Partial domain wall partition functions

O. Foda^a and M. Wheeler^b

^a*Department of Mathematics and Statistics, University of Melbourne,
Parkville, Victoria 3010, Australia*

^b*Laboratoire de Physique Théorique et Hautes Energies, CNRS UMR 7589 and
Université Pierre et Marie Curie - Paris 6,
4 place Jussieu, 75252 Paris cedex 05, France*

E-mail: omar.foda@unimelb.edu.au, mwheeler@lpthe.jussieu.fr

ABSTRACT: We consider six-vertex model configurations on an $(n \times N)$ lattice, $n \leq N$, that satisfy a variation on domain wall boundary conditions that we define and call *partial domain wall boundary conditions*. We obtain two expressions for the corresponding *partial domain wall partition function*, as an $(N \times N)$ -determinant and as an $(n \times n)$ -determinant. The latter was first obtained by I Kostov. We show that the two determinants are equal, as expected from the fact that they are partition functions of the same object, that each is a discrete KP τ -function, and, recalling that these determinants represent tree-level structure constants in $\mathcal{N} = 4$ SYM, we show that introducing 1-loop corrections, as proposed by N Gromov and P Vieira, preserves the determinant structure.

KEYWORDS: Lattice Integrable Models, Integrable Hierarchies

ARXIV EPRINT: [1205.4400](https://arxiv.org/abs/1205.4400)

Contents

1	Introduction	1
1.1	Partial domain wall boundary conditions	2
1.2	A brief history of partial domain wall configurations	2
1.3	Outline of contents	3
1.4	Glossary of frequently used notation	4
2	Six-vertex model configurations	4
2.1	Lines, orientations and rapidity variables	4
2.2	Segments, arrows and vertices	4
2.3	Weights, configurations and partition functions	4
2.4	Six vertices that conserve arrow flow	5
2.5	Limiting form of the weights	6
2.6	The domain wall partition function, DWPF	6
2.7	Izergin’s determinant	6
2.8	The scalar product	7
2.9	Slavnov’s determinant	7
2.10	The partial domain wall partition function, pDWPF	8
2.11	Deleting lines from opposite boundaries	9
3	Domain wall partition function in the infinite-rapidity limit	10
3.1	One rapidity becomes infinite	10
3.2	$(N - n)$ rapidities become infinite	10
3.3	Limit of Izergin’s determinant as one rapidity becomes infinite	11
3.4	Limit of Izergin’s determinant as $(N - n)$ rapidities become infinite	11
3.5	A ‘ <i>partial Vandermonde</i> ’ way to write the determinant	13
3.6	Towards the trigonometric pDWPF	13
3.7	The trigonometric pDWPF as an $(N \times N)$ -determinant	14
4	Slavnov scalar product in the infinite-rapidity limit	14
4.1	n rapidities become infinite	14
4.2	The infinite rapidity limit of Slavnov’s determinant	15
4.3	Kostov’s determinant	17
4.4	Writing $Z_2(\{x\}_n \{y\}_N)$ as a sum over partitions	17
4.5	The trigonometric version of Kostov’s determinant	18
5	Equivalence of determinants	19
5.1	Set of properties which characterize the pDWPF	19
5.2	$Z_{N \times N}(\{x\}_n \{y\}_N)$ satisfies the four properties	20
5.3	$Z_{n \times n}(\{x\}_n \{y\}_N)$ satisfies the four properties	21
5.4	$Z_{N \times N}(\{x\}_n \{y\}_N)$ and $Z_{n \times n}(\{x\}_n \{y\}_N)$ are equal	22

6	Casorati determinants and discrete KP hierarchy	22
6.1	Notation related to sets of variables	22
6.2	The complete symmetric function $h_i\{x\}$	23
6.3	Useful identities for $h_i\{x\}$	23
6.4	Discrete derivatives	23
6.5	The discrete KP hierarchy	23
6.6	Casoratian matrices and determinants	24
6.7	Notation for column vectors and determinants	24
6.8	Identities for Casoratian determinants	25
6.9	Casoratian determinants are discrete KP τ -functions	25
6.10	$Z_{N \times N}(\{x\}_n \{y\}_N)$ is a discrete KP τ -function in $\{y\}_N$	26
6.11	$Z_{n \times n}(\{x\}_n \{y\}_N)$ is a discrete KP τ -function in $\{x\}_n$	26
7	The Gromov-Vieira polynomial version of partial domain wall partition functions	27
7.1	The Gromov-Vieira mapping	27
7.2	Aim of this section	27
7.3	The GV mapping in terms of symmetric functions	28
7.4	Action of H_2 , and G_2	28
7.5	Remarks on notation	28
7.6	Degree-2 terms in the determinant	29
7.7	Action of H_2 on equation (7.10)	29
7.8	Action of G_2 on equation (7.10)	30
7.9	Determinant expression for $[Z_{N \times N}(\{x\}_n \{y\}_N)]_y$	30
8	Remarks	31
8.1	Summary of results	31
8.2	Taking the free variables to infinity in Slavnov's determinant	32
8.3	Asymptotics	32
8.4	Higher rank scalar products	33
8.5	Combinatorics and counting	33

1 Introduction

The discovery of classical and quantum integrable structures on both sides of the anti-de Sitter/conformal field theory correspondence, AdS/CFT, in the late 1990's and early 2000's, culminating in [1, 2] and in the intensive rapid developments that followed these seminal works, has been beneficial to all subjects involved.¹ On the one hand, integrability is widely considered to be a viable approach to proving the AdS/CFT correspondence. On the other, ideas and insights from AdS/CFT will continue to enrich the subject of integrability.

¹For a comprehensive introduction to applications of classical and quantum integrability in gauge and string theories, we refer the reader to [3] and references therein.

1.1 Partial domain wall boundary conditions

The purpose of this note is to study six-vertex model configurations on a rectangular lattice with n horizontal and N vertical lines, $n \leq N$, as in figure 1, that satisfy $(n \times N)$ *partial domain wall boundary conditions*, pDWBC's. In our conventions,² these are defined as follows.

1. All arrows on the left and right boundaries point inwards,
2. n_u (n_l) arrows on the upper (lower) boundary, such that $n_u + n_l = N - n$, also point inwards,
3. The remaining $n + N$ arrows on the upper and lower boundaries point outwards,
4. The locations of the inward-pointing arrows on the upper and lower boundaries, with n_u and n_l fixed, are summed over.

The corresponding partition function is an $(n \times N)$ *partial domain wall partition function*, pDWPF. For $n = N$, and $n_u = n_l = 0$, we recover Korepin's domain wall boundary conditions, DWBC's [4], and the pDWPF reduces to Izergin's domain wall partition function, DWPF [5, 6].

1.2 A brief history of partial domain wall configurations

The configurations considered in this work were first introduced in the work of Bogoliubov, Pronko and Zvonarev in their study of boundary correlation functions in the presence of DWBC's [7], and subsequent works [8–11]. However, in these works, they were camouflaged by the fact that they were paired with complementary configurations to produce $(N \times N)$ configurations with conventional DWBC's, and that only a subset of the positions that the inverted n_u (n_l) arrows on the upper (lower) boundaries can take were included in the statistical sum, as one does in computations of boundary correlation functions.

In [12, 13], Escobedo, Gromov, Sever and Vieira studied 3-point functions of three gauge-invariant single-trace length- N_i operators, $\mathcal{O}_i(x_i)$, $i \in \{1, 2, 3\}$, that are composed of elementary scalars in $SU(2)$ subsectors of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory.³ Using the connection with quantum integrable models, Escobedo et al. obtained a sum expression for the studied structure constants. In [14], Gromov, Sever and Vieira considered the same structure constants as in [12, 13] in the special case where $\mathcal{O}_1$ and $\mathcal{O}_3$, in the conventions of [14], are BPS operators. That is, the rapidity variables $\{x_1\}$ and $\{x_3\}$ that characterize these two operators are taken to infinity. In this special case, the sum expression of [12] simplifies to a sum expression for a quantity that they refer to as $\mathcal{A}\{x_2\}$.

²Our conventions include choosing $n \leq N$, all $2n$ arrows on the left and right boundaries and $(N - n)$ arrows on the upper and lower boundaries point inwards, while all other arrows on the upper and lower boundaries point outwards. All these choices could have been reversed.

³In [12, 13], $\{\mathcal{O}_i\}$, $i \in \{1, 2, 3\}$ are chosen such that their lengths L_i satisfy *non-extremal length* conditions, $L_i < L_j + L_k$ for any distinct $\{i, j, k\}$, and further, they are characterized by rapidity variables $\{x_i\}$, such that they are non-BPS ($\{x_i\}$ has finitely many elements that are finite, rather than infinite), and have well-defined conformal dimensions (the elements of $\{x_i\}$ satisfy Bethe equations).

In [15], the sum expression of [12, 13] was evaluated in determinant form. This determinant is essentially Slavnov's determinant expression for the scalar product of a Bethe eigenstate and a generic state in a periodic spin- $\frac{1}{2}$ XXX chain. In vertex model terms, Slavnov's determinant is the partition function of rational six-vertex configurations with $2n$ horizontal lines (auxiliary spaces), N vertical lines (quantum spaces), and boundary conditions that are specified in [15, 16]. The limit used in [14] to produce $\mathcal{A}\{x_2\}$ is achieved in six-vertex model terms by simply deleting the n horizontal lines that represent the Bethe eigenstate, as explained in the sequel. The resulting configurations are the partial domain wall configurations discussed in this note.

1.3 Outline of contents

In section 2, we recall basic definitions related to the six-vertex model. In section 3, we start from $(N \times N)$ domain wall configurations and delete $N - n$ horizontal lines to obtain $(n \times N)$ partial domain wall configurations. For simplicity we consider the case $n_u = N - n$ and $n_l = 0$. Once this case is understood, the general case is straightforward to obtain. The corresponding $(n \times N)$ pDWPF is obtained starting from Izergin's $(N \times N)$ determinant expression for the $(N \times N)$ DWPF of the initial domain wall configuration, taking the rapidity variables of the lines that are deleted to infinity, and normalizing appropriately to obtain the pDWPF $Z_{N \times N}$ in $(N \times N)$ determinant form.

In section 4, we start from the $(2n \times N)$ configurations that describe the scalar product of an n -magnon Bethe eigenstate and an n -magnon generic state, on an N -site periodic spin- $\frac{1}{2}$ chain. We delete the n horizontal lines that describe the Bethe eigenstate to obtain $(n \times N)$ partial domain wall configurations with $n_u = N - n$ and $n_l = 0$. The corresponding $(n \times N)$ pDWPF is obtained starting from Slavnov's $(n \times n)$ determinant expression for the scalar product, taking the rapidity variables of the lines that we deleted (which are the Bethe roots) to infinity, and normalizing appropriately to obtain the pDWPF $Z_{n \times n}$ in $(n \times n)$ determinant form. $Z_{n \times n}$ was first derived by Kostov [17–19]. Expanding $Z_{n \times n}$, one obtains the sum expression of Gromov et al. [12]. Starting from the trigonometric Slavnov scalar product, we also derive the trigonometric version of $Z_{n \times n}$.

Each determinant, $Z_{N \times N}$ and $Z_{n \times n}$, is a function of the set $\{x\}$ of cardinality n , associated with the n horizontal lines, and the set $\{y\}$ of cardinality N , associated with the N vertical lines. In section 5, as an independent check of the correctness of our expressions for $Z_{N \times N}$ and $Z_{n \times n}$, we show that they can be written as polynomials in each of their variables x_i , with the same bound on their degree, and that they satisfy the same recursion relations and initial condition. This proves that they are equal, as expected from the fact that they are different expressions for the same partition function. In section 6, we recall basic facts regarding Casorati determinants (the discrete analogues of Wronskians) and discrete KP τ -functions, then we show that pDWPF's are discrete KP τ -functions in the $\{x\}$ as well as in the $\{y\}$ variables.

In section 7, we recall a mapping that Gromov and Vieira use in [20, 21] to introduce 1-loop corrections into the 0-loop expressions of certain structure constants in $\mathcal{N}=4$ supersymmetric Yang-Mills theory, and show that the $(n \times N)$ pDWPF remains a determinant under this mapping. In section 8, we include remarks on recent developments.

1.4 Glossary of frequently used notation

$\{x\}$ ($\{y\}$) is a set of rapidity variables that do not satisfy Bethe equations and that flow along horizontal (vertical) lines. We always take $\{x\}$ and $\{y\}$ to be free variables. $\{b\}$ is a set of rapidity variables that *do* satisfy the Bethe equations and that flow along horizontal lines. When a set $\{x\}$ has cardinality N , we sometimes indicate this by writing $\{x\}_N$. At times we also use the notation

$$\Delta\{x\}_N = \prod_{1 \leq i < j \leq N} [x_j - x_i], \quad \Delta\{-x\}_N = \prod_{1 \leq i < j \leq N} [x_i - x_j] \quad (1.1)$$

for Vandermonde determinants in the variables $\{x\}_N$. $[x - y] = x - y$ in the rational case, and $[x - y] = \sinh(x - y)$ in the trigonometric case.

2 Six-vertex model configurations

In this section we recall basic definitions related to the six-vertex model on an $(n \times N)$ square lattice, $n \leq N$, including vertex model descriptions of Korepin's domain wall configurations on an $(N \times N)$ lattice [4], Slavnov's scalar product configurations on a $(2n \times N)$ lattice [16], and the determinant expressions for these objects [5, 22]. Finally, we define the partial domain wall configurations and the corresponding partial domain wall partition functions.

2.1 Lines, orientations and rapidity variables

Consider a square lattice with n horizontal lines and N vertical lines that intersect at $(n \times N)$ points, $n \leq N$. We order the horizontal lines from bottom to top and assign the i -th line an orientation from left to right and a rapidity variable x_i . We order the vertical lines from left to right and assign the j -th line an orientation from bottom to top and a rapidity variable y_j . See figure 1. The orientations that we assign to the lattice lines are matters of convention and are meant to make the vertices of the six-vertex model, that we introduce shortly, unambiguous.

2.2 Segments, arrows and vertices

Each lattice line is divided into segments by all other lines that are perpendicular to it. *Bulk segments* are attached to two intersection points. *Boundary segments* are attached to one intersection point only. Assign each segment an arrow that can point in either direction, and define the vertex v_{ij} as the union of the intersection point of the i -th horizontal line and the j -th vertical line, the four line segments attached to this intersection point, and the arrows on these segments.

2.3 Weights, configurations and partition functions

Assign every vertex v_{ij} a weight w_{ij} that depends on the specific orientations of its arrows, and the rapidities x_i and y_j that flow through it. Any lattice configuration with a definite assignment of arrows is assigned a weight equal to the product of the weights of its vertices. The partition function of the lattice in figure 1 is the sum of the weights of all lattice configurations which respect the boundary conditions that we impose.

