Published for SISSA by Springer

Received: May 11, 2017
Accepted: May 31, 2017
Published: June 9, 2017

Twisted partition functions and H-saddles

JHEP06(2017)045

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Abstract: While studying supersymmetric G-gauge theories, one often observes that a zero-radius limit of the twisted partition function Ω^G is computed by the partition function $\mathcal{Z}^G$ in one less dimensions. We show how this type of identification fails generically due to integrations over Wilson lines. Tracing the problem, physically, to saddles with reduced effective theories, we relate Ω^G to a sum of distinct $\mathcal{Z}^H$'s and classify the latter, dubbed H-saddles. This explains why, in the context of pure Yang-Mills quantum mechanics, earlier estimates of the matrix integrals $\mathcal{Z}^G$ had failed to capture the recently constructed bulk index $\mathcal{I}^G_{\text{bulk}}$. The purported agreement between 4d and 5d instanton partition functions, despite such subtleties also present in the ADHM data, is explained.

Keywords: D-branes, M(atrix) Theories, Supersymmetric Gauge Theory, Supersymmetry and Duality

ArXiv ePrint: 1704.08285

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Article funded by SCOAP^3.

doi:10.1007/JHEP06(2017)045

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1 Index and bulk index

Perhaps the simplest of the topological observables, available for supersymmetric theories, is the twisted partition function,

$$\Omega(\beta; z) \equiv \mathrm{Tr}\left[(-1)^{\mathcal{F}}\, e^{zF}\, e^{-\beta\mathcal{H}}\right] \tag{1.1}$$

where the trace is taken over the physical Hilbert space and e^{zF} denotes, collectively, all admissible chemical potential terms. Although we will mostly display results with the chemical potential turned off in this note, these chemical potentials are implicitly assumed and often indispensable part of the computation.

A naive expectation about this quantity is the β-independence, which would allow evaluation of $\Omega(\beta; z)$ in the $\beta \to 0$ limit. This is supported by the familiar one-to-one mapping between bosonic and fermionic states, which at least naively follows from the superalgebra

$$\mathcal{H} = \mathcal{Q}^2\,, \qquad \{(-1)^{\mathcal{F}}, \mathcal{Q}\} = 0\,, \qquad [e^{zF}, \mathcal{Q}] = 0\,. \tag{1.2}$$

The same reasoning would imply that the twisted partition functions count supersymmetric objects, and hence are inherently integral. As such, the twisted partition function would compute the Witten index [1], or its various refined generalizations.

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As with any powerful and sweeping argument, however, this comes with caveats. One finds that β-dependence can actually survive unless the Hilbert space is completely discrete. Instead, the $\beta \to 0$ limit produces an object called the bulk index,

$$\mathcal{I}_{\text{bulk}}(z) \equiv \lim_{\beta \to 0} \Omega(\beta; z)\,, \tag{1.3}$$

which can be sometimes an interesting physical object by itself. If we are aiming at the (refined) Witten index, a more appropriate limit is

$$\mathcal{I}(z) = \lim_{\beta \to \infty} \Omega(\beta; z)\,. \tag{1.4}$$

The difference between $\mathcal{I}$ and $\mathcal{I}_{\text{bulk}}$, denoted by $\delta\mathcal{I}$, may be in some cases computed separately and combined to give the integral Witten index,

$$\mathcal{I}(z) = \mathcal{I}_{\text{bulk}}(z) + \delta\mathcal{I}(z)\,. \tag{1.5}$$

Unlike the bulk index, the continuum contribution $\delta\mathcal{I}$ has no convenient and universal computational tools. For pure Yang-Mills quantum mechanics and also for $\mathcal{N} = 4$ quiver quantum mechanics, nevertheless, a general pattern has been uncovered [2] and both $\mathcal{I}_{\text{bulk}}$ and $\delta\mathcal{I}$ for wide classes of theories have been computed [2, 3].

On the other hand, the usual localization procedure, which seemingly computes the twisted partition function $\Omega(\beta; z)$ at some finite and arbitrary β, usually computes $\mathcal{I}_{\text{bulk}}(z)$ [2]. A good hint of this is that the resulting Ω has no β-dependence, regardless of specifics of the theory. As we noted above, the β-dependence does in general persist for theories with continuum sectors, so the localization must be, in secret, computing a limit of $\Omega(\beta; z)$. The only two logical possibilities are either $\beta \to 0$ or $\beta \to \infty$. However, given that the final expressions are integrals of some local functions, $\beta \to \infty$ is hardly possible, hence we can anticipate

$$\mathcal{I}_{\text{bulk}}(z) \;=\; \Omega(z) \;\equiv\; \Omega\,\Big|_{\text{localization}}\,. \tag{1.6}$$

We will later give a more explicit argument supporting this for gauged quantum mechanics.

Since β can be thought of as the Euclidean time interval, a dimensional reduction to one less dimension is natural. Indeed, in supersymmetric quantum mechanics recast of index theorems, for instance as in Alvarez-Gaume's 1d path-integral derivation of Euler index [4], the contributing saddle localizes to constant configurations, and the 1d path integral reduces to ordinary integral over the target manifold. Something like this also happens with supersymmetric gauged quantum mechanics, where $\beta \to 0$ limit reduces the twisted partition function to ordinary integrals over Lie Algebra and matter representations thereof, which we collectively call the matrix integral.

The lore is, as such, that one can take an additional scaling limit of the chemical potential $z = \beta z'$ with vanishing β and finite z' and find

$$\Omega^G(z) \quad \to \quad \mathcal{Z}^G(z')\,. \tag{1.7}$$

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The right hand side means the matrix integral with the exponent of the measure given by the dimensional reduction to 0d of the Euclidean action of the 1d theory. Note that, here, $z = \beta z'$ limit is taken after the usual $\beta \to 0$ limit of $\Omega(\beta; z)$ was taken as in (1.3).

However, this natural expectation proves to be false for general gauge theories, and in particular for 1d gauged quantum mechanics. When one compactifies a gauge theory on a circle, the Wilson line emerges as natural low energy degrees of freedom, associated with the Cartan torus. The path integral would involve integrations over such Wilson line variables, yet it is clear that one will lose their periodic nature if the $\beta \to 0$ limit is taken first before performing the integration; the Cartan torus is replaced by the Cartan subalgebra of infinite extension. Do we then lose a contributing sector, say, from somewhere on the opposite side of the Cartan torus? As we will show, the answer is yes: one generically loses contributing saddles, or loses poles if a localization is employed, by taking $\beta \to 0$ limit casually.

The phenomenon is quite general for twisted partition functions, regardless of details of the theory or of the spacetime dimensions, as long as there is a circle $\mathbb{S}^1$ and the associated Wilson-line variables to integrate over. Whenever one tries to relate a twisted partition function on $\mathbb{S}^1 \times \mathbb{M}$ to the partition function of the dimensionally reduced theory on (compact) $\mathbb{M}$, one must worry about such extra saddles. In retrospect, the same mechanism can be seen to be responsible for how 2d elliptic genus generically fails to compute 1d Witten index via an appropriate limit [5]. In this note, however, we will confine ourselves to 1d/0d examples, and derive precise relations between the two sides. The same reasoning and derivation are easily applicable to higher dimensions, especially when the twisted partition function is computed by a residue formula in the space of (complexified) Wilson lines.

2 Preliminary: an old story

This finding will also resolve an old mystery surrounding Witten index computations of supersymmetric Yang-Mills quantum mechanics (SYMQ). These are $\mathcal{N} = 4, 8, 16$ SYMQ, respectively obtained from the dimensional reduction of minimally supersymmetric Yang-Mills theory in D-dimensions with $D = 4, 6, 10$ [6, 7]. Let us start with a review of the argument for $\mathcal{I}^G_{\text{bulk}} \to \mathcal{Z}^G$ for this simplest class of supersymmetric gauged quantum mechanics. Interestingly enough, this subtlety does not plague SU(N) cases, namely that of N D0 branes in the type IIA theory [8], for which this identification was originally derived [9, 10]. The content of this section is borrowed from ref. [9].

We start with $\mathcal{N} \geq 4$ pure Yang-Mills quantum mechanics for arbitrary simple group G, whose dimension is denoted as g. $D-1$ bosonic X_i and their canonical conjugates, in the adjoint of G, obey

$$[\pi^a_i, X^b_j] = -i\delta^{ab}\delta_{ij}\,. \tag{2.1}$$

The spinor consists of $\mathcal{N}$ adjoint (real) fermions Ψ^a_β obeying

$$\{\Psi^a_\alpha, \Psi^b_\beta\} = \delta^{ab}\delta_{\alpha\beta}\,, \tag{2.2}$$

which form $(g\mathcal{N})$-dimensional Clifford algebra. The Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\pi^a_i\pi^a_i - \frac{1}{2}X^a_i K^a_i + \frac{1}{4}[X_i, X_j]^2\,, \tag{2.3}$$

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where the sum is over $i, j = 1, 2, \ldots, D-1$ as well as gauge indices $a = 1, 2, \ldots, g$ and the fermion bilinears, K_i^a, are part of

$$K_\mu^a = i f_{abc} \Psi^b \gamma_\mu \Psi^c \,, \qquad \mu = 1, \ldots, D \,, \tag{2.4}$$

with appropriate Dirac matrices γ_μ.

The $\mathcal{N}$ supersymmetries are generated by

$$\mathcal{Q}_\alpha = \gamma_{i\alpha\beta} \Psi_\beta^a \pi_i^a - \frac{1}{2} \gamma_{ij\alpha\beta} f^{abc} \Psi_\beta^a X_i^b X_j^c \,, \tag{2.5}$$

and the adjoint thereof, and lead to the twisted partition function,

$$\Omega^G(\beta) \equiv \mathrm{Tr}\left[(-1)^{\mathcal{F}} e^{zF} e^{-\beta\mathcal{H}}\right] = \int dX \, \langle X| \, \mathrm{tr}\, (-1)^{\mathcal{F}} e^{zF} e^{-\beta\mathcal{H}} \mathcal{P}_{G/Z_G} \, |X\rangle \,. \tag{2.6}$$

The projection to the gauge-singlet sector is instituted by an insertion of

$$\mathcal{P}_{G/Z_G} = \frac{1}{\mathrm{vol}(G/Z_G)} \oint_{G/Z_G} d\theta \, e^{i\theta^a G^a} \tag{2.7}$$

with the Gauss constraints

$$G^a = f^{abc} X_i^b \pi_i^c - \frac{i}{2} K_D^a \,. \tag{2.8}$$

We chose to integrate not over G but over G/Z_G, as the center Z_G acts trivially on the adjoint representation. An important subtlety related to this projector will be revisited in next section.

