Boundary conditions for General Relativity on AdS_3 and the KdV hierarchy

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ABSTRACT: It is shown that General Relativity with negative cosmological constant in three spacetime dimensions admits a new family of boundary conditions being labeled by a nonnegative integer k . Gravitational excitations are then described by “boundary gravitons” that fulfill the equations of the k -th element of the KdV hierarchy. In particular, $k = 0$ corresponds to the Brown-Henneaux boundary conditions so that excitations are described by chiral movers. In the case of $k = 1$, the boundary gravitons fulfill the KdV equation and the asymptotic symmetry algebra turns out to be infinite-dimensional, abelian and devoid of central extensions. The latter feature also holds for the remaining cases that describe the hierarchy ($k > 1$). Our boundary conditions then provide a gravitational dual of two noninteracting left and right KdV movers, and hence, boundary gravitons possess anisotropic Lifshitz scaling with dynamical exponent $z = 2k + 1$. Remarkably, despite spacetimes solving the field equations are locally AdS, they possess anisotropic scaling being induced by the choice of boundary conditions. As an application, the entropy of a rotating BTZ black hole is precisely recovered from a suitable generalization of the Cardy formula that is compatible with the anisotropic scaling of the chiral KdV movers at the boundary, in which the energy of AdS spacetime with our boundary conditions depends on z and plays the role of the central charge. The extension of our boundary conditions to the case of higher spin gravity and its link with different classes of integrable systems is also briefly addressed.

KEYWORDS: Black Holes, Classical Theories of Gravity, Gauge-gravity correspondence

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1 Introduction

In the study of the asymptotic structure of spacetime it is customary and natural to assume that the values of the lapse and shift functions, describing the deformations of spacelike surfaces, are held fixed to be constant at infinity [1–4]. Indeed, this choice ensures that observables, as the conserved charges, are measured with respect to fixed time and length scales. Although, this is certainly a reasonable and useful practice, it is not strictly a necessary one. Here we explore some of the consequences of choosing the time and length scales at the boundary in a non standard way, so that the corresponding Lagrange multipliers are fixed at infinity by a precise functional dependence on the dynamical fields. Afterwards, we focus in the case of General Relativity with negative cosmological constant in three spacetime dimensions. The standard analysis in this case was performed in the metric formalism by Brown and Henneaux [3]. In order to extend these results it turns out to be simpler to work in terms of two independent $sl(2, \mathbb{R})$ gauge fields, $A^\pm = \omega \pm \frac{e}{\ell}$, where ω and e stand for the spin connection and the dreibein, respectively, so that General Relativity can be formulated as a Chern-Simons theory [5, 6]. Following the lines of [7] the asymptotic fall-off of the gauge fields can be written as

$$A^\pm = g_\pm^{-1} (d + a^\pm) g_\pm, \quad (1.1)$$

so that the group elements $g_{\pm} = e^{\pm \log(r/\ell)L_0}$ entirely capture the radial dependence, and the components of the auxiliary connections $a^{\pm} = a_{\varphi}^{\pm}d\varphi + a_t^{\pm}dt$, depend only on time and the angular coordinate. According to the analysis in [8, 9], the asymptotic behaviour of the gauge fields is generically determined by

$$a_{\varphi}^{\pm} = L_{\pm 1} - \frac{1}{4}\mathcal{L}_{\pm}L_{\mp 1}; \quad a_t^{\pm} = \pm\Lambda^{\pm}[\mu^{\pm}], \quad (1.2)$$

where

$$\Lambda^{\pm}[\mu^{\pm}] = \mu^{\pm} \left(L_{\pm 1} - \frac{1}{4}\mathcal{L}_{\pm}L_{\mp 1} \right) \mp \mu^{\pm'}L_0 + \frac{1}{2}\mu^{\pm''}L_{\mp 1}, \quad (1.3)$$

and $\mathcal{L}^{\pm}, \mu^{\pm}$ stand for arbitrary functions of t, φ .¹

The asymptotic form of the field equations $F^{\pm} = dA^{\pm} + A^{\pm} \wedge A^{\pm} = 0$, then reduces to

$$\dot{\mathcal{L}}_{\pm} := \pm\mathcal{D}^{\pm}\mu^{\pm}, \quad (1.4)$$

with

$$\mathcal{D}^{\pm} := (\partial_{\varphi}\mathcal{L}_{\pm}) + 2\mathcal{L}_{\pm}\partial_{\varphi} - 2\partial_{\varphi}^3. \quad (1.5)$$