Figure 1. A square ($n \times N$) lattice with oriented lines and rapidity variables. The lines are assigned orientations indicated by the white arrows.

Figure 2. Assignment of weights to vertices.

2.4 Six vertices that conserve arrow flow

Since every arrow can point in either direction, there are $2^4 = 16$ possible (types of) vertices. We are interested in models with ‘*conservation of arrow flow*’. That is, the only vertices with non-zero weights are those such that the number of arrows that point toward the intersection point of the vertex is equal to the number of arrows that point away from it. These are six such vertices shown in figure 2. The remaining vertices have zero weights.

In this work, we study the rational and the trigonometric six-vertex model. The former is a special case of the latter. For the rational six-vertex model, we use the weights

$$a_{\pm}(x, y) = 1, \quad b_{\pm}(x, y) = \frac{x - y}{x - y + 1}, \quad c_{\pm}(x, y) = \frac{1}{x - y + 1} \quad (2.1)$$

Figure 3. Lattice definition of $Z(\{x\}_N|\{y\}_N)$. The boundary segments have the definite arrow assignments shown, and all bulk segments are summed over.

For the trigonometric six-vertex model, we use the weights

$$a_{\pm}(x, y) = 1, \quad b_{\pm}(x, y) = e^{\pm\gamma} \frac{[x - y]}{[x - y + \gamma]}, \quad c_{\pm}(x, y) = e^{\pm(x-y)} \frac{[\gamma]}{[x - y + \gamma]} \quad (2.2)$$

where $[x] \equiv \sinh(x)$. The weights in equations (2.1) and (2.2) satisfy the Yang-Baxter equations and unitarity. The parametrization in the trigonometric case is not unique. The reason for using the parametrization in equation (2.2) is explained in subsection 3.6.

2.5 Limiting form of the weights

From equations (2.1) and (2.2), as $x \rightarrow \infty$, the rational weights become

$$a_{\pm}(x, y) \rightarrow 1, \quad b_{\pm}(x, y) \rightarrow 1, \quad c_{\pm}(x, y) \rightarrow \frac{1}{x} \quad (2.3)$$

while the trigonometric weights become

$$a_{\pm}(x, y) \sim b_{+}(x, y) \rightarrow 1, \quad b_{-}(x, y) \rightarrow e^{-2\gamma}, \quad c_{+}(x, y) \rightarrow e^{-\gamma}[\gamma], \quad c_{-}(x, y) \rightarrow \frac{e^{-\gamma}[\gamma]}{e^{2(x-y)}} \quad (2.4)$$

2.6 The domain wall partition function, DWPF

This standard object is defined in six-vertex model terms as the partition function of the configurations in figure 3, [4, 6]. It depends on two sets of variables $\{x\}_N = \{x_1, \dots, x_N\}$ and $\{y\}_N = \{y_1, \dots, y_N\}$, and we denote it by $Z(\{x\}_N|\{y\}_N)$.

2.7 Izergin's determinant

Following [5], in the rational parametrization of equation (2.1) the DWPF is given by

$$Z(\{x\}_N|\{y\}_N) = \frac{\prod_{i,j=1}^N (x_i - y_j)}{\Delta\{x\}_N \Delta\{-y\}_N} \det \left(\frac{1}{(x_i - y_j)(x_i - y_j + 1)} \right)_{1 \leq i,j \leq N} \quad (2.5)$$

Figure 4. Lattice representation of $S(\{x\}_n, \{b\}_n | \{y\}_N)$. There are two sets of horizontal rapidities $\{x\}_n$ and $\{b\}_n$, and one set of vertical rapidities $\{y\}_N$. The variables $\{b\}_n$ satisfy Bethe equations.

In the trigonometric parametrization of equation (2.2), the DWPF is given by

$$Z(\{x\}_N | \{y\}_N) = \frac{e^{|x|-|y|} \prod_{i,j=1}^N [x_i - y_j]}{\Delta\{x\}_N \Delta\{-y\}_N} \det \left(\frac{[\gamma]}{[x_i - y_j][x_i - y_j + \gamma]} \right)_{1 \leq i,j \leq N} \quad (2.6)$$

where we use the notation $|x| = \sum_{k=1}^N x_k$.

2.8 The scalar product

This is another standard object that is defined in this work in six-vertex model terms⁴ as the partition function of the configuration in figure 4, [16, 22, 23]. It depends on three sets of variables $\{x\}_n = \{x_1, \dots, x_n\}$, $\{b\}_n = \{b_1, \dots, b_n\}$, $\{y\}_N = \{y_1, \dots, y_N\}$, where $n \leq N$. We denote it by $S(\{x\}_n, \{b\}_n | \{y\}_N)$.

2.9 Slavnov's determinant

Following [22] we assume that one set of variables in figure 4, $\{b\}_n$, obeys the Bethe equations.⁵ In the rational and trigonometric parametrizations they are given by

$$\prod_{j=1}^N \left(\frac{b_i - y_j + 1}{b_i - y_j} \right) = \prod_{j \neq i}^n \left(\frac{b_i - b_j + 1}{b_i - b_j - 1} \right),$$

$$\prod_{j=1}^N \frac{[b_i - y_j + \gamma]}{[b_i - y_j]} = e^{N\gamma} \prod_{j \neq i}^n \frac{[b_i - b_j + \gamma]}{[b_i - b_j - \gamma]}, \quad \forall 1 \leq i \leq n \quad (2.7)$$

⁴The scalar product is usually defined as a vacuum expectation value of algebraic Bethe Ansatz operators. For our purposes, this formalism is unnecessary.

⁵In this work, all rapidity variables denoted by b_i are assumed to obey Bethe equations, while all rapidity variables denoted by x_i or y_j are free.

Figure 5. On the left, lattice representation of $Z_1(\{x\}_n|\{y\}_N)$. On the right, lattice representation of $Z_2(\{x\}_n|\{y\}_N)$. The number of horizontal rapidities x_i is less than the number of vertical rapidities y_j . The top boundary segments are without arrows to indicate summation at these points.

respectively. Assuming that the Bethe equations hold, in the rational parametrization the scalar product has the determinant representation

$$S(\{x\}_n, \{b\}_n | \{y\}_N) = \Delta^{-1} \{x\}_n \Delta^{-1} \{-b\}_n \quad (2.8)$$

$$\times \det \left[\frac{\prod_{k \neq j}^n (b_k - x_i - 1) \prod_{k=1}^N \left(\frac{x_i - y_k}{x_i - y_k + 1} \right) - \prod_{k \neq j}^n (b_k - x_i + 1)}{x_i - b_j} \right]_{1 \leq i, j \leq n}$$

In the trigonometric parametrization, it is given by

$$S(\{x\}_n, \{b\}_n | \{y\}_N) = [\gamma]^n e^{|b|-|x|} \Delta^{-1} \{x\}_n \Delta^{-1} \{-b\}_n \quad (2.9)$$

$$\times \det \left[\frac{e^{(N-n)\gamma} \prod_{k \neq j}^n [b_k - x_i - \gamma] \prod_{k=1}^N \frac{[x_i - y_k]}{[x_i - y_k + \gamma]} - e^{-n\gamma} \prod_{k \neq j}^n [b_k - x_i + \gamma]}{[x_i - b_j]} \right]_{1 \leq i, j \leq n}$$

2.10 The partial domain wall partition function, pDWPF

Let n be an integer satisfying $1 \leq n \leq N$. Consider the partition function generated by deleting the top $(N-n)$ rows from the lattice in figure 3, or the top n rows from the lattice in figure 4, and whose top boundary is summed over all arrow configurations. We denote these objects by $Z_1(\{x\}_n|\{y\}_N)$ and $Z_2(\{x\}_n|\{y\}_N)$ respectively, and represent them by the lattices in figure 5.

We emphasize that, unlike the usual domain wall configurations, the top boundary segments in figure 5 are *not* fixed to definite arrow configurations but are summed over just as the bulk segments.

As we will see in subsection 3.2, up to a numerical coefficient, $Z_1(\{x\}_n|\{y\}_N)$ is the leading term in $Z(\{x\}_N|\{y\}_N)$ as $x_N, \dots, x_{n+1} \rightarrow \infty$. In this limit, the contribution from

Figure 6. An alternative lattice representation of $Z_1(\{x\}_n|\{y\}_N)$. Horizontal lines get removed from the top and bottom of the DWPF lattice. Both the top and bottom boundary segments are without arrows to indicate summation at these points. The top boundary is summed over all configurations which have exactly $(m - n)$ downward facing arrows, while the bottom boundary is summed over all configurations with exactly $(N - m)$ upward facing arrows. This lattice sum is equal to the one on the left of figure 5, up to an overall factor.

the top $(N - n)$ rows of figure 3 becomes trivial, and we are left with the lattice shown in figure 5. For this reason, $Z_1(\{x\}_n|\{y\}_N)$ is a *partial domain wall partition function*, pDWPF.

One can also calculate the pDWPF $Z_2(\{x\}_n|\{y\}_N)$ as the leading term of the scalar product $S(\{x\}_n, \{b\}_n|\{y\}_N)$ as $b_n, \dots, b_1 \rightarrow \infty$. In this limit, the contribution from the top n rows of figure 4 becomes trivial, and we are left with the lattice shown in figure 5. This is discussed in subsection 4.1.

In this paper we calculate $Z_1(\{x\}_n|\{y\}_N)$ and $Z_2(\{x\}_n|\{y\}_N)$ by taking the two limits described above. The starting points for these calculations are, respectively, Izergin's determinant formula for $Z(\{x\}_N|\{y\}_N)$ and Slavnov's determinant formula for $S(\{x\}_n, \{b\}_n|\{y\}_N)$. In the case of the rational six-vertex model, whose vertex weights are invariant under the reversal of all arrows, the two quantities $Z_1(\{x\}_n|\{y\}_N)$ and $Z_2(\{x\}_n|\{y\}_N)$ are in fact equal (which is easily verified by comparing the two lattices in figure 5). Therefore in the rational parametrization we obtain two different determinant expressions for the same object.

2.11 Deleting lines from opposite boundaries

By symmetry of $Z(\{x\}_N|\{y\}_N)$ in the variables $\{x\}_N$, we are free to distribute the rapidities $x_N, \dots, x_{n+1}$ over the horizontal lines of the lattice in any way we wish, prior to taking the limit $x_N, \dots, x_{n+1} \rightarrow \infty$.

For example, we can choose to place the variables $x_N, \dots, x_{m+1}$ on the lowest lines of the lattice and $x_m, \dots, x_{n+1}$ on the highest, where m is some integer satisfying $n \leq m \leq N$. In the limit $x_N, \dots, x_{n+1} \rightarrow \infty$, the bottom $(N - m)$ and the top $(m - n)$ rows become trivial, and we obtain the lattice shown in figure 6. The lattice sum in figure 6 is equal to the one on the left of figure 5, up to multiplication by an overall factor. In the rational six-vertex model, this factor is the binomial coefficient $\binom{N-n}{N-m}$.

3 Domain wall partition function in the infinite-rapidity limit

In this section, we obtain a determinant expression for the pDWPF starting from Izergin's formula for the DWPF, equation (2.5), and taking appropriate limits. The $(N \times N)$ determinant that we obtain is 'hybrid' in the sense that it contains n rows of the type in Izergin's formula, and $(N - n)$ rows of Vandermonde determinant-type.

3.1 One rapidity becomes infinite

Consider the rational DWPF. Due to the domain wall boundary conditions, and the conservation of arrow flow, the top row of the lattice always contains precisely one c_+ vertex, while all remaining vertices in that row are of the type a_+ or b_+ . Using the asymptotic behaviour of the vertex weights in equation (2.3) and the definition of $Z_1(\{x\}_{N-1}|\{y\}_N)$, it is easy to see that

$$Z_1(\{x\}_N|\{y\}_N) \rightarrow \frac{Z_1(\{x\}_{N-1}|\{y\}_N)}{x_N}, \quad \text{as } x_N \rightarrow \infty \quad (3.1)$$

and the pDWPF $Z_1(\{x\}_{N-1}|\{y\}_N)$ can be computed from the DWPF as

$$Z_1(\{x\}_{N-1}|\{y\}_N) = \lim_{x_N \rightarrow \infty} \left(x_N Z_1(\{x\}_N|\{y\}_N) \right) \quad (3.2)$$

3.2 $(N - n)$ rapidities become infinite

Consider the lattice representation of the pDWPF $Z_1(\{x\}_i|\{y\}_N)$, for some $1 \leq i \leq N$. The top boundary of this lattice consists of a sum over $\binom{N}{i}$ possible arrow configurations. From now on, we make remarks which apply to the internal part of this lattice, assuming that the boundary is fixed to any one of these $\binom{N}{i}$ configurations.

Consider the x_i -row of vertices.⁶ Any configuration that this row takes must contain at least one c_+ vertex and m pairs of $\{c_+, c_-\}$ vertices, $m = 0, 1, 2, \dots$. In the large x_i limit, the leading contribution to $Z_1(\{x\}_i|\{y\}_N)$ corresponds to $m = 0$. Taking multiple counting into consideration, we obtain

$$Z_1(\{x\}_i|\{y\}_N) \rightarrow (N - i + 1) \frac{Z_1(\{x\}_{i-1}|\{y\}_N)}{x_i}, \quad \text{as } x_i \rightarrow \infty \quad (3.3)$$

Iterating this result through $i = \{N, \dots, n + 1\}$ we obtain

$$Z_1(\{x\}_n|\{y\}_N) = \frac{1}{(N - n)!} \lim_{x_N, \dots, x_{n+1} \rightarrow \infty} \left(x_{n+1} \cdots x_N Z_1(\{x\}_N|\{y\}_N) \right) \quad (3.4)$$

where the limits are to be taken sequentially, starting with x_N .

⁶All vertices through which the x_i rapidity variable flows.