With the chemical potential turned off, $z = 0$, for simplicity, the heat kernel expansion suffices,

$$\langle X| \, e^{-\beta\mathcal{H}} \, |X'\rangle = \frac{1}{(2\pi\beta)^{g(D-1)/2}} e^{-(X'-X)^2/2\beta} e^{-\beta(V+H_F)} \left(1 + O(\beta)\right) \,, \tag{2.9}$$

where V is the bosonic potential while $H_F \equiv -X_i^a K_i^a/2$. The Gauss constraint rotates $|X\rangle$ to $|X(\theta)\rangle$, so a small β limit of

$$\frac{1}{\mathrm{vol}(G/Z_G)(2\pi\beta)^{g(D-1)/2}} \int dX \oint d\theta \; \mathrm{tr}_\Psi \, (-1)^{\mathcal{F}} e^{-(X(\theta)-X)^2/2\beta} e^{-\beta\,(V+H_F)} e^{\theta^a K_D^a/2} \tag{2.10}$$

needs to be evaluated and thus it is sufficient to consider regions $X(\theta) \sim X$. An obvious thing to do is to expand θ as

$$\theta = \beta\xi \,, \tag{2.11}$$

whereby one finds

$$\frac{\beta^g}{\mathrm{vol}(G/Z_G)(2\pi\beta)^{g(D-1)/2}} \int dX \int [d\xi] \; \mathrm{tr}_\Psi \, (-1)^{\mathcal{F}} e^{-\beta\,[\xi,X]^2/2 - \beta\,V} e^{-\beta\,(H_F - \xi^a K_D^a/2)} \,. \tag{2.12}$$

Identifying ξ with X_D, i.e. the Euclideanized A_0, we find that this limit is computed by, with $\beta^{1/4} X \to X$,

$$\lim_{\tilde\beta \to 0} \frac{1}{\mathrm{vol}(G/Z_G)} \frac{(2\pi)^{g/2}}{(2\pi)^{gD/2} \tilde\beta^{g\mathcal{N}/2}} \int dX \; e^{-[X_\mu, X_\nu]^2/4} \; \mathrm{tr}_\Psi \, (-1)^{\mathcal{F}} e^{\tilde\beta X_\mu^a K_\mu^a/2} \,. \tag{2.13}$$

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Keeping only the leading power in small $\tilde{\beta} \equiv \beta^{3/4}$, we find a G-matrix integral with $\mathcal{N} = 2(D-2)$ supersymmetries,

$$\mathcal{Z}^G \equiv \frac{1}{\mathrm{vol}(G/Z_G)} \frac{(2\pi)^{g/2}}{(2\pi)^{gD/2}} \int dX\, d\Psi\; e^{-[X_\mu, X_\nu]^2/4 + X_\mu^a K_\mu^a/2} \,. \tag{2.14}$$

This line of reasoning has led to the expectation, with chemical potentials restored,

$$\mathcal{I}^G_{\text{bulk}}(z) \quad \rightarrow \quad \mathcal{Z}^G(z')\,, \tag{2.15}$$

which has been successfully used for $G = \mathrm{SU}(N)$.

For other simple gauge groups, however, we will see that $\mathcal{Z}^G$ captures only part of $\mathcal{I}^G_{\text{bulk}}$. Computation of $\mathcal{Z}^G$ was performed for $G = \mathrm{SU}(2)$ by Yi [9] and also by Sethi and Stern [10] and for $\mathrm{SU}(N)$ by Moore, Nekrasov, and Shatashvili [11]. The latter, in particular, introduced a 0d localization method, to be here-in referred to as the MNS method, which was then generalized to arbitrary simple groups by Staudacher [12] and also by Pestun [13]. On the other hand, a localization method for twisted partition function of general gauge theories was derived from the first principle by Hori, Kim, and Yi (HKY) [5, 14] and, with this, $\mathcal{I}^G_{\text{bulk}}$ was recently computed [2]. These two sets of answers disagree, except for $\mathrm{SU}(N)$'s.

One logical possibility is that the MNS method with its ad hoc, if elegant, prescription of the final contour integrals, is inadequate for general G and fails to compute the matrix integral correctly. However, we have confirmed that the first-principle contour derived along the line of HKY also give the same set of numbers for $\mathcal{Z}^G$, which motivated the study in this note. In the following, we will show why the above expectation fails, how the $\mathrm{SU}(N)$ case evaded this subtlety, and also recover $\mathcal{I}^G_{\text{bulk}}$ as a sum of $\mathcal{Z}^H$'s with H certain subgroups of G.

3 Localizations and missing residues

It is instructive to recall the gauge projector

$$\mathcal{P}_{G/Z_G} = \frac{1}{\mathrm{vol}(G/Z_G)} \oint_{G/Z_G} d\theta\, e^{i\theta^a G^a}\,, \tag{3.1}$$

which is hardly a unique choice. Since the center Z_G of G acts trivially on adjoints, we can equally use

$$\mathcal{P}_G = \frac{1}{\mathrm{vol}(G)} \oint_G d\theta\, e^{i\theta^a G^a}\,. \tag{3.2}$$

Since both project out all unphysical states and keep all gauge invariant states, there should be no difference between the two such choices. How does such an ambiguity manifest in actual evaluation? As we outlined above, one canonical way to evaluate relies on the straightforward $\beta \rightarrow 0$ limit with an identification

$$\theta = \beta X_D\,. \tag{3.3}$$

As such, one obtains a G-matrix integral such as (2.14), whose integrand is seemingly oblivious to whether one started with $\mathcal{P}_G$ or $\mathcal{P}_{G/Z_G}$. This cannot be right, however, since these

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two give an identical integral but with different overall factors, $1/\mathrm{vol}(G)$ or $1/\mathrm{vol}(G/Z_G)$, respectively.

The resolution to this is obvious, though. When one uses $\mathcal{P}_G$ in place of $\mathcal{P}_{G/Z_G}$, the β expansion (3.3) is not the only possible one. Rather one must also consider

$$\theta = \Theta_G + \beta X_D \tag{3.4}$$

where $e^{i\Theta_G}$ is an element of Z_G. No dynamical variable is affected by the center Z_G in pure Yang-Mills, so we have $|Z_G|$-many gauge-equivalent saddle points. Individual saddles are infinitely separated from each other in the limit $\beta \to 0$, yet we must sum them up since we are computing a limit of 1d quantity rather than 0d quantity. This sum recovers the extra overall factor of $|Z_G|$, only to be canceled by the same factor in $\mathrm{vol}(G) = |Z_G| \cdot \mathrm{vol}(G/Z_G)$.

In fact, this ambiguity goes beyond G vs G/Z_G. Sometimes, we find it simpler to parameterize the group by a multiple cover. A main example of this for us would be F_4, which has no center and whose Cartan torus has the natural volume $(2\pi)^4/8$. Yet, it is easier to deal with F_4 if we pretend that each Cartan parameters span $[0, 2\pi)$ independently. If we choose to do this, the integral range in the projector become 8-fold larger and must be compensated by the factor $1/(8 \cdot \mathrm{vol}(F_4))$. Therefore, the required list of $e^{i\Theta_G}$ is determined as much by how one parameterizes G as by the abstract group structure and the field content of the theory.

For such reasons, the normalization issue is generally more subtle with the matrix integral than its 1d counterpart, if one wishes to recover $\mathcal{I}^G_{\mathrm{bulk}}$ via matrix integral. Thankfully, this has been worked out in the past for SYMQ. A particularly powerful version is via a 0d localization which leaves only rank-many contour integrals as [11–13]

$$\mathcal{Z}^G(z') = \frac{|\det(Q^G_{ab})|}{|W_G|} \int_{\mathcal{C}'} \frac{d^{r_G} u'}{(2\pi i)^{r_G}}\, f_G(u'; z') \tag{3.5}$$

with the Weyl group W_G and $r_G = \mathrm{rank}(G)$. The generalization of $|Z_G|$ factor sits in $\det(Q^G_{ab})$ where $|Q^G_{ab}|$ is the matrix spanned by the simple roots. Potential rescaling of u's in the measure is counteracted by this determinant, and, for example, $|\det(Q^G_{ab})|$ equals $|Z_G|$ for classical gauge groups with the defining representations normalized to have "unit" charges.

Let us come back to 1d and discuss how this normalization factor shows up in the 1d localization procedure. In the latter the electric coupling constant, e^2, instead of β, is taken to zero,

$$\Omega^G(z) \equiv \lim_{e^2 \to 0} \Omega^G(\beta; z)\,. \tag{3.6}$$

For theories with no other parameters, $e^{2/3}\beta$ is the only dimensionless parameter, so clearly the localization process $e^2 \to 0$ has to compute the $\beta \to 0$ limit,

$$\Omega^G(z) = \mathcal{I}^G_{\mathrm{bulk}}(z)\,. \tag{3.7}$$

Fayet-Iliopoulos constants could have complicated this identificaton, but it has been observed that FI constants need to be scaled to infinite, ahead of any e^2 scaling, in order to

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minimize continuum contributions from the flat Coulombic directions [5]. The 1d localization assumes such a limit, hence the above identification remains valid despite FI constants. Indeed, a perfect match between Ω^G as the bulk index [2] and independently computed defect terms $\delta\mathcal{I}^G$ [2, 9, 15, 16] has been established, such that the combination $\mathcal{I}^G = \Omega^G + \delta\mathcal{I}^G$ is integral as the true Witten index should be. The resulting Witten indices matched M-theory predictions [8, 17] not only for $G = \mathrm{SU}(N)$ but also for $\mathrm{Sp}(N)$, and $O(N)$ [3].

The 1d localization for $\Omega^G = \mathcal{I}^G_{\text{bulk}}$ also gives a contour integral [5],

$$\Omega^G(z) = \frac{1}{|W_G|}\int_{\mathcal{C}} \frac{d^{r_G}u}{(2\pi i)^{r_G}}\, g_G(e^u; e^z)\,. \tag{3.8}$$

The contour $\mathcal{C}$ and the integrand g_G reduces to $\mathcal{C}'$ and $f_G \cdot \beta^{-r_G}$, respectively, by a dimensional reduction process with

$$u = \beta u'\,, \qquad z = \beta z' \tag{3.9}$$

in the limit of $\beta \to 0$ while maintaining finite u' and z'. This brings (3.8) to (3.5), with a caveat. Because of the scaling, 0d localization contour $\mathcal{C}'$ can keep only a subset of $\mathcal{C}$, namely only those that can be shrunken to an infinitesimal neighborhood near $e^u = 1$. Any other part of $\mathcal{C}$, say, with nonzero phases of e^u, cannot survive the limit.[1]

Now we are ready to discuss how the overall factor $|\det(Q^G)|$ of (3.5) is hidden in (3.8). The gauge-variables u are, unlike u' that live in $\mathbb{C}^r$, periodic variables living in $(\mathbb{C}^*)^r$. As such, the 1d localization formula secretly assumes $2\pi i$ periodic u-variables, independent of one another, and ignores possible discrete division, such as by Z_G. Therefore, the poles of $g_G(e^u; e^z)$, contours around which would constitute $\mathcal{C}$, come in $|\det(Q^G)|$-many multiplets, separated from one another by $e^{i\Theta_G}$ shifts. In the 1d localization, therefore, one must sum over these identical residues, and this emulates the factor $|\det(Q^G)|$. In the 0d localization, on the other hand, these poles are located infinitely far away from each other in u' planes, so $\mathcal{Z}^G$ will miss these residues, which is then remedied by the overall numerical factor as in (3.5).