It is thus clear that in the reduced phase space, $\mathcal{L}_{\pm}$ describe the dynamical fields while $\mu^{\pm}$ are Lagrange multipliers.

It must be emphasized here that the set of boundary conditions is not yet specified at this step, because in order to do that, one needs to provide the precise form in which the Lagrange multipliers $\mu^{\pm}$ are fixed at infinity.

In the standard approach [7] the Lagrange multipliers are chosen to be fixed at the boundary according to $\mu^{\pm} = 1$, in agreement with Brown and Henneaux [3]. These boundary conditions can also be slightly generalized as in [8, 9] so that the Lagrange multipliers are chosen as $\mu^{\pm} = \mu_0^{\pm}(t, \varphi)$, where $\mu_0^{\pm}$ stand for arbitrary functions of t, φ that are held fixed at the boundary without variation, i.e., $\delta\mu_0^{\pm} = 0$.

In the next section we explore the set of different possible choices of Lagrange multipliers $\mu^{\pm}$ that are allowed by consistency of the action principle.

2 Specifying generic boundary conditions

The action principle for General Relativity in terms of $sl(2, \mathbb{R})$ gauge fields acquires the form

$$I = I_{\text{CS}}[A^{+}] - I_{\text{CS}}[A^{-}], \quad (2.1)$$

where $I_{\text{CS}}[A^{\pm}]$ stands for the Chern-Simons action. For the remaining analysis it is useful to split the connection along the spacelike and timelike components, $A^{\pm} = A_i^{\pm}dx^i + A_t^{\pm}dt$, so that

$$I_{\text{CS}}[A^{\pm}] = -\frac{\kappa}{4\pi} \int dt d^2x \varepsilon^{ij} \left\langle A_i^{\pm} \dot{A}_j^{\pm} - A_t^{\pm} F_{ij}^{\pm} \right\rangle + B_{\infty}^{\pm}. \quad (2.2)$$

Here the bracket corresponds to the trace in the fundamental representation of $sl(2, \mathbb{R})$, and $B_{\infty}^{\pm}$ stand for suitable boundary terms that are needed in order to ensure that the

¹Here L_i span each copy of $sl(2, \mathbb{R})$, while ℓ, G denote the AdS radius and the Newton constant, respectively, so that $\kappa = \frac{\ell}{4G}$. Hereafter, dot and prime stand for derivatives with respect to t and φ .

action principle attains an extremum everywhere. Taking the variation of the action with respect to the gauge fields, one finds that it vanishes provided the curvatures $F^\pm$ also do in the bulk, while for the asymptotic fall-off described by (1.1), (1.2), the variation of the boundary terms is found to be given by

$$\delta B_\infty^\pm = \mp \frac{\kappa}{8\pi} \int dt d\varphi \mu^\pm \delta \mathcal{L}_\pm.$$

Therefore, since $sl(2, \mathbb{R})$ gauge fields are assumed to be independent, the action principle attains a bona fide extremum provided the following integrability conditions are fulfilled:

$$\delta^2 B_\infty^\pm = \mp \frac{\kappa}{8\pi} \int dt d\varphi \delta \mu^\pm \wedge \delta \mathcal{L}_\pm = 0. \quad (2.3)$$