3.3 Limit of Izergin's determinant as one rapidity becomes infinite

Starting from the expression in equation (2.5) for $Z_1(\{x\}_N|\{y\}_N)$, it is simple to take the limit specified in equation (3.2). Absorbing the factor $x_N \prod_{j=1}^N (x_N - y_j) / \prod_{j=1}^{N-1} (x_N - x_j)$ into the final row of the determinant in equation (2.5), writing $x_N = 1/\epsilon$ and taking $\epsilon \rightarrow 0$, we get

$$\lim_{x_N \rightarrow \infty} \left(x_N Z_1 \left(\{x\}_N \middle| \{y\}_N \right) \right) = \prod_{i=1}^{N-1} \prod_{j=1}^N (x_i - y_j) \Delta^{-1} \{x\}_{N-1} \Delta^{-1} \{-y\}_N \quad (3.5)$$

$$\times \lim_{\epsilon \rightarrow 0} \begin{vmatrix} \frac{1}{(x_1 - y_1)(x_1 - y_1 + 1)} & \cdots & \frac{1}{(x_1 - y_N)(x_1 - y_N + 1)} \\ \vdots & & \vdots \\ \frac{1}{(x_{N-1} - y_1)(x_{N-1} - y_1 + 1)} & \cdots & \frac{1}{(x_{N-1} - y_N)(x_{N-1} - y_N + 1)} \\ f_1^{(N)}(\epsilon) & \cdots & f_N^{(N)}(\epsilon) \end{vmatrix}$$

where for all $1 \leq i \leq N$ we have defined the function

$$f_j^{(i)}(\epsilon) = \frac{\prod_{k=1}^N (1 - y_k \epsilon)}{(1 - y_j \epsilon)(1 - \bar{y}_j \epsilon) \prod_{k=1}^{i-1} (1 - x_k \epsilon)} \quad (3.6)$$

and set $\bar{y}_j = y_j - 1$ for convenience. In the limit, every entry of the final row goes to 1, hence

$$\lim_{x_N \rightarrow \infty} \left(x_N Z_1 \left(\{x\}_N \middle| \{y\}_N \right) \right) = \prod_{i=1}^{N-1} \prod_{j=1}^N (x_i - y_j) \Delta^{-1} \{x\}_{N-1} \Delta^{-1} \{-y\}_N \quad (3.7)$$

$$\times \begin{vmatrix} \frac{1}{(x_1 - y_1)(x_1 - y_1 + 1)} & \cdots & \frac{1}{(x_1 - y_N)(x_1 - y_N + 1)} \\ \vdots & & \vdots \\ \frac{1}{(x_{N-1} - y_1)(x_{N-1} - y_1 + 1)} & \cdots & \frac{1}{(x_{N-1} - y_N)(x_{N-1} - y_N + 1)} \\ 1 & \cdots & 1 \end{vmatrix}$$

3.4 Limit of Izergin's determinant as $(N - n)$ rapidities become infinite

Lemma 1. *If $h_i(y_j, \bar{y}_j)$ is the i -th complete symmetric function in two variables $y_j, \bar{y}_j$, given by the generating series*

$$\sum_{i=0}^{\infty} h_i(y_j, \bar{y}_j) \epsilon^i = \frac{1}{(1 - y_j \epsilon)(1 - \bar{y}_j \epsilon)} \quad (3.8)$$

then

$$\lim_{x_N, \dots, x_{n+1} \rightarrow \infty} \left(x_{n+1} \cdots x_N Z_1 \left(\{x\}_N \middle| \{y\}_N \right) \right) = \prod_{i=1}^n \prod_{j=1}^N (x_i - y_j) \Delta^{-1} \{x\}_n \Delta^{-1} \{-y\}_N \begin{vmatrix} \frac{1}{(x_1 - y_1)(x_1 - y_1 + 1)} & \cdots & \frac{1}{(x_1 - y_N)(x_1 - y_N + 1)} \\ \vdots & & \vdots \\ \frac{1}{(x_n - y_1)(x_n - y_1 + 1)} & \cdots & \frac{1}{(x_n - y_N)(x_n - y_N + 1)} \\ h_{N-n-1}(y_1, \bar{y}_1) & \cdots & h_{N-n-1}(y_N, \bar{y}_N) \\ \vdots & & \vdots \\ h_0(y_1, \bar{y}_1) & \cdots & h_0(y_N, \bar{y}_N) \end{vmatrix} \quad (3.9)$$

Proof. Let $\mathcal{P}_{N-n}$ denote the proposition that equation (3.9) is true. Based on equation (3.7) for the one-rapidity case, we see that $\mathcal{P}_1$ is true. Let us assume that $\mathcal{P}_{N-n}$ is true and show that this implies $\mathcal{P}_{N-n+1}$. Multiplying equation (3.9) by x_n , making the change of variables $x_n = 1/\epsilon$ and taking the limit $\epsilon \rightarrow 0$ gives

$$\lim_{x_N, \dots, x_n \rightarrow \infty} \left(x_n \cdots x_N Z_1 \left(\{x\}_N \middle| \{y\}_N \right) \right) = \prod_{i=1}^{n-1} \prod_{j=1}^N (x_i - y_j) \Delta^{-1} \{x\}_{n-1} \Delta^{-1} \{-y\}_N \times \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon^{N-n}} \begin{vmatrix} \frac{1}{(x_1 - y_1)(x_1 - y_1 + 1)} & \cdots & \frac{1}{(x_1 - y_N)(x_1 - y_N + 1)} \\ \vdots & & \vdots \\ \frac{1}{(x_{n-1} - y_1)(x_{n-1} - y_1 + 1)} & \cdots & \frac{1}{(x_{n-1} - y_N)(x_{n-1} - y_N + 1)} \\ f_1^{(n)}(\epsilon) & \cdots & f_N^{(n)}(\epsilon) \\ h_{N-n-1}(y_1, \bar{y}_1) & \cdots & h_{N-n-1}(y_N, \bar{y}_N) \\ \vdots & & \vdots \\ h_0(y_1, \bar{y}_1) & \cdots & h_0(y_N, \bar{y}_N) \end{vmatrix} \quad (3.10)$$

Consider the functions $f_j^{(n)}(\epsilon)$ in the n -th row of the above determinant, which is defined in equation (3.6). Since ϵ is small, they can be expanded in powers of ϵ using the definition of the elementary and complete symmetric functions

$$f_j^{(n)}(\epsilon) = \sum_{k=0}^{\infty} \sum_{l=0}^k \sum_{m=0}^l (-)^{k-l} e_{k-l}(y_1, \dots, y_N) h_{l-m}(x_1, \dots, x_{n-1}) h_m(y_j, \bar{y}_j) \epsilon^k \quad (3.11)$$

where $e_k(y_1, \dots, y_N)$ and $h_k(x_1, \dots, x_{n-1})$ are elementary and complete symmetric functions, given respectively by the generating functions

$$\sum_{k=0}^{\infty} e_k(y_1, \dots, y_N) \epsilon^k = \prod_{k=1}^N (1 + y_k \epsilon), \quad \sum_{k=0}^{\infty} h_k(x_1, \dots, x_{n-1}) \epsilon^k = \prod_{k=1}^{n-1} \frac{1}{(1 - x_k \epsilon)} \quad (3.12)$$

Using the series expression in equation (3.11) for the row of entries $f_j^{(n)}(\epsilon)$, one can see that all terms in the first sum with $0 \leq k \leq (N - n - 1)$ give no contribution to the determinant

since they are linear combinations of the lower $(N - n)$ rows, and the first sum starts at $k = (N - n)$. Taking the limit $\epsilon \rightarrow 0$, all higher order terms in this sum vanish.

Studying the $k = (N - n)$ term in the series in equation (3.11), it is clear that many of its sub-terms do not contribute to the determinant either. In fact, only the sub-term corresponding to $l = m = (N - n)$ survives, and this is identically $h_{N-n}(y_j, \bar{y}_j)$. Therefore we obtain

$$\lim_{x_N, \dots, x_n \rightarrow \infty} \left(x_n \cdots x_N Z_1 \left(\{x\}_N \middle| \{y\}_N \right) \right) = \frac{\prod_{i=1}^{n-1} \prod_{j=1}^N (x_i - y_j)}{\Delta \{x\}_{n-1} \Delta \{-y\}_N} \begin{vmatrix} \frac{1}{(x_1 - y_1)(x_1 - y_1 + 1)} & \cdots & \frac{1}{(x_1 - y_N)(x_1 - y_N + 1)} \\ \vdots & & \vdots \\ \frac{1}{(x_{n-1} - y_1)(x_{n-1} - y_1 + 1)} & \cdots & \frac{1}{(x_{n-1} - y_N)(x_{n-1} - y_N + 1)} \\ h_{N-n}(y_1, \bar{y}_1) & \cdots & h_{N-n}(y_N, \bar{y}_N) \\ h_{N-n-1}(y_1, \bar{y}_1) & \cdots & h_{N-n-1}(y_N, \bar{y}_N) \\ \vdots & & \vdots \\ h_0(y_1, \bar{y}_1) & \cdots & h_0(y_N, \bar{y}_N) \end{vmatrix} \quad (3.13)$$

which proves $\mathcal{P}_{N-n+1}$. This completes the proof of equation (3.9) for all $0 \leq n \leq N - 1$, by induction. $\square$

3.5 A ‘partial Vandermonde’ way to write the determinant

A simple check shows that the highest order term in $h_{N-i}(y_j, \bar{y}_j)$ is $(N - i + 1)y_j^{N-i}$. Using row operations to cancel all terms of lower order and extracting an overall factor of $(N - n)!$ from the determinant in equation (3.9), we obtain

$$Z_1 \left(\{x\}_n \middle| \{y\}_N \right) = \frac{\prod_{i=1}^n \prod_{j=1}^N (x_i - y_j)}{\Delta \{x\}_n \Delta \{-y\}_N} \begin{vmatrix} \frac{1}{(x_1 - y_1)(x_1 - y_1 + 1)} & \cdots & \frac{1}{(x_1 - y_N)(x_1 - y_N + 1)} \\ \vdots & & \vdots \\ \frac{1}{(x_n - y_1)(x_n - y_1 + 1)} & \cdots & \frac{1}{(x_n - y_N)(x_n - y_N + 1)} \\ y_1^{N-n-1} & \cdots & y_N^{N-n-1} \\ \vdots & & \vdots \\ y_1^0 & \cdots & y_N^0 \end{vmatrix} \quad (3.14)$$

Equation (3.14) is our $(N \times N)$ determinant expression for the pDWPF. As previously mentioned, the top n rows are of Izergin determinant-type, whereas the lower $(N - n)$ rows are of Vandermonde determinant-type.

3.6 Towards the trigonometric pDWPF

Using the asymptotic behaviour of the trigonometric weights given in equation (2.4), we can repeat the procedure of subsection 3.2 to derive the relation

$$\begin{aligned} Z_1 \left(\{x\}_i \middle| \{y\}_N \right) &\sim (1 + \cdots + e^{-2\gamma(N-i)}) e^{-\gamma[\gamma]} Z_1 \left(\{x\}_{i-1} \middle| \{y\}_N \right) \\ &\sim (1 - e^{-2\gamma(N-i+1)}) Z_1 \left(\{x\}_{i-1} \middle| \{y\}_N \right), \quad \text{as } x_i \rightarrow \infty \end{aligned} \quad (3.15)$$

between trigonometric partial domain wall partition functions. Iterating this result through $i = \{N, \dots, n+1\}$, we obtain

$$Z_1 \left(\{x\}_n \middle| \{y\}_N \right) = \frac{e^{(N-n)\gamma}}{[\gamma]^{N-n} [N-n]_q!} \lim_{x_N, \dots, x_{n+1} \rightarrow \infty} Z_1 \left(\{x\}_N \middle| \{y\}_N \right) \quad (3.16)$$

where

$$q = e^{-2\gamma}, \quad [k]_q = \frac{1 - q^k}{1 - q}, \quad [j]_q! = \prod_{k=1}^j [k]_q \quad (3.17)$$

Equation (3.16) is the trigonometric analogue of equation (3.4).

3.7 The trigonometric pDWPF as an $(N \times N)$ -determinant

Starting from Izergin's trigonometric determinant, equation (2.6), and taking the limits in equation (3.16), it is straightforward to show that

$$Z_1 \left(\{x\}_n \middle| \{y\}_N \right) = \frac{\prod_{i=1}^n \prod_{j=1}^N [x_i - y_j]}{[\gamma]^n e^{(N-n+1)(|x|-|y|)} \Delta\{x\}_n \Delta\{-y\}_N} \begin{vmatrix} \frac{1}{[x_1 - y_1][x_1 - y_1 + \gamma]} & \cdots & \frac{1}{[x_1 - y_N][x_1 - y_N + \gamma]} \\ \vdots & & \vdots \\ \frac{1}{[x_n - y_1][x_n - y_1 + \gamma]} & \cdots & \frac{1}{[x_n - y_N][x_n - y_N + \gamma]} \\ e^{2y_1(N-n)} & \cdots & e^{2y_N(N-n)} \\ \vdots & & \vdots \\ e^{2y_1} & \cdots & e^{2y_N} \end{vmatrix} \quad (3.18)$$

We omit the details since they are similar to those in the rational case.

4 Slavnov scalar product in the infinite-rapidity limit

In this section we obtain an alternative expression for the pDWPF, by starting from Slavnov's formula for the scalar product (equation (2.8)) and taking appropriate limits. The resulting expression is an $(n \times n)$ determinant.

4.1 n rapidities become infinite

Using the six-vertex model representation of the scalar product, figure 4, it is possible to calculate the partial domain wall partition function in the alternative way

$$Z_2 \left(\{x\}_n \middle| \{y\}_N \right) = \frac{1}{n!} \lim_{b_n, \dots, b_1 \rightarrow \infty} \left(b_1 \cdots b_n S \left(\{x\}_n, \{b\}_n \middle| \{y\}_N \right) \right) \quad (4.1)$$

where the limits are sequentially, starting with b_n . The argument which underlies equation (4.1) is the same as the one that underlies equation (3.4).

4.2 The infinite rapidity limit of Slavnov's determinant

To obtain an alternative determinant expression for the pDWPF, we start from Slavnov's determinant in equation (2.8) and perform the limits specified in equation (4.1). We do this using induction, by proving the following result.