Also this comparison neatly resolves the apparent normalization ambiguity of the measure on the 0d side. The 1d Hilbert space trace has no such ambiguity and the simplest convention is to demand gauge charges normalized so that an individual u variable has $2\pi i$ period. For classical groups, it is easy to show the numerical factor $|\det(Q^G)|$ equals $|Z_G|$, while for F_4, e.g., the same normalization requires the roots to be of the length-squared 4 and 8, resulting in $\det(Q^{F_4}_{ab}) = 8$. The latter number has precisely the same origin as 8 mentioned earlier, and manifests in the 1d localization as 8-fold degeneracy of poles.[2]

The reduction from the 1d twisted partition function to the 0d matrix side is fraught with other dangers, however; once we accept the possibility of additional saddles, shifted by $e^{i\Theta_G}$, we must also ask whether there might be a different type of saddles, not gauge-equivalent to the one at origin but contributing to $\mathcal{I}^G_{\text{bulk}}$. In view of how $e^{i\Theta_G}$-shifted saddles

[1] Strictly speaking, the original contour prescription by MNS generally differs from this limit, $\mathcal{C}'$, when $\mathcal{C}$ is the one derived from the first principle [5, 18, 19]. For examples shown in this note, however, this difference does not seem to matter.

[2] Ref. [13] used a different Q^{F_4} normalization, which was nevertheless countered correctly by the measure, du'.

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correspond to missing residues in 0d, it is easy to imagine that such new type of saddles, if any, will also manifest as missing residues when we reduce (3.8) to (3.5).

As is evident from the above 0d vs 1d comparison, some of the poles are lost in the process of $\beta \to 0$ limit. Consider a pair of poles for pure Sp(1) theory, related by a center $Z_{\mathrm{Sp}(1)} = \mathbb{Z}_2$. If they are located at $e^{u_*} = e^z, -e^z$, the first of the two would survive the limit but not the latter, as we rescale $z = \beta z'$ and $u = \beta u'$ with finite z' and u'. This particular loss of poles is innocuous since, as we saw above, it can be corrected by a factor 2 associated with $|Z_{\mathrm{Sp}(1)}| = 2$. However, given a doublet of poles in the 1d localization, there is no guarantee that one of them does survive the 0d limit. Suppose that the pair happen to sit at $e^{u_*} = ie^z, -ie^z$; both of them would have been pushed out to infinity as we go over to the u', z' variables. While this does not actually happen for pure Sp(1), something similar does happen generically for Sp($N \geq 2$) and many other simple groups of rank two and higher.[3]

This means that 0d localization computation of a G-matrix integral $\mathcal{Z}^G$ will generally miss residues which would have contributed to the 1d computation, $\mathcal{I}^G_{\text{bulk}}$, and the loss cannot be compensated by an overall numerical factor. Indeed, a recent computation [2] of 1d twisted partition functions for general simple gauge group G gave answers different from those of the 0d matrix integral, with the exception of $G = \mathrm{SU}(N)$ theories. In table 1, we list these two sets of numbers for $\mathcal{N} = 4$ SYMQ. The numbers in the second column are borrowed from ref. [2] which worked out the bulk index $\mathcal{I}^G_{\text{bulk}} = \Omega^G$ and the Witten index $\mathcal{I}^G$, for $\mathcal{N} = 4, 8, 16$ and general G. Here, we took the unrefined limit, for the comparison with $\mathcal{Z}^G$. The numbers in the third column are newly computed using 0d localization of $\mathcal{Z}^G$ whose z'-dependence drops out for $\mathcal{N} = 4$.

The latter set of numbers agree with older analytical results by Staudacher [12] and by Pestun [13] as well as with the Monte Carlo estimates by Krauth and Staudacher [20] within the latter's error bars. Interestingly, $\mathcal{Z}^G$ is consistently smaller than $\mathcal{I}^G_{\text{bulk}}$ whenever the two disagree. In next section, we will dig deeper and unravel the precise physical reason behind such disagreements. It is worthwhile to repeat here that such disagreements are not confined to SYMQ but prevalent phenomena for twisted partition functions of gauge theories in various dimensions. We will also discover the correct identities between $\mathcal{I}^G_{\text{bulk}}$'s and $\mathcal{Z}^G$'s, in next section, and test them against explicit 0d and 1d computations in section 5.

4 *H*-saddles

The statement that a limit of $\mathcal{I}^G_{\text{bulk}}$ is computed by $\mathcal{Z}^G$, although widely accepted in the community, must be therefore revised. We will presently find, in general,

$$\mathcal{I}^G_{\text{bulk}}(z = \beta z')\Big|_{\beta \to 0} = \mathcal{Z}^G(z') + \cdots \tag{4.1}$$

where the ellipsis denotes additional contributions due to saddles which are not gauge-equivalent to the one at origin. See eq. (4.31) for the precise formula. For the rest of this note, we will study and catalog such additional contributions, to be called H-saddles.

[3]With fundamental matters present, a similar mismatch of poles between 0d and 1d can happen even for rank-one theories, as we will encounter later in the ADHM examples.

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$\mathcal{N}=4$	$\mathcal{I}^G_{\text{bulk}}(0)=\Omega^G(0)$	$\mathcal{Z}^G$
SU(N)	$\frac{1}{N^2}$	$\frac{1}{N^2}$
Sp(2)	$\frac{5}{32}$	$\frac{9}{64}$
Sp(3)	$\frac{15}{128}$	$\frac{51}{512}$
Sp(4)	$\frac{195}{2048}$	$\frac{1275}{16384}$
Sp(5)	$\frac{663}{8192}$	$\frac{8415}{131072}$
Sp(6)	$\frac{4641}{65536}$	$\frac{115005}{2097152}$
Sp(7)	$\frac{16575}{262144}$	$\frac{805035}{16777216}$
SO(7)	$\frac{15}{128}$	$\frac{25}{256}$
SO(8)	$\frac{59}{1024}$	$\frac{117}{2048}$
SO(9)	$\frac{195}{2048}$	$\frac{613}{8192}$
SO(10)	$\frac{27}{512}$	$\frac{53}{1024}$
SO(11)	$\frac{663}{8192}$	$\frac{1989}{32768}$
SO(12)	$\frac{1589}{32768}$	$\frac{6175}{131072}$
SO(13)	$\frac{4641}{65536}$	$\frac{26791}{524288}$
SO(14)	$\frac{1471}{32768}$	$\frac{5661}{131072}$
SO(15)	$\frac{16575}{262144}$	$\frac{92599}{2097152}$
G_2	$\frac{35}{144}$	$\frac{151}{864}$
F_4	$\frac{30145}{165888}$	$\frac{493013}{3981312}$

Table 1. $\mathcal{I}^G_{\text{bulk}}$ vs. $\mathcal{Z}^G$, with SO(3) $\simeq$ SU(2) $\simeq$ Sp(1), SO(4) $\simeq$ SU(2) $\times$ SU(2), SO(5) $\simeq$ Sp(2), and SO(6) $\simeq$ SU(4) understood.

To understand when and how such saddles appear, let us step back to the expression (2.10) for the unrefined $\mathcal{I}^G_{\text{bulk}}$,

$$\sim \beta^{-g(D-1)/2} \int dX \oint d\theta\; e^{-(X(\theta)-X)^2/2\beta-\beta V} \text{tr}_\Psi \,(-1)^{\mathcal{F}} e^{-\beta H_F+\theta^a K^a_D/2} \tag{4.2}$$

and the subsequent expansion of θ, (2.11), around $\theta=0$. Could there be other saddles in this $\beta\to 0$ limit? We already noted that gauge-equivalent saddles at, $\theta=\Theta_G+\beta X_D$, infinitely far away from the 0d perspective, must be summed over if we had chosen to use $\mathcal{P}_G$ instead of $\mathcal{P}_{G/Z_G}$. Each of these saddles leads to the same matrix integral as (4.2), so an overall numerical factor was invented and effectively took care of them, instead. Let us

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write similarly,

$$\theta = \Theta + \beta X_D \tag{4.3}$$

and ask for what other Θ's can there be a contribution to $\mathcal{I}^G_{\text{bulk}}$.

4.1 The 0d limit and H-saddles

Turning on Θ is analogous to a Wilson line symmetry breaking, so we split the Lie Algebra $\mathfrak{g}$ of G into $\mathfrak{h}$ of the unbroken subgroup $H \subset G$ and the rest $\mathfrak{j}$. The commutators obey

$$[\mathfrak{h},\mathfrak{h}] \sim \mathfrak{h}\,, \qquad [\mathfrak{h},\mathfrak{j}] \sim \mathfrak{j}\,, \qquad [\mathfrak{j},\mathfrak{j}] \sim \mathfrak{h}+\mathfrak{j}\,. \tag{4.4}$$

Then, we split the variables into two parts as

$$\begin{aligned} X_\mu &= Z^{\mathfrak{h}}_\mu + Y^{\mathfrak{j}}_\mu\,, \\ \Psi &= \Phi^{\mathfrak{h}} + \Lambda^{\mathfrak{j}}\,. \end{aligned} \tag{4.5}$$

The superscripts $\mathfrak{h}$ and $\mathfrak{j}$ will be henceforth suppressed.

The fermion part of the exponent can be schematically written as

$$-\beta H_F + \theta^a K^a_D/2 \sim \Theta\Lambda\Lambda + \beta Z\Phi\Phi + \beta Z\Lambda\Lambda + \beta Y\Phi\Lambda\,, \tag{4.6}$$

which we need to bring down $g\mathcal{N}/2$ times to saturate the fermionic trace. Two immediate facts follow from this rough form. First, since all of Λ's couple to Θ, terms of type $\beta Z\Lambda\Lambda$ are irrelevant for $\beta \to 0$ limit,

$$-\beta H_F + \theta^a K^a_D/2 \sim \Theta\Lambda\Lambda + \beta Y\Lambda\Phi + \beta Z\Phi\Phi\,. \tag{4.7}$$

Second, if H contains a U(1) factor, not all of Φ can appear in $Z\Phi\Phi$. Then, for each such decoupled Φ, the fermionic trace has to bring down one factor of $\beta Y\Lambda\Phi$. This means, in turn, the Λ trace will cost an extra power of βY than otherwise. Combined with the β power counting in the subsequent Y and Z integration, we find that $\beta \to 0$ will kill this expansion, regardless of the detail.

