Diffusion and butterfly velocity at finite density

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ABSTRACT: We study diffusion and butterfly velocity (v_B) in two holographic models, linear axion and axion-dilaton model, with a momentum relaxation parameter (β) at finite density or chemical potential (μ). Axion-dilaton model is particularly interesting since it shows linear- T -resistivity, which may have something to do with the universal bound of diffusion. At finite density, there are two diffusion constants $D_{\pm}$ describing the coupled diffusion of charge and energy. By computing $D_{\pm}$ exactly, we find that in the incoherent regime ($\beta/T \gg 1$, $\beta/\mu \gg 1$) D_+ is identified with the charge diffusion constant (D_c) and D_- is identified with the energy diffusion constant (D_e). In the coherent regime, at very small density, $D_{\pm}$ are ‘maximally’ mixed in the sense that $D_+(D_-)$ is identified with $D_e(D_c)$, which is opposite to the case in the incoherent regime. In the incoherent regime $D_e \sim C_- \hbar v_B^2 / k_B T$ where $C_- = 1/2$ or 1 so it is universal independently of β and μ . However, $D_c \sim C_+ \hbar v_B^2 / k_B T$ where $C_+ = 1$ or $\beta^2 / 16\pi^2 T^2$ so, in general, C_+ may not saturate to the lower bound in the incoherent regime, which suggests that the characteristic velocity for charge diffusion may not be the butterfly velocity. We find that the finite density does not affect the diffusion property at zero density in the incoherent regime.

KEYWORDS: Holography and condensed matter physics (AdS/CMT), Gauge-gravity correspondence

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1 Introduction

Strongly correlated systems display exotic properties compared to weakly correlated ones. More interestingly, some of exotic properties appear in various materials with a remarkable degree of universality [1]. For example, in diverse strange metals (cuprates, pnictides, heavy fermions etc.), resistivity (ρ) is linear in temperature (T)

$$\rho \sim T, \quad (1.1)$$

unlike ordinary metals where it is quadratic in temperature as explained by the Fermi liquid theory. In many high temperature superconductors, an empirical universal property so called Homes' law¹ have been observed [2, 3]. While such interesting phenomena in strongly correlated systems are not easy to analyze theoretically, gauge/gravity duality or holographic methods [1, 4, 5] have been providing new and effective ways to study them. For linear- T -resistivity, there have been a lot of works and we refer to [1]. For the Homes' law see [6–9]. In this paper, we investigate another universal property, universal bounds of charge and energy diffusion constants of strongly correlated systems from perspective of holographic methods.

¹It is a universal relation between critical temperature (T_c), DC conductivity near the critical temperature ($\sigma_{DC}(T_c)$), and superfluid density at zero temperature $\rho_s(T=0)$:

$$\rho_s(T=0) = C\sigma_{DC}(T_c)T_c, \quad (1.2)$$

where C is a material independent universal number.

It is interesting that these universal diffusion bounds may have something to do with aforementioned linear- T -resistivity via a fundamental universal relaxation timescale (τ_P) so called ‘Planckian’ time scale [3, 10]

$$\tau_P \sim \frac{\hbar}{k_B T}. \quad (1.3)$$

A basic idea is as follows. Charge diffusion constant (D_c) is proposed to be related to a relaxation time (τ_P) [11]:

$$D_c \gtrsim v^2 \tau_P \gtrsim v^2 \frac{\hbar}{k_B T}, \quad (1.4)$$

with a characteristic velocity scale. Transport in *incoherent* metals, where momentum is relaxed quickly, may be governed by diffusive physics of charge and energy rather than momentum. To make a connection between transport and diffusion one can use the Einstein relation $\sigma = D_c \chi$, where χ is the charge susceptibility. The Einstein relation with (1.4) yields

$$\sigma \gtrsim \chi v^2 \frac{\hbar}{k_B T}. \quad (1.5)$$

If D_c saturates to the bound and χv^2 is temperature independent, $\sigma \sim 1/T$ hence linear- T -resistivity is explained. Even though the universality of diffusion bounds itself is important it becomes more appealing because of the relation to another universality, the linear- T -resistivity.

At finite density, there are two diffusion constants $D_{\pm}$ describing the coupled diffusion of charge and energy so the Einstein relation need to be generalized [11] as follows.²

$$D_+ D_- = \frac{\sigma}{\chi} \frac{\kappa}{c_\rho}, \quad (1.6)$$

$$D_+ + D_- = \frac{\sigma}{\chi} + \frac{\kappa}{c_\rho} + \frac{T(\zeta\sigma - \chi\alpha)^2}{c_\rho \chi^2 \sigma}, \quad (1.7)$$

where σ, α, κ are the electric, thermoelectric and thermal conductivity respectively. χ is the compressibility, c_ρ is the specific heat at fixed charge density and ζ is the thermoelectric susceptibility. If the charge density is zero, since $\alpha = \zeta = 0$, $D_{\pm}$ are decoupled and D_+ and D_- can be identified with the charge diffusion constant (D_c) and the energy diffusion constant (D_e) respectively.

In [11], it was proposed the diffusion constants are bounded as

$$D_{\pm} \gtrsim v^2 \frac{\hbar}{k_B T}, \quad (1.8)$$

where v is an unknown characteristic velocity. It was conjectured by noticing that the KSS(Kovtun, Son and Starinets) bound of shear viscosity per entropy ratio (η/s) at zero chemical potential in a relativistic system may be expressed as $D \gtrsim c^2 \frac{\hbar}{k_B T}$, where c is the speed of light and D is the momentum diffusion constant. Eq. (1.8) suggests that diffusion is governed by the Planckian time scale (1.3) independently of the charge density

²The conductivities may be diagonalized as in [12].

and the mechanism of momentum relaxation. Recently, this time scale has been observed in the scattering rates of materials showing a linear T resistivity [13] and in the thermal diffusivity [14].

To investigate this conjectured bounds further, we first need to identify what the characteristic velocity (v) is in (1.8). An interesting candidate is the butterfly velocity (v_B), the speed at which the chaos spatially propagates through the system [15]. It implies that there is some connection between transport properties at strong coupling and quantum chaos.³ This idea was first tested at zero density in a class of holographic model with an scaling infrared geometry in [15, 19], where concrete examples supporting the bound (1.8) with the butterfly velocity were provided. More evidence for the energy diffusion bound (D_e/v_B^2) was shown in holographic models that flow to $\text{AdS}_2 \times R^d$ fixed points in the infrared in [20] and in the Sachdev-Ye-Kitaev (SYK) models [21–23].⁴ However, it was shown that charge diffusion (D_c/v_B^2) may not have a universal lower bound in striped holographic matter [24] and in the SYK model [22]. When the higher derivative correction is added the energy diffusion (D_e/v_B^2) still can have a lower bound while the charge diffusion (D_c/v_B^2) may vanish depending on the higher derivative couplings [25]. However, recently, it was shown that the energy diffusion (D_e/v_B^2) also may not have a universal lower bound in an inhomogeneous SYK model [26].