Lemma 2. *For all $0 \leq m \leq n-1$, we have*

$$\lim_{b_n, \dots, b_{m+1} \rightarrow \infty} \frac{b_{m+1} \dots b_n S(\{x\}_n, \{b\}_n | \{y\}_N)}{(n-m)!} = \frac{\prod_{i=1}^n \prod_{j=1}^m (b_j - x_i - 1)}{\Delta\{-x\}_n \Delta\{b\}_m} \quad (4.2)$$

$$\times \det \left(\underbrace{\frac{t_i^{(m)}}{(b_j - x_i)(b_j - x_i + 1)} - \frac{s_i}{(b_j - x_i)(b_j - x_i - 1)}}_{1 \leq j \leq m} \middle| \underbrace{\frac{x_i^{j-1} t_i^{(m)} - (x_i + 1)^{j-1} s_i}{n-m \geq j \geq 1}}_{1 \leq i \leq n} \right)$$

where

$$s_i = \prod_{k=1}^N \left(\frac{x_i - y_k}{x_i - y_k + 1} \right), \quad t_i^{(m)} = \prod_{k=1}^m \left(\frac{b_k - x_i + 1}{b_k - x_i - 1} \right) \quad (4.3)$$

Proof. Let equation (4.2) be a proposition, $\mathcal{P}_{n-m}$. We begin with the proof of $\mathcal{P}_1$. Using equation (2.8) for the scalar product, we have

$$\lim_{b_n \rightarrow \infty} \left(b_n S(\{x\}_n, \{b\}_n | \{y\}_N) \right) = \lim_{b_n \rightarrow \infty} \frac{b_n \prod_{i=1}^n \prod_{j=1}^n (b_j - x_i - 1)}{\Delta\{-x\}_n \Delta\{b\}_n} \quad (4.4)$$

$$\times \det \left(\underbrace{\frac{t_i^{(n)}}{(b_j - x_i)(b_j - x_i + 1)} - \frac{s_i}{(b_j - x_i)(b_j - x_i - 1)}}_{1 \leq j \leq n} \right)_{1 \leq i \leq n}$$

In the limit being considered, we have $t_i^{(n)} \rightarrow t_i^{(n-1)}$. The only other place where the determinant in equation (4.4) depends on b_n is in its final column, and we may absorb all terms in the prefactor of equation (4.4) which depend on b_n into the final column of the determinant, and take the limit easily. The result is

$$\lim_{b_n \rightarrow \infty} \left(b_n S(\{x\}_n, \{b\}_n | \{y\}_N) \right) = \frac{\prod_{i=1}^n \prod_{j=1}^{n-1} (b_j - x_i - 1)}{\Delta\{-x\}_n \Delta\{b\}_{n-1}} \quad (4.5)$$

$$\times \det \left(\underbrace{\frac{t_i^{(n-1)}}{(b_j - x_i)(b_j - x_i + 1)} - \frac{s_i}{(b_j - x_i)(b_j - x_i - 1)}}_{1 \leq j \leq n-1} \middle| \frac{t_i^{(n-1)} - s_i}{n-1} \right)_{1 \leq i \leq n}$$

which proves $\mathcal{P}_1$. Now we assume that $\mathcal{P}_{n-m}$ is true and show that this implies $\mathcal{P}_{n-m+1}$. Taking equation (4.2) as our starting point, we find that

$$\frac{\lim_{b_n, \dots, b_m \rightarrow \infty} \left(b_m \dots b_n S \left(\{x\}_n, \{b\}_n \middle| \{y\}_N \right) \right)}{(n-m+1)!} = \frac{\lim_{b_m \rightarrow \infty} b_m \prod_{i=1}^n \prod_{j=1}^m (b_j - x_i - 1)}{(n-m+1) \Delta\{-x\}_n \Delta\{b\}_m} \quad (4.6)$$

$$\times \det \left(\underbrace{\frac{t_i^{(m)}}{(b_j - x_i)(b_j - x_i + 1)} - \frac{s_i}{(b_j - x_i)(b_j - x_i - 1)}}_{1 \leq j \leq m} \middle| \underbrace{\frac{x_i^{j-1} t_i^{(m)} - (\bar{x}_i)^{j-1} s_i}{n-m \geq j \geq 1}}_{n-m \geq j \geq 1} \right)_{1 \leq i \leq n}$$

where we define as usual $\bar{x}_i = x_i + 1$. Let $1/b_m = \epsilon$, where ϵ is small in view of the limit being taken. Then we can write

$$\frac{b_m}{(b_m - x_i)(b_m - x_i + 1)} = \sum_{k=0}^{\infty} h_k(x_i, x_i - 1) \epsilon^{k+1} \quad (4.7)$$

$$\frac{b_m}{(b_m - x_i)(b_m - x_i - 1)} = \sum_{k=0}^{\infty} h_k(\bar{x}_i, \bar{x}_i - 1) \epsilon^{k+1} \quad (4.8)$$

These expressions can be substituted into the m -th column of the determinant in equation (4.6), thereby cancelling all terms in the sums in equations (4.7), (4.8) of degree $\leq (n-m)$ in ϵ , since these are linear combinations of the last $(n-m)$ columns of the determinant, and we obtain

$$\frac{\lim_{b_n, \dots, b_m \rightarrow \infty} \left(b_m \dots b_n S \left(\{x\}_n, \{b\}_n \middle| \{y\}_N \right) \right)}{(n-m+1)!} = \frac{\lim_{b_m \rightarrow \infty} \prod_{i=1}^n \prod_{j=1}^m (b_j - x_i - 1)}{(n-m+1) \Delta\{-x\}_n \Delta\{b\}_m} \times \quad (4.9)$$

$$\left| \underbrace{\dots}_{1 \leq j \leq m-1} \middle| \sum_{k=n-m}^{\infty} \left(h_k(x_i, x_i - 1) t_i^{(m)} - h_k(\bar{x}_i, \bar{x}_i - 1) s_i \right) b_m^{-k-1} \middle| \underbrace{\frac{x_i^{j-1} t_i^{(m)} - (\bar{x}_i)^{j-1} s_i}{n-m \geq j \geq 1}}_{n-m \geq j \geq 1} \right|$$

where we abbreviate $\det(\cdot)_{1 \leq i \leq n}$ by $|\cdot|$, and where the first $(m-1)$ columns are as before, so we do not write them. Now we can move all terms in the prefactor which depend on b_m inside the m -th column, and take the limit. This gives us

$$\frac{\lim_{b_n, \dots, b_m \rightarrow \infty} \left(b_m \dots b_n S \left(\{x\}_n, \{b\}_n \middle| \{y\}_N \right) \right)}{(n-m+1)!} = \frac{\prod_{i=1}^n \prod_{j=1}^{m-1} (b_j - x_i - 1)}{(n-m+1) \Delta\{-x\}_n \Delta\{b\}_{m-1}} \times \quad (4.10)$$

$$\left| \underbrace{\dots}_{1 \leq j \leq m-1} \middle| h_{n-m}(x_i, x_i - 1) t_i^{(m-1)} - h_{n-m}(\bar{x}_i, \bar{x}_i - 1) s_i \middle| \underbrace{\frac{x_i^{j-1} t_i^{(m-1)} - (\bar{x}_i)^{j-1} s_i}{n-m \geq j \geq 1}}_{n-m \geq j \geq 1} \right|$$

where we have again abbreviated $\det(\cdot)_{1 \leq i \leq n}$ by $|\cdot|$. Finally, we observe that the highest order terms in $h_{n-m}(x_i, x_i - 1)$ and $h_{n-m}(\bar{x}_i, \bar{x}_i - 1)$ are $(n-m+1)x_i^{n-m}$ and $(n-m+1)\bar{x}_i^{n-m}$ respectively, while all other terms cancel with the last $(n-m)$ columns of the determinant,

therefore

$$\lim_{b_n, \dots, b_m \rightarrow \infty} \frac{\left(b_m \dots b_n S \left(\{x\}_n, \{b\}_n \middle| \{y\}_N \right) \right)}{(n-m+1)!} = \frac{\prod_{i=1}^n \prod_{j=1}^{m-1} (b_j - x_i - 1)}{\Delta\{-x\}_n \Delta\{b\}_{m-1}} \quad (4.11)$$

$$\times \det \left(\underbrace{\frac{t_i^{(m-1)}}{(b_j - x_i)(b_j - x_i + 1)} - \frac{s_i}{(b_j - x_i)(b_j - x_i - 1)}}_{1 \leq j \leq m-1} \middle| \underbrace{x_i^{j-1} t_i^{(m-1)} - (\bar{x}_i)^{j-1} s_i}_{n-m+1 \geq j \geq 1} \right)_{1 \leq i \leq n}$$

thus $\mathcal{P}_{n-m+1}$ is true. This proves equation (4.2) for all $0 \leq m \leq n-1$, by induction. $\square$

4.3 Kostov's determinant

For $m = 0$, equation (4.2) leads to

$$\frac{1}{n!} \lim_{b_n, \dots, b_1 \rightarrow \infty} \left(b_1 \dots b_n S \left(\{x\}_n, \{b\}_n \middle| \{y\}_N \right) \right) = \Delta^{-1}\{-x\}_n \det \left(\underbrace{x_i^{j-1} - (x_i + 1)^{j-1} s_i}_{n \geq j \geq 1} \right)_{1 \leq i \leq n} \quad (4.12)$$

where we have used $t_i^{(0)} \equiv 1$. Combining equations (4.1) and (4.12), we see that the partial domain wall partition function can be written as

$$Z_2 \left(\{x\}_n \middle| \{y\}_N \right) = \Delta^{-1}\{x\}_n \det \left(x_i^{j-1} - (x_i + 1)^{j-1} \prod_{k=1}^N \frac{(x_i - y_k)}{(x_i - y_k + 1)} \right)_{1 \leq i, j \leq n} \quad (4.13)$$

where we have simultaneously reversed the order of variables in the Vandermonde and the order of columns in the determinant. In contrast to the determinant in equation (3.14), which is $(N \times N)$, the determinant in equation (4.13) is $(n \times n)$. equation (4.13) is due to I Kostov [17–19].

4.4 Writing $Z_2(\{x\}_n | \{y\}_N)$ as a sum over partitions

To conclude the section, we show that the determinant in equation (4.13) can be expanded as a certain sum, which is the precise form in which it appears in [14]. Let us define the functions

$$X_j(x_i) = x_i^{j-1}, \quad Y_j(x_i) = \bar{x}_i^{j-1} \prod_{k=1}^N \left(\frac{x_i - y_k}{\bar{x}_i - y_k} \right) \quad (4.14)$$

Then we have

$$Z_2 \left(\{x\}_n \middle| \{y\}_N \right) = \Delta^{-1}\{x\}_n \det \left(X_j(x_i) - Y_j(x_i) \right)_{1 \leq i, j \leq n} \quad (4.15)$$

Using Laplace's formula for the determinant of a sum of two matrices, we write the expression in equation (4.15) as a sum over all partitions of the integers $\{1, \dots, n\}$ into disjoint sets $\{\alpha\}_{n-m} = \{\alpha_1 < \dots < \alpha_{n-m}\}$, $\{\beta\}_m = \{\beta_1 < \dots < \beta_m\}$. The result is

$$Z_2\left(\{x\}_n \middle| \{y\}_N\right) = \Delta^{-1}\{x\}_n \sum_{\substack{\{1, \dots, n\} = \\ \{\alpha\}_{n-m} \cup \{\beta\}_m}} (-)^{\text{sgn}(P)+m} \begin{vmatrix} X_1(x_{\alpha_1}) & \cdots & X_n(x_{\alpha_1}) \\ \vdots & & \vdots \\ X_1(x_{\alpha_{n-m}}) & \cdots & X_n(x_{\alpha_{n-m}}) \\ Y_1(x_{\beta_1}) & \cdots & Y_n(x_{\beta_1}) \\ \vdots & & \vdots \\ Y_1(x_{\beta_m}) & \cdots & Y_n(x_{\beta_m}) \end{vmatrix} \quad (4.16)$$

where we have abbreviated $\prod_{1 \leq i < j \leq n} (x_j - x_i) = \Delta\{x\}_n$ and $\text{sgn}(P)$ denotes the sign of the permutation $P\{1, \dots, n\} = \{\alpha_1, \dots, \alpha_{n-m}, \beta_1, \dots, \beta_m\}$. It is possible to extract common factors from the determinant in the sum in equation (4.16), and write it as

$$\begin{vmatrix} X_1(x_{\alpha_1}) & \cdots & X_n(x_{\alpha_1}) \\ \vdots & & \vdots \\ X_1(x_{\alpha_{n-m}}) & \cdots & X_n(x_{\alpha_{n-m}}) \\ Y_1(x_{\beta_1}) & \cdots & Y_n(x_{\beta_1}) \\ \vdots & & \vdots \\ Y_1(x_{\beta_m}) & \cdots & Y_n(x_{\beta_m}) \end{vmatrix} = \prod_{i=1}^m \prod_{k=1}^N \left(\frac{x_{\beta_i} - y_k}{\bar{x}_{\beta_i} - y_k} \right) \begin{vmatrix} x_{\alpha_1}^0 & \cdots & x_{\alpha_1}^{n-1} \\ \vdots & & \vdots \\ x_{\alpha_{n-m}}^0 & \cdots & x_{\alpha_{n-m}}^{n-1} \\ \bar{x}_{\beta_1}^0 & \cdots & \bar{x}_{\beta_1}^{n-1} \\ \vdots & & \vdots \\ \bar{x}_{\beta_m}^0 & \cdots & \bar{x}_{\beta_m}^{n-1} \end{vmatrix} \\ = \prod_{i=1}^m \prod_{k=1}^N \frac{(x_{\beta_i} - y_k)}{(\bar{x}_{\beta_i} - y_k)} \Delta\{x_{\alpha}\}_{n-m} \prod_{i=1}^{n-m} \prod_{j=1}^m (\bar{x}_{\beta_j} - x_{\alpha_i}) \Delta\{\bar{x}_{\beta}\}_m$$

where $\prod_{1 \leq i < j \leq n-m} (x_{\alpha_j} - x_{\alpha_i}) = \Delta\{x_{\alpha}\}_{n-m}$ and $\prod_{1 \leq i < j \leq m} (\bar{x}_{\beta_j} - \bar{x}_{\beta_i}) = \Delta\{\bar{x}_{\beta}\}_m = \Delta\{x_{\beta}\}_m$. Putting this expression back into equation (4.16), we get

$$Z_2\left(\{x\}_n \middle| \{y\}_N\right) = \sum_{\{1, \dots, n\} = \{\alpha\}_{n-m} \cup \{\beta\}_m} (-)^m \prod_{i=1}^m \prod_{k=1}^N \left(\frac{x_{\beta_i} - y_k}{\bar{x}_{\beta_i} - y_k} \right) \left(\prod_{i=1}^{n-m} \prod_{j=1}^m \frac{x_{\alpha_i} - \bar{x}_{\beta_j}}{x_{\alpha_i} - x_{\beta_j}} \right) \quad (4.17)$$

Up to simple changes in variables, this is the sum expression in [14].

4.5 The trigonometric version of Kostov's determinant

For completeness, we give the trigonometric version of the pDWPF obtained taking limits of Slavnov's scalar product in equation (2.9). The starting point in the calculation is the relation

$$Z_2\left(\{x\}_n \middle| \{y\}_N\right) = \frac{e^{n\gamma}}{[\gamma]^n [n]_q!} \lim_{b_n, \dots, b_1 \rightarrow \infty} S\left(\{x\}_n, \{b\}_n \middle| \{y\}_N\right) \quad (4.18)$$

which arises from the lattice version of the scalar product in figure 4 and that of the pDWPF on the right of figure 5, and the asymptotic behaviour of the weights in equation (2.4).

