C_T for higher derivative conformal fields and anomalies of $(1, 0)$ superconformal 6d theories

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ABSTRACT: In [8] we proposed the linear relations between the Weyl anomaly c_1, c_2, c_3 coefficients and the 4 coefficients in the chiral anomaly polynomial for $(1,0)$ superconformal 6d theories. These relations were determined up to one free parameter ξ and its value was then conjectured using some additional assumptions. A different value for ξ was recently suggested in arXiv:1702.03518 using an alternative method. Here we confirm that this latter value is indeed the correct one by providing an additional data point: the Weyl anomaly coefficient c_3 for the higher derivative $(1,0)$ superconformal 6d vector multiplet. This multiplet contains the 4-derivative conformal gauge vector, 3-derivative fermion and 2-derivative scalar. We find the corresponding value of c_3 which is proportional to the coefficient C_T in the 2-point function of stress tensor using its relation to the first derivative of the Rényi entropy or the second derivative of the free energy on the product of thermal circle and 5d hyperbolic space. We present some general results of the computation of the Rényi entropy and C_T from the partition function on $S^1 \times \mathbb{H}^{d-1}$ for higher derivative conformal scalars, spinors and vectors in even dimensions. We also give an independent derivation of the conformal anomaly coefficients of the 6d higher derivative vector multiplet from the Seeley-DeWitt coefficients on an Einstein background.

KEYWORDS: AdS-CFT Correspondence, Conformal Field Theory, Supergravity Models

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1 Introduction

The conformal anomaly of a classically Weyl invariant theory in 6d depends on 4 independent coefficients a, c_1, c_2, c_3 [1–4]

$$(4\pi)^3 \langle T_\mu^\mu \rangle = -a E_6 + c_1 I_1 + c_2 I_2 + c_3 I_3 , \quad (1.1)$$

where E_6 is the 6d Euler density and the three Weyl invariants are $I_1 = C_{\alpha\mu\nu\beta} C^{\mu\rho\sigma\nu} C_{\rho\alpha\beta\sigma}$, $I_2 = C_{\alpha\beta\mu\nu} C^{\mu\nu\rho\sigma} C_{\rho\sigma\alpha\beta}$, $I_3 = C_{\mu\nu\alpha\beta} \nabla^2 C^{\mu\nu\alpha\beta} + \dots$. As (1.1) appears in the log UV divergent part of the effective action, c_3 can be determined from the 2-point function of

stress tensor $\langle TT \rangle$, c_2 and c_3 — from the 3-point function¹ and the a-coefficient — from the 4-point function.

In the presence of $(1,0)$ supersymmetry one expects that the Weyl invariants I_i are bosonic parts of only two possible 6d superinvariants, i.e. the coefficients c_i should satisfy one linear relation. As discussed in [8], free-theory calculations [3], strong-coupling (holography) arguments [9], and studies in other contexts [10–12] indicate that this relation is²

$$c_3 = -\frac{1}{6}(c_1 - 2c_2) . \quad (1.2)$$

The 6d chiral $(SU(2)$ R-symmetry and gravitational) anomalies are encoded in the 8-form polynomial parametrized by 4 numerical coefficients $(\alpha, \beta, \gamma, \delta)$.³ The 6d chiral and Weyl anomalies belong to a supersymmetry multiplet [17–19] and as in the 4d case [20–23] one expects to find linear relations between their coefficients, i.e. between (a, c_1, c_2, c_3) and $(\alpha, \beta, \gamma, \delta)$. Ref. [24] derived such relation for the a-coefficient using supersymmetry and the results from the background supergravity couplings⁴

$$a = -\frac{1}{72}(\alpha - \beta + \gamma + \frac{3}{8}\delta) . \quad (1.3)$$

The 4 coefficients in (1.3) could have been fixed also from the a-anomalies for the 4 multiplets: free tensor, free hyper, the $(2,0)$ multiplet at large N and the higher derivative vector multiplet (the a-anomaly of which was found in [8, 13] after [24] already appeared).

Assuming that similar linear relations exist also for c_1, c_2, c_3 , in [8] we attempted to fix their form using the available data about c-anomalies of particular $(1,0)$ superconformal theories. The linear relations for c_1 and c_2 in terms of $(\alpha, \beta, \gamma, \delta)$ contain, in general, 8 coefficients (c_3 is given by (1.2)). We first used the values of anomaly coefficients for free scalar $S^{(1,0)}$ and tensor $T^{(1,0)}$ multiplets

$$S^{(1,0)} = 4\varphi + 2\psi^-, \quad T^{(1,0)} = \varphi + 2\psi^- + T^-, \quad T^{(2,0)} = S^{(1,0)} + T^{(1,0)} \quad (1.4)$$

built out of the standard 2-derivative real scalar φ , Majorana-Weyl (MW) spinor ψ and (anti) selfdual rank 2 tensor T with known Weyl anomalies [3]. This gave 4 coefficients out of 8. One more coefficient was fixed by considering the 4-derivative vector multiplet $V^{(1,0)}$ (see (1.8) below) on a Ricci-flat background when its Weyl anomalies can be readily computed. Two more coefficients were found from the known anomalies of interacting $(2,0)$ A_N theory (see [25, 26] and refs. there). As a result, we were able to find the form of the

¹ $\langle TTT \rangle$ in 6d depends on three parameters, but one of them is related to the 2-point function or c_3 by a conformal Ward identity [5–7].

²In the case of $(2,0)$ supersymmetry, the three invariants I_i are parts of a single superinvariant (6d conformal supergravity action [13–15]) and thus c_i obey the additional constraint $c_1 - 4c_2 = 0$. Then there is only one independent c-coefficient: $c_1 = 4c_2 = -12c_3$. This relation holds for the free $(2,0)$ tensor multiplet [3] as well as for the large N strong coupling limit of the interacting $(2,0)$ theory described by supergravity in AdS₇ [16].

³Explicitly, $\mathcal{I}_8 = \frac{1}{4!}(\alpha c_2^2 + \beta c_2 p_1 + \gamma p_1^2 + \delta p_2)$, $c_2 = \text{tr } F^2$, $p_1 = -\frac{1}{2} \text{tr } R^2$, $p_2 = -\frac{1}{4} \text{tr } R^4 + \frac{1}{8} (\text{tr } R^2)^2$.

⁴In our normalization the a-anomaly of $(2,0)$ tensor multiplet is $a(T^{(2,0)}) = -\frac{7}{1152}$.

relations for c_i in terms of $(\alpha, \beta, \gamma, \delta)$ up to one undetermined parameter ξ , i.e.

$$\begin{aligned} c_1 &= -\frac{4}{3}\alpha + \left(\frac{146}{63} + \frac{6}{7}\xi\right)\beta + \left(-\frac{80}{63} + \frac{4}{7}\xi\right)\gamma + \xi\delta, \\ c_2 &= -\frac{1}{3}\alpha + \left(\frac{11}{126} - \frac{3}{14}\xi\right)\beta + \left(-\frac{22}{63} - \frac{1}{7}\xi\right)\gamma + \left(-\frac{1}{2} - \frac{1}{4}\xi\right)\delta, \end{aligned} \quad (1.5)$$

with c_3 given by (1.2). We then conjectured that the value of ξ should be

$$\xi_{\text{BT}} = -\frac{31}{27}. \quad (1.6)$$

This particular choice was motivated [8] by certain special features in the rank dependence of the c-anomaly of particular interacting (1,0) superconformal theories and also by potential relation between 6d and 4d anomalies for certain (1, 0) theories compactified on 2-torus [27].

Recently, the same expressions for c_i (1.5),(1.2) but with a different value of ξ

$$\xi_{\text{YZ}} = -\frac{8}{9}, \quad (1.7)$$

were found in [28] in a different approach using the assumption that the supersymmetric Rényi entropy for (1,0) superconformal 6d theory should be a cubic polynomial in inverse of its argument.

In this paper we will settle the question about the right value of ξ in our original approach of [8] by using an additional information about the anomalies of the free (1,0) vector supermultiplet. We will confirm that the value (1.7) suggested in [28] is indeed the correct one.

This multiplet is the higher-derivative (non-unitary) superconformal 6d (1,0) vector multiplet $V^{(1,0)}$ that contains the 4-derivative gauge vector $V_\mu^{(4)}$ (with action $\sim \int F_{\mu\nu} \partial^2 F^{\mu\nu}$), the 3-derivative MW spinors $\psi^{(3)}$ and the 2-derivative scalars φ [8, 13–15, 29] (cf. (1.4))

$$V^{(1,0)} = 3\varphi + 2\psi^{(3),+} + V^{(4)}. \quad (1.8)$$

The anomaly polynomial for this multiplet has coefficients⁵

$$(\alpha, \beta, \gamma, \delta) = \left(-1, -\frac{1}{2}, -\frac{7}{240}, \frac{1}{60}\right), \quad (1.9)$$

so that using (1.5),(1.2), we should thus expect to find

$$c_1 = \frac{40}{189} - \frac{3}{7}\xi, \quad c_2 = \frac{551}{1890} + \frac{3}{28}\xi, \quad c_3 = \frac{13}{210} + \frac{3}{28}\xi. \quad (1.10)$$

The direct computation of Weyl anomalies c_i for $V^{(1,0)}$ on a general curved background is challenging as it requires the knowledge of 6d Seeley-DeWitt coefficients for the corresponding higher derivative vector and spinor operators. In the two special cases — of a sphere and a Ricci flat space — that were discussed in [8, 13] the higher derivative

⁵The corresponding a-anomaly is $a(V^{(1,0)}) = -\frac{251}{210}$ [8, 13], in agreement with (1.3).

operators factorize and the anomalies can be readily computed using the expressions for the Seeley-DeWitt coefficients of 2nd order Laplacians. This fixes 3 out of 4 coefficients in (1.1) and thus does not allow to determine ξ . In fact, the higher derivative scalar, vector and (squared) spinor operators discussed below factorize also on a general Einstein space $R_{\mu\nu} = \frac{1}{d}g_{\mu\nu}R$ (on which the curvature invariants in (1.1) remain independent) and thus their 6d anomalies may be computed using the 6d Seeley coefficient of 2nd order Laplacians as was done in [3] for the standard scalar, spinor and 2-form fields.⁶ We will use this observation below in section 6.

To fix the value of ξ it is sufficient, according to (1.10), to compute just c_3 which itself is determined by the coefficient C_T appearing in the 2-point function of stress tensor in flat background. In fact, as C_T for the scalar and the 4-derivative vector is already known [30, 31], it remains only to compute it for the 3-derivative spinor field.⁷

In more detail, the 6d Weyl anomaly coefficient c_3 in (1.1) is given by⁸

$$c_3 = \frac{5}{3 \times 7!} C_{T,6}, \quad (1.11)$$

where $C_{T,d}$ is the coefficient in the 2-point function of stress tensor in a d -dimensional CFT

$$\begin{aligned} \langle T^{\mu\nu}(x) T^{\rho\sigma}(0) \rangle &= \frac{C_{T,d}}{[V_{S^{d-1}}]^2 (x^2)^d} \mathcal{I}^{\mu\nu\rho\sigma}(x), \\ V_{S^{d-1}} &= \frac{2\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}, \quad \mathcal{I}^{\mu\nu\rho\sigma} = \frac{1}{2}(I^{\mu\sigma} I^{\nu\rho} + I^{\mu\rho} I^{\nu\sigma}) - \frac{1}{d} \eta^{\mu\nu} \eta^{\sigma\rho}, \quad I^{\mu\nu} = \eta^{\mu\nu} - \frac{2}{x^2} x^\mu x^\nu. \end{aligned} \quad (1.12)$$

The coefficient $C_{T,d}$ is known for several unitary and non-unitary conformal theories [5, 30–35]. In particular, for the standard real conformal scalar and spin 1/2 fermion one has

$$C_{T,d}(\varphi) = \frac{d}{d-1}, \quad C_{T,d}(\psi) = \frac{1}{2} n_f d, \quad (1.13)$$

where n_f is the (complex) dimension of the spinor space ($n_f = 2^{\frac{d}{2}-1}$ for Majorana and $2^{\frac{d}{2}-2}$ for MW case). For example, for a 4d Majorana fermion $C_{T,4}(\psi) = 4$, while for a 6d MW fermion $C_{T,6}(\psi) = 6$.⁹

Using the scalar value in (1.13) and the known value of C_T for the 4-derivative gauge vector $V^{(4)}$ [30, 31]

$$C_{T,6}(V^{(4)}) = -90, \quad (1.14)$$

we find that C_T for the 4-derivative vector multiplet (1.8) is given by

$$C_{T,6}(V^{(1,0)}) = 3 \times \frac{6}{5} + 2 \times C_{T,6}(\psi^{(3)}) - 90. \quad (1.15)$$

⁶We thank D. Diaz for this remark.

⁷We thank Ying-Hsuan Lin and Chi-Ming Chang for this suggestion.

⁸In 4 dimensions the coefficient c of the Weyl-squared term in the trace anomaly is given by $c = \frac{1}{160} C_{T,4}$.

⁹Let us add also that for an (anti) self-dual 6d tensor one finds $C_{T,6}(T^+) = 54$. The corresponding values of c_3 in (1.11) are $c_3(\varphi) = \frac{1}{2520}$, $c_3(\psi) = \frac{1}{504}$, $c_3(T^+) = \frac{1}{56}$.

