PSU(2, 2|4) character of quasiclassical AdS/CFT

Nikolay Gromov,^{a,b,c} Vladimir Kazakov^{d,e,1} and Zengo Tsuboi^f

^a*DESY Theory,
Hamburg, Germany*

^b*II. Institut für Theoretische Physik Universität,
Hamburg, Germany*

^c*St.Petersburg INP,
St.Petersburg, Russia*

^d*Ecole Normale Supérieure, LPT, 75231 Paris CEDEX-5, France*

^e*L'Université Paris-VI, France*

^f*Okayama Institute for Quantum Physics,
Okayama 1-9-1, Okayama 700-0015, Japan*

E-mail: nikgromov@gmail.com, kazakov@lpt.ens.fr, ztsuboi@gmail.com

ABSTRACT: We solve the recently proposed T- and Y-systems (Hirota equation) for the exact spectrum of AdS/CFT in the strong coupling scaling limit for an arbitrary quasiclassical string state. The corresponding T-functions appear to be super-characters of the SU(2, 2|4) group in unitary representations with a highest weight, with the classical $\text{AdS}_5 \times S^5$ superstring monodromy matrix as the group element. We propose a concise first Weyl-type formula for these characters and show that they correctly reproduce the results of quasiclassical one-loop quantization in all sectors of the superstring, under some natural assumptions. We also speculate about possible relation between the T-functions and the quantum monodromy matrix.

KEYWORDS: AdS-CFT Correspondence, Bethe Ansatz, Integrable Field Theories, Sigma Models

ARXIV EPRINT: [1002.3981](https://arxiv.org/abs/1002.3981)

¹Member of Institut Universitaire de France.

Contents

1	Introduction	1
2	Y-system of AdS/CFT and the characters of $U(2, 2 4)$	4
2.1	Y-system for AdS/CFT: equations and definitions	4
2.2	The Q-system limit and its $U(2, 2 4)$ “character” solution	6
2.3	New determinant formulae	8
3	Description of representations of $U(2, 2 4)$ for the AdS/CFT Y-system	9
3.1	Description of the general construction	10
3.2	Unitarity	12
3.3	Comparing highest weight with the equation for characters	13
3.4	Example: representations of $GL(2)$	13
3.4.1	Quantum $GL(1 + 1)$	15
4	Solution of Y-system in the scaling limit and quasiclassical strings	16
4.1	Asymptotic solution in scaling limit	16
4.2	Asymptotic T-functions for the entire T-hook	18
4.3	Classical limit	19
4.4	Asymptotic solution as a limit of $U(2, 2 4)$ super-characters	21
4.5	Fixing the parameters of the general solution	22
4.6	Magic products	23
4.7	One-loop energy and quasiclassical quantization	24
4.7.1	Quasiclassical quantization from the algebraic curve	24
4.7.2	One-loop energy from Y-system	26
5	Conclusions	29
A	Mathematica expressions for general solution of Q-system	30
B	Derivation of the modified Bethe equations	31

1 Introduction

Recently, two of the authors and P.Vieira proposed an infinite set of equations, the so called Y-system for AdS/CFT for the exact spectrum of anomalous dimensions of all gauge invariant local operators in planar $\mathcal{N} = 4$ super-Yang-Mills theory (SYM) at arbitrary value of the 't Hooft coupling λ [1]. These equations were inspired by similar Y-systems in relativistic and lattice models [2–9, 13–16], by the asymptotic Bethe ansatz (ABA) [17–24], by the idea of the mirror theory proposed in [25, 26] with a non-relativistic single impurity

dispersion relation in $\mathcal{N} = 4$ SYM theory [27, 28] and by the structure of the leading finite size correction [29–31]. In [32] these equations were written in a form convenient for numerical solution and the anomalous dimension of the lowest lying Konishi state [33–35] was found in a broad range of coupling constants $0 < \lambda < 700$ in [36]. For the first time, the energy of a non-protected by super-symmetry low lying state in a 4D gauge theory was found as a function of coupling in the planar limit. The Y-system has passed a few important checks. The correct 4-loop result of direct Feynman graph calculation of the Konishi anomalous dimension $\Delta_K(\lambda)$ [37, 38] was reproduced from the Y-system in [1].¹ A similar comparison for the length 3 operators at 5-loops also confirms the validity of the Y-system [42]. The extrapolation of numerical results of [36] to the strong coupling was found to be approximately $\Delta_K \simeq 2.0004\lambda^{1/4} + 1.99/\lambda^{1/4} + \dots$ where the leading coefficient agrees with the string prediction $2\sqrt{n} \lambda^{1/4}$ for $n = 1$ [43].² Also the general asymptotic solution of the Y-system for long operators was constructed [1] and its consistency with the asymptotic Bethe ansatz (ABA) of [23, 24] was shown. As we will see this asymptotic solution plays a fundamental role in the whole construction since it defines the analytic and asymptotic properties of the exact solution of the Y-system for a given physical state at a finite L . In particular it was used in [32] to write the integral equation for excited states. Another important test was done in [32, 49–51] where the Y-system was obtained from the Al. Zamolodchikov thermodynamic Bethe ansatz (TBA) approach for the BMN ground state.³

Whereas the classical finite gap solutions of Metsaev-Tseytlin sigma model [53, 54] stemming from the world sheet integrability [55] and calculating the dimensions of long operators in a strong coupling regime follow from the ABA equations [24] in a direct and simple way, the quasiclassical one-loop corrections, also available for an arbitrary finite gap solution [56–60], are already in a severe disagreement with the ABA even at infinite length in the scaling $L \sim \sqrt{\lambda} \rightarrow \infty$ [61].⁴ Recently it was shown [65] that the Y-system cures this one-loop disagreement generically for any classical states in $AdS_3 \times S^1$ and thus takes

¹Initially the 4-loop perturbative results were reproduced in [29–31] using the conjectured Lüscher-like formulas for the world sheet theory. A similar test was successfully performed in the ABJM model - the 3-dimensional integrable analogue of the $\mathcal{N} = 4$ SYM theory [39]. These formulas presumably capture the leading finite-size corrections and can be used up to 7-loops for the Konishi state. 5-loops were computed using this approach in [40]. Recently the result was shown to be consistent with the Y-system approach numerically [41].

²The subleading terms are still a challenge for string theorists. Two different values for $1/\lambda^{1/4}$ coefficient were obtained on the string side [44–46] on the basis of rather bold assumptions. The results [46] are obtained for a truncated model where quantum contribution of some string modes are ignored whereas in [44, 45] the applicability of the quasi-classics in the small charge limit was assumed. It is also important to mention that the prediction of [36] for that coefficient was subject to the explicit assumption that the first expansion coefficients are of order of 1 and do not grow too rapidly so that the numerical results for $\lambda \lesssim 700$ allow for this prediction. In [47] a warning was raised that this assumption could be not correct. Hopefully the direct world sheet computation in the Metsaev-Tseytlin superstring sigma model [48] along the lines of other approaches mentioned in [44, 45] will be done soon and will lift this uncertainty.

³The ground state by itself is protected and has zero anomalous dimension. Nevertheless the TBA equations capture some important structural information about the Y-system.

⁴The ABA agrees with the one-loop corrections when $L/\sqrt{\lambda} \gg 1$ [62–64].

Figure 1. (Left: figure 1a; Right: figure 1b) **T**-shaped “fat hook” (**T**-hook) uniting two $SU(2|2)$ fat hooks, see [1] for this **T**-hook and its generalization [52].

into account infinitely many finite size wrapping contributions.⁵ Similar results were also obtained for several subsectors of the ABJM theory [66, 67] where the Y-system was also conjectured [1, 67, 68]. These results deeply test the structure of the Y-systems since all wrapping contributions are crucially important in that case.

In this paper we construct the complete solution of the AdS/CFT Y-system in the strong coupling limit generalizing [65] and reproduce from this solution the equations arising for the quasiclassical one-loop corrections of [57–60]. In this limit, the finite difference operators w.r.t the spectral parameter disappear from the Y-system and it reduces to a simplified system of equations called in the mathematical literature the Q-system:⁶

$$T_{a,s}^2 = T_{a+1,s}T_{a-1,s} + T_{a,s+1}T_{a,s-1}.$$

This equation is related to the Y-system in the considered limit by $Y_{a,s} = \frac{T_{a,s+1}T_{a,s-1}}{T_{a+1,s}T_{a-1,s}}$. The Q-systems are frequently used as the defining equations for the characters of representations of the underlying symmetry groups (see for example [69–71, 73–76]; a Q-system for $sl(M|N)$ related models appeared in [73] in the sense that the Q-system is a degeneration of the T-system. A physical application of this Q-system can be seen in the following paper: [72]). Here we constructed the general solution of such a Q-system with the T-hook boundary conditions with respect to the representational indices a, s (see figure 4). We will argue that the solution is given in terms of characters of certain unitary representations of $SU(2, 2|4)$ group. It demonstrates in a nontrivial way that the full global symmetry of AdS/CFT is present in the Y-system in spite of the original $SU(2|2) \times SU(2|2)$ setting due to the choice of the light cone gauge. Then, using the asymptotic solution, we fix the parameters of the general solution. This leads to a very simple result

$$T_{a,s} = \text{Str}_{a,s} \Omega(x), \quad (1.1)$$

where $\Omega(x)$ is the classical monodromy matrix [55] and the trace is taken in some particular representations labeled by a, s . We also speculate that a similar relation should hold at

⁵By wrapping contributions we mean essentially all the finite size corrections to the ABA. Historically the term refers to specific Feynman graphs of SYM running around all the legs of a local operator under study.

⁶A terminological note: One should not confuse the Q-system with the Baxter Q-operators. Q-system is a simplified version of Hirota equation 2.6. We often call the last one as T-system. Baxter equations can be called TQ-relations and the particle-hole duality relations among supersymmetric Baxter functions are called QQ-relations.

the quantum level. Let us also note that the quantum generalization of this character solution of the AdS/CFT Y-system would reduce the problem at any coupling to a finite set of non-linear integral equations similar to Destri-DeVega equations known for some relativistic sigma models. This should be possible due to the underlying integrable discrete Hirota dynamics of the Y-system [16] (see [52, 77] for the first steps). The general strong coupling solution we present here can be an important step on this way.

In section 2 we present the general character solution of the Q-system (or of the related Y-system), for the T-hook boundary conditions reflecting the global $SU(2, 2|4)$ symmetry of AdS/CFT problem. At the end of section 2 these characters will be presented in a new, explicit and concise form in terms of determinants of 2×2 and 4×4 matrices reminding the 1-st Weyl formula; a detailed description of these representations is given in the section 3. In section 4 we compare this solution of Y-system and Q-system to the quasiclassical (one-loop) spectrum of the theory. The section 5 will summarize our results and propose some new possible directions in testing and simplifying the AdS/CFT Y-system.

2 Y-system of AdS/CFT and the characters of $U(2, 2|4)$

In this section we will remind the formulation of the AdS/CFT Y-system and construct the complete solution in an important particular case: when the dependence on the spectral parameter is slow and the finite shifts can be neglected. In this case, the Y-system, or the equivalent T-system (Hirota bilinear difference equation), does not contain the shifts in u and is usually called the Q-system. The Q-system, a finite difference equation with respect to a couple of discrete variables, can be interpreted as an equation for characters of particular irreducible representations (usually with $a \times s$ rectangular Young diagrams, with a and s being its size in the antisymmetric and symmetric directions, respectively) for a given symmetry group. A specific group enters the Q-system only through the boundary conditions w.r.t. the discrete variables parameterizing the representation space. We will give in this section the full solution of such a Q-system with the T-hook boundary conditions (see figure 1a), relevant to the AdS/CFT Y-system [1], in an explicit and concise form and interpret them as super-characters of some unitary representations of the $U(2, 2|4)$ group.

2.1 Y-system for AdS/CFT: equations and definitions

Y-system encoding the spectrum of all local operators in planar AdS/CFT correspondence [1] is a set of functional equations⁷

$$\frac{Y_{a,s}^+ Y_{a,s}^-}{Y_{a+1,s} Y_{a-1,s}} = \frac{(1 + Y_{a,s+1})(1 + Y_{a,s-1})}{(1 + Y_{a+1,s})(1 + Y_{a-1,s})}. \quad (2.1)$$

The functions $Y_{a,s}(u)$ are defined only on the nodes marked by gray and white circles in figure 1b. Solutions of Y-system with appropriate analytic properties define the energy of

⁷We shall always denote $f^\pm = f(u \pm i/2)$ or even more generally $f^{[\pm a]} = f(u \pm a i/2)$. In this equation we choose the Y-functions to have branch cuts going to infinity. The analytic properties of the Y-functions could be seen from the asymptotic solution described below.

a state (anomalous dimension of an operator in $\mathcal{N} = 4$ SYM) through the formula

$$E = \sum_j \epsilon_1^{\text{ph}}(u_{4,j}) + \sum_{a=1}^{\infty} \int_{-\infty}^{\infty} \frac{du}{2\pi i} \frac{\partial \epsilon_a}{\partial u} \log(1 + Y_{a,0}(u)), \quad (2.2)$$

where the physical dispersion relation between the energy ϵ_a and the momentum p_a is parameterized in terms of the rapidity (spectral parameter) u as follows

$$\epsilon_a(u) = a + \frac{2ig}{x^{[+a]}} - \frac{2ig}{x^{[-a]}} \quad (2.3)$$

where $g = \frac{\sqrt{\lambda}}{4\pi}$. We consider two different branches of the double valued function $x(u)$

$$x^{\text{ph}}(u) = \frac{1}{2} \left(\frac{u}{g} + \sqrt{\frac{u}{g} - 2} \sqrt{\frac{u}{g} + 2} \right), \quad x^{\text{mir}}(u) = \frac{1}{2} \left(\frac{u}{g} + i \sqrt{4 - \frac{u^2}{g^2}} \right). \quad (2.4)$$

They coincide above the real axis. $x^{\text{ph}}(u)$ is defined to have a finite branch cut between $\pm 2g$ whereas in $x^{\text{mir}}(u)$ the cut is chosen to go through infinity along the real axes. If it is not stated otherwise we always define $x(u) = x^{\text{mir}}(u)$. The rapidities $u_{4,j}$ are fixed by the exact Bethe ansatz equations for any size L of the SYM operators

$$Y_{1,0}^{\text{ph}}(u_{4,j}) = -1. \quad (2.5)$$

The Y-system for AdS/CFT can be rewritten as Hirota bilinear difference equation (T-system) [1]⁸

$$T_{a,s}^+ T_{a,s}^- = T_{a+1,s} T_{a-1,s} + T_{a,s+1} T_{a,s-1}, \quad (2.6)$$

where

$$Y_{a,s} = \frac{T_{a,s+1} T_{a,s-1}}{T_{a+1,s} T_{a-1,s}}. \quad (2.7)$$

The functions $T_{a,s}(u)$ are non-zero only on the 2D lattice drawn on figure 1a.

$$T_{a,s} = 0 \text{ if } a < 0 \text{ or } a, |s| > 2. \quad (2.8)$$

There is a “gauge” freedom in (2.6)

$$T_{a,s}(u) \rightarrow G_1 \left(u + i \frac{a+s}{2} \right) G_2 \left(u + i \frac{a-s}{2} \right) G_3 \left(u - i \frac{a+s}{2} \right) G_4 \left(u - i \frac{a-s}{2} \right) T_{a,s}(u) \quad (2.9)$$

which maps one solution of the T-system to another but leaves the $Y_{a,s}$ intact.

⁸In a sense the T-system is more fundamental than the Y-system. Any explicit solution of Y-system looks simpler in terms of T's. Moreover for the T-hook, two equations, for $(a, s) = (2, 2)$ and $(2, -2)$, are missing in the Y-system. One cannot write the equations for these nodes in terms of Y-functions in a “local” functional form. However, these equations are present in a local form in T-system. We will clearly see this while solving the T-system for strong coupling.

Figure 2. “Fat hook” where the representations of a $SU(M|N)$ symmetric super-spin chain live. See [73–76] for the details on fat hooks and T -functions for spin chains related to superalgebras.

