Direct and semi-direct approaches to lepton mixing with a massless neutrino

Stephen F. King and Patrick Otto Ludl

School of Physics and Astronomy, University of Southampton,
Southampton, SO17 1BJ, U.K.

E-mail: king@soton.ac.uk, P.Ludl@soton.ac.uk

ABSTRACT: We discuss the possibility of enforcing a massless Majorana neutrino in the direct and semi-direct approaches to lepton mixing, in which the PMNS matrix is partly predicted by subgroups of a discrete family symmetry, extending previous group searches up to order 1535. We find a phenomenologically viable scheme for the semi-direct approach based on $Q(648)$ which contains $\Delta(27)$ and the quaternion group as subgroups. This leads to novel predictions for the first column of the PMNS matrix corresponding to a normal neutrino mass hierarchy with $m_1 = 0$, and sum rules for the mixing angles and phase which are characterised by the solar angle being on the low side $\theta_{12} \sim 31°$ and the Dirac (oscillation) CP phase δ being either about $\pm 45°$ or $\pm \pi$.

Contents

1 Introduction

Neutrino mass and lepton mixing differs markedly from that of quarks in several ways. The extreme smallness of neutrino mass, together with large lepton mixing provide fascinating clues in the search for a theory of flavour. One idea is that lepton mixing may be governed by a discrete family symmetry group G_f which controls the Majorana neutrino and charged lepton mass matrices leading to lepton mixing predictions [1, 2]. The three possible implementations of flavour symmetries are known as "direct", "semi-direct" and "indirect" [1, 2].

     According to the "direct" approach, the Klein symmetry $G_\nu = \mathbb{Z}_2 \times \mathbb{Z}_2$ of the Majorana neutrino mass matrix and the symmetry G_ℓ which fixes the form of the lepton mass matrix are both subgroups of G_f, resulting in a prediction for all the lepton mixing angles and Dirac phase. The advantage of the direct approach is that the prediction arises purely from symmetry and does not require any detailed knowledge of the model. However, the direct approach requires a rather large group [3–6], and the only viable mixing pattern is the trimaximal mixing, with δ being either zero or $\pm\pi$.

     In the "semi-direct" approach, the symmetry of the neutrino mass matrix is typically reduced to $\mathbb{Z}_2$ for Majorana neutrinos, which constrains only the second column of the

PMNS matrix to be $(1, 1, 1)^T/\sqrt{3}$, or the first column to be $(2, 1, 1)^T/\sqrt{6}$ (up to phases), and the reactor angle θ_{13} can be accommodated with a small discrete group such as S_4. In the "indirect" approach, the flavour symmetry is completely broken such that the observed neutrino flavour symmetry emerges indirectly as an accidental symmetry, and the predictions are model dependent (for a recent review see [7]).

The above direct and semi-direct approaches usually assume three non-degenerate Majorana neutrino masses. There has recently been some discussion of how this picture changes if one of the Majorana neutrino masses is zero [8, 9]. In this case the phase of the massless neutrino field is undetermined resulting in one of the $\mathbb{Z}_2$ factors being replaced by $\mathbb{Z}_n$, with the consequence that the determinant of the family symmetry G_f need not be ± 1, i.e. it is a subgroup of U(3) rather than SU(3). Although this opens up the possibility that a new type of viable direct model being found, in fact only a no-go theorem results from such searches up to order 511 [8, 9].

In the present paper we extend the reach of such searches for direct models with one massless neutrino up to order 1535, but without phenomenological success. On the other hand we also perform a new type of search for semi-direct models up to order 1535, and find a successful example of this kind, based on the group,

$$G_f \simeq (\Delta(27) \rtimes Q_8) \rtimes \mathbb{Z}_3 \tag{1.1}$$

where Q_8 denotes the quaternion group of order 8 and the group is therefore of order $3^4 \times 8 = 648$. We denote this group as $Q(648)$. This leads to a successful prediction for the first column of the PMNS matrix,

$$U_{\text{PMNS}} = \begin{pmatrix} U_{e1} & - & - \\ U_{\mu1} & - & - \\ U_{\tau1} & - & - \end{pmatrix} \tag{1.2}$$

corresponding to a normal neutrino mass hierarchy with $m_1 = 0$ and sum rules for the mixing angles and phase which are characterised by the solar angle being on the low side $\theta_{12} \sim 31°$ and the Dirac (oscillation) CP phase δ either about $\pm 45°$ or $\pm \pi$ (e.g. the recent hint of $\delta \sim -\pi/2$ is not allowed).

Before we outline the details of our analysis, we would like to briefly comment on dynamical settings enforcing $m_1 = 0$. The most prominent scenarios of this kind are type-I seesaw models with two right-handed neutrinos [10]. Such models necessarily imply the mass of the lightest neutrino to vanish, independent of any imposed flavour symmetries. In the same way, the discrete residual symmetries discussed in this paper lead to $m_1 = 0$ independent of the number of right-handed neutrino fields. Therefore, although not directly connected, the two approaches can easily be combined. For example if the three left-handed lepton doublets L transform under a triplet representation $\underline{\mathbf{3}}$ of the flavour group G_f and three Higgs-doublets (or alternatively three flavons) ϕ also transform under a triplet representation $\underline{\mathbf{3}}'$ of G_f, and the tensor product $\underline{\mathbf{3}}^* \otimes \underline{\mathbf{3}}'$ contains one or two-dimensional representations, a Dirac Yukawa coupling $\overline{L} \phi \nu_R$ to two right-handed neutrinos is compatible with G_f.

The remainder of the paper is laid out as follows. In section 2 we review residual symmetries with a massless Majorana neutrino and describe our strategy and results of group searches for direct and indirect models, for groups up to order 1535. In section 3 we present the results of a numerical phenomenological analysis, and show that while the direct models are excluded by current data, there is a unique group (up to this order) which yields acceptable results in the semi-direct approach, leading to mixing sum rules and phenomenological predictions. Section 4 concludes the paper.

2 Group searches with a massless neutrino

2.1 Residual symmetries in the lepton mass matrices

The main question around which the framework of residual symmetries in the fermion mass matrices [3, 5, 11–17] has been constructed is the question: *When do symmetries of a 3×3-matrix (partly) fix the matrix which diagonalises it?*

The answer to this question is fairly simple. Consider unitary matrices S_i and T_j which leave a complex symmetric matrix M or a Hermitian matrix H invariant,[1] i.e.

$$S_i^T M S_i = M \quad \forall i, \tag{2.1a}$$

$$T_j^\dagger H T_j = H \quad \forall j. \tag{2.1b}$$

The set of all matrices S_i forms a group G_M of unitary 3×3-matrices. In the same way, also the matrices T_j form a unitary group G_H. If the matrices M and H have non-degenerate singular values[2] the groups G_M and G_H must be *Abelian*. For this case one can show that the matrices U_M and U_H which (simultaneously) bring all S_i and T_j to diagonal form, i.e.

$$U_M^\dagger S_i U_M = \hat{S}_i \quad \forall i, \tag{2.2a}$$

$$U_H^\dagger T_j U_H = \hat{T}_j \quad \forall j, \tag{2.2b}$$

also diagonalise H and M via[3] [12]

$$U_M^T M U_M = \hat{M}, \tag{2.4a}$$

$$U_H^\dagger H U_H = \hat{H}, \tag{2.4b}$$

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[1] For application to the fermion mass problem the study of Hermitian and complex symmetric matrices is sufficient. For the lepton sector we will later have $M = M_\nu$ and $H = M_\ell M_\ell^\dagger$.