Repeating the ideas already developed in this section and working from Slavnov's determinant in equation (2.9), equation (4.18) ultimately leads us to the expression

$$Z_2 \left(\{x\}_n \middle| \{y\}_N \right) = e^{-(n-1)|x|} \Delta^{-1} \{x\}_n \times \det \left[e^{2x_i(j-1)} \left(1 - e^{N\gamma} \prod_{k=1}^N \frac{[x_i - y_k]}{[x_i - y_k + \gamma]} e^{-2\gamma(n-j)} \right) \right]_{1 \leq i, j \leq n} \quad (4.19)$$

for the trigonometric pDWPF.

5 Equivalence of determinants

In this section we show directly that the determinants in equations (3.14) and (4.13) are equal. We do this for completeness and as a check of our limit calculations, because on the surface it is not apparent that the two expressions coincide. We restrict our attention to the determinants obtained from the rational six-vertex model, because as we have already mentioned the two lattice sums in figure 5 are *not* equivalent in the trigonometric parametrization.

Our approach is to convert both equations (3.14) and (4.13) to polynomials, by multiplying them by an overall factor (precisely the factor present in the denominator of the b and c weights). We distinguish the resulting expressions by calling them $Z_{N \times N}(\{x\}_n | \{y\}_N)$ and $Z_{n \times n}(\{x\}_n | \{y\}_N)$, in reference to the size of the determinants in question, and show that both $Z_{N \times N}(\{x\}_n | \{y\}_N)$ and $Z_{n \times n}(\{x\}_n | \{y\}_N)$ satisfy a list of conditions which the pDWPF must itself obey. This proves that they are equal, since the conditions only admit a unique solution.

5.1 Set of properties which characterize the pDWPF

Let $Z(\{x\}_n | \{y\}_N)$ be the partition function of either of the lattices in figure 5, but whose weights are given by

$$a_{\pm}(x, y) = x - y + 1, \quad b_{\pm}(x, y) = x - y, \quad c_{\pm}(x, y) = 1 \quad (5.1)$$

This polynomial version of $Z(\{x\}_n | \{y\}_N)$ is obtained from the rational version by multiplying by $\prod_{i=1}^n \prod_{j=1}^N (x_i - y_j + 1)$. Following Korepin [4], one can show that

- A. $Z(\{x\}_n | \{y\}_N)$ is a polynomial in x_n of degree bounded by $(2N - 1)$.
- B. $Z(\{x\}_n | \{y\}_N)$ is symmetric in the set of variables $\{y\}_N = \{y_1, \dots, y_N\}$.
- C. For all $n \geq 2$, $Z(\{x\}_n | \{y\}_N)$ satisfies

$$Z \left(\{x\}_n \middle| \{y\}_N \right) \Big|_{x_n = y_N} = \prod_{k=1}^{N-1} (y_N - y_k + 1) \prod_{k=1}^{n-1} (x_k - \bar{y}_N) Z \left(\{x\}_{n-1} \middle| \{y\}_{N-1} \right) \quad (5.2)$$

$$Z \left(\{x\}_n \middle| \{y\}_N \right) \Big|_{x_n = \bar{y}_N} = \prod_{k=1}^{N-1} (y_N - y_k - 1) \prod_{k=1}^{n-1} (x_k - y_N) Z \left(\{x\}_{n-1} \middle| \{y\}_{N-1} \right) \quad (5.3)$$

where $\bar{y}_N = y_N - 1$.

D. $Z(x_1|\{y\}_N)$ is known explicitly for all $N \geq 1$, and is given by

$$Z(x_1|\{y\}_N) = \sum_{l=1}^N \prod_{1 \leq k < l} (x_1 - y_k) \prod_{l < k \leq N} (x_1 - y_k + 1) = \prod_{k=1}^N (x_1 - y_k + 1) - \prod_{k=1}^N (x_1 - y_k) \quad (5.4)$$

The second equality in equation (5.4) evaluates the sum over l , and can be established by induction on N .

These four properties uniquely determine the functions $Z(\{x\}_n|\{y\}_N)$, for all $1 \leq n \leq N$. This is because, from **A**, $Z(\{x\}_n|\{y\}_N)$ is a polynomial in x_n , and from **B** and **C**, it is known at more points than its degree.

5.2 $Z_{N \times N}(\{x\}_n|\{y\}_N)$ satisfies the four properties

Consider the polynomial version of the pDWPF in equation (3.14), obtained by multiplying by $\prod_{i=1}^n \prod_{j=1}^N (x_i - y_j + 1)$. Let us denote this by $Z_{N \times N}(\{x\}_n|\{y\}_N)$. Using $\bar{y} = y - 1$, it is given by

$$Z_{N \times N}(\{x\}_n|\{y\}_N) \equiv \mathcal{P}(\{x\}_n, \{y\}_N) \det \mathcal{M}(\{x\}_n, \{y\}_N) \\ = \frac{\prod_{i=1}^n \prod_{j=1}^N (x_i - y_j)(x_i - \bar{y}_j)}{\Delta\{x\}_n \Delta\{-y\}_N} \begin{vmatrix} \frac{1}{(x_1 - y_1)(x_1 - \bar{y}_1)} & \cdots & \frac{1}{(x_1 - y_N)(x_1 - \bar{y}_N)} \\ \vdots & & \vdots \\ \frac{1}{(x_n - y_1)(x_n - \bar{y}_1)} & \cdots & \frac{1}{(x_n - y_N)(x_n - \bar{y}_N)} \\ y_1^{N-n-1} & \cdots & y_N^{N-n-1} \\ \vdots & & \vdots \\ y_1^0 & \cdots & y_N^0 \end{vmatrix} \quad (5.5)$$

Due to the cancellation of the Vandermonde $\prod_{1 \leq i < j \leq n} (x_j - x_i)$ with trivial zeros of the determinant, $Z_{N \times N}(\{x\}_n|\{y\}_N)$ is a polynomial in x_n and the highest degree it can obtain in this variable is $(2N - n - 1)$. Therefore property **A** is satisfied. Interchanging the variables y_i and y_j leaves $Z_{N \times N}(\{x\}_n|\{y\}_N)$ invariant, therefore property **B** is satisfied. Setting $x_n = \{y_N, \bar{y}_N\}$, we see that only the minor

$$\det \mathcal{M}(\{x\}_{n-1}, \{y\}_{N-1}) = \begin{vmatrix} \frac{1}{(x_1 - y_1)(x_1 - \bar{y}_1)} & \cdots & \frac{1}{(x_1 - y_{N-1})(x_1 - \bar{y}_{N-1})} \\ \vdots & & \vdots \\ \frac{1}{(x_{n-1} - y_1)(x_{n-1} - \bar{y}_1)} & \cdots & \frac{1}{(x_{n-1} - y_{N-1})(x_{n-1} - \bar{y}_{N-1})} \\ y_1^{N-n-1} & \cdots & y_{N-1}^{N-n-1} \\ \vdots & & \vdots \\ y_1^0 & \cdots & y_{N-1}^0 \end{vmatrix} \quad (5.6)$$

survives in the Laplace expansion of $\det \mathcal{M}(\{x\}_n, \{y\}_N)$ down the right-most column. The prefactor $\mathcal{P}(\{x\}_n, \{y\}_N)$ in equation (5.5) satisfies

$$\frac{(-)^{n+N} \mathcal{P}(\{x\}_n, \{y\}_N)}{(x_n - y_N)(x_n - \bar{y}_N)} = \begin{cases} \prod_{i=1}^{n-1} (x_i - \bar{y}_N) \prod_{j=1}^{N-1} (y_N - y_j + 1) \mathcal{P}(\{x\}_{n-1}, \{y\}_{N-1}), & x_n = y_N \\ \prod_{i=1}^{n-1} (x_i - y_N) \prod_{j=1}^{N-1} (y_N - y_j - 1) \mathcal{P}(\{x\}_{n-1}, \{y\}_{N-1}), & x_n = \bar{y}_N \end{cases} \quad (5.7)$$

Combining results, we see that property **C** is satisfied. Finally, when $n = 1$, we have

$$Z_{N \times N} \left(x_1 \middle| \{y\}_N \right) = \Delta^{-1} \{-y\}_N \prod_{j=1}^N (x_1 - y_j)(x_1 - \bar{y}_j) \begin{vmatrix} \frac{1}{(x_1 - y_1)(x_1 - \bar{y}_1)} & \cdots & \frac{1}{(x_1 - y_N)(x_1 - \bar{y}_N)} \\ y_1^{N-2} & \cdots & y_N^{N-2} \\ \vdots & & \vdots \\ y_1^0 & \cdots & y_N^0 \end{vmatrix} \quad (5.8)$$

Laplace expanding this determinant along the first row and using the Vandermonde determinant identity, we obtain

$$Z_{N \times N} \left(x_1 \middle| \{y\}_N \right) = \sum_{j=1}^N \prod_{k \neq j}^N \frac{(x_1 - y_k)(x_1 - \bar{y}_k)}{(y_j - y_k)} \quad (5.9)$$

Comparing this polynomial in x_1 with the polynomial in equation (5.4) at the points $x_1 = \{y_1, \bar{y}_1, \dots, y_N, \bar{y}_N\}$, we find that they are equal, and property **D** is satisfied.

5.3 $Z_{n \times n}(\{x\}_n | \{y\}_N)$ satisfies the four properties

Consider the polynomial version of the pDWPF in equation (4.13), obtained by multiplying the expression in that equation by $\prod_{i=1}^n \prod_{j=1}^N (x_i - y_j + 1)$. We denote this by $Z_{n \times n}(\{x\}_n | \{y\}_N)$. Using $\bar{x} = x + 1$, $\bar{y} = y - 1$, it is given by

$$\begin{aligned} Z_{n \times n} \left(\{x\}_n \middle| \{y\}_N \right) &= \Delta^{-1} \{x\}_n \det \left(x_i^{j-1} \prod_{k=1}^N (x_i - \bar{y}_k) - (\bar{x}_i)^{j-1} \prod_{k=1}^N (x_i - y_k) \right)_{1 \leq i, j \leq n} \\ &\equiv \Delta^{-1} \{x\}_n \det \left(\mathcal{M}_j(x_i, \{y\}_N) \right)_{1 \leq i, j \leq n} \end{aligned} \quad (5.10)$$

Clearly $Z_{n \times n}(\{x\}_n | \{y\}_N)$ is a polynomial in x_n , and the highest possible degree it can obtain in this variable is N . So property **A** is satisfied. Since all $\{y\}_N$ dependence is in the products $\prod_{k=1}^N (x_i - \bar{y}_k)$ and $\prod_{k=1}^N (x_i - y_k)$, property **B** is satisfied. Setting $x_n = \{y_N, \bar{y}_N\}$, the entries of the final row of the determinant in equation (5.10) become

$$\mathcal{M}_j \left(x_n, \{y\}_N \right) = \begin{cases} y_N^{j-1} \prod_{k=1}^{N-1} (y_N - y_k + 1), & x_n = y_N \\ y_N^{j-1} \prod_{k=1}^{N-1} (y_N - y_k - 1), & x_n = \bar{y}_N \end{cases} \quad (5.11)$$

Rearranging the entries of the top $(n-1)$ rows to write them as

$$\mathcal{M}_j(x_i, \{y\}_N) = -\frac{(x_i - y_N)(x_i - \bar{y}_N)}{y_N} \left(\frac{x_i^{j-1} \prod_{k=1}^{N-1} (x_i - \bar{y}_k)}{(1 - x_i/y_N)} - \frac{(\bar{x}_i)^{j-1} \prod_{k=1}^{N-1} (x_i - y_k)}{(1 - \bar{x}_i/y_N)} \right) \quad (5.12)$$

Extracting factors which are common to each row of the determinant, we obtain

$$Z_{n \times n}(\{x\}_n | \{y\}_N) = \Delta^{-1} \{x\}_{n-1} \det \begin{pmatrix} \mathcal{N}_j(x_i, \{y\}_N) \\ \mathcal{N}_j(y_N) \end{pmatrix}_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \times \begin{cases} \prod_{i=1}^{n-1} (x_i - \bar{y}_N) \prod_{k=1}^{N-1} (y_N - y_k + 1), & x_n = y_N \\ \prod_{i=1}^{n-1} (x_i - y_N) \prod_{k=1}^{N-1} (y_N - y_k - 1), & x_n = \bar{y}_N \end{cases} \quad (5.13)$$

where we have defined the matrix entries

$$\mathcal{N}_j(x_i, \{y\}_N) = \frac{x_i^{j-1} \prod_{k=1}^{N-1} (x_i - \bar{y}_k)}{(1 - x_i/y_N)} - \frac{(\bar{x}_i)^{j-1} \prod_{k=1}^{N-1} (x_i - y_k)}{(1 - \bar{x}_i/y_N)}, \quad \mathcal{N}_j(y_N) = y_N^{j-n} \quad (5.14)$$

Subtracting $(\text{column } j+1)/y_N$ from $(\text{column } j)$ for all $1 \leq j < n$, it is easy to show that

$$\begin{aligned} \det \begin{pmatrix} \mathcal{N}_j(x_i, \{y\}_N) \\ \mathcal{N}_j(y_N) \end{pmatrix}_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} &= \det \begin{pmatrix} \mathcal{M}_j(x_i, \{y\}_{N-1}) & \mathcal{N}_n(x_i, \{y\}_N) \\ 0 & 1 \end{pmatrix}_{1 \leq i, j \leq n-1} \\ &= \det \left(\mathcal{M}_j(x_i, \{y\}_{N-1}) \right)_{1 \leq i, j \leq n-1} \end{aligned} \quad (5.15)$$

Substituting this into equation (5.13), we have verified that property **C** is satisfied. Finally, when $n = 1$, observe that $Z_{n \times n}(x_1 | \{y\}_N)$ is identically equal to the right hand side of equation (5.4), so property **D** is satisfied.