2.2 The Q-system limit and its $U(2, 2|4)$ “character” solution

In the strong coupling limit $\lambda \rightarrow \infty$ which we shall study in this paper, we notice that the λ -dependence can be scaled out from the formulas like (2.4) by simple rescaling $u \rightarrow 2gz$ with $g = \frac{\sqrt{\Lambda}}{4\pi}$. Then the u -shifts in the Y-system (2.1) and T-system (2.6) become negligible and it can be now written as a Q-system⁹

$$T_{a,s}^2 = T_{a+1,s} T_{a-1,s} + T_{a,s+1} T_{a,s-1}, \quad (2.10)$$

with the AdS/CFT T-hook boundary conditions (2.8). The gauge transformations for the the Q-system are reduced to

$$T_{a,s} \rightarrow g_1 g_2^a g_3^s g_4^{as} T_{a,s}. \quad (2.11)$$

We shall now construct the general solution of this Q-system in terms of certain “characters” of $GL(4|4)$ group. We will show in the next section that these characters lead, after an appropriate identification of their parameters with the quasi-momenta of the classical monodromy matrix, to exactly the same expressions as in the full quasiclassical (one-loop) solution of MT string sigma-model [64] thus generalizing the results of [65] for the $SL(2)$ subsector. The representations corresponding to the solution in T-hook are certain infinite dimensional representations described in section 3. We first start from a more standard example of finite dimensional representations of $GL(4|4)$ group. The super characters for “symmetric” representations $T_{1,s}$, or supersymmetric Schur polynomials, for the group $GL(M|N)$ are defined through the following generating function (see e.g. [78] which will be relevant to our discussion):

$$w_{M|N}(t; g) = \text{Sdet} \frac{1}{1 - gt} = \frac{\prod_{n=1}^N (1 - y_n t)}{\prod_{m=1}^M (1 - x_m t)} = \sum_{s=1}^{\infty} t^s T_{1,s}^{(M|N)}[g], \quad (2.12)$$

⁹The shifts in the spectral parameter cannot be neglected close to the branch cuts of $T_{a,s}$ going along the real axis with $|u| > 2g$.

where $(x_1, \dots, x_M | y_1, \dots, y_N)$ are the eigenvalues of a group element $g \in \text{GL}(M|N)$.¹⁰ The rest of the characters for “rectangular” representations (for which the Young diagram contains a columns and s -rows) can be calculated using the Jacobi-Trudi formula:

$$T_{a,s} = \det_{1 \leq i, j \leq a} T_{1, s+i-j} . \quad (2.13)$$

These characters are non-zero only in a “fat hook”, or $[M|N]$ -hook presented on the figure 2. The generating function of $\text{GL}(4|4)$ characters can be represented as

$$w_{4|4}(t; g) = \frac{(1 - y_1 t)(1 - y_2 t)}{(1 - x_1 t)(1 - x_2 t)} \times \frac{(1 - y_3 t)(1 - y_4 t)}{(1 - x_3 t)(1 - x_4 t)} \quad (2.14)$$

i.e.,

$$T_{1,s}^{(4|4)}[g] = \oint_{C_0} \frac{dt w_{4|4}(t; g)}{2\pi i} t^{-s-1} . \quad (2.15)$$

Notice that the integrand has also poles at $t = 1/x_j$ in addition to the pole at the origin. To get (2.12) one should encircle only the point $t = 0$ and leave outside all other poles. Using that $w_{4|4}(t; g^L \otimes g^R) = w_{2|2}(t; g^L) \times w_{2|2}(t; g^R)$, where $g^L, g^R \in \text{GL}(2|2)$ we can represent $T_{1,s}^{(4|4)}$ in a specific form

$$T_{1,s}^{(4|4)}[g^L \otimes g^R] = \sum_{j=0}^s T_{1,s-j}^{(2|2)}[g^L] \times T_{1,j}^{(2|2)}[g^R] . \quad (2.16)$$

What would be the analog of these characters satisfying the Q-system (2.10) and the T-hook boundary conditions (2.8)? A natural definition appears to be the same eq. (2.15), after a simple change of the integration contour: we encircle this time $t = 0$ together with the poles $\frac{1}{x_3}, \frac{1}{x_4}$ corresponding to the second subgroup $\text{GL}(2|2)$, leaving outside the poles $\frac{1}{x_1}, \frac{1}{x_2}$ corresponding to the first subgroup $\text{GL}(2|2)$. This amounts to expanding the first factor in (2.14) in t and the second one in $1/t$ and picking the power t^s :

$$T_{1,s}[g^L \otimes g^R] = \frac{y_3 y_4}{x_3 x_4} \sum_{j=\max(0, -s)}^{\infty} T_{1,s+j}^{(2|2)}[g^L] \times T_{1,j}^{(2|2)}[1/g^R] . \quad (2.17)$$

Note that unlike the finite-dimensional representations (2.15), the r.h.s of (2.17) contains infinite number of terms and thus such characters correspond to some infinite dimensional unitary representations of the group $\text{U}(2, 2|4)$. Then one can see that the Jacobi-Trudi formula (2.13) gives the full solution of the Q-system for the T-hook figure 1a with the boundary conditions (2.8)!

Let us mention here that this method of construction of characters, or even quantum transfer matrices of infinite dimensional representations of $u(2, 2|4)$ was already mentioned in [88] where it was applied to the AdS/CFT one-loop Bethe ansatz. It was proposed there (see appendix B) to expand some of the monomials of the generating functional (the analogue of our $w_{4|4}(t; g)$; proposed by Krichever et al. in [15] and generalized to

¹⁰We consider the generic diagonalized group element.

the supersymmetric case in [73, 74]) in positive powers, and some - in negative powers of the shift operator $D = e^{-i\partial_u/2}$. Note that this method is enough to generate the general solution of the full quantum Hirota equation (2.6) within the $(2, 2|4)$ $\mathbb{T}$ -hook by means of the Bazhanov-Reshetikhin type formula

$$T_{a,s} = \det_{1 \leq j, k \leq a} T_{1,s+j-k} \left(u - \frac{i}{2}(a - k - j + 1) \right), \quad (2.18)$$

where the T-functions $T_{1,s}(u)$ are generated by the following generating functional (the direct generalization of (2.14))

$$W = \left[(1 - DY_1 D) \frac{1}{1 - DX_1 D} \frac{1}{1 - DX_2 D} (1 - DY_2 D) \right]_+ \times \left[(1 - DY_3 D) \frac{1}{1 - DX_3 D} \frac{1}{1 - DX_4 D} (1 - DY_4 D) \right]_- = \sum_{s=-\infty}^{\infty} D^s T_{1,s} D^s.$$

Here $\{Y_1(u)|X_1(u), X_2(u)|Y_2(u), Y_3(u)|X_3(u), X_4(u)|Y_4(u)\}$ are 8 arbitrary functions of the spectral u parameterizing the general solution. One expands here in positive powers of the operator D inside the bracket $[\dots]_+$ corresponding to the $u(2|2)_R$ subalgebra, and in negative powers of D inside the bracket $[\dots]_-$ corresponding to the $u(2|2)_L$ subalgebra. We hope to explore this general solution in the future for the construction of finite system of non-linear integral equations for the AdS/CFT spectrum.¹¹

In the next section, we will write the above $T_{a,s}$ for any a, s in a concise and explicit form, similar to the first Weyl formula for characters, through Wronskian-like determinant expressions of certain 2×2 and 4×4 matrices.

2.3 New determinant formulae

The sum in (2.17) can be calculated explicitly and the result can be presented in the following remarkable determinant form:

$$T_{a,s} = \begin{cases} (-1)^{(a+1)s} \left(\frac{x_3 x_4}{y_1 y_2 y_3 y_4} \right)^{s-a} \frac{\det \left(S_i^{\theta_{j,s+2}} y_i^{j-4-(a+2)\theta_{j,s+2}} \right)_{1 \leq i, j \leq 4}}{\det \left(S_i^{\theta_{j,0+2}} y_i^{j-4-(0+2)\theta_{j,0+2}} \right)_{1 \leq i, j \leq 4}}, & a \geq |s| \\ \frac{\det \left(Z_i^{(1-\theta_{j,a})} x_i^{2-j+(s-2)(1-\theta_{j,a})} \right)_{1 \leq i, j \leq 2}}{\det \left(Z_i^{(1-\theta_{j,0})} x_i^{2-j+(0-2)(1-\theta_{j,0})} \right)_{1 \leq i, j \leq 2}}, & s \geq +a \end{cases} \quad (2.19)$$

where

$$S_i = \frac{(y_i - x_3)(y_i - x_4)}{(y_i - x_1)(y_i - x_2)} \quad (2.20)$$

$$Z_i = \frac{(x_i - y_1)(x_i - y_2)(x_i - y_3)(x_i - y_4)}{(x_i - x_3)(x_i - x_4)}, \quad (2.21)$$

¹¹ A solution in the $\mathbb{T}$ -hook was already constructed in [52] on the base of the Bäcklund transformations of [75, 76].

Figure 3. Dynkin diagram for the Lie superalgebra $\mathfrak{gl}(4|4)$ with the Dynkin indexes corresponding to the characters in (2.19).

and

$$\theta_{j,s} = \begin{cases} 1, & j > s \\ 0, & j \leq s \end{cases}. \quad (2.22)$$

The other T 's can be obtained using the wing-exchange symmetry which is related to an outer automorphism of the Dynkin diagram of $\mathfrak{gl}(4|4)$

$$T_{a,s}(x_1, \dots, x_4 | y_1, \dots, y_4) = \left(\frac{y_1 y_2 y_3 y_4}{x_1 x_2 x_3 x_4} \right)^a T_{a,-s} \left(\frac{1}{x_4}, \dots, \frac{1}{x_1} \middle| \frac{1}{y_4}, \dots, \frac{1}{y_1} \right). \quad (2.23)$$

Note that for the $\mathfrak{sl}(4|4)$ case the first factor in the r.h.s. is absent.

These formulae are summarized in the appendix A in the *Mathematica* form. Note that the upper part of the T-hook is represented by a 4×4 determinant reminding the 1-st Weyl formula for $GL(4)$ characters (it would be them if all S_i were equal to 1). The left and right wings are presented by 2×2 determinants similar to $GL(2)$ characters. Hence, we can identify the variables y_1, y_2, y_3, y_4 as the eigenvalues from $U(4)$ subgroup of $SU(2, 2|4)$, the variables x_1, x_2 as the eigenvalues of the $U_R(2)$ subgroup and x_3, x_4 as the eigenvalues of the $U_L(2)$ subgroup. Our solution of the Q-system is symmetric under any permutations of y_i and the permutations of $x_1 \leftrightarrow x_2$ and $x_3 \leftrightarrow x_4$. However the solution is not invariant under the full Weyl group of $\mathfrak{gl}(4|4)$ which includes arbitrary permutations of x_i . The origin of this “symmetry breaking” will go back to the fact that $T_{a,s}$ is a super-character of an *infinite* dimensional representation. Another important property, under the rescaling of eigenvalues, reads

$$T_{a,s}(\alpha x_a, \alpha y_a) = \alpha^{as} T_{a,s}(x_a, y_a), \quad (2.24)$$

which means that $T_{a,s}$ is a homogeneous function of degree as .

3 Description of representations of $U(2, 2|4)$ for the AdS/CFT Y-system

In this section we describe in details the representations of $U(2, 2|4)$ group corresponding to the super-characters from the previous section.

3.1 Description of the general construction

In (2.19) we presented a formal solution of Hirota (Q-system) equations with the T-hook boundary conditions. At the same time it is known that the Q-systems with various boundary conditions can be often solved by characters of representations with rectangular Young diagrams. In this section, we describe a class of representations of the superalgebra $\mathfrak{gl}(4|4)$ leading to the super-character solution (2.19).¹² Provided the Y-system is indeed a set of equations encoding the exact AdS/CFT spectrum of the theory, these representations clearly have some physical importance. These representations live in the mirror space of the theory and reflect the $SU(2,2|4)$ symmetry properties of the mirror “particles”. In analogy to the spin chain terminology, we would call the mirror space as auxiliary space, in contradistinction to the physical space where the physical “particles” live.

Comparing our formula (2.19) with similar formulas for characters of the refs. [79, 80] we find that we deal with a class of representations called “unitarizable irreducible $\mathfrak{gl}(M_1|N|M_2)$ -modules”.¹³ We denote them as $W(M_1|N|M_2; \lambda)$. They are infinite dimensional irreducible highest weight representations of $\mathfrak{gl}(M|N)$ where $M = M_1 + M_2$ with respect to a (non-standard) Borel subalgebra corresponding to the $(M_1|N|M_2)$ grading described below. They are the unitary representations of $\mathfrak{u}(2,2|4)$.

The algebra is generated by standard super-generators E_{ab} . In fundamental representation $(E_{ab})_{ij} = \delta_{ia}\delta_{jb}$ and they obey the following super-commutation relation

$$[E_{ab}, E_{cd}] = \delta_{bc}E_{ad} - (-1)^{(p_a+p_b)(p_c+p_d)}\delta_{da}E_{cb}, \quad (3.1)$$

where the grading $p_i = 1$ for $i = M_1 + 1, M_1 + 2, \dots, M_1 + N$ and is 0 otherwise.

We define the Cartan subalgebra $\mathfrak{h}$ and Borel subalgebra $\mathfrak{b}$ as follows

$$\mathfrak{h} = \sum_{a=1}^{N+M} \mathbb{C}E_{aa} \quad , \quad \mathfrak{b} = \sum_{a \leq b} \mathbb{C}E_{ab} \quad . \quad (3.2)$$

Note that for the definition of the Borel subalgebra the ordering of indexes is crucial and we chose M_1 bosonic components, followed by N fermionic and then again by M_2 bosonic.

Then we introduce the space $\mathfrak{h}^*$ dual to the Cartan subalgebra $\mathfrak{h}$. Let ε_i be a graded basis of the dual space $\mathfrak{h}^*$ of the Cartan subalgebra $\mathfrak{h}$ such that $\varepsilon_i(E_{jj}) = \delta_{ij}$. We define a bilinear form $(\cdot|\cdot)$ in $\mathfrak{h}^*$:

$$(\varepsilon_i|\varepsilon_j) = (-1)^{p_i}\delta_{ij} \quad . \quad (3.3)$$

The simple root system in this basis is given as follows:

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad , \quad i = 1, 2, \dots, N + M - 1 \quad . \quad (3.4)$$

The class of representations we would like to describe is parameterized by a generalized partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_a)$ where $\lambda_1, \lambda_2, \dots, \lambda_a \in \mathbb{Z}$ (not necessarily positive) and ordered $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_a$ (see figure 4). Let λ be a generalized partition such that

¹²V.K. thanks N.Beisert for inspiring discussions on this subject.

¹³or $\mathfrak{gl}_{M_1+M_2|0+N}$ in the notations of [79].

Figure 4. “T- hook” and the highest weight components arranged as a generalized Young diagram living inside the T-hook.

$-N \leq \lambda_{a-M_1}, \lambda_{M_2+1} \leq 0$. Then $W(M_1|N|M_2; \lambda)$ is defined as the representation with the highest weight

$$\Lambda = - \sum_{i=1}^{M_1} (\langle \lambda_{i+a-M_1}^- - N \rangle + a) \varepsilon_i - \sum_{i=M_1+1}^{M_1+N} ((\lambda^-)'_{N+M_1+1-i} - a) \varepsilon_i + \sum_{i=N+M_1+1}^{N+M} \lambda_{i-N-M_1}^+ \varepsilon_i, \quad (3.5)$$

where we introduced the symbols

$$\langle x \rangle := \max(x, 0) \quad , \quad \lambda^+ := (\langle \lambda_1 \rangle, \langle \lambda_2 \rangle, \dots, \langle \lambda_a \rangle) \quad , \quad \lambda^- := (\langle -\lambda_1 \rangle, \langle -\lambda_2 \rangle, \dots, \langle -\lambda_a \rangle) \quad ,$$

and by $(\lambda^\pm)'$ we denote a conjugate partition obtained from the usual partition $\lambda^\pm$, with only positive entries, by the reflection of the associated Young diagram w.r.t. its main diagonal.¹⁴

The character formulae of these representations are given in [79]. In our case we have to take $M_1 = M_2 = 2$, $N = 4$ and the generalized partition is represented by a rectangular $a \times (s-2)$ Young diagram

$$\lambda = (\underbrace{s-2, \dots, s-2}_a). \quad (3.6)$$

On the level of representation, the Q-system for these representations follows from the decomposition of the tensor product of representations in the Theorem 6.1 in [79].

Now let us consider the case $\mathfrak{gl}(2|4|2)$ which is of the prime importance for us.

For the rectangular diagram (3.6) the weight (3.5) is

$$\Lambda = \begin{cases} a(-\varepsilon_1 - \varepsilon_2) + (s+2) \sum_{i=7}^{a+6} \varepsilon_i, & s < -2, \quad 0 \leq a \leq 2 \\ a(-\varepsilon_1 - \varepsilon_2 + \sum_{i=3}^{s+4} \varepsilon_i), & -2 \leq s \leq 2, \quad 0 \leq a \\ a(-\varepsilon_1 - \varepsilon_2 + \sum_{i=3}^6 \varepsilon_i) + (s-2) \sum_{i=7}^{a+6} \varepsilon_i, & 2 < s, \quad 0 \leq a \leq 2 \end{cases}. \quad (3.7)$$

¹⁴formally defined as $(\lambda^\pm)'_j = \text{Card}\{k | (\lambda^\pm)_k \geq j\}$.

The Kac-Dynkin labels¹⁵ can be easily calculated (see figure 3). The parameters $(x_1, \dots, x_4)$ and $(y_1, \dots, y_4)$ entering (2.19) can be defined in our notations as formal exponentials

$$x_3 = e^{\varepsilon_1}, x_4 = e^{\varepsilon_2} \mid y_1 = e^{\varepsilon_3}, y_2 = e^{\varepsilon_4}, y_3 = e^{\varepsilon_5}, y_4 = e^{\varepsilon_6} \mid x_1 = e^{\varepsilon_7}, x_2 = e^{\varepsilon_8}. \quad (3.8)$$

The way we identify x_i with ε_i 's is somewhat nontrivial. This notation means that for example x_3 for a given element h of the Cartan subalgebra returns the first eigenvalue $x_3(h)$ of the corresponding group element in the fundamental 8 dimensional representation.¹⁶

One may want to transform the Dynkin labels to different gradings. For that one can use the Weyl reflection with respect to the odd simple roots ([81]; V.V. Serganova, appendix to the paper: [82]) (see figure 5).

Figure 5. Transformation property of the Dynkin labels under the fermionic duality. The duality transform the diagram in one grading to another. The dotted lines correspond to the fermionic grading whereas the solid lines represent bosonic grading.

Q-system is a set of functional relations among characters of representations of Yangians, or quantum affine algebras. Thus a solution of the Q-system is in general a linear combination of characters of the Lie algebra. But for the super Yangian $Y(\mathfrak{gl}(M|N))$, it is just a super-character of $\mathfrak{gl}(M|N)$ since the evaluation map from $Y(\mathfrak{gl}(M|N))$ to $\mathfrak{gl}(M|N)$ allows one to lift the representations of $\mathfrak{gl}(M|N)$ to those of $Y(\mathfrak{gl}(M|N))$. We find that this is also the case with our AdS/CFT Q-system.

3.2 Unitarity

As we mentioned, the class of representations described above is unitarizable, which means that for a particular choice of the real form the representation is unitary. One can show [79] (section 3.2) that for this type of representations the generators have the following Hermitian conjugation properties

$$\eta E_{ab}^\dagger \eta = E_{ba} \quad (3.9)$$

where

$$\eta = \text{diag}(-1_{M_1}, +1_N, +1_{M_2}). \quad (3.10)$$

We see that the representations described above are indeed the representations of $SU(M_1, M_2|N)!$ We will examine this property below in section 3.4 for an explicit example.

¹⁵Here we define the Kac-Dynkin labels as $b_j = (\lambda|\alpha_j)$. In the mathematical literature it is usually normalized as $b_j = 2(\lambda|\alpha_j)/(\alpha_j|\alpha_j)$ for $(\alpha_j|\alpha_j) \neq 0$.