[2] The singular values are the elements of the diagonalised matrix. This condition is necessarily fulfilled in all relevant applications, because the singular values of H will be identified with the charged-lepton masses squared and the singular values of M will be identified with the three light-neutrino masses.

[3] Let us, as an illustration, give the derivation of this fact for a complex symmetric matrix M. From equation (2.4a) we have $M u_k = m_k u_k^*$ (no summation over k) for the columns u_k of U_M, i.e. the u_k are singular vectors of M with singular values m_k. From equation (2.1a) we then have

$$S_i^T M S_i u_k = m_k u_k^* \Rightarrow M(S_i u_k) = m_k (S_i u_k)^*. \tag{2.3}$$

Since the singular values m_i are assumed to be non-degenerate, the corresponding singular vectors are unique up to multiplication with a constant, i.e. $S_i u_k = s_i u_k$. Therefore, the u_k are simultaneous eigenvectors to all S_i, which implies equation (2.2a).

where hatted matrices are diagonal matrices. In other words *the set of symmetries S_i of a matrix M fixes its diagonalising matrix U_M*. The same statement holds for T_j and H.

Let us now apply these mathematical considerations to the case of residual symmetries in the lepton sector. The assumption of residual symmetries is that there is a flavour symmetry group G_f (acting on the lepton fields) which is spontaneously broken to a symmetry group G_ν in the neutrino sector,

$$S_i^T M_\nu S_i = M_\nu, \quad S_i \in G_\nu, \tag{2.5}$$

and a symmetry group G_ℓ in the charged-lepton sector,

$$T_j^\dagger M_\ell M_\ell^\dagger T_j = M_\ell M_\ell^\dagger, \quad T_j \in G_\ell. \tag{2.6}$$

Identifying $M = M_\nu$ and $H = M_\ell M_\ell^\dagger$ in the above discussion, we immediately find $G_\nu \subset G_M$ and $G_\ell \subset G_H$. Therefore, the symmetry groups G_ℓ and G_ν potentially (but not necessarily) impose constraints on the matrices U_ν and U_ℓ, which diagonalise M_ν and $M_\ell M_\ell^\dagger$ via

$$U_\nu^T M_\nu U_\nu = \mathrm{diag}(m_1, m_2, m_3), \tag{2.7}$$

$$U_\ell^\dagger M_\ell M_\ell^\dagger U_\ell = \mathrm{diag}(m_e^2, m_\mu^2, m_\tau^2), \tag{2.8}$$

and consequently also on the mixing matrix $U_{\mathrm{PMNS}} = U_\ell^\dagger U_\nu$.

In order to know how G_ν and G_ℓ constrain U_ν and U_ℓ we have to understand to which degree an Abelian unitary $3 \times 3-$matrix group A determines the unitary matrix U which simultaneously diagonalises all elements of A. There are only three possibilities:

- None of the common eigenvectors of the elements of A is unique.[4] $\Rightarrow A = \{\mathbb{1}_3\}$ and U is an arbitrary unitary matrix.

- One of the common eigenvectors of the elements of A is unique. $\Rightarrow$ One column of U is proportional to this eigenvector. $\Rightarrow$ One column of U is fixed by A up to a rephasing.

- All three of the common eigenvectors of the elements of A are unique. $\Rightarrow$ Each column of U is proportional to one of these eigenvectors. $\Rightarrow U$ is fixed by A up to rephasing and reordering of its columns.

According to this finding, one classifies models based on residual symmetries into two categories:

(A) <u>Direct models:</u> G_ν fixes U_ν and G_ℓ fixes U_ℓ (up to reordering and rephasing of the columns). $\Rightarrow U_{\mathrm{PMNS}}$ is fixed (up to reordering and rephasing of rows and columns).

(B) <u>Semidirect models:</u> in one sector the full diagonalising matrix is fixed, in the other sector only a column is fixed.

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[4]For eigenvectors *unique* here means unique up to multiplication with a complex number.

(B1) G_ℓ fixes U_ℓ, G_ν fixes a column of U_ν. $\Rightarrow$ One column of U_{PMNS} is fixed up to permutation of its elements. One may choose (if not determined by a concrete model) which column of U_{PMNS} is fixed.

(B2) G_ℓ fixes one column of U_ℓ, G_ν fixes U_ν. $\Rightarrow$ One row of U_{PMNS} is fixed up to permutation of its elements. One may choose (if not determined by a concrete model) which row of U_{PMNS} is fixed.

In this work we will only study the cases A and B1. Case B2 has for example been studied in [18]. One could in principle also consider a third very weakly restrictive case C for which in each sector only one column is fixed (up to rephasing). In this case only one element of U_{PMNS} would be fixed. Due to its low predictive power, this scenario is usually not studied, and also we will not study it here.

2.2 Residual symmetries enforcing one massless neutrino

The residual symmetry groups G_ν and G_ℓ are Abelian groups of unitary 3×3-matrices. Therefore, they are subgroups of $\text{U}(1) \times \text{U}(1) \times \text{U}(1)$. For the charged lepton sector, this is also the maximal symmetry group, i.e. $G_H = \text{U}(1) \times \text{U}(1) \times \text{U}(1)$. For Majorana neutrinos the situation is different. If all neutrinos are massive, the maximal symmetry group is $G_M = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, while, if one neutrino is massless, also $G_M = \text{U}(1) \times \mathbb{Z}_2 \times \mathbb{Z}_2$ is allowed. The case of one massless neutrino has been studied in [8, 9].