5.4 $Z_{N \times N}(\{x\}_n | \{y\}_N)$ and $Z_{n \times n}(\{x\}_n | \{y\}_N)$ are equal

Since $Z_{N \times N}(\{x\}_n | \{y\}_N)$ and $Z_{n \times n}(\{x\}_n | \{y\}_N)$ satisfy the properties **A–D**, which admit a unique solution, we have proved that

$$Z_{N \times N}(\{x\}_n | \{y\}_N) = Z_{n \times n}(\{x\}_n | \{y\}_N), \quad \text{for all } 1 \leq n \leq N \quad (5.16)$$

6 Casorati determinants and discrete KP hierarchy

6.1 Notation related to sets of variables

In this section we use $\{x\}$ for a set of finitely many variables, and $\{\hat{x}_m\}$ for $\{x\}$ with the element x_m omitted. If a variable x_i is repeated m_i times, we use the superscript (m_i) to indicate the multiplicity of x_i . For example, $\{x_1^{(1)}, x_2^{(3)}, x_3^{(2)}, x_4^{(1)}, \dots\}$ is the same as $\{x_1, x_2, x_2, x_2, x_3, x_3, x_4, \dots\}$ and $f\{x_i^{(m_i)}\}$ indicates that f depends on m_i distinct variables all of which are set to the same value x_i . Often we write x_i instead of $x_i^{(1)}$.

6.2 The complete symmetric function $h_i\{x\}$

Let $\{x\}$ denote the set $\{x_1, \dots, x_N\}$. The complete symmetric function $h_i\{x\}$ is the coefficient of k^i in the power series expansion

$$\prod_{i=1}^N \frac{1}{1 - x_i k} = \sum_{i=0}^{\infty} h_i\{x\} k^i \quad (6.1)$$

For example, $h_0\{x\} = 1$, $h_1(x_1, x_2) = x_1 + x_2$, $h_2(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3$. By definition, $h_i\{x\} = 0$ for $i < 0$.

6.3 Useful identities for $h_i\{x\}$

From equation (6.1), it follows that

$$h_i\{x\} = h_i\{\widehat{x}_m\} + x_m h_{i-1}\{x\} \quad (6.2)$$

From equation (6.2), one obtains

$$(x_m - x_n) h_{i-1}\{x\} = h_i\{\widehat{x}_n\} - h_i\{\widehat{x}_m\} \quad (6.3)$$

$$(x_m - x_n) h_i\{x\} = x_m h_i\{\widehat{x}_n\} - x_n h_i\{\widehat{x}_m\} \quad (6.4)$$

6.4 Discrete derivatives

The discrete derivative $\Delta_m h_i\{x\}$ of $h_i\{x\}$ with respect to $x_m \in \{x\}$ is defined using equation (6.2) as

$$\Delta_m h_i\{x\} = \frac{h_i\{x\} - h_i\{\widehat{x}_m\}}{x_m} = h_{i-1}\{x\} \quad (6.5)$$

Note that by applying Δ_m to a degree i complete symmetric function, $h_i\{x\}$, one obtains a complete symmetric function $h_{i-1}\{x\}$ of degree $i - 1$, in the same set of variables $\{x\}$.

6.5 The discrete KP hierarchy

Discrete KP is an infinite hierarchy of integrable partial *difference* equations in an infinite set of continuous Miwa variables $\{x_1, x_2, \dots\}$ with multiplicities $\{m_1, m_2, \dots\}$. Time evolution is obtained by changing the multiplicities of the Miwa variables. In this work, we take the number of non-zero Miwa variables to be finite, and set all continuous Miwa variables apart from $\{x_1, \dots, x_N\}$ to zero. In this case, the discrete KP hierarchy can be written in bilinear form as the $(n \times n)$ determinant equations

$$\det \begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-2} & x_1^{n-2} \tau_{+1}\{x\} \tau_{-1}\{x\} \\ 1 & x_2 & \cdots & x_2^{n-2} & x_2^{n-2} \tau_{+2}\{x\} \tau_{-2}\{x\} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & \cdots & x_n^{n-2} & x_n^{n-2} \tau_{+n}\{x\} \tau_{-n}\{x\} \end{pmatrix} = 0 \quad (6.6)$$

where $3 \leq n \leq N$, and

$$\begin{aligned} \tau_{+i}\{x\} &= \tau\{x_1^{(m_1)}, \dots, x_i^{(m_i+1)}, \dots, x_N^{(m_N)}\} \\ \tau_{-i}\{x\} &= \tau\{x_1^{(m_1+1)}, \dots, x_i^{(m_i)}, \dots, x_N^{(m_N+1)}\} \end{aligned} \quad (6.7)$$

In words, if $\tau\{x\}$ has m_i copies of the variable x_i , then $\tau_{+i}\{x\}$ has $(m_i + 1)$ copies of x_i and the multiplicities of all other variables remain the same, while $\tau_{-i}\{x\}$ has one more copy of each variable except x_i . In the simpler notation

$$\begin{aligned}\tau_{+i}\{x\} &= \tau\{m_1, \dots, (m_i + 1), \dots, m_N\} \\ \tau_{-i}\{x\} &= \tau\{(m_1 + 1), \dots, m_i, \dots, (m_N + 1)\}\end{aligned}\tag{6.8}$$

the simplest discrete KP bilinear difference equation is

$$\begin{aligned}& x_i(x_j - x_k)\tau\{m_i + 1, m_j, m_k\}\tau\{m_i, m_j + 1, m_k + 1\} \\ & + x_j(x_k - x_i)\tau\{m_i, m_j + 1, m_k\}\tau\{m_i + 1, m_j, m_k + 1\} \\ & + x_k(x_i - x_j)\tau\{m_i, m_j, m_k + 1\}\tau\{m_i + 1, m_j + 1, m_k\} = 0\end{aligned}\tag{6.9}$$

where $\{x_i, x_j, x_k\} \in \{x\}$ and $\{m_i, m_j, m_k\} \in \{m\}$ are any three continuous Miwa variables and their corresponding multiplicities.

6.6 Casoratian matrices and determinants

Ω is a Casoratian matrix if and only if its matrix elements ω_{ij} satisfy

$$\omega_{i,j+1}\{x\} = \Delta_m \omega_{ij}\{x\}\tag{6.10}$$

where Δ_m is the discrete derivative with respect to any variable $x_m \in \{x\}$. It is redundant to choose a specific variable x_m , since $\omega_{ij}\{x\}$ is symmetric in $\{x\}$.

From the definition of Δ_m , the elements ω_{ij} of Casoratian matrices satisfy

$$\omega_{ij}\{x_1, \dots, x_m^{(2)}, \dots, x_N\} = \omega_{ij}\{x_1, \dots, x_N\} + x_m \omega_{i,j+1}\{x_1, \dots, x_m^{(2)}, \dots, x_N\}\tag{6.11}$$

which gives the identity

$$\begin{aligned}& (x_r - x_s) \omega_{ij}\{x_1, \dots, x_r^{(2)}, \dots, x_s^{(2)}, \dots, x_N\} = \\ & x_r \omega_{ij}\{x_1, \dots, x_r^{(2)}, \dots, x_N\} - x_s \omega_{ij}\{x_1, \dots, x_s^{(2)}, \dots, x_N\}\end{aligned}\tag{6.12}$$

If Ω is a Casoratian matrix, then $\det \Omega$ is a Casoratian determinant. Casoratian determinants are discrete analogues of Wronskian determinants.

6.7 Notation for column vectors and determinants

We introduce the column vector notation

$$\vec{\omega}_j = \begin{pmatrix} \omega_{1j}\{x_1^{(m_1)}, \dots, x_N^{(m_N)}\} \\ \omega_{2j}\{x_1^{(m_1)}, \dots, x_N^{(m_N)}\} \\ \vdots \\ \omega_{Nj}\{x_1^{(m_1)}, \dots, x_N^{(m_N)}\} \end{pmatrix}\tag{6.13}$$

and

$$\vec{\omega}_j^{[k_1, \dots, k_n]} = \begin{pmatrix} \omega_{1j} \{x_1^{(m_1)}, \dots, x_{k_1}^{(m_{k_1}+1)}, \dots, x_{k_n}^{(m_{k_n}+1)}, \dots, x_N^{(m_N)}\} \\ \omega_{2j} \{x_1^{(m_1)}, \dots, x_{k_1}^{(m_{k_1}+1)}, \dots, x_{k_n}^{(m_{k_n}+1)}, \dots, x_N^{(m_N)}\} \\ \vdots \\ \omega_{Nj} \{x_1^{(m_1)}, \dots, x_{k_1}^{(m_{k_1}+1)}, \dots, x_{k_n}^{(m_{k_n}+1)}, \dots, x_N^{(m_N)}\} \end{pmatrix} \quad (6.14)$$

for the corresponding column vector where the multiplicity of the subset of variables $x_{k_1}, \dots, x_{k_n}$ is increased by 1. We introduce the determinant notation

$$\tau = \det \begin{pmatrix} \vec{\omega}_1 & \vec{\omega}_2 & \cdots & \vec{\omega}_N \end{pmatrix} = | \vec{\omega}_1 \ \vec{\omega}_2 \ \cdots \ \vec{\omega}_N | \quad (6.15)$$

and

$$\tau^{[k_1, \dots, k_n]} = | \vec{\omega}_1^{[k_1, \dots, k_n]} \ \vec{\omega}_2^{[k_1, \dots, k_n]} \ \cdots \ \vec{\omega}_N^{[k_1, \dots, k_n]} | \quad (6.16)$$

for the determinant with shifted multiplicities.

6.8 Identities for Casoratian determinants

Following [24], equations (6.11) and (6.12) can be used to perform column operations in the determinant expressions for $\tau^{[1]}$ and $\tau^{[1, \dots, n]}$, to obtain the two identities

$$x_1^{n-2} \tau^{[1]} = | \vec{\omega}_1 \ \vec{\omega}_2 \ \cdots \ \vec{\omega}_{N-1} \ \vec{\omega}_{N-n+2}^{[1]} | \quad (6.17)$$

$$\prod_{1 \leq r < s \leq n} (x_r - x_s) \tau^{[1, \dots, n]} = | \vec{\omega}_1 \ \cdots \ \vec{\omega}_{N-n} \ \vec{\omega}_{N-n+1}^{[n]} \ \vec{\omega}_{N-n+1}^{[n-1]} \ \cdots \ \vec{\omega}_{N-n+1}^{[1]} | \quad (6.18)$$

6.9 Casoratian determinants are discrete KP τ -functions

Following [25], consider the $(2N \times 2N)$ determinant

$$\det \begin{pmatrix} \vec{\omega}_1 & \cdots & \vec{\omega}_{N-1} & \vec{\omega}_{N-n+2}^{[1]} & 0_1 & \cdots & 0_{N-n+1} & \vec{\omega}_{N-n+2}^{[n]} & \cdots & \vec{\omega}_{N-n+2}^{[2]} \\ 0_1 & \cdots & 0_{N-1} & \vec{\omega}_{N-n+2}^{[1]} & \vec{\omega}_1 & \cdots & \vec{\omega}_{N-n+1} & \vec{\omega}_{N-n+2}^{[n]} & \cdots & \vec{\omega}_{N-n+2}^{[2]} \end{pmatrix} = 0 \quad (6.19)$$

which is identically zero. For notational clarity, we have used subscripts to label the position of columns of zeros. Laplace expanding the left hand side of equation (6.19) in $(N \times N)$ minors along the top $(N \times 2N)$ block, we obtain

$$\sum_{k=1}^n (-)^{k-1} | \vec{\omega}_1 \ \cdots \ \vec{\omega}_{N-1} \ \vec{\omega}_{N-n+2}^{[k]} | \times | \vec{\omega}_1 \ \cdots \ \vec{\omega}_{N-n+1} \ \vec{\omega}_{N-n+2}^{[n]} \ \cdots \ \vec{\omega}_{N-n+2}^{[k+1]} \ \vec{\omega}_{N-n+2}^{[k-1]} \ \cdots \ \vec{\omega}_{N-n+2}^{[1]} | = 0 \quad (6.20)$$

From equations (6.17) and (6.18), equation (6.20) can be written as

$$\sum_{k=1}^n (-)^{k-1} x_k^{n-2} \tau^{[k]} \prod_{\substack{1 \leq r < s \leq n \\ r, s \neq k}} (x_r - x_s) \tau^{[1, \dots, \hat{k}, \dots, n]} = 0 \quad (6.21)$$

Using the Vandermonde determinant identity

$$\det \begin{pmatrix} 1 & x_1 & \cdots & x_1^{n-2} \\ \vdots & \vdots & & \vdots \\ \langle 1 & x_k & \cdots & x_k^{n-2} \rangle \\ \vdots & \vdots & & \vdots \\ 1 & x_n & \cdots & x_n^{n-2} \end{pmatrix} = \prod_{\substack{1 \leq r < s \leq n \\ r, s \neq k}} (x_r - x_s) \quad (6.22)$$

with $\langle 1 \ x_k \ \cdots \ x_k^{n-2} \rangle$ denoting the omission of the k -th row of the matrix, we see that equation (6.21) is the cofactor expansion of the determinant in equation (6.6) along its last column. Hence we conclude that Casoratian determinants satisfy the bilinear difference equations of discrete KP.

6.10 $Z_{N \times N}(\{x\}_n | \{y\}_N)$ is a discrete KP τ -function in $\{y\}_N$

In this subsection we show that $Z_{N \times N}(\{x\}_n | \{y\}_N)$, in equation (5.5), is a Casoratian determinant. The discrete derivatives are taken with respect to any of the variables y_j . From the above discussion, this is sufficient to show that $Z_{N \times N}(\{x\}_n | \{y\}_N)$ is a τ -function of discrete KP in $\{y\}_N$.

The first step is to rearrange equation (5.5) by bringing the numerator of the prefactor $\mathcal{P}(\{x\}_n, \{y\}_N)$ inside the determinant. We do this by multiplying the j -th column of the determinant by $\prod_{k=1}^n (x_k - y_j)(x_k - \bar{y}_j)$, for all $1 \leq j \leq N$. The j -th column of the resulting determinant has entries which are polynomial in y_j . After a routine calculation, we obtain

$$Z_{N \times N}(\{x\}_n | \{y\}_N) = \Delta^{-1}\{x\}_n \Delta^{-1}\{-y\}_N \det \left(\sum_{k=1}^{N+n} c_{ik}\{x\} y_j^{k-1} \right)_{1 \leq i, j \leq N} \quad (6.23)$$

where the coefficients $c_{ik}\{x\}$ depend on the row of the matrix and are given by

$$c_{ik}\{x\} = \begin{cases} e_{2n-k-1} \left(\{-x, -\bar{x}\} \setminus \{-x_i, -\bar{x}_i\} \right), & 1 \leq i \leq n \\ e_{2n-k+N-i+1} \{-x, -\bar{x}\}, & n+1 \leq i \leq N \end{cases} \quad (6.24)$$

It remains to take the Vandermonde $\Delta\{-y\}_N$ inside the determinant of equation (6.23). This is essentially the same as proving the Jacobi-Trudi identity for Schur functions, see [26]. The final result is

$$Z_{N \times N}(\{x\}_n | \{y\}_N) = \Delta^{-1}\{x\}_n \det \left(\sum_{k=1}^{N+n} c_{ik}\{x\} h_{k-j}\{y\} \right)_{1 \leq i, j \leq N} \quad (6.25)$$

Up to the Vandermonde factor in the denominator, which is a constant in $\{y\}_N$, this is clearly a Casoratian determinant.