Comparing this to (1.10), (1.11) we conclude that the two suggested values of ξ in (1.6) and (1.7) correspond to

$$\xi_{\text{BT}} = -\frac{31}{27} \quad \rightarrow \quad C_{T,6}(\psi^{(3)}) = -\frac{246}{5}, \quad (1.16)$$

$$\xi_{\text{YZ}} = -\frac{8}{9} \quad \rightarrow \quad C_{T,6}(\psi^{(3)}) = -\frac{36}{5}. \quad (1.17)$$

As we shall find below, it is the second value (1.17) that is the correct result for the C_T of the 3-derivative 6d MW fermion.

To find C_T for a free conformal field one may follow the standard route of first determining the explicit form of the stress tensor $T_{\mu\nu}$ as a conformal primary or obtaining it from the metric variation of a Weyl-invariant action in curved background and then using (1.12).¹⁰ An alternative approach that we shall follow below is to exploit the relation between C_T and the Rényi entropy [36]. As we shall demonstrate, this second approach turns out to be more efficient in the case of the higher-derivative conformal fields.

Given a CFT in flat even-dimensional space one has the following relation between the first derivative of the Rényi entropy $\mathbb{S}_q$ (which is a function of q defined in the next section) at $q = 1$ and the coefficient $C_{T,d}$ in (1.12) [36]

$$\mathbb{S}'_1 = -\bar{V}_{\mathbb{H}^{d-1}} \frac{\pi^{\frac{d}{2}+1} \Gamma(\frac{d}{2}) (d-1)}{(d+1)! [V_{S^{d-1}}]^2} C_{T,d}. \quad (1.18)$$

Here $V_{S^{d-1}}$ is the volume of the sphere as in (1.12) and $\bar{V}_{\mathbb{H}^{d-1}}$ is the finite coefficient in the regularized volume of the odd-dimensional unit-radius hyperbolic space $\mathbb{H}^{d-1}$ (Λ_{IR} is an IR cutoff)

$$V_{\mathbb{H}^{d-1}} \equiv \bar{V}_{\mathbb{H}^{d-1}} \log \Lambda_{\text{IR}}, \quad \bar{V}_{\mathbb{H}^{d-1}} = (-1)^{\frac{d}{2}-1} \frac{2\pi^{\frac{d}{2}-1}}{\Gamma(\frac{d}{2})}. \quad (1.19)$$

In particular,

$$\bar{V}_{\mathbb{H}^3} = -2\pi, \quad \bar{V}_{\mathbb{H}^5} = \pi^2, \quad \bar{V}_{\mathbb{H}^7} = -\frac{\pi^3}{3}, \quad \dots \quad (1.20)$$

and thus

$$C_{T,4} = 80 \mathbb{S}'_1, \quad C_{T,6} = -504 \mathbb{S}'_1, \quad C_{T,8} = 2880 \mathbb{S}'_1, \quad \dots \quad (1.21)$$

We shall start in section 2 with defining the Rényi entropy in terms of the free energy $\mathcal{F}_q$ on $S^1_q \times \mathbb{H}^{d-1}$, i.e. the product of a thermal circle (with length $\beta = 2\pi q$) and hyperbolic space, thus relating C_T to the second derivative of the free energy at $q = 1$. We will then describe our method of computing this free energy using heat kernel representation.

To illustrate this method of computing free energy and C_T in section 3 we will consider the examples of the 4- and 6-derivative conformal scalars in even number of dimensions. In section 4 we will discuss the case of the 4-derivative conformal gauge vector in 6d reproducing the value (1.14) of its C_T obtained earlier by other methods.

¹⁰Some simplification is that in computing $\langle TT \rangle$ in (1.12) one can drop total derivative (descendant) parts in one of the two T -factors [30].

Section 5 will contain a similar computation of free energy and thus Rényi entropy and C_T for the 3-derivative conformal fermion in $d = 4$ and $d = 6$. For higher derivative operators the computation of C_T turns out to be subtle: surprisingly, a naive approach (discussed in appendix A) leads to the value in (1.16) while the correct evaluation gives (1.17).

In section 6 we will provide an independent derivation of the conformal anomalies of the vector multiplet (1.10) with ξ given by (1.7) by directly computing the Seeley-DeWitt coefficients of the higher derivative operators involved using the fact of their factorization on a generic Einstein background.

In appendix A we will supplement the discussion in section 5 by explaining a different method of computing the free energy on $S_q^1 \times \mathbb{H}^{d-1}$. In appendix B we will compute C_T for the non-unitary 2-derivative conformal vector theory which has no gauge invariance in $d \neq 4$. Finally, in appendix C, we shall present the result for the conformal anomalies for a family of vector multiplets generalizing (1.8) that shows again the agreement with the relations (1.2), (1.5) with ξ given by (1.7).

2 Free energy for conformal fields on $S_q^1 \times \mathbb{H}^{d-1}$ and Rényi entropy

The Rényi entropy $\mathbb{S}_q$ is a measure of generalized quantum entanglement and can be computed from traces of the reduced density matrix raised to a power $q \geq 0$. For a d -dimensional CFT, the Rényi entropy across S^{d-2} may be equivalently extracted from the partition function on q -cover of the sphere S^d or from the thermal partition function on $S_q^1 \times \mathbb{H}^{d-1}$ (see [37–39] and refs. there).¹¹ Here $\mathbb{H}^{d-1}$ is real hyperbolic space (of curvature radius $r = 1$) and the length of the thermal circle $x_0 = q\tau$ or the inverse temperature is $\beta = 2\pi q$.

2.1 General relations

Here we shall use the latter definition of $\mathbb{S}_q$ in terms of the partition function or free energy on $S_q^1 \times \mathbb{H}^{d-1}$ for even d . Given a free real conformal field Φ with the action

$$I = \frac{1}{2} \int d^d x \sqrt{g} \Phi \mathcal{O} \Phi, \quad (2.1)$$

where $\mathcal{O}$ is a (possibly higher order) covariant differential operator including curvature terms needed to ensure the Weyl invariance of (2.1) in a general curved background, the corresponding free energy on $S_q^1 \times \mathbb{H}^{d-1}$ is

$$F_q = -\log Z_q = \frac{1}{2} \log \det \mathcal{O}. \quad (2.2)$$

¹¹The metrics of the two spaces are related by a singular conformal rescaling

$$ds_{q\text{-sphere}}^2 = \sin^2 \theta q^2 d\tau^2 + d\theta^2 + \cos^2 \theta d\Omega_{d-2}^2 = \sin^2 \theta (q^2 d\tau^2 + d\rho^2 + \sinh^2 \rho d\Omega_{d-2}^2) = \cosh^{-2} \rho ds_{S_q^1 \times \mathbb{H}^{d-1}}^2$$

Here $\tau \in (0, 2\pi)$ and $\sinh \rho = \cot \theta$. This transformation maps the subspace S^{d-2} to the boundary of $\mathbb{H}^{d-1}$. For $q = 1$ the space $S_q^1 \times \mathbb{H}^{d-1}$ becomes conformal to regular S^d and thus also to $\mathbb{R}^d$ as $ds^2 = dz^2 + z^2 dx_0^2 + dx_n dx_n = z^2 (dx_0^2 + \frac{dz^2 + dx_n dx_n}{z^2})$.

In the present case of a homogeneous space F_q is proportional to its volume, i.e. to $2\pi q V_{\mathbb{H}^{d-1}}$ in (1.19). Extracting the IR divergent factor, we may define the IR finite “free energy” $\mathcal{F}_q$ by

$$F_q \equiv \mathcal{F}_q \log \Lambda_{\text{IR}} . \quad (2.3)$$

For even d the free energy on $S_q^1 \times \mathbb{H}^{d-1}$ does not contain logarithmic UV divergences¹² while the non-universal power divergent part of $\mathcal{F}_q$ (which is proportional to the volume and is thus linear in q) should be subtracted using some regularization prescription.

The finite Rényi entropy is then given by

$$\mathbb{S}_q \equiv \frac{q \mathcal{F}_1 - \mathcal{F}_q}{1 - q}, \quad \mathcal{F}_q = q \mathcal{F}_1 + (q - 1) \mathbb{S}_q . \quad (2.4)$$

Note that under a linear in q and constant shift of the free energy we have

$$\mathcal{F}_q \rightarrow \mathcal{F}_q + k_1 q + k_2 \quad \rightarrow \quad \mathbb{S}_q \rightarrow \mathbb{S}_q + k_2 . \quad (2.5)$$

As all power UV divergent terms in $\mathcal{F}_q$ are linear in q they drop out of $\mathbb{S}_q$ which is thus UV finite. The $q = 1$ value of the Rényi entropy which is the entanglement entropy

$$\mathbb{S}_1 = \mathcal{F}'_1 - \mathcal{F}_1 \quad (2.6)$$

is sensitive to the constant (q -independent) part of $\mathcal{F}_q$. $\mathbb{S}_1$ is expected to be proportional to the a-anomaly coefficient of the d -dimensional CFT, e.g.,¹³

$$d = 4 : \quad \mathbb{S}_1 = -4a, \quad d = 6 : \quad \mathbb{S}_1 = -96a, \quad (2.7)$$

as that happens when F_d is computed on the q -cover of the sphere S^d [37, 40–44].¹⁴ However, the transformation between the q -cover of the S^d and $S_q^1 \times \mathbb{H}^{d-1}$ is a non-trivial Weyl rescaling (cf. footnote 11) and thus the two free energies may a priori differ by a Weyl-anomaly term. It was observed that for fields with gauge invariance $\mathbb{S}_1$ computed on $S_q^1 \times \mathbb{H}^{d-1}$ is not automatically proportional to the Weyl anomaly a -coefficient (see [45, 46] for 4d vectors and [47] for 6d antisymmetric tensors), but one can achieve this by shifting $\mathcal{F}_q$ by a constant (that may be interpreted as an edge mode contribution).

The C_T coefficient which is proportional to the first derivative of the Rényi entropy (1.18) may be expressed in terms of the second derivative of the free energy $\mathcal{F}_q$ and thus is not sensitive to the shifts in (2.5). Explicitly,

$$C_{T,d} = \frac{(-1)^{\frac{d}{2}} (d+1)!}{(d-1) \left[\left(\frac{d}{2} - 1 \right)! \right]^2} \mathcal{F}_1'' , \quad \mathbb{S}'_1 = \frac{1}{2} \mathcal{F}_1'' . \quad (2.8)$$

In particular (see (1.21), (1.11) and 8)

$$d = 4 : \quad C_{T,4} = 160 c = 40 \mathcal{F}_1'' , \quad d = 6 : \quad C_{T,6} = 3024 c_3 = -252 \mathcal{F}_1'' . \quad (2.9)$$

Thus to compute C_T we need to find the free energy $\mathcal{F}_q$ on $S_q^1 \times \mathbb{H}^{d-1}$.

¹²Since S_q^1 factor is flat and $\mathbb{H}^{d-1}$ is conformally flat, all logarithmic divergent terms containing the Weyl tensor vanish, while the Euler density in d dimensions vanishes when evaluated on $\mathbb{H}^{d-1}$.

¹³In 4 dimensions (cf. (1.1)) $(4\pi)^2 T_\mu^\mu = -a R^* R^* + c C^{\mu\nu\lambda\rho} C_{\mu\nu\lambda\rho}$.

¹⁴One expects that the log UV divergent part of free energy on q -cover of the S^d should be matching the log IR part of free energy on $S_q^1 \times \mathbb{H}^{d-1}$, and that was checked on specific examples, though a general proof of this statement appears to be missing in the literature.

2.2 Computational scheme

The covariant kinetic operator $\mathcal{O}$ specified to $S_q^1 \times X^{d-1}$ where X^{d-1} is a symmetric space like S^{d-1} or $\mathbb{H}^{d-1}$ will be a polynomial in derivatives ∂_0 along the “euclidean time” direction S^1 and the covariant derivatives $D_i \equiv \mathbf{D}_i$ on X^{d-1} , i.e. symbolically $\mathcal{O} = P(i\partial_0, -\mathbf{D}^2)$ (with X^{d-1} curvature factors translating into the coefficients of lower-order terms in P). In the case of $X^{d-1} = \mathbb{H}^{d-1}$ the free energy $\mathcal{F}_q$ in (2.2),(2.3) will have the following structure

$$\mathcal{F}_q = \frac{1}{2} \bar{V}_{\mathbb{H}^{d-1}} \sum_n \int d\mu(\lambda) \log P_{\mathbb{H}}\left(\frac{n}{q}, \lambda\right), \quad (2.10)$$

where $\frac{n}{q}$ is the eigenvalue of $i\partial_0$ and $d\mu(\lambda)$ is the spectral measure for the continuous eigenvalue λ of the spatial operator $-\mathbf{D}^2 + \dots$ (a particular definition of λ will depend on a type of the field Φ in (2.1), see below). The summation index n takes values in $\mathbb{Z}$ for bosons and in $\mathbb{Z} + \frac{1}{2}$ for fermions.¹⁵

It turns out that for conformal fields the kinetic operators $\mathcal{O}$ restricted to $S_q^1 \times X^{d-1}$, i.e. $P(i\partial_0, -\mathbf{D}^2)$, have special factorized structure, i.e. are given by a product of simple two-derivative factors.¹⁶ A particular reason for this can be understood by observing that the operators on $S^1 \times \mathbb{H}^{d-1}$ and $S^1 \times S^{d-1}$ are formally related by an analytic continuation changing the sign of the curvature. The thermal partition function on $S^1 \times S^{d-1}$ is expressed in terms of characters of conformal group and this in turn is related to factorization of the (higher-derivative) kinetic operator discussed in detail in [48]. In the case of $S_q^1 \times S^{d-1}$ we get

$$F_q = \frac{1}{2} V_{S^{d-1}} \sum_n \sum_m \mu(m) \log P_S\left(\frac{n}{q}, m\right), \quad (2.11)$$

where the sum over m is over the discrete spectrum of $-\mathbf{D}^2 + \dots$ on S^{d-1} and $\mu(m)$ is the multiplicity factor of the eigenvalue with label m . The higher-derivative Weyl-covariant operators $\mathcal{O} = D^{2p} + \dots$ turn out to factorize [48] into simple factors so that the corresponding eigenvalues on $S_q^1 \times S^{d-1}$ are

$$P_S = \prod_{k=1}^p \left[\frac{n^2}{q^2} + \frac{1}{r^2} (m + \ell_k)^2 \right], \quad (2.12)$$

where r is the radius of S^{d-1} . In this case, the standard free energy F_q in (2.2) is expressed in terms of the single-particle partition function $\mathcal{Z}(x)$ that has a simple structure

$$F_q = \sum_{n=1}^{\infty} \frac{1}{n} \mathcal{Z}(x^n), \quad \mathcal{Z}(x) = \sum_{k=1}^p \sum_m \mu(m) x^{m+\ell_k}, \quad x \equiv e^{-2\pi q}. \quad (2.13)$$

Here $m + \ell_k$ correspond to the single-particle energies or integer dimensions of conformal operators in $\mathbb{R}^d$ built out of Φ and its derivatives.