Here we will further elaborate on the case of a massless neutrino within the framework of residual symmetries. The case of direct models has been studied in [9] for all suitable finite groups up to order 511. In the present paper we extend this analysis to order 1535. Moreover, we discuss semidirect models (of type B1) with a massless neutrino, also up to order 1535. Before we discuss the details of the group searches we have performed, we want to have a look on the generic requirements potential flavour groups G_f with a massless neutrino have to fulfill. As outlined in detail in appendix A, viable groups G_f

- must possess a faithful three-dimensional irreducible representation,

- must not be of the form $G_f \simeq G_f' \times \mathbb{Z}_n$ $(n > 1)$ and

- must not be of the form of the following theorem by Joshipura and Patel [9]: let G be a group of 3×3 matrices which contains only elements of the form "*diagonal matrix of phases times permutation matrix*", where the six permutation matrices are given by

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$
$$P_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad P_6 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{2.9}$$

Then, if such a group G is used to build models enforcing a massless neutrino, the column vector of the mixing matrix associated to the massless neutrino must be

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \; \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \qquad (2.10)$$

or permutations thereof (i.e. permutations of the elements of an individual column.)

As a starting point for our analysis, we need a list of groups fulfilling these criteria. We used the library SmallGroups[5] [19, 20] and the computer algebra system GAP [21] to find all groups of order smaller than 1536 which fulfill these minimal criteria.[6] As a result of this scan we have found 22 groups of order smaller than 1536 fulfilling the minimal criteria.[7] They are shown in equation (A.7) in appendix A.

2.2.1 Direct models

Let us now investigate the requirements for direct models enforcing a massless neutrino. The requirement for the charged-lepton sector is the usual one: any group G_ℓ which uniquely determines the diagonalising matrix U_ℓ is sufficient. The same also holds for semidirect models of type B1. In the neutrino sector, we require a residual symmetry group G_ν which completely fixes U_ν (as always up to rephasing and reordering of the columns) and which enforces one neutrino mass to vanish. The requirement of a vanishing neutrino mass implies that G_ν is a subgroup of $U(1) \times \mathbb{Z}_2 \times \mathbb{Z}_2$ instead of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. Therefore, there exists a basis in which all elements of G_ν have the form

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} \qquad (2.11)$$

with $\lambda \in U(1)$, $\alpha, \beta \in \{-1, +1\}$. Moreover, in order to enforce a vanishing neutrino mass, for at least one element of G_ν we must have $\lambda \neq \pm 1$. This element may be of four forms:

$$\begin{aligned} S_1 &= \mathrm{diag}(\lambda, +1, +1), \quad S_2 = \mathrm{diag}(\lambda, +1, -1), \\ S_3 &= \mathrm{diag}(\lambda, -1, +1), \quad S_4 = \mathrm{diag}(\lambda, -1, -1). \end{aligned} \qquad (2.12)$$

Every group G_ν capable of fixing the complete matrix U_ν and enforcing a massless neutrino contains at least one element of the form S_2 or S_3. Namely, if it did not contain such an

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[5]Throughout the paper we will use the SmallGroups ID to identify groups. This ID consists of two numbers in square brackets, i.e. $[g, n]$, g being the group order and n being a label. Two groups with different SmallGroups IDs are non-isomorphic.

[6]Since we require groups which have a three-dimensional irreducible representation, the group order must be divisible by 3. Up to order 1535 there are 1342632 groups whose order is divisible by three. For the group order 1536 alone there are 408641062 groups. Therefore, we had to stop our searches at order 1535.

[7]In this paper we follow the approach of scanning over a set of eligible groups, in the end discarding which are incompatible with experiment. The opposite approach of constructing eligible groups directly from experimental data on the mixing matrices has e.g. been used in [22, 23].

element, in order to fulfill all requirements it would have to contain at least one element of the form

$$\text{diag}(\lambda, \pm 1, \pm 1) \tag{2.13}$$

and one further element with non-degenerate 22 and 33 elements, i.e.

$$\text{diag}(\pm 1, \pm 1, \mp 1) \quad \text{or} \quad \text{diag}(\pm 1, \mp 1, \pm 1). \tag{2.14}$$

However, the product of the matrices of equations (2.13) and (2.14) is of the form S_2 or S_3, which proves that G_ν always contains an element of this form. The matrices S_2 and S_3, since they have non-degenerate eigenvalues, on their own already fix the complete matrix U_ν. Therefore, we can restrict the analysis to groups $G'_\nu = \langle\langle S_2 \rangle\rangle \subset G_\nu$ generated by S_2 (or $G'_\nu = \langle\langle S_3 \rangle\rangle \subset G_\nu$ generated by S_3.)[8] Thus, the requirement on G_f is:

- G_f has a faithful three-dimensional irreducible representation which has at least one element with one eigenvalue $\lambda \neq \pm 1$, one eigenvalue $+1$ and one eigenvalue -1. This element generates the residual symmetry group $G'_\nu \simeq \mathbb{Z}_n$ (n even) and determines U_ν up to rephasing and reordering of the columns.

Thus, among the groups of equation (A.7) we search for those which have a faithful three-dimensional irreducible representation $\underline{\mathbf{3}}$ (defining a matrix group $\underline{\mathbf{3}}(G_f)$ isomorphic to G_f) fulfilling the following criteria:

- $\underline{\mathbf{3}}(G_f)$ contains a matrix S with eigenvalues $\{\lambda, +1, -1\}$, $\lambda \neq \pm 1$. This is a basis independent property and may easily be checked by testing $\text{Tr}\, S + \det S = 0$, $\text{Tr}\, S \neq \pm 1$. The matrix S is then a candidate for a generator of G'_ν.

- $\underline{\mathbf{3}}(G_f)$ contains an Abelian subgroup G_ℓ which can completely fix its diagonalising matrix U_ℓ.

- There must be choices of G'_ℓ and G'_ν such that G'_ν is not a subgroup of G'_ℓ and vice versa. (Otherwise the mixing matrix U_{PMNS} would be trivial!) Also, none of the generators of G'_ℓ must commute with the generator S of G'_ν —see appendix B.

- Moreover, the groups G'_ℓ and G'_ν must together generate the whole matrix group $\underline{\mathbf{3}}(G_f)$. Namely, if they do not, we can restrict ourselves to the subgroup $G'_f \equiv \langle\langle G'_\ell, G'_\nu \rangle\rangle \subset \underline{\mathbf{3}}(G_f)$. If G'_f is an irreducible matrix group, G'_f fulfills all criteria for our search and predicts the same mixing matrix as $\underline{\mathbf{3}}(G_f)$. If $\underline{\mathbf{3}}(G'_f)$ is reducible, the mixing matrix has two vanishing mixing angles and is therefore not compatible with experiment.

Performing a group search with GAP, we find that only seven groups of order smaller than 1536 fulfill all these criteria:

$$[108, 15], [324, 111], [432, 239], [648, 533], [864, 675], [972, 411], [1296, 1995]. \tag{2.15}$$

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[8]In this paper the symbol $\langle\langle \ldots \rangle\rangle$ means "generated by ...".

