbf{P}_1, \mathbf{P}_3$ as

$$\tilde{\mathbf{P}}_1 + \tilde{\mathbf{P}}_3 = -(\mathbf{P}_1 + \mathbf{P}_3) \tag{B.6}$$

$$\tilde{\mathbf{P}}_1 - \tilde{\mathbf{P}}_3 = \mathbf{P}_1 - \mathbf{P}_3. \tag{B.7}$$

This, together with the asymptotics (2.10) implies $\mathbf{P}_1 = \epsilon x^{-J/2}$, $\mathbf{P}_3 = -\epsilon x^{J/2}$ where ϵ is a constant. Let us note that these $\mathbf{P}_1, \mathbf{P}_3$ contain half-integer powers of x, and the analytic continuation around the branch points at $\pm 2g$ replaces $\sqrt{x} \to 1/\sqrt{x}$. Now, taking the sum and difference of the equations for $\mathbf{P}_2$, $\mathbf{P}_4$ we get

$$\tilde{\mathbf{P}}_2 + \tilde{\mathbf{P}}_4 + \mathbf{P}_2 + \mathbf{P}_4 = -a_1 \left(x^{J/2} + x^{-J/2} \right) \cosh 2\pi u \tag{B.8}$$

$$\tilde{\mathbf{P}}_2 - \tilde{\mathbf{P}}_4 - (\mathbf{P}_2 - \mathbf{P}_4) = a_1 \left(x^{J/2} - x^{-J/2} \right) \cosh 2\pi u \tag{B.9}$$

We can split the expansion

$$\cosh 2\pi u = \sum_{k=-\infty}^{\infty} I_{2k} x^{2k} \tag{B.10}$$

into the positive and negative parts according to

$$\cosh 2\pi u = \cosh_- + \cosh_+ + I_0 \tag{B.11}$$

where

$$\cosh_+ = \sum_{k=1}^{\infty} I_{2k} x^{2k}, \qquad \cosh_- = \sum_{k=1}^{\infty} I_{2k} x^{-2k}. \tag{B.12}$$

Then we can write

$$\mathbf{P}_2 + \mathbf{P}_4 = -a_1(x^{J/2} + x^{-J/2}) \cosh_- -a_1 I_0 x^{-J/2} + Q, \tag{B.13}$$

$$\mathbf{P}_2 - \mathbf{P}_4 = -a_1(x^{J/2} - x^{-J/2}) \cosh_- +a_1 I_0 x^{-J/2} + P, \tag{B.14}$$

where Q and P are some polynomials in $\sqrt{x}, 1/\sqrt{x}$ satisfying

$$\tilde{Q} = -Q, \ \tilde{P} = P. \tag{B.15}$$

We get

$$\mathbf{P}_2 = -a_1 x^{J/2} \cosh_- + \frac{Q+P}{2}, \tag{B.16}$$

$$\mathbf{P}_4 = a_1 x^{-J/2} \cosh_- -a_1 I_0 x^{-J/2} + \frac{Q-P}{2}. \tag{B.17}$$

Now imposing the correct asymptotics of $\mathbf{P}_2$ we find

$$\frac{P+Q}{2} = a_1 x^{J/2} \sum_{k=1}^{\frac{J-1}{2}} I_{2k} x^{-2k} \tag{B.18}$$

Due to ([B.15](#)) this relation fixes Q and P completely, and we obtain the solution given in section [2.1](#),

$$\mu_{12} = 1, \quad \mu_{13} = 0, \quad \mu_{14} = 0, \quad \mu_{24} = \cosh(2\pi u), \quad \mu_{34} = 1, \tag{B.19}$$

$$\mathbf{P}_1 = a_1 x^{-J/2}, \tag{B.20}$$

$$\mathbf{P}_2 = -a_1 x^{J/2} \sum_{k=-\infty}^{-\frac{J+1}{2}} I_{2k} x^{2k}, \tag{B.21}$$

$$\mathbf{P}_3 = -a_1 x^{J/2}, \tag{B.22}$$

$$\mathbf{P}_4 = a_1 x^{-J/2} \cosh_- - a_1 x^{-J/2} \sum_{k=1}^{\frac{J-1}{2}} I_{2k} x^{2k} - a_1 I_0 x^{-J/2}. \tag{B.23}$$

Notice that the branch point at infinity is absent from the product of any two $\mathbf{P}$'s, as it should be [19, 20]. One can check that this solution gives again the correct result ([3.35](#)) for the slope function.

C NLO solution of $\mathbf{P}\mu$ system: details

In this appendix we will provide more details on the solution of the $\mathbf{P}\mu$-system and calculation of curvature function for $J = 2, 3, 4$ which was presented in the main text in sections [4.1](#), [4.5](#), [C.4](#).

C.1 NLO corrections to μ_{ab} for $J = 2$

Here we present some details of calculation of NLO corrections to μ_{ab} for $J = 2$ omitted in the main text. As described in section [4.2](#), $\mu_{ab}^{(1)}$ are found as solutions of ([4.5](#)) with appropriate asymptotics. The general solution of this equation consists of a general solution of the corresponding homogeneous equation (which can be reduced to one-parametric form ([4.26](#))) and a particular solution of the inhomogeneous one. The latter can be taken to be

$$\mu_{ab}^{\text{disc}} = \Sigma \cdot \left(\mathbf{P}_a^{(1)} \tilde{\mathbf{P}}_b^{(1)} - \mathbf{P}_b^{(1)} \tilde{\mathbf{P}}_a^{(1)} \right). \tag{C.1}$$