In this paper, our goal is to study the bounds (1.8) at *finite density*. Most studies so far have focused on the case i) at zero density or ii) D_c and D_e at finite density instead of $D_{\pm}$ by ignoring the mixing term, the third term in (1.7), and/or by taking small temperature limit. Unlike the previous studies, we first study $D_{\pm}$ at finite density *without any approximation* and deduce the property of D_c and D_e in the incoherent regime. We consider two holographic models: the linear axion model [27] and one of the axion-dilaton models [28, 29] based on the Gubser-Rocha model [30]. We choose these models because both allow the analytic solutions and they are related in the sense that at zero density the axion-dilaton model undergoes the phase transition to the linear axion model if the momentum relaxation is weak. The axion-dilaton model is particularly interesting because this model exhibits linear- T -resistivity: it will be interesting to see if there is any relation between the universal bound of the charge diffusion and linear- T -resistivity such as (1.5).

This paper is organized as follows. In section 2, we summarize the methods and formulas we will use to compute the diffusion constants and the butterfly velocity. In section 3, we study the linear axion model at finite density, focusing on i) the relation between $D_{\pm}$ and $D_c(D_e)$ and ii) the effect of finite density to diffusion and the butterfly velocity. In section 4, we first analyze the phase structure of a axion-dilaton theory based on the Gubser-Rocha model. We find that there are two branches of classical solutions. After figuring out the ground state, we study the diffusion constants and butterfly velocity both at zero and finite density. In section 5, we conclude.

³While this connection between transport properties and chaos was first proposed in the holographic models, it has been also observed in condensed matter systems [14, 16–18].

⁴In [23], it was also shown that the diffusion constant may vanish across the phase transition, implying a dynamical transition to an many-body localization phase.

2 Methods

In this section, we briefly summarize the method and formulas we will use in our computation in section 3 and 4. Our goal is to study the universal lower bound (1.8), which can be written as

$$D_{\pm} = \frac{C_{\pm}}{2\pi} v_B^2 \frac{1}{T}, \quad (2.1)$$

where $C_{\pm}$ is expected to be universal and 2π is introduced for later convenience. From here, we set $\hbar = k_B = 1$. In other words, our main objects are

$$C_{\pm} = \frac{2\pi T D_{\pm}}{v_B^2}, \quad 2\pi T D_{\pm} \text{ or } v_B^2, \quad (2.2)$$

which will be computed and displayed in section 3 and 4.

In this paper, we are mainly interested in the incoherent regime, $\beta/T \gg 1$ and $\beta/\mu \gg 1$, where β is the strength of momentum relaxation, T is temperature and μ is chemical potential, because in this regime momentum is relaxed quickly and we expect the transport is governed by diffusion of charge and energy [11].

2.1 Diffusion constants

From (1.6) and (1.7) two diffusion constants are computed as

$$D_{\pm} = \frac{c_2 \pm \sqrt{c_2^2 - 4c_1}}{2}, \quad (2.3)$$

where

$$c_1 \equiv \frac{\sigma}{\chi} \frac{\kappa}{c_{\rho}}, \quad c_2 \equiv \frac{\sigma}{\chi} + \frac{\kappa}{c_{\rho}} + \mathcal{M}, \quad \mathcal{M} \equiv \frac{T(\zeta\sigma - \chi\alpha)^2}{c_{\rho}\chi^2\sigma}. \quad (2.4)$$

Six variables defining c_1 and c_2 belong to two classes: thermodynamic susceptibilities (χ, ζ, c_{ρ}) and conductivities (σ, κ, α).

First, thermodynamic susceptibilities are defined as

$$\begin{aligned} \chi &\equiv -\frac{\partial^2 G}{\partial \mu^2} = \left(\frac{\partial \rho}{\partial \mu} \right)_T, \\ \zeta &\equiv -\frac{\partial^2 G}{\partial T \partial \mu} = \left(\frac{\partial \rho}{\partial T} \right)_{\mu} = \left(\frac{\partial s}{\partial \mu} \right)_T, \\ c_{\rho} &\equiv c_{\mu} - \frac{\zeta^2 T}{\chi}, \quad c_{\mu} \equiv -T \frac{\partial^2 G}{\partial T^2} = T \left(\frac{\partial s}{\partial T} \right), \end{aligned} \quad (2.5)$$

with the thermodynamic potential density at fixed chemical potential and temperature:

$$G = \epsilon - sT - \mu\rho. \quad (2.6)$$

Once the thermodynamic potential is computed by the gravity on-shell action according to the AdS/CFT duality, the susceptibilities are computed by (2.5) following standard thermodynamics.

Next, the DC conductivities of a class of holographic models with momentum relaxation may be expressed in terms of black hole horizon data [31]. For the action

$$S = \int d^4x \sqrt{-g} \left[R - \frac{Z(\phi)}{4} F^2 - \frac{1}{2} (\partial\phi)^2 - V(\phi) - \frac{1}{2} \sum_{I=1}^2 (\Phi_I(\phi) \partial\psi_I)^2 \right], \quad (2.7)$$

with the ansatz

$$\begin{aligned} ds^2 &= -U dt^2 + \frac{dr^2}{U} + V_1 dx^2 + V_2 dy^2, \\ A &= a dt, \quad \psi_I = \beta_I \delta_{Ii} x^i, \end{aligned} \quad (2.8)$$

the DC electric (σ), thermal ($\bar{\kappa}$), and thermoelectric (α) conductivities along the x -direction can be computed at the black hole horizon (r_h) as follows.⁵

$$\begin{aligned} \sigma &= \left[\frac{Z(\phi)s}{4\pi V_1} + \frac{4\pi\rho^2}{\beta_1^2 \Phi_1(\phi)s} \right]_{r=r_h}, \\ \bar{\kappa} &= \left[\frac{4\pi s T}{\beta_1^2 \Phi_1(\phi)} \right]_{r=r_h}, \\ \alpha &= \left[\frac{4\pi\rho}{\beta_1^2 \Phi_1(\phi)} \right]_{r=r_h}, \end{aligned} \quad (2.9)$$

The thermal conductivity with open circuit boundary conditions, which is the usual thermal conductivity, is

$$\kappa = \bar{\kappa} - \frac{\alpha^2 T}{\sigma}. \quad (2.10)$$

2.2 Butterfly velocity

In this section we briefly review on the butterfly velocity in strongly correlated systems and its holographic dual. For more details, we refer to [15, 19, 37–43].

The butterfly effect as chaotic behaviour refers to the exponential growth of a small perturbation to a quantum system. It can be diagnosed by certain out-of-time-order (four-point) correlation function (OTOC) of two generic Hermitian operators $W(t, \vec{x})$ and $V(0, 0)$, or the following average of the commutator squared:

$$C(t, \vec{x}) \equiv -\langle [W(t, \vec{x}), V(0, 0)]^2 \rangle_\beta, \quad (2.11)$$

where $\beta = 1/T$ (the inverse temperature) and $\langle \dots \rangle_\beta$ denotes thermal average. The function $C(t, \vec{x})$ quantifies the effect of a perturbation $V(0, 0)$ on $W(t, \vec{x})$ and characterizes the strength of the butterfly effect at $\vec{x}$ at time t induced by a perturbation at $t = \vec{x} = 0$. In general, $C(t, \vec{x})$ takes the following form

$$C(t, \vec{x}) = e^{\lambda_L(t - t_* - \frac{|\vec{x}|}{v_B})} + \dots. \quad (2.12)$$

⁵These DC formulas have been confirmed by computing the optical conductivities and taking the zero frequency limit [32–34]. See also [35, 36] which were the first papers developing the techniques to calculate the electric conductivity in terms of the black hole horizon data in massive gravity.