2.2.2 Semidirect models

The requirements for semidirect models of type (B1) are identical with only two differences:

- The requirement on S now becomes: $\underline{\mathbf{3}}(G_f)$ contains a matrix S with eigenvalues $\{\lambda, \pm 1, \pm 1\}$, $\lambda \neq \pm 1$, respectively (i.e. two degenerate eigenvalues in each case). This ensures that one mass is set to zero, but only one column of U_ν is fixed.

- The argument used to prove that none of the generators of G_ℓ must commute with the generator S of G_ν — see appendix B — does not hold for semidirect models. Thus, one of the generators of G'_ℓ is allowed to commute with S.

Also for the semidirect case the groups G'_ℓ and G'_ν have to generate the full group $\underline{\mathbf{3}}(G_f)$. However, the argument for this is different to the case of direct models. Again, if $\underline{\mathbf{3}}(G'_f)$ is irreducible, we may replace G_f by $\underline{\mathbf{3}}(G'_f)$. If $\underline{\mathbf{3}}(G'_f)$ is reducible one can (by the same argument as used in appendix A to show that $\underline{\mathbf{3}}(G_f)$ must be irreducible) show that a reducible $\underline{\mathbf{3}}(G'_f)$ leads to predictions incompatible with experiment.

Doing a group search with GAP, one finds that there is only one single group of order smaller than 1536 which meets all requirements for a semidirect model. This group has the identification number $[648, 533]$ in the SmallGroups library.

3 Numerical analysis and phenomenology

3.1 Direct models

In order to test the seven candidate groups for direct models with a massless neutrino, we computed all faithful three-dimensional irreducible representations of the groups, computed all Abelian subgroups and listed all possible combinations (G_ℓ, G_ν). For each of these combinations, the possible mixing matrices have been computed — see [3] for a detailed description of this procedure. In order to compare the predictions for the mixing matrix with experiment we fitted the three mixing angles to the global fit data of [24]. As χ^2-function we used

$$\chi^2(\theta_{12}, \theta_{23}, \theta_{13}) \equiv \sum_{ij=12,23,13} \left(\frac{\sin^2\theta_{ij}^{\mathrm{exp}} - \sin^2\theta_{ij}^{\mathrm{pred}}}{\sigma(\sin^2\theta_{ij})} \right)^2, \tag{3.1}$$

which has three degrees of freedom. For the errors $\sigma(\sin^2\theta_{ij})$ we used the values given in [24] (in case of an asymmetric error distribution we used the larger error). The resulting minimal values of χ^2 are listed in table 1. Evidently, none of the candidate groups is compatible with the experimental data. For groups up to order 511 this result has been found earlier in [9].

3.2 Semidirect models

For the semidirect models, there is a unique candidate group $[648, 533]$. It has six faithful three-dimensional irreducible representations, each of which can predict (the same) 19 different patterns for a column of the mixing matrix. It is therefore sufficient to study only

Group	$\chi^2_{\min}$ (normal spectrum)	$\chi^2_{\min}$ (inverted spectrum)
$[108, 15]$	1.80×10^2	4.52×10^2
$[324, 111]$	1.80×10^2	4.52×10^2
$[432, 239]$	1.07×10^4	1.56×10^4
$[648, 533]$	1.30×10^2	4.19×10^2
$[864, 675]$	1.07×10^4	1.56×10^4
$[972, 411]$	1.80×10^2	4.52×10^2
$[1296, 1995]$	1.07×10^4	1.56×10^4

Table 1. The minimal values of χ^2 for the seven candidate groups for direct models. Groups which lead to the same $\chi^2_{\min}$ predict the same values for the elements $|U_{ij}|$ of the mixing matrix.

one of the faithful three-dimensional irreducible representations constructed with GAP, i.e. we pick one of them and use it to define the group $[648, 533]$ as a matrix group. In this representation the group is generated by the two matrices

$$S = \frac{1}{3} \begin{pmatrix} 1 + \epsilon^4 - \epsilon^6 - \epsilon^8 & -\epsilon^2 + \epsilon^7 & -\epsilon^2 + \epsilon^4 \\ \epsilon - \epsilon^5 & 1 + \epsilon - \epsilon^2 - \epsilon^6 & \epsilon^4 - \epsilon^5 \\ \epsilon - \epsilon^8 & \epsilon^7 - \epsilon^8 & 1 - \epsilon^5 - \epsilon^6 + \epsilon^7 \end{pmatrix} \tag{3.2}$$

and

$$T = \begin{pmatrix} \epsilon^2 & 0 & 0 \\ 0 & \epsilon^5 & 0 \\ 0 & 0 & \epsilon^8 \end{pmatrix}, \tag{3.3}$$

where

$$\epsilon \equiv \exp(2\pi i/9), \tag{3.4}$$

i.e.

$$G_f = \langle\langle S, T \rangle\rangle \simeq (((\mathbb{Z}_3 \times \mathbb{Z}_3) \rtimes \mathbb{Z}_3) \rtimes Q_8) \rtimes \mathbb{Z}_3 \simeq (\Delta(27) \rtimes Q_8) \rtimes \mathbb{Z}_3. \tag{3.5}$$

Q_8 here denotes the quaternion group of order 8. This group corresponds to $[648, 533]$ which we denote as $Q(648)$.

The two only columns predictable by $Q(648)$ being compatible with experiment emerge from the choice

$$G_\nu = \langle\langle S \rangle\rangle \simeq \mathbb{Z}_3, \quad G_\ell = \langle\langle T \rangle\rangle \simeq \mathbb{Z}_9. \tag{3.6}$$

We are already in a basis where all elements of G_ℓ are diagonal. Therefore, the eigenvector of S with eigenvalue $\neq \pm 1$ is the predicted column of U_{PMNS}. Indeed, S has two eigenvalues $+1$ and an eigenvalue $\omega \equiv \exp(2\pi i/3) = \epsilon^3$ with the corresponding eigenvector

$$u = \frac{1}{3} \begin{pmatrix} 1 + \epsilon + \epsilon^8 \\ \epsilon^4 + \epsilon^6 + \epsilon^8 \\ \epsilon^3 + \epsilon^7 + \epsilon^8 \end{pmatrix}. \tag{3.7}$$

The absolute values of the entries of this vector are

$$|u_1| = \frac{1}{3}\left(1 + 2\cos\frac{2\pi}{9}\right) \approx 0.844, \tag{3.8}$$

$$|u_2| = \frac{1}{3}\left(1 + 2\cos\frac{4\pi}{9}\right) \approx 0.449, \tag{3.9}$$

$$|u_3| = -\frac{1}{3}\left(1 + 2\cos\frac{8\pi}{9}\right) \approx 0.293. \tag{3.10}$$