Therefore, a saddle contribution around (4.3) may contribute only if the unbroken H is either a simple group or a product of simple groups. Such Θ is possible only at discrete points, and we must in general sum up such H-saddles if we wish to express $\mathcal{I}^G_{\text{bulk}}$ as matrix integrals. It remains to show, though, that such an "H-saddle" with nonvanishing fermionic trace does contribute to $\mathcal{I}^G_{\text{bulk}}$. Let us first see how the massive bosonic degrees of freedom, associated with the broken part $\mathfrak{j}$, contribute. The fermionic trace over Λ produces no factors of β or X in the leading terms, and leaves only Φ trace. Since we can consider Y and Λ "fast" variables, its integration will lead to an additive contribution to $\mathcal{I}^G_{\text{bulk}}$ so that

$$\mathcal{I}^G_{\text{bulk}} \quad\rightarrow\quad \sum_{H\subset G}\int dZ\, d\Phi\; \mathcal{O}_{G;H}(Z)\, e^{-[Z,Z]^2/4+Z_\mu K_\mu(\Phi)/2}\,, \tag{4.8}$$

for some operator $\mathcal{O}_{G;H}(Z)$ of fixed degree. We will collect the power of β by starting with (4.2) and show that such H-saddle contributions generically survive $\beta \to 0$ limit.

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The explicit factor of β in front of (4.2) can be conveniently split into, with $h \equiv \dim H$,

$$\frac{\beta^h}{\beta^{h(D-1)/2}} \cdot \frac{1}{\beta^{(g-h)(D-1)/2}} \cdot \beta^{g-h} \,. \tag{4.9}$$

We already saw that the fermionic part contributes power of β only via Φ trace, which together with the first factor above cancels out in the transition to an H-matrix integral. It remains to count the powers of β generated by Y integration. The bosonic part of the exponent can be schematically grouped as follows,

$$\begin{aligned}\frac{1}{\beta}(X(\theta)_i - X_i)^2 &\to \frac{1}{\beta}\left(\Delta_\Theta Y_i + \beta[Y_D, Y_i] + \beta[Y_D, Z_i] + \beta[Z_D, Y_i] + \beta[Z_D, Z_i]\right)^2 \,,\\ \beta[X_i, X_j]^2 &\to \beta\left([Y_i, Y_j] + [Y_i, Z_j] + [Z_i, Y_j] + [Z_i, Z_j]\right)^2 \,.\end{aligned} \tag{4.10}$$

Since $\Delta_\Theta Y_i \equiv Y(\Theta)_i - Y_i$ is of order β^0, we can drop some of higher order Y_i terms, leaving behind,

$$\begin{aligned}\frac{1}{\beta}(X(\theta)_i - X_i)^2 &\to \frac{1}{\beta}\left(\Delta_\Theta Y_i + \beta[Y_D, Z_i]\right)^2 + \beta[Z_D, Z_i]^2 \,,\\ \beta[X_i, X_j]^2 &\to \beta\left([Y_i, Z_j] + [Z_i, Y_j]\right)^2 + \beta[Z_i, Z_j]^2 \,.\end{aligned} \tag{4.11}$$

Terms involving Z_μ variable are needed to constitute H-matrix integral, so we only need to consider terms with Y, and the integration thereof.

Y_i integration generates $\beta^{(g-h)(D-1)/2}$ which cancels the second factor of (4.9), and also replaces Y_i inside $[Y_i, Z_j]$ by $\beta\Delta_\Theta^{-1}[Z_i, Y_D]$. If the action of Δ_Θ on Y_i is diagonal, we can further organize

$$\beta\left([Y_i, Z_j] + [Z_i, Y_j]\right)^2 \sim \beta^3\left([[Z_i, Z_j], Y_D]\right)^2 \,. \tag{4.12}$$

The subsequent integration of Y_D generates prefactors, which, combined with the third factor in (4.9), produce

$$\sim \frac{1}{\beta^{(g-h)/2}} \cdot \frac{1}{Z^{2(g-h)}} \,. \tag{4.13}$$

With the standard rescaling $\beta^{1/4} Z \to Z$, we see finally that the extra power of β cancels out, and

$$\mathcal{O}_{G;H}(Z) \sim \frac{\beta^0}{Z^{2(g-h)}} \,. \tag{4.14}$$

The resulting H-matrix integral has no reason to vanish whatsoever, and thus must contribute to $\mathcal{I}^G_{\rm bulk}$ additively. We conclude that, in general, $\mathcal{Z}^G$ cannot by itself compute the unrefined limit of $\mathcal{I}^G_{\rm bulk}$.

4.2 Recovering $\Omega^G(z)$ from H-saddles

While we have demonstrated how H-saddles can contribute additively to $\Omega^G = \mathcal{I}^G_{\rm bulk}$, their evaluation is another matter. Such H-saddles must account for the difference, e.g., for $1/64$ for $\mathcal{N} = 4$ Sp(2),

$$\Omega^{\rm Sp(2)}\Big|_{\rm unrefined} = \mathcal{Z}^{\rm Sp(2)} + \frac{1}{64} \,. \tag{4.15}$$

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Can we account for such differences precisely by evaluating H-saddle contributions, saddle by saddle? On the other hand, we already noted how the such additive difference manifests as the missing residue phenomena between the 1d localization and the 0d localization. It is thus natural to ask if one can establish a precise relation between these two and thereby compute individual H-saddles via localization.

For gauged quantum mechanics with at least two supersymmetries, HKY derived the residue formula for Ω,

$$\Omega = \frac{1}{|W_G|}\ \text{JK-Res}_\eta\ \frac{g(t;\mathbf{y},\cdots)}{\prod_s t_s}\ \mathrm{d}^r t\,, \tag{4.16}$$

where $(t_1,\ldots,t_r)$ parameterize the Cartan torus, $(\mathbb{C}^*)^r$. For $\mathcal{N}\geq 4$, with $\mathrm{SU}(2)\times\mathrm{U}(1)$ R-symmetry, the functional determinant g takes the form,

$$\begin{aligned} g(t;\mathbf{y},\cdots) = {} & \left(\frac{1}{\mathbf{y}-\mathbf{y}^{-1}}\right)^r \prod_\alpha \frac{t^{-\alpha/2}-t^{\alpha/2}}{t^{\alpha/2}\mathbf{y}^{-1}-t^{-\alpha/2}\mathbf{y}} \\ & \times \prod_i \frac{t^{-Q_i/2}x^{-F_i/2}\mathbf{y}^{-(R_i/2-1)}-t^{Q_i/2}x^{F_i/2}\mathbf{y}^{R_i/2-1}}{t^{Q_i/2}x^{F_i/2}\mathbf{y}^{R_i/2}-t^{-Q_i/2}x^{-F_i/2}\mathbf{y}^{-R_i/2}} \end{aligned} \tag{4.17}$$

where α runs over the roots of the gauge group and i labels the individual chiral multiplets, with $\mathrm{U}(1)$ R-charge R_i, the gauge charge Q_i under the Cartan. The chemical potential terms asscoiated with Cartan of the flavor group are denoted collectively as x^{F_i}. For detailed derivation and description of this JK residue [14] formula as well as for how to select and use the auxiliary parameters η, please see the section 4 of ref. [5].

This arises from the $e^2\to 0$ limit of the path integral version of $\Omega(\beta;z)$,

$$\Omega(\beta;z) = \int [d\mathcal{A}_0\, d\mathcal{X}_i\,\cdots]\ \exp\left(-\int_0^\beta \mathcal{L}_{\text{Euclidean}}(\mathcal{A}_0,\mathcal{X}_i,\cdots;\mathbf{z}\equiv 2\log\mathbf{y},\cdots)\right) \tag{4.18}$$

where, as we already noted, the β-dependence is implicitly removed by this localization process. On the other hand, the naive $\beta\to 0$ limit of this path integral is

$$\mathcal{Z} = \int dX_D\, dX_i\,\cdots\ \exp\left(-\mathcal{L}_{\text{Euclidean}}(X_D,X_i,\cdots;\mathbf{z}',\cdots)\right) \tag{4.19}$$

obtained by restricting the fields to the constant configurations

$$\mathcal{A}_0\to X_D\,,\quad \mathcal{X}_i\to X_i\,,\quad \cdots \tag{4.20}$$

and expanding the chemical potentials as

$$\log\mathbf{y} = \beta\mathbf{z}'/2 \tag{4.21}$$

with $\mathbf{z}'$ kept finite, and similarly for flavor chemical potentials x.

Evaluating the latter matrix integral, one obtains the 0d contour integral formula referred to in the previous section, as follows: with the 1d localization formula, take $\beta\to 0$ limit on the integrand first,

$$\frac{1}{|W_G|}\ \text{JK-Res}_\eta\ \lim_{\beta\to 0}\beta^r g_G(t=e^{\beta u'};\mathbf{y}=e^{\beta\mathbf{z}'/2},\cdots)\ \mathrm{d}^r u'$$

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$$\to \frac{1}{|W_G|}\,\text{JK-Res}'_\eta\; f_G(u';\mathbf{z}',\cdots)\;\mathrm{d}^r u'\,, \tag{4.22}$$

where we took care to put a prime in the latter JK-Res to emphasize that not all available poles of g survive this limit. This misses the other gauge-equivalent saddles, (3.4), so more generally we must also include contributions from

$$\mathcal{A}_0 \to \frac{\Theta_G}{\beta} + X_D\,, \quad \mathcal{X}_i \to X_i\,, \quad \cdots \tag{4.23}$$

for the Wilson lines $e^{i\Theta_G}$. We sum over these and find

$$\begin{aligned} \mathcal{Z}^G &= \sum_{\Theta_G} \frac{1}{|W_G|}\,\text{JK-Res}_\eta \lim_{\beta\to 0} \beta^r g_G(t=e^{i\Theta_G}e^{\beta u'};\mathbf{y}=e^{\beta\mathbf{z}'/2},\cdots)\;\mathrm{d}^r u' \\ &= \frac{|\det(Q^G)|}{|W_G|}\,\text{JK-Res}'_\eta\; f_G(u';\mathbf{z}',\cdots)\;\mathrm{d}^r u'\,, \end{aligned} \tag{4.24}$$

since g_G is invariant under such shifts. This is the 0d formula (3.5) of the previous section.

It is quite clear that nontrivial H-saddles are no different than Θ_G saddles, in that they are merely different kinds of Wilson lines,

$$\mathcal{A}_0 \to \frac{\Theta}{\beta} + X_D\,, \quad \mathcal{X}_i \to X_i\,, \quad \cdots \tag{4.25}$$

around which a reduced H theory resides. Taking these into account as well, one finds

$$\begin{aligned} &\Omega^G(\beta z')\Big|_{\beta\to 0} \\ &= \mathcal{Z}^G + \sum_{\Theta} \frac{1}{|W_G|}\,\text{JK-Res}'_\eta \lim_{\beta\to 0} \beta^r g_G(t=e^{i\Theta}e^{\beta u'};\mathbf{y}=e^{\beta\mathbf{z}'/2},\cdots)\;\mathrm{d}^r u'\,. \end{aligned} \tag{4.26}$$

If a pole for Ω^G, say, at $t_* = h(\mathbf{y},x)$, survives the 0d limit, the pole at $t_* = e^{-i\Theta}h(\mathbf{y},x)$ would be missed by $\mathcal{Z}^G$ but could be a contributing pole in the Θ summand. This way, the latter sum compute H-saddles individually via a 0d localization.