Here, λ_L is called a “Lyapunov” exponent following the classical chaos terminology. It measures the rate at which the system becomes scrambled and lose memory of its initial state. t_* is the scrambling time at which $C(t, 0)$ becomes order one. For $\vec{x} \neq 0$ there is a spatial delay in scrambling characterized by v_B , so called the “butterfly velocity”. $C(t, \vec{x})$ grows to be order one at $t = t_* + \frac{|\vec{x}|}{v_B}$, which defines an effective light cone for chaos, a “butterfly effect cone”. Outside of the cone $C(t, \vec{x}) \ll 1$, even if the operators V and W are time-like separated with respect to the causal light cone $t = \vec{x}$. Inside the cone, $C(t, \vec{x})$ grows quickly. Therefore, the butterfly velocity v_B characterizes the speed at which the chaos spatially propagates through the system.

It has been shown that the Lyapunov exponent is bounded by the temperature

$$\lambda_L \leq 2\pi k_B T / \hbar = 2\pi / \tau_P, \quad (2.13)$$

and saturates to the bound $2\pi / \tau_P$ for thermal systems that have a dual holographic black hole description of which near horizon geometry is described by Einstein gravity. Notice that the Planckian time scale appears as a time scale of the growth of chaos in time. Therefore, it is quite appealing to use the butterfly velocity as a characteristic velocity in (1.8). With this identification, the bound of diffusion constant can be written only in terms of the characteristic parameters in quantum chaos

$$D_{\pm} \gtrsim v_B^2 \tau_P \gtrsim 2\pi v_B^2 / \lambda_L. \quad (2.14)$$

Like the Lyapunov exponent, the butterfly velocity also can be computed holographically by considering shockwave geometries. For systems that have dual holographic model with an infrared geometry

$$ds_{d+2}^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} + V(r)d\vec{x}_d^2. \quad (2.15)$$

The Lyapunov exponent and the butterfly velocity are given by [15, 19]

$$\lambda_L = \frac{2\pi}{\tau_P}, \quad v_B^2 = \frac{2\pi T}{V'(r_h)}, \quad (2.16)$$

where r_h is the location of horizon. While the Lyapunov exponent is universal, saturating the bound (2.13), the butterfly velocity is not.

3 Linear axion model

We first consider a simple holographic model of momentum relaxation, the four dimensional linear scalar model [27]. This is the Einstein-Maxwell action coupled to massless scalars:

$$S = \int d^4x \sqrt{-g} \left[R + \frac{6}{L^2} - \frac{1}{4}F^2 - \frac{1}{2} \sum_{I=1,2} (\partial\psi_I)^2 \right], \quad (3.1)$$

where we have chosen a unit such that the gravitational constant $16\pi G$ is equal to 1. The second term is nothing but -2Λ with the negative cosmological constant, $\Lambda = -\frac{3}{L^2}$.

$F = dA$ is the field strength for a U(1) gauge field A . Because we want to consider a system at finite density we assume $A = A_t(r)dt$ as a background solution. In the last term, two massless scalar fields of the form $\psi_I = \beta_{Ii}x^i = \beta\delta_{Ii}x^i$ are introduced to break the translational symmetry in an isotropic way in the x - y space, where a constant β is interpreted as the strength of momentum relaxation. This term induces momentum relaxation effect so makes conductivity finite [27]. An advantage of this ansatz for massless scalars is that we can still have a homogeneous metric background solution, even though the translational symmetry is broken. Under these assumption for A and ψ_I , a classical solution of the action (3.1) is

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{dr^2}{f(r)} + \frac{r^2}{L^2}(dx^2 + dy^2), \\ f(r) &= \frac{r^2}{L^2} \left(1 + \frac{L^2\mu^2 r_h^2}{4r^4} - \frac{L^4\beta^2}{2r^2} - \frac{r_h^3}{r^3} \left(1 + \frac{L^2\mu^2}{4r_h^2} - \frac{L^4\beta^2}{2r_h^2} \right) \right), \\ A &= \mu \left(1 - \frac{r_h}{r} \right) dt, \quad \psi_I = \beta\delta_{Ii}x^i, \end{aligned} \quad (3.2)$$

where $f(r)$ is the emblackening factor and r_h denotes the black hole horizon. From here we set $L = 1$.

From the solution (3.2), the thermodynamic quantities read as follows. μ is interpreted as the chemical potential of the boundary field theory, $\mu = \lim_{r \rightarrow \infty} A_t$. The temperature is

$$T = \frac{f'(r_h)}{4\pi} = \frac{1}{4\pi} \left(3r_h - \frac{\mu^2 + 2\beta^2}{4r_h} \right), \quad (3.3)$$

so r_h is expressed in terms of T, μ and β :

$$r_h = \frac{1}{6} \left(4\pi T + \sqrt{6\beta^2 + 16\pi^2 T^2 + 3\mu^2} \right). \quad (3.4)$$

The entropy density is

$$s = 4\pi r_h^2 = \frac{\pi}{9} \left(4\pi T + \sqrt{6\beta^2 + 16\pi^2 T^2 + 3\mu^2} \right)^2, \quad (3.5)$$

by the Bekenstein-Hawking formula and the expectation value of the charge density reads

$$\rho = \mu r_h = \frac{\mu}{6} \left(4\pi T + \sqrt{6\beta^2 + 16\pi^2 T^2 + 3\mu^2} \right). \quad (3.6)$$

The butterfly velocity (2.16) is

$$v_B^2 = \frac{\pi T}{r_h} = \frac{6\pi T}{4\pi T + \sqrt{6\beta^2 + 16\pi^2 T^2 + 3\mu^2}}, \quad (3.7)$$

from the metric (3.2) and (3.4).