This leads to two patterns compatible with the first column[9] of U_{PMNS}, namely:

$$\begin{pmatrix} |U_{e1}| \\ |U_{\mu 1}| \\ |U_{\tau 1}| \end{pmatrix} = \begin{pmatrix} |u_1| \\ |u_2| \\ |u_3| \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} |U_{e1}| \\ |U_{\mu 1}| \\ |U_{\tau 1}| \end{pmatrix} = \begin{pmatrix} |u_1| \\ |u_3| \\ |u_2| \end{pmatrix}. \tag{3.11}$$

Note that we here have used the permutation freedom of the elements of the predicted column, which comes from the fact that the residual symmetries cannot fix any mass orderings. The equations

$$|U_{e1}| = c_{12}\,c_{13}, \tag{3.12a}$$

$$|U_{\mu 1}| = |s_{12}c_{23} + c_{12}s_{13}s_{23}e^{i\delta}|, \tag{3.12b}$$

where $s_{ij} \equiv \sin\theta_{ij}$ and $c_{ij} \equiv \cos\theta_{ij}$, give relations between the mixing angles and the Dirac phase δ. These relations can be used to predict θ_{12} as a function of θ_{13} and $\cos\delta$ as a function of θ_{13} and θ_{23}:

$$s_{12}^2 = 1 - |U_{e1}|^2(1 + t_{13}^2), \tag{3.13a}$$

$$\cos\delta = \frac{|U_{\mu 1}|^2\left(1 + t_{23}^2\right) + |U_{e1}|^2\left(t_{23}^2 - \left(1 + t_{13}^2\right)\left(t_{23}^2 - 1\right)\right) - 1}{2\,|U_{e1}|\,t_{13}\,t_{23}\sqrt{1 - |U_{e1}|^2\left(1 + t_{13}^2\right)}}. \tag{3.13b}$$

Here $t_{ij} \equiv \tan\theta_{ij}$. Using the values of the global fit of ref. [24],

$$\sin^2\theta_{23} = 0.452^{+0.052}_{-0.028}, \quad \sin^2\theta_{13} = 0.0218^{+0.0010}_{-0.0010}, \tag{3.14}$$

we find (best-fit)

$$\sin^2\theta_{12} = 0.272, \qquad \theta_{12} = 31.4°, \tag{3.15a}$$

$$\cos\delta = 0.697, \qquad \delta = \pm 45.8° \tag{3.15b}$$

for $|U_{e1}| = |u_1|$, $|U_{\mu 1}| = |u_2|$. The other possibility with $|U_{e1}| = |u_1|$, $|U_{\mu 1}| = |u_3|$ turns out to be incompatible with the best-fit values of equation (3.14),

$$\sin^2\theta_{12} = 0.272, \quad \theta_{12} = 31.4°, \tag{3.16a}$$

$$\cos\delta = -1.07 \rightarrow \text{ inconsistent}, \tag{3.16b}$$

---

[9]Since the neutrino mass associated with the predicted column vanishes, we have $m_1 = 0$ which requires a normal neutrino mass spectrum.
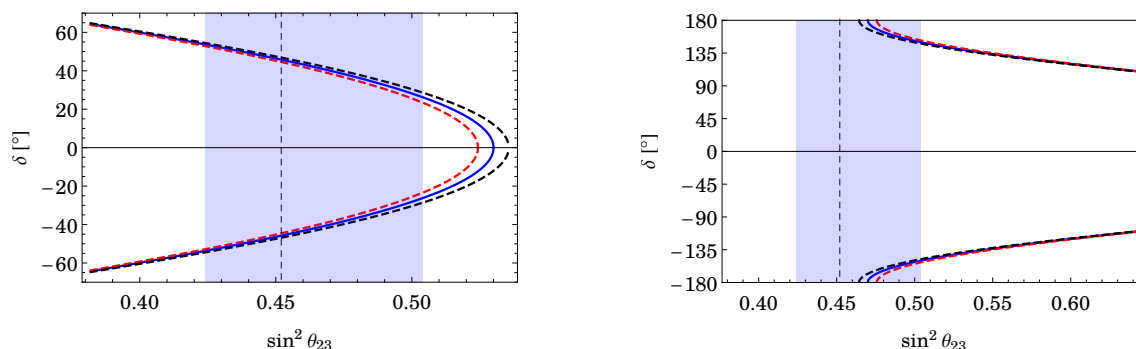
Figure 1. The value of the Dirac phase δ as a function of $\sin^2\theta_{23}$ for different values of $\sin^2\theta_{13}$ according to the global fit of [24]: best-fit (blue), 1σ lower bound (red dashed), 1σ upper bound (black dashed). The dashed vertical line indicates the best-fit value $\sin^2\theta_{23} = 0.452$ and the blue shaded area is the 1σ region for $\sin^2\theta_{23}$. Left plot: $(|U_{e1}|, |U_{\mu1}|) = (|u_1|, |u_2|)$, right plot: $(|U_{e1}|, |U_{\mu1}|) = (|u_1|, |u_3|)$.

but remains consistent with experiment for $s_{23}^2 \gtrsim 0.47$. Using the best-fit values of the mass-squared differences and $\sin^2\theta_{13}$ from [24] as input parameters, the predicted range for $m_{\beta\beta}$ is $(1.22 \div 3.38)$ meV, i.e. several meV (as for every model with a normal neutrino mass spectrum and vanishing m_1). Since $m_{\beta\beta}$ depends only on the sum of δ and one of the (unconstrained) Majorana phases, our model puts no stronger constraint on $m_{\beta\beta}$.

In total, the two discussed column patterns are compatible with the global fit values of [24] at about 2–3 sigma. The reason for tension is the too small value of the solar mixing angle predicted by the group using the reactor angle as an input. The global-fit result for the solar mixing angle is

$$\sin^2\theta_{12} = 0.304^{+0.013}_{-0.012}, \quad \theta_{12} = (33.48^{+0.78}_{-0.75})^\circ. \tag{3.17}$$

The main prediction of the model is a value of δ of about $\pm45^\circ$ or $\pm\pi$ (for the best-fit values). The value of δ for different values of s_{23}^2 and s_{13}^2 is shown in figure 1.

4 Conclusions

In this paper we have discussed the possibility of enforcing a massless Majorana neutrino in the direct and semi-direct approaches to lepton mixing, in which the PMNS matrix is partly predicted by subgroups of a discrete family symmetry. Our analysis extends previous group searches for direct models from order 511 up to 1535, and provides the first analysis of semi-direct models with a massless neutrino up to this order. Our results confirm and extend the no-go results of Joshipura and Patel up to order 1535 for the direct approach.

However, we find a new phenomenologically viable scheme for the semi-direct approach based on $Q(648)$ which contains $\Delta(27)$ and the quaternion group as subgroups. This leads to novel predictions for the first column of the PMNS matrix corresponding to a normal neutrino mass hierarchy with $m_1 = 0$, and sum rules for the mixing angles and phase which are characterised by the solar angle being on the low side $\theta_{12} \sim 31^\circ$ and the Dirac (oscillation) CP phase δ being either about $\pm45^\circ$ or $\pm\pi$.