¹⁶In [79], the authors consider characters, while we are dealing with supercharacters. Thus one has to change the sign of y_i to compare our formulae with the character formulae in [79].

3.3 Comparing highest weight with the equation for characters

We can easily check (3.7) by extracting the highest weight from our expression for characters (2.19). We want the descendants of the highest weight to be suppressed which implies $e^{-\alpha_i} \ll 1$. From (3.4) and (3.8) we see that this can be achieved in the limit $|x_3| \gg |x_4| \gg |y_1| \gg |y_2| \gg |y_3| \gg |y_4| \gg |x_1| \gg |x_2|$. In this limit, we find from (2.19)

$$T_{1,+s} \simeq \frac{x_1^{s-2} y_1 y_2 y_3 y_4}{x_3 x_4}, \quad T_{1,-s} \simeq \frac{1}{x_3^{s-2} x_4^2}, \quad s \geq 2 \quad (3.11)$$

and

$$T_{a,+2} \simeq \frac{y_1^a y_2^a y_3^a y_4^a}{x_3^a x_4^a}, \quad T_{a,+1} \simeq \frac{y_1^a y_2^a y_3^a}{x_3^a x_4^a}, \quad T_{a,+0} \simeq \frac{y_1^a y_2^a}{x_3^a x_4^a}, \quad T_{a,-1} \simeq \frac{y_1^a}{x_3^a x_4^a}, \quad T_{a,-2} \simeq \frac{1}{x_3^a x_4^a} \quad (3.12)$$

which is in complete agreement with the highest weight (3.7) after the identification (3.8).

3.4 Example: representations of $GL(2)$

Let us first study this type of representations on the simplest example of $\mathfrak{gl}(1|0|1)$, where the simple root is given as $\alpha = \varepsilon_1 - \varepsilon_2$. The corresponding characters can be calculated from (2.12), similarly to (2.17), as follows:

$$T_{1,s} \equiv T_s^{(1+1)} = \frac{1}{x_2} \sum_{j=\max(0,-s)}^{\infty} T_{s+j}^{(1)}(x_1) T_j^{(1)}(1/x_2) = \sum_{j=\max(0,-s)}^{\infty} x_1^{s+j} x_2^{-j-1} \quad (3.13)$$

A simple calculation gives:

$$T_{1,s} \equiv T_s^{(1+1)}(x_1, x_2) = \begin{cases} \frac{x_1^s}{x_2 - x_1}, & s > 0 \\ \frac{x_2^s}{x_2 - x_1}, & s \leq 0 \end{cases} \quad (3.14)$$

It is interesting to notice that if we want to satisfy the Q-system (2.10) with $T_{1,s} \equiv T_s^{(1+1)}$ we have to add, after fixing the gauge $T_{0,s} = 1$, $-\infty < s < \infty$, another set of representations

$$T_{a,0} = \frac{x_2^{1-a}}{x_2 - x_1}. \quad (3.15)$$

Curiously, although we are not dealing here with a supergroup, the characters as solutions of the Q-system live in a T-hook, though having zero width in the vertical strip (see figure 6).

From (3.5) with $M_1 = M_2 = 1$ and $N = 0$ we have for these representations the highest weight

$$\Lambda = \begin{cases} -\varepsilon_1 + |s|\varepsilon_2, & s > 0, \quad a = 1 \\ -\varepsilon_1 - |s|\varepsilon_1, & s < 0, \quad a = 1 \\ -a\varepsilon_1, & s = 0, \quad a > 0 \end{cases} \quad (3.16)$$

The $s = 0, a > 0$ case is described by the same representation as $s < 0, a = 1$ under the identification $a = 1 - s$ which is consistent with (3.15) and (3.14). In what follows we

Figure 6. A degenerate “T-hook” for the $\mathfrak{gl}(1|0|1)$ representations.

consider only $a = 1$ case. Note that for both representations the Dynkin label is $-|s| - 1$. For the unimodular case $x_1 = 1/x_2$ the characters (3.14) are indeed equal $T_{1,s} = T_{1,-s}$ because $\varepsilon_1 + \varepsilon_2 = 0$. To build the highest weight representation we follow the standard procedure. We introduce the following combinations of generators

$$h_3 \equiv E_{11} - E_{22} \quad , \quad h_0 \equiv E_{11} + E_{22} \quad (3.17)$$

$$h_+ \equiv E_{12} \quad , \quad h_- \equiv -E_{21} \quad (3.18)$$

where h_0 commutes with all generators and the other commutation relations are

$$[h_3, h_{\pm}] = \pm 2h_{\pm} \quad , \quad [h_-, h_+] = h_3 \quad . \quad (3.19)$$

Since $\Lambda(h_0) = s - 1$ we have $h_0 = (s - 1)\text{id}$. Then $\Lambda(h_3) = -|s| - 1$ which implies the following infinite matrix representations of the generators

$$h_3 = \begin{pmatrix} -|s| - 1 & 0 & 0 & \cdots \\ 0 & -|s| - 3 & 0 & \cdots \\ 0 & 0 & -|s| - 5 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix} \quad , \quad (3.20)$$

and

$$h_+ = \begin{pmatrix} 0 & a_0 & 0 & 0 & \cdots \\ 0 & 0 & a_1 & 0 & \cdots \\ 0 & 0 & 0 & a_2 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \quad , \quad h_- = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ -a_0 & 0 & 0 & 0 & \cdots \\ 0 & -a_1 & 0 & 0 & \cdots \\ 0 & 0 & -a_2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix} \quad ,$$

$$a_j = i\sqrt{(j+1)(j+|s|+1)} \quad (3.21)$$

One can see that the $\mathfrak{gl}(2)$ commutation relations (3.19) are satisfied. We notice that we are dealing with the unitary representations of $\mathfrak{u}(1,1)$ since we have¹⁷

$$h_+^s = (h_-^s)^\dagger \quad . \quad (3.22)$$

¹⁷For $\mathfrak{u}(2)$ one should have $h_+^s = -(h_-^s)^\dagger$.

Now we can compute the character for these representations. Using the identification (3.8) we get

$$\begin{aligned} \text{tr exp}(\log x_2 E_{11} + \log x_1 E_{22}) &= \text{tr exp} \left(\frac{h_0}{2} \log x_1 x_2 + \frac{h_3}{2} \log \frac{x_2}{x_1} \right) \\ &= \sum_{j=1}^{\infty} \exp \left(\frac{s-1}{2} \log x_1 x_2 - \frac{|s|+2j-1}{2} \log \frac{x_2}{x_1} \right) \\ &= \frac{x_2^{\frac{s-|s|}{2}} x_1^{\frac{s+|s|}{2}}}{x_2 - x_1}, \end{aligned} \quad (3.23)$$

which perfectly agrees with (3.14) and with the character formula in [79]. More formally, one can write the character as

$$\sum_{n=0}^{\infty} e^{\Lambda - n\alpha} = \frac{e^{\Lambda}}{1 - e^{-\alpha}} \quad (3.24)$$

and then put $x_1 = e^{\varepsilon_2}$ and $x_2 = e^{\varepsilon_1}$ similarly to (3.8) to get (3.23).

Now let us consider the case $\mathfrak{gl}(0|2|0)$ with $M_1 = M_2 = 0$ and $N = 2$. From (3.5) we have

$$\Lambda_s = \begin{cases} a\varepsilon_1 + a\varepsilon_2, & s = 0 \\ a\varepsilon_1, & s = -1 \\ 0, & s = -2 \end{cases} \quad (3.25)$$

and $\alpha = \varepsilon_1 - \varepsilon_2$. The normalized Dynkin label $2(\Lambda|\alpha)/(\alpha|\alpha)$ is a for $s = -1$ and 0 for $s = 0, -2$. This means that the $s = -1$ case is the $a+1$ dimensional unitary representation of $\mathfrak{u}(2)$.

3.4.1 Quantum $GL(1+1)$

We can easily generalize this character solution of the $GL(2)$ Hirota equation (2.10) with the T-hook boundary conditions figure 6 to the full spectral parameter dependent solution of (2.6). Let us choose a gauge $T_{0,s} = 1$, $-\infty < s < \infty$, $T_{1,0} = 1$ and denote $T_{1,s>0}(u) = \frac{\Phi_R^{[s]}}{\Phi_R^{[-s]}}$, $T_{1,s<0}(u) = \frac{\Phi_L^{[s]}}{\Phi_L^{[-s]}}$, $T_{2,0} = \frac{\Psi^+}{\Psi^-}$. The only non-trivial Hirota equation expresses Ψ through two independent functions parameterizing the solution:

$$\xi(u) := \frac{\Psi^+}{\Psi^-} = 1 - \frac{\Phi_R^+ \Phi_L^+}{\Phi_R^- \Phi_L^-} \quad (3.26)$$

which can be solved as

$$\Psi(u + iK) = \Psi(u) \prod_{k=0}^{K-1} \xi(u + i(k + 1/2)) \quad (3.27)$$

The rest of the non-trivial T-functions are easily expressed from the Hirota equation (2.6) which has in these cases only two terms and represents the discrete Laplace equation:

$$T_{a>0,0}(u) = \frac{\Psi(u + i(a-1)/2)}{\Psi(u - i(a-1)/2)}, \quad (3.28)$$

$$T_{1,s>0}(u) = \frac{\Phi_R(u + is/2)}{\Phi_R(u - is/2)}, \quad (3.29)$$

$$T_{1,s<0}(u) = \frac{\Phi_L(u + is/2)}{\Phi_L(u - is/2)}. \quad (3.30)$$

Thus we parameterized the general solution in terms of two functions $\Phi_{L,R}(u)$. It would be interesting to understand whether this system describes any integrable sigma model, with appropriate analyticity properties for these functions.

4 Solution of Y-system in the scaling limit and quasiclassical strings

In this section we recall the construction [1] of the general solution of the AdS/CFT Y-system for an arbitrary state, to the leading wrapping order $\mathcal{O}(e^{-cL})$. An important feature of this solution is that it is in one to one correspondence with the large L spectrum of the theory given by asymptotic Bethe ansatz of [24]. It can be used to establish a link between an exact solution of the Y-system, or Hirota equation, and the corresponding state of the theory. Then we use this asymptotic solution of the Y-system to identify our $SU(2,2|4)$ character solution (2.19) with the full quasiclassical result of [64] containing the complete one-loop approximation of the Metsaev-Tseytlin sigma-model around any classical finite gap solution. In [65] it was proposed to read off the quantum numbers of a state by matching the behavior at large a and large s with its asymptotic large L solution. In this section we apply this procedure for a general classical state, using our new general character solution of the Y-system in the strong coupling limit. For each classical finite gap solution we find the corresponding solution of Y-system. Then we show that this general classical Y-system solution has the same structure as the one arising in the direct one-loop quasiclassical string analysis.¹⁸

4.1 Asymptotic solution in scaling limit

Let us remind the asymptotic large L solution of the Y-system [1] compatible with the ABA of [23, 24]. In the context of ABA the states are parameterized by 7 types of Bethe roots (one for each node of the $\mathfrak{psu}(2,2|4)$ Dynkin diagram). We denote them $u_{a,j}$ where the roots of each type $a = 1, \dots, 7$ are labeled by the index $j = 1, \dots, K_a$. The auxiliary roots $(u_{1,j}, u_{2,j}, u_{3,j})$ and $(u_{5,j}, u_{6,j}, u_{7,j})$ describe the “magnons” of the left and right $\mathfrak{su}(2|2)$ subalgebras. The middle node roots $u_{4,j}$ are momentum carrying and can be viewed as the

¹⁸Though the structures stemming from these two very different approaches, the Y-system on the one hand and the quasiclassical quantization of the finite gap algebraic curve on the other hand (confirmed by the direct one-loop computations in the string functional integral) will be convincingly identical, for the complete comparison one should study the exact Bethe equations for auxiliary roots $u_{a,j}$ (see below). We postpone the detailed analysis for the future work.

rapidities of the inhomogeneities of an $\mathfrak{su}_L(2|2) \oplus \mathfrak{su}_R(2|2)$ spin chain. The corresponding $\mathfrak{su}_L(2|2)$ S-matrix, satisfying the Yang-Baxter equation, was obtained in [83]. For a given K_4 one can construct the full $\mathfrak{psu}(2,2|4)$ transfer matrix from this S-matrix. Its eigenvalues are also parameterized by the auxiliary roots $(u_{1,j}, u_{2,j}, u_{3,j})$ for the left wing and $(u_{5,j}, u_{6,j}, u_{7,j})$ for the right wing. In the simplest fundamental $\mathfrak{su}_R(2|2)$ representation for the auxiliary transfer-matrix the eigenvalues are [1]

$$\mathbf{T}_{1,1}^R(u) = -\mathbf{T}_{1,1}^{R,1}(u) + \mathbf{T}_{1,1}^{R,2}(u) + \mathbf{T}_{1,1}^{R,3}(u) - \mathbf{T}_{1,1}^{R,4}(u) \quad (4.1)$$

where

$$\begin{aligned} \mathbf{T}_{1,1}^{R,1}(u) &= \frac{Q_1^-}{Q_1^+} \prod_{j=1}^{K_4} \frac{1 - 1/(x^+ x_{4,j}^-)}{1 - 1/(x^+ x_{4,j}^+)} \frac{x^- - x_{4,j}^-}{x^- - x_{4,j}^+}, & \mathbf{T}_{1,1}^{R,2}(u) &= \frac{Q_1^- Q_2^{++}}{Q_1^+ Q_2} \prod_{j=1}^{K_4} \frac{x^- - x_{4,j}^-}{x^- - x_{4,j}^+}, \\ \mathbf{T}_{1,1}^{R,3}(u) &= \frac{Q_2^- Q_3^+}{Q_2 Q_3^-} \prod_{j=1}^{K_4} \frac{x^- - x_{4,j}^-}{x^- - x_{4,j}^+}, & \mathbf{T}_{1,1}^{R,4}(u) &= \frac{Q_3^+}{Q_3^-} \end{aligned}$$

with the Q -functions and Zhukowsky $x(u)$ variables defined as follows

$$\begin{aligned} Q_a &= Q_a(u) = \prod_j^{K_a} (u - u_{a,j}) \\ x &= x(u) = \frac{1}{2} \left(u/g + i\sqrt{4 - u^2/g^2} \right) \\ x_{a,j} &= \frac{1}{2} \left(u_{a,j}/g + \sqrt{u_{a,j}/g - 2} \sqrt{u_{a,j}/g + 2} \right) . \end{aligned} \quad (4.2)$$

Similar formulas are true for $\mathfrak{su}_L(2|2)$ transfer matrix $\mathbf{T}_{1,-1}^L$ and we omitted the R, L superscripts by Q -functions. For more general representations in auxiliary space with rectangular $a \times s$ Young diagrams one can use the following generating function [1, 73–76, 84]

$$\mathcal{W}_R = \left[1 - \mathbf{T}_{1,1}^{R,1} D \right] \cdot \left[1 - \mathbf{T}_{1,1}^{R,2} D \right]^{-1} \cdot \left[1 - \mathbf{T}_{1,1}^{R,3} D \right]^{-1} \cdot \left[1 - \mathbf{T}_{1,1}^{R,4} D \right] , \quad D = e^{-i\partial_u} \quad (4.3)$$

To generate the corresponding $\mathbf{T}_{a,1}$ and $\mathbf{T}_{1,s}$ one should expand the above functional in a formal series in D and commute all D 's to the right

$$\mathcal{W}_R = \sum_{s=0}^{\infty} \mathbf{T}_{1,s}^R \left(u + i\frac{1-s}{2} \right) D^s , \quad \mathcal{W}_R^{-1} = \sum_{a=0}^{\infty} (-1)^a \mathbf{T}_{a,1}^R \left(u + i\frac{1-a}{2} \right) D^a . \quad (4.4)$$

In our gauge $\mathbf{T}_{a,0}^R = \mathbf{T}_{0,s}^R = 1$ and the boundary $\mathbf{T}_{a,2}^R, \mathbf{T}_{2,s}^R$ (corresponding to typical representations) can be easily found from the Hirota equation (2.6). One can also write similar equations for the left wing, replacing (Q_1, Q_2, Q_3) by (Q_7, Q_6, Q_5) . In order to match the ABA equations (valid at large L) it was proposed in [1] to relate these eigenvalues of transfer matrices to the Y-functions in the following way

$$\mathbf{Y}_{\Delta_a} = \frac{\mathbf{T}_{a,1}^+ \mathbf{T}_{a,1}^-}{\mathbf{T}_{a+1,1} \mathbf{T}_{a-1,1}} - 1 , \quad 1/\mathbf{Y}_{\bigcirc_s} = \frac{\mathbf{T}_{1,s}^+ \mathbf{T}_{1,s}^-}{\mathbf{T}_{1,s+1} \mathbf{T}_{1,s-1}} - 1 , \quad (4.5)$$

$$\mathbf{Y}_{\bullet_a} \simeq \mathbf{T}_{a,1}^R \mathbf{T}_{a,1}^L \prod_{n=-\frac{a-1}{2}}^{\frac{a-1}{2}} \Phi(u + in) \quad (4.6)$$

where

$$\begin{aligned} \Phi(u) = & \left(\frac{x^-}{x^+}\right)^L \prod_{j=1}^{K_4} \sigma^2(u, u_{4,j}) \frac{x_{4,j}^+ (1/x^+ - x_{4,j}^-)(x^- - x_{4,j}^+)}{x_{4,j}^- (1/x^- - x_{4,j}^+)(x^+ - x_{4,j}^-)} \times \\ & \times \prod_{j=1}^{K_1} \frac{1/x^+ - x_{1,j}}{1/x^- - x_{1,j}} \prod_{j=1}^{K_3} \frac{1/x^- - x_{3,j}}{1/x^+ - x_{3,j}} \prod_{j=1}^{K_5} \frac{1/x^- - x_{5,j}}{1/x^+ - x_{5,j}} \prod_{j=1}^{K_7} \frac{1/x^+ - x_{7,j}}{1/x^- - x_{7,j}}. \end{aligned} \quad (4.7)$$