¹⁵The antiperiodicity of fermions in “thermal” circle is related to the original definition of partition function on q -cover of S^d .

¹⁶This applies to bosonic operators and squared fermionic operators.

The factorization of the higher-derivative Weyl-covariant kinetic operator $\mathcal{O}$ on $S^1 \times \mathbb{H}^{d-1}$ is thus intimately related to its factorization on $S^1 \times S^{d-1}$ which in turn is related to integrality of dimensions of the CFT operators in $\mathbb{R}^d$.¹⁷

One may also consider the analytic continuation between S^{d-1} and $\mathbb{H}^{d-1}$ at the level of the spectrum (see [49, 50] and appendix C of [51]). For example, for a 2nd order Laplacian acting on symmetric traceless rank s tensors on a homogeneous space one has the following spectrum on S^{d-1} with radius r

$$-\mathbf{D}_{S^{d-1}}^2 \varphi_s = \omega_m \varphi_s, \quad \omega_m = \frac{1}{r^2} \left[\left(m + \frac{d-2}{2} \right)^2 - \left(\frac{d-2}{2} \right)^2 - s \right], \quad m = s, s+1, \dots \quad (2.14)$$

The eigenvalues ω_λ of the same operator on $\mathbb{H}^{d-1}$ with curvature radius r are obtained by replacing

$$m \rightarrow i\sqrt{\lambda} - \frac{d-2}{2}, \quad r \rightarrow ir, \quad \omega_m \rightarrow \omega_\lambda = \frac{1}{r^2} \left[\lambda + \left(\frac{d-2}{2} \right)^2 + s \right]. \quad (2.15)$$

Here $0 \leq \lambda < \infty$ is the eigenvalue of the following operator on $\mathbb{H}^{d-1}$ (here and in what follows we set the radius of $\mathbb{H}^{d-1}$ to be $r = 1$) [49]

$$\Delta_s \varphi_s = \lambda \varphi_s, \quad \Delta_s = -\mathbf{D}_{\mathbb{H}^{d-1}}^2 - \left(\frac{d-2}{2} \right)^2 - s. \quad (2.16)$$

The analytical continuation (2.15) then translates the factorization (2.12) into the one on $S^1 \times \mathbb{H}^{d-1}$.

In addition, we need to replace the sum $\sum_m \mu(m)$ in (2.11) by $\int d\mu(\lambda)$ in (2.10) with a definite correspondence between the discrete multiplicity on S^{d-1} and the spectral measure on $\mathbb{H}^{d-1}$. The latter is the Plancherel measure for the transverse traceless symmetric rank s field on $\mathbb{H}^{d-1}$ corresponding to the spectrum (2.16) [49]

$$d\mu_{s,d-1} = \frac{(2s+d-4)(s+d-5)!}{(d-4)! s!} \frac{\lambda + (s + \frac{d-4}{2})^2}{2^{d-2} \pi^{\frac{d-1}{2}} \Gamma(\frac{d-1}{2})} \left| \frac{\Gamma(i\sqrt{\lambda} + \frac{d-4}{2})}{\Gamma(i\sqrt{\lambda})} \right|^2 d\sqrt{\lambda}. \quad (2.17)$$

Having $\mathcal{O}$ factorized into a product of second-derivative factors, the polynomial $P_{\mathbb{H}}$ in (2.10) may be written in the product form which is the counterpart of (2.12),

$$P_{\mathbb{H}} = \prod_{k=1}^p \left[\frac{n^2}{q^2} + (\sqrt{\lambda} + i\alpha_k)^2 \right], \quad (2.18)$$

where α_k are real constants (appearing in $\pm$ conjugate pairs so that $P_{\mathbb{H}}$ is real). Then $\log P_{\mathbb{H}}$ in (2.10) becomes the sum of p terms. Using the proper-time representation separately for each log term in the sum we then get (in bosonic case)

$$\begin{aligned} \mathcal{F}_q &= -\frac{1}{2} \bar{V}_{\mathbb{H}^{d-1}} \int_0^\infty \frac{dt}{t} K_{S^1}(t) \bar{K}_{\mathbb{H}^{d-1}}(t), \quad K_{S^1}(t) = \sum_{n \in \mathbb{Z}} e^{-t \frac{n^2}{q^2}}, \quad (2.19) \\ \bar{K}_{\mathbb{H}^{d-1}}(t; \alpha_k) &= \int_0^\infty d\mu(\lambda) e^{-t(\sqrt{\lambda} + i\alpha_k)^2}, \quad \bar{K}_{\mathbb{H}^{d-1}}(t) = \sum_{k=1}^p \bar{K}_{\mathbb{H}^{d-1}}(t; \alpha_k). \end{aligned} \quad (2.20)$$

¹⁷Similar factorization is found also for $\mathcal{O}$ defined on S^d or $\mathbb{H}^d$.

Here K_{S^1} is the trace of the heat kernel of $-\partial_0^2$ on S^1 while $\overline{K}_{\mathbb{H}^{d-1}}(t; \alpha)$ may be interpreted as the heat kernel corresponding to the operator $(\sqrt{\Delta_s} + i\alpha)^2$ on $\mathbb{H}^{d-1}$ (cf. (2.16)). Using the Poisson resummation¹⁸ we may represent $K_{S^1}(t)$ as

$$K_{S^1}(t) = \frac{2\pi q}{(4\pi t)^{1/2}} \sum_{n \in \mathbb{Z}} e^{-\frac{\pi^2 n^2 q^2}{t}}. \quad (2.21)$$

Similarly, in the fermion (antiperiodic) case one finds

$$K_{S^1}^f(t) = \sum_{n \in \mathbb{Z} + \frac{1}{2}} e^{-t \frac{n^2}{q^2}} = \frac{2\pi q}{(4\pi t)^{1/2}} \sum_{n \in \mathbb{Z}} (-1)^n e^{-\frac{\pi^2 n^2 q^2}{t}}. \quad (2.22)$$

Assuming $t > 0$ the integral over λ in (2.20) is convergent, i.e. the relevant real part of $\overline{K}_{\mathbb{H}^{d-1}}(t; \alpha)$ is proportional to a finite polynomial in t , i.e.

$$\begin{aligned} \overline{K}_{\mathbb{H}^{d-1}}(t; \alpha) + \overline{K}_{\mathbb{H}^{d-1}}(t; -\alpha) &= \int_0^\infty d\mu(\lambda) [e^{-t(\sqrt{\lambda}+i\alpha)^2} + e^{-t(\sqrt{\lambda}-i\alpha)^2}] \\ &= 2 \int_0^\infty d\mu(\lambda) e^{-t(\lambda-\alpha^2)} \cos(2\alpha t \sqrt{\lambda}) \\ &= \frac{1}{(4\pi t)^{\frac{d-1}{2}}} \sum_{j \geq 0} \nu_j t^j, \end{aligned} \quad (2.23)$$

where ν_j are numerical constants depending on α , dimension d and spin of the field. The integral over t in (2.19) is then power-divergent at $t = 0$ for $n = 0$ term in (2.21) or (2.22). Subtracting these power divergences as a proper-time regularization prescription corresponds to omitting the $n = 0$ term in the sum. As a result, we are left with a finite sum over $n \geq 1$ expressing $\mathcal{F}_q$ as a finite polynomial in q^{-1} with coefficients proportional to the Riemann zeta-function values.¹⁹

To summarize, the computation of the free energy $\mathcal{F}_q$ will contain the following sequence of steps: (i) integration over the eigenvalue λ ; (ii) integration over the proper time t with $t \rightarrow 0$ power divergences subtracted; (iii) performing the remaining finite sum over $n \neq 0$. We shall illustrate this procedure in detail on several examples below. Having found $\mathcal{F}_q$ one can then compute the Rényi entropy in (2.4) and C_T in (2.8).

3 Scalar fields

To illustrate the relation (1.18),(2.8) in this section we will use it to compute C_T for free higher-derivative conformal scalar theories in even dimension d , reproducing the results obtained previously by other methods in a novel way.

¹⁸ In general,

$$\sum_{n \in \mathbb{Z}} e^{-t \frac{(n+\gamma)^2}{q^2}} = \frac{2\pi q}{(4\pi t)^{1/2}} \sum_{n \in \mathbb{Z}} e^{-\frac{\pi^2 n^2 q^2}{t} + 2\pi i \gamma n}.$$

¹⁹ In the antiperiodic case one has

$$\zeta_{2k} = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} \rightarrow \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{2k}} = (2^{1-2k} - 1) \zeta_{2k}.$$

3.1 ∂^2 scalar

The standard action for the conformally coupled scalar is

$$I = \frac{1}{2} \int d^d x \sqrt{g} \varphi \left[-D^2 + \frac{d-2}{4(d-1)} R \right] \varphi. \quad (3.1)$$

The corresponding free energy on $S_q^1 \times \mathbb{H}^{d-1}$ is given by ($R(\mathbb{H}^{d-1}) = -(d-1)(d-2)$)

$$F_q = \frac{1}{2} \log \det(-\partial_0^2 + \Delta_0), \quad \Delta_0 \equiv -D^2 - \frac{(d-2)^2}{4}. \quad (3.2)$$

The spectrum of the operator Δ_0 (i.e. the $s = 0$ case of (2.16)) is $\frac{n^2}{q^2} + \lambda$ where $n \in \mathbb{Z}$ and $\lambda \geq 0$. The spectral measure is given by the $s = 0$ case of (2.17), in particular, in $d = 4$ and $d = 6$,

$$d\mu_{0,3} = \frac{1}{4\pi^2} \sqrt{\lambda} d\lambda, \quad d\mu_{0,5} = \frac{1}{24\pi^3} \sqrt{\lambda} (1 + \lambda) d\lambda. \quad (3.3)$$

In $d = 4$ we get from (2.20)

$$\bar{K}_{\mathbb{H}^3}(t) = \int_0^\infty d\lambda \frac{\sqrt{\lambda}}{4\pi^2} e^{-t\lambda} = \frac{1}{(4\pi t)^{3/2}}. \quad (3.4)$$

Then using (2.21),(1.20) we find

$$\mathcal{F}_q = \frac{1}{4} q \sum_{n=1}^\infty \int_0^\infty \frac{dt}{t^3} e^{-\frac{\pi^2 n^2 q^2}{t}} = \frac{1}{4} q \sum_{n=1}^\infty \frac{1}{n^4 \pi^4 q^4} = \frac{1}{360 q^3}, \quad (3.5)$$

where we omitted the $n = 0$ mode which corresponds to subtracting the Λ^4 UV divergence ($t = \varepsilon = \Lambda^{-2} \rightarrow 0$). The resulting Rényi entropy and the Weyl anomaly coefficients have indeed the standard values (see (2.9))

$$\mathbb{S}_q = -\frac{(1+q)(1+q^2)}{360 q^3}, \quad \mathfrak{a} = -\frac{1}{4} \mathbb{S}_1 = \frac{1}{360}, \quad C_{T,4} = 160 \mathfrak{c} = 80 \mathbb{S}'_1 = \frac{4}{3}. \quad (3.6)$$

Similarly, in $d = 6$

$$\bar{K}_{\mathbb{H}^5}(t) = \int_0^\infty d\lambda \frac{\sqrt{\lambda}(1+\lambda)}{24\pi^3} e^{-t\lambda} = \frac{3+2t}{3(4\pi t)^{5/2}}, \quad (3.7)$$

$$\begin{aligned} \mathcal{F}_q &= -\frac{1}{96} q \sum_{n=1}^\infty \int_0^\infty \frac{dt}{t^4} (3+2t) e^{-\frac{\pi^2 n^2 q^2}{t}} \\ &= -\frac{1}{48\pi^6 q^5} \sum_{n=1}^\infty \frac{3+n^2 \pi^2 q^2}{n^6} = -\frac{2+7q^2}{30240 q^5}, \end{aligned} \quad (3.8)$$

$$\begin{aligned} \mathbb{S}_q &= \frac{(1+q)(1+3q^2)(2+3q^2)}{30240 q^5}, \\ \mathfrak{a} &= -\frac{1}{96} \mathbb{S}_1 = -\frac{5}{72 \times 7!}, \quad C_{T,6} = -504 \mathbb{S}'_1 = \frac{6}{5}, \end{aligned} \quad (3.9)$$

where we again dropped the $n = 0$ term in the sum corresponding to subtracting the Λ^6 and Λ^4 UV divergences. The above values for $C_{T,d}$ are in agreement with the general expression in (1.13).