The integrability conditions are then solved by

$$\mu^\pm = \frac{\delta H^\pm}{\delta \mathcal{L}_\pm}, \quad (2.4)$$

where $H^\pm$ can be assumed to correspond to arbitrary functionals of $\mathcal{L}_\pm$ and their derivatives, i.e., $H^\pm = \int d\phi \mathcal{H}^\pm [\mathcal{L}_\pm, \mathcal{L}'_\pm, \mathcal{L}''_\pm, \dots]$, and hence, the boundary terms integrate as

$$B_\infty^\pm = \mp \frac{\kappa}{8\pi} \int dt d\varphi \mathcal{H}^\pm.$$

One then concludes that the boundary conditions become completely determined once the functionals $H^\pm$ are specified at the boundary. Consequently, the asymptotic form of the Lagrange multipliers $\mu^\pm$ is determined by eq. (2.4), which guarantees the integrability of the boundary term required by consistency of the action principle.

2.1 Asymptotic symmetries and conserved charges

Once the boundary conditions are generically specified through the choice of $H^\pm$, one already possesses all what is needed in order to study the asymptotic structure. By virtue of (1.1) the analysis of the asymptotic symmetries can be directly performed in terms of the auxiliary connections $a^\pm$. We then look for the subset of gauge transformations $\delta a^\pm = d\eta^\pm + [a^\pm, \eta^\pm]$ that preserve their form, given by (1.2). The asymptotic form of $a_\varphi^\pm$ is maintained for gauge transformations spanned by $\eta^\pm = \Lambda^\pm [\varepsilon^\pm]$, where $\varepsilon^\pm = \varepsilon^\pm(t, \varphi)$, and $\Lambda^\pm$ is defined in (1.3), provided the transformation law of the dynamical fields is given by

$$\delta \mathcal{L}_\pm = \mathcal{D}^\pm \varepsilon^\pm. \quad (2.5)$$

Preserving the asymptotic form of $a_t^\pm$ then implies the following conditions:

$$\delta \mu^\pm = \pm \dot{\varepsilon}^\pm + \varepsilon^\pm \mu^{\pm'} - \mu^\pm \varepsilon^{\pm'}. \quad (2.6)$$

Therefore, since the form of the Lagrange multipliers $\mu^\pm$ is determined by $H^\pm$ according to (2.4), the latter condition implies that the time derivatives of the parameters $\varepsilon^\pm$ associated to the asymptotic symmetries have to fulfill

$$\dot{\varepsilon}^\pm = \pm \frac{\delta}{\delta \mathcal{L}_\pm} \int d\phi \frac{\delta H^\pm}{\delta \mathcal{L}_\pm} \mathcal{D}^\pm \varepsilon^\pm, \quad (2.7)$$

which implies that $\varepsilon^\pm$ generically acquire a nontrivial dependence on the dynamical fields $\mathcal{L}_\pm$ and their derivatives.

Besides, in the canonical approach [1], the variation of the generators of the asymptotic symmetries is readily found to be given by

$$\delta Q^\pm [\varepsilon^\pm] = -\frac{\kappa}{8\pi} \int d\varphi \varepsilon^\pm \delta \mathcal{L}_\pm . \quad (2.8)$$

It is then worth pointing out that eq. (2.7) guarantees that the variation of the canonical generators is conserved in time ($\delta \dot{Q}^\pm = 0$) on-shell.

However, in order to integrate the variation of the canonical generators of the asymptotic symmetries in (2.8), one needs to know the general solution of eq. (2.7), which for a generic choice of boundary conditions specified by $H^\pm$, turns out to be a very hard task.

Nonetheless, this can always be done for the particular cases of asymptotic Killing vectors ∂_φ or ∂_t , when they belong to the asymptotic symmetries. Indeed, in that cases, the angular momentum reads

$$J = Q [\partial_\varphi] = \frac{\kappa}{8\pi} \int d\varphi (\mathcal{L}_+ - \mathcal{L}_-) , \quad (2.9)$$

while the variation of the total energy, given by

$$\delta E = \delta Q [\partial_t] = \frac{\kappa}{8\pi} \int d\varphi (\mu^+ \delta \mathcal{L}_+ + \mu^- \delta \mathcal{L}_-) , \quad (2.10)$$

by virtue of (2.4), integrates as

$$E = \frac{\kappa}{8\pi} (H^+ + H^-) . \quad (2.11)$$

In order to carry out the complete analysis of the asymptotic structure, concrete choices of boundary conditions have then to be given.