Here $\sigma(u, v)$ is the dressing factor of [24]. This asymptotic solution of the Y-system was constructed in such a way as to fit the ABA equations of [23, 24], which take for the middle nodes the form

$$\mathbf{Y}_{\bullet_a}^{\text{ph}}(u_{4,j}) \simeq -1. \quad (4.8)$$

4.2 Asymptotic T-functions for the entire T-hook

To do the comparison of our character solution (2.19) to the asymptotic solution of the AdS/CFT Y-system described in the previous section, we have to find the T-functions not only for the $\mathfrak{su}_{L,R}(2|2)$ wings but also for the middle nodes. As was mentioned in [1] it is possible to pack these two sets of $\mathbf{T}_{a,s}^R, \mathbf{T}_{a,s}^L$ with a, s belonging to the $\mathfrak{su}_{L,R}(2|2)$ fat hooks (figure 2) into one $\mathfrak{psu}(2, 2|4)$ T-hook (figure 1a). This, however, necessarily involves a change of the gauge for T-functions so that at least one of the wings would be exponentially suppressed at large length L , similarly to the example of $\text{SU}(2)$ principal chiral field considered in [16, 85]. We define

$$\begin{aligned} \mathbf{T}_{a,+s} &= \mathbf{T}_{a,s}^R \prod_{m=-\frac{s-1}{2}}^{\frac{s-1}{2}} \prod_{n=-\frac{a-1}{2}}^{\frac{a-1}{2}} \Phi^R(u + in + im), \quad s > 0 \\ \mathbf{T}_{a,+0} &= 1, \\ \mathbf{T}_{a,-s} &= \mathbf{T}_{a,s}^L \prod_{m=-\frac{s-1}{2}}^{\frac{s-1}{2}} \prod_{n=-\frac{a-1}{2}}^{\frac{a-1}{2}} \Phi^L(u + in + im), \quad s > 0 \end{aligned} \quad (4.9)$$

where we split the factor $\Phi(u)$ into two

$$\begin{aligned} \Phi^R(u) &= \left(\frac{x^-}{x^+}\right)^{\frac{L}{2}} \prod_{j=1}^{K_4} \sigma(u, u_{4,j}) \sqrt{\frac{x_{4,j}^+ (\frac{1}{x^+} - x_{4,j}^-)(x^- - x_{4,j}^+)}{x_{4,j}^- (\frac{1}{x^-} - x_{4,j}^+)(x^+ - x_{4,j}^-)}} \prod_{j=1}^{K_1} \frac{\frac{1}{x^+} - x_{1,j}}{\frac{1}{x^-} - x_{1,j}} \prod_{j=1}^{K_3} \frac{\frac{1}{x^-} - x_{3,j}}{\frac{1}{x^+} - x_{3,j}}, \\ \Phi^L(u) &= \left(\frac{x^-}{x^+}\right)^{\frac{L}{2}} \prod_{j=1}^{K_4} \sigma(u, u_{4,j}) \sqrt{\frac{x_{4,j}^+ (\frac{1}{x^+} - x_{4,j}^-)(x^- - x_{4,j}^+)}{x_{4,j}^- (\frac{1}{x^-} - x_{4,j}^+)(x^+ - x_{4,j}^-)}} \prod_{j=1}^{K_7} \frac{\frac{1}{x^+} - x_{7,j}}{\frac{1}{x^-} - x_{7,j}} \prod_{j=1}^{K_5} \frac{\frac{1}{x^-} - x_{5,j}}{\frac{1}{x^+} - x_{5,j}}, \end{aligned} \quad (4.10)$$

so that $\Phi(u) = \Phi^R(u)\Phi^L(u)$. Notice that for large L both of these factors are exponentially small. For $s > 0$ and $s < 0$ the new $\mathbf{T}_{a,s}$ are equivalent up to a gauge transformation to the old ones. This implies that Hirota equation is satisfied exactly for $s > 0$ and $s < 0$. Moreover, it is easy to check that now Hirota equation is approximately satisfied even for

$s = 0$, though with an exponential precision in large L (for a fixed coupling). Finally, now we can write the middle node Y-functions $Y_{\bullet_a}$ in terms of these $\mathbf{T}$'s in a standard way

$$Y_{\bullet_a} = \frac{\mathbf{T}_{a,+1} \mathbf{T}_{a,-1}}{\mathbf{T}_{a+1,0} \mathbf{T}_{a-1,0}} \simeq \mathbf{T}_{a,+1} \mathbf{T}_{a,-1} . \quad (4.11)$$

4.3 Classical limit

Now we will take the classical limit in the asymptotic large L solution of the AdS/CFT Y-system described above. We remind that the ABA equations were constructed in [23, 86] to reproduce the correct finite gap algebraic curve of [54] in the scaling limit $L \sim K_a \sim \sqrt{\lambda}$. In this limit the Bethe roots are densely distributed along some linear stretches in the complex u -plane. These stretches can be interpreted as branch cuts of some Riemann surface connecting in various ways 8 sheets of this surface. We denote the corresponding 8-valued function as (λ_a, μ_a) where $a = 1, 2, 3, 4$. They are the eigenvalues of the monodromy matrix of the classical worldsheet theory. Using notations of [54] one can define

$$\lambda_a \equiv e^{-i\hat{p}_a}, \quad \mu_a \equiv e^{-i\tilde{p}_a} \quad (4.12)$$

where

$$\begin{aligned} \hat{p}_1 &= + \frac{Lx/(2g) + \mathcal{Q}_2 x}{x^2 - 1} + H_1 + \bar{H}_3 - \bar{H}_4 \\ \tilde{p}_1 &= + \frac{Lx/(2g) - \mathcal{Q}_1}{x^2 - 1} + H_1 - H_2 - \bar{H}_2 + \bar{H}_3 \\ \tilde{p}_2 &= + \frac{Lx/(2g) - \mathcal{Q}_1}{x^2 - 1} + H_2 - H_3 - \bar{H}_1 + \bar{H}_2 \\ \hat{p}_2 &= + \frac{Lx/(2g) + \mathcal{Q}_2 x}{x^2 - 1} - H_3 + H_4 - \bar{H}_1 \\ \hat{p}_3 &= - \frac{Lx/(2g) + \mathcal{Q}_2 x}{x^2 - 1} + H_5 - H_4 + \bar{H}_7 \\ \tilde{p}_3 &= - \frac{Lx/(2g) - \mathcal{Q}_1}{x^2 - 1} - H_6 + H_5 + \bar{H}_7 - \bar{H}_6 \\ \tilde{p}_4 &= - \frac{Lx/(2g) - \mathcal{Q}_1}{x^2 - 1} - H_7 + H_6 + \bar{H}_6 - \bar{H}_5 \\ \hat{p}_4 &= - \frac{Lx/(2g) - \mathcal{Q}_2 x}{x^2 - 1} - H_7 - \bar{H}_5 + \bar{H}_4 . \end{aligned} \quad (4.13)$$

Here the Bethe root resolvents H_a are

$$H_a = \sum_{j=1}^{K_a} \frac{x^2}{x^2 - 1} \frac{1}{x - x_{a,j}} , \quad \bar{H}_a(x) = H_1(1/x) . \quad (4.14)$$

Expanding (4.3) in the scaling limit one gets¹⁹

$$\mathcal{W}^R = \frac{(1 - d^R \lambda_1)(1 - d^R \lambda_2)}{(1 - d^R \mu_1)(1 - d^R \mu_2)} , \quad \mathcal{W}^L = \frac{(1 - d^L/\lambda_4)(1 - d^L/\lambda_3)}{(1 - d^L/\mu_4)(1 - d^L/\mu_3)} \quad (4.15)$$

¹⁹For this expansion we assume the spectral parameter to be in the upper half plane. For the other values an analytical continuation should be performed. See for example see [87] for more details of this procedure.

where $d^{\text{R,L}}$ are new formal expansion parameters related to the old D in the following way

$$d^{\text{R}} = D \exp \left[-i \left(-\frac{Lx/(2g) + xQ_2}{x^2 - 1} - H_4 + \bar{H}_1 - \bar{H}_3 \right) \right] , \quad (4.16)$$

$$d^{\text{L}} = D \exp \left[-i \left(-\frac{Lx/(2g) + xQ_2}{x^2 - 1} - H_4 + \bar{H}_7 - \bar{H}_5 \right) \right] . \quad (4.17)$$

The transfer-matrix eigenvalues look in new notations in this limit as follows

$$\begin{aligned} \mathbf{T}_{1,s}^{\text{R}} &= \frac{\mu_1^{s-1}(\mu_1 - \lambda_1)(\mu_1 - \lambda_2) - \mu_2^{s-1}(\mu_2 - \lambda_1)(\mu_2 - \lambda_2)}{\mu_1 - \mu_2} \left(\frac{d^{\text{R}}}{D} \right)^s \\ \mathbf{T}_{a,1}^{\text{R}} &= (-1)^a \frac{\lambda_1^{a-1}(\lambda_1 - \mu_1)(\lambda_1 - \mu_2) - \lambda_2^{a-1}(\lambda_2 - \mu_1)(\lambda_2 - \mu_2)}{\lambda_1 - \lambda_2} \left(\frac{d^{\text{R}}}{D} \right)^a \end{aligned} \quad (4.18)$$

and similarly for $\mathbf{T}^{\text{L}}$. Noticing that

$$\Phi^{\text{R}}(u) \simeq \exp \left[-i \left(\frac{xL/(2g) + xQ_2}{x^2 - 1} + H_4 - \bar{H}_1 + \bar{H}_3 \right) \right] = \frac{D}{d^{\text{R}}} , \quad (4.19)$$

$$\Phi^{\text{L}}(u) \simeq \exp \left[-i \left(\frac{xL/(2g) + xQ_2}{x^2 - 1} + H_4 - \bar{H}_7 + \bar{H}_5 \right) \right] = \frac{D}{d^{\text{L}}} \quad (4.20)$$

we see that the global $\mathbf{T}$'s defined in the previous section are functions of λ_a only!

$$\begin{aligned} \mathbf{T}_{2,+s} &= (\lambda_1 - \mu_1)(\lambda_1 - \mu_2)(\lambda_2 - \mu_1)(\lambda_2 - \mu_2) \mu_1^{s-2} \mu_2^{s-2} , \quad s > 1 \\ \mathbf{T}_{1,+s} &= \frac{\mu_1^{s-1}(\mu_1 - \lambda_1)(\mu_1 - \lambda_2) - \mu_2^{s-1}(\mu_2 - \lambda_1)(\mu_2 - \lambda_2)}{\mu_1 - \mu_2} , \quad s > 0 \\ \mathbf{T}_{+0,s} &= 1 \\ \mathbf{T}_{a,+0} &= 1 , \quad a > 0 \\ \mathbf{T}_{a,+1} &= (-1)^a \frac{\lambda_1^{a-1}(\lambda_1 - \mu_1)(\lambda_1 - \mu_2) - \lambda_2^{a-1}(\lambda_2 - \mu_1)(\lambda_2 - \mu_2)}{\lambda_1 - \lambda_2} , \quad a > 0 \\ \mathbf{T}_{a,+2} &= (\lambda_1 - \mu_1)(\lambda_1 - \mu_2)(\lambda_2 - \mu_1)(\lambda_2 - \mu_2) \lambda_1^{a-2} \lambda_2^{a-2} , \quad a > 1 \end{aligned} \quad (4.21)$$

and for $\mathbf{T}_{a,-s}$, $s > 0$ one should replace $\lambda_a \rightarrow 1/\lambda_{5-a}$ and $\mu_a \rightarrow 1/\mu_{5-a}$ in $\mathbf{T}_{a,+s}$. We recognize in these formulae the $\text{U}(2|2)$ super-characters generated by the formula (2.12) with $x_1 = \lambda_1, x_2 = \lambda_2, y_1 = \mu_1, y_2 = \mu_2$. This also implies that all asymptotic $\mathbf{Y}$ -functions can be written solely in terms of λ_j, μ_j in the scaling limit. Hirota equation is satisfied exactly for all nodes except the middle ones, for $s = 0$. Notice that $\lambda_1, \lambda_2, \mu_1, \mu_2 \sim \Delta = e^{-\frac{Lx}{2g(x^2-1)}}$ are exponentially small for large L/g whereas $\lambda_3, \lambda_4, \mu_3, \mu_4 \sim 1/\Delta$ are exponentially large. One can see that at $s = 0$ Hirota equation is satisfied only with Δ^{2a} precision for the a -th middle node. Thus for large a 's this solution should share the same behaviors with the exact solution. Now we will compare the classical solution (4.21) with our $\text{SU}(2, 2|4)$ characters (2.19).

4.4 Asymptotic solution as a limit of $U(2, 2|4)$ super-characters

Before relating the above asymptotic solution to (2.19), we first perform the following gauge transformation

$$\tilde{T}_{a,s} = \left(\frac{x_3 x_4}{y_3 y_4} \right)^a T_{a,s}. \quad (4.22)$$

In this gauge, the factor $\frac{y_3 y_4}{x_3 x_4}$ in (2.17) will disappear. Then under a natural identification which will be formally confirmed in the next section

$$x_i = \mu_i, \quad y_i = \lambda_i, \quad i = 1, \dots, 4 \quad (4.23)$$

we get for $L/g \gg 1$, when $\{\mu_1, \mu_2 | \lambda_1, \lambda_2\} \sim \Delta \ll 1$ and $\{\mu_3, \mu_4 | \lambda_3, \lambda_4\} \sim 1/\Delta \gg 1$,

$$\tilde{T}_{a,s} \simeq \mathbf{T}_{a,s} + \mathcal{O}(\Delta^{|s|+2}) \quad (4.24)$$

and $\mathbf{T}_{a,s} \sim \Delta^{|s|}$. Notice that the map (4.23) in particular implies that the $T_{a,s}$ have a very simple interpretation from the worldsheet theory point of view. We remind that $\{\mu_1, \dots, \mu_4 | \lambda_1, \dots, \lambda_4\}$ are the eigenvalues of the classical monodromy matrix

$$\Omega = \text{P exp} \oint \mathcal{L} d\sigma \quad (4.25)$$

where $\mathcal{L}$ is the classical Lax connection constructed as a linear combination of the worldsheet spacial and temporal components of the classical $SU(2, 2|4)$ current with the coefficients depending on a spectral parameter [55]. Thus $T_{a,s}$ are simply the characters of the monodromy matrix!²⁰

$$T_{a,s} = \text{Str}_{a,s} \Omega. \quad (4.26)$$

$T_{a,s}$, as well as the monodromy matrix, is thus an explicit functional of the elementary fields of Metsaev-Tseytlin superstring. One can speculate that at the quantum level a similar relation exists. Namely, we expect

$$T_{a,s} = \langle \text{state} | \text{Str}_{a,s} \hat{\Omega} | \text{state} \rangle. \quad (4.27)$$

Usually Hirota equation follows for this kind of objects automatically provided a Yang-Baxter relation is satisfied for the quantum analog of the Lax connection. Of course the details of this identification could be complicated²¹ [93].

²⁰This kind of relations is similar to the way the asymptotic solution is constructed as eigenvalues of transfer matrices of the asymptotic spin-chain. See also [88], appendix B and [54], eq. (2.40).

²¹It would be interesting to study this kind of relation at weak coupling where the spectrum is governed by a $\mathfrak{su}(2, 2|4)$ generalization of the rational Heisenberg spin chain whose space of state is infinite dimensional. For that one may evaluate the universal R -matrix for evaluation representations of the the super Yangian $Y(\mathfrak{gl}(4|4))$ based on infinite dimensional representations of $\mathfrak{u}(2, 2|4)$ mentioned in section 3 and use supersymmetric extension ([77, 89, 90]; see also the following paper from a different approach: [91]) of the Bazhanov-Lukyanov-Zamolodchikov construction of T and Q-operators [92]. This should lead to the weak coupling limit of $T_{a,s}$ in physical kinematics.