One can go further, in fact. The poles missed by (4.24) are all such that the argument of Sinh functions in the denominator of (4.17) are either intact or shifted by some finite angle, due to

$$e^{i\Theta}E_\alpha e^{-i\Theta} = e^{i\phi_\alpha}E_\alpha \tag{4.27}$$

for some $\phi_\alpha \in (0,2\pi)$. For pure gauge theories, then, the contributing determinant factors in g_G fall in two distinct categories. For roots belonging to the unbroken group H, we merely take the 0d scaling limit as

$$\frac{t^{-\alpha/2}-t^{\alpha/2}}{t^{\alpha/2}\mathbf{y}^{-1}-t^{-\alpha/2}\mathbf{y}} \quad\Rightarrow\quad \frac{e^{-\beta u'\cdot\alpha/2}-e^{\beta u'\cdot\alpha/2}}{e^{\beta(u'\cdot\alpha-\mathbf{z}')/2}-e^{-\beta(u'\cdot\alpha-\mathbf{z}')/2}} \quad\to\quad -\frac{\alpha\cdot u'}{\alpha\cdot u'-\mathbf{z}'}\,. \tag{4.28}$$

The broken ones suffer a common and nonzero shift of the phase both in the numerator and in the denominator, and, thanks to this, reduces to -1 universally in the 0d scaling limit,

$$\frac{t^{-\alpha/2}-t^{\alpha/2}}{t^{\alpha/2}\mathbf{y}^{-1}-t^{-\alpha/2}\mathbf{y}} \quad\Rightarrow\quad \frac{e^{-\beta u'\cdot\alpha/2}e^{-i\phi_\alpha/2}-e^{\beta u'\cdot\alpha/2}e^{i\phi_\alpha/2}}{e^{\beta(u'\cdot\alpha-\mathbf{z}')/2}e^{i\phi_\alpha/2}-e^{-\beta(u'\cdot\alpha-\mathbf{z}')/2}e^{-i\phi_\alpha/2}} \quad\to\quad -1\,. \tag{4.29}$$

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For contributions from the adjoint chirals, the same happens, producing -1's for the latter class in particular. Therefore, the integrand reduces, at such shifted saddles, to

$$\lim_{\beta\to 0}\beta^r g_G(t=e^{i\Theta}e^{\beta u'};\mathbf{y}=e^{\beta\mathbf{z}'/2},\cdots)=\lim_{\beta\to 0}\beta^r g_H(t=e^{\beta u'};\mathbf{y}=e^{\beta\mathbf{z}'/2},\cdots)\,. \tag{4.30}$$

The saddle contribution at $e^{i\Theta}$ is therefore nothing but the canonical H-matrix integral, except that the overall group theory factor in front is that of G rather than that of H, which, amazingly, must be the sole effect of the complicated operator insertion $\mathcal{O}(G;H)$ in (4.8).

After careful account of H-saddles and their Weyl copies, we arrive at the following universal formula for $\mathcal{N}=4,8,16$ SYMQ,

$$\Omega^G(\beta z')\Big|_{\beta\to 0} \;=\; \mathcal{Z}^G(z')+\sum_H d_{G:H}\,\frac{|\det(Q^G)|/|W_G|}{|\det(Q^H)|/|W_H|}\,\mathcal{Z}^H(z')\,. \tag{4.31}$$

The integer $d_{G:H}$ counts the number of Wilson lines that leave H unbroken; it counts Weyl copies of $e^{i\Theta}$ as distinct, modulo $e^{i\Theta_G}$ shift by left multiplication. Note that, since the whole formula started with the G theory, the normalization of the charge matrix Q^H of simple roots for H must be the one inherited from the G root system. For classical group G, this further simplifies to

$$\Omega^G(\beta z')\Big|_{\beta\to 0} \;=\; \mathcal{Z}^G(z')+\sum_H \frac{d_{G:H}}{2}\,\frac{|W_H|}{|W_G|}\,\mathcal{Z}^H(z')\,. \tag{4.32}$$

Again, the integer $d_{G:H}$ counts the number of Wilson lines, up to the action of Z_G by left multiplication. We will show that such a H-saddle sum (and its analog) happen to be absent for SU(N) SYMQ and for U(k) ADHM, but otherwise generically present for gauged quantum mechanics.

While we concentrated on 1d theories in this note, it is pretty clear that the phenomena of the missing residues are prevalent whenever we consider gauge theory on a vanishing circle, i.e., when we compute the twisted partition function of a supersymmetric gauge theory on $\mathbb{S}^1\times\mathbb{M}$ and try to relate its limit to partition functions on $\mathbb{M}$. In fact, the derivation here is easily extendible, regardless of the details of the theory or even of the spatial dimension, dim($\mathbb{M}$), as long as a residue formula involving the $\mathbb{S}^1$ Wilson line variables is available.

5 Classifying H-saddles

For Yang-Mills theories with adjoint representations only, a Wilson line, $e^{i\Theta}\in G$, gives a contributing saddle if and only if it preserves H a product of simple subgroups of G. One convenient parametrization of the Wilson line is

$$\Theta = 2\pi\sum_s \frac{2k_s^{(\Theta)}}{|\beta_s|^2}\,\vec{\mu}_s\cdot\vec{\mathbb{H}} \tag{5.1}$$

with Cartan generators $\vec{\mathbb{H}}$, simple roots $\vec{\beta}_s$ and the associated fundamental weights $\vec{\mu}_s$. A general positive root $\vec{\alpha}_{\{n\}}=\sum_s n_s\vec{\beta}_s$ of G is in the H root system if and only if

$$\sum_s k_s^{(\Theta)}n_s \;=\; 0 \;\text{ mod }\; \mathbb{Z}\,. \tag{5.2}$$

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For generic values for $k_s^{(\Theta)}$'s, it is clear that U(1) generated by Θ itself will be a free U(1) in H. Only at discrete choices of Θ, we expect to find contributing saddles.

5.1 Classical G

$G = \mathrm{SU}(N)$ is the simplest to analyze since possible values of n_s are either 1 or 0. With H a proper subgroup of SU(N), there has to be at least one root that fails (5.2), and we can use the Weyl transformation to bring it to the form

$$n = (1, 0, 0, \ldots, 0)$$

with $k_1^{(\Theta)} \notin \mathbb{Z}$. If all other $k_s^{(\Theta)}$ are integral, $H \simeq \mathrm{SU}(N-1) \times \mathrm{U}(1)$, so such a Wilson line does not contribute. Suppose that there is exactly one more nonintegral $k_s^{(\Theta)}$. If $s > 2$, the U(1) persists and the saddle is irrelevant. If $s = 2$, and if $k_1^{(\Theta)} + k_2^{(\Theta)} \notin \mathbb{Z}$, the unbroken group H has one more U(1) factor. Finally, if $k_1^{(\Theta)}+k_2^{(\Theta)} \in \mathbb{Z}$, $H \simeq \mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{SU}(N-2)$, hence again irrelevant. Proceeding similarly, it is easy to see that, for $G = \mathrm{SU}(N)$, an unbroken proper subgroup H always contains at least one U(1) factor. Thus,

$$\mathcal{I}_{\text{bulk}}^{\mathrm{SU}(N)}(\beta z')\Big|_{\beta\to 0} = \mathcal{Z}^{\mathrm{SU}(N)}(z')\,, \tag{5.3}$$

as is consistent with table 1.

The same table also suggests, however, that *for no other simple Lie Group, such an equality will hold.* For the other classical groups, say Sp(K) and SO(N), we find a large class of H-saddles, corresponding to

$$H \simeq \mathrm{Sp}(m) \times \mathrm{Sp}(K-m)\,, \qquad H \simeq \mathrm{SO}(2m) \times \mathrm{SO}(N-2m)\,. \tag{5.4}$$

The respective Wilson lines can be written compactly as

$$\begin{aligned} \Theta^{\mathrm{Sp}(K)\to\mathrm{Sp}(m)\times\mathrm{Sp}(K-m)} &= \pi \sum_{s=1}^{m} \mathbb{H}_s\,, \\ \Theta^{\mathrm{SO}(N)\to\mathrm{SO}(2m)\times\mathrm{SO}(N-2m)} &= \pi \sum_{s=1}^{m} \mathbb{H}_s\,, \end{aligned} \tag{5.5}$$

which can be universally written as

$$k_l^{(\Theta)} = \frac{1}{2}\delta_{lm} \tag{5.6}$$

with the canonical choice of simple roots,

$$\beta_1 = e_1 - e_2\,, \quad \beta_2 = e_2 - e_3\,, \quad \cdots\,. \tag{5.7}$$

Absence of an unbroken U(1) factor and $H \neq G$ further demand $1 \leq m \leq K-1$ for Sp(K), $4 \leq 2m \leq N-4$ for even SO(N), and $4 \leq 2m \leq N-1$ for odd SO(N).

With these, the identity (4.31) simplifies to

$$\mathcal{I}_{\text{bulk}}^{\mathrm{Sp}(K)}(\beta z')\Big|_{\beta\to 0} = \mathcal{Z}^{\mathrm{Sp}(K)}(z') + \sum_{m=1}^{K-1} \frac{1}{4}\, \mathcal{Z}^{\mathrm{Sp}(m)\times\mathrm{Sp}(K-m)}(z')\,, \tag{5.8}$$

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and similarly,

$$\mathcal{I}^{\mathrm{SO}(N)}_{\mathrm{bulk}}(\beta z')\Big|_{\beta\to 0} = \mathcal{Z}^{\mathrm{SO}(N)}(z') + \sum_{m=2}^{N/2-2} \frac{1}{8}\,\mathcal{Z}^{\mathrm{SO}(2m)\times\mathrm{SO}(N-2m)}(z') \tag{5.9}$$

for even N, and

$$\mathcal{I}^{\mathrm{SO}(N)}_{\mathrm{bulk}}(\beta z')\Big|_{\beta\to 0} = \mathcal{Z}^{\mathrm{SO}(N)}(z') + \sum_{m=2}^{(N-1)/2} \frac{1}{4}\,\mathcal{Z}^{\mathrm{SO}(2m)\times\mathrm{SO}(N-2m)}(z') \tag{5.10}$$

for odd N.