The thermodynamic susceptibilities (2.5) are computed by using (3.5) and (3.6) as follows. The compressibility and the thermoelectric susceptibility are

$$\begin{aligned} \chi &= \left(\frac{\partial \rho}{\partial \mu} \right)_T = \frac{1}{6} \left(4\pi T + \frac{6\beta^2 + 16\pi^2 T^2 + 6\mu^2}{\sqrt{6\beta^2 + 16\pi^2 T^2 + 3\mu^2}} \right), \\ \zeta &= \left(\frac{\partial \rho}{\partial T} \right)_\mu = \left(\frac{\partial s}{\partial \mu} \right)_T = \frac{2\pi\mu}{3} \left(1 + \frac{4\pi T}{\sqrt{6\beta^2 + 16\pi^2 T^2 + 3\mu^2}} \right). \end{aligned} \quad (3.8)$$

The specific heat at fixed chemical potential or fixed charge density are

$$c_\mu = T \left(\frac{\partial s}{\partial T} \right) = \frac{8\pi^2 T \left(4\pi T + \sqrt{6\beta^2 + 16\pi^2 T^2 + 3\mu^2} \right)^2}{9\sqrt{6\beta^2 + 16\pi^2 T^2 + 3\mu^2}}, \quad (3.9)$$

$$c_\rho = c_\mu - \frac{\zeta^2 T}{\chi} = \frac{8\pi^2 T \left(4\pi T + \sqrt{6\beta^2 + 16\pi^2 T^2 + 3\mu^2} \right)^3}{9(6\beta^2 + 6\mu^2 + 4\pi T(4\pi T + \sqrt{6\beta^2 + 16\pi^2 T^2 + 3\mu^2}))}.$$

To compute the conductivity we use the general formula (2.9). In our model (3.1)

$$\phi = 0, \quad \Phi_I = 1, \quad Z = 1, \quad V = -6, \quad \beta_I = \beta, \quad (3.10)$$

so the electrical, thermal, thermoelectric conductivities are

$$\sigma = 1 + \frac{\mu^2}{\beta^2},$$

$$\bar{\kappa} = \frac{4\pi s T}{\beta^2} = \frac{4\pi^2 T}{9\beta^2} \left(4\pi T + \sqrt{6\beta^2 + 16\pi^2 T^2 + 3\mu^2} \right)^2, \quad (3.11)$$

$$\alpha = \frac{4\pi\rho}{\beta^2} = \frac{2\pi\mu}{3\beta^2} \left(4\pi T + \sqrt{6\beta^2 + 16\pi^2 T^2 + 3\mu^2} \right),$$

and

$$\kappa = \bar{\kappa} - \frac{\alpha^2 T}{\sigma} = \frac{4\pi^2 T}{9(\beta^2 + \mu^2)} \left(4\pi T + \sqrt{6\beta^2 + 16\pi^2 T^2 + 3\mu^2} \right)^2. \quad (3.12)$$

Now we are ready to compute diffusion constants $D_\pm$, (2.3) and (2.4), which we rewrite here for convenience,

$$D_\pm = \frac{c_2 \pm \sqrt{c_2^2 - 4c_1}}{2}, \quad (3.13)$$

where

$$c_1 = \frac{\sigma}{\chi} \frac{\kappa}{c_\rho}, \quad c_2 = \frac{\sigma}{\chi} + \frac{\kappa}{c_\rho} + \mathcal{M}, \quad \mathcal{M} = \frac{T(\zeta\sigma - \chi\alpha)^2}{c_\rho \chi^2 \sigma}. \quad (3.14)$$

The analytic formulas of $D_\pm$ can be obtained by plugging (3.8), (3.9), (3.11), and (3.12) into (3.14). Because the final expressions are complicated and not very illuminating we show their plots in figure 1(a). The blue curve displays $2\pi T D_+$ and the red curve displays $2\pi D_-$. The green curve means v_B^2 (3.7). As β/T increases both $2\pi T D_\pm$ and v_B^2 go to zero, but all of them behave as $1/(\beta/T)$. Thus, $2\pi T D_\pm/v_B^2$ saturate the finite lower bound as shown in figure 1(b).

These bounds in the incoherent regime ($\beta/T \gg 1$ and $\beta/\mu \gg 1$) can be read also from the analytic expression of $D_\pm$ and v_B^2 at large β :

$$D_+ = \frac{\sqrt{6}}{\beta} - \frac{4\pi T}{\beta^2} + \frac{3\sqrt{6}\mu^2 + 16\pi^2\sqrt{6}T^2}{12\beta^3} \dots, \quad (3.15)$$

$$D_- = \frac{\sqrt{\frac{3}{2}}}{\beta} + \frac{16\sqrt{6}\pi^2 T^2 - 3\sqrt{6}\mu^2}{24\beta^3} \dots, \quad (3.16)$$

$$v_B^2 = \sqrt{6}\pi \frac{T}{\beta} - \frac{4\pi^2 T^2}{\beta^2} + \frac{16\sqrt{6}\pi^3 T^3 - 3\sqrt{6}\pi\mu^2 T}{12\beta^3}, \quad (3.17)$$

Figure 1. Diffusion constants of the linear axion model at finite density. $\mu/T = 0.1, 1, 5$ from left to right. The blue curve is for D_+ and the red curve is for D_- . The green curve displays v_B^2 . $2\pi T D_{\pm}/v_B^2$ saturates the universal value in the incoherent regime: $\beta/T \gg 1$ and $\beta/\mu \gg 1$.

which yield

$$C_+ = \frac{2\pi T D_+}{v_B^2} = 2 + \frac{\mu^2}{\beta^2} + \frac{2\pi\sqrt{\frac{2}{3}}\mu^2 T}{\beta^3} + \dots, \quad (3.18)$$

$$C_- = \frac{2\pi T D_-}{v_B^2} = 1 + \frac{2\pi\sqrt{\frac{2}{3}}T}{\beta} + \frac{8\pi^2 T^2}{3\beta^2} + \frac{16\sqrt{6}\pi^3 T^3 - 9\sqrt{6}\pi\mu^2 T}{18\beta^3} + \dots. \quad (3.19)$$

We find that $C_{\pm}$ saturate the bounds $C_+ = 2$ and $C_- = 1$ in the incoherent regime. Indeed, in this regime, D_+ and D_- can be identified with D_c and D_e respectively because the mixing term $\mathcal{M}$ in (3.14) vanishes as explained in the following paragraph. Thus the bounds $C_+ = 2$ and $C_- = 1$ at zero density [19] still hold at finite density in the incoherent regime.

To see the effect of the mixing term $\mathcal{M}$ in (3.14), we may drop it in (3.14) and simply identify $D_c \equiv \sigma/\chi$ and $D_e \equiv \kappa/c_\rho$. These are shown as the triangles in figure 2, where the solid curves in figure 1 are displayed together for comparison. For $\beta/4\pi T \lesssim 1$ the mixing term is important and for $\beta/4\pi T \gtrsim 1$ it is negligible. For small $\mu/T = 0.1$, this mixing effect is ‘maximal’ in the sense D_- is identified with D_c and D_+ is identified with D_e , which is opposite to the case in the incoherent regime. However, even for small β/T , the effect of the mixing term decreases as μ/T increases. The effect of μ/T also can be checked by large μ/T expansion of $T\mathcal{M}$:

$$T\mathcal{M} = \frac{4\pi^2 T^3}{\sqrt{3}\mu\beta^2} + \dots, \quad (3.20)$$

Figure 2. Diffusion constants of the linear axion model with/without a mixing term. $\mu/T = 0.1, 1, 5$ from left to right. The blue curve is for D_c and the red curve is for D_e . The triangles display the results without the mixing term $\mathcal{M}$ in (3.14). For comparison, we also display the results with the mixing term, the solid curves in figure 1.

which implies that the mixing term may be small if μ/T is large even for small β/T . In conclusion, in the incoherent regime, the mixing term, $\mathcal{M}$ in (3.14), between the charge and energy diffusions are negligible so D_+ and D_- may be identified with D_c and D_e respectively. (Because $\mu/T \leq 5$ in figure 2, $\beta/4\pi T \sim 5$ is the incoherent regime.)