A   Generic requirements on flavour symmetries in the framework of residual symmetries with one massless neutrino.

In order to be potentially phenomenologically viable in the framework of residual symmetries in the lepton sector, a flavour group G_f must fulfill two generic requirements:

- G_f must have a *faithful three-dimensional irreducible representation*. The requirement for a three-dimensional representation comes from the fact that there are three generations of leptons. This representation must be faithful because otherwise we could restrict ourselves to the smaller group defined by the non-faithful matrix representation. Moreover, the three-dimensional faithful representation under consideration must also be irreducible. Namely, if it was reducible, there would be a basis in which the matrices of G_ν and G_ℓ are simultaneously block-diagonal. In direct models this would mean that also the mixing matrix is block-diagonal, thus implying two vanishing mixing angles, which is clearly not compatible with experimental observations. For the case of semidirect models we consider the example of models of type B1. The arguments for case B2 are analogous. In the block-diagonal basis we have

$$G_\nu : \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix}, \dots \quad G_\ell : \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix}, \dots, \tag{A.1}$$

where $\times$ stands for a non-zero entry. We can now make a further basis transformation (a unitary 23-rotation in our example) which makes all elements of G_ℓ diagonal, i.e.

$$G_\nu : \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix}, \dots \quad G_\ell : \begin{pmatrix} \times & 0 & 0 \\ 0 & \times & 0 \\ 0 & 0 & \times \end{pmatrix}, \dots . \tag{A.2}$$

In this basis, the column of U_{PMNS} which is fixed by the semidirect model is a common eigenvector of the matrices of G_ν. But all matrices of G_ν are still block-diagonal, which means that this common eigenvector can only be of the form

$$\begin{pmatrix} \times \\ 0 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 \\ \times \\ \times \end{pmatrix}. \tag{A.3}$$

Thus, U_{PMNS} would contain at least one vanishing element, which is phenomenologically not viable.

- We can *discard all groups of the form* $G_f = G'_f \times \mathbb{Z}_n$ $(n > 1)$. Namely, since the relevant representation of G_f must be irreducible, the elements of $\mathbb{Z}_n$ are represented as matrices proportional to $\mathbb{1}_3$. Such symmetries cannot constrain the mixing matrix U_{PMNS}. Therefore, it is sufficient to confine the study to the smaller group G'_f.

There are 384 groups of order smaller than 1536 which fulfill these two criteria, they are shown in equation (A.8). This list extends the list of finite subgroups of U(3) found in [25] to order 1535.

In the case of massless neutrinos there is a third constraint:

- Consider matrix groups which have only elements of the form "*diagonal matrix of phases times permutation matrix*", where by permutation matrices we mean the six matrices

$$
P_1 = \begin{pmatrix} 1\ 0\ 0 \\ 0\ 1\ 0 \\ 0\ 0\ 1 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0\ 1\ 0 \\ 1\ 0\ 0 \\ 0\ 0\ 1 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0\ 0\ 1 \\ 0\ 1\ 0 \\ 1\ 0\ 0 \end{pmatrix},
$$
$$
P_4 = \begin{pmatrix} 1\ 0\ 0 \\ 0\ 0\ 1 \\ 0\ 1\ 0 \end{pmatrix}, \quad P_5 = \begin{pmatrix} 0\ 0\ 1 \\ 1\ 0\ 0 \\ 0\ 1\ 0 \end{pmatrix}, \quad P_6 = \begin{pmatrix} 0\ 1\ 0 \\ 0\ 0\ 1 \\ 1\ 0\ 0 \end{pmatrix}. \tag{A.4}
$$

It has been shown by Joshipura and Patel in [9] that models based on such groups with a massless neutrino (enforced by the residual symmetry) can only lead to the following columns of the mixing matrix (absolute values of the entries of the fixed column of U_{PMNS}) associated to the massless neutrino:

$$
\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \ \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \tag{A.5}
$$

and permutations thereof (i.e. permutations of the elements of an individual column.) The only phenomenologically viable case here is

$$
\frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \tag{A.6}
$$

which is called TM$_2$ in the literature, since it fits the second column of the lepton mixing matrix. However, this scenario would predict $m_2 = 0$, which is excluded by experiment. Therefore, we can exclude also all groups which are of the form discussed in the theorem by Joshipura and Patel.

There are only 22 groups of order smaller than 1536 which also fulfill the third requirement:

$$[60, 5], [108, 15], [168, 42], [216, 25], [216, 88], [324, 111], [432, 57], [432, 239], [432, 273],$$
$$[648, 352], [648, 531], [648, 532], [648, 533], [648, 551], [864, 194], [864, 675], [864, 737], \quad \text{(A.7)}$$
$$[972, 411], [1080, 260], [1296, 1239], [1296, 1995], [1296, 2203].$$

Therefore, for the study of massless neutrinos in the framework of residual symmetries, confining oneself to flavour symmetry groups of order smaller than 1536, it is sufficient to study the 22 groups of equation (A.7).