One can get rid of the operation Σ, expressing μ_{ab}^{disc} in terms of Γ' and p_a'. This procedure is based on two facts: the definition ([4.11](#)) of p_a' and the statement that on functions decaying at infinity Σ coincides with Γ' defined by ([4.9](#)). After a straightforward but long calculation we find

$$\mu_{31}^{\text{disc}} = \epsilon^2 \Sigma \left(\frac{1}{x^2} - x^2 \right) = -\epsilon^2 \ \left(\Gamma \cdot x^2 + p_2 \right), \tag{C.2}$$

$$\mu_{41}^{\text{disc}} = \epsilon^2 \left[-2I_1 p_1 - 4I_1 \Gamma \cdot x + \sinh(2\pi u) \left(\Gamma \cdot x^2 + p_0 \right) + \Gamma \cdot \sinh_- \left(x - \frac{1}{x} \right)^2 \right], \tag{C.3}$$

$$\mu_{43}^{\text{disc}} = -2\epsilon^2 \left[-2I_1 p_1 - 4I_1 \Gamma \cdot x + \sinh(2\pi u)(p_2 - p_0) + \Gamma \cdot \sinh_- \left(x - \frac{1}{x} \right)^2 \right], \tag{C.4}$$

$$\mu_{21}^{\text{disc}} = \epsilon^2 \left[2I_1 \Gamma \cdot x - \sinh(2\pi u) \, \Gamma \cdot x^2 - \Gamma \cdot \sinh_- \left(x^2 + \frac{1}{x^2} \right) \right], \tag{C.5}$$

$$\mu_{24}^{\text{disc}} = \epsilon^2 \left[2I_1 \Gamma \cdot \sinh_- \left(x + \frac{1}{x} \right) + I_1^2 p_0 + \right. \tag{C.6}$$

$$\left. + \sinh(2\pi u)\Gamma \cdot \sinh_- \left(x^2 - \frac{1}{x^2} \right) - \Gamma \cdot \sinh_-^2 \left(x^2 - \frac{1}{x^2} \right) \right]. \tag{C.7}$$

Here we write Γ and p_a instead of Γ' and p'_a taking into account the discussion between equations (4.31)–(4.36).

C.2 NLO solution of the Pμ-system at $J = 3$

In this appendix we present some intermediate formulas for the calculation of curvature function for $J = 3$ in section 4.5 omitted in the main text.

- The particular solution of the inhomogeneous equation (4.5) which we construct as $\mu_{31}^{\text{disc}} = \Sigma \cdot \left(\mathbf{P}_a^{(1)} \tilde{\mathbf{P}}_b^{(1)} - \mathbf{P}_b^{(1)} \tilde{\mathbf{P}}_a^{(1)} \right)$ can be written using the operation Γ and p_a defined by (4.36) and (4.34)[19]

$$\mu_{31}^{\text{disc}} = \Sigma \cdot (\mathbf{P}_3 \tilde{\mathbf{P}}_1 - \mathbf{P}_1 \tilde{\mathbf{P}}_3) = -2\epsilon^2 \left[\Gamma x^3 + p_3 \right], \tag{C.8}$$

$$\mu_{41}^{\text{disc}} = -\epsilon^2 \left[2p_2 I_2 + 2I_2 \Gamma x^2 + 2\Gamma \cdot \cosh_- + (I_0 - \cosh(2\pi u))p_0 \right], \tag{C.9}$$

$$\mu_{34}^{\text{disc}} = \epsilon^2 \left[2I_2 \Gamma x + I_0 \Gamma x^3 - \Gamma \cdot (x^3 + x^{-3}) \cosh_- + \cosh(2\pi u)(2p_3 + \Gamma x^3) \right], \tag{C.10}$$

$$\mu_{21}^{\text{disc}} = \epsilon^2 \left[2I_2 \Gamma x + (I_0 - \cosh(2\pi u))\Gamma x^3 - \Gamma \left((x^3 + x^{-3}) \cosh(2\pi u) \right) \right], \tag{C.11}$$

$$\mu_{24}^{\text{disc}} = -2\epsilon^2 \left[-\frac{1}{2}\Gamma \cdot \cosh_-^2 \left(x^3 - x^{-3} \right) + \left(\frac{\cosh(2\pi u)}{2} - I_0 \right) \Gamma \cdot \frac{\cosh_-}{x^3} \right. \tag{C.12}$$

$$-I_2 \Gamma \cdot \left(x + \frac{1}{x} \right) \cosh_- - \frac{1}{2} \cosh(2\pi u)\Gamma \cdot x^3 \cosh_- + \tag{C.13}$$

$$\left. + \frac{I_0}{2} \left(I_0 - \cosh(2\pi u) \right) \Gamma \cdot x^3 + \frac{I_1 I_2}{2\pi g}\Gamma x - I_2^2 p_1 \right]. \tag{C.14}$$

- The zero mode of the system (4.52)–(4.55), which we added to the solution in eqs. (4.58)–(4.61) to ensure correct asymptotics, is

$$\mathbf{P}_1^{\text{zm}} = L_1 x^{-1/2} + L_3 x^{1/2}, \tag{C.15}$$

$$\mathbf{P}_2^{\text{zm}} = -L_1 x^{1/2}\text{ch}_- + L_2 x^{-1/2} - L_3 x^{-1/2} \left(\text{ch}_- + \frac{1}{2}I_0 \right) + L_4 \left(x^{1/2} - x^{-1/2} \right),$$

$$\mathbf{P}_3^{\text{zm}} = -L_1 x^{1/2} - L_3 x^{1/2},$$

$$\mathbf{P}_4^{\text{zm}} = -L_1 \left(I_0 x^{-1/2} + x^{-1/2} \cosh_- \right) - L_2 x^{1/2} + L_4(x^{1/2} - x^{-1/2})$$

$$-L_3 x^{1/2} \left(\text{ch}_- + \frac{1}{2}I_0 \right).$$

---

[19]Alternatively one can use p'_a and Γ' instead of p_a and Γ- see the discussion between the equations (4.31)–(4.36).