3 Selected choices of boundary conditions

A sensible criterium to fix the explicit form of $H^\pm$ turns out to allow as much asymptotic symmetries as possible, which amounts to know the general solution of (2.7) for arbitrary values of the dynamical fields and their derivatives. Indeed, an infinite number of asymptotic symmetries is certainly welcome because it helps in order to explicitly find the space of solutions that fulfill the boundary conditions. These criteria are certainly met in the cases that $H^\pm$ define integrable systems. In what follows, we provide a few (but still infinite) number of examples with the desired features.

3.1 $k = 0$: chiral movers (Brown-Henneaux)

One of the simplest possible choices of boundary conditions corresponds to the ones of Brown and Henneaux [3]. As aforementioned, these boundary conditions are specified by choosing $\mu_{(0)}^\pm = 1$, which according to (2.4), amounts to set $H_{(0)}^\pm = \int d\varphi \mathcal{H}_{(0)}^\pm$, with

$$\mathcal{H}_{(0)}^\pm := \mathcal{L}_\pm . \quad (3.1)$$

In this case the field equations (1.4) reduce to

$$\dot{\mathcal{L}}_{\pm} = \pm \mathcal{L}'_{\pm}, \quad (3.2)$$

describing chiral movers. Analogously, eq. (2.7) reads

$$\dot{\varepsilon}^{\pm} = \pm \varepsilon^{\pm'}, \quad (3.3)$$

so that the parameters that describe the asymptotic symmetries do not depend on the dynamical fields. Therefore, the variation of the canonical generators in eq. (2.8) directly integrates as

$$Q^{\pm}[\varepsilon^{\pm}] = -\frac{\kappa}{8\pi} \int d\varphi \varepsilon^{\pm} \mathcal{L}_{\pm}. \quad (3.4)$$

The algebra of the global charges can then be readily obtained from $\{Q[\varepsilon_1], Q[\varepsilon_2]\} = \delta_{\varepsilon_2} Q[\varepsilon_1]$, which by virtue of the transformation law of the dynamical fields in (2.5) reduces to two independent copies of the Virasoro algebra with the Brown-Henneaux central extension.

3.2 $k = 1$: KdV movers

A different simple choice of boundary conditions is given by $\mu_{(1)}^{\pm} = \mathcal{L}_{\pm}$, which corresponds to setting $H_{(1)}^{\pm} = \int d\varphi \mathcal{H}_{(1)}^{\pm}$

$$\mathcal{H}_{(1)}^{\pm} := \frac{1}{2} \mathcal{L}_{\pm}^2. \quad (3.5)$$

For this case the field equations (1.4) imply that left and right movers are described by the KdV equation, i.e.,

$$\dot{\mathcal{L}}_{\pm} = \pm (3\mathcal{L}_{\pm} \mathcal{L}'_{\pm} - 2\mathcal{L}_{\pm}'''). \quad (3.6)$$

The parameters associated to the asymptotic symmetries are subject to fulfill eq. (2.7), which here reduces to

$$\dot{\varepsilon}^{\pm} = \pm (3\mathcal{L}_{\pm} \partial_{\varphi} \varepsilon^{\pm} - 2\partial_{\varphi}^3 \varepsilon^{\pm}), \quad (3.7)$$

and since the KdV equation corresponds to an integrable system, we know its general solution assuming that $\varepsilon^{\pm}$ are local functions of $\mathcal{L}_{\pm}$ and their spatial derivatives. It is given by a linear combination of the form

$$\varepsilon^{\pm} = \sum_{j=0}^{\infty} \eta_{(j)}^{\pm} R_{(j)}^{\pm}, \quad (3.8)$$

where $\eta_{(j)}^{\pm}$ are constants and $R_{(j)}^{\pm}$ stand for the Gelfand-Dikii polynomials. They can be defined through the following recursion relation:²

$$\partial_{\varphi} R_{(j+1)}^{\pm} = \frac{j+1}{2j+1} \mathcal{D}^{\pm} R_{(j)}^{\pm}, \quad (3.9)$$