$$
\begin{aligned}
&[12,3], [21,1], [24,12], [27,3], [27,4], [36,3], [39,1], [48,3], [48,30], [54,8], \\
&[57,1], [60,5], [63,1], [75,2], [81,6], [81,7], [81,8], [81,9], [81,10], [81,14], [84,11], \\
&[93,1], [96,64], [96,65], [108,3], [108,11], [108,15], [108,19], [108,21], [108,22], [111,1], \\
&[117,1], [129,1], [144,3], [147,1], [147,5], [150,5], [156,14], [162,10], [162,12], [162,14], \\
&[162,44], [168,42], [171,1], [183,1], [189,1], [189,4], [189,5], [189,7], [189,8], \\
&[192,3], [192,182], [192,186], [201,1], [216,17], [216,25], [216,88], [216,95], [219,1], \\
&[225,3], [228,11], [237,1], [243,16], [243,19], [243,20], [243,24], [243,25], [243,26], \\
&[243,27], [243,50], [243,55], [252,11], [273,3], [273,4], [279,1], [291,1], [294,7], [300,13], \\
&[300,43], [309,1], [324,3], [324,13], [324,15], [324,17], [324,43], [324,45], [324,49], \\
&[324,50], [324,51], [324,60], [324,102], [324,111], [324,128], [327,1], [333,1], [336,57], \\
&[351,1], [351,4], [351,5], [351,7], [351,8], [363,2], [372,11], [381,1], [384,568], [384,571], \\
&[384,581], [387,1], [399,3], [399,4], [417,1], [432,3], [432,33], [432,57], [432,100], \\
&[432,102], [432,103], [432,239], [432,260], [432,273], [441,1], [441,7], [444,14], [453,1], \\
&[468,14], [471,1], [486,26], [486,28], [486,61], [486,125], [486,164], [489,1], [507,1], \\
&[507,5], [513,1], [513,5], [513,6], [513,8], [513,9], [516,11], [525,5], [543,1], [549,1], \\
&[567,1], [567,4], [567,5], [567,7], [567,12], [567,13], [567,14], [567,23], [567,36], [576,3], \\
&[579,1], [588,11], [588,16], [588,60], [597,1], [600,45], [600,179], [603,1], [624,60], \\
&[633,1], [648,19], [648,21], [648,23], [648,244], [648,259], [648,260], [648,266], [648,352], \\
&[648,531], [648,532], [648,533], [648,551], [648,563], [651,3], [651,4], [657,1], [669,1], \\
&[675,5], [675,9], [675,11], [675,12], [684,11], [687,1], [711,1], [723,1], [726,5], [729,62], \\
&[729,63], [729,64], [729,80], [729,86], [729,94], [729,95], [729,96], [729,97], [729,98], \\
&[729,284], [729,393], [729,397], [732,14], [741,3], [741,4], [756,11], [756,113], [756,114], \\
&[756,116], [756,117], [768,1083477], [768,1085333], [768,1085335], [768,1085351], [777,3], [777,4], \\
&[804,11], [813,1], [819,3], [819,4], [831,1], [837,1], [837,4], [837,5], [837,7], [837,8], \\
&[849,1], [864,69], [864,194], [864,675], [864,701], [864,703], [864,737], [867,2], [873,1], \\
&[876,14], [900,66], [903,5], [903,6], [912,57], [921,1], [927,1], [939,1], [948,11], [972,3], \\
&[972,29], [972,31], [972,64], [972,117], [972,121], [972,122], [972,123], [972,147], [972,152], \\
&[972,153], [972,170], [972,309], [972,348], [972,411], [972,520], [972,550], [975,5], [981,1], \\
&[993,1], [999,1], [999,5], [999,6], [999,8], [999,9], [1008,57], [1011,1], [1014,7], [1029,6], \\
&[1029,9], [1047,1], [1053,16], [1053,25], [1053,26], [1053,27], [1053,29], [1053,32], [1053,35], \\
&[1053,37], [1053,47], [1080,260], [1083,1], [1083,5], [1089,3], [1092,68], [1092,69], [1101,1], \\
&[1116,11], [1119,1], [1137,1], [1143,1], [1161,6], [1161,9], [1161,10], [1161,11], [1161,12], \\
&[1164,14], [1176,57], [1176,243], [1191,1], [1197,3], [1197,4], [1200,183], [1200,384], \\
&[1200,682], [1209,3], [1209,4], [1227,1], [1236,11], [1251,1], [1263,1], [1281,3], [1281,4], \\
&[1296,3], [1296,35], [1296,37], [1296,39], [1296,220], [1296,222], [1296,226], [1296,227], \\
&[1296,228], [1296,237], [1296,647], [1296,688], [1296,689], [1296,699], [1296,1239], [1296,1499], \\
&[1296,1995], [1296,2113], [1296,2203], [1299,1], [1308,14], [1317,1], [1323,1], [1323,4], \\
&[1323,5], [1323,7], [1323,8], [1323,14], [1323,40], [1323,42], [1323,43], [1332,14], [1344,393], \\
&[1350,46], [1359,1], [1371,1], [1389,1], [1404,14], [1404,137], [1404,138], [1404,140], \\
&[1404,141], [1407,3], [1407,4], [1413,1], [1425,5], [1443,3], [1443,4], [1452,11], [1452,34], \\
&[1458,615], [1458,618], [1458,659], [1458,663], [1458,666], [1458,1095], [1458,1354], [1458,1371], \\
&[1461,1], [1467,1], [1488,57], [1497,1], [1521,1], [1521,7], [1524,11], [1533,3], [1533,4].
\end{aligned}
\tag{A.8}
$$

B Relations between G_ℓ and G_ν

In this appendix we will show that for direct models, in order to obtain a phenomenologically viable mixing matrix U_{PMNS}, the generator S of G_ν must not commute with any of the generators of G_ℓ. Note that we here exclude generators of G_ℓ which are proportional to $\mathbb{1}_3$, because such elements do not restrict U_ℓ and are therefore superfluous.

Suppose a generator T of G_ℓ commutes with S. Since we have excluded generators T which are proportional to $\mathbb{1}_3$, T must have at least two different eigenvalues. Therefore, T alone fixes one column u of U_ℓ. Since S commutes with T, u is also an eigenvector of S. However, since all eigenvectors of S are unique (because S fixes U_ν), u is also a column of U_ν. Therefore, U_ℓ and U_ν have two equal columns and U_{PMNS} is block-diagonal. Thus two mixing angles vanish, which is phenomenologically excluded.

C The character table of the group $Q(648)$

For completeness, we show the character table of the group $Q(648) = [648, 533]$ constructed with GAP in table 2.

Table of the character table of the group $Q(648) = [648, 533]$:

	$1C^1$	$1C^3$	$1C^3$	$24C^3$	$9C^2$	$9C^6$	$9C^6$	$54C^4$	$54C^{12}$	$54C^{12}$	$12C^3$	$12C^3$	$12C^3$	$72C^9$	$36C^6$	$36C^6$	$36C^6$	$12C^3$	$12C^3$	$12C^3$	$72C^9$	$36C^6$	$36C^6$	$36C^6$
$\mathbf{1}_1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}_2$	1	1	1	1	1	1	1	1	1	1	ω	ω	ω	ω	ω	ω	ω	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2
$\mathbf{1}_3$	1	1	1	1	1	1	1	1	1	1	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω^2	ω	ω	ω	ω	ω	ω	ω
$\mathbf{2}_1$	2	2	2	2	-2	-2	-2	0	0	0	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
$\mathbf{2}_2$	2	2	2	2	-2	-2	-2	0	0	0	$-\omega$	$-\omega$	$-\omega$	$-\omega$	$-\omega$	$-\omega$	$-\omega$	$-\omega^2$	$-\omega^2$	$-\omega^2$	$-\omega^2$	$-\omega^2$	$-\omega^2$	$-\omega^2$
$\mathbf{2}_3$	2	2	2	2	-2	-2	-2	0	0	0	$-\omega^2$	$-\omega^2$	$-\omega^2$	$-\omega^2$	$-\omega^2$	$-\omega^2$	$-\omega^2$	$-\omega$	$-\omega$	$-\omega$	$-\omega$	$-\omega$	$-\omega$	$-\omega$
$\mathbf{3}_1$	3	3	3	3	3	3	3	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{3}_2$	3	$3\omega^2$	3ω	0	-1	$-\omega^2$	$-\omega$	1	ω^2	ω	$2\omega+\omega^2$	$\omega+2\omega^2$	$-\omega-2\omega^2$	0	$-\omega^2$	$-\omega$	$-\omega^2$	$2\omega+\omega^2$	$\omega+2\omega^2$	$-\omega-2\omega^2$	0	$-\omega$	$-\omega^2$	$-\omega^2$
$\mathbf{3}_3$	3	$3\omega^2$	3ω	0	-1	$-\omega^2$	$-\omega$	1	ω^2	ω	$\omega+2\omega^2$	$2\omega+\omega^2$	$-2\omega-\omega^2$	0	$-\omega^2$	$-\omega$	$-\omega^2$	$\omega+2\omega^2$	$2\omega+\omega^2$	$-2\omega-\omega^2$	0	$-\omega$	$-\omega^2$	$-\omega^2$
$\mathbf{3}_4$	3	$3\omega^2$	3ω	0	-1	$-\omega^2$	$-\omega$	1	ω^2	ω	$-\omega-2\omega^2$	$-2\omega-\omega^2$	$\omega+2\omega^2$	0	$-\omega^2$	$-\omega$	$-\omega^2$	$-\omega-2\omega^2$	$-2\omega-\omega^2$	$\omega+2\omega^2$	0	$-\omega$	$-\omega^2$	$-\omega^2$
$\mathbf{3}_5$	3	3ω	$3\omega^2$	0	-1	$-\omega$	$-\omega^2$	1	ω	ω^2	$2\omega+\omega^2$	$\omega+2\omega^2$	$-\omega-2\omega^2$	0	$-\omega$	$-\omega^2$	$-\omega$	$2\omega+\omega^2$	$\omega+2\omega^2$	$-\omega-2\omega^2$	0	$-\omega^2$	$-\omega$	$-\omega$
$\mathbf{3}_6$	3	3ω	$3\omega^2$	0	-1	$-\omega$	$-\omega^2$	1	ω	ω^2	$\omega+2\omega^2$	$2\omega+\omega^2$	$-2\omega-\omega^2$	0	$-\omega$	$-\omega^2$	$-\omega$	$\omega+2\omega^2$	$2\omega+\omega^2$	$-2\omega-\omega^2$	0	$-\omega^2$	$-\omega$	$-\omega$
$\mathbf{3}_7$	3	3ω	$3\omega^2$	0	-1	$-\omega$	$-\omega^2$	1	ω	ω^2	$-\omega-2\omega^2$	$-2\omega-\omega^2$	$\omega+2\omega^2$	0	$-\omega$	$-\omega^2$	$-\omega$	$-\omega-2\omega^2$	$-2\omega-\omega^2$	$\omega+2\omega^2$	0	$-\omega^2$	$-\omega$	$-\omega$
$\mathbf{6}_1$	6	$6\omega^2$	6ω	0	2	$2\omega^2$	2ω	0	0	0	$2\omega+\omega^2$	$-\omega-2\omega^2$	$2\omega+\omega^2$	0	-1	$-\omega^2$	$-\omega$	$2\omega+\omega^2$	$-\omega-2\omega^2$	$2\omega+\omega^2$	0	-1	$-\omega$	$-\omega$
$\mathbf{6}_2$	6	$6\omega^2$	6ω	0	2	$2\omega^2$	2ω	0	0	0	$2\omega+\omega^2$	$-2\omega-\omega^2$	$\omega+2\omega^2$	0	-1	$-\omega^2$	$-\omega$	$2\omega+\omega^2$	$-2\omega-\omega^2$	$\omega+2\omega^2$	0	-1	$-\omega$	$-\omega$
$\mathbf{6}_3$	6	$6\omega^2$	6ω	0	2	$2\omega^2$	2ω	0	0	0	$-2\omega-\omega^2$	$\omega+2\omega^2$	$-\omega-2\omega^2$	0	-1	$-\omega^2$	$-\omega$	$-2\omega-\omega^2$	$\omega+2\omega^2$	$-\omega-2\omega^2$	0	-1	$-\omega$	$-\omega$
$\mathbf{6}_4$	6	6ω	$6\omega^2$	0	2	2ω	$2\omega^2$	0	0	0	$2\omega+\omega^2$	$-\omega-2\omega^2$	$2\omega+\omega^2$	0	-1	$-\omega$	$-\omega^2$	$2\omega+\omega^2$	$-\omega-2\omega^2$	$2\omega+\omega^2$	0	-1	$-\omega^2$	$-\omega^2$
$\mathbf{6}_5$	6	6ω	$6\omega^2$	0	2	2ω	$2\omega^2$	0	0	0	$-\omega-2\omega^2$	$-\omega-2\omega^2$	$\omega+2\omega^2$	0	-1	$-\omega$	$-\omega^2$	$-\omega-2\omega^2$	$-\omega-2\omega^2$	$\omega+2\omega^2$	0	-1	$-\omega^2$	$-\omega^2$
$\mathbf{6}_6$	6	6ω	$6\omega^2$	0	2	2ω	$2\omega^2$	0	0	0	$-2\omega-\omega^2$	$\omega+2\omega^2$	$-\omega-2\omega^2$	0	-1	$-\omega$	$-\omega^2$	$-2\omega-\omega^2$	$\omega+2\omega^2$	$-\omega-2\omega^2$	0	-1	$-\omega^2$	$-\omega^2$
$\mathbf{8}_1$	8	8	8	-1	0	0	0	0	0	0	2	2	2	-1	0	0	0	2	2	2	-1	0	0	0
$\mathbf{8}_2$	8	8	8	-1	0	0	0	0	0	0	$2\omega^2$	$2\omega^2$	$2\omega^2$	$-\omega^2$	0	0	0	2ω	2ω	2ω	$-\omega$	0	0	0
$\mathbf{8}_3$	8	8	8	-1	0	0	0	0	0	0	2ω	2ω	2ω	$-\omega$	0	0	0	$2\omega^2$	$2\omega^2$	$2\omega^2$	$-\omega^2$	0	0	0
$\mathbf{9}_1$	9	$9\omega^2$	9ω	0	-3	$-3\omega^2$	-3ω	$-\omega^2$	$-\omega^2$	$-\omega^2$	0	0	0	$-\omega^2$	0	0	0	0	0	0	$-\omega^2$	0	0	0
$\mathbf{9}_2$	9	9ω	$9\omega^2$	0	-3	-3ω	$-3\omega^2$	$-\omega$	$-\omega$	$-\omega$	0	0	0	$-\omega$	0	0	0	0	0	0	$-\omega$	0	0	0

Table 2. The character table of the group $Q(648) = [648, 533]$ constructed with GAP. The notation for the conjugacy classes of the group is NC^n, where N denotes the number of elements of the conjugacy class and n is the order of the elements of the class. ($\omega = \exp(2\pi i/3)$.)

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