C.3 NLO solution of the Pμ-system at $J = 4$

Solution of the Pμ system at NLO for $J = 4$ is completely analogous to the case of $J = 2$. The starting point is the LO solution (3.19)–(3.22). As described in section C.1, from LO $\mathbf{P}_a$ we can find μ_{ab} at NLO. Its discontinuous part is

$$\mu_{31}^{\mathrm{disc}} = -\epsilon^2 \ \left(\Gamma \cdot x^4 + p_4\right), \tag{C.16}$$

$$\mu_{41}^{\mathrm{disc}} = \frac{1}{2}\epsilon^2 \left(\sinh(2\pi u)\left(p_0 + \Gamma \cdot x^4\right) + 2\left(I_1 p_1 + I_3 p_3\right) + \right.$$
$$\left. + \Gamma \cdot \sinh_- \left(x^2 - \frac{1}{x^2}\right)^2 - 2\left(I_1 + I_3\right)\left(\Gamma \cdot x^3 + \Gamma \cdot x\right)\right), \tag{C.17}$$

$$\mu_{43}^{\mathrm{disc}} = \epsilon^2 \left(\left(p_4 - p_0\right)\sinh(2\pi u) + 2\left(I_1 p_1 + I_3 p_3\right) - \right.$$
$$\left. - \Gamma \cdot \sinh_- \left(x^2 - \frac{1}{x^2}\right)^2 + 2\left(I_1 + I_3\right)\left(\Gamma \cdot x^3 + \Gamma \cdot x\right)\right), \tag{C.18}$$

$$\mu_{21}^{\mathrm{disc}} = \epsilon^2 \left(-\frac{1}{2}\sinh(2\pi u)\Gamma \cdot x^4 + I_1 p_3 + I_3 p_1 - \right.$$
$$\left. - \frac{1}{2}\Gamma \cdot \sinh_- \left(x^4 + \frac{1}{x^4}\right) + I_1\Gamma \cdot x^3 + I_3\Gamma \cdot x\right), \tag{C.19}$$

$$\mu_{24}^{\mathrm{disc}} = \epsilon^2 \left(\frac{1}{2}\sinh(2\pi u)\Gamma \cdot \sinh_- \left(x^4 - \frac{1}{x^4}\right) + I_3^2 p_2 + I_1 I_3 p_0 - \right.$$
$$- \frac{1}{2}\Gamma \cdot \sinh_-^2 \left(x^4 - \frac{1}{x^4}\right) + I_1\Gamma \cdot \sinh_- \left(x^3 + \frac{1}{x^3}\right) +$$
$$\left. + I_3\Gamma \cdot \sinh_- \left(x + \frac{1}{x}\right) + \left(I_3^2 - I_1^2\right)\Gamma \cdot x^2\right), \tag{C.20}$$

and as discussed for $J = 2$ the zero mode can be brought to the form

$$\pi_{12} = 0, \qquad \pi_{13} = 0, \qquad \pi_{14} = 0, \tag{C.21}$$

$$\pi_{24} = c_{1,24}\cosh 2\pi u, \qquad \pi_{34} = 0. \tag{C.22}$$

After that, we calculate r_a by formula (4.31) and solve the expanded to NLO Pμ system for $\mathbf{P}_a^{(1)}$ as

$$\mathbf{P}_3^{(1)} = H \cdot r_3, \tag{C.23}$$

$$\mathbf{P}_1^{(1)} = \frac{1}{2}\mathbf{P}_3^{(1)} + K \cdot \left(r_1 - \frac{1}{2}r_3\right), \tag{C.24}$$

$$\mathbf{P}_4^{(1)} = K \cdot \left[(H \cdot r_3)\sinh(2\pi u) + r_4 - \frac{1}{2}r_3\sinh(2\pi u)\right] - C(x + 1/x), \tag{C.25}$$

$$\mathbf{P}_2^{(1)} = H \cdot \left[-\mathbf{P}_4^{(1)} - \mathbf{P}_1^{(1)}\sinh(2\pi u) + r_2\right] + C/x, \tag{C.26}$$

where C is a constant which is fixed by requiring correct asymptotics of $\mathbf{P}_2$. Finally we find leading coefficients A_a of $\mathbf{P}_a^{(1)}$ and use expanded up to $\mathcal{O}(S^2)$ formulas (2.12), (2.13) in the same way as in section 4.4 to obtain the result (C.27).

C.4 Result for $J = 4$

The final result for the curvature function at $J = 4$ reads

$$
\gamma_{J=4}^{(2)} = \oint \frac{du_x}{2\pi i} \oint \frac{du_y}{2\pi i} \frac{1}{ig^2(I_3 - I_5)^3} \Bigg[\tag{C.27}
$$

$$
\frac{2\left(\mathrm{sh}_-^x\right)^2 y^4 \left(I_3\left(x^{10} + 1\right) - I_5 x^2 \left(x^6 + 1\right)\right)}{x^4\left(x^2 - 1\right)} - \frac{2\left(\mathrm{sh}_-^y\right)^2 x^4\left(y^8 - 1\right)\left(I_3 x^2 - I_5\right)}{\left(x^2 - 1\right) y^4} +
$$

$$
+ \frac{4\mathrm{sh}_-^x \mathrm{sh}_-^y \left(x^4 y^4 - 1\right)\left(I_3 + I_3 x^6 y^4 - I_5 x^2\left(x^2 y^4 + 1\right)\right)}{x^4\left(x^2 - 1\right) y^4}
$$