3.2 ∂^4 scalar

The Weyl-invariant action for the 4-derivative scalar in curved 4d space is given by [52]

$$I = \frac{1}{2} \int d^4x \sqrt{g} \left[D^2 \varphi D^2 \varphi - 2 \left(R^{\mu\nu} - \frac{1}{3} R g^{\mu\nu} \right) D_\mu \varphi D_\nu \varphi \right]. \quad (3.10)$$

The generalization of the D^4 operator in (3.10) to any $d > 4$ is the Paneitz operator [53]

$$\begin{aligned} \mathcal{O}^{(4)} &= D^4 + \frac{4}{d-2} R^{\mu\nu} D_\mu D_\nu + k_d R D^2 + \frac{d-4}{2} (n_d R_{\mu\nu} R^{\mu\nu} + m_d R^2) + O(D^2 R), \quad (3.11) \\ k_d &= -\frac{d^2 - 4d + 8}{2(d-1)(d-2)}, \quad n_d = -\frac{2}{(d-2)^2}, \quad m_d = \frac{d^3 - 4d^2 + 16d - 16}{8(d-1)^2(d-2)^2}. \end{aligned}$$

We did not write explicitly the $D^2 R$ term as we will be interested in the homogeneous $S^1 \times X^{d-1}$ background with

$$R = -(d-1)(d-2)\epsilon, \quad R_{0i} = 0, \quad R_{ij} = \frac{1}{d-1} R g_{ij} = -(d-2)\epsilon g_{ij}, \quad (3.12)$$

where we introduced the curvature sign factor ϵ which is $+1$ for $X^{d-1} = \mathbb{H}^{d-1}$ and -1 for $X^{d-1} = S^{d-1}$. Then (3.11) is found to factorize in either of the following two d -independent ways

$$\begin{aligned} \mathcal{O}^{(4)} &= D^4 + \frac{1}{2} \epsilon (d^2 - 4d + 8) D^2 - 4\epsilon \mathbf{D}^2 + \frac{1}{16} \epsilon^2 d^2 (d-4)^2 \\ &= [(i\partial_0 + \sqrt{\epsilon})^2 + \mathbf{\Delta}_0] [(i\partial_0 - \sqrt{\epsilon})^2 + \mathbf{\Delta}_0] \end{aligned} \quad (3.13)$$

$$= [(i\partial_0)^2 + (\sqrt{\mathbf{\Delta}_0} + i\sqrt{\epsilon})^2] [(i\partial_0)^2 + (\sqrt{\mathbf{\Delta}_0} - i\sqrt{\epsilon})^2], \quad (3.14)$$

where $\mathbf{D}^2 \equiv D^i D_i$ and $\mathbf{\Delta}_0 = -\mathbf{D}^2 - \frac{(d-2)^2}{4} \epsilon$ is the conformal scalar Laplacian as in (3.2). This factorization was already observed on $S^1 \times S^{d-1}$ where $\epsilon = -1$ (see eq. (B.22) in [51] for $d = 4$).

The eigenvalues of $\mathcal{O}^{(4)}$ are thus naturally expressed in terms of the eigenvalue λ of the conformal scalar Laplacian on $\mathbb{H}^{d-1}$ in (2.16)

$$\mathcal{O}^{(4)} \rightarrow \left[\left(\frac{n}{q} + 1 \right)^2 + \lambda \right] \left[\left(\frac{n}{q} - 1 \right)^2 + \lambda \right] = \left[\frac{n^2}{q^2} + (\sqrt{\lambda} + i)^2 \right] \left[\frac{n^2}{q^2} + (\sqrt{\lambda} - i)^2 \right]. \quad (3.15)$$

This is thus the special case of (2.18) with $\alpha_k = \pm 1$ so that the corresponding free energy can be computed as in (2.19)–(2.23). Explicitly, we find that in this case $\bar{K}_{\mathbb{H}^{d-1}}(t)$ is given by (2.23) with $\alpha = 1$ so that for $d = 4$ (cf. (3.5), (3.6))

$$\bar{K}_{\mathbb{H}^3}(t) = 2 \int_0^\infty d\lambda \frac{\sqrt{\lambda}}{4\pi^2} e^{-t(\lambda-1)} \cos(2t\sqrt{\lambda}) = \frac{2-4t}{(4\pi t)^{3/2}}, \quad (3.16)$$

$$\mathcal{F}_q = \frac{1-30q^2}{180q^3}, \quad \mathbb{S}_q = \frac{(1+q)(-1+29q^2)}{180q^3}, \quad (3.17)$$

$$a = -\frac{1}{4} \mathbb{S}_1 = -\frac{7}{90}, \quad C_{T,4} = 160c = 80\mathbb{S}'_1 = -\frac{32}{3}. \quad (3.18)$$

These values of the Weyl anomaly coefficients a and c for the 4-derivative scalar agree with the result of the direct computation in [52, 54].

In $d = 6$ get (cf. (3.7)–(3.9))

$$\bar{K}_{\mathbb{H}^5}(t) = 2 \int_0^\infty d\lambda \frac{\sqrt{\lambda}(1+\lambda)}{24\pi^3} e^{-t(\lambda-1)} \cos(2t\sqrt{\lambda}) = \frac{2(1-10t)}{3(4\pi t)^{5/2}}, \quad (3.19)$$

$$\mathcal{F}_q = \frac{-2+35q^2}{15120q^5}, \quad \mathbb{S}_q = \frac{(1+q)(2-33q^2-33q^4)}{15120q^5}, \quad (3.20)$$

$$a = -\frac{1}{96} \mathbb{S}_1 = \frac{4}{9 \times 7!}, \quad C_{T,6} = 3024 c_3 = -504 \mathbb{S}'_1 = -6. \quad (3.21)$$

The value of a in (3.21) agrees with the one found in [13] (see table 1 there).

The above values of C_T in (3.18) and (3.21) are in agreement with the general expression for the 4-derivative conformal scalar in dimension d found in [30, 55]

$$C_{T,d}(\varphi^{(4)}) = -\frac{2d(d+4)}{(d-1)(d-2)}. \quad (3.22)$$

3.3 ∂^6 scalar

The general expression for the Weyl-covariant 6-derivative scalar operator in curved background can be found, e.g., in [56]. Ignoring terms with derivatives of the curvature and specifying to $d = 6$ it can be written as

$$\begin{aligned} \mathcal{O}^{(6)} = & -D^6 - (16P^{\mu\nu} - 6g^{\mu\nu}P)D_\mu D_\nu D^2 \\ & + 8(4P^{\mu\nu}P - g^{\mu\nu}P_{\kappa\rho}P^{\kappa\rho})D_\mu D_\nu + 8(P_{\mu\nu}P^{\mu\nu} - P^2)D^2, \end{aligned} \quad (3.23)$$

where the Schouten tensor $P_{\mu\nu}$ and its trace P are in general defined as

$$P_{\mu\nu} = \frac{1}{d-2} \left(R_{\mu\nu} - \frac{1}{2(d-1)} R \right), \quad P = P^\mu_\mu = \frac{1}{2(d-1)} R. \quad (3.24)$$

Using the properties (3.12) of the curvature of $S^1 \times \mathbb{H}^5$ we find

$$\mathcal{O}^{(6)} = -D^6 + 16\mathbf{D}^2 D^2 - 20D^4 + 64\mathbf{D}^2 - 64D^2. \quad (3.25)$$

Like the 4-derivative scalar operator (3.13),(3.14) (where $\epsilon = 1$ for $\mathbb{H}^{d-1}$) this operator may be factorized in the two possible ways

$$\mathcal{O}^{(6)} = [(i\partial_0)^2 + \mathbf{\Delta}_0] [(i\partial_0 + 2)^2 + \mathbf{\Delta}_0] [(i\partial_0 - 2)^2 + \mathbf{\Delta}_0] \quad (3.26)$$

$$= [(i\partial_0)^2 + \mathbf{\Delta}_0] [(i\partial_0)^2 + (\sqrt{\mathbf{\Delta}_0} + 2i)^2] [(i\partial_0)^2 + (\sqrt{\mathbf{\Delta}_0} - 2i)^2], \quad (3.27)$$

so that the corresponding eigenvalues are given by (2.18) with $\alpha_1 = 0$, $\alpha_2 = 2$, $\alpha_3 = -2$. We thus get a combination of the standard 2-derivative scalar and a conjugate pair of operators with the shift parameter $\alpha = 2$. The heat kernel for the latter is given by (2.23)

and as a result we find in $d = 6$ (cf. (3.19)–(3.21))

$$\overline{K}_{\mathbb{H}^5}(t) = \int_0^\infty d\lambda \frac{\sqrt{\lambda}(1+\lambda)}{24\pi^3} e^{-t\lambda} [1 + 2e^{4t} \cos(4t\sqrt{\lambda})] = \frac{3 - 30t + 32t^2}{(4\pi t)^{5/2}}, \quad (3.28)$$

$$\mathcal{F}_q = \frac{2 - 105q^2 + 1680q^4}{10080q^5}, \quad \mathbb{S}_q = \frac{(1+q)(2 - 103q^2 + 1577q^4)}{10080q^5}, \quad (3.29)$$

$$a = -\frac{1}{96} \mathbb{S}_1 = -\frac{123}{8 \times 7!}, \quad C_{T,6} = -504 \mathbb{S}'_1 = 54. \quad (3.30)$$

The value of a -coefficient agrees with the one following from the partition function of 6-order GJMS operator on S^6 [57] while the value of $C_{T,6}$ agrees with the $d = 6$ case of the general expression for ∂^6 conformal scalar in [30]

$$C_{T,d}(\varphi^{(6)}) = \frac{3d(d+4)(d+6)}{(d-1)(d-2)(d-4)}. \quad (3.31)$$

4 Conformal vector fields

Conformal generalization of the Maxwell theory to general dimension d has a higher derivative Lagrangian $L = F_{\mu\nu}(\partial^2)^{\frac{d-4}{2}} F^{\mu\nu}$. In particular, in 6 dimensions this gives a 4-derivative non-unitary vector gauge theory that we shall consider below. The computation of C_T for 2-derivative non gauge invariant conformal vector theory in generic d (reducing to Maxwell theory for $d = 4$) will be discussed in appendix B.

4.1 ∂^2 gauge vector in $d = 4$

It useful to start with recalling the computation of free energy of the Maxwell theory on $S_q^1 \times \mathbb{H}^3$. The closely related case of $S_q^1 \times S^3$ background was discussed, e.g., in section 2.2 of [48]. Starting with $I = -\frac{1}{4} \int d^4x \sqrt{g} F^{\mu\nu} F_{\mu\nu}$ where $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ and fixing the $V_0 = 0$ gauge one ends up with the free energy expressed in terms of the operator defined on transverse 3-vector (cf. (3.2))

$$F_q = \frac{1}{2} \log \det(-\partial_0^2 + \Delta_1), \quad \Delta_1 = -D^2 - 2, \quad (4.1)$$

where Δ_1 is the $d = 4$, $s = 1$ case of the operator in (2.16) with the eigenvalue λ . The corresponding spectral density is the $s = 1$ case of (2.17), i.e. in $d = 4$ and $d = 6$ it reads

$$d\mu_{1,3} = \frac{1+\lambda}{2\pi^2\sqrt{\lambda}} d\lambda, \quad d\mu_{1,5} = \frac{\sqrt{\lambda}(4+\lambda)}{6\pi^3} d\lambda. \quad (4.2)$$

As a result, the $\mathbb{H}^3$ part of heat kernel in (2.19) is (cf. (3.4))

$$\overline{K}_{\mathbb{H}^3} = \int_0^\infty d\lambda \frac{1+\lambda}{2\pi^2\sqrt{\lambda}} e^{-t\lambda} = \frac{2(1+2t)}{(4\pi t)^{3/2}}, \quad (4.3)$$

and thus integrating over t , dropping quartic and quadratic divergences and summing over n as in (3.5) we get

$$\mathcal{F}_q = \frac{1+30q^2}{180q^3}, \quad \mathbb{S}_q = -\frac{(1+q)(1+31q^2)}{180q^3}, \quad (4.4)$$

$$a = -\frac{1}{4} S_1 = \frac{4}{45}, \quad C_{T,4} = 160c = 80S'_1 = 16. \quad (4.5)$$

This reproduces the correct value of C_T or c-coefficient for the Maxwell field but not the standard value of the a-coefficient that should be $\frac{31}{180} = \frac{4}{45} + \frac{1}{12}$. As mentioned in section 2, this matching need not be expected to follow automatically when free energy is computed on $S_q^1 \times \mathbb{H}^3$ but one can formally enforce the relation between the $\mathbb{S}_1$ and the Weyl anomaly a-coefficient by shifting $\mathcal{F}_q$ and thus $\mathbb{S}_q$ by a constant as in (2.5):

$$\mathcal{F}_q \rightarrow \mathcal{F}_q - \frac{1}{3} = \frac{1 + 30q^2 - 60q^3}{180q^3}, \quad \mathbb{S}_q \rightarrow \mathbb{S}_q - \frac{1}{3} = -\frac{1 + q + 31q^2 + 91q^3}{180q^3}. \quad (4.6)$$