²For later convenience, we have chosen the factor in (3.9) such that the polynomials become normalized according to $R_{(j)} = \mathcal{L}^j + \dots$, where the ellipsis refers to terms that depend on derivatives of $\mathcal{L}$.

and they fulfill

$$R_{(j)}^{\pm} = \frac{\delta H_{(j)}^{\pm}}{\delta \mathcal{L}_{\pm}}. \quad (3.10)$$

In particular, according to eqs. (3.1) and (3.5), $R_{(0)}^{\pm} = \mu_{(0)}^{\pm} = 1$, and $R_{(1)}^{\pm} = \mu_{(1)}^{\pm} = \mathcal{L}_{\pm}$.

Therefore, for an arbitrary asymptotic symmetry, being spanned by (3.8), the variation of the canonical generators integrates as

$$Q^{\pm} [\varepsilon^{\pm}] = -\frac{\kappa}{8\pi} \sum_{j=0}^{\infty} \eta_{(j)}^{\pm} H_{(j)}^{\pm}. \quad (3.11)$$

The algebra of the canonical generators is then found to be an abelian one and devoid of central extensions, which goes by hand with the well-known fact that the conserved charges of an integrable system, as it is the case of KdV, are in involution.

In particular, the first four conserved charges of the series are explicitly given by

$$\begin{aligned} H_{(0)}^{\pm} &= \int d\varphi \mathcal{L}_{\pm}, & H_{(1)}^{\pm} &= \int d\varphi \frac{1}{2} \mathcal{L}_{\pm}^2, & H_{(2)}^{\pm} &= \int d\varphi \frac{1}{3} (\mathcal{L}_{\pm}^3 + 2\mathcal{L}_{\pm}'^2), \\ H_{(3)}^{\pm} &= \int d\varphi \frac{1}{4} \left(\mathcal{L}_{\pm}^4 + 8\mathcal{L}_{\pm} \mathcal{L}_{\pm}'^2 + \frac{16}{5} \mathcal{L}_{\pm}''^2 \right), \end{aligned} \quad (3.12)$$

and it is also worth pointing out that, for this choice of boundary conditions, according to (2.11), the total energy of a gravitational configuration is given by the sum of the energies of left and right KdV movers, i.e.,

$$E = \frac{\kappa}{16\pi} \int d\varphi (\mathcal{L}_{+}^2 + \mathcal{L}_{-}^2). \quad (3.13)$$

3.3 Generic k : KdV hierarchy

One is then naturally led to extend the previous analysis through a new family of boundary conditions that is labeled by a nonnegative integer k , so that the Brown-Henneaux boundary conditions as well as the ones that describes KdV movers are recovered for $k = 0, 1$, respectively. The boundary conditions are proposed to be such that the Lagrange multipliers are given by

$$\mu_{(k)}^{\pm} = R_{(k)}^{\pm} = \frac{\delta H_{(k)}^{\pm}}{\delta \mathcal{L}_{\pm}}, \quad (3.14)$$

so that the field equations (1.4) now describe left and right movers that evolve according to the k -th representative of the KdV hierarchy:

$$\dot{\mathcal{L}}_{\pm} = \pm \mathcal{D}^{\pm} R_{(k)}^{\pm}. \quad (3.15)$$

The asymptotic symmetries are then spanned by parameters $\varepsilon^{\pm}$ that fulfill (2.7) with $H^{\pm} = H_{(k)}^{\pm}$.