$$
+ \mathrm{sh}_-^y \Bigg(\left(y^4 + y^{-4}\right) x^{-1}\left(\left(I_1 I_5 - I_3^2\right)\left(3x^4 + 1\right) - 2I_1 I_3 x^6\right) +
$$

$$
+ \frac{2I_3 x^2\left(I_5\left(x^2 + 1\right) x^2 + I_1\left(1 - x^2\right)\right) - I_1 I_5\left(x^2 - 1\right)^2 + I_3^2\left(-2x^6 + x^4 + 1\right)}{x(x^2 - 1)} +
$$

$$
+ 2\left(y^3 + y^{-3}\right) \frac{I_1 I_3 x^6 - I_1 I_5 x^4 - I_3^2\left(x^2 - 1\right)}{x^2 - 1} -
$$

$$
- 2I_3\left(y + y^{-1}\right) \frac{I_1\left(x^2 - 1\right) - I_3\left(x^6 - x^2 + 1\right) + I_5\left(x^4 - x^2 + 1\right)}{x^2 - 1} \Bigg) +
$$

$$
+ \frac{4x^6 y^2 I_3\left(I_3^2 - I_1^2\right)}{x^2 - 1} + \frac{4xy I_1\left(I_3 y^2 + I_1\right)\left(I_3 + I_5\right)}{x^2 - 1} +
$$

$$
\frac{2y^4\left(I_1 + I_3\right)\left(I_1 I_5 - I_3^2\right)}{x^2 - 1} - \frac{2y\left(y^2 + 1\right)\left(I_1 + I_3\right)\left(I_1 I_5 - I_3^2\right)}{x\left(x^2 - 1\right)} -
$$

$$
- \frac{2x^3 y\left(I_1 + I_3\right)\left(I_1\left(2I_3 + \left(3\,y^2 + 1\right) I_5\right) - I_3\left(2I_5 y^2 + \left(y^2 + 3\right) I_3\right)\right)}{x^2 - 1}
$$

$$
+ \frac{2x^2 y^4\left(-I_3^3 - I_1\left(3I_3 + I_5\right) I_3 + I_1^2 I_5\right)}{x^2 - 1} + \frac{2x^4 y\left(I_1^2\left(2yI_5 - 2y^3 I_3\right) - 2y\left(y^2 + 1\right) I_3^2 I_5\right)}{x^2 - 1} +
$$

$$
+ \frac{4x^5 y I_3\left(2I_1^2 y^2 + I_3\left(I_5 - I_3\right) y^2 + I_1\left(I_3 + I_5\right)\right)}{x^2 - 1} \Bigg] \frac{1}{4\pi i} \partial_u \log \frac{\Gamma\left(iu_x - iu_y + 1\right)}{\Gamma\left(1 - iu_x + iu_y\right)}
$$

where, similarly to $J = 2, 3$, the integrals go around the branch between $-2g$ and $2g$.

D   Weak coupling expansion — details

First, we give the expansion of our results for the slope-to slope functions $\gamma_J^{(2)}$ to 10 loops. We start with $J = 2$:

$$
\gamma_{J=2}^{(2)} = -8g^2\zeta_3 + g^4\left(140\zeta_5 - \frac{32\pi^2\zeta_3}{3}\right) + g^6\left(200\pi^2\zeta_5 - 2016\zeta_7\right) \tag{D.1}
$$

$$
+ g^8\left(-\frac{16\pi^6\zeta_3}{45} - \frac{88\pi^4\zeta_5}{9} - \frac{9296\pi^2\zeta_7}{3} + 27720\zeta_9\right)
$$

$$
+ g^{10}\left(\frac{208\pi^8\zeta_3}{405} + \frac{160\pi^6\zeta_5}{27} + 144\pi^4\zeta_7 + 45440\pi^2\zeta_9 - 377520\zeta_{11}\right)
$$

$$+g^{12}\left(-\frac{7904\pi^{10}\zeta_3}{14175}-\frac{17296\pi^8\zeta_5}{4725}-\frac{128\pi^6\zeta_7}{15}-\frac{6312\pi^4\zeta_9}{5}\right.$$

$$\left.-653400\pi^2\zeta_{11}+5153148\zeta_{13}\right)$$

$$+g^{14}\left(\frac{1504\pi^{12}\zeta_3}{2835}+\frac{106576\pi^{10}\zeta_5}{42525}-\frac{18992\pi^8\zeta_7}{405}-\frac{16976\pi^6\zeta_9}{15}\right.$$

$$\left.+\frac{25696\pi^4\zeta_{11}}{9}+\frac{28003976\pi^2\zeta_{13}}{3}-70790720\zeta_{15}\right)$$

$$+g^{16}\left(-\frac{178112\pi^{14}\zeta_3}{382725}-\frac{239488\pi^{12}\zeta_5}{127575}+\frac{2604416\pi^{10}\zeta_7}{42525}+\frac{8871152\pi^8\zeta_9}{4725}\right.$$

$$\left.+\frac{30157072\pi^6\zeta_{11}}{945}+\frac{8224216\pi^4\zeta_{13}}{45}-133253120\pi^2\zeta_{15}+979945824\zeta_{17}\right)$$

$$+g^{18}\left(\frac{147712\pi^{16}\zeta_3}{382725}+\frac{940672\pi^{14}\zeta_5}{637875}-\frac{490528\pi^{12}\zeta_7}{8505}-\frac{358016\pi^{10}\zeta_9}{189}\right.$$

$$\left.-\frac{37441312\pi^8\zeta_{11}}{945}-\frac{9616256\pi^6\zeta_{13}}{15}-\frac{16988608\pi^4\zeta_{15}}{3}\right.$$

$$\left.+1905790848\pi^2\zeta_{17}-13671272160\zeta_{19}\right)$$

$$+g^{20}\left(-\frac{135748672\pi^{18}\zeta_3}{442047375}-\frac{103683872\pi^{16}\zeta_5}{88409475}+\frac{1408423616\pi^{14}\zeta_7}{29469825}\right.$$

$$\left.+\frac{2288692288\pi^{12}\zeta_9}{1403325}+\frac{34713664\pi^{10}\zeta_{11}}{945}+\frac{73329568\pi^8\zeta_{13}}{105}\right.$$

$$\left.+\frac{305679296\pi^6\zeta_{15}}{27}+121666688\pi^4\zeta_{17}-27342544320\pi^2\zeta_{19}+192157325360\zeta_{21}\right)$$