For $C_{\pm}$ to be universal it is important to consider the incoherent regime ($\beta/T \gg 1$ and $\beta/\mu \gg 1$). To see it more clearly let us consider a different case $\mu/T \gg 1$ and $\mu/\beta \gg 1$, which yield the following expansions:

$$D_+ = \frac{\sqrt{3}\mu}{\beta^2} - \frac{2\pi T}{\beta^2}, \dots, \quad D_- = \frac{\sqrt{3}}{\mu} - \frac{2\pi T}{\mu^2} + \dots, \quad v_B^2 = \frac{2\sqrt{3}\pi T}{\mu} - \frac{8\pi^2 T^2}{\mu^2} + \dots, \quad (3.21)$$

and

$$C_+ = \frac{2\pi T D_+}{v_B^2} = \frac{\mu^2}{\beta^2} + \frac{2\pi\mu T}{\sqrt{3}\beta^2}, + \dots, \quad (3.22)$$

$$C_- = \frac{2\pi T D_-}{v_B^2} = 1 + \frac{2\pi T}{\sqrt{3}\mu} + \dots. \quad (3.23)$$

Thus C_+ is not universal while C_- is.

Finally, let us compare our results with [20] where *low temperature limit* was considered in a class of holographic models that flow to $AdS_2 \times R^d$ fixed points in the infrared. Our model belongs to that class and serves as a concrete example. Low temperature expansion

of our solution is as follows.

$$D_+ = \frac{\sqrt{3}\sqrt{2\beta^2 + \mu^2}}{\beta^2} - \frac{2\pi(2\beta^2 + \mu^2)}{\beta^2(\beta^2 + \mu^2)}T + \dots, \quad (3.24)$$

$$D_- = \frac{\sqrt{3}}{\sqrt{2\beta^2 + \mu^2}} - \frac{2\pi\mu^2}{2\beta^4 + 3\beta^2\mu^2 + \mu^4}T + \dots, \quad (3.25)$$

$$v_B^2 = \frac{2\sqrt{3}\pi}{\sqrt{\mu^2 + 2\beta^2}}T + \dots, \quad (3.26)$$

where $\sqrt{2\beta^2 + \mu^2}/\sqrt{3}$ corresponds to c_h^0 in [20]. The mixing term between the charge and energy diffusion is expanded as

$$\mathcal{M} = \frac{\mu^2}{\beta^2} \frac{4\pi^2 \sqrt{2\beta^2 + \mu^2}}{\sqrt{3}(\beta^2 + \mu^2)^2} T^2 + \dots, \quad (3.27)$$

which can be ignored at low temperature as argued in [20]. Thus we may identify $D_+ = D_c$ and $D_- = D_e$ and

$$C_+ = \frac{2\pi T D_c}{v_B^2} = \left(2 + \frac{\mu^2}{\beta^2}\right) + \frac{\mu^2}{\beta^2} \frac{2\pi \sqrt{2\beta^2 + \mu^2}}{\sqrt{3}(\beta^2 + \mu^2)} T + \dots, \quad (3.28)$$

$$C_- = \frac{2\pi T D_e}{v_B^2} = 1 + \frac{2\pi \sqrt{2\beta^2 + \mu^2}}{\sqrt{3}(\beta^2 + \mu^2)} T + \dots. \quad (3.29)$$

The thermal diffusion constant has a universal coefficient C_- irrespective of β and μ , which agree to [20]. The coefficient $C_+ = 2 + \mu^2/\beta^2$ for charge diffusion constant is not universal and a function of μ and β , but in the incoherent regime ($\mu/\beta \ll 1$) it becomes universal. i.e. $C_+ = 2$.

4 Axion-dilaton model

Next, let us consider a four dimensional Einstein-Maxwell-Axion-Dilaton theory. It is based on the Einstein-Maxwell-Dilaton model so called the Gubser-Rocha model [30]. To include momentum relaxation effect, a graviton mass term breaking translational invariance was added to the action and the linear- T -resistivity was observed in this model [44]. Because the universal bound of the diffusion constant may be related to the linear- T -resistivity it will be interesting to investigate the diffusion constants in this model. Here, we modify the original Gubser-Rocha model by adding the scalar fields ψ_I instead of a graviton mass term to induce the momentum relaxation.⁶ It makes possible a direct comparison with the linear axion model in the previous section.

The action is

$$S = \int d^4x \sqrt{-g} \left[R - \frac{1}{4} e^\phi F^2 - \frac{3}{2} (\partial\phi)^2 + \frac{6}{L^2} \cosh \phi - \frac{1}{2} \sum_{I=1}^2 (\partial\psi_I)^2 \right], \quad (4.1)$$

⁶The conductivities of this model was also studied in [45], focusing on the slow momentum relaxation in low frequency approximation.

which belongs to (2.7) and is reduced to (3.1) if $\phi = 0$. An analytic classical solution is

$$\begin{aligned}
 ds^2 &= \frac{r^2 g(r)}{L^2} (-h(r) dt^2 + dx^2 + dy^2) + \frac{L^2}{r^2 g(r) h(r)} dr^2, \\
 h(r) &= 1 - \frac{L^4 \beta^2}{2(Q+r)^2} - \frac{(Q+r_h)^3}{(Q+r)^3} \left(1 - \frac{L^4 \beta^2}{2(Q+r_h)^2} \right), \quad g(r) = \left(1 + \frac{Q}{r} \right)^{\frac{3}{2}}, \\
 A &= \sqrt{\frac{3Q(Q+r_h)}{L^2} \left(1 - \frac{L^4 \beta^2}{2(Q+r_h)^2} \right)} \left(1 - \frac{Q+r_h}{Q+r} \right) dt, \\
 \phi &= \frac{1}{3} \log(g(r)), \quad \psi_I = \beta \delta_{Ii} x^i.
 \end{aligned} \tag{4.2}$$

where $r_h > 0$ and $r_h > -Q$ to have a regular solution. Note that Q can be negative. From here we set $L = 1$.