4.2 ∂^4 gauge vector in $d = 6$

Defined on a curved background, the 6d conformal vector gauge theory has the following Weyl-invariant action [13]

$$I = \int d^6x \sqrt{g} \left[D_\lambda F^{\lambda\mu} D^\nu F_{\nu\mu} - \left(R_{\mu\nu} - \frac{1}{5} R g_{\mu\nu} \right) F^{\mu\lambda} F^\nu{}_\lambda \right], \quad (4.7)$$

where $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$. To compute the corresponding free energy on $S_q^1 \times \mathbb{H}^{d-1}$ it is convenient to choose again the temporal gauge $V_0 = 0$. This leads to the ghost factor $(\det \partial_0^2)^{1/2}$ in the partition function. Using (3.12) the Lagrangian in (4.7) then becomes (here $i, j, \dots = 1, \dots, 5$ are indices of $\mathbb{H}^5$)

$$\mathcal{L} = (\partial_0^2 V_i + D^k F_{ki})^2 + (\partial_0 D^i V_i)^2 - 4 (\partial_0 V_i)^2. \quad (4.8)$$

Change of the variables $V_i \rightarrow (V_i^\perp, \varphi)$,

$$V_i = V_i^\perp + \partial_i \varphi, \quad D^i V_i^\perp = 0, \quad (4.9)$$

introduces the Jacobian factor $(\det \mathbf{D}^2)^{1/2}$ ($\mathbf{D}^2 \equiv D^i D_i$) in the path integral, while the Lagrangian (4.8) takes the form

$$\mathcal{L} = [(-\partial_0^2 + \mathbf{\Delta}'_1) V_i^\perp]^2 - 4 (\partial_0 V_i^\perp)^2 + \varphi \partial_0^2 \mathbf{D}^2 (-\partial_0^2 - \mathbf{D}^2 - 4) \varphi, \quad (4.10)$$

$$\mathbf{\Delta}'_1 V_i^\perp \equiv (-\mathbf{D}^2 g_{ij} + R_{ij}) V^{\perp j} = (-\mathbf{D}^2 - 4) V_i^\perp = (\mathbf{\Delta}_1 + 1) V_i^\perp, \quad (4.11)$$

where $\mathbf{\Delta}_1$ is the $d = 6$ case of the operator defined in (2.16). Integrating over φ we get a factor $[\det(\partial_0^2 \mathbf{D}^2)]^{-1/2}$ (which cancels against the previously mentioned ghost and Jacobian factors) as well as the contribution of the conformal 6d scalar $[\det(-\partial_0^2 - \mathbf{D}^2 - 4)]^{-1/2}$ (see (3.2)).

The remaining 4-derivative operator acting on $V_i^\perp$ in (4.10) factorizes exactly as in the 4-derivative scalar case (3.13) (with $\mathbf{\Delta}_0 \rightarrow \mathbf{\Delta}_1 = \mathbf{\Delta}'_1 - 1$)

$$\mathcal{O}_1^{(4)} = (-\partial_0^2 + \mathbf{\Delta}'_1)^2 + 4\partial_0^2 = [(i\partial_0 + 1)^2 + \mathbf{\Delta}_1] [(i\partial_0 - 1)^2 + \mathbf{\Delta}_1] \quad (4.12)$$

$$= [(i\partial_0)^2 + (\sqrt{\mathbf{\Delta}_1} + i)^2] [(i\partial_0)^2 + (\sqrt{\mathbf{\Delta}_1} - i)^2]. \quad (4.13)$$

As in (3.13), the same factorization as in (4.12),(4.13) is found if one considers the theory (4.7) on $S^1 \times S^5$.²⁰ Using that Δ_1 in (2.16) has the eigenvalue λ , the polynomial $P_{\mathbb{H}}$ in (2.18) is again given by (3.15). The only difference compared to the 4-derivative 6d scalar case is that now the spectral measure is given by the $s = 1$ case of (2.17), i.e. by $d\mu_{1,5}$ in (4.2).

As a result, the free energy $\mathcal{F}_q$ and thus the Rényi entropy and C_T for the 4-derivative vector theory (4.7) is given by the sum of the contribution of the transverse spatial vector (with the kinetic operator $\mathcal{O}_1^{(4)}$) and of the standard 6d conformal scalar, i.e.

$$\mathcal{F}_q(V^{(4)}) = \mathcal{F}_q(\mathbf{V}_{\perp}^{(4)}) + \mathcal{F}_q(\varphi) . \quad (4.14)$$

The scalar contribution was already given in (3.7)–(3.9). The total $\mathbb{H}^5$ heat kernel factor in the resulting free energy is then (cf. (3.28))

$$\overline{K}_{\mathbb{H}^5}(t) = \int_0^\infty d\lambda \frac{\sqrt{\lambda}(4+\lambda)}{6\pi^3} e^{-t\lambda} [1 + 2e^t \cos(2t\sqrt{\lambda})] = \frac{9 - 10t - 32t^2}{(4\pi t)^{5/2}} , \quad (4.15)$$

and thus finally

$$\mathcal{F}_q = \frac{-6 + 35q^2 + 1680q^4}{10080q^5}, \quad \mathbb{S}_q = -\frac{(1+q)(-6 + 29q^2 + 1709q^4)}{10080q^5}, \quad (4.16)$$

$$a = -\frac{1}{96} \mathbb{S}_1 = \frac{433}{24 \times 7!}, \quad C_{T,6} = 3024c_3 = -504\mathbb{S}'_1 = -90 . \quad (4.17)$$

The value of $C_{T,6}$ is the same (1.14) as quoted in the Introduction, found earlier by other methods in [31, 56]. To also reproduce the correct value $a = \frac{275}{8 \times 7!}$ [13] of the a -anomaly for the 4-derivative 6d vector field one needs, as in the $d = 4$ vector case (4.6), to shift $\mathcal{F}_q$ and thus $\mathbb{S}_q$ by the constant term $-\frac{14}{45}$.

5 Fermionic fields

Finally, let us discuss the fermionic fields. We shall first review the computation of free energy and C_T for the standard Dirac fermion and then consider the conformal 3-derivative fermion which is part of the 6d superconformal vector multiplet (1.8).

5.1 $\not{D}$ fermion

The curved space Weyl-invariant action for a standard massless fermion

$$I = i \int d^d x \sqrt{g} \bar{\psi} \not{D} \psi, \quad (5.1)$$

²⁰Due to the change of the sign of the curvature of the spatial part, here $\Delta'_1 = -D^2 + 4$ that has discrete eigenvalues on the sphere, i.e. $\Delta'_1 \rightarrow m^2 + 6m + 8$ with integer $m \geq 0$. The $4\partial_0^2$ term in (4.12) here has flipped sign (as it came from the curvature term in (4.7)) and thus we find that on $S^1 \times S^5$

$$\begin{aligned} \left(-\partial_0^2 + \Delta'_1\right)^2 - 4\partial_0^2 &\rightarrow \left[\left(\frac{n}{q} + i\right)^2 + (m+3)^2\right] \left[\left(\frac{n}{q} - i\right)^2 + (m+3)^2\right] \\ &= \left[\frac{n^2}{q^2} + (m+2)^2\right] \left[\frac{n^2}{q^2} + (m+4)^2\right]. \end{aligned}$$

This leads to the thermal free energy corresponding to the spectrum of dimensions $w_m = m + 2$, $m + 4$ expected from the operator counting on $\mathbb{R} \times S^5$ (as explicitly discussed in [48] in the 4d case).

leads to the following formal expression for its free energy on $S_q^1 \times \mathbb{H}^{d-1}$ in terms of the eigenvalues of the squared operator $(i\mathcal{D})^2 = -\partial_0^2 + (i\mathcal{D})^2$ [58, 59]

$$\mathcal{F}_q = -\text{tr} \log(i\mathcal{D}) = -\frac{1}{2} n_f \bar{V}_{\mathbb{H}^{d-1}} \sum_{n \in \mathbb{Z} + \frac{1}{2}} \int_0^\infty d\mu_{\frac{1}{2}, d-1}(\lambda) \log\left(\frac{n^2}{q^2} + \lambda\right). \quad (5.2)$$

Here n_f is the complex dimension of the spinor space (e.g., $n_f = 2$ for a Weyl fermion in $d = 4$ or MW fermion in $d = 6$) and the sum over half-integer n corresponds to the antiperiodic boundary conditions on the “thermal” circle S^1 . λ is the eigenvalue of the operator $(i\mathcal{D})^2 = -\mathbf{D}^2 + \frac{1}{4}R$ equal to $-\mathbf{D}^2 - \frac{1}{4}(d-1)(d-2)$ on $\mathbb{H}^{d-1}$ (cf. (2.16)). The corresponding spin 1/2 Plancherel measure for even d is [58, 59] (cf. (2.17))

$$d\mu_{\frac{1}{2}, d-1} = \frac{1}{2^{d-2} \pi^{\frac{d-1}{2}} \Gamma(\frac{d-1}{2})} \left| \frac{\Gamma(i\sqrt{\lambda} + \frac{d-1}{2})}{\Gamma(i\sqrt{\lambda} + \frac{1}{2})} \right|^2 d\sqrt{\lambda}, \quad (5.3)$$

$$d\mu_{\frac{1}{2}, 3} = \frac{1+4\lambda}{16\pi^2\sqrt{\lambda}} d\lambda, \quad d\mu_{\frac{1}{2}, 5} = \frac{(1+4\lambda)(9+4\lambda)}{384\pi^3\sqrt{\lambda}} d\lambda, \quad \dots \quad (5.4)$$

Using the proper time representation for the log in (5.2) we get as in (2.19), (2.22)

$$\mathcal{F}_q = -\frac{1}{2} n_f \bar{V}_{\mathbb{H}^{d-1}} \int_0^\infty \frac{dt}{t} K_{S^1}^f(t) \bar{K}_{\mathbb{H}^{d-1}}(t), \quad (5.5)$$

$$K_{S^1}^f(t) = \frac{2\pi q}{(4\pi t)^{1/2}} \sum_{n \in \mathbb{Z}} (-1)^n e^{-\frac{\pi^2 n^2 q^2}{t}}, \quad \bar{K}_{\mathbb{H}^{d-1}}(t) = \int_0^\infty d\mu_{\frac{1}{2}, d-1}(\lambda) e^{-t\lambda}. \quad (5.6)$$

The power UV divergences in the proper time integral should be again subtracted by omitting the $n = 0$ term in the sum. Explicitly, one finds in $d = 4$

$$\bar{K}_{\mathbb{H}^3}(t) = \int_0^\infty d\lambda \frac{1+4\lambda}{4\pi^2\sqrt{\lambda}} e^{-t\lambda} = \frac{2+t}{2(4\pi t)^{3/2}}, \quad (5.7)$$

$$\mathcal{F}_q = \frac{7+30q^2}{2880q^3} n_f, \quad \mathbb{S}_q = -\frac{(1+q)(7+37q^2)}{2880q^3} n_f, \quad (5.8)$$

$$a = -\frac{1}{4} S_1 = \frac{11}{1440} n_f, \quad C_{T,4} = 160 c = 80 S'_1 = 2 n_f, \quad (5.9)$$

and in $d = 6$

$$\bar{K}_{\mathbb{H}^5}(t) = \int_0^\infty d\lambda \frac{\sqrt{\lambda}(1+4\lambda)(9+4\lambda)}{384\pi^3\sqrt{\lambda}} e^{-t\lambda} = \frac{2 - \frac{10}{3}t + \frac{3}{2}t^2}{2(4\pi t)^{5/2}}, \quad (5.10)$$

$$\mathcal{F}_q = -\frac{31+245q^2+945q^4}{483840q^5} n_f, \quad \mathbb{S}_q = \frac{(1+q)(31+276q^2+1221q^4)}{483840q^5} n_f, \quad (5.11)$$

$$a = -\frac{1}{96} S_1 = -\frac{191}{576 \times 7!} n_f, \quad C_{T,6} = 3024 c_3 = -504 S'_1 = 3 n_f. \quad (5.12)$$

Eqs. (5.9), (5.12) give the correct known values of the a and c Weyl anomaly coefficients in $d = 4$ [60] and in $d = 6$ [3] and the values of C_T also agree with the general expression for $C_{T,d}(\psi)$ given in (1.13). The expressions for the Rényi entropy agree with [38, 47].

5.2 Conformal $\not{D}^3$ fermion

The Weyl-invariant operator for a 3-derivative fermion was first found in the context of extended conformal supergravity [61] in $d = 4$ [52] (for Majorana fermions)

$$I = i \int d^4x \sqrt{g} \bar{\psi} \left[\not{D}^3 + \left(R^{\mu\nu} - \frac{1}{6} R g^{\mu\nu} \right) \gamma_\mu D_\nu \right] \psi. \quad (5.13)$$

In $d = 6$ the analogous 3-derivative operator was recently found in [15]²¹

$$\mathcal{O}^{(3)} = \not{D}^3 + \frac{1}{2} \left(R^{\mu\nu} - \frac{1}{10} R g^{\mu\nu} \right) \gamma_\mu D_\nu + \frac{1}{10} \gamma^\mu D_\mu R. \quad (5.14)$$

The generalization of (5.13), (5.14) to any d reads

$$\mathcal{O}^{(3)} = \not{D}^3 + 2P^{\mu\nu} \gamma_\mu D_\nu + \gamma^\mu D_\mu P, \quad (5.15)$$

where $P_{\mu\nu}$ is the Schouten tensor as in (3.24). On $S^1 \times \mathbb{H}^{d-1}$ we can use (3.12) to get the following explicit form of (5.15)

$$\mathcal{O}^{(3)} = \not{D}^3 + \not{D} - 2\not{D} = (\gamma^0 \partial_0 + \not{D})^3 + \gamma^0 \partial_0 - \not{D}. \quad (5.16)$$

As a result, its square factorizes in a d -independent manner just like in the 4-derivative scalar case in (3.13) (cf. also (4.12))²²

$$\begin{aligned} (i\mathcal{O}^{(3)})^2 &= -\partial_0^2 (\partial_0^2 + \not{D}^2 + 1)^2 - \not{D}^2 (\partial_0^2 + \not{D}^2 - 1)^2 \\ &= (-\partial_0^2 - \not{D}^2) [(i\partial_0 + 1)^2 - \not{D}^2] [(i\partial_0 - 1)^2 - \not{D}^2] \end{aligned} \quad (5.17)$$

$$= (-\partial_0^2 - \not{D}^2) [(i\partial_0)^2 + (i\not{D} + i)^2] [(i\partial_0)^2 + (i\not{D} - i)^2]. \quad (5.18)$$

The corresponding eigenvalue polynomial (2.18) is then the product of the standard fermion part and the same factor as in the ∂^4 scalar case in (3.15) (and also has a similar structure as the result in the 4-derivative vector case in (4.13)).