Next, for $J=3$,

$$\gamma^{(2)}_{J=3}=-2g^2\zeta_3+g^4\left(12\zeta_5-\frac{4\pi^2\zeta_3}{3}\right)+g^6\left(\frac{2\pi^4\zeta_3}{45}+8\pi^2\zeta_5-28\zeta_7\right)\qquad\text{(D.2)}$$

$$+g^8\left(-\frac{4\pi^6\zeta_3}{45}-\frac{4\pi^4\zeta_5}{15}-528\zeta_9\right)$$

$$+g^{10}\left(\frac{934\pi^8\zeta_3}{14175}+\frac{8\pi^6\zeta_5}{9}-\frac{82\pi^4\zeta_7}{9}-900\pi^2\zeta_9+12870\zeta_{11}\right)$$

$$+g^{12}\left(-\frac{572\pi^{10}\zeta_3}{14175}-\frac{104\pi^8\zeta_5}{175}-\frac{256\pi^6\zeta_7}{45}+\frac{2476\pi^4\zeta_9}{9}+\frac{57860\pi^2\zeta_{11}}{3}-208208\zeta_{13}\right)$$

$$+g^{14}\left(\frac{2878\pi^{12}\zeta_3}{127575}+\frac{404\pi^{10}\zeta_5}{1215}+\frac{326\pi^8\zeta_7}{75}+\frac{3352\pi^6\zeta_9}{135}\right.$$

$$\left.-\frac{80806\pi^4\zeta_{11}}{15}-316316\pi^2\zeta_{13}+2994992\zeta_{15}\right)$$

$$+g^{16}\left(-\frac{159604\pi^{14}\zeta_3}{13395375}-\frac{257204\pi^{12}\zeta_5}{1488375}-\frac{14836\pi^{10}\zeta_7}{6075}-\frac{71552\pi^8\zeta_9}{2025}\right.$$

$$\left.+\frac{4948\pi^6\zeta_{11}}{189}+\frac{4163068\pi^4\zeta_{13}}{45}+\frac{14129024\pi^2\zeta_{15}}{3}-41116608\zeta_{17}\right)$$

$$+g^{18}\left(\frac{494954\pi^{16}\zeta_3}{81860625}+\frac{156368\pi^{14}\zeta_5}{1819125}+\frac{6796474\pi^{12}\zeta_7}{5457375}+\frac{332\pi^{10}\zeta_9}{15}\right.$$

$$+\frac{1745318\pi^8\zeta_{11}}{4725}-\frac{868088\pi^6\zeta_{13}}{315}-\frac{22594208\pi^4\zeta_{15}}{15}$$

$$\left.-67084992\pi^2\zeta_{17}+553361016\zeta_{19}\right)$$

$$+g^{20}\left(-\frac{940132\pi^{18}\zeta_3}{315748125}-\frac{244456\pi^{16}\zeta_5}{5893965}-\frac{29637008\pi^{14}\zeta_7}{49116375}-\frac{11808196\pi^{12}\zeta_9}{1002375}\right.$$

$$-\frac{2265364\pi^{10}\zeta_{11}}{8505}-\frac{68767984\pi^8\zeta_{13}}{14175}+\frac{480208\pi^6\zeta_{15}}{9}$$

$$\left.+\frac{71785288\pi^4\zeta_{17}}{3}+934787840\pi^2\zeta_{19}-7390666360\zeta_{21}\right)$$

Finally, for $J=4$,

$$\gamma^{(2)}_{J=4}=g^2\left(-\frac{14\zeta_3}{5}+\frac{48\zeta_5}{\pi^2}-\frac{252\zeta_7}{\pi^4}\right)\tag{D.3}$$

$$+g^4\left(-\frac{22\pi^2\zeta_3}{25}+\frac{474\zeta_5}{5}-\frac{8568\zeta_7}{5\pi^2}+\frac{8316\zeta_9}{\pi^4}\right)$$

$$+g^6\left(\frac{32\pi^4\zeta_3}{875}+\frac{3656\pi^2\zeta_5}{175}-\frac{56568\zeta_7}{25}+\frac{196128\zeta_9}{5\pi^2}-\frac{185328\zeta_{11}}{\pi^4}\right)$$

$$+g^8\left(-\frac{4\pi^6\zeta_3}{175}-\frac{68\pi^4\zeta_5}{75}-\frac{55312\pi^2\zeta_7}{125}+\frac{1113396\zeta_9}{25}-\frac{3763188\zeta_{11}}{5\pi^2}+\frac{3513510\zeta_{13}}{\pi^4}\right)$$

$$+g^{10}\left(\frac{176\pi^8\zeta_3}{16875}+\frac{2488\pi^6\zeta_5}{7875}+\frac{2448\pi^4\zeta_7}{125}+\frac{209532\pi^2\zeta_9}{25}-\frac{3969878\zeta_{11}}{5}\right.$$