Note that in the case of boundary conditions with $k > 1$, the field equations (3.15) as well as the conditions on the parameters in (2.7) become severely modified as compared with the case of $k = 1$ (see eqs. (3.6) and (3.7)). Nonetheless, the remarkable properties

of the Gelfand-Dikii polynomials imply that the general solution of (2.7) for $k > 1$ is described precisely by the same series as in the case of $k = 1$, i.e., if $\varepsilon^\pm$ are assumed to be local functions of $\mathcal{L}_\pm$ and their spatial derivatives, the parameters are given by (3.8). Consequently, the corresponding canonical generators are precisely given by eq. (3.11), that possess an infinite dimensional abelian algebra with no central extensions. However, for the choice of boundary conditions described here, a gravitational configuration possesses a total energy that corresponds to the sum of the energies of left and right movers that evolve according to the k -th representative of the KdV hierarchy, given by $E = E_+ + E_-$, with

$$E_\pm = \frac{\kappa}{8\pi} H_{(k)}^\pm. \quad (3.16)$$

An interesting remark is in order. For a generic choice of the integer that labels the boundary conditions, given by k , the “boundary gravitons” that fulfill the field equations (3.15) possess an anisotropic Lifshitz scaling that is characterized by a dynamical exponent

$$z = 2k + 1. \quad (3.17)$$

This is because our boundary conditions make the field equations (3.15) to be invariant under³

$$t \rightarrow \lambda^z t, \quad \varphi \rightarrow \lambda \varphi, \quad \mathcal{L}_\pm \rightarrow \lambda^{-2} \mathcal{L}_\pm. \quad (3.18)$$

It is then worth highlighting that, although spacetimes that solve the field equations are locally AdS, they remarkably inherit an anisotropic scaling that is induced by our choice of boundary conditions in (3.14). Indeed, the corresponding line elements are manifestly invariant under (3.18), provided the radial coordinate scales as $r \rightarrow \lambda^{-1} r$. This can be explicitly seen in section 6.

As it is explained in the next section, the anisotropic Lifshitz scaling yields interesting consequences concerning the asymptotic growth of the number of states in the context of black hole entropy.

4 BTZ black hole with selected boundary conditions: global charges and thermodynamics

The BTZ black hole [10, 11] fits within the choice of boundary conditions in (3.14) for an arbitrary nonnegative integer k . Indeed, this class of configurations is described by constant $\mathcal{L}_\pm$, which trivially solves the field equations (3.15) that correspond to the k -th representative of KdV hierarchy. It is worth noting that once the spacetime metric is reconstructed from (1.1) and expressed in an ADM decomposition, it acquires a similar form as in the standard case, but where the lapse and shift are now described in terms of $\mu_{(k)}^\pm = \mathcal{L}_\pm^k$ (see section 6). AdS spacetime is then recovered for $\mathcal{L}_\pm = -1$.

³In order to explicitly check the invariance of the field equations under the anisotropic scaling, it is useful to take into account that under (3.18), the Gelfand-Dikii polynomials scale as $R_{(k)} \rightarrow \lambda^{-2k} R_{(k)}$.

Consequently, the energy associated to left and right movers in the case of the BTZ black hole can be directly obtained from (3.16), which in terms of the dynamical exponent (3.17), reads

$$E_{\pm} = \frac{\kappa}{2} \frac{1}{z+1} \mathcal{L}_{\pm}^{\frac{z+1}{2}}. \quad (4.1)$$

Note that for our boundary conditions the corresponding left and right energies of AdS spacetime manifestly depend on the dynamical exponent, so that they are given by

$$E_0^{\pm}[z] = \frac{\kappa}{2} \frac{1}{z+1} (-1)^{\frac{z+1}{2}}, \quad (4.2)$$

and turn out to be positive or negative in the case of even or odd values of k , respectively.