4.5 Fixing the parameters of the general solution

In this section we derive the map (4.23) in a direct way, similarly to how it was done for the $\mathfrak{sl}(2)$ sector in [65]. The map (4.23) could be established by comparing the large a and s asymptotics of T -functions (4.21) and the characters (2.19). First of all we formally treat (2.19) as a general solution of Hirota equation in the T-hook figure 1a. Indeed it has 7 independent gauge invariant parameters (due to (2.24) we can always set, say, $x_1 = 1$) and this is the maximal number of independent parameters for a solution in the T-hook up to the gauge transformations.²² Together with the gauge transformations we have 11 parameters to fix. We can always choose the gauge $T_{0,s} = 1$ which fixes $g_1 = g_3 = 1$ leaving us with only 9 parameters. We will fix these parameters by comparing the large a and s behavior of the exact $T_{a,s}$ with the asymptotic $\mathbf{T}_{a,s}$ and get the map between $(x_1, \dots, x_4, y_1, \dots, y_4, g_2)$ and $(\lambda_1, \dots, \lambda_4, \mu_1, \dots, \mu_4)$. We have for that precisely 9 equations

$$\lim_{a \rightarrow +\infty} \frac{1}{a} \log \frac{T_{a,s}^g}{\mathbf{T}_{a,s}} = 0 \quad , \quad s = -2, -1, 0, 1, 2 \quad (4.28)$$

$$\lim_{s \rightarrow +\infty} \frac{1}{s} \log \frac{T_{a,s}^g}{\mathbf{T}_{a,s}} = 0 \quad , \quad a = 1, 2 \quad (4.29)$$

$$\lim_{s \rightarrow -\infty} \frac{1}{s} \log \frac{T_{a,s}^g}{\mathbf{T}_{a,s}} = 0 \quad , \quad a = 1, 2 \quad (4.30)$$

where

$$T_{a,s}^g = g_2^a T_{a,s}. \quad (4.31)$$

Using the explicit expression (2.19) one finds

$$\lim_{a \rightarrow +\infty} \frac{1}{a} \log \frac{T_{a,+2}^g}{\mathbf{T}_{a,+2}} = \log \left(\frac{y_1 y_2 y_3 y_4}{x_3 x_4} \frac{g_2}{\lambda_1 \lambda_2} \right) \quad (4.32)$$

as well as

$$\lim_{a \rightarrow +\infty} \frac{1}{a} \log \frac{T_{a,0}^g}{\mathbf{T}_{a,0}} = \log \left(\frac{y_3 y_4}{x_3 x_4} g_2 \right) \quad (4.33)$$

We had to assume here that

$$|y_3 y_4| > |y_2 y_4|, |y_1 y_4|, |y_2 y_3|, |y_1 y_3|, |y_1 y_2| \quad (4.34)$$

which is the case asymptotically because $y_3, y_4 \sim 1/\Delta$, $y_1, y_2 \sim \Delta$, where Δ is exponentially small for large L . From (4.32) and (4.33) we obtain

$$g_2 = \frac{x_3 x_4}{y_3 y_4}, \quad y_1 y_2 = \lambda_1 \lambda_2. \quad (4.35)$$

Similarly

$$\lim_{a \rightarrow +\infty} \frac{1}{a} \log \frac{T_{a,+1}^g}{\mathbf{T}_{a,+1}} = \lim_{a \rightarrow +\infty} \frac{1}{a} \log \left(\frac{A_1 y_1^a + A_2 y_2^a}{B_1 \lambda_1^a + B_2 \lambda_2^a} \right) = 0 \quad (4.36)$$

²²For example, one can specify $T_{3,s}$, $T_{4,s}$, $s = -2, -1, 0, 1, 2$ and $T_{0,1}$ and reconstruct all the others $T_{a,s}$ from these $5 + 5 + 1$ functions by means of the Hirota equation. It follows from Hirota equation that $T_{2,s} = (T_{3,s}^+ T_{3,s}^+ - T_{3,s-1} T_{3,s+1}) / T_{4,s}$. Repeating this procedure it is easy to reconstruct all $T_{a,s}$ for any s and $a \geq |s|$. Then, along with $T_{0,1}$, one finds in the same way $T_{a,\pm s}$ for any a and $s > a$. One should extract then 4 gauge transformations $T_{a,s} \rightarrow g_1 g_2^a g_3^s g_4^{as} T_{a,s}$ to get $11 - 4 = 7$ parameters.

since generically we have $|\lambda_1| > |\lambda_2|$ or $|\lambda_2| > |\lambda_1|$ and the expression for $T_{a,s}^g$ is symmetric under exchange $y_1 \leftrightarrow y_2$. We can assume $|y_1| > |y_2|$ then from (4.36) and (4.32) we get $y_1 = \lambda_1$, $y_2 = \lambda_2$ or $y_1 = \lambda_2$, $y_2 = \lambda_1$. Again due to the symmetry we can always choose

$$y_1 = \lambda_1, y_2 = \lambda_2. \quad (4.37)$$

In the same way, from the other equations one fixes uniquely (up to a trivial interchange symmetries) the rest of relations, to obtain finally

$$y_i = \lambda_i, \quad x_i = \mu_i, \quad i = 1, \dots, 4 \quad (4.38)$$

which is the map we conjectured in the previous section on the basis of the asymptotic solution. Notice that in contrast to [65] we fixed the parameters of the general solution by comparing the large a, s limit of the T-functions rather than the Y-functions. It has an advantage because one does not need to rely on TBA equations *at all* in this consideration. From this point of view the TBA approach is only one of the numerous tests of the fundamental Hirota dynamics. The asymptotic solution and the T-hook boundary conditions for Hirota dynamics of [1] are the only two fundamental blocks needed to find the exact T-functions and thus to find the AdS/CFT spectrum. Moreover, in the framework of the integral TBA equations each state should be subjected to a case-by-case study since the “driving” terms could be very different for different states and could change when a singularity of the integrand crosses the integration contours at some values of parameters [10–12, 36, 47] leaving unaffected the equations in the functional form.²³ Technically, however, one should not ignore completely the TBA equations since they give a convenient framework for the study of the spectrum. In the next section we speculate about the possible extension of the TBA equations for excited states to the other sectors, using the known exact solution built above for strong coupling.

4.6 Magic products

$T_{a,s}$ are rather complicated functions. Moreover they contain a gauge ambiguity. It was shown in [65] that some particularly important gauge invariant combinations of them are relatively simple functions of λ_i, μ_i . They are some infinite products of $Y_{a,s}$ called in [65] as “magic” products. Here we give their generalizations from the $\mathfrak{sl}(2)$ sector to the full theory. All of them can be verified simply using the Hirota equation (2.10). There exists a couple of relatively simple “magic” products

$$e^{\mathcal{M}_F^+} \equiv \frac{1}{Y_{1,+1}Y_{2,+2}} \prod_{a=1}^{\infty} (1 + Y_{a,0}) = \frac{\lambda_1 \lambda_2}{\mu_1 \mu_2}, \quad (4.39)$$

$$e^{\mathcal{M}_F^-} \equiv \frac{1}{Y_{1,-1}Y_{2,-2}} \prod_{a=1}^{\infty} (1 + Y_{a,0}) = \frac{\mu_3 \mu_4}{\lambda_3 \lambda_4} \quad (4.40)$$

²³There are usually several equivalent forms integral equations. They could have a different structures of singularities which indicates their artificial, non-physical nature.

and a bit more complicated ones

$$e^{\mathcal{M}_0} \equiv \prod_{a=1}^{\infty} (1 + Y_{a,0})^a = e^{\mathcal{N}_*} \quad , \quad \mathcal{N}_* \equiv \sum_{i=1,2} \sum_{j=3,4} \log \frac{(1 - \mu_i/\lambda_j)(1 - \lambda_i/\mu_j)}{(1 - \mu_i/\mu_j)(1 - \lambda_i/\lambda_j)} \quad , \quad (4.41)$$

and

$$e^{-\mathcal{M}_+^+} \equiv e^{-\mathcal{M}_0} \prod_{a=1}^{\infty} \left(\frac{1 + Y_{a,+1}}{1 + \mathbf{Y}_{a,+1}} \right)^a = e^{-\mathcal{N}_{2*}} \quad , \quad |\lambda_1| < |\lambda_2| \quad , \quad (4.42)$$

$$e^{-\mathcal{M}_+^-} \equiv e^{-\mathcal{M}_0} \prod_{a=1}^{\infty} \left(\frac{1 + Y_{a,-1}}{1 + \mathbf{Y}_{a,-1}} \right)^a = e^{-\mathcal{N}_{\hat{3}*}} \quad , \quad |\lambda_3| < |\lambda_4| \quad (4.43)$$

where

$$\mathcal{N}_{\hat{i}*} \equiv \sum_{j=3,4} \log \frac{(1 - \lambda_i/\mu_j)}{(1 - \lambda_i/\lambda_j)} \quad , \quad \mathcal{N}_{\tilde{i}*} \equiv \sum_{j=3,4} \log \frac{(1 - \mu_i/\lambda_j)}{(1 - \mu_i/\mu_j)} \quad , \quad (4.44)$$

$$\mathcal{N}_{*j} \equiv \sum_{i=1,2} \log \frac{(1 - \mu_i/\lambda_j)}{(1 - \lambda_i/\lambda_j)} \quad , \quad \mathcal{N}_{*\hat{j}} \equiv \sum_{i=1,2} \log \frac{(1 - \lambda_i/\mu_j)}{(1 - \mu_i/\mu_j)} \quad .$$

For $|\lambda_1| > |\lambda_2|$ one can use the $\lambda_1 \leftrightarrow \lambda_2$ symmetry.²⁴ And finally we define

$$e^{-\mathcal{M}_-^+} \equiv \frac{1 + 1/Y_{2,+2}}{1 + 1/\mathbf{Y}_{2,+2}} \prod_{a=2}^{\infty} \left(\frac{1 + \mathbf{Y}_{a,+1}}{1 + Y_{a,+1}} \right)^{a-2} = e^{-\mathcal{N}_{\hat{1}*}} \quad , \quad |\lambda_1| < |\lambda_2| \quad (4.45)$$

$$e^{-\mathcal{M}_-^-} \equiv \frac{1 + 1/Y_{2,-2}}{1 + 1/\mathbf{Y}_{2,-2}} \prod_{a=2}^{\infty} \left(\frac{1 + \mathbf{Y}_{a,-1}}{1 + Y_{a,-1}} \right)^{a-2} = e^{-\mathcal{N}_{\hat{4}*}} \quad , \quad |\lambda_3| < |\lambda_4| \quad . \quad (4.46)$$

In the next section we will see that these products appear in the equations for the spectrum at the one-loop level and also discuss them in the context of quasiclassical quantization.

4.7 One-loop energy and quasiclassical quantization

In this section we show that the leading classical solution of the Y-system allows to make a rather nontrivial comparison with the direct worldsheet quasiclassical one-loop quantization, technically entirely based on the algebraic curve of the classical finite gap solution of [54].

4.7.1 Quasiclassical quantization from the algebraic curve

In this section we briefly remind the idea of one-loop quantization from the algebraic curve. The algebraic curve allows one to classify the quasi-periodic classical solutions of the worldsheet sigma model in a transparent and covariant way. The algebraic curve is an 8-sheet Riemann surface constructed out of the eigenvalues $(\lambda_1, \dots, \lambda_4 | \mu_1, \dots, \mu_4)$ of

²⁴It is clear that $e^{\mathcal{M}_+^+}$ is not analytic for real values of the spectral parameter since it follows from (4.13) that on the real axis (between the branch points) $|\lambda_1| = |\lambda_2|$ due to the $x \rightarrow 1/x$ symmetry. Similar statement is true for $e^{\mathcal{M}_+^-}$ which is symmetric under the exchange of λ_3 and λ_4 . The analytic continuation under the real axis is given by $\mathcal{M}_-^{\pm}$.

the monodromy matrix $\Omega(x)$. The BMN vacuum corresponds to the “empty” curve where the eigenvalues have only singularities at $x = \pm 1$. In general, there are some branch cuts connecting the sheets. The ABA is equivalent at the classical level to the algebraic curve and the cuts can be thought of as condensates of the Bethe roots. The precise map between the configuration of the Bethe roots and the curve is given by (4.13), where λ_i and μ_i are related to 8 quasi-momenta $p_i = \{\hat{p}_i, \tilde{p}_i\}$ by (4.12) where $\mathbf{i}$ can take the values $\hat{1}, \dots, \hat{4}$ or $\tilde{1}, \dots, \tilde{4}$.

As was proposed in [56] the one-loop corrections to the classical energies can be computed by studying the spectrum of quadratic fluctuations $\delta E_n^{\mathbf{ij}}$ around a given classical state

$$\delta E = \frac{1}{2} \sum_{(\mathbf{ij})} \sum_n (-1)^{F_{\mathbf{ij}}} \delta E_n^{\mathbf{ij}} \quad (4.47)$$

where $\mathbf{ij}$ labels the polarization and can take the following values for bosonic fluctuations ($F_{\mathbf{ij}} = 0$)

$$(\mathbf{ij}) = (\tilde{1}, \tilde{3}), (\tilde{2}, \tilde{3}), (\tilde{1}, \tilde{4}), (\tilde{2}, \tilde{4}) , \quad (\hat{1}, \hat{3}), (\hat{2}, \hat{3}), (\hat{1}, \hat{4}), (\hat{2}, \hat{4}) , \quad (4.48)$$

and for fermionic fluctuations ($F_{\mathbf{ij}} = 1$)

$$(\mathbf{ij}) = (\tilde{1}, \hat{3}), (\tilde{2}, \hat{3}), (\tilde{1}, \hat{4}), (\tilde{2}, \hat{4}) , \quad (\hat{1}, \tilde{3}), (\hat{2}, \tilde{3}), (\hat{1}, \tilde{4}), (\hat{2}, \tilde{4}) . \quad (4.49)$$

The quadratic fluctuations have a natural interpretation in terms of some additional small cuts on the curve in the background of the macroscopic cuts corresponding to the initial classical finite gap solution. For example the fluctuation $(\tilde{1}, \tilde{3})$ correspond to additional small cuts connecting $\tilde{p}_1$ and $\tilde{p}_3$. The values of the spectral parameter x where one can add small cuts are not arbitrary. At these points the sheets of the Riemann surface should touch each other which imposes

$$p_{\mathbf{i}}(x_n^{\mathbf{ij}}) - p_{\mathbf{j}}(x_n^{\mathbf{ij}}) = 2\pi n . \quad (4.50)$$

Addition of these extra cuts has two different effects: firstly, the small cuts by themselves carry an energy, secondly, the roots belonging to the big cuts are now slightly displaced which also affects the energy

$$\delta E = \frac{1}{2} \sum_{(\mathbf{ij})} \sum_n (-1)^{F_{\mathbf{ij}}} \omega(x_n^{\mathbf{ij}}) + \int_C \omega(x) \delta \rho(x) dx , \quad (4.51)$$

where $\omega(x) \equiv \frac{x^2+1}{x^2-1}$. Notice that we can convert the sum in (4.51) into an integral as follows

$$\frac{1}{2} \oint \frac{dx}{2\pi i} \omega(x) \partial_x \mathcal{N}_* . \quad (4.52)$$

Indeed $\partial_x \mathcal{N}_*$ defined in (4.41) has simple poles with residues ± 1 when the condition (4.50) is satisfied and thus leads precisely to the sum in (4.51). Next we deform the contour in the integral above so that it goes around the unit circle $|x| = 1$. The integral around the unit circle in Zhukovski parameterization $2z = x + 1/x$ is given as follows

$$\delta E_{\text{direct}} = \int_{-1}^1 \frac{dz}{2\pi} \frac{z}{\sqrt{1-z^2}} \partial_z \mathcal{N}_* . \quad (4.53)$$

It is easy to see that $\mathcal{N}_*$ is exponentially small when $L/\sqrt{\lambda} \gg 1$ and to match the ABA one should drop it. The sum in (4.51) is however not necessarily exponentially small in this limit. This is because while deforming the contour one should also take into account that the quasi-momenta in $\mathcal{N}_*$ could have various quadratic branch cuts. These branch cuts get also caught into the deformed contour and they give rise to generically non-vanishing in this limit contributions. However, these contributions could be absorbed into a redefinition of $\delta\rho$ [64] so that the one-loop correction becomes²⁵

$$\delta E = \delta E_{\text{direct}} + \int_{\mathcal{C}} \omega(x) \delta \varrho(x) dx . \quad (4.54)$$

The modified density $\varrho(x)$ is precisely the density of the momentum carrying roots $u_{4,j}$ in the asymptotic limit $L/\sqrt{\lambda} \gg 1$ [64]. In our case $L/\sqrt{\lambda} \sim 1$ but we can show that ϱ is also a density of the roots $u_{4,j}$ but for corrected ABA equations (see appendix B for more details)

$$\begin{aligned} 1 = & - \left(\frac{x_{4,k}^-}{x_{4,k}^+} \right)^L \prod_{j=1}^{K_4} \sigma^2(u_{4,k}, u_{4,j}) \frac{x_{4,j}^+ (1/x_{4,k}^+ - x_{4,j}^-)(x_{4,k}^- - x_{4,j}^+)}{x_{4,j}^- (1/x_{4,k}^- - x_{4,j}^+)(x_{4,k}^+ - x_{4,j}^-)} \times \\ & \times \prod_{j=1}^{K_1} \frac{1/x_{4,k}^+ - x_{1,j}}{1/x_{4,k}^- - x_{1,j}} \prod_{j=1}^{K_3} \frac{x_{4,k}^+ - x_{3,j}}{x_{4,k}^- - x_{3,j}} \prod_{j=1}^{K_5} \frac{x_{4,k}^+ - x_{5,j}}{x_{4,k}^- - x_{5,j}} \prod_{j=1}^{K_7} \frac{1/x_{4,k}^+ - x_{7,j}}{1/x_{4,k}^- - x_{7,j}} \\ & \times \exp \left[- \int_{-1}^{+1} (r(x_{4,k}, z)(\mathcal{N}_{2*} + \mathcal{N}_{*3}) - r(1/x_{4,k}, z)(\mathcal{N}_{1*} + \mathcal{N}_{*4}) + 2u(x_{4,k}, z)\mathcal{N}_*) dz \right] \end{aligned} \quad (4.55)$$

where we use the following kernels

$$\begin{aligned} r(y, z) &= \frac{y^2}{y^2 - 1} \frac{\partial_z}{2\pi g} \frac{1}{y - x(z)} , \quad u(y, z) = \frac{y}{y^2 - 1} \frac{\partial_z}{2\pi g} \frac{1}{x^2(z) - 1} \\ q(y, z) &= \frac{1}{y^2 - 1} \frac{\partial_z}{2\pi g} \frac{x(z)}{x^2(z) - 1} . \end{aligned}$$

The last factor in (4.55) is exponentially small in this scaling limit and contains the information about all wrappings for $L \sim \sqrt{\lambda}$. Equations for the auxiliary roots should be modified as well. These modified equations are presented in the appendix B.

Now we will see how naturally these structures appear in the expression for the energy of a state and for the exact finite volume Bethe equations.

4.7.2 One-loop energy from Y-system

In this subsection we will show how the energy can be computed from the Y-system with the one-loop accuracy. Expanding (2.2) at strong coupling we get

$$E = \sum_{j=1}^{K_4} \frac{x_{4,j}^2 + 1}{x_{4,j}^2 - 1} - \sum_{a=1}^{\infty} \int_{-\infty}^{\infty} \frac{dz}{2\pi} \partial_z \frac{z}{\sqrt{1 - z^2}} \log(1 + Y_{a,0})^a + \mathcal{O}(1/\sqrt{\lambda}) , \quad (4.56)$$

²⁵For the $\mathfrak{su}(2)$ and $\mathfrak{sl}(2)$ subsectors the modification is trivial $\varrho = \rho$.

where we introduce a rescaled spectral parameter $z = u/(2g)$. In the scaling limit the number of Bethe roots K_4 should scale as $\sqrt{\lambda}$. We see that the first term is of the order $\sqrt{\lambda}$ and thus contains the classical part of the energy which scales as $\sqrt{\lambda}$ and also some part of the one loop correction which scales as λ^0 . The second term is already of the order λ^0 and contributes only to the one-loop correction. To evaluate the second term one only needs to know the leading order $Y_{a,0}$ found in the previous sections. As it can be seen from the asymptotic expressions for $Y_{a,0}$ in the considered limit, they are supported on the interval $(-1,1)$. We assume that the exact $Y_{a,0}$ are suppressed outside this interval and one can integrate only over this interval in (4.56). Using (4.41) we immediately recognize that the integral term is precisely δE_{direct} of (4.53) from the algebraic curve computation (and hence related to ABA).