These identities are checked affirmatively by table 1, and their analog for $\mathcal{N}=8$ SYMQ. For classical groups of general ranks, a conjectural formula is available for the numerical limit of $\mathcal{Z}$ for $\mathcal{N}=4,8$ [13], while its 1d counterpart Ω's has been also computed [2, 16]. The comparison of these two sets of numbers via the above identities is given in the appendix. We also confirmed these for $\mathcal{N}=16$ SYQM up to rank 3.

5.2 Exceptional G

G_2's root system is generated by the simple roots $\vec{\beta}_{1,2}$ with $|\beta_2|^2 = 3|\beta_1|^2$ and $2\vec{\beta}_1\cdot\vec{\beta}_2 = -3|\beta_1|^2$. The three short positive roots, $\{\beta_1, \beta_1+\beta_2, 2\beta_1+\beta_2\}$, and the three long positive roots, $\{\beta_2, 3\beta_1+\beta_2, 3\beta_1+2\beta_2\}$, each span an SU(3) root system. SU(3)-saddles come from Wilson lines Θ with

$$k^{(\Theta)} = \left(\pm\frac{1}{3}, 0\right), \tag{5.11}$$

that leave the long roots unbroken, while there are also $\mathrm{SU}(2)\times\mathrm{SU}(2)$-saddles at

$$k^{(\Theta)} = \left(0, \frac{1}{2}\right), \quad k^{(\Theta)} = \left(\frac{1}{2}, 0\right), \quad k^{(\Theta)} = \left(\frac{1}{2}, \frac{1}{2}\right) \tag{5.12}$$

preserving $\{E_{\beta_1}, E_{3\beta_1+2\beta_2}\}$, $\{E_{2\beta_1+\beta_2}, E_{\beta_2}\}$, and $\{E_{\beta_1+\beta_2}, E_{3\beta_1+\beta_2}\}$, respectively. Both classes of saddles will contribute, and (4.31) becomes

$$\mathcal{I}^{G_2}_{\mathrm{bulk}}(\beta z')\Big|_{\beta\to 0} = \mathcal{Z}^{G_2}(z') + \frac{1}{3}\,\mathcal{Z}^{\mathrm{SU}(3)}(z') + \frac{1}{2}\,\mathcal{Z}^{\mathrm{SU}(2)\times\mathrm{SU}(2)}(z'), \tag{5.13}$$

after we carefully keep track of the charge normalization factors. This is, again, verified by table 1.

For F_4, the root system is a combination of SO(9) roots and the 16 spinor weights thereof,

$$\begin{aligned} &\pm 2e_s \pm 2e_t\,, \qquad s,t=1,2,3,4\,, \qquad s\neq t\,,\\ &\pm 2e_s\,, \qquad s=1,2,3,4\,,\\ &\pm e_1 \pm e_2 \pm e_3 \pm e_4\,. \end{aligned} \tag{5.14}$$

The following 2^3 Wilson lines,

$$e^{i\Theta_{F_4}} = e^{i\sum_s \phi_s \mathbb{H}_s}\,, \qquad \{e^{i\phi_s}\} = \{\pm1, \pm1, \pm1, \pm1\} \tag{5.15}$$

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with an even number of -1's leave the entire F_4 unbroken. These saddles must be taken into account if we take the integration range of each θ_s to be $[0, 2\pi)$. As noted before, this has something to do with the inherent ambiguity between

$$\mathcal{P}_{F_4} = \frac{1}{\mathrm{vol}(F_4)} \int_{F_4} e^{i\theta_a G_a} \tag{5.16}$$

and

$$\mathcal{P}'_{F_4} = \frac{1}{8\mathrm{vol}(F_4)} \int_{8F_4} e^{i\theta_a G_a} \tag{5.17}$$

where, in the latter, the Cartan torus is taken to have an artificially enlarged volume $(2\pi)^4$. The latter choice of the projector is implicitly used for the localization computation of $\mathcal{I}^{F_4}_{\mathrm{bulk}}$, the factor 8 in the volume is correctly counteracted by this eight-fold degeneracy of the $e^{i\Theta_{F_4}}$ saddles.

Thus, we have the associated 8-fold gauge-equivalence, $(\mathbb{Z}_2)^3$, from the left multiplication by $e^{i\Theta_{F_4}}$'s, up to which we classify the H-saddles and count $d_{F_4:H}$'s. The easiest to spot are a triplet of SO(9)-saddles, modulo $(\mathbb{Z}_2)^3$, sitting at

$$\{e^{i\phi_s}\} = \{1, 1, 1, -1\}\,, \qquad \{i, i, i, i\}\,, \qquad \{i, i, i, -i\} \tag{5.18}$$

respectively. The first removes the SO(9) spinor weights, while the latter two removes SO(8) vector weights and, respectively, chiral or anti-chiral SO(8) spinor weights. Thanks to the SO(8) triality, all of these preserve an SO(9). The Wilson lines

$$\{e^{i\phi_s}\} = \{1, 1, i, i\} \tag{5.19}$$

and the Weyl copies thereof, modulo $(\mathbb{Z}_2)^3$, produce 12 $\mathrm{Sp}(3) \times \mathrm{Sp}(1)$ saddles. Similarly,

$$\{e^{i\phi_s}\} = \{\omega, \omega, \omega, \omega^3\} \tag{5.20}$$

with $\omega = e^{\pi i/4}$, produces 24 $\mathrm{SU}(4) \times \mathrm{SU}(2)$ saddles, and

$$\{e^{i\phi_s}\} = \{1, \lambda, \lambda, \lambda^2\} \tag{5.21}$$

with $\lambda = e^{\pi i/3}$, produces 32 $\mathrm{SU}(3) \times \mathrm{SU}(3)$ saddles. These altogether imply, with (4.31),

$$\begin{aligned} \mathcal{I}^{F_4}_{\mathrm{bulk}}(\beta z')\Big|_{\beta\to 0} &= \mathcal{Z}^{F_4}(z') + \frac{1}{2}\,\mathcal{Z}^{\mathrm{SO}(9)}(z') + \frac{1}{2}\,\mathcal{Z}^{\mathrm{Sp}(3)\times\mathrm{Sp}(1)}(z') \\ &\quad + \frac{1}{4}\,\mathcal{Z}^{\mathrm{SU}(4)\times\mathrm{SU}(2)}(z') + \frac{1}{3}\,\mathcal{Z}^{\mathrm{SU}(3)\times\mathrm{SU}(3)}(z')\,, \end{aligned} \tag{5.22}$$

which is, again, easily confirmed by table 1.

This leaves $G = E_{6,7,8}$. Since the above examples illustrated and confirmed H-saddles and their consequences amply, we will merely demonstrate existence of H-saddles for these remaining cases. The root system of E_8 is the combination of 112 roots and 128 chiral spinor weights of SO(16),

$$\pm 2e_s \pm 2e_t\,, \qquad 1 \le s, t \le 8\,, \quad s \ne t\,,$$

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$$\pm e_1 \pm e_2 \cdots \pm e_8\,, \qquad \text{even number of } +\text{'s}\,. \tag{5.23}$$

A simple Wilson line,

$$\Theta = \pi\, \mathbb{H}_8 \tag{5.24}$$

breaks the 128 entirely, while preserving SO(16) subgroup, so there are nontrivial $H \simeq$ SO(16) saddles. For E_7, a useful representation of root systems is

$$\begin{aligned} &\pm 2e_s \pm 2e_t\,, \qquad 1 \le s,t \le 6\,, \quad s \ne t\,, \\ &\pm e_1 \pm e_2 \cdots \pm e_6 \pm \sqrt{2}\, e_7\,, \qquad \text{even number of } +\text{'s for } e_{1,2,\dots,6}\,, \\ &\pm 2\sqrt{2}\, e_7\,, \end{aligned} \tag{5.25}$$

so the Wilson line

$$\Theta = \frac{\pi}{\sqrt{2}}\, \mathbb{H}_7 \tag{5.26}$$

preserves $H \simeq \mathrm{SO}(12) \times \mathrm{SU}(2)$. Finally, the E_6 root system is

$$\begin{aligned} &\pm 2e_s \pm 2e_t\,, \qquad 1 \le s,t \le 5\,, \quad s \ne t\,, \\ &\pm e_1 \pm e_2 \cdots \pm e_5 \pm \sqrt{3}\, e_6\,, \qquad \text{odd number of } +\text{'s}\,, \end{aligned} \tag{5.27}$$

and the Wilson line

$$\Theta = \frac{\pi}{4} \left(\mathbb{H}_1 + \cdots + \mathbb{H}_5 - \sqrt{3}\, \mathbb{H}_6 \right) \tag{5.28}$$

preserves $H \simeq \mathrm{SU}(6) \times \mathrm{SU}(2)$. So, there is at least one class of contributing H-saddles for each of $E_{6,7,8}$ and,

$$\mathcal{I}^{E_{6,7,8}}_{\text{bulk}}(\beta z')\bigg|_{\beta\to 0} = \mathcal{Z}^{E_{6,7,8}}(z') + \cdots\,. \tag{5.29}$$

Again, one cannot resort to $\mathcal{Z}^{E_{6,7,8}}$ alone, *a priori*, for the computation $\mathcal{I}^{E_{6,7,8}}_{\text{bulk}}$.

6 ADHM and 4d/5d instanton partition functions

Although we have so far considered SYMQ with the adjoint representation only, the same kind of problems in going over from 1d to 0d can be expected generally. Perhaps another most notable class is the ADHM-type that describes D0 dynamics in D4 background, possibly with the additional ingredients of Orientifold 4-planes or 8-planes and also D8-branes. The 0d version enters instanton partition functions for 4d supersymmetric Yang-Mills theories on Omega-deformed $\mathbb{R}^4$, such as the Nekrasov partition functions, while the 1d version enters its 5d analog on $\mathbb{S}^1 \times \mathbb{R}^4$.

It is clear that the phenomena of disappearing residues will persist when we compare the 0d and the 1d localization computations, since existence of H-saddles originates in the vector multiplet; the additional chiral or hypermultiplets can only make such saddles more diverse than otherwise. As with pure Yang-Mills cases, an H-saddle for twisted partition functions of general gauged quantum mechanics means a Wilson line $e^{i\Theta}$ at which the theory breaks up into heavy and light parts and the resulting saddle point integral survives the $\beta \to 0$ limit. One difference is that this does not always require breaking of G to a

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smaller group; it may be that some of the matter fields can become heavy instead. Thus, an H-saddle should generally refer to contributing saddles at which the effective theory is smaller than the one we started with.

This means that, sector by sector labeled by the instanton number, the 0d and 1d partition functions of the ADHM data cannot generally agree with each other. On the other hand, we anticipate the instanton partition functions are themselves very physical quantities, easily expected to be continuous in the zero radius limit of $\mathbb{S}^1$. This is a qualitatively different issue than the one we addressed so far: the new problem emerges because $\mathcal{Z}^{\mathrm{ADHM}}$'s apparently have their own physical meanings, and the 4d/5d field theory interpretations seem to require a continuity of some kind with its 1d analog without having to add H-saddle contributions.