It is also worth highlighting that the Bekenstein-Hawking entropy

$$S = \frac{A}{4G} = \pi\kappa \left(\sqrt{\mathcal{L}_+} + \sqrt{\mathcal{L}_-} \right), \quad (4.3)$$

once expressed in terms of the extensive variables, given by the left and right energies in (4.1), reads

$$S = \pi\kappa \left(\frac{2}{\kappa} (z+1) \right)^{\frac{1}{z+1}} \left(E_+^{\frac{1}{z+1}} + E_-^{\frac{1}{z+1}} \right). \quad (4.4)$$

In terms of left and right temperatures $T_{\pm} = \beta_{\pm}^{-1}$, given by⁴

$$\beta_{\pm} = \frac{\partial S}{\partial E_{\pm}} = 2\pi \left(\frac{2}{\kappa} (z+1) E_{\pm} \right)^{-\frac{z}{z+1}}, \quad (4.5)$$

the black hole entropy acquires the form

$$S = \frac{\kappa}{2} (2\pi)^{1+\frac{1}{z}} \left(T_+^{\frac{1}{z}} + T_-^{\frac{1}{z}} \right). \quad (4.6)$$

Remarkably, the black hole entropy not only acquires the expected dependence on the energy or the temperature of noninteracting left and right movers of a field theory with Lifshitz scaling in two dimensions, see e.g., [12–17], but it can actually be precisely recovered from a suitable generalization of the Cardy formula in the case of anisotropic scaling, along the lines of [18].

In the next section, we show how the results in [18] extend to the case of left and right movers with the same Lifshitz scaling, whose corresponding ground state energies are allowed to depend on the dynamical exponent z .

5 Asymptotic growth of the number of states from anisotropic modular invariance

As explained in [18], thermal field theories with Lifshitz scaling in two dimensions, defined on a torus parametrized by $0 \leq \varphi < 2\pi$, $0 \leq t_E < \beta$, where t_E is the Euclidean time, naturally possess a duality between low and high temperatures, given by

$$\frac{\beta}{2\pi} \rightarrow \left(\frac{2\pi}{\beta} \right)^{\frac{1}{z}}, \quad (5.1)$$

⁴Left and right temperatures are related to the Hawking temperature according to $\frac{1}{T} = \frac{1}{2} \left(\frac{1}{T_+} + \frac{1}{T_-} \right)$.

so that the partition function can be assumed to be invariant under

$$Z[\beta; z] = Z\left[\frac{(2\pi)^{1+\frac{1}{z}}}{\beta^{\frac{1}{z}}}; \frac{1}{z}\right]. \quad (5.2)$$

In the case of independent noninteracting left and right movers with the same dynamical exponent z , the theory is defined on a torus with modular parameter

$$\tau = i\frac{\beta}{2\pi}, \quad (5.3)$$

where β stands for the complexification of left and right temperatures $\beta_{\pm}$. The high/low temperature duality relation then reads

$$\tau \rightarrow \frac{i^{1+\frac{1}{z}}}{\tau^{\frac{1}{z}}}, \quad (5.4)$$

and therefore, one can assume that the partition function fulfills

$$Z[\tau; z] = Z\left[i^{1+\frac{1}{z}}\tau^{-\frac{1}{z}}; z^{-1}\right]. \quad (5.5)$$

This is the anisotropic version of the well known S-modular invariance for chiral movers, which for $z = 1$ reduces to standard one in conformal field theory [19, 20].

If one assumes that the spectrum of left and right movers possesses a gap, the high/low temperature duality allows to obtain the precise value of the asymptotic growth of the number of states for fixed left and right energies $\Delta_{\pm}$. The existence of a gap ensures that at low temperatures the partition function turns out to be dominated by the ground state. It is also assumed that the ground state is not degenerate, being such that its left and right energies are negative and given by $-\Delta_0^{\pm}[z]$, which generically depend on the dynamical exponent. Therefore, the partition function at low temperature approximates as

$$Z[\tau; z] \approx e^{-2\pi i(\tau\Delta_0[z] - \bar{\tau}\bar{\Delta}_0[z])}, \quad (5.6)$$

so that in the high temperature regime, by virtue of (5.5), the partition function reads

$$Z[\tau; z] \approx e^{2\pi\left((-i\tau)^{-\frac{1}{z}}\Delta_0[z^{-1}] + (i\bar{\tau})^{-\frac{1}{z}}\bar{\Delta}_0[z^{-1}]\right)}. \quad (5.7)$$