Using the expression for the spin 1/2 spectral measure in (5.4) and starting with (5.5) we then find in $d = 4$ (cf. (3.16)–(3.18))

$$\bar{K}_{\mathbb{H}^3}(t) = \int_0^\infty d\lambda \frac{1 + 4\lambda}{4\pi^2 \sqrt{\lambda}} e^{-t\lambda} [1 + 2e^t \cos(2t\sqrt{\lambda})] = \frac{6 - 5t}{2(4\pi t)^{3/2}}, \quad (5.19)$$

$$\mathcal{F}_q = \frac{7 - 50q^2}{960q^3} n_f, \quad \mathbb{S}_q = \frac{(1 + q)(-7 + 43q^2)}{960q^3} n_f, \quad (5.20)$$

$$a = -\frac{1}{4} \mathbb{S}_1 = -\frac{3}{160} n_f, \quad C_{T,4} = 160c = 80 \mathbb{S}'_1 = -\frac{2}{3} n_f. \quad (5.21)$$

The values of $a = -\frac{3}{80}$ and $c = -\frac{1}{120}$ for a Majorana fermion ($n_f = 2$) agree with the ones found by direct computation in [52, 54].²³

²¹We thank D. Butter for pointing this out to us and a clarifying discussion.

²²The factorization of the operator (5.13) on $S^1 \times S^3$ was observed in [51].

²³In the notation of table 6.1 in [54] one has for the 3-derivative Λ -spinor: $\beta_1 = \frac{7}{240}$, $\beta_2 = -\frac{1}{60}$ with $a = -\beta_1 + \frac{1}{2}\beta_2$, $c = \frac{1}{2}\beta_2$. To compare, for a real 4-derivative scalar $\beta_1 = \frac{1}{90}$, $\beta_2 = -\frac{2}{15}$, giving $a = -\frac{7}{90}$, $c = -\frac{1}{15}$ in agreement with values given earlier in (3.18) (note that the ∂^4 scalar φ in table 6.1 is complex).

In 6 dimensions we get (cf. (3.19)–(3.21)) and (5.10)–(5.12))

$$\overline{K}_{\mathbb{H}^5}(t) = \int_0^\infty d\lambda \frac{\sqrt{\lambda}(1+4\lambda)(9+4\lambda)}{384\pi^3\sqrt{\lambda}} e^{-t\lambda} [1 + 2e^t \cos(2t\sqrt{\lambda})] = \frac{3-3t-t^2}{(4\pi t)^{5/2}}, \quad (5.22)$$

$$\mathcal{F}_q = \frac{-31 + 147q^2 + 735q^4}{161280q^5} n_f, \quad \mathbb{S}_q = -\frac{(1+q)(-31+116q^2+851q^4)}{161280q^5} n_f, \quad (5.23)$$

$$a = -\frac{1}{96} S_1 = \frac{39}{64 \times 7!} n_f, \quad C_{T,6} = 3024 c_3 = -504 S'_1 = -\frac{18}{5} n_f. \quad (5.24)$$

Thus for a 6d MW fermion with $n_f = 2$ we get $a = \frac{39}{32 \times 7!}$ in agreement with the value found in [13] while

$$C_{T,6}(\psi^{(3)}) = -\frac{36}{5}. \quad (5.25)$$

This confirms the value corresponding to ξ_{vz} in (1.17).

To emphasize that (5.25) is a result of a rather non-trivial computation, in appendix A we shall present an alternative way of arriving at (5.25) based on the approach that does not use the proper time representation and utilizes the first way (5.17) of factorizing the square of the 3-derivative spinor operator (5.16). Surprisingly, a naive application of this alternative approach leads precisely to the value of $C_{T,6}$ in (1.16) corresponding to ξ_{BT} in (1.6).

It is possible to generalize the $d = 4$ (5.21) and $d = 6$ (5.24) expressions for C_T of the 3-derivative conformal fermion to any dimension d obtaining the following counterpart of the general d expressions for C_T of the standard scalar and spinor (1.13), 4-derivative scalar (3.22) and 6-derivative scalar (3.31)

$$C_{T,d}(\psi^{(3)}) = -n_f \frac{d(d^2 + d - 18)}{2(d-1)(d-2)} = -\frac{d^2 + d - 18}{(d-1)(d-2)} C_{T,d}(\psi). \quad (5.26)$$

It would be interesting to reproduce (5.26) in alternative flat-space approaches like the ones used in the higher-derivative scalar cases in [55] and [30].

6 Conformal anomaly of 6d higher derivative vector multiplet from Seeley-DeWitt coefficient

Let us now rederive the above results for the c_3 coefficient in (1.1) for the fields of the vector multiplet (1.8) by the same direct method as used in [3] to compute the conformal anomalies of the standard fields in the (2,0) tensor multiplet — using the general expression [62] for the $b_6 = \langle T_\mu^\mu \rangle$ Seeley-DeWitt coefficient of the 2nd order Laplace-type operator $\Delta = -D^2 + X$.

The two key observations that allow one to do this are:²⁴ (i) like the higher derivative conformal scalar operators [63], the 4-derivative vector operator in (4.7) and the square of the 3-derivative spinor operator in (5.14) factorize into a product of 2nd order Laplacians on an Einstein space $R_{\mu\nu} = \frac{1}{6} R g_{\mu\nu}$ and thus their anomalies can be readily computed; (ii) considering a general Einstein background is sufficient to determine all the 4 anomaly coefficients a , c_i in (1.1). The special cases were already considered before — the 6-sphere (allowing to find the a -coefficient [13]) and the Ricci-flat space (allowing to fix the c_i up

²⁴We are grateful to D. Diaz for suggesting this approach to us.

to one free parameter [8]). On a general Einstein background one may have both the scalar curvature and the Weyl tensor non-zero so that one may capture the $RC^{\mu\nu\lambda\rho}C_{\mu\nu\lambda\rho}$ terms in the expression for b_6 and thus determine one more combination of the Weyl anomaly coefficients.

As a result, in addition to the anomaly coefficients for the conformally coupled 2-derivative 6d scalar ($\Delta = -D^2 + \frac{1}{5}R$) obtained in [3]

$$a = -\frac{5}{72 \times 7!}, \quad c_1 = -\frac{28}{3 \times 7!}, \quad c_2 = \frac{5}{3 \times 7!}, \quad c_3 = \frac{2}{7!}, \quad (6.1)$$

we find for the 4-derivative conformal vector (4.7)

$$a = \frac{275}{8 \times 7!}, \quad c_1 = \frac{2716}{7!}, \quad c_2 = \frac{911}{7!}, \quad c_3 = -\frac{150}{7!}, \quad (6.2)$$

and 3-derivative conformal MW spinor

$$a = \frac{39}{32 \times 7!}, \quad c_1 = \frac{448}{3 \times 7!}, \quad c_2 = \frac{110}{3 \times 7!}, \quad c_3 = -\frac{12}{7!}. \quad (6.3)$$

As a result, the anomaly coefficients for the higher derivative vector multiplet (1.8) are found to be²⁵

$$a = \frac{1757}{48 \times 7!}, \quad c_1 = \frac{8960}{3 \times 7!}, \quad c_2 = \frac{2968}{3 \times 7!}, \quad c_3 = -\frac{168}{7!}. \quad (6.4)$$

The values of a-coefficient were found already in the special case of S^6 background in [13]. The (1,0) supersymmetry constraint $c_1 - 2c_2 + 6c_3 = 0$ in (1.2) and the relation $c_1 + 4c_2 = \frac{62}{45}$ were obtained by considering the Ricci-flat background in [8]. The coefficients c_3 in (6.2) and (6.3) (or $C_{T,6} = 3024c_3$ in (1.11)) are the same as the ones found above in (4.17) and (5.24) from the computation of free energy on $S^1 \times \mathbb{H}^5$. The values of c_i in (6.4) thus agree with (1.10) for the value of $\xi = -\frac{8}{9}$ found in [28] providing another independent confirmation of (1.7).

In appendix C we shall present the extension of the computation presented in this section to more general (1,0) superconformal multiplets with maximal spin 1 with the results that are again in agreement with (1.5) with (1.7).

Below we shall follow the notation in [3] and use that for an Einstein background one has $D_\mu R = 0$ and $D^\mu C_{\mu\nu\lambda\rho} = 0$ so that many terms in the general expression in b_6 simplify. The E_6 and $I_{1,2,3}$ invariants in (1.1) defined in [3] take the form²⁶

$$\begin{aligned} E_6 &= -\frac{16}{75}R^3 - \frac{8}{5}RC_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + 64C_\alpha{}^\mu{}_\gamma{}^\nu C^{\alpha\beta\gamma\delta}C_{\beta\mu\delta\nu} - 32C_{\alpha\beta}{}^{\mu\nu}C^{\alpha\beta\gamma\delta}C_{\gamma\delta\mu\nu}, \\ I_1 &= -C_\alpha{}^\mu{}_\gamma{}^\nu C^{\alpha\beta\gamma\delta}C_{\beta\mu\delta\nu}, \quad I_2 = C_{\alpha\beta}{}^{\mu\nu}C^{\alpha\beta\gamma\delta}C_{\gamma\delta\mu\nu}, \end{aligned} \quad (6.5)$$

²⁵This corrects the expressions for c_i in eq. (2.3) in [8] that assumed the wrong value of ξ in (1.6).

²⁶The only non zero total derivative terms among $C_{1,\dots,7}$ in [3] here are

$$\begin{aligned} C_5 &= C^{\alpha\beta\gamma\delta}D^2C_{\alpha\beta\gamma\delta} + (D_\mu C_{\alpha\beta\gamma\delta})^2 = D_\mu(C_{\alpha\beta\gamma\delta}D^\mu C^{\alpha\beta\gamma\delta}), \\ C_7 &= \frac{1}{12}RC_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} - C_\alpha{}^\mu{}_\gamma{}^\nu C^{\alpha\beta\gamma\delta}C_{\beta\mu\delta\nu} - \frac{1}{4}C_{\alpha\beta}{}^{\mu\nu}C^{\alpha\beta\gamma\delta}C_{\gamma\delta\mu\nu} + \frac{1}{4}(D_\mu C_{\alpha\beta\gamma\delta})^2. \end{aligned}$$

$$I_3 = -\frac{6}{5}RC_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + 8C_{\alpha}^{\mu}{}_{\gamma}{}^{\nu}C^{\alpha\beta\gamma\delta}C_{\beta\mu\delta\nu} + 2C_{\alpha\beta}{}^{\mu\nu}C^{\alpha\beta\gamma\delta}C_{\gamma\delta\mu\nu} \\ + 6C^{\alpha\beta\gamma\delta}D^2C_{\alpha\beta\gamma\delta} + 3(D_{\mu}C_{\alpha\beta\gamma\delta})^2.$$

Given a general scalar Laplacian

$$\Delta_0(\kappa) \equiv -D^2 + \kappa\bar{R}, \quad \bar{R} = \frac{1}{d(d-1)}R = \frac{1}{30}R, \quad (6.6)$$

the corresponding b_6 coefficient computed as in [3] is found to be ($\bar{b}_6 \equiv (4\pi)^3 b_6$)

$$7!\bar{b}_6[\Delta_0(\kappa)] = \left(\frac{2278}{675} + \frac{56}{25}\kappa + \frac{7}{15}\kappa^2 + \frac{7}{225}\kappa^3\right)R^3 \\ + \left(\frac{194}{45} + \frac{14}{15}\kappa\right)RC_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + \frac{80}{9}C_{\alpha}^{\mu}{}_{\gamma}{}^{\nu}C^{\alpha\beta\gamma\delta}C_{\beta\mu\delta\nu} \\ + \frac{44}{9}C_{\alpha\beta}{}^{\mu\nu}C^{\alpha\beta\gamma\delta}C_{\gamma\delta\mu\nu} + 12C^{\alpha\beta\gamma\delta}D^2C_{\alpha\beta\gamma\delta} + 9(D_{\mu}C_{\alpha\beta\gamma\delta})^2. \quad (6.7)$$

Expressing this in terms of the invariants in (1.1) using (6.5) and ignoring the total derivative terms we find that in the special case of the conformally coupled scalar when $\kappa = \frac{1}{4}d(d-2) = 6$ we reproduce the coefficients in (6.1).