4.1 Thermodynamics and transport coefficients

By the same method as in the linear-axion model, the temperature, the entropy density, chemical potential and charge density are

$$T = \frac{[r_h^2 g(r_h) h(r_h)]'}{4\pi} = \frac{\sqrt{r_h} (6(Q+r_h)^2 - \beta^2)}{8\pi(Q+r_h)^{3/2}}, \tag{4.3}$$

$$s = \frac{r_h^2 g(r_h)}{4G} = 4\pi \sqrt{r_h} (Q+r_h)^{3/2}, \tag{4.4}$$

$$\mu = \sqrt{3Q(Q+r_h) \left(1 - \frac{\beta^2}{2(Q+r_h)^2} \right)}, \tag{4.5}$$

$$\rho = (Q+r_h) \sqrt{3Q(Q+r_h) \left(1 - \frac{\beta^2}{2(Q+r_h)^2} \right)}. \tag{4.6}$$

Thanks to a scaling symmetry of the solution, it is convenient to define the following variables:

$$\tilde{t} = t r_h, \quad \tilde{x} = x r_h, \quad \tilde{y} = y r_h, \quad \tilde{r} = \frac{r}{r_h}, \quad \tilde{\beta} = \frac{\beta}{r_h}, \quad \tilde{Q} = \frac{Q}{r_h}, \tag{4.7}$$

Consequently, we may define the field theory variables as:

$$\tilde{T} = \frac{T}{r_h}, \quad \tilde{s} = \frac{s}{r_h^2}, \quad \tilde{\mu} = \frac{\mu}{r_h}, \quad \tilde{\rho} = \frac{\rho}{r_h^2}. \tag{4.8}$$

In other words, we may set $r_h = 1$ in (4.2) and replace all variables with the ‘tilde’ variables. In (4.2) $\tilde{Q}$ and $\tilde{\beta}$ look natural independent variables but in our analysis, from the perspective of the dual field theory, μ/T and β/T will be used as independent variables. By the relations (4.3) and (4.5), $\tilde{Q}$ and $\tilde{\beta}$ are expressed in terms of μ/T and β/T .

To find a possible range of $\tilde{Q}$, we impose physical condition $T \geq 0$, $\mu \geq 0$ and $\tilde{Q} > -1$. Without loss of generality, we can take $\beta \geq 0$. All these inequalities imply

$$\left\{ \begin{array}{ll} \frac{\beta}{4\pi T} = 0 & \Rightarrow \tilde{Q} \geq 0 \\ 0 < \frac{\beta}{4\pi T} \leq \frac{1}{\sqrt{2}} & \Rightarrow \left\{ \begin{array}{l} -1 < \tilde{Q} \leq -1 + 2 \left(\frac{\beta}{4\pi T} \right)^2 \\ 0 \leq \tilde{Q} \end{array} \right. \\ \frac{1}{\sqrt{2}} \leq \frac{\beta}{4\pi T} & \Rightarrow \left\{ \begin{array}{l} -1 < \tilde{Q} \leq 0 \\ -1 + 2 \left(\frac{\beta}{4\pi T} \right)^2 \leq \tilde{Q} \end{array} \right. \end{array} \right. . \quad (4.9)$$

Here we find that there may be two branches of the solutions: positive $\tilde{Q}$ and negative $\tilde{Q}$. They correspond to the green region in figure 3. The boundary of the green region is nothing but the condition for $\mu = 0$. Indeed, by using (4.3) and (4.5) we can obtain two solutions. For example, for $\mu/T = 0.1$, and 5, they are shown in figure 3: the red curve for positive $\tilde{Q}$ and the blue curve for negative $\tilde{Q}$. To determine which one corresponds to the ground state solution, we compare the grand potential density of two solutions. The grand potential density (G) is the on-shell action divided by the spatial volume and temperature [29]

$$\begin{aligned} G &= \epsilon - Ts - \mu\rho \\ &= -\frac{\beta^2 Q}{2} + \frac{2\rho^2}{3Q} - Ts - \mu\rho = -(Q + r_h)^3 - \frac{r_h \beta^2}{2} . \end{aligned} \quad (4.10)$$

As shown in the right panel of figure 3 the positive $\tilde{Q}$ solutions always correspond to the ground state, where $\delta G = G(\tilde{Q} > 0) - G(\tilde{Q} < 0)$. At zero density, there is no positive $\tilde{Q}$ for $\beta/4\pi T < 1/\sqrt{2}$. In this case $\tilde{Q} = 0$ is the ground state.

To compute the diffusion constants $D_{\pm}$ we first need to compute thermodynamic susceptibilities (2.5) as a function of T and μ . Because the charge density ρ (4.6) and entropy s (4.4) are functions of r_h and Q it will be convenient to express them as a function of T and μ . In principle, r_h and Q can be expressed in terms of T and μ from (4.3) and (4.5) but their analytic expressions are very complicated except for $\mu = 0$. Therefore we will not present their expressions here and show some plots in the following subsections.

From the general formula (2.9), the electrical, thermal, and thermoelectric conductivities read

$$\begin{aligned} \sigma &= e^{\phi(r_h)} + \frac{4\pi\rho^2}{\beta^2 s} = \sqrt{1 + \frac{Q}{r_h}} \left(1 + \frac{\mu^2}{\beta^2} \right) , \\ \bar{\kappa} &= \frac{4\pi s T}{\beta^2} , \quad \alpha = \frac{4\pi\rho}{\beta^2} . \end{aligned} \quad (4.11)$$

and from (2.16) the butterfly velocity is

$$v_B^2 = \frac{4\pi T}{Q + 4r_h} \sqrt{\frac{r_h}{Q + r_h}} . \quad (4.12)$$

Figure 3. The left panel: two values of $\tilde{Q}$ for given $\mu/T = 0.1$ (a) and $\mu/T = 5$ (b). The red curve is for positive $\tilde{Q}$ and the blue curve is for negative $\tilde{Q}$. The right panel: the difference of the grand potential ($\delta G = G(\tilde{Q} > 0) - G(\tilde{Q} < 0)$) is shown. The positive $\tilde{Q}$ solution is always thermodynamically preferred. The green region represent two branches in (4.9).

4.2 Zero density

Let us first consider a neutral case, i.e. $\mu = \rho = 0$. As shown in the previous subsection and (4.5), there are two solutions: $\tilde{Q} = 0$ and $\tilde{Q} = -1 + \frac{\beta}{\sqrt{2}}$.

For $\tilde{Q} = 0$, the dilaton field ϕ vanishes and the solution (4.2) is reduced to (3.2) with $\mu = 0$. This solution has been considered in [19]. In this case, the transport coefficients and susceptibilities are given as

$$\sigma = 1, \quad \kappa = \frac{4\pi^2 T}{9\beta^2} \left(4\pi T + \sqrt{6\beta^2 + 16\pi^2 T^2} \right)^2, \quad (4.13)$$

$$\chi = \frac{1}{6} \left(4\pi T + \sqrt{6\beta^2 + 16\pi^2 T^2} \right), \quad c_\rho = \frac{8\pi^2 T \left(4\pi T + \sqrt{6\beta^2 + 16\pi^2 T^2} \right)^2}{9\sqrt{6\beta^2 + 16\pi^2 T^2}}, \quad (4.14)$$

with $\alpha = 0$ and $\zeta = 0$. Thus the charge and energy diffusion constants are⁷

$$D_c = \frac{\sigma}{\chi} = \frac{6}{4\pi T + \sqrt{6\beta^2 + 16\pi^2 T^2}},$$

$$D_e = \frac{\kappa}{c_\rho} = \frac{\sqrt{6\beta^2 + 16\pi^2 T^2}}{2\beta^2}. \quad (4.15)$$

⁷The energy diffusion constant was also computed in [46].