The resolution of this is already implicit in literatures. It is well-known among practitioners that the ADHM partition functions sometimes compute more than what are needed for the instantons. The instanton moduli space is equivalent to the Higgs phase, while the ADHM themselves contain the Coulomb phase as well. Since the latter is often factored out neatly, one logical possibility is that the 1d/0d discontinuity between $\mathcal{I}_{\mathrm{bulk}}$ and $\mathcal{Z}$ resides entirely in the latter Coulomb side and that way becomes irrelevant for the field theory quantities. To see if this is actually case, we will consider two distinct classes of ADHM data. $\mathrm{U}(k)$ ADHM data for $\mathrm{U}(N)$ instantons and $\mathrm{Sp}(k)$ ADHM data for $\mathrm{SO}(N)$ instantons.

6.1 Missing residues, again

Although $\mathrm{U}(k)$ looks similar to $\mathrm{SU}(k)$ superficially, the two are very different. First of all, the distribution of the adjoint poles in the 1d localization is such that the k-fold degeneracy we have encountered in the adjoint-only $\mathrm{SU}(k)$ theory no longer appears. Indeed, when we compute ADHM via 1d localization, it is clear that there is no residue that would be dropped when we take the strict 0d limit. As such, the 1d $\mathrm{U}(k)$ ADHM partition functions are continuously connected to 0d $\mathrm{U}(k)$ ADHM partition functions.

This can be seen more directly from H-saddle classification. $\mathrm{U}(k)$ ADHM field content differs from $\mathcal{N}=16$ $\mathrm{SU}(N)$ theory in two respects: the overall $\mathrm{U}(1)$ gauge group, always unbroken under the Wilson line Θ and the $\mathrm{U}(k)$ fundamental matter. Since the latter couples to the $\mathrm{U}(1)$ factors generically, an unbroken $\mathrm{U}(1)$ gaugino can easily find a Yukawa coupling with another light fermion, apparently evading the condition on allowed H-saddle. However, we need to recall that a nontrivial Wilson line Θ always breaks $\mathrm{SU}(k)$ part of $\mathrm{U}(k)$ with at least one factor of $\mathrm{U}(1)$, so that the unbroken group under Θ will break $\mathrm{U}(k)$ to an H with at least two factors of $\mathrm{U}(1)$'s. Furthermore, it is clear that light fermions in the fundamental matter multiplet can couple to only one linear combination of these two light gauginos. For example, take $\mathrm{U}(k)\to\mathrm{U}(k')\times\mathrm{U}(k-k')$ due to

$$e^{i\Theta}=\mathrm{diag}_{k\times k}(-1,-1,\ldots,-1,1,\ldots,1)\,.$$

The first k' fermions in the fundamental become massive while the other $k-k'$ remain massless. There are two light $\mathrm{U}(1)$ gauginos, associated with each of the two blocks; while the one associated with the $\mathrm{U}(k-k')$ block can couple to the light fundamental fermions, the other

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associated with $\mathrm{U}(k')$ can only couple to the heavy ones. The same is true of the case $k = k'$; the single $\mathrm{U}(1)$ gaugino couples to heavy fermions only, and the saddle does not survive $\beta \to 0$ limit. The same mechanism that killed potential H-saddles of pure $\mathrm{SU}(N)$ theory repeats itself for potential H-saddle for $\mathrm{U}(k)$ ADHM. The contributing saddle is possible only when H contains no more than one $\mathrm{U}(1)$, or equivalently when $H = \mathrm{U}(k) = G$. Therefore, as we already noted based on the localization comparison above, the matrix integral limit, $\mathcal{Z}^{\mathrm{U}(k)}_{\mathrm{ADHM}}$, is continuously connected to the respective 1d partition function $\Omega^{\mathrm{U}(k)}_{\mathrm{ADHM}}$.

Let us now turn to $\mathrm{Sp}(k)$ ADHM. The missing residue problem appears already with $\mathrm{Sp}(1)$, instead of starting with $\mathrm{Sp}(2)$, if fundamental matter is present. Proceeding with 1d localization of $\mathrm{Sp}(1)$ theory, we can expect to find pairs of singularities, collectively denoted as $S^{(+)}$ and $S^{(-)}$, where the adjoint charge becomes massless and ones, say, P, involving fundamental becoming massless. The pairs $S^{(\pm)}$ have positions mutually displaced by the $\mathbb{Z}_2$ center; if we take $S^{(+)}$ to be the ones that can be scaled to near the origin of u-space, $S^{(-)}$ would be relatively displaced by -1 in e^u coordinate. These two would have different residues for the ADHM case, since the fundamental matter contributes differently. The second type, P, would not be in pairs, as it is entirely due to the fundamental matters. The passage to 0d will then lose, $S^{(-)}$, while $S^{(+)}$ and P can be scaled to fit in u' planes. Note that, even though $S^{(-)}$ are related to $S^{(+)}$ by $Z_{\mathrm{Sp}(1)} = \mathbb{Z}_2$, the unavoidable omission of $S^{(-)}$ cannot be cured by an overall numerical factor of two on 0d side, unlike the adjoint only $\mathrm{Sp}(1)$ theory. With higher rank $\mathrm{Sp}(k)$, this problem will get only worse.

Such mismatches between the 1d and the 0d localizations also manifest via contributing H-saddles in $\mathrm{Sp}(1)$ ADHM theory. What is the extra saddle? Consider the Wilson line,

$$e^{i\Theta} = \mathrm{diag}_{2\times 2}(-1,-1) \tag{6.1}$$

which is an element of $Z_{\mathrm{Sp}(1)} = \mathbb{Z}_2$. The unbroken group H equals $G = \mathrm{Sp}(1)$ at this saddle, yet the fundamental multiplets become massive and the theory reduces to a smaller one. This is clearly another form of the H-saddle we discussed earlier in the section.

For $\mathrm{Sp}(k)$ ADHM theories, therefore, the mismatch between $\Omega^{\mathrm{Sp}(k)}_{\mathrm{ADHM}}$ and $\mathcal{Z}^{\mathrm{Sp}(k)}_{\mathrm{ADHM}}$ is unavoidable, already starting with rank 1. Adding H-saddle contributions to the latter will restore the limiting form of $\Omega^{\mathrm{Sp}(k)}_{\mathrm{ADHM}}$, but the problem at hand is that $\mathcal{Z}^{\mathrm{Sp}(k)}_{\mathrm{ADHM}}$ might be by itself a physical quantity. The continuity between 5d and 4d instanton partition functions demands a different kind of continuity between $\Omega^{\mathrm{Sp}(k)}_{\mathrm{ADHM}}$ and $\mathcal{Z}^{\mathrm{Sp}(k)}_{\mathrm{ADHM}}$, to which we turn next.

6.2 Relating 4d and 5d

As noted already, the crux of the matter lies in the fact that ADHM partition function may compute more than what are needed for the 4d/5d field theory instanton partition functions. When D(-1) or D0 reside in D3/D4, they are the field theory instantons. ADHM, however, knows about directions transverse to D3/D4 as well, in the form of the “Coulomb branch.” Since we are dealing with low dimensional path integrals, the “branches” are not superselection sectors and have to be integrated over as well. Therefore, the right thing to do is to factor out the latter’s contribution from the ADHM partition functions, which should leave behind 4d/5d field theory quantities.

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For U(k) ADHM, it is known that the "Coulomb" contributions to the partition functions are absent. This can be understood from the fact that the flavorless U(k) theory comes with a free U(1). A free vector multiplet forces $\mathcal{I}$, Ω, and $\mathcal{Z}$ to vanish altogether. This implies the desired continuity,

$$\left[\sum q^k \Omega^{\mathrm{U}(k)}_{\mathrm{ADHM}}\right] \quad \to \quad \left[\sum q^k \mathcal{Z}^{\mathrm{U}(k)}_{\mathrm{ADHM}}\right] , \tag{6.2}$$

also reflected in the absence of H-saddle, or equivalently in the absence of missing residue phenomena, as discussed already. On the other hand, for Sp(k) and $O(m)$ ADHM data, the continuity is lost. The Coulombic contributions in such cases have been dealt with in the past in the context of 5d instantons [21, 22], so what we need to do here is to check whether its 4d analog works as well. For example, is the limit

$$\left[\sum q^k \Omega^{\mathrm{Sp}(k)}_{\mathrm{ADHM}}\right]\left[\sum q^k \Omega^{\mathrm{Sp}(k)}_{\mathrm{Coulomb}}\right]^{-1} \quad \to \quad \left[\sum q^k \mathcal{Z}^{\mathrm{Sp}(k)}_{\mathrm{ADHM}}\right]\left[\sum q^k \mathcal{Z}^{\mathrm{Sp}(k)}_{\mathrm{Coulomb}}\right]^{-1} \tag{6.3}$$

continuous? Below, the simplest example of this for 4d $\mathcal{N}=2^*$ and for 5d $\mathcal{N}=1^*$ theories is outlined for an illustration. A first-principle computations of $\Omega^{\mathrm{Sp}(k)}_{\mathrm{ADHM}}$'s were already given in ref. [22], while their 4d counterpart, $\mathcal{Z}^{\mathrm{Sp}(k)}_{\mathrm{ADHM}}$'s, are newly computed here for the above comparison.

The twisted partition function of the Sp(k) ADHM quantum mechanics is

$$\Omega^{\mathrm{Sp}(k)}_{\mathrm{ADHM}} = \mathrm{Tr}\left[(-1)^{\mathcal{F}} e^{-\beta\{\mathcal{Q},\tilde{\mathcal{Q}}\}} t^{2(J_{1R}+J_{2R})} u^{2J_{1L}} v^{2J_{2L}} \prod_{a=1}^{N_f} w_a^{2\Pi_a}\right] , \tag{6.4}$$

where the supercharges are inherited from the 5d ones as $\mathcal{Q} \leftarrow Q^{A=1}_{\dot\alpha=\dot 1}$ and $\tilde{\mathcal{Q}} \leftarrow Q^{A=2}_{\dot\alpha=\dot 2}$, while J_{1R}, J_{1L}, J_{2R}, and J_{2L} are the Cartan generators of SU$(2)_{1R}$, SU$(2)_{1L}$, SU$(2)_{2R}$, and SU$(2)_{2L}$ R-symmetries, respectively. These SU(2)'s sit inside the SO$(4)_1$ little group and SO$(5)_2$ R-symmetry of the 5d theory in question. Π_a's are the Cartan generators of the flavor symmetry, inherited from the 5d gauge symmetry SO($2N_f$).