The 4-derivative vector operator in (4.7) restricted to an Einstein background factorizes in the same way as in the sphere case discussed in [13]: the action depends only on the transverse part $V_{\mu}^{\perp}$ of the vector and reduces to the integral of $V^{\perp}\Delta_{1\perp}(7)\Delta_{1\perp}(5)V^{\perp}$. The resulting partition function is then given by

$$Z(V^{(4)}) = \left[\frac{\det\Delta_0(0)}{\det\Delta_{1\perp}(7)\det\Delta_{1\perp}(5)}\right]^{1/2} = \left[\frac{\det\Delta_0(2)[\det\Delta_0(0)]^2}{\det\Delta_1(7)\det\Delta_1(5)}\right]^{1/2}, \quad (6.8)$$

where like in (6.6) we defined $\Delta_1(\kappa)V_{\mu} \equiv (-D^2 + \kappa\bar{R})V_{\mu}$ and $\Delta_{1\perp}$ is Δ_1 restricted to $V_{\mu}^{\perp}$. The standard vector Laplacian on an Einstein background is $(-D^2g_{\mu\nu} + R_{\mu\nu})V^{\nu} = \Delta_1(5)V_{\mu}$. In (6.8) we used that $\det\Delta_1(\kappa) = \det\Delta_{1\perp}(\kappa)\det\Delta_0(\kappa-5)$.²⁷ To find the b_6 coefficient for the vector Laplacian $\Delta_1(\kappa)$ from the general expressions in [3, 62] one is to note that here the covariant derivative contains the extra vector connection part with the “internal” curvature $(\mathcal{F}_{ij})_{\alpha}{}^{\beta} = R_{ij\alpha}{}^{\beta}$. The analog of (6.7) then reads

$$7!\bar{b}_6[\Delta_1(\kappa)] = \left(\frac{3394}{225} + \frac{938}{75}\kappa + \frac{14}{5}\kappa^2 + \frac{14}{75}\kappa^3\right)R^3 \\ - \frac{36}{5}\left(6 + \frac{7}{6}\kappa\right)RC_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + \frac{80}{9}C_{\alpha}^{\mu}{}_{\gamma}{}^{\nu}C^{\alpha\beta\gamma\delta}C_{\beta\mu\delta\nu} \\ - \frac{164}{3}C_{\alpha\beta}{}^{\mu\nu}C^{\alpha\beta\gamma\delta}C_{\gamma\delta\mu\nu} - 96C^{\alpha\beta\gamma\delta}D^2C_{\alpha\beta\gamma\delta} - 58(D_{\mu}C_{\alpha\beta\gamma\delta})^2. \quad (6.9)$$

This generalizes the expression found in [3] in the special case of the standard vector Laplacian (corresponding to $\kappa = 5$).

²⁷If $V_{\mu} = V_{\mu}^{\perp} + \partial_{\mu}\varphi$, then on an Einstein background one has (dropping total derivatives) $V^{\mu}\Delta_1(\kappa)V_{\mu} = V^{\mu\perp}\Delta_{1\perp}(\kappa)V_{\mu}^{\perp} + \varphi\Delta_0(\kappa-5)\varphi$.

Eqs. (6.7) and (6.9) are all we need to compute the anomalies of the 4-derivative conformal vector since according to (6.8)

$$b_6(V^{(4)}) = b_6[\Delta_1(7)] + b_6[\Delta_1(5)] - b_6[\Delta_0(2)] - 2b_6[\Delta_0(0)] . \quad (6.10)$$

As a result, one finds the coefficients given in (6.2).

The 3-derivative conformal spinor operator (5.14) restricted to an Einstein background becomes $\mathcal{O}^{(3)} = \not{D}^3 + \frac{1}{30}R\not{D}$ so that its square factorizes as

$$(i\mathcal{O}^{(3)})^2 = \left(-D^2 + \frac{1}{4}R\right) \left(-D^2 + \frac{13}{60}R\right) = \Delta_{\frac{1}{2}}\left(\frac{15}{2}\right) \left[\Delta_{\frac{1}{2}}\left(\frac{13}{2}\right)\right]^2, \quad (6.11)$$

where $\Delta_{\frac{1}{2}}(\kappa) \equiv -D^2 + \kappa\bar{R}$ acting on spinors has D being spinor covariant derivative with the corresponding “internal” curvature $\mathcal{F}_{ij} = \frac{1}{4}R_{ijab}\gamma^{ab}$. The counterpart of (6.7) and (6.9) in the spinor case is then found to be²⁸

$$\begin{aligned} 7!(n_f)^{-1}\bar{b}_6[\Delta_{\frac{1}{2}}(\kappa)] = & -\left(\frac{7369}{2700} + \frac{637}{300}\kappa + \frac{7}{15}\kappa^2 + \frac{7}{225}\kappa^3\right)R^3 \\ & + \left(\frac{389}{90} + \frac{49}{60}\kappa\right)R C_{\alpha\beta\gamma\delta}C^{\alpha\beta\gamma\delta} + \frac{109}{9}C_{\alpha}^{\mu}{}_{\gamma}{}^{\nu}C^{\alpha\beta\gamma\delta}C_{\beta\mu\delta\nu} \\ & + \frac{101}{18}C_{\alpha\beta}{}^{\mu\nu}C^{\alpha\beta\gamma\delta}C_{\gamma\delta\mu\nu} + 9C^{\alpha\beta\gamma\delta}D^2C_{\alpha\beta\gamma\delta} + 5(D_{\mu}C_{\alpha\beta\gamma\delta})^2. \end{aligned} \quad (6.12)$$

For the standard spinor field with the squared operator being $\Delta_{\frac{1}{2}}(\kappa)$ with $\kappa = \frac{15}{2}$ eq. (6.12) reproduces the coefficients in (1.1) found in [3], i.e. for MW spinor with $n_f = 2$ we get $a = -\frac{191}{288 \times 7!}$, $c_1 = -\frac{224}{3 \times 7!}$, $c_2 = -\frac{8}{7!}$, $c_3 = \frac{10}{7!}$. Using that for the 3-derivative spinor we have from (6.11) $b_6(\psi^{(3)}) = b_6[\Delta_{\frac{1}{2}}(\frac{15}{2})] + b_6[\Delta_{\frac{1}{2}}(\frac{13}{2})]$, we find that the corresponding conformal anomaly coefficients are given by (6.3).

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²⁸Here we included the -1 fermion sign factor. n_f is the complex dimension of the spinor space as in (1.13) equal to 2 in the 6d MW spinor case.

A Alternative computational scheme for free energy and C_T of 3-derivative conformal spinor field

Let us start with the standard fermion case and compute the corresponding free energy without using the proper time representation for the log factor in (5.2) and doing the sum over n first and the integral over λ last. The sum over n requires a regularization prescription and we shall adopt the same one as used, e.g., in [38]²⁹

$$\sum_{n=-\infty}^{\infty} \log \left[\frac{(n+\gamma)^2}{q^2} + m^2 \right] \Big|_{\text{reg}} = \log [2 \cosh(2\pi q m) - 2 \cos(2\pi \gamma)] . \quad (\text{A.1})$$

It is important to stress that because of the regularization involved this relation directly applies for $\gamma < 1$ (and $q^2 m^2 > -1$ if $\gamma = 0$); the expressions found using (A.1) with parameters outside that range should be defined by an analytic continuation. The case of half-integer summation in (5.2) corresponds to the legitimate values $\gamma = \frac{1}{2}$, $m^2 = \lambda \geq 0$. As a result, we get

$$\mathcal{F}_q = -\frac{1}{2} n_f \bar{V}_{\mathbb{H}^{d-1}} \int_0^\infty d\mu_{\frac{1}{2}, d-1}(\lambda) \log [2 \cosh(2\pi q \sqrt{\lambda}) + 2] . \quad (\text{A.2})$$

The integral over λ is divergent at large λ ; omitting the power divergent part proportional to q one reproduces the same $d = 4$ and $d = 6$ expressions as in (5.8) and (5.11). The second q -derivative of $\mathcal{F}_q$ is always finite and using (2.8), we then reproduce from (A.2) the standard result for $C_{T,d}(\psi) = \frac{1}{2} n_f d$ given in (1.13).

In the case of the 3-derivative spinor we may use the factorized expression (5.17) for the square of its kinetic operator leading to the following expression for the free energy that generalizes (5.2)

$$\mathcal{F}_q = -\text{tr} \log(i \mathcal{O}^{(3)}) = -\frac{1}{2} n_f \bar{V}_{\mathbb{H}^{d-1}} \int_0^\infty d\mu_{\frac{1}{2}, d-1}(\lambda) K(\lambda, q) , \quad (\text{A.3})$$

$$K(\lambda, q) \equiv \sum_{n \in \mathbb{Z} + \frac{1}{2}} \log \left(\left[\frac{n^2}{q^2} + \lambda \right] \left[\frac{(n+q)^2}{q^2} + \lambda \right] \left[\frac{(n-q)^2}{q^2} + \lambda \right] \right) . \quad (\text{A.4})$$

Computing the sum in $K(\lambda, q)$ using the prescription (A.1) (with γ equal to $\frac{1}{2}$, $q + \frac{1}{2}$, $q - \frac{1}{2}$) we find the following analog of (A.2)

$$\mathcal{F}_q = -\frac{1}{2} n_f \bar{V}_{\mathbb{H}^{d-1}} \int_0^\infty d\mu_{\frac{1}{2}, d-1}(\lambda) \left(\log [2 \cosh(2\pi q \sqrt{\lambda}) + 2] + 2 \log [2 \cosh(2\pi q \sqrt{\lambda}) + 2 \cos(2\pi q)] \right) . \quad (\text{A.5})$$

Taking the second derivative of (A.5) over q at $q = 1$ and computing the resulting finite integral over λ we find, according to (2.8), the following expression for the C_T coefficient

²⁹This relation may be justified, e.g., by first taking the derivative over m , then doing the convergent sum using $\sum_{n=-\infty}^{\infty} \frac{1}{[(n+\gamma)^2 + q^2 m^2]} = \frac{\pi \sinh(2\pi q m)}{q m [\cosh(2\pi q m) - \cos(2\pi \gamma)]}$, and finally integrating back over m (assuming also that $\sum_{n=-\infty}^{\infty} c = 0$). Note that the choice of integration constants or regularization involved may break the formal invariance under the integer shifts of γ .

for the 3-derivative conformal fermion in d dimensions (with $n_f = 2$ in $d = 4$ and $d = 6$)

$$C_{T,d}(\psi^{(3)}) = -n_f \frac{d(5d+11)}{2(d-1)}, \quad \text{i.e.} \quad C_{T,4}(\psi^{(3)}) = -\frac{124}{3}, \quad C_{T,6}(\psi^{(3)}) = -\frac{246}{5}. \quad (\text{A.6})$$

Remarkably, the $d = 6$ value is precisely the one in (1.16) corresponding to ξ_{BT} in (1.6). However, a warning sign is that the $d = 4$ value disagrees with the correct one $C_{T,4} = -\frac{4}{3}$ in (5.21) corresponding to $c = -\frac{1}{120}$ found by direct computation in [52, 54].

This suggests some problem with the above computation. Indeed, while the representation (A.2) for the free energy of the standard fermion is true for any q , the expression (A.5) that was obtained using (A.1) with $\gamma = q \pm \frac{1}{2}$ is formally valid only for $0 \leq |q| < \frac{1}{2}$. It cannot thus be differentiated directly at $q = 1$ and this is the reason why the resulting values of $C_T \sim \mathcal{F}_1''$ or (A.6) are not correct.

The correct procedure is to first evaluate (A.5) for $0 \leq |q| < \frac{1}{2}$, then analytically extend the resulting expression for $\mathcal{F}_q$ to all values of q and finally differentiate it over q obtaining, in particular, the corresponding Rényi entropy (2.4) and $C_T \sim \mathcal{F}_1''$ in (2.8). The results will then agree with (5.20), (5.21) and (5.23), (5.24) found using the heat-kernel regularization approach used in the main text.

To see this explicitly let us note that the $+2 \cos(2\pi q) = -2 \cos[2\pi(q \pm \frac{1}{2})]$ term in (A.5) originated from the $\gamma = q \pm \frac{1}{2}$ shifts in (A.1). To make the use of (A.1) legitimate we may first formally replace this shift by $\gamma = \frac{q}{k} \pm \frac{1}{2}$, evaluate (A.5) for $k > 2$ and then analytically continue $k \rightarrow 1$ in the final result. Replacing $\cos(2\pi q) \rightarrow \cos(2\pi \frac{q}{k})$ in (A.5) we find after computing the second derivative of (A.5) at $q = 1$ in $d = 4, 6$ for $k > 2$ (see (2.9))

$$C_{T,4}(\psi^{(3)}) = 40\mathcal{F}_1'' = \left(6 - \frac{20}{3k^2}\right) n_f, \quad C_{T,6}(\psi^{(3)}) = -252\mathcal{F}_1'' = \left(9 - \frac{161}{10k^2} + \frac{7}{2k^4}\right) n_f. \quad (\text{A.7})$$

These expressions indeed reproduce the correct values $C_{T,4} = -\frac{2}{3}n_f$ in (5.21) and $C_{T,6} = -\frac{18}{5}n_f$ in (5.21) after the analytic continuation to $k = 1$.³⁰

B 2-derivative non-gauge conformal vector

Here we shall follow [56, 64] and consider a non-unitary theory described by 2-derivative vector field with conformal but not gauge-invariant action for $d \neq 4$.³¹ The corresponding Weyl-invariant curved space action is

$$I = - \int d^d x \sqrt{g} \left[D^\mu V^\nu D_\mu V_\nu - \frac{4}{d} (D^\mu V_\mu)^2 + \frac{2}{d-2} R^{\mu\nu} V_\mu V_\nu + \frac{d(d-4)}{4(d-1)(d-2)} R V^\mu V_\mu \right]. \quad (\text{B.1})$$

It is equivalent to the standard Maxwell action for $d = 4$. The corresponding C_T coefficient found in [56] is

$$C_{T,d \neq 4}(V^{(2)}) = \frac{d^2}{d-1}, \quad C_{T,4}(V^{(2)}) = 16. \quad (\text{B.2})$$

³⁰Note that for $k \rightarrow \infty$ the expressions in (A.7) become 3 times the standard fermion values in (5.9) and (5.12). The reason for this is that in this limit the $\pm q \rightarrow \pm \frac{q}{k}$ shifts of n in (A.4) disappear and we get the 3rd power of the standard fermion expression under the log.