6.11 $Z_{n \times n}(\{x\}_n | \{y\}_N)$ is a discrete KP τ -function in $\{x\}_n$

We can repeat the above procedure to write $Z_{n \times n}(\{x\}_n | \{y\}_N)$ as a Casoratian determinant, whose discrete derivatives are with respect to any of the variables x_i . Starting from

equation (5.10), we already have $Z_{n \times n}(\{x\}_n | \{y\}_N)$ as a determinant whose i -th row entries are polynomials in x_i . Expanding these polynomials in powers of x_i , we obtain

$$Z_{n \times n}(\{x\}_n | \{y\}_N) = \Delta^{-1}\{x\}_n \det \left(\sum_{k=1}^{N+n} x_i^{k-1} d_{kj}\{y\} \right)_{1 \leq i, j \leq n} \quad (6.26)$$

where the coefficients $d_{kj}\{y\}$ depend on the column of the matrix and are given by

$$d_{kj}\{y\} = \sum_{l=0}^{N+j-k} \left[\binom{N-l}{k-j} - \binom{j-1}{k-N+l-1} \right] e_l\{-y\} \quad (6.27)$$

Taking the Vandermonde $\Delta\{x\}_n$ inside the determinant of equation (6.26), we have

$$Z_{n \times n}(\{x\}_n | \{y\}_N) = \det \left(\sum_{k=1}^{N+n} h_{k-i}\{x\} d_{kj}\{y\} \right)_{1 \leq i, j \leq n} \quad (6.28)$$

Hence $Z_{n \times n}(\{x\}_n | \{y\}_N)$ is a Casoratian determinant, and satisfies the discrete KP equations in $\{x\}_n$.

7 The Gromov-Vieira polynomial version of partial domain wall partition functions

Following [14], partial domain wall configurations are (the essential part of) 3-point functions of tree-level single-trace operators in the $SU(2)$ sector of SYM_4 , with two BPS and one non-BPS operators. In [20, 21], Gromov and Vieira showed that 1-loop corrections can be introduced using the mapping discussed in this section. In the sequel, we show that the determinant form of these objects at tree-level is preserved under the GV mapping, thus the corresponding 1-loop corrected objects in SYM_4 can also be expressed as determinants.

7.1 The Gromov-Vieira mapping

In [20, 21], Gromov and Vieira define the following mapping on any function $f(\theta_1, \dots, \theta_N)$ of the variables $\{\theta_1, \dots, \theta_N\}$,

$$f \mapsto [f]_\theta = f \Big|_{\theta_1, \dots, \theta_N \rightarrow 0} + \frac{g^2}{2} \sum_{i=1}^N (\partial_{\theta_i} - \partial_{\theta_{i+1}})^2 f \Big|_{\theta_1, \dots, \theta_N \rightarrow 0} + O(g^4) \quad (7.1)$$

where $\partial_{\theta_{N+1}} \equiv \partial_{\theta_1}$. Note that the mapping is defined to $O(g^2)$ in some small expansion parameter g .

7.2 Aim of this section

Our aim is to show that, up to $O(g^2)$, the GV mapping acts on a Casoratian determinant $Z_{N \times N}(\{x\}_n | \{y\}_N)$ to return a new determinant. We show this by explicitly evaluating $[Z_{N \times N}(\{x\}_n | \{y\}_N)]_y$.

7.3 The GV mapping in terms of symmetric functions

We need the following degree-2 cyclically symmetric function in $\{y\} = \{y_1, y_2, \dots, y_N\}$,

$$m_2\{y\} = y_1y_2 + y_2y_3 + \dots + y_{(N-1)}y_N + y_Ny_1 \quad (7.2)$$

Using $m_2\{y\}$ and the definition of the complete symmetric functions in equation (6.1), one can write

$$\sum_{i=1}^N (y_i - y_{i+1})^2 = 4h_2\{y\} - 2h_1^2\{y\} - 2m_2\{y\} \quad (7.3)$$

where we assume the periodicity $y_{(N+1)} \equiv y_1$. In terms of the corresponding differential operators,

$$\begin{aligned} \sum_{i=1}^N (\partial_{y_i} - \partial_{y_{i+1}})^2 &= 4h_2\{\partial_y\} - 2h_1^2\{\partial_y\} - 2m_2\{\partial_y\} \\ &= 4h_2\{\partial_y\} - 2\left(h_1^2\{\partial_y\} + m_2\{\partial_y\}\right) = 4h_2\{\partial_y\} - 2g_2\{\partial_y\} \end{aligned} \quad (7.4)$$

where we have defined

$$g_2\{\partial_y\} = h_1^2\{\partial_y\} + m_2\{\partial_y\} \quad (7.5)$$

We are interested in computing $h_2\{\partial_y\}f\{y\}|_{y_1, \dots, y_N \rightarrow 0}$, and $g_2\{\partial_y\}f\{y\}|_{y_1, \dots, y_N \rightarrow 0}$, for generic symmetric functions $f\{y\}$, so for convenience we adopt the shorthand

$$h_2\{\partial_y\}f\{y\}|_{y_1, \dots, y_N \rightarrow 0} \equiv H_2f, \quad g_2\{\partial_y\}f\{y\}|_{y_1, \dots, y_N \rightarrow 0} \equiv G_2f, \quad (7.6)$$

7.4 Action of H_2 , and G_2

Let $(h_1)^{m_1}(h_2)^{m_2} \dots (h_L)^{m_L}$ be an arbitrary monomial in the complete symmetric functions. Using the definitions in equation (7.6), we obtain

$$H_2 \left((h_1)^{m_1} (h_2)^{m_2} \dots (h_L)^{m_L} \right) = \begin{cases} N(N+1), & m_1=2, m_2=0, m_3=\dots=m_L=0 \\ N(N+3)/2, & m_1=0, m_2=1, m_3=\dots=m_L=0 \\ 0, & \text{otherwise} \end{cases} \quad (7.7)$$

$$G_2 \left((h_1)^{m_1} (h_2)^{m_2} \dots (h_L)^{m_L} \right) = \begin{cases} 2N(N+1), & m_1=2, m_2=0, m_3=\dots=m_L=0 \\ N(N+2), & m_1=0, m_2=1, m_3=\dots=m_L=0 \\ 0, & \text{otherwise} \end{cases} \quad (7.8)$$

Both H_2 and G_2 act trivially on any monomial whose degree $d = m_1 + 2m_2 + \dots + Lm_L \neq 2$, which greatly simplifies the action of the GV mapping on $Z_{N \times N}(\{x\}_n | \{y\}_N)$.

7.5 Remarks on notation

Henceforth we reserve i and j for the row and column indices of a determinant, respectively, and assume that they range over all values $1 \leq i, j \leq N$. For example, we write the determinant in equation (6.25) as

$$\left| \sum_{k=1}^{N+n} c_{ik}\{x\}h_{k-j}\{y\} \right| = \left| \sum_{k=1}^{N+n} c_{jk}\{x\}h_{k-i}\{y\} \right| = \begin{vmatrix} c_{jk}h_{k-1} \\ \vdots \\ c_{jk}h_{k-N} \end{vmatrix} \quad (7.9)$$

where the first equality follows from the invariance of the determinant under matrix transposition. In the second equality we suppress arguments and the summation symbol, but show the j -th row explicitly. In the rest of this section, all calculations will change determinant on a row-by-row basis.

7.6 Degree-2 terms in the determinant

Since the only terms which survive under the action of H_2 and G_2 are degree-2 monomials in the complete symmetric functions, we focus on these terms by expanding our determinant as follows

$$\begin{vmatrix} c_{j,k}h_{k-1} \\ \vdots \\ c_{j,k}h_{k-N} \end{vmatrix} = \sum_{l=1}^N \begin{vmatrix} c_{j,1}h_0 \\ \vdots \\ c_{j,l-1}h_0 \\ c_{j,l+2}h_2 \\ c_{j,l+1}h_0 \\ \vdots \\ c_{j,N}h_0 \end{vmatrix} + \sum_{1 \leq l_1 < l_2 \leq N} \begin{vmatrix} c_{j,1}h_0 \\ \vdots \\ c_{j,l_1-1}h_0 \\ c_{j,l_1+1}h_1 \\ \vdots \\ c_{j,l_2+1}h_1 \\ c_{j,l_2+1}h_0 \\ \vdots \\ c_{j,N}h_0 \end{vmatrix} + \text{degree-}(d \neq 2) \text{ terms} \quad (7.10)$$

where we maintain the symbol h_0 for clarity, despite the fact that $h_0 = 1$.

7.7 Action of H_2 on equation (7.10)

Acting on equation (7.10) with H_2 and using equation (7.7), we find

$$H_2 \begin{vmatrix} c_{j,k}h_{k-1} \\ \vdots \\ c_{j,k}h_{k-N} \end{vmatrix} = \frac{N(N+3)}{2} \begin{vmatrix} c_{j,1} \\ \vdots \\ c_{j,N-2} \\ c_{j,N+1} \\ c_{j,N} \end{vmatrix} + \frac{N(N+3)}{2} \begin{vmatrix} c_{j,1} \\ \vdots \\ c_{j,N-1} \\ c_{j,N+2} \end{vmatrix} + N(N+1) \begin{vmatrix} c_{j,1} \\ \vdots \\ c_{j,N-2} \\ c_{j,N} \\ c_{j,N+1} \end{vmatrix} \quad (7.11)$$

The first two terms come from the sum $\sum_{l=1}^N$ in equation (7.10), while the final term comes from the sum $\sum_{1 \leq l_1 < l_2 \leq N}$. All other terms vanish under the action of H_2 , either because they have the wrong degree or give rise to a determinant with two equivalent rows.

Combining the first and third determinant in equation (7.11), which are the same up to the ordering of their rows, we obtain

$$H_2 \begin{vmatrix} c_{j,k}h_{k-1} \\ \vdots \\ c_{j,k}h_{k-N} \end{vmatrix} = -\frac{N(N-1)}{2} \begin{vmatrix} c_{j,1} \\ \vdots \\ c_{j,N-2} \\ c_{j,N+1} \\ c_{j,N} \end{vmatrix} + \frac{N(N+3)}{2} \begin{vmatrix} c_{j,1} \\ \vdots \\ c_{j,N-1} \\ c_{j,N+2} \end{vmatrix} \quad (7.12)$$

7.8 Action of G_2 on equation (7.10)

Acting on equation (7.10) with G_2 and using equation (7.8), we find

$$G_2 \begin{vmatrix} c_{j,k} h_{k-1} \\ \vdots \\ c_{j,k} h_{k-N} \end{vmatrix} = N(N+2) \begin{vmatrix} c_{j,1} \\ \vdots \\ c_{j,N-2} \\ c_{j,N+1} \\ c_{j,N} \end{vmatrix} + N(N+2) \begin{vmatrix} c_{j,1} \\ \vdots \\ c_{j,N-1} \\ c_{j,N+2} \end{vmatrix} + 2N(N+1) \begin{vmatrix} c_{j,1} \\ \vdots \\ c_{j,N-2} \\ c_{j,N} \\ c_{j,N+1} \end{vmatrix} \quad (7.13)$$

The first two terms come from the sum $\sum_{l=1}^N$ in equation (7.10), while the final term comes from the sum $\sum_{1 \leq l_1 < l_2 \leq N}$. All other terms vanish under the action of G_2 , either because they have the wrong degree or give rise to a determinant with two equivalent rows.

Combining the first and third determinant in equation (7.13), which are the same up to the ordering of their rows, we obtain

$$G_2 \begin{vmatrix} c_{j,k} h_{k-1} \\ \vdots \\ c_{j,k} h_{k-N} \end{vmatrix} = -N^2 \begin{vmatrix} c_{j,1} \\ \vdots \\ c_{j,N-2} \\ c_{j,N+1} \\ c_{j,N} \end{vmatrix} + N(N+2) \begin{vmatrix} c_{j,1} \\ \vdots \\ c_{j,N-1} \\ c_{j,N+2} \end{vmatrix} \quad (7.14)$$

7.9 Determinant expression for $[Z_{N \times N}(\{x\}_n | \{y\}_N)]_y$

We are ready to express the action of the GV mapping on $Z_{N \times N}(\{x\}_n | \{y\}_N)$ as a single determinant. Firstly, using equation (6.25) it is trivial to calculate

$$Z_{N \times N}(\{x\}_n | \{y\}_N) \Big|_{y_1, \dots, y_N \rightarrow 0} = \Delta^{-1} \{x\}_n \begin{vmatrix} c_{j,1} \\ \vdots \\ c_{j,N} \end{vmatrix} \quad (7.15)$$

where the column index j ranges over all values $1 \leq j \leq N$, as usual. For the second part of the GV mapping in equation (7.1), we wish to calculate

$$\sum_{i=1}^N (\partial_{y_i} - \partial_{y_{i+1}})^2 Z_{N \times N}(\{x\}_n | \{y\}_N) \Big|_{y_1, \dots, y_N \rightarrow 0} = \frac{(4H_2 - 2G_2)}{\Delta \{x\}_n} \begin{vmatrix} c_{j,k} h_{k-1} \\ \vdots \\ c_{j,k} h_{k-N} \end{vmatrix} \quad (7.16)$$

Putting together the results of the previous subsections, namely equations (7.12) and (7.14), we find that

$$\frac{(4H_2 - 2G_2)}{\Delta \{x\}_n} \begin{vmatrix} c_{j,k} h_{k-1} \\ \vdots \\ c_{j,k} h_{k-N} \end{vmatrix} = \frac{2N}{\Delta \{x\}_n} \begin{vmatrix} c_{j,1} \\ \vdots \\ c_{j,N-2} \\ c_{j,N+1} \\ c_{j,N} \end{vmatrix} + \frac{2N}{\Delta \{x\}_n} \begin{vmatrix} c_{j,1} \\ \vdots \\ c_{j,N-1} \\ c_{j,N+2} \end{vmatrix} \quad (7.17)$$

Using equations (7.15)–(7.17) we obtain

$$\begin{aligned}
\left[Z_{N \times N} \left(\{x\}_n \middle| \{y\}_N \right) \right]_y &\equiv Z_{N \times N} \Big|_{y_1, \dots, y_N \rightarrow 0} + \frac{g^2}{2} \sum_{i=1}^N (\partial_{y_i} - \partial_{y_{i+1}})^2 Z_{N \times N} \Big|_{y_1, \dots, y_N \rightarrow 0} + O(g^4) \\
&= \frac{1}{\Delta\{x\}_n} \begin{vmatrix} c_{j,1} \\ \vdots \\ c_{j,N} \end{vmatrix} + \frac{g^2 N}{\Delta\{x\}_n} \begin{vmatrix} c_{j,1} \\ \vdots \\ c_{j,N-2} \\ c_{j,N+1} \\ c_{j,N} \end{vmatrix} + \frac{g^2 N}{\Delta\{x\}_n} \begin{vmatrix} c_{j,1} \\ \vdots \\ c_{j,N-1} \\ c_{j,N+2} \end{vmatrix} + O(g^4)
\end{aligned} \tag{7.18}$$

The first three terms of equation (7.18) can actually be combined into a single determinant, which is correct up to $O(g^2)$. Our final result is

$$\left[Z_{N \times N} \left(\{x\}_n \middle| \{y\}_N \right) \right]_y = \frac{1}{\Delta\{x\}_n} \begin{vmatrix} c_{j,1} \\ \vdots \\ c_{j,N-2} \\ c_{j,N-1} + g^2 N c_{j,N+1} \\ c_{j,N} + g^2 N c_{j,N+2} \end{vmatrix} + O(g^4) \tag{7.19}$$

The result in equation (7.19) is such a simple modification of the original expression, obtained by setting $g^2 \rightarrow 0$, that we expect that higher derivative versions of the GV mapping will also preserve the determinant form of the pDWPF.⁷

Since the GV mapping is an expansion around the homogeneous limit at which all variables $y_i = 0$, we cannot consider the determinant in equation (7.19) to be a discrete KP τ -function in the $\{y\}$ variables. On the other hand, according to the methods of section 7, the determinant in equation (7.19) is not in Casorati form, hence we cannot conclude that it is a discrete KP τ -function in the $\{x\}$ variables.