$$\left.+\frac{13213200\zeta_{13}}{\pi^2}-\frac{61261200\zeta_{15}}{\pi^4}\right)$$

$$+g^{12}\left(-\frac{88072\pi^{10}\zeta_3}{20671875}-\frac{449816\pi^8\zeta_5}{4134375}-\frac{327212\pi^6\zeta_7}{65625}-\frac{338536\pi^4\zeta_9}{875}\right.$$

$$\left.-\frac{129520798\pi^2\zeta_{11}}{875}+\frac{66969474\zeta_{13}}{5}-\frac{220540320\zeta_{15}}{\pi^2}+\frac{1017636048\zeta_{17}}{\pi^4}\right)$$

$$+g^{14}\left(\frac{795136\pi^{12}\zeta_3}{487265625}+\frac{522784\pi^{10}\zeta_5}{13921875}+\frac{4021288\pi^8\zeta_7}{2953125}+\frac{1869152\pi^6\zeta_9}{21875}\right.$$

$$+\frac{18573952\pi^4\zeta_{11}}{2625}+\frac{62633272\pi^2\zeta_{13}}{25}-\frac{1092799344\zeta_{15}}{5}$$

$$\left.+\frac{17844607872\zeta_{17}}{5\pi^2}-\frac{16405526592\zeta_{19}}{\pi^4}\right)$$

$$+g^{16}\left(-\frac{30581888\pi^{14}\zeta_3}{51162890625}-\frac{43988768\pi^{12}\zeta_5}{3410859375}-\frac{446380184\pi^{10}\zeta_7}{1136953125}\right.$$

$$-\frac{20108936\pi^8\zeta_9}{984375}-\frac{31755036\pi^6\zeta_{11}}{21875}-\frac{321449336\pi^4\zeta_{13}}{2625}$$

$$\left.-\frac{1031925232\pi^2\zeta_{15}}{25}+\frac{87296960712\zeta_{17}}{25}-\frac{283092985656\zeta_{19}}{5\pi^2}+\frac{259412389236\zeta_{21}}{\pi^4}\right)$$

$$+g^{18}\left(\frac{6706432\pi^{16}\zeta_3}{31672265625}+\frac{816838192\pi^{14}\zeta_5}{186232921875}+\frac{2004636572\pi^{12}\zeta_7}{17054296875}\right.$$

$$+\frac{1950592976\pi^{10}\zeta_9}{378984375}+\frac{2220222512\pi^8\zeta_{11}}{6890625}+\frac{20963856\pi^6\zeta_{13}}{875}$$

$$+\frac{254959316\pi^4\zeta_{15}}{125}+\frac{584553371616\pi^2\zeta_{17}}{875}$$

$$\left.-\frac{1375388084412\zeta_{19}}{25}+\frac{4432313039616\zeta_{21}}{5\pi^2}-\frac{4049650420200\zeta_{23}}{\pi^4}\right)$$

$$+g^{20}\left(-\frac{15308976272\pi^{18}\zeta_3}{209512037109375}-\frac{1764947984\pi^{16}\zeta_5}{1197211640625}-\frac{18667123736\pi^{14}\zeta_7}{517313671875}\right.$$

$$-\frac{538293689008\pi^{12}\zeta_9}{399070546875}-\frac{657466372\pi^{10}\zeta_{11}}{8859375}-\frac{119709052\pi^8\zeta_{13}}{23625}$$

$$-\frac{9095498848\pi^6\zeta_{15}}{23625}-\frac{260407748416\pi^4\zeta_{17}}{7875}-\frac{1869110789976\pi^2\zeta_{19}}{175}$$

$$\left.+\frac{4293062840352\zeta_{21}}{5}-\frac{13755955395600\zeta_{23}}{\pi^2}+\frac{62673161265000\zeta_{25}}{\pi^4}\right)$$

For future reference we have also computed[20] the weak coupling expansion of the anomalous dimensions at order S^3, using the known predictions from ABA which are available for any spin at $J=2$ and $J=3$. For $J=2$ we have computed the expansion to three loops.[21]

$$\gamma_{J=2}^{(3)}=g^2\frac{4}{45}\pi^4+g^4\left(40\zeta_3^2-\frac{28\pi^6}{405}\right)$$

$$+g^6\left(\frac{192}{5}\zeta_{5,3}-\frac{6992\zeta_3\zeta_5}{5}+\frac{280\pi^2\zeta_3^2}{3}+\frac{6962\pi^8}{212625}\right)+\mathcal{O}(g^8)$$

(D.4)

Compared to the S^2 part, a new feature is the appearance of multiple zeta values — here we have $\zeta_{5,3}$, which is defined by

$$\zeta_{a_1,a_2,\ldots,a_k}=\sum_{0<n_1<n_2<\ldots<n_k<\infty}\frac{1}{n_1^{a_1}n_2^{a_2}\ldots n_k^{a_k}}$$

(D.5)

and cannot be reduced to simple zeta values ζ_n.

For $J=3$ we have obtained the expansion to four loops:

$$\gamma_{J=3}^{(3)}=\frac{1}{90}\pi^4g^2+g^4\left(4\zeta_3^2+\frac{\pi^6}{1890}\right)+g^6\left(4\zeta_{5,3}+4\pi^2\zeta_3^2-72\zeta_3\zeta_5-\frac{2\pi^8}{675}\right)$$

$$+g^8\left(-112\zeta_{2,8}+\frac{20}{3}\pi^2\zeta_{5,3}+728\zeta_3\zeta_7+448\zeta_5^2-\frac{224}{3}\pi^2\zeta_3\zeta_5\right.$$

$$\left.+\frac{4\pi^4\zeta_3^2}{5}-\frac{41\pi^{10}}{133650}\right)+\mathcal{O}(g^{10})$$

(D.6)

---

[20]As described in the main text (see section 5), in the calculations we used several Mathematica packages for dealing with harmonic sums.