Now we have to define the positions of roots $x_{4,j}$ with a precision $\mathcal{O}(\lambda^0)$. They can be expanded in $1/\sqrt{\lambda}$

$$x_{4,j} = x_{4,j}^0 + x_{4,j}^1/\sqrt{\lambda} + \dots \quad (4.57)$$

The asymptotic values $x_{4,j}^0$ lead to the classical energy from the first term in (4.56) whereas the corrections $x_{4,j}^1$ contribute at one-loop. Below we show that the equation for $x_{4,j}$ coincides with the Bethe equation (4.55) corrected by virtual fluctuations. For that one can speculate about a possible form of TBA equations for excited states outside the $\mathfrak{sl}(2)$ sector²⁶ and extend the consideration of [65] to the full theory.

For the $\mathfrak{sl}(2)$ sector the TBA equation for the $Y_{1,0}$ was proposed in [32] to be of the form

$$\log Y_{1,0} = \sum_{m=1}^{\infty} \mathcal{T}_{1,m} * \log(1 + Y_{m,0}) + \sum_{m=1}^{\infty} 2\mathcal{R}^{(10)} \circledast K_{m-1} * \log(1 + Y_{m,1}) + i\Phi \quad (4.58)$$

where $*$ and $\circledast$ denote some convolutions, $\mathcal{T}_{1,m}$, $\mathcal{R}^{(10)}$ are the kernels and Φ is a potential term, all defined in [32]. This equation is especially important for us since it allows to find the corrected Bethe equation for the middle node by doing analytic continuation to the physical sheet, like in [36]. For the $\mathfrak{sl}(2)$ sector one had $Y_{a,+s} = Y_{a,-s}$ and thus a natural generalization is

$$\log Y_{1,0} = \sum_{m=1}^{\infty} \mathcal{T}_{1,m} * \log(1 + Y_{m,0}) + \sum_{m=1}^{\infty} \mathcal{R}^{(10)} \circledast K_{m-1} * \log(1 + Y_{m,+1})(1 + Y_{m,-1}) + i\tilde{\Phi}.$$

In [32] the potential terms were inferred by recovering the ABA type contributions from the kernels of TBA equations for the vacuum [32, 49–51]. As was shown in [65], technically it is very convenient (and should be also useful for numerics) to subtract the equation satisfied by the asymptotic solution from the above exact equation, to cancel the potential terms²⁷

$$\log \frac{Y_{1,0}}{\mathbf{Y}_{1,0}} = \sum_{m=1}^{\infty} \mathcal{T}_{1,m} * \log(1 + Y_{m,0}) + \sum_{m=1}^{\infty} \mathcal{R}^{(10)} \circledast K_{m-1} * \log \left(\frac{1 + Y_{m,+1}}{1 + \mathbf{Y}_{m,+1}} \frac{1 + Y_{m,-1}}{1 + \mathbf{Y}_{m,-1}} \right). \quad (4.59)$$

²⁶We would like to thank P.Vieira for the discussion on some of the points raised in this section.

²⁷Strictly speaking the driving terms in the exact and asymptotic solution could be slightly different. We assume that this difference is suppressed.

Another advantage of this trick at strong coupling is that the Y 's outside the interval $[-1, 1]$ in the rescaled variable $z = u/(2g)$ where $g = \frac{\sqrt{\lambda}}{4\pi}$ should coincide with the asymptotic $\mathbf{Y}$'s since the middle nodes $Y_{a,0}$ are exponentially small for these values of the spectral parameter in the mirror kinematics. Thus in all convolutions we can restrict the integrations to the interval $[-1, 1]$ where our solution of Q-system is valid. Making the analytic continuation to the physical sheet we get [36]

$$\begin{aligned} \log \frac{Y_{1,0}^{\text{ph}}}{\mathbf{Y}_{1,0}^{\text{ph}}} &= \sum_{m=1}^{\infty} \mathcal{T}_{1,m}^{\text{ph,mir}} * \log(1 + Y_{m,0}) + \sum_{m=1}^{\infty} K_{m-1}^- * \log \left(\frac{1 + Y_{m,+1}}{1 + \mathbf{Y}_{m,+1}} \frac{1 + Y_{m,-1}}{1 + \mathbf{Y}_{m,-1}} \right) \\ &+ \sum_{m=2}^{\infty} \left(\mathcal{R}^{(10)\text{ph,mir}} - \mathcal{B}^{(10)\text{ph,mir}} \right) * K_{m-1} * \log \left(\frac{1 + Y_{m,+1}}{1 + \mathbf{Y}_{m,+1}} \frac{1 + Y_{m,-1}}{1 + \mathbf{Y}_{m,-1}} \right) \\ &+ \mathcal{R}^{(10)\text{ph,mir}} * \log \left(\frac{1 + Y_{1,+1}}{1 + \mathbf{Y}_{1,+1}} \frac{1 + Y_{1,-1}}{1 + \mathbf{Y}_{1,-1}} \right) \\ &- \mathcal{B}^{(10)\text{ph,mir}} * \log \left(\frac{1 + 1/Y_{2,+2}}{1 + 1/\mathbf{Y}_{2,+2}} \frac{1 + 1/Y_{2,-2}}{1 + 1/\mathbf{Y}_{2,-2}} \right) \end{aligned}$$

where the last two terms appeared from converting the convolution around the B-cycle $\circledast$ into two usual integrals $*$ over $[-1, 1]$. Now we simply use the strong coupling expansion of the kernels at large g from [65]²⁸

$$\begin{aligned} \mathcal{R}^{(10)\text{ph,mir}}(z_k, w) &\simeq r(x_k, w) , \\ \mathcal{B}^{(10)\text{ph,mir}}(z_k, w) &\simeq r(1/x_k, w) , \\ \mathcal{T}_{1,m}^{\text{ph,mir}}(z_k, w) &\simeq -m [2r(x_k, w) + 2u(x_k, w)] , \\ K_m(z_k - w) &\simeq \delta(w - z_k) + m [r(x_k, w) + r(1/x_k, w)] , \end{aligned} \tag{4.60}$$

which leads to

$$\begin{aligned} \log \frac{Y_{1,0}^{\text{ph}}}{\mathbf{Y}_{1,0}^{\text{ph}}} &= r(x_k, w) \log \prod_{m=1}^{\infty} \left(\frac{1}{(1 + Y_{m,0})^2} \frac{1 + Y_{m,+1}}{1 + \mathbf{Y}_{m,+1}} \frac{1 + Y_{m,-1}}{1 + \mathbf{Y}_{m,-1}} \right)^m \\ &+ r(1/x_k, w) \log \frac{1 + 1/\mathbf{Y}_{2,+2}}{1 + 1/Y_{2,+2}} \frac{1 + 1/\mathbf{Y}_{2,-2}}{1 + 1/Y_{2,-2}} \prod_{m=1}^{\infty} \left(\frac{1 + Y_{m,+1}}{1 + \mathbf{Y}_{m,+1}} \frac{1 + Y_{m,-1}}{1 + \mathbf{Y}_{m,-1}} \right)^{m-2} \\ &- 2u(x_k, w) \log(1 + Y_{m,0})^m . \end{aligned}$$

Using the relations (4.41), (4.42), (4.45) for this kind of products of Y-functions and assuming that the contours of integration are displaced so that $|\lambda_1| < |\lambda_2|$ and $|\lambda_3| < |\lambda_4|$ we get

$$\log \frac{Y_{1,0}^{\text{ph}}}{\mathbf{Y}_{1,0}^{\text{ph}}} \simeq \int_{-1}^1 \left(-r(x_k, z) (\mathcal{N}_{\hat{2}*} + \mathcal{N}_{*\hat{3}}) + r(1/x_k, z) (\mathcal{N}_{\hat{1}*} + \mathcal{N}_{*\hat{4}}) - 2u(x_k, z) \mathcal{N}_* \right) dz .$$

²⁸Here we use the equivalent form of TBA equations for the massive nodes which differs by zero total momentum from the ones in [65]. This requires simultaneous redefinition of the kernel $\mathcal{T}_{1,m}^{\text{ph,mir}}(z_k, w)$ and the free terms like in [67]. As a result the strong coupling expansion of $\mathcal{T}_{1,m}^{\text{ph,mir}}(z_k, w)$ is a bit different from [67].

Since the exact Bethe equation for the momentum carrying node is $Y_{1,0}^{\text{ph}}(u_{4,k}) = -1$ we get the modified asymptotic Bethe equation (4.55) from the previous section.

We see that precisely the same equations for the spectrum appear in both the Y-system and the quasiclassical quantization of algebraic curve. This is a striking confirmation of the correctness of the AdS/CFT Y-system [1]. To complete the prove that both approaches lead to the same result one should also study the auxiliary Bethe equations. Whereas it is easy to obtain the modified form of the auxiliary Bethe equations from the quasiclassical quantization, it is usually more complicated to see them directly from the Y-system (see e.g. [16]). It should follow from some analyticity conditions, which can be read off from the asymptotic solution of the Y-system. Indeed the asymptotic solution is constructed in terms of the transfer-matrices and the auxiliary Bethe equations are simply the conditions of pole cancellations. We postpone the detailed analysis of the auxiliary equations for the future work.

5 Conclusions

One of the main motivations for this work was to find a good test for the Y-system conjectured in [1], in the situation when the asymptotic Bethe ansatz is essentially inaccurate due to the presence of all multiple windings, but the Y-system is still treatable analytically. The strong coupling limit considered here, in the situation when $L \sim \sqrt{\lambda} \rightarrow \infty$, gives such an opportunity.

On the one hand, the all-wrapping corrections originated from the discreteness of the sum over the fluctuation frequencies [94, 95] are present already in the one-loop energy; on the other hand, the dependence on the spectral parameter u becomes slow²⁹ in this scaling limit and hence the finite difference operator could be neglected in Y-system and in the related Hirota equation. In this case, the Hirota equation simplifies to the form (2.10) and becomes similar to the one solved by the (super)-characters of irreducible representations with $(a \times s)$ rectangular Young diagrams. This simplified Hirota equation is often called as “Q-system” in the mathematical literature.

Another motivation of this work was to understand the nature of the $\text{PSU}(2, 2|4)$ representations which enter through the a, s variables into the Q-system. The full superconformal group was usually difficult to identify because of the light cone gauge in which this superstring theory is studied. On this way, we managed to find a general solution of such a Y-system with no shifts with respect to the spectral parameter, with the AdS/CFT type boundary conditions: Y-functions are defined within the T-hook figure 1b in the representation space of $(a \times s)$ rectangular Young diagrams. The solution we found appears to define super-characters of certain unitary infinite-dimensional representations of $\text{SU}(2, 2|4)$. Comparing the $a, s \rightarrow \infty$ asymptotics of these super-characters with those of the asymptotic solution we uniquely identified the parameters of these super-characters with the eigenvalues of the classical monodromy matrix or equivalently with 8 quasi-momenta of an arbitrary finite gap solution. Importantly, to build the classical solution of the Y-system we

²⁹this approximation is not valid in the vicinity of the real axis for $|u| > 2g$ but should give the right result on the rest of the complex plane.

use only the data from [1] with no input from the TBA approach. This reduces drastically the number of assumptions we have to adopt.

Then we show that using this leading order strong coupling solution of the Y-system we reproduce the equations arising in the worldsheet quasiclassical quantization procedure. We observe that precisely the same structures involving different combinations of quasi-momenta follow from the Y-system in a very nontrivial way. At this stage we assume some natural generalization of the TBA equations for the excited states originally proposed for the $\mathfrak{sl}(2)$ subsector [32].

Probably one of the most interesting problems left is the derivation of the explicit “quantum” generalization of our solution (2.19), now for the full u -dependent T-system (2.6), in terms of certain Wronskian-type determinant expressions. Unlike the solution given in [52] using the Bäcklund techniques of [75, 76], the one we mean here would not contain, similarly to (2.19), any infinite sums, in analogy to the solution of [77] for the super-spin chains. Then one should fix the parameters of this solution for each given state of the theory. As was demonstrated in this paper, such a solution should be more convenient for fixing explicitly the large a and s asymptotics. This would be an important step in construction of a finite system of non-linear integral equations of a Destri-DeVega type (in analogy with [16]) for the AdS/CFT spectrum problem, also exact for any operator and at any coupling λ . Apart from its obvious advantages for a numeric analysis, such a finite set of equations, extending the observations of this paper to the quantum level, would allow us to better understand the full $\text{PSU}(2, 2|4)$ integrability structure of both sides of the AdS/CFT correspondence which is somewhat hidden due to the original light cone gauge and the related $\text{SU}(2|2) \times \text{SU}(2|2)$ setup.

Acknowledgments

The work of NG was partly supported by the German Science Foundation (DFG) under the Collaborative Research Center (SFB) 676 and RFFI project grant 06-02-16786. The work of VK was partly supported by the ANR grants INT-AdS/CFT (BLAN-06-0124-03) and GranMA (BLAN-08-1-313695) and the grant RFFI 08-02-00287. The work of ZT is supported by Grant-in-Aid for Young Scientists, B #19740244 from The Ministry of Education, Culture, Sports, Science and Technology in Japan. We thank N.Beisert, C.Candu, M.R.Douglas, S.Leurent, F.Levkovich-Maslyuk, L.Mazzucato, S.Schafer-Nameki, M.Staudacher, D.Serban, A.Tseytlin, P.Vieira, D.Volin and K.Zarembo for discussions. NG and VK thank the Simons Center for Geometry and Physics, where a part of the work was done, for the kind hospitality. ZT thanks Ecole Normale Supérieure, LPT, where a part of this work was done, for the hospitality.

A Mathematica expressions for general solution of Q-system

```

s[i_]=((y[i]-x[3])(y[i]-x[4]))/((y[i]-x[1])(y[i]-x[2]));
z[i_]=((x[i]-y[1])(x[i]-y[2])(x[i]-y[3])(x[i]-y[4]))/((x[i]-x[3])(x[i]-x[4]));
t[j_,s_]=Boole[j>s];

```

```

M4[a_,s_]:=Table[S[i]^t[j,s+2]y[i]^(j-4-(a+2)t[j,s+2]),{i,4},{j,4}];
M2[a_,s_]:=Table[Z[i]^(1-t[j,a])x[i]^(2-j+(s-2)(1-t[j,a])),{i,2},{j,2}];
M2n[a_,s_]:=M2[a,-s]/.{x[i_]->1/x[5-i],y[i_]->1/y[i]};
T[a_,s_]:=0/;(a>2&&Abs[s]>2)||a<0;
T[a_,s_]:=(((1)^-(a+s)((x[3]x[4])/(y[1]y[2]y[3]y[4]))^-(s-a)
    Det[M4[a,s]])/Det[M4[0,0]])/;a>=Abs[s];
T[a_,s_]:=Det[M2[a,s]]/Det[M2[0,s]]/;s>=a;
T[a_,s_]:=((y[1]y[2]y[3]y[4])/(x[1]x[2]x[3]x[4]))^a Det[M2n[a,s]]/Det[M2n[0,s]]/;s<=-a;
    
```

One can see indeed that Hirota equation is satisfied (the code is quite time consuming)

```

Hir[a_?NumericQ,s_?NumericQ]:=T[a,s]^2-T[a+1,s]T[a-1,s]-T[a,s+1]T[a,s-1]
Table[Hir[a,s]]//Factor,{a,0,3},{s,-4,4}
    
```

For the completeness we add also the asymptotic solution from section 4.3

```

Tb[2,s_]:= (1t[1]-1h[1])(1t[2]-1h[1])(1t[1]-1h[2])(1t[2]-1h[2]) 1t[1]^(s-2) 1t[2]^(s-2)/;s>1
Tb[1,s_]:= ((1t[1]^(s-1)(1t[1]-1h[1])(1t[1]-1h[2])-1t[2]^(s-1)(1t[2]-1h[1])(1t[2]-1h[2]))
    /(1t[1]-1t[2]))/;s>0
Tb[a_,2]:= (1h[1]-1t[1])(1h[1]-1t[2])(1t[1]-1h[2])(1t[2]-1h[2]) 1h[1]^(a-2) 1h[2]^(a-2)/;a>1
Tb[a_,1]:= ((-1)^a (1h[1]^(a-1)(1h[1]-1t[1])(1h[1]-1t[2])-1h[2]^(a-1)(1h[2]-1t[1])(1h[2]-1t[2]))
    /(1h[1]-1h[2]))/;a>0
Tb[0,s_]:=1; Tb[a_,0]:=1/;a >= 0;
Tb[a_,s_]:=0/;(Abs[s]>2&&a>2)||a < 0;
Tb[a_,s_]:= (Tb[a,-s]/.{1h[c_]->1/1h[5-c],1t[c_]->1/1t[5-c]})/;s < 0
    
```

B Derivation of the modified Bethe equations

In this section we show how the effect from virtual fluctuations, corresponding to the one-loop quantum corrections, can be absorbed into a certain modification of the asymptotic Bethe equations. The algebraic curve can be described by a set of integral equations for the “densities”, or discontinuities of the quasi-momenta $p_i = \{\hat{p}_i, \tilde{p}_i\}$ in the following way

$$p_i - p_j = 2\pi k \quad , \quad x \in \mathcal{C}_{ij}^k \quad (\text{B.1})$$

where $p_i \equiv (p_i(x+i0) + p_i(x-i0))/2$. The fluctuations modify the curve by extra poles at

$$x = x_n^{ij} : \quad p_i(x_n^{ij}) - p_j(x_n^{ij}) = 2\pi n \quad (\text{B.2})$$

with the residues

$$\alpha(x) = \frac{4\pi}{\sqrt{\lambda}} \frac{x^2}{x^2 - 1} \quad . \quad (\text{B.3})$$