Figure 4. Diffusion constants of the axion-dilaton model at zero density for $\tilde{Q} = 0$ (a), for $\tilde{Q} = -1 + \frac{\tilde{\beta}}{\sqrt{2}}$ (b), and for the ground state (c). The blue curve is for D_c and the red curve is for D_e . The green curve displays v_B^2 .

The butterfly velocity is

$$v_B^2 = \frac{6\pi T}{4\pi T + \sqrt{6\beta^2 + 16\pi^2 T^2}}, \quad (4.16)$$

so

$$\begin{aligned} \frac{2\pi T D_c}{v_B^2} &= 2, \\ \frac{2\pi T D_e}{v_B^2} &= 1 + \frac{2\pi T}{3\beta^2} \left(4\pi T + \sqrt{6\beta^2 + 16\pi^2 T^2} \right). \end{aligned} \quad (4.17)$$

We show the plots for the diffusion constants and the butterfly velocity in figure 4(a).

For $\tilde{Q} = -1 + \frac{\tilde{\beta}}{\sqrt{2}}$, there is a nontrivial ϕ and $r_h = 2^{5/2}\pi^2 T^2/\beta$. The transport coefficients and susceptibilities are⁸

$$\sigma = \frac{\beta}{2\sqrt{2}\pi T}, \quad \kappa = \frac{16\sqrt{2}\pi^3 T^2}{\beta}, \quad (4.18)$$

$$\chi = \beta/\sqrt{2}, \quad c_\rho = 4\sqrt{2}\pi^2 T\beta, \quad (4.19)$$

with $\alpha = 0$ and $\zeta = 0$. Thus the charge and energy diffusion constants are⁹

$$D_c = \frac{\sigma}{\chi} = \frac{1}{2\pi T}, \quad D_e = \frac{\kappa}{c_\rho} = \frac{4\pi T}{\beta^2}. \quad (4.20)$$

The butterfly velocity is

$$v_B^2 = \frac{16\pi^2 T^2}{24\pi^2 T^2 + \beta^2} \rightarrow \frac{16\pi^2 T^2}{\beta^2} \text{ for } \beta/T \gg 1, \quad (4.21)$$

so

$$\begin{aligned} \frac{2\pi T D_c}{v_B^2} &= \frac{\beta^2}{16\pi^2 T^2} + \frac{3}{2} \rightarrow \frac{\beta^2}{16\pi^2 T^2} \text{ for } \beta/T \gg 1, \\ \frac{2\pi T D_e}{v_B^2} &= \frac{1}{2} + \frac{12\pi^2 T^2}{\beta^2} \rightarrow \frac{1}{2} \text{ for } \beta/T \gg 1. \end{aligned} \quad (4.22)$$

We show the plots for the diffusion constants and the butterfly velocity in figure 4(b).

By comparing the grand potential of two cases, $\tilde{Q} = 0$ and $\tilde{Q} = -1 + \tilde{\beta}/\sqrt{2}$, we find that the $\tilde{Q} = 0$ case is the ground state for $\beta/(4\pi T) < 1/\sqrt{2}$ and the $\tilde{Q} = -1 + \tilde{\beta}/\sqrt{2}$ case is the ground state for $\beta/(4\pi T) > 1/\sqrt{2}$, similarly to figure 3(a). Taking this phase transition into account we show the the final results in figure 4(c).

Let us summarize the asymptotic behavior in the incoherent regime $\beta/T \gg 1$. For $\tilde{Q} = 0$, $D_c, D_e \sim T/\beta$ and $v_B^2 \sim T/\beta$, so both D_c/v_B^2 and D_e/v_B^2 saturate their bounds. For $\tilde{Q} = -1 + \tilde{\beta}/\sqrt{2}$, $D_c \sim 1/T$ and $D_e \sim T/\beta^2$ while $v_B^2 \sim T^2/\beta^2$, so only D_e/v_B^2 saturates its bound. However, in the incoherent regime, $\tilde{Q} = -1 + \tilde{\beta}/\sqrt{2}$ case is the stable state. Note that, in the incoherent regime, the charge diffusion ($2\pi T D_c$) saturates the bound but $2\pi T D_c/v_B^2$ diverges, while $2\pi T D_e/v_B^2$ saturates the bound.

4.3 Finite density

The horizon position r_h can be expressed in terms of T and μ by eliminating Q in (4.3) and (4.5). However, unlike the previous cases, if $\mu \neq 0$ the expression of r_h is long and complicated. Furthermore, to compute thermodynamic susceptibilities we need to perform differentiations where a few steps of implicit differentiations and chain rules are involved. As a result, all the final results are analytic but they are not so illuminating. Therefore, instead of presenting analytic expressions we show the plots in figure 5. At finite density

Figure 5. Diffusion constants of the axion-dilaton model at finite density. $\mu/T = 0.1, 1, 5$ from left to right. The blue curve is for D_+ and the red curve is for D_- . The green curve displays v_B^2 . $2\pi D_+$ and $2\pi TD_-/v_B^2$ saturates the universal value in the incoherent regime: $\beta/T \gg 1$ and $\beta/\mu \gg 1$.

($\mu \neq 0$) there are two branches, positive Q and negative Q , as explained in (4.9) and figure 3. Because the positive Q branch is the stable solution we showed it in figure 5.

Even though the exact analytic formulas are complicated their large β limit, which we are interested in for the universal bound, can be read off analytically. For large β in the positive Q branch, $r_h/\beta \rightarrow 0$ and $Q/\beta \rightarrow 1/\sqrt{2}$ from (4.3) and (4.5), which yield

$$\begin{aligned} \sigma &\sim \frac{\beta}{2\sqrt{2}\pi T}, & \kappa &\sim \frac{16\sqrt{2}\pi^3 T^2}{\beta}, & \alpha &\sim \frac{4\pi\mu}{\sqrt{2}\beta}, \\ \chi &\sim \beta/\sqrt{2}, & c_\rho &\sim 4\sqrt{2}\pi^2 T\beta, & \zeta &\sim \frac{64\sqrt{2}\pi^4 T^3 \mu}{\beta^3}. \end{aligned} \quad (4.23)$$

Therefore, the charge and energy diffusion constants are

$$D_+ \sim \frac{1}{2\pi T}, \quad D_- \sim \frac{4\pi T}{\beta^2}. \quad (4.24)$$

Together with the butterfly velocity at large $\beta \gg \mu$

$$v_B^2 \sim \frac{16\pi^2 T^2}{\beta^2}, \quad (4.25)$$

we have

$$\frac{2\pi TD_+}{v_B^2} \sim \frac{\beta^2}{16\pi^2 T^2}, \quad \frac{2\pi TD_e}{v_B^2} \sim \frac{1}{2} + \frac{12\pi^2 T^2}{\beta^2}. \quad (4.26)$$

⁸The electric conductivity was also computed in [47].

⁹A similar result was obtained in [48], where the graviton mass term was added instead of the linear axion term.