³¹Partition function of a similar 2-derivative spin 2 theory was discussed in [57].

Specifying to the case of the $S^1 \times \mathbb{H}^{d-1}$ background and separating V_0 and $V_i = V_i^\perp + \partial_i \varphi$ components as in (4.9) we find that the corresponding partition function has two contributions: one from $V_i^\perp$ and one corresponding to the ∂^4 conformal scalar in $d \neq 4$. The scalar part is absent in $d = 4$ due to gauge invariance that is then present in (B.1) (cf. section 4.1 and [48]).

From (B.1) we get the following mixed Lagrangian for $\chi \equiv V_0$, φ and $V_i^\perp$

$$\begin{aligned} \mathcal{L} = & (D_\mu \chi)^2 + (D_\mu V_i^\perp + D_\mu D_i \varphi)^2 - \frac{4}{d} (\partial_0 \chi + \mathbf{D}^2 \varphi)^2 \\ & - 2 (V_i^\perp + D_i \varphi)^2 - \frac{d(d-4)}{4} [\chi^2 + (V_i^\perp + D_i \varphi)^2] \equiv \mathcal{L}(\chi, \varphi) + \mathcal{L}(\mathbf{V}_\perp^{(2)}), \end{aligned} \quad (\text{B.3})$$

where

$$\mathcal{L}(\mathbf{V}_\perp^{(2)}) = V_i^\perp (-\partial_0^2 + \Delta_1) V_i^\perp, \quad \Delta_1 = -\mathbf{D}^2 - \frac{(d-2)^2}{4} - 1. \quad (\text{B.4})$$

Using that on $S^1 \times \mathbb{H}^{d-1}$ we have $D_i D^2 D_i \varphi = \partial_0^2 \mathbf{D}^2 + (\mathbf{D}^2)^2 - (d-2) \mathbf{D}^2$, we obtain for the scalar part of (B.3)

$$\begin{aligned} \mathcal{L}(\chi, \varphi) = & \chi \left[\frac{d-4}{d} \partial_0^2 - \mathbf{D}^2 - \frac{d(d-4)}{4} \right] \chi + \frac{8}{d} \chi \partial_0 \mathbf{D}^2 \varphi \\ & + \varphi \mathbf{D}^2 \left[\partial_0^2 + \frac{d-4}{d} \mathbf{D}^2 + \frac{(d-4)^2}{4} \right] \varphi. \end{aligned} \quad (\text{B.5})$$

Integrating over χ and φ in the path integral, their contribution can be represented in terms of the determinant of the following 6-order scalar operator

$$\mathcal{O}^{(6)} = -\frac{d-4}{d} \mathbf{D}^2 \left[\partial_0^4 + \frac{d^2 - 4d + 8}{2} \partial_0^2 + \left(2\partial_0^2 + \frac{d(d-4)}{2} \right) \mathbf{D}^2 + (\mathbf{D}^2)^2 \right]. \quad (\text{B.6})$$

The determinant of the $\mathbf{D}^2$ factor cancels against the Jacobian of the change of variables $V_i \rightarrow V_i^\perp + \partial_i \varphi$, while the remaining 4-order scalar operator is equivalent to the conformal ∂^4 one which factorizes as in (3.13) with the eigenvalues given in (3.15).

As a result, we find that

$$C_{T,4}(V^{(2)}) = C_{T,4}(\mathbf{V}_\perp^{(2)}), \quad C_{T,d \neq 4}(V^{(2)}) = C_{T,d}(\mathbf{V}_\perp^{(2)}) + C_{T,d}(\varphi^{(4)}). \quad (\text{B.7})$$

In view of the expression (3.22) for $C_{T,d}(\varphi^{(4)})$, to match (B.2) we should thus get

$$C_{T,4}(\mathbf{V}_\perp^{(2)}) = 16, \quad C_{T,d \neq 4}(\mathbf{V}_\perp^{(2)}) = \frac{d(d^2 + 8)}{(d-1)(d-2)}. \quad (\text{B.8})$$

The transverse spatial vector part of the free energy that follows from (B.4) is given by (see (2.16) for $s = 1$, (2.17) and (A.1))³²

$$\begin{aligned} \mathcal{F}_q(\mathbf{V}_\perp^{(2)}) &= \frac{1}{2} \bar{V}_{\mathbb{H}^{d-1}} \sum_{n=-\infty}^{\infty} \int_0^\infty d\mu_{1,d-1}(\lambda) \log \left(\frac{n^2}{q^2} + \lambda \right) \\ &= \bar{V}_{\mathbb{H}^{d-1}} \int_0^\infty d\mu_{1,d-1}(\lambda) \log [2 \sinh(\pi q \sqrt{\lambda})]. \end{aligned} \quad (\text{B.9})$$

³²Here we use (A.1) with $\gamma = 0$ so the result is equivalent to the one in the heat kernel approach used in the main text.

Field	SO(6)	SU(2) _R	Δ'
φ	(0, 0, 0)	p + 1	2p
ψ ⁺	($\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$)	p	2p + $\frac{1}{2}$
V _μ	(1, 0, 0)	p − 1	2p + 1
ψ [−]	($\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}$)	p − 2	2p + $\frac{3}{2}$
φ'	(0, 0, 0)	p − 3	2p + 2

Table 1. Short multiplets $V_p^{(1,0)}$ of $\text{OSp}(2,6|2)$ that appear in tensor product of p copies of $(1,0)$ doubleton (hypermultiplet) representation.

Computing the corresponding C_T according to (2.8) we find

$$C_{T,4}(\mathbf{V}_\perp^{(2)}) = 16, \quad C_{T,6}(\mathbf{V}_\perp^{(2)}) = \frac{66}{5}, \quad C_{T,8}(\mathbf{V}_\perp^{(2)}) = \frac{96}{7}, \quad \text{etc.}, \quad (\text{B.10})$$

in agreement with (B.8). This provides an alternative derivation of (B.2).

C Conformal anomalies of general higher derivative short superconformal 6d vector multiplets

The calculation of the full 6d conformal anomaly of the $V^{(1,0)}$ multiplet from the Seeley-DeWitt coefficients on an Einstein background presented in section 6 may be generalized to other $(1,0)$ superconformal vector multiplets. $V^{(1,0)} \equiv V_{p=2}^{(1,0)}$ is the lowest member of a family of multiplets $V_p^{(1,0)}$ ($p = 2, 3, 4, \dots$) that contain scalars, spinors and vectors with p -dependent higher-derivative kinetic terms.

In terms of $\text{OSp}(2,6|2)$ representations [65–67] the hypermultiplet $S^{(1,0)}$ is a doubleton ultra-short representation [68]. New (possibly massive) conformal representations are obtained by tensoring p copies of $S^{(1,0)}$. The resulting multiplets $V_p^{(1,0)}$ are short with the maximal spin equal to 1. The structure of these multiplets [69] is shown in table 1. Here Δ' is the scaling dimension of the conformal group $\text{SO}(2,6)$ related to the canonical dimension of the corresponding 6d field Φ by $\dim \Phi = 6 - \Delta'$. We indicated also the $\text{SU}(2)$ R-symmetry representations. $\psi^\pm$ are positive/negative chirality MW spinors while φ and φ' are scalars. The vector V_μ has conformal but not gauge invariant action for $p > 2$. From the canonical dimensions one can determine the number of derivatives in kinetic terms in the corresponding 6d Lagrangian that has the following schematic form

$$\mathcal{L} = \varphi \square^{2p-3} \varphi + \bar{\psi}^+ \not{\partial}^{4p-5} \psi^+ + V_\mu \square^{2p-2} V_\mu + \bar{\psi}^- \not{\partial}^{4p-3} \psi^- + \varphi' \square^{2p-1} \varphi', \quad (\text{C.1})$$

where fields transform under $\text{SU}(2)_R$ according to table 1.

The higher derivative operators in (C.1) should have a covariant and Weyl-invariant generalization to curved background. Remarkably, just as for the $p = 2$ case discussed in section 6 all these operators for conformal fields in table 1 factorize on an Einstein space background. Let us denote by $\Phi^{(n)}$ a field with n derivatives in the kinetic term. For

higher derivative conformal scalar $\varphi^{(2n)}$ operators (GJMS operators) the factorization on an Einstein space reads [63]

$$\prod_{k=1}^n \Delta_0(6 - k(k-1)), \quad (\text{C.2})$$

where $\Delta_0(\kappa)$ is the scalar Laplacian defined in (6.6). Similarly, for the $(2n+1)$ -derivative spinor $\psi^{(2n+1)}$, the square of the corresponding conformal operator factorizes as [70]

$$\Delta_{\frac{1}{2}}\left(\frac{15}{2}\right) \prod_{k=1}^n \left[\Delta_{\frac{1}{2}}\left(\frac{15-2k^2}{2}\right) \right]^2, \quad (\text{C.3})$$

which generalizes the $n=1$ expression in (6.11). Finally, for the vectors $V^{(2n)}$, with $n > 2$, the factorization on a general Einstein space looks the same as on the 6-sphere and can be found from (A.17) of [13] for the massive conformal representation $[\Delta, \mathbf{h}] = [3+n, (1, 0, 0)]$. The corresponding partition function can be written as (cf. (6.8))

$$Z(V^{(2n)}) = \left[\frac{\Delta_0(-(n+3)(n-2)) \prod_{k=1}^n \Delta_0(-(k+1)(k-2))}{\prod_{k=1}^{n+1} \Delta_0(-(k+2)(k-3)) \prod_{k=1}^n \Delta_1(7+k-k^2)} \right]^{1/2}. \quad (\text{C.4})$$

These factorizations into 2nd order Laplacians allow us to compute the corresponding conformal anomalies using the same method as in the $V^{(1,0)}$ (i.e. $p=2$) case in section 6. Using the expressions for the Seeley–DeWitt coefficients b_6 in (6.7), (6.9), (6.12) we can compute the anomalies of the fields in the $V_p^{(1,0)}$ multiplet with the results summarized below³³

$$\begin{aligned} \varphi^{(2n)} : \quad & 7! a = -\frac{1}{144} n^3 (3n^4 - 21n^2 + 28), \\ & 7! c_1 = -\frac{2}{9} n (3n^2 - 5) (3n^4 - 16n^2 - 8), \\ & 7! c_2 = -\frac{1}{18} n (9n^6 - 63n^4 + 112n^2 - 88), \\ & 7! c_3 = \frac{1}{6} n (n^6 - 7n^4 + 18), \\ \psi^{(2n+1)} : \quad & 7! a = \frac{1}{288} (2n+1) (12n^6 + 36n^5 - 102n^4 - 264n^3 + 244n^2 + 382n - 191), \\ & 7! c_1 = \frac{2}{9} (2n+1) (18n^6 + 54n^5 - 153n^4 - 396n^3 + 415n^2 + 622n - 336), \\ & 7! c_2 = \frac{1}{18} (2n+1) (18n^6 + 54n^5 - 153n^4 - 396n^3 + 317n^2 + 524n - 144), \\ & 7! c_3 = -\frac{1}{6} (2n+1) (2n^6 + 6n^5 - 17n^4 - 44n^3 + 57n^2 + 80n - 60), \\ V_m^{(2n)} : \quad & 7! a = -\frac{1}{8} n^3 (n^4 - 14n^2 + 21), \\ & 7! c_1 = -\frac{4}{3} n (9n^6 - 126n^4 + 231n^2 - 2), \\ & 7! c_2 = -\frac{1}{3} n (9n^6 - 126n^4 + 147n^2 + 80), \\ & 7! c_3 = n (n^6 - 14n^4 + 35n^2 - 10). \end{aligned} \quad (\text{C.5}) \quad (\text{C.6}) \quad (\text{C.7})$$

³³Our discussion is formal as curved-space higher-derivative operators in a given dimension (here $d=6$) may exist only to some critical order (as is well known in the scalar GJMS case).

The total results for the multiplet $V_p^{(1,0)}$ are then

$$\begin{aligned} V_p^{(1,0)} : \quad 7! a &= 70(p-1)^4 - 35(p-1)^2 + \frac{77}{48}, \\ 7! c_1 &= 6720(p-1)^4 - 3920(p-1)^2 + \frac{560}{3}, \\ 7! c_2 &= 1680(p-1)^4 - 700(p-1)^2 + \frac{28}{3}, \\ 7! c_3 &= -560(p-1)^4 + 420(p-1)^2 - 28. \end{aligned} \quad (\text{C.8})$$

The expressions in (C.8) are in perfect agreement with (1.3),(1.2),(1.5) with (1.7) as one can see using that for general p the coefficients in the anomaly polynomial are [8]

$$(\alpha, \beta, \gamma, \delta) = \left(-(p-1)^4, -\frac{1}{2}(p-1)^2, -\frac{7}{240}, \frac{1}{60} \right). \quad (\text{C.9})$$

Remarkably, these expressions continue to hold also for $p = 2$ where (C.4) should be replaced by (6.8).³⁴

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³⁴This agreement is not accidental. For $p = 2$ the vector $V^{(4)}$ has gauge invariant action (it is described by the conformal representation $[5, (1, 0, 0)] - [6, (0, 0, 0)]$ where the subtraction takes into account the gauge invariance), while the scalar φ' is absent in the multiplet $V^{(1,0)}$. In the general expression for the partition function for the multiplet $V_p^{(1,0)}$ continued formally to $p = 2$ the field φ' enters effectively with a negative multiplicity and thus contributes precisely like the ghost scalar factor in (6.8).

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