There are some additional constraints on the asymptotics of the resulting quasi-momenta due to the $x \rightarrow 1/x$ symmetry. As is explained in great detail in the paper [96] the equations (B.1) are modified by fluctuations x_n^{ij} induced by extra potentials in the following way

$$p_1 - p_p + V_1^{ij,n} - V_p^{ij,n} = 2\pi k \quad , \quad x \in \mathcal{C}_{ip}^k \quad (\text{B.4})$$

where indexes $\mathbf{i}, \mathbf{j}$ characterize the polarization of the fluctuation, n is roughly a Fourier mode of the fluctuation. These potentials are given by

$$\begin{pmatrix} V_1^{\mathbf{ij},n} \\ V_2^{\mathbf{ij},n} \\ V_3^{\mathbf{ij},n} \\ V_4^{\mathbf{ij},n} \end{pmatrix} = \begin{pmatrix} +1 \\ +1 \\ -1 \\ -1 \end{pmatrix} \frac{x}{x^2-1} \frac{\alpha(x_n^{\mathbf{ij}})}{(x_n^{\mathbf{ij}})^2} + \begin{pmatrix} +\delta_{1\mathbf{i}} \\ +\delta_{2\mathbf{i}} \\ -\delta_{3\mathbf{j}} \\ -\delta_{4\mathbf{j}} \end{pmatrix} \frac{\alpha(x)}{x-x_n^{\mathbf{ij}}} - \begin{pmatrix} +\delta_{2\mathbf{i}} \\ +\delta_{1\mathbf{i}} \\ -\delta_{4\mathbf{j}} \\ -\delta_{3\mathbf{j}} \end{pmatrix} \frac{\alpha(1/x)}{1/x-x_n^{\mathbf{ij}}}, \quad (\text{B.5})$$

and

$$\begin{pmatrix} V_1^{\mathbf{ij},n} \\ V_2^{\mathbf{ij},n} \\ V_3^{\mathbf{ij},n} \\ V_4^{\mathbf{ij},n} \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ +1 \\ +1 \end{pmatrix} \frac{1}{x^2-1} \frac{\alpha(x_n^{\mathbf{ij}})}{x_n^{\mathbf{ij}}} - \begin{pmatrix} +\delta_{1\mathbf{i}} \\ +\delta_{2\mathbf{i}} \\ -\delta_{3\mathbf{j}} \\ -\delta_{4\mathbf{j}} \end{pmatrix} \frac{\alpha(x)}{x-x_n^{\mathbf{ij}}} + \begin{pmatrix} +\delta_{2\mathbf{i}} \\ +\delta_{1\mathbf{i}} \\ -\delta_{4\mathbf{j}} \\ -\delta_{3\mathbf{j}} \end{pmatrix} \frac{\alpha(1/x)}{1/x-x_n^{\mathbf{ij}}}. \quad (\text{B.6})$$

For the one-loop shift one should introduce the fluctuations with all possible polarizations listed in (4.48) and (4.49) and sum over all Fourier modes with a factor $1/2$. As in (4.51) we rewrite the sum as a contour integral around all fluctuations $x_n^{\mathbf{ij}}$, $n = -\infty, \dots, \infty$

$$p_1 - p_{\mathbf{p}} + \sum_{\mathbf{ij}} \frac{1}{2} \oint \frac{dy}{2\pi i} \partial_y \log \sin \frac{p_{\mathbf{i}}(y) - p_{\mathbf{j}}(y)}{2} \left(V_1^{\mathbf{ij}}(x, y) - V_{\mathbf{p}}^{\mathbf{ij}}(x, y) \right) = 2\pi k, \quad x \in \mathcal{C}_{\mathbf{lp}}^k \quad (\text{B.7})$$

where $V_{\mathbf{p}}^{\mathbf{ij}}(x, y)$ is $V_{\mathbf{p}}^{\mathbf{ij},n}$ with $x_n^{\mathbf{ij}}$ replaced by y . Next we deform the integration contour to pass around the unit circle $|y| = 1$. Since the quasi-momenta in general have branch points and the potentials have poles at $y = x$ we also have to add the contributions from all these singularities. Some of these contributions can be absorbed into redefinitions of the quasi-momenta $p_{\mathbf{i}} \rightarrow q_{\mathbf{i}}$ so that $q_{\mathbf{i}}$ has the same analytic properties as $p_{\mathbf{i}}$. The contributions which cannot be absorbed into the redefinition of $p_{\mathbf{i}}$ are called “Anomaly” (by historical reasons, see [63, 88, 97–100]). Now let us use

$$\partial_y \log \sin \frac{p_{\mathbf{i}} - p_{\mathbf{j}}}{2} = \frac{p'_{\mathbf{i}} - p'_{\mathbf{j}}}{2} + \partial_y \log (1 - e^{-ip_{\mathbf{i}} + ip_{\mathbf{j}}}) . \quad (\text{B.8})$$

As was shown in [96] the first term reflects the contribution of Hernandez-Lopez (HL) [101] phase in ABA and we get

$$q_1 - q_{\mathbf{p}} + \text{Anomaly} + \text{HL} - 2\pi k = \sum_{\mathbf{ij}} \oint_{\mathbb{U}^+} \frac{dy}{2\pi i} \partial_y \log (1 - e^{-ip_{\mathbf{i}} + ip_{\mathbf{j}}}) \left(V_1^{\mathbf{ij}} - V_{\mathbf{p}}^{\mathbf{ij}} \right), \quad (\text{B.9})$$

where the integration contour $\mathbb{U}^+$ goes along the upper part of the unit circle $|y| = 1$. The last equation is an integral equation for the “densities” $\varrho = \frac{q(x-i0) - q(x+i0)}{2\pi i}$.

At the same time, it was shown in [63, 64] that the ABA can be written with a one-loop accuracy as the following equation for the density of the momentum-carrying roots ϱ

$$q_1 - q_{\mathbf{p}} + \text{Anomaly} + \text{HL} - 2\pi k = 0 . \quad (\text{B.10})$$

Now we clearly see that the discrepancy between the exact one-loop energies and the prediction of the ABA is due to the integral in the r.h.s. of (B.9). This last term is responsible to all-wrapping contributions. We can easily modify the Bethe equations so that the two results agree again at the one-loop level. For example the equation for the middle node $u_{4,k}$ in the scaling limit becomes

$$\text{ABA}_{u_4} = \exp [i (\not{q}_2 - \not{q}_3 + \text{Anomaly} + \text{HL})] \quad (\text{B.11})$$

and to correct it we simply add an extra phase to the r.h.s.

$$\text{ABA}_{u_4} = \exp \left[\sum_{\mathbf{ij}} \oint_{\mathbb{U}^+} \frac{dy}{2\pi} \partial_y \log (1 - e^{-ip_i + ip_j}) (V_2^{\mathbf{ij}} - V_3^{\mathbf{ij}}) \right]. \quad (\text{B.12})$$

Using this kind of expression with the explicit form of the potentials (B.6) one gets

$$\begin{aligned} & \prod_{j=1}^{K_2} \frac{u_{1,k} - u_{2,j} + \frac{i}{2}}{u_{1,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{1 - 1/(x_{1,k} x_{4,j}^+)}{1 - 1/(x_{1,k} x_{4,j}^-)} \\ &= \exp \left[\int_{-1}^{+1} \left(r(1/x_{1,k}, z) (\mathcal{N}_{\hat{2}*} + \mathcal{N}_{\hat{2}*}) \right. \right. \\ & \quad \left. \left. - r(x_{1,k}, z) (\mathcal{N}_{\hat{1}*} + \mathcal{N}_{\hat{1}*}) - u(x_{1,k}, z) \mathcal{N}_* - q(x_{1,k}, z) \mathcal{N}_* \right) dz \right], \end{aligned} \quad (\text{B.13})$$

$$\begin{aligned} & - \prod_{j=1}^{K_2} \frac{u_{2,k} - u_{2,j} - \frac{i}{2}}{u_{2,k} - u_{2,j} + \frac{i}{2}} \prod_{j=1}^{K_1} \frac{u_{2,k} - u_{1,j} + \frac{i}{2}}{u_{2,k} - u_{1,j} - \frac{i}{2}} \prod_{j=1}^{K_3} \frac{u_{1,k} - u_{3,j} + \frac{i}{2}}{u_{1,k} - u_{3,j} - \frac{i}{2}} \\ &= \exp \left[\int_{-1}^{+1} \left(r(x_{2,k}, z) + r(1/x_{2,k}, z) \right) (\mathcal{N}_{\hat{1}*} - \mathcal{N}_{\hat{2}*}) dz \right], \end{aligned} \quad (\text{B.14})$$

$$\begin{aligned} & \prod_{j=1}^{K_2} \frac{u_{3,k} - u_{2,j} + \frac{i}{2}}{u_{3,k} - u_{2,j} - \frac{i}{2}} \prod_{j=1}^{K_4} \frac{x_{3,k} - x_{4,j}^+}{x_{3,k} - x_{4,j}^-} \\ &= \exp \left[\int_{-1}^{+1} \left(r(x_{3,k}, z) (\mathcal{N}_{\hat{2}*} + \mathcal{N}_{\hat{2}*}) \right. \right. \\ & \quad \left. \left. - r(1/x_{3,k}, z) (\mathcal{N}_{\hat{1}*} + \mathcal{N}_{\hat{1}*}) + u(x_{3,k}, z) \mathcal{N}_* + q(x_{3,k}, z) \mathcal{N}_* \right) dz \right], \end{aligned} \quad (\text{B.15})$$

$$\begin{aligned} & - \left(\frac{x_{4,k}^+}{x_{4,k}^-} \right)^L \prod_{j=1}^{K_4} \sigma^{-2}(u_{4,k}, u_{4,j}) \frac{(1 - 1/(x_{4,k}^- x_{4,j}^+))(x_{4,k}^+ - x_{4,j}^-)}{(1 - 1/(x_{4,k}^+ x_{4,j}^-))(x_{4,k}^- - x_{4,j}^+)} \times \\ & \times \prod_{j=1}^{K_1} \frac{1 - 1/(x_{4,k}^- x_{1,j})}{1 - 1/(x_{4,k}^+ x_{1,j})} \prod_{j=1}^{K_3} \frac{x_{4,k}^- - x_{3,j}}{x_{4,k}^+ - x_{3,j}} \prod_{j=1}^{K_5} \frac{x_{4,k}^- - x_{5,j}}{x_{4,k}^+ - x_{5,j}} \prod_{j=1}^{K_7} \frac{1 - 1/(x_{4,k}^- x_{7,j})}{1 - 1/(x_{4,k}^+ x_{7,j})} \\ &= \exp \left[\int_{-1}^{+1} \left(r(1/x_{4,k}, z) (\mathcal{N}_{\hat{1}*} + \mathcal{N}_{\hat{4}*}) \right. \right. \\ & \quad \left. \left. - r(x_{4,k}, z) (\mathcal{N}_{\hat{2}*} + \mathcal{N}_{\hat{3}*}) - 2u(x_{4,k}, z) \mathcal{N}_* \right) dz \right] \end{aligned} \quad (\text{B.16})$$

where we use the following kernels

$$r(y, z) = \frac{y^2}{y^2 - 1} \frac{\partial_z}{2\pi g} \frac{1}{y - x(z)} , \quad u(y, z) = \frac{y}{y^2 - 1} \frac{\partial_z}{2\pi g} \frac{1}{x^2(z) - 1} ,$$

$$q(y, z) = \frac{1}{y^2 - 1} \frac{\partial_z}{2\pi g} \frac{x(z)}{x^2(z) - 1}$$

and $\mathcal{N}$'s are some combinations of the quasi-momenta defined in (4.44). The equations for $u_{5,k}, u_{6,k}, u_{7,k}$ could be easily written down following the obvious pattern of modifying phases. Together with the corrected equation for the energy

$$E = \sum_{j=1}^{K_4} \left(1 + \frac{2ig}{x_{4,j}^+} - \frac{2ig}{x_{4,j}^-} \right) - \int_{-1}^1 \frac{dz}{2\pi} \partial_z \frac{z}{\sqrt{1-z^2}} \mathcal{N}_* \quad (\text{B.17})$$

these equations define the quasiclassical energy including one-loop contributions with all wrapping corrections included.³⁰

Open Access. This article is distributed under the terms of the Creative Commons Attribution Noncommercial License which permits any noncommercial use, distribution, and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] N. Gromov, V. Kazakov and P. Vieira, *Exact spectrum of anomalous dimensions of planar $N = 4$ supersymmetric Yang-Mills theory*, *Phys. Rev. Lett.* **103** (2009) 131601 [[arXiv:0901.3753](#)] [[SPIRES](#)].
- [2] C.-N. Yang and C.P. Yang, *One-dimensional chain of anisotropic spin-spin interactions. I: proof of Bethe's hypothesis for ground state in a finite system*, *Phys. Rev.* **150** (1966) 321 [[SPIRES](#)].
- [3] A.B. Zamolodchikov, *On the thermodynamic Bethe ansatz equations for reflectionless ADE scattering theories*, *Phys. Lett. B* **253** (1991) 391 [[SPIRES](#)].
- [4] N. Dorey, *Magnon bound states and the AdS/CFT correspondence*, *J. Phys. A* **39** (2006) 13119 [[hep-th/0604175](#)] [[SPIRES](#)].
- [5] M. Takahashi, *Thermodynamics of one-dimensional solvable models*, Cambridge University Press, Cambridge U.K. (1999).
- [6] F.H.L. Essler, H.Frahm, F.Göhmman, A. Klümper and V. Korepin, *The one-dimensional Hubbard model*, Cambridge University Press, Cambridge U.K. (2005).
- [7] D. Fioravanti, A. Mariottini, E. Quattrini and F. Ravanini, *Excited state Destri-De Vega equation for sine-Gordon and restricted sine-Gordon models*, *Phys. Lett. B* **390** (1997) 243 [[hep-th/9608091](#)] [[SPIRES](#)].
- [8] A.G. Bytsko and J. Teschner, *Quantization of models with non-compact quantum group symmetry: modular XXZ magnet and lattice sinh-Gordon model*, *J. Phys. A* **39** (2006) 12927 [[hep-th/0602093](#)] [[SPIRES](#)].

³⁰In the derivation of these equations usually one requires the filling fractions K_a/L to be sufficiently small and also the modifying phases are not too large. The case of arbitrary filling fractions could be described by an appropriate analytic continuation.

- [9] H. Saleur and B. Pozsgay, *Scattering and duality in the 2 dimensional OSP(2|2) Gross Neveu and σ -models*, *JHEP* **02** (2010) 008 [[arXiv:0910.0637](#)] [[SPIRES](#)].
- [10] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, *Quantum field theories in finite volume: Excited state energies*, *Nucl. Phys. B* **489** (1997) 487 [[hep-th/9607099](#)] [[SPIRES](#)].
- [11] P. Dorey and R. Tateo, *Excited states by analytic continuation of TBA equations*, *Nucl. Phys. B* **482** (1996) 639 [[hep-th/9607167](#)] [[SPIRES](#)].
- [12] P. Dorey and R. Tateo, *Excited states in some simple perturbed conformal field theories*, *Nucl. Phys. B* **515** (1998) 575 [[hep-th/9706140](#)] [[SPIRES](#)].
- [13] P.A. Pearce and A. Kluemper, *Finite size corrections and scaling dimensions of solvable lattice models: an analytic method*, *Phys. Rev. Lett.* **66** (1991) 974 [[SPIRES](#)].
- [14] A. Kuniba, T. Nakanishi and J. Suzuki, *Functional relations in solvable lattice models. 1: functional relations and representation theory*, *Int. J. Mod. Phys. A* **9** (1994) 5215 [[hep-th/9309137](#)] [[SPIRES](#)].
- [15] I. Krichever, O. Lipan, P. Wiegmann and A. Zabrodin, *Quantum integrable models and discrete classical Hirota equations*, *Commun. Math. Phys.* **188** (1997) 267 [[hep-th/9604080](#)] [[SPIRES](#)].
- [16] N. Gromov, V. Kazakov and P. Vieira, *Finite volume spectrum of 2D field theories from Hirota dynamics*, *JHEP* **12** (2009) 060 [[arXiv:0812.5091](#)] [[SPIRES](#)].
- [17] J.A. Minahan and K. Zarembo, *The Bethe-ansatz for $N = 4$ super Yang-Mills*, *JHEP* **03** (2003) 013 [[hep-th/0212208](#)] [[SPIRES](#)].
- [18] N. Beisert, C. Kristjansen and M. Staudacher, *The dilatation operator of $N = 4$ super Yang-Mills theory*, *Nucl. Phys. B* **664** (2003) 131 [[hep-th/0303060](#)] [[SPIRES](#)].
- [19] R.A. Janik, *The $AdS_5 \times S^5$ superstring worldsheet S -matrix and crossing symmetry*, *Phys. Rev. D* **73** (2006) 086006 [[hep-th/0603038](#)] [[SPIRES](#)].
- [20] M. Staudacher, *The factorized S -matrix of CFT/ AdS* , *JHEP* **05** (2005) 054 [[hep-th/0412188](#)] [[SPIRES](#)].
- [21] N. Beisert, *The $SU(2|2)$ dynamic S -matrix*, *Adv. Theor. Math. Phys.* **12** (2008) 945 [[hep-th/0511082](#)] [[SPIRES](#)].
- [22] N. Beisert, R. Hernandez and E. Lopez, *A crossing-symmetric phase for $AdS_5 \times S^5$ strings*, *JHEP* **11** (2006) 070 [[hep-th/0609044](#)] [[SPIRES](#)].
- [23] N. Beisert and M. Staudacher, *Long-range $PSU(2,2|4)$ Bethe ansatzes for gauge theory and strings*, *Nucl. Phys. B* **727** (2005) 1 [[hep-th/0504190](#)] [[SPIRES](#)].
- [24] N. Beisert, B. Eden and M. Staudacher, *Transcendentality and crossing*, *J. Stat. Mech.* (2007) P01021 [[hep-th/0610251](#)] [[SPIRES](#)].
- [25] J. Ambjorn, R.A. Janik and C. Kristjansen, *Wrapping interactions and a new source of corrections to the spin-chain/string duality*, *Nucl. Phys. B* **736** (2006) 288 [[hep-th/0510171](#)] [[SPIRES](#)].
- [26] G. Arutyunov and S. Frolov, *On string S -matrix, bound states and TBA*, *JHEP* **12** (2007) 024 [[arXiv:0710.1568](#)] [[SPIRES](#)].
- [27] A. Santambrogio and D. Zanon, *Exact anomalous dimensions of $N = 4$ Yang-Mills operators with large R charge*, *Phys. Lett. B* **545** (2002) 425 [[hep-th/0206079](#)] [[SPIRES](#)].
- [28] N. Beisert, V. Dippel and M. Staudacher, *A novel long range spin chain and planar $N = 4$ super Yang-Mills*, *JHEP* **07** (2004) 075 [[hep-th/0405001](#)] [[SPIRES](#)].