Figure 6. Diffusion constants of the axion-dilaton model with/without a mixing term. $\mu/T = 0.1, 1, 5$ from left to right. The blue curve is for D_c and the red curve is for D_e . The triangles display the results without the mixing term $\mathcal{M}$ in (3.14). For comparison, we also display the results with the mixing term, the solid curves in figure 5.

Thus we find that the bounds for $D_{\pm}$ at zero density (4.22) still hold at finite density. The correction by μ is at higher order and the results are robust in the incoherent regime.

Like the linear axion model, in the incoherent regime, the mixing term between the charge and energy diffusion, $\mathcal{M}$ in (3.14), is negligible so D_+ and D_- may be identified with D_c and D_e respectively. To see the effect of the mixing term, we show another plot for D_c and D_e in figure 6. Like figure 2, the triangles are the results without the mixing term, where we simply identify $D_c \equiv \sigma/\chi$ and $D_e \equiv \kappa/c_\rho$. Like the linear axion model, below $\beta/4\pi T \lesssim 1$ the mixing term is important and this mixing effect decreases as μ/T increases. Note that, for small $\mu/T = 0.1$, this mixing effect is ‘maximal’ in the sense D_- is identified with D_c and D_+ is identified with D_e .

In this model, the relation between the linear- T -resistivity and the universality of diffusion (1.5) is not realized. Even though $\sigma \sim 1/T$, C_+ is not bounded because $\chi v_B^2 \sim T^2/\beta^2$. It is possible that there is a more relevant velocity scale than the butterfly velocity for the charge diffusion. If that velocity scale is independent of temperature, (1.5) can be realized because χ is temperature-independent in this model.

5 Conclusion

In this paper, we have studied two diffusion constants and the butterfly velocity at *finite density* in two holographic models: linear axion model and axion-dilaton model. In both cases, the axion field is of the form $\psi_I = \beta \delta_{Ii} x^i$ and plays a role of momentum relaxation, where large β means large momentum relaxation. At zero density, the axion-dilaton model undergoes a phase transition to the linear axion model when momentum relaxation is weak.

	$D_+(\approx D_c)$	$D_-(\approx D_e)$	v_B	$C_+ \left(\frac{2\pi T D_+}{v_B^2} \right)$	$C_- \left(\frac{2\pi T D_-}{v_B^2} \right)$
Linear axion	$\frac{\sqrt{6}}{\beta}$	$\frac{\sqrt{3/2}}{\beta}$	$\sqrt[4]{6}\sqrt{\pi}\sqrt{\frac{T}{\beta}}$	2	1
Axion-dilaton	$\frac{1}{2\pi T}$	$\frac{4\pi T}{\beta^2}$	$\frac{4\pi T}{\beta}$	$\frac{\beta^2}{16\pi^2 T^2}$	$\frac{1}{2}$

Table 1. Summary of the results: diffusion constants for the linear axion model and axion-dilaton model in the incoherent regime $\beta/T \gg 1$ and $\beta/\mu \gg 1$.

At finite density, the axion-dilaton model has two branches of solutions so we have to choose a ground state by comparing their grand potentials.

There are two diffusion constants $D_{\pm}$ describing the coupled diffusion of charge and energy. We have showed the exact relation between $D_{\pm}$ and (D_c, D_e) in figure 2 and figure 6. In the incoherent regime, the mixing between charge and thermal diffusion is suppressed so D_+ and D_- can be identified with D_c and D_e respectively. However, in the coherent regime the effect of the mixing term becomes strong so D_c and D_e can not be decoupled and $D_{\pm}$ should be considered. In particular, at very small μ/T , this mixing effect is ‘maximal’ in the sense that D_- is identified with D_c and D_+ is identified with D_e , which is opposite to the case in the incoherent regime. In table 1 the diffusion constants and the butterfly velocity of two models at finite density in the incoherent regime are summarized.

The thermal diffusion constant at finite density in the incoherent regime can be written as

$$D_- \approx D_e \approx C_- \frac{v_B^2}{2\pi T}, \quad (5.1)$$

where $C_- = 1$ for the linear axion model and $C_- = 1/2$ for the axion-dilaton model. These agree to the values at zero density and C_- is universal independently of density. In [20], it was shown that the holographic models with IR geometry of $\text{AdS}_2 \times R^d$, which can be supported by finite density and/or axion field, give $1/2 < C_- \leq 1$ at low temperature limit. Because the IR geometry of the axion model is $\text{AdS}_2 \times R^d$, $C_- = 1$ can be anticipated from [20], where the low temperature limit of the linear axion model was considered. Here, we analysed the model at *any temperature* for both the coherent and the incoherent regime. The IR geometry of the axion-dilaton model in this paper is conformal to $\text{AdS}_2 \times R^d$ so the model does not belong to the class (the models with IR geometry of $\text{AdS}_2 \times R^d$) studied in [20]. However, our result $C_- = 1/2$ yields the lower bound of the range that the class allows i.e. $1/2 < C_- \leq 1$.

The charge diffusion constant is written as

$$D_+ \approx D_c \approx C_+ \frac{v_B^2}{2\pi T}, \quad (5.2)$$

where $C_+ = 2$ for the linear axion model and $C_+ = \beta^2/16\pi^2 T^2$ for the axion-dilaton model. These agree to the results at zero density. C_+ of the axion model is universal at any finite

density but C_+ in the axion-dilaton model increases as β/T increases. However, the charge diffusion constant of the axion-dilaton model itself saturates to the universal lower bound:

$$D_+ \approx D_c \approx \frac{1}{2\pi T}. \quad (5.3)$$

Thus, if there is more relevant velocity scale than the butterfly velocity and if it is temperature-independent, the relation between the universality of charge diffusion and linear- T -resistivity (1.5) may be realized in this model (χ is independent of temperature in this model). Indeed, it may be possible that there is another velocity scale for charge diffusion by the following observations.

To understand the universality of energy diffusion, it is important to note that two susceptibilities, χ and ζ , cannot be written in terms of the horizon data while c_ρ can be written only in terms of the horizon data as shown in [20]. To compute χ and ζ , the full bulk solution is needed. On the other hand, all conductivities and the butterfly velocity are written in terms of the horizon data as shown in (2.9) and (2.16). Because the energy diffusion constant *in the incoherent regime* is the ratio of κ to c_ρ , it can be written only in terms of horizon data, while the charge diffusion constant can not. Therefore, it is plausible that the energy diffusion constant in the incoherent regime may be universal thanks to a universal property of the black hole horizon. In the coherent regime, the mixing term $\mathcal{M}$ (2.4) is important and (ζ, χ) should be taken into account so the energy diffusion constant in the coherent regime will not be universal. By the same reason the charge diffusion constant may not be universal and this non-universality was also observed in [22, 24, 25] and it seems that chaos is only connected to energy diffusion [18, 22]. Therefore, it is an interesting future direction to search a velocity scale for charge diffusion. For energy diffusion, the bound based on the butterfly velocity seems to be more robust than charge diffusion. However, recently a counter example was found in [26] and it will be also interesting to understand the extent to which the energy diffusion bound with the butterfly velocity is robust [49].

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