[21]We remind that in our notation the anomalous dimension is written as $\gamma=\gamma^{(1)}S+\gamma^{(2)}S^2+\gamma^{(3)}S^3+\ldots$.

E Higher mode numbers

E.1 Slope function for generic filling fractions and mode numbers

Let us extend the discussion of section 3.1 by considering the state corresponding to a solution of the asymptotic Bethe equations with arbitrary mode numbers and filling fractions.[22] We expect that in the $\mathbf{P}\mu$ system this should correspond to[23]

$$\mu_{24} = \sum_{n=-\infty}^{\infty} C_n e^{2\pi n u}.$$ (E.1)

As an example, for the ground state twist operator we have $\mu_{24} = \sinh(2\pi u)$, which is reproduced by choosing $C_{-1} = -1/2, C_1 = 1/2$ and all other C's set to 0.

It is straightforward to solve the $\mathbf{P}\mu$ system in the same way as in section 3.1, and we find the energy

$$\gamma = \frac{\sqrt{\lambda}}{J} \frac{\sum_n C_n I_{J+1}(n\sqrt{\lambda})}{\sum_n C_n I_J(n\sqrt{\lambda})/n} S,$$ (E.2)

which can also be written in a more familiar form as

$$\gamma = \sum_n \alpha_n \frac{n\sqrt{\lambda}}{J} \frac{I_{J+1}(n\sqrt{\lambda})}{I_J(n\sqrt{\lambda})} S,$$ (E.3)

where

$$\alpha_n = \frac{C_n I_J(n\sqrt{\lambda})/n}{\sum_m C_m I_J(m\sqrt{\lambda})/m}$$ (E.4)

are the filling fractions.

The coefficients C_n are additionally constrained by

$$\sum_n C_n I_J(n\sqrt{\lambda}) = 0,$$ (E.5)

which ensures that the $\mathbf{P}_a$ functions have correct asymptotics. This constraint implies a relation between the filling fractions,

$$\sum_n n\,\alpha_n = 0,$$ (E.6)

which is also familiar from the asymptotic Bethe ansatz.

E.2 Curvature function and higher mode numbers

In the main text we discussed the NLO solutions to the $\mathbf{P}\mu$ system which are based on the leading order solutions (3.19)–(3.22) or (3.24)–(3.27). One of the assumptions for constructing the leading order solution was to allow μ_{ab} to have only $e^{\pm 2\pi u}$ in asymptotics at infinity (we recall that this led to all μ's being constatnt except μ_{24} which is equal to

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[22]For simplicity we also consider even J here.

[23]We no longer expect μ_{24} to be either even or odd, since in the Bethe ansatz description of the state with generic mode numbers and filling fractions the Bethe roots are not distributed symmetrically.

sinh $(2\pi u)$ or cosh $(2\pi u)$), while in principle requiring μ_{ab} to be periodic one could also allow to have $e^{2n\pi u}$ with any integer n. Thus a natural generalization of the leading order solution is to consider $\mu_{24} = \sinh(2\pi n u)$ or $\mu_{24} = \cosh(2\pi n u)$, where n is an arbitrary integer. As discussed above (see the end of section 3.1 and appendix E.1), we believe that at the leading order in S such solutions correspond to states with mode numbers equal to n, and they reproduce the slope function for this case.

Proceeding to order S^2, the calculation of the curvature function $\gamma^{(2)}(g)$ with $\mu_{24} = \sinh(2\pi n u)$ or $\mu_{24} = \cosh(2\pi n u)$ can be done following the same steps as for $n = 1$. The final results for $J = 2, 3$ and 4 are given by exactly the same formulas as for $n = 1$ ((4.46), (4.63) and (C.27) respectively) — the only difference is that now one should set in those expressions

$$I_k = I_k(4\pi n g), \tag{E.7}$$

$$\text{sh}_-^x = [\sinh(2\pi n u_x)]_- , \tag{E.8}$$

$$\text{sh}_-^y = [\sinh(2\pi n u_y)]_- , \tag{E.9}$$

$$\text{ch}_-^x = [\cosh(2\pi n u_x)]_- , \tag{E.10}$$

$$\text{ch}_-^y = [\cosh(2\pi n u_y)]_- . \tag{E.11}$$

It would be natural to assume that this solution of the $\mathbf{P}\mu$ system describes anomalous dimensions for states with mode number n at order S^2. However we found some peculiarities in the strong coupling expansion of the result. The strong coupling data available for comparison in the literature for states with $n > 1$ also relies on some conjectures (see [23, 64]), so the interpretation of this solution is not fully clear to us.

The weak coupling expansion for this case turns out to be related in a simple way to the $n = 1$ case. One should just replace $\pi \to n\pi$ in the expansions for $n = 1$ which were given in (D.1), (D.2), (D.3). For example,

$$\gamma_{J=2}^{(2)} = -8g^2\zeta_3 + g^4\left(140\zeta_5 - \frac{32n^2\pi^2\zeta_3}{3}\right) + g^6\left(200n^2\pi^2\zeta_5 - 2016\zeta_7\right) + \dots \tag{E.12}$$

It would be interesting to compare these weak coupling predictions to results obtained from the asymptotic Bethe ansatz (or by other means) as it was done for $n = 1$ in section 5.