- [29] R.A. Janik and T. Lukowski, *Wrapping interactions at strong coupling — The giant magnon*, *Phys. Rev. D* **76** (2007) 126008 [[arXiv:0708.2208](#)] [[SPIRES](#)].
- [30] M.P. Heller, R.A. Janik and T. Lukowski, *A new derivation of Lüscher F-term and fluctuations around the giant magnon*, *JHEP* **06** (2008) 036 [[arXiv:0801.4463](#)] [[SPIRES](#)].
- [31] Z. Bajnok and R.A. Janik, *Four-loop perturbative Konishi from strings and finite size effects for multiparticle states*, *Nucl. Phys. B* **807** (2009) 625 [[arXiv:0807.0399](#)] [[SPIRES](#)].
- [32] N. Gromov, V. Kazakov, A. Kozak and P. Vieira, *Exact spectrum of anomalous dimensions of planar $N = 4$ supersymmetric Yang-Mills theory: TBA and excited states*, *Lett. Math. Phys.* **91** (2010) 265 [[arXiv:0902.4458](#)] [[SPIRES](#)].
- [33] K. Konishi, *Anomalous supersymmetry transformation of some composite operators in SQCD*, *Phys. Lett. B* **135** (1984) 439 [[SPIRES](#)].
- [34] M. Bianchi, S. Kovacs, G. Rossi and Y.S. Stanev, *Properties of the Konishi multiplet in $N = 4$ SYM theory*, *JHEP* **05** (2001) 042 [[hep-th/0104016](#)] [[SPIRES](#)].
- [35] B. Eden, C. Jarczak, E. Sokatchev and Y.S. Stanev, *Operator mixing in $N = 4$ SYM: the Konishi anomaly revisited*, *Nucl. Phys. B* **722** (2005) 119 [[hep-th/0501077](#)] [[SPIRES](#)].
- [36] N. Gromov, V. Kazakov and P. Vieira, *Exact spectrum of planar $N = 4$ supersymmetric Yang-Mills theory: Konishi dimension at any coupling*, *Phys. Rev. Lett.* **104** (2010) 211601 [[arXiv:0906.4240](#)] [[SPIRES](#)].
- [37] F. Fiamberti, A. Santambrogio, C. Sieg and D. Zanon, *Finite-size effects in the superconformal beta-deformed $N = 4$ SYM*, *JHEP* **08** (2008) 057 [[arXiv:0806.2103](#)] [[SPIRES](#)].
- [38] V.N. Velizhanin, *Leading transcendentality contributions to the four-loop universal anomalous dimension in $N = 4$ SYM*, *Phys. Lett. B* **676** (2009) 112 [[arXiv:0811.0607](#)] [[SPIRES](#)].
- [39] J.A. Minahan, O. Ohlsson Sax and C. Sieg, *Anomalous dimensions at four loops in $N = 6$ superconformal Chern-Simons theories*, [arXiv:0912.3460](#) [[SPIRES](#)].
- [40] Z. Bajnok, A. Hegedus, R.A. Janik and T. Lukowski, *Five loop Konishi from AdS/CFT*, *Nucl. Phys. B* **827** (2010) 426 [[arXiv:0906.4062](#)] [[SPIRES](#)].
- [41] G. Arutyunov, S. Frolov and R. Suzuki, *Five-loop Konishi from the mirror TBA*, *JHEP* **04** (2010) 069 [[arXiv:1002.1711](#)] [[SPIRES](#)].
- [42] F. Fiamberti, A. Santambrogio and C. Sieg, *Five-loop anomalous dimension at critical wrapping order in $N = 4$ SYM*, *JHEP* **03** (2010) 103 [[arXiv:0908.0234](#)] [[SPIRES](#)].
- [43] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *A semi-classical limit of the gauge/string correspondence*, *Nucl. Phys. B* **636** (2002) 99 [[hep-th/0204051](#)] [[SPIRES](#)].
- [44] R. Roiban and A.A. Tseytlin, *Quantum strings in $AdS_5 \times S^5$: strong-coupling corrections to dimension of Konishi operator*, *JHEP* **11** (2009) 013 [[arXiv:0906.4294](#)] [[SPIRES](#)].
- [45] A.A. Tseytlin, *Quantum strings in $AdS_5 \times S^5$ and AdS/CFT duality*, *Int. J. Mod. Phys. A* **25** (2010) 319 [[arXiv:0907.3238](#)] [[SPIRES](#)].
- [46] G. Arutyunov and S. Frolov, *Uniform light-cone gauge for strings in $AdS_5 \times S^5$: Solving SU(1|1) sector*, *JHEP* **01** (2006) 055 [[hep-th/0510208](#)] [[SPIRES](#)].
- [47] G. Arutyunov, S. Frolov and R. Suzuki, *Exploring the mirror TBA*, *JHEP* **05** (2010) 031 [[arXiv:0911.2224](#)] [[SPIRES](#)].
- [48] R.R. Metsaev and A.A. Tseytlin, *Type IIB superstring action in $AdS_5 \times S^5$ background*, *Nucl. Phys. B* **533** (1998) 109 [[hep-th/9805028](#)] [[SPIRES](#)].

- [49] D. Bombardelli, D. Fioravanti and R. Tateo, *Thermodynamic Bethe ansatz for planar AdS/CFT: a proposal*, *J. Phys. A* **42** (2009) 375401 [[arXiv:0902.3930](#)] [[SPIRES](#)].
- [50] G. Arutyunov and S. Frolov, *String hypothesis for the $AdS_5 \times S^5$ mirror*, *JHEP* **03** (2009) 152 [[arXiv:0901.1417](#)] [[SPIRES](#)].
- [51] G. Arutyunov and S. Frolov, *Thermodynamic Bethe ansatz for the $AdS_5 \times S^5$ mirror model*, *JHEP* **05** (2009) 068 [[arXiv:0903.0141](#)] [[SPIRES](#)].
- [52] A. Hegedus, *Discrete Hirota dynamics for AdS/CFT*, *Nucl. Phys. B* **825** (2010) 341 [[arXiv:0906.2546](#)] [[SPIRES](#)].
- [53] V.A. Kazakov, A. Marshakov, J.A. Minahan and K. Zarembo, *Classical/quantum integrability in AdS/CFT*, *JHEP* **05** (2004) 024 [[hep-th/0402207](#)] [[SPIRES](#)].
- [54] N. Beisert, V.A. Kazakov, K. Sakai and K. Zarembo, *The algebraic curve of classical superstrings on $AdS_5 \times S^5$* , *Commun. Math. Phys.* **263** (2006) 659 [[hep-th/0502226](#)] [[SPIRES](#)].
- [55] I. Bena, J. Polchinski and R. Roiban, *Hidden symmetries of the $AdS_5 \times S^5$ superstring*, *Phys. Rev. D* **69** (2004) 046002 [[hep-th/0305116](#)] [[SPIRES](#)].
- [56] S. Frolov and A.A. Tseytlin, *Semiclassical quantization of rotating superstring in $AdS_5 \times S^5$* , *JHEP* **06** (2002) 007 [[hep-th/0204226](#)] [[SPIRES](#)].
- [57] N. Gromov, *Integrability in AdS/CFT correspondence: quasi-classical analysis*, *J. Phys. A* **42** (2009) 254004.
- [58] P. Vieira, *Integrability in AdS/CFT*, Ph.D. Thesis, École Normale Supérieure, Paris (2008).
- [59] N. Gromov and P. Vieira, *Constructing the AdS/CFT dressing factor*, *Nucl. Phys. B* **790** (2008) 72 [[hep-th/0703266](#)] [[SPIRES](#)].
- [60] N. Gromov and P. Vieira, *The $AdS_5 \times S^5$ superstring quantum spectrum from the algebraic curve*, *Nucl. Phys. B* **789** (2008) 175 [[hep-th/0703191](#)] [[SPIRES](#)].
- [61] S. Schäfer-Nameki, M. Zamaklar and K. Zarembo, *How accurate is the quantum string Bethe ansatz?*, *JHEP* **12** (2006) 020 [[hep-th/0610250](#)] [[SPIRES](#)].
- [62] S. Schäfer-Nameki, M. Zamaklar and K. Zarembo, *Quantum corrections to spinning strings in $AdS_5 \times S^5$ and Bethe ansatz: a comparative study*, *JHEP* **09** (2005) 051 [[hep-th/0507189](#)] [[SPIRES](#)].
- [63] N. Beisert and A.A. Tseytlin, *On quantum corrections to spinning strings and Bethe equations*, *Phys. Lett. B* **629** (2005) 102 [[hep-th/0509084](#)] [[SPIRES](#)].
- [64] N. Gromov and P. Vieira, *Complete 1-loop test of AdS/CFT*, *JHEP* **04** (2008) 046 [[arXiv:0709.3487](#)] [[SPIRES](#)].
- [65] N. Gromov, *Y-system and quasi-classical strings*, *JHEP* **01** (2010) 112 [[arXiv:0910.3608](#)] [[SPIRES](#)].
- [66] O. Aharony, O. Bergman, D.L. Jafferis and J. Maldacena, *$N = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals*, *JHEP* **10** (2008) 091 [[arXiv:0806.1218](#)] [[SPIRES](#)].
- [67] N. Gromov and F. Levkovich-Maslyuk, *Y-system, TBA and quasi-classical strings in $AdS_4 \times CP^3$* , *JHEP* **06** (2010) 088 [[arXiv:0912.4911](#)] [[SPIRES](#)].
- [68] D. Bombardelli, D. Fioravanti and R. Tateo, *TBA and Y-system for planar AdS_4/CFT_3* , *Nucl. Phys. B* **834** (2010) 543 [[arXiv:0912.4715](#)] [[SPIRES](#)].

- [69] A.N. Kirillov and N.Yu. Reshetikhin, *Representations of Yangians and multiplicities of the inclusion of the irreducible components of the tensor product of representations of simple Lie algebras*, *J. Soviet Math.* **52** (1990) 3156.
- [70] A. Kuniba, T. Nakanishi and Z. Tsuboi, *The canonical solutions of the Q-systems and the Kirillov-Reshetikhin conjecture*, *Commun. Math. Phys.* **227** (2002) 155 [[math/0105145](#)].
- [71] P. Di Francesco and R. Kedem, *Q-systems as cluster algebras II: Cartan matrix of finite type and the polynomial property*, *Lett. Math. Phys.* **89** (2009) 183 [[arXiv:0803.0362](#)] [[SPIRES](#)].
- [72] Z. Tsuboi, *Nonlinear integral equations and high temperature expansion for the $U_q(\hat{sl}(r+1|s+1))$ Perk-Schultz model*, *Nucl. Phys. B* **737** (2006) 261 [[cond-mat/0510458](#)] [[SPIRES](#)].
- [73] Z. Tsuboi, *Analytic Bethe ansatz and functional equations for Lie superalgebra $sl(r+1|s+1)$* , *J. Phys. A* **30** (1997) 7975 [[arXiv:0911.5386](#)].
- [74] Z. Tsuboi, *Analytic Bethe Ansatz and functional equations associated with any simple root systems of the Lie superalgebra $sl(r+1|s+1)$* , *Physica A* **252** (1998) 565 [[arXiv:0911.5387](#)] [[SPIRES](#)].
- [75] V. Kazakov, A.S. Sorin and A. Zabrodin, *Supersymmetric Bethe ansatz and Baxter equations from discrete Hirota dynamics*, *Nucl. Phys. B* **790** (2008) 345 [[hep-th/0703147](#)] [[SPIRES](#)].
- [76] A. Zabrodin, *Backlund transformations for difference Hirota equation and supersymmetric Bethe ansatz*, [arXiv:0705.4006](#) [[SPIRES](#)].
- [77] Z. Tsuboi, *Solutions of the T-system and Baxter equations for supersymmetric spin chains*, *Nucl. Phys. B* **826** (2010) 399 [[arXiv:0906.2039](#)] [[SPIRES](#)].
- [78] J.B. Conrey, D.W. Farmer and M.R. Zirnbauer, *Howe pairs, supersymmetry, and ratios of random characteristic polynomials for the unitary groups U_N* , [math-ph/0511024](#).
- [79] S.-J. Cheng, N. Lam and R.B. Zhang, *Character formula for infinite dimensional unitarizable modules of the general linear superalgebra*, *J. Algebra* **273** (2004) 780 [[math.RT/0301183](#)].
- [80] J.-H. Kwon, *Rational semistandard tableaux and character formula for the Lie superalgebra $\hat{\mathfrak{gl}}_{\infty|\infty}$* , *Adv. Math.* **217** (2008) 713 [[math.RT/0605005](#)].
- [81] V.K. Dobrev and V.B. Petkova, *On the group-theoretical approach to extended conformal supersymmetry: function space realizations and invariant differential operators*, *Fortschr. Phys.* **35** (1987) 537 [*Progr. Phys.* **35** (1987) 537].
- [82] V.D.A. Leites, M.V. Saveliev and V.V. Serganova, *Embeddings of $OSP(N/2)$ and the associated nonlinear supersymmetric equations*, in the proceedings of *Group Theoretical Methods in Physics*, M.A. Markov et al. eds., VNU Science Press, Utrecht The Netherlands (1986), see pag 255.
- [83] N. Beisert, *The $SU(2|2)$ dynamic S-matrix*, *Adv. Theor. Math. Phys.* **12** (2008) 945 [[hep-th/0511082](#)] [[SPIRES](#)].
- [84] N. Beisert, *The analytic Bethe ansatz for a chain with centrally extended $\mathfrak{su}(2|2)$ symmetry*, *J. Stat. Mech.* (2007) P01017 [[nlin/0610017](#)] [[SPIRES](#)].
- [85] J. Balog and A. Hegedus, *TBA equations for excited states in the $O(3)$ and $O(4)$ nonlinear σ -model*, *J. Phys. A* **37** (2004) 1881 [[hep-th/0309009](#)] [[SPIRES](#)].

- [86] G. Arutyunov, S. Frolov and M. Staudacher, *Bethe ansatz for quantum strings*, *JHEP* **10** (2004) 016 [[hep-th/0406256](#)] [[SPIRES](#)].
- [87] N. Gromov and V. Kazakov, *Double scaling and finite size corrections in $sl(2)$ spin chain*, *Nucl. Phys. B* **736** (2006) 199 [[hep-th/0510194](#)] [[SPIRES](#)].
- [88] N. Beisert, V.A. Kazakov, K. Sakai and K. Zarembo, *Complete spectrum of long operators in $N = 4$ SYM at one loop*, *JHEP* **07** (2005) 030 [[hep-th/0503200](#)] [[SPIRES](#)].
- [89] P.P. Kulish and A.M. Zeitlin, *Superconformal field theory and SUSY $N = 1$ KdV hierarchy. II: the Q -operator*, *Nucl. Phys. B* **709** (2005) 578 [[hep-th/0501019](#)] [[SPIRES](#)].
- [90] V.V. Bazhanov and Z. Tsuboi, *Baxter's Q -operators for supersymmetric spin chains*, *Nucl. Phys. B* **805** (2008) 451 [[arXiv:0805.4274](#)] [[SPIRES](#)].
- [91] A.V. Belitsky, S.E. Derkachov, G.P. Korchemsky and A.N. Manashov, *Baxter Q -operator for graded $SL(2|1)$ spin chain*, *J. Stat. Mech.* (2007) P01005 [[hep-th/0610332](#)] [[SPIRES](#)].
- [92] V.V. Bazhanov, S.L. Lukyanov and A.B. Zamolodchikov, *Integrable structure of conformal field theory. III: the Yang-Baxter relation*, *Commun. Math. Phys.* **200** (1999) 297 [[hep-th/9805008](#)] [[SPIRES](#)].
- [93] A. Mikhailov and S. Schäfer-Nameki, *Algebra of transfer-matrices and Yang-Baxter equations on the string worldsheet in $AdS_5 \times S^5$* , *Nucl. Phys. B* **802** (2008) 1 [[arXiv:0712.4278](#)] [[SPIRES](#)].
- [94] S. Schäfer-Nameki, *Exact expressions for quantum corrections to spinning strings*, *Phys. Lett. B* **639** (2006) 571 [[hep-th/0602214](#)] [[SPIRES](#)].
- [95] S. Schäfer-Nameki, M. Zamaklar and K. Zarembo, *How accurate is the quantum string Bethe ansatz?*, *JHEP* **12** (2006) 020 [[hep-th/0610250](#)] [[SPIRES](#)].
- [96] N. Gromov and P. Vieira, *Constructing the AdS/CFT dressing factor*, *Nucl. Phys. B* **790** (2008) 72 [[hep-th/0703266](#)] [[SPIRES](#)].
- [97] J.-M. Daul and V.A. Kazakov, *Wilson loop for large- N Yang-Mills theory on a two-dimensional sphere*, *Phys. Lett. B* **335** (1994) 371 [[hep-th/9310165](#)] [[SPIRES](#)].
- [98] D.V. Boulatov, *Wilson loop on a sphere*, *Mod. Phys. Lett. A* **9** (1994) 365 [[hep-th/9310041](#)] [[SPIRES](#)].
- [99] N. Beisert, A.A. Tseytlin and K. Zarembo, *Matching quantum strings to quantum spins: One-loop vs. finite-size corrections*, *Nucl. Phys. B* **715** (2005) 190 [[hep-th/0502173](#)] [[SPIRES](#)].
- [100] R. Hernandez, E. Lopez, A. Perianez and G. Sierra, *Finite size effects in ferromagnetic spin chains and quantum corrections to classical strings*, *JHEP* **06** (2005) 011 [[hep-th/0502188](#)] [[SPIRES](#)].
- [101] R. Hernandez and E. Lopez, *Quantum corrections to the string Bethe ansatz*, *JHEP* **07** (2006) 004 [[hep-th/0603204](#)] [[SPIRES](#)].