8 Remarks

8.1 Summary of results

Rational and trigonometric partial domain wall partition functions, pDWPF's, are partition functions of six-vertex model configurations on lattices with unequal numbers of horizontal lines L_h and vertical lines L_v . They can be regarded as less restrictive variations on Korepin's rational and trigonometric domain wall partition functions, DWPF's, which require $L_h = L_v$, but can be deduced from them, as well as from configurations that describe scalar products, by taking some of the rapidities to infinity.

In this work, we gave explicit derivations of the determinant expressions for pDWPF's as limits of Izergin's DWPF determinant, as well as of Slavnov's determinant for the scalar product of a Bethe eigenstate and a generic state, in the rational and trigonometric cases,

⁷An earlier draft of this work contained an incorrect version of equation (7.3) that led to a more complicated version of equation (7.19). We thank D Serban for pointing this out.

and studied some of their properties. The rational pDWPF was first derived from Slavnov's determinant by I Kostov [17]. We showed how the two determinants obtained as limits of Izergin's determinant and of Slavnov's determinant are different (one is $(N \times N)$ while the other is $(n \times n)$, where $n < N$), but can be directly related, that they are KP τ -functions in each of two sets of variables, and that they remain determinants under the mapping of Gromov and Vieira.⁸

8.2 Taking the free variables to infinity in Slavnov's determinant

In section 5, following Kostov [17], we derived pDWPF's from Slavnov's scalar products. We kept the free rapidity variables $\{x\}$ finite, and took the rapidity variables that satisfy Bethe equations, $\{b\}$, to infinity. The result is finite and non-trivial.

If we would have kept the Bethe roots $\{b\}$ finite and took $\{x\}$ to infinity, the result would have been zero. The reason is that this limit corresponds to the scalar product of a Bethe eigenstate, labeled by $\{b\}$, and a descendant of the reference state (the result of the action of spin-lowering operators on the reference state, that lower the net spin but do not introduce Bethe roots [12]). Since the scalar product of the Bethe eigenstate $|\{b\}\rangle$ and the reference state vanishes, the scalar product of $|\{b\}\rangle$ with a descendant of the reference state also vanishes. In other words, a pDWPF with auxiliary space (horizontal line) rapidities that obey Bethe equations vanishes.

8.3 Asymptotics

In [14, 17–19], pDWPF's were studied in the thermodynamic limit $L_v \rightarrow \infty$, such that the ratios L_h/L_v and x_i/L_v , $i \in \{1, \dots, L_h\}$, remain finite, where L_v (L_h) is the number of vertical (horizontal) lattice lines, $L_h < L_v$, and $\{x\}$ are the rapidities of the horizontal lines.⁹ While, strictly speaking, the variables $\{x\}$ are free, in applications, such as computations of 3-point functions of two BPS and one non-BPS operators in the scalar sector of SYM_4 , they are restricted to obey the Bethe equations of a spin chain of length L , such that $L \neq L_v$. For that reason, Bethe Ansatz asymptotics apply, but the pDWPF is nonetheless non-vanishing. This is the set-up used in [14, 18, 19].

Following [14, 18], in the above thermodynamic limit, the variables $\{x\}$, which are solutions of Bethe equations of a spin chain of length $L > L_v$, $L \sim L_v$, condense on a set of contours $\Gamma = \bigcup_k \Gamma_k$, with linear density $\rho\{x\}$, $\rho \sim \mathcal{O}(1)$, $x_i \sim \mathcal{O}(L_v)$. In the homogeneous limit, $y_i = 0$, $i \in \{1, \dots, L_v\}$, the asymptotic pDWPF can be expressed as an exponential of a contour integral over a dilogarithm function

$$\exp \left(\oint_{\mathcal{C}} \frac{dz}{2\pi} \text{Li}_2 \left(e^{iq(z)} \right) \right), \quad \text{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad (8.1)$$

⁸It is likely that the determinant expression is preserved under the action of higher derivative versions of the GV mapping. We did not pursue this since, at this stage, the relation between the higher derivative versions and the inclusion of higher loop corrections to the 3-point functions is not clear. However, in [27], D Serban argued that this is indeed the case, at least in the limit $L_i \rightarrow \infty$, $i \in \{1, 2, 3\}$. That is, when all three operators are represented by asymptotically long spin chain states.

⁹Because of the condition that x_i/L_v , $i \in \{1, \dots, L_h\}$, remains finite, this limit is also known as the ‘Sutherland limit’ [28, 29].

where $\mathcal{C}$ encircles Γ counter-clockwise, and

$$q(z) = -i \log \left(f(z) \right) + \int_{\Gamma} dy \frac{\rho(y)}{z-y}, \quad f(z) = \left(\frac{z-i/2}{z+i/2} \right)^L \quad (8.2)$$

The point we wish to mention here is that in the same limit, the Slavnov scalar product factorizes into a product of terms that are either the asymptotic pDWPF in equation (8.1), or simple variations of it [18, 19]. Thus, at least asymptotically, pDWPF's are building blocks of scalar products.

8.4 Higher rank scalar products

In [30, 31], pDWPF's appear as factors in certain degenerations of the SU(3)-analogue of Slavnov's scalar product. One starts from sum expressions for the SU(3)-analogue of Slavnov's scalar product, takes both sets of Bethe roots to infinity [30], or either one (there are two sets of Bethe roots in SU(3)-invariant spin chains) [31], only to find that the sum expression factorizes into determinants that inevitably include one or more pDWPF. This factorization, and the appearance of pDWPF's as factors, is expected on general grounds to remain the case for SU(N)-analogues, $N \geq 4$, of Slavnov's scalar product. Since no determinant expression is known for the SU(N)-analogues of Slavnov's scalar product, we hope that a deeper understanding of the properties of building blocks, such as pDWPF's, will help solve this problem.

8.5 Combinatorics and counting

Six-vertex model configurations with domain wall boundary conditions are in one-to-one correspondence with alternating sign matrices, ASM's [32]. Using this observation, Kuperberg counted $(N \times N)$ ASM's by evaluating Izergin's determinant at the combinatoric value of the crossing parameter $\gamma = 2\pi i/3$ [32].

This leads one to expect that similar arguments can be applied to the pDWPF's to count more general objects than ASM's. This is not the case, or at least not in an obvious way, because the trigonometric weights that we needed to derive determinant expressions for the trigonometric pDWPF's, equation (2.2), contain phases that vary from configuration to configuration and thereby rule out any (straightforward) 1-counting as in the DWPF case.

Acknowledgments

OF wishes to thank J. Caetano, N. Gromov, I. Kostov, D. Serban and P. Vieira for discussions, H. Saleur and V. Schomerus for hospitality at the Institut Henri Poincare, Paris, where this work started, and I Kostov and D. Serban for remarks that helped us improve the manuscript. We thank the Australian Research Council for financial support.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, *Phys. Rev. D* **69** (2004) 046002 [[hep-th/0305116](#)] [[INSPIRE](#)].
- [2] J. Minahan and K. Zarembo, *The Bethe ansatz for $N = 4$ super Yang-Mills*, *JHEP* **03** (2003) 013 [[hep-th/0212208](#)] [[INSPIRE](#)].
- [3] N. Beisert et al., *Review of AdS/CFT Integrability: An Overview*, *Lett. Math. Phys.* **99** (2012) 3 [[arXiv:1012.3982](#)] [[INSPIRE](#)].
- [4] V. Korepin, *Calculation of norms of Bethe wave functions*, *Commun. Math. Phys.* **86** (1982) 391 [[INSPIRE](#)].
- [5] A.G. Izergin, *Partition function of the six-vertex model in a finite volume*, *Sov. Phys. Dokl.* **32** (1987) 878.
- [6] V.E. Korepin, N.M. Bogoliubov and A.G. Izergin, *Quantum inverse scattering method and correlation functions*, Cambridge University Press, Cambridge U.K. (1993).
- [7] N.M. Bogoliubov, A.G. Pronko and M.B. Zvonarev, *Boundary correlation functions of the six-vertex model*, *J. Phys. A* **35** (2002) 5525 [[math-ph/0203025](#)].
- [8] O. Foda and I. Preston, *On the correlation functions of the domain wall six vertex model*, *J. Stat. Mech.* **0411** (2004) P11001 [[math-ph/0409067](#)] [[INSPIRE](#)].
- [9] F. Colomo and A. Pronko, *On the partition function of the six vertex model with domain wall boundary conditions*, *J. Phys. A* **37** (2004) 1987 [[math-ph/0309064](#)] [[INSPIRE](#)].
- [10] F. Colomo and A.G. Pronko, *On two-point boundary correlations in the six-vertex model with domain wall boundary conditions*, *J. Stat. Mech.* **5** (2005) 10 [[math-ph/0503049](#)].
- [11] F. Colomo and A.G. Pronko, *An approach for calculating correlation functions in the six-vertex model with domain wall boundary conditions*, *Theor. Math. Phys.* **171** (2012) 641 [[arXiv:1111.4353](#)].
- [12] J. Escobedo, N. Gromov, A. Sever and P. Vieira, *Tailoring three-point functions and integrability*, *JHEP* **09** (2011) 028 [[arXiv:1012.2475](#)] [[INSPIRE](#)].
- [13] J. Escobedo, N. Gromov, A. Sever and P. Vieira, *Tailoring three-point functions and integrability II. Weak/strong coupling match*, *JHEP* **09** (2011) 029 [[arXiv:1104.5501](#)] [[INSPIRE](#)].
- [14] N. Gromov, A. Sever and P. Vieira, *Tailoring three-point functions and integrability III. Classical tunneling*, *JHEP* **07** (2012) 044 [[arXiv:1111.2349](#)] [[INSPIRE](#)].
- [15] O. Foda, *$N = 4$ SYM structure constants as determinants*, *JHEP* **03** (2012) 096 [[arXiv:1111.4663](#)] [[INSPIRE](#)].
- [16] M. Wheeler, *An Izergin-Korepin procedure for calculating scalar products in six-vertex models*, *Nucl. Phys. B* **852** (2011) 468 [[arXiv:1104.2113](#)] [[INSPIRE](#)].
- [17] I. Kostov, private communication.
- [18] I. Kostov, *Classical limit of the three-point function from integrability*, [arXiv:1203.6180](#) [[INSPIRE](#)].
- [19] I. Kostov, *Three-point function of semiclassical states at weak coupling*, [arXiv:1205.4412](#) [[INSPIRE](#)].

- [20] N. Gromov and P. Vieira, *Quantum integrability for three-point functions*, [arXiv:1202.4103](#) [[INSPIRE](#)].
- [21] N. Gromov and P. Vieira, *Tailoring three-point functions and integrability IV. Theta-morphism*, [arXiv:1205.5288](#) [[INSPIRE](#)].
- [22] N.A. Slavnov, *Calculation of scalar products of wave functions and form factors in the framework of the algebraic Bethe Ansatz*, *Theor. Math. Phys.* **79** (1989) 502.
- [23] N. Kitanine, J.M. Maillet and V. Terras, *Form factors of the XXZ Heisenberg spin-1/2 finite chain*, *Nucl. Phys. B* **554** (1999) 647 [[math-ph/9807020](#)].
- [24] O. Foda and G. Schrader, *XXZ scalar products, Miwa variables and discrete KP*, in *New Trends in Quantum Integrable Systems*, B. Feigin, M. Jimbo and M. Okado eds., World Scientific, Singapore (2010), pg. 61–80 [[arXiv:1003.2524](#)].
- [25] Y. Ohta, R. Hirota, S. Tsujimoto and T. Inami, *Casorati and discrete Gram type determinant representations of solutions to the discrete KP hierarchy*, *J. Phys. Soc. Japan* **62** (1993) 1872.
- [26] I.G. Macdonald, *Symmetric functions and Hall polynomials*, Oxford University Press, Oxford U.K. (1995).
- [27] D. Serban, *A note on the eigenvectors of long-range spin chains and their scalar products*, [arXiv:1203.5842](#) [[INSPIRE](#)].
- [28] B. Sutherland, *Low-Lying Eigenstates of the One-Dimensional Heisenberg Ferromagnet for any Magnetization and Momentum*, *Phys. Rev. Lett.* **74** (1995) 816 [[INSPIRE](#)].
- [29] A. Dhar and B. Sriram Shastry, *Bloch Walls and Macroscopic String States in Bethe's Solution of the Heisenberg Ferromagnetic Linear Chain*, *Phys. Rev. Lett.* **85** (2000) 2813 [[INSPIRE](#)].
- [30] J. Caetano and P. Vieira, private communication.
- [31] M. Wheeler, *Scalar products in generalized models with SU(3)-symmetry*, [arXiv:1204.2089](#) [[INSPIRE](#)].
- [32] G. Kuperberg, *Another proof of the alternating sign matrix conjecture*, *Int. Math. Res. Notices* **3** (1996) 139 [[math/9712207](#)].