Let us now discuss the strong coupling expansion. According to Basso's conjecture [23] (see also [64]), the structure of the expansion may be obtained from

$$\Delta^2 = J^2 + S\left(A_1\sqrt{\mu} + A_2 + \dots\right) + S^2\left(B_1 + \frac{B_2}{\sqrt{\mu}} + \dots\right) + S^3\left(\frac{C_1}{\mu^{1/2}} + \frac{C_2}{\mu^{3/2}} + \dots\right) + \mathcal{O}(S^4), \tag{E.13}$$

where $\mu = n^2\lambda$. This gives

$$\Delta = J + \frac{S}{2J}\left(A_1 n\sqrt{\lambda} + A_2 + \frac{A_3}{n\sqrt{\lambda}} + \dots\right) \tag{E.14}$$

$$+ S^2\left(-\frac{A_1^2}{8J^3}n^2\lambda - \frac{A_1 A_2}{4J^3}n\sqrt{\lambda} + \left[\frac{B_1}{2J} - \frac{A_2^2 + 2A_1 A_3}{8J^3}\right]\right.$$

$$\left. + \left[\frac{B_2}{2J} - \frac{A_2 A_3 + A_1 A_4}{4J^3}\right]\frac{1}{n\sqrt{\lambda}} + \dots\right) + \mathcal{O}(S^3).$$

where A_i are known from Basso's slope function. Substituting them, we find

$$\gamma_J^{(2)}(g) = -\frac{8\pi^2 g^2 n^2}{J^3} + \frac{2\pi g n}{J^3} + \frac{B_1 - 1}{2J} + \frac{8B_2 J^2 - 4J^2 + 1}{64\pi g J^3 n} + \dots \qquad \text{(E.15)}$$

However, already in [64] some inconsistencies were found if one assumes this structure for $n > 1$. Let us extend that analysis by comparing the prediction (E.15) to our results from the $\mathbf{P}\mu$-system. To compute the expansion of our results, similarly to the $n = 1$ case, we evaluated $\gamma_J^{(2)}(g)$ numerically for many values of g, and then fitted the result by powers of g. As for $n = 1$ we found with high precision (about $\pm 10^{-16}$) that the first several coefficients involve only rational numbers and powers of π. Our results for $n = 2, 3$ and $J = 2, 3, 4$ are summarized below:

$$\gamma_{J=2,n=2}^{(2)}(g) = -4\pi^2 g^2 + \frac{\pi g}{2} + \frac{17}{8} - \frac{0.29584877037648771(2)}{g} + \dots \qquad \text{(E.16)}$$

$$\gamma_{J=3,n=2}^{(2)}(g) = -\frac{32}{27}\pi^2 g^2 + \frac{4\pi g}{27} + \frac{17}{12} - \frac{0.2928304112866493(9)}{g} + \dots \qquad \text{(E.17)}$$

$$\gamma_{J=4,n=2}^{(2)}(g) = -\frac{1}{2}\pi^2 g^2 + \frac{\pi g}{16} + \frac{17}{16} - \frac{0.319909936615448(9)}{g} + \dots \qquad \text{(E.18)}$$

$$\gamma_{J=2,n=3}^{(2)}(g) = -9\pi^2 g^2 + \frac{3\pi g}{4} + \frac{23}{4} - \frac{0.8137483(9)}{g} + \dots \qquad \text{(E.19)}$$

$$\gamma_{J=3,n=3}^{(2)}(g) = -\frac{8}{3}\pi^2 g^2 + \frac{2\pi g}{9} + \frac{23}{6} - \frac{0.892016609(2)}{g} + \dots \qquad \text{(E.20)}$$

$$\gamma_{J=4,n=3}^{(2)}(g) = -\frac{9}{8}\pi^2 g^2 + \frac{3\pi g}{32} + \frac{23}{8} - \frac{1.035945580(6)}{g} + \dots \qquad \text{(E.21)}$$

Here in the coefficient of $\frac{1}{g}$ the digit in brackets is the last known one within our precision.[24]

Comparing to (E.15) we find full agreement in the first two terms (of order g^2 and of order g). The next term in (E.15) (of order g^0) is determined by B_1, which in [64] was found to be

$$B_1 = \frac{3}{2} \qquad \text{(E.22)}$$

for all n, J, based on consistency with the classical energy. However, comparing our results with (E.15) we find a different value:

$$B_1 = \frac{19}{2} \quad \text{for} \quad n = 2 \,, \qquad \text{(E.23)}$$
$$B_1 = 23 \quad \text{for} \quad n = 3 \,.$$

For both $n = 2$ and $n = 3$ this prediction for B_1 is independent of J.

The next term is of order $\frac{1}{g}$ and is determined by B_2, which in [64] was fixed to

$$B_2 = \begin{cases} -3\,\zeta_3 + \frac{3}{8} \,, & n = 1 \\ -24\,\zeta_3 - \frac{13}{8} \,, & n = 2 \\ -81\,\zeta_3 - \frac{24}{8} \,, & n = 3 \end{cases} \qquad \text{(E.24)}$$

---

[24]We did not seek to achieve high precision in this coefficient for $n = 3$.

However, this does not agree with our numerical predictions for $n = 2$ and 3. Furthermore, for $n = 2$ we extracted the coefficient of $\frac{1}{g}$ with high precision (about 10^{-17}, see (E.16)) but were unable to fit it as a combination of simple zeta values using the EZ-Face calculator [63].

Thus our results appear to disagree with the values of B_1 and B_2 obtained in [64], but how to interpret this is not clear to us. Although our solution of the $\mathbf{P}\mu$-system for $n > 1$ looks fine at order S, it may be that to capture anomalous dimensions at order S^2 some other solution should be used. Another option is that the ansatz for the structure of anomalous dimensions at strong coupling may need to be modified when $n > 1$ (as already suspected in [64]), and our results may help provide some guidance in this case.

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