With $N_f = 0$, this theory is a D0 quantum mechanics in the presence of the O4$^-$-plane, and the partition function here captures the "Coulombic" part of the ADHM data for $N_f \geq 1$. The generating function for 1d ADHM partition functions has been proposed and confirmed up to $k = 3$ [22] as

$$\sum_{k=0}^{\infty} q^k \Omega^{\mathrm{Sp}(k)}_{\mathrm{ADHM,SO(0)}} = \mathrm{PE}\left[\frac{1}{2}\frac{t^2(v+v^{-1}-u-u^{-1})(t+t^{-1})}{(1-tu)(1-tu^{-1})(1+tv)(1+tv^{-1})}\frac{q}{1-q}\right] . \tag{6.5}$$

where PE stands for the Plethystic exponential [23]. Taking $\beta \to 0$ limit of this expression, with the fugacities scaling as

$$t = e^{-\beta\epsilon_+}, \quad u = e^{-\beta\epsilon_-}, \quad v = e^{-\beta m}, \quad w_i = e^{-\beta z_i} , \tag{6.6}$$

we find

$$\left.\sum_{k=0}^{\infty} q^k \Omega^{\mathrm{Sp}(k)}_{\mathrm{ADHM,SO(0)}}\right|_{\beta\to 0} = \exp\left[\frac{1}{4}\frac{m^2-\epsilon_-^2}{\epsilon_+^2-\epsilon_-^2}\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1-q^n}\right] . \tag{6.7}$$

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On the other hand, our ADHM matrix integrals suggest

$$\sum_{k=0}^{\infty} q^k \mathcal{Z}^{\mathrm{Sp}(k)}_{\mathrm{ADHM,SO}(0)} = \exp\left[\frac{1}{8}\frac{m^2-\epsilon_-^2}{\epsilon_+^2-\epsilon_-^2}\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1-q^n}\right], \tag{6.8}$$

instead. It is clear that the $\beta \to 0$ limit of (6.5) does not match (6.8), as anticipated from our H-saddle story.

Similarly, the 1d twisted partition functions for $N_f = 1$ are also proposed and checked up to $k = 3$ by the authors of ref. [22] as

$$\sum_{k=0}^{\infty} q^k \Omega^{\mathrm{Sp}(k)}_{\mathrm{ADHM,SO}(2)} = \mathrm{PE}\left[\frac{1}{2}\frac{t^2(v+v^{-1}-u-u^{-1})(2v+2v^{-1}+3t+3t^{-1})}{(1-tu)(1-tu^{-1})(1+tv)(1+tv^{-1})}\frac{q}{1-q}\right] \tag{6.9}$$

with $\beta \to 0$ limit,

$$\left.\sum_{k=0}^{\infty} q^k \Omega^{\mathrm{Sp}(k)}_{\mathrm{ADHM,SO}(2)}\right|_{\beta\to 0} = \exp\left[\frac{5}{4}\frac{m^2-\epsilon_-^2}{\epsilon_+^2-\epsilon_-^2}\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1-q^n}\right]. \tag{6.10}$$

We find that the ADHM matrix integrals are consistent with

$$\sum_{k=0}^{\infty} q^k \mathcal{Z}^{\mathrm{Sp}(k)}_{\mathrm{ADHM,SO}(2)} = \exp\left[\frac{9}{8}\frac{m^2-\epsilon_-^2}{\epsilon_+^2-\epsilon_-^2}\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1-q^n}\right]. \tag{6.11}$$

so, again, there is a discrepancy between the two.

If this discrepancy lies entirely in the "Coulombic" part of the ADHM data, the continuity would be restored by taking a ratio between $N_f = 1$ and $N_f = 0$ generating functions. Indeed, from the comparison of the above four generating functions, it is pretty clear that this is the case with

$$\left[\sum q^k \mathcal{Z}^{\mathrm{Sp}(k)}_{\mathrm{ADHM,SO}(2)}\right]\left[\sum q^k \mathcal{Z}^{\mathrm{Sp}(k)}_{\mathrm{ADHM,SO}(0)}\right]^{-1} = \exp\left[\frac{m^2-\epsilon_-^2}{\epsilon_+^2-\epsilon_-^2}\sum_{n=1}^{\infty}\frac{1}{n}\frac{q^n}{1-q^n}\right]. \tag{6.12}$$

We have performed a similar check for $\mathrm{SO}(N \leq 5)$ instantons, again up to $k = 3$, confirming the proposed continuity convincingly.

We need to emphasize one important normalization issue that enters the comparison. Recall that for pure Yang-Mills cases, the matrix side must be multiplied by an overall factor $|\det(Q^G)|$, which corrects the problem that the 0d localization misses the pole locations shifted by Z_G. On the other hand, with ADHM data, there is no such obvious numerical factor. Instead, as we saw in the Sp(1) case, a Z_G-shifted missing residue is inequivalent to the one at origin, and should be considered a nontrivial H-saddle. This example suggests that the multiplicative factor for $\mathcal{Z}^G_{\mathrm{ADHM}}$ should be the one associated with the fundamental charges, say, $|\det(Q^G_F)| = 1$ for classical G's. It is with this normalization choice that the proposed continuity between 5d and 4d works.

An interesting exercise would be to study how these features manifest in the instanton partition functions for general 4d $\mathcal{N} = 2$ theories and for general 5d $\mathcal{N} = 1$ theories [24–26]. Computations for these objects have been carried out extensively in the past, yet some ambiguity seems to persist for the symplectic and the orthogonal cases. A more thorough investigation of 4d/5d instanton partition functions in view of our new findings will appear elsewhere.

Acknowledgments

We would like to thank Richard Eager for bringing our attention to ref. [20], and also Joonho Kim and Jaewon Song for useful discussions. P.Y. is grateful to Seung-Joo Lee, previous collaborations with whom motivated the main question of this note.

A Ω^G and $\mathcal{Z}^G$ for high rank classical groups

A general and algebraic expression for Ω^G with $\mathcal{N} = 4, 8$ and any simple Lie group G is known in terms of the Weyl group W_G as follows [2],

$$
\begin{aligned}
\Omega^G_{\mathcal{N}=4} &= \frac{1}{|W_G|} \sum_w{}' \frac{1}{\det\left(\mathbf{y}^{-1} - \mathbf{y}\cdot w\right)}\,, \\
\Omega^G_{\mathcal{N}=8} &= \frac{1}{|W_G|} \sum_w{}' \frac{1}{\det\left(\mathbf{y}^{-1} - \mathbf{y}\cdot w\right)} \cdot \frac{\det\left(x^{F/2}\mathbf{y}^{-1} - x^{-F/2}\mathbf{y}\cdot w\right)}{\det\left(x^{F/2} - x^{-F/2}\cdot w\right)}\,,
\end{aligned}
\tag{A.1}
$$

with the common unrefined limit [9, 15, 16]

$$
\Omega^G = \frac{1}{|W_G|} \sum_w{}' \frac{1}{\det\left(1 - w\right)}\,, \tag{A.2}
$$

where the sum is restricted to the so-called elliptic Weyl elements, defined by $\det(1-w) \neq 0$.

The formula (A.2) has been further evaluated for classical groups as [13],

$$
\begin{aligned}
\Omega^{\mathrm{Sp}(r)} &= \Omega^{\mathrm{SO}(2r+1)} = \frac{1}{2^{2r} r!} \prod_{j=0}^{r-1} (4j+1)\,, \\
\Omega^{\mathrm{SO}(2r)} &= \frac{1}{2^{r-1} r!}\, 2^{-r-1} \left(\prod_{j=0}^{r-1} (4j+1) + \prod_{j=0}^{r-1} (4j-1) \right).
\end{aligned}
\tag{A.3}
$$

On the other hand, the same reference offered conjectural formulae for the matrix integral counterpart.

$$
\begin{aligned}
\mathcal{Z}^{\mathrm{Sp}(r)} &= \frac{1}{2^{3r-1} r!} \prod_{j=1}^{r-1} (8j+1)\,, \\
\mathcal{Z}^{\mathrm{SO}(2r+1)} &= \frac{1}{2^r r!} \sum_{l=1}^{r} 2^{r+1-3l} F_r^l b_l\,, \\
\mathcal{Z}^{\mathrm{SO}(2r)} &= \frac{2}{2^{r-1} r!} \sum_{l/2=1}^{[r/2]} 2^{r+1-3l} F_r^l b_l\,.
\end{aligned}
\tag{A.4}
$$

F_r^l is the absolute value of the Stirling number of the first kind, or equivalently

$$
\sum_{l=1}^{r} F_r^l s^l = s(s+1)(s+2)\cdots(s+r-1)\,, \tag{A.5}
$$

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while b_l is a sequence of integers defined by

$$b_{2k} = b_{2k-1} = (-1)^{k+1} 2^{k-1} \beta_k \tag{A.6}$$

where β_k is from the k-th expansion coefficient of $\sqrt{\cos x}$:

$$\sqrt{\cos x} = 1 - \sum_{k=0}^{\infty} \frac{\beta_k x^{2k}}{2^k (2k)!} \,. \tag{A.7}$$

For SO's, by the way, our expressions actually differ a little from those in ref. [13], in part due to various typos in the latter. The above expressions (A.3) and (A.4) are consistent with low rank numbers in table 1.

Now we are ready to check the identities (5.8)–(5.10) against for classical groups of general ranks. For SO groups, we have confirmed (5.9)–(5.10) numerically against (A.3) and (A.4), up to the rank 100, going well beyond table 1. For the symplectic cases, a general and elementary proof follows once we rewrite (5.8) as

$$\Omega^{\mathrm{Sp}(K)} = \frac{1}{4} \sum_{r=0}^{K} \mathcal{Z}^{\mathrm{Sp}(r)} \cdot \mathcal{Z}^{\mathrm{Sp}(K-r)} \,, \tag{A.8}$$

with $\mathcal{Z}^{\mathrm{Sp}(0)} \equiv 2$ understood as a natural extrapolation from (A.4). Numerical limits of both $\mathcal{Z}$'s and Ω's are proportional to binomial coefficients as

$$\begin{aligned} \frac{\mathcal{Z}^{\mathrm{Sp}(r)}}{2} &= \frac{\Gamma(\frac{1}{8}+r)}{r!\Gamma(\frac{1}{8})} = (-1)^r \begin{pmatrix} -\frac{1}{8} \\ r \end{pmatrix} \,, \\ \Omega^{\mathrm{Sp}(r)} &= \frac{\Gamma(\frac{1}{4}+r)}{r!\Gamma(\frac{1}{4})} = (-1)^r \begin{pmatrix} -\frac{1}{4} \\ r \end{pmatrix} \,, \end{aligned} \tag{A.9}$$

and, as such, the identity (A.8) follows immediately from the trivial equality,

$$(1+s)^{-1/8} \cdot (1+s)^{-1/8} = (1+s)^{-1/4} \,,$$

i.e., from the binomial expansions thereof.

Recall that we have derived the identities (5.8)–(5.10) rigorously from the path integral while the general formulae (A.1), also confirmed by explicit localization computations up to rank 7, have strong physical motivations [2, 9, 16]. Given these, it is perhaps more sensible to view these checks as a confirmation of the numerics in (A.4).

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