Hence, at fixed energies $\Delta_{\pm} \gg \Delta_0^{\pm}[z]$, the asymptotic growth of the number of states can be directly obtained by evaluating (5.7) in the saddle point approximation, which is described by an entropy given by

$$S = 2\pi(z+1)\left[\left(\frac{\Delta_0[z^{-1}]}{z}\right)^z \Delta\right]^{\frac{1}{z+1}} + 2\pi(z+1)\left[\left(\frac{\bar{\Delta}_0[z^{-1}]}{z}\right)^z \bar{\Delta}\right]^{\frac{1}{z+1}}. \quad (5.8)$$

Note that the Cardy formula is recovered in the case of $z = 1$, where the role of the central charges is expressed through the lowest eigenvalues of the shifted Virasoro operators $L_0 \rightarrow L_0 - \frac{c}{24}$, see e.g., [19, 21–23].

In terms of the (Lorentzian) left and right energies, the entropy reads

$$S = 2\pi (z+1) \left[\left(\frac{|\Delta_0^+ [z^{-1}]|}{z} \right)^z \Delta_+ \right]^{\frac{1}{z+1}} + 2\pi (z+1) \left[\left(\frac{|\Delta_0^- [z^{-1}]|}{z} \right)^z \Delta_- \right]^{\frac{1}{z+1}}, \quad (5.9)$$

and from the first law in the canonical ensemble, $dS = \beta_+ d\Delta_+ + \beta_- d\Delta_-$, one finds that left and right movers follow an anisotropic version of the Stefan-Boltzmann law, given by

$$\Delta_{\pm} = \frac{1}{z} (2\pi)^{1+\frac{1}{z}} |\Delta_0^{\pm} [z^{-1}]| T_{\pm}^{1+\frac{1}{z}}, \quad (5.10)$$

which reduces to the standard one for $z = 1$.

In terms of left and right temperatures the entropy (5.9) then reads

$$S = (2\pi)^{1+\frac{1}{z}} \left(1 + \frac{1}{z} \right) \left(|\Delta_0^+ [z^{-1}]| T_+^{\frac{1}{z}} + |\Delta_0^- [z^{-1}]| T_-^{\frac{1}{z}} \right). \quad (5.11)$$

It is then very remarkable that the Bekenstein-Hawking entropy of the BTZ black hole, once expressed in terms of the energies associated to left and right movers that evolve according to the field equations of the k -th KdV hierarchy, given by (4.4), is precisely reproduced from (5.9). Indeed, this is the case if one identifies the left and right energies of the field theory with the ones of the black hole, i.e., $\Delta_{\pm} = E_{\pm}$, provided the ground state energies correspond to the ones of AdS spacetime with our boundary conditions, $\Delta_0^{\pm} [z] = -E_0^{\pm} [z]$, where $E_0^{\pm} [z]$ is given by (4.2).

Analogously, the anisotropic Stefan-Boltzmann law (5.10) agrees with (4.5), as well as eq. (5.11) does with (4.6).

6 Summary of results in terms of the spacetime metric

For a generic choice of boundary conditions, specified by $\mu^{\pm}$ in (2.4), the asymptotic structure of the spacetime metric can be reconstructed from the asymptotic form of the $sl(2, \mathbb{R})$ gauge fields given by (1.1), (1.2). The fall-off of the metric in the asymptotic region, $r \gg \ell$, reads

$$\begin{aligned} g_{tt} &= -(\mathcal{N}^2 - \ell^2 \mathcal{N}^{\varphi 2}) \frac{r^2}{\ell^2} + f_{tt} + \mathcal{O}(r^{-1}), \\ g_{tr} &= -\mathcal{N}^{\varphi'} \frac{\ell^2}{r} + \mathcal{O}(r^{-4}), \\ g_{t\varphi} &= \mathcal{N}^{\varphi} r^2 + f_{t\varphi} + \mathcal{O}(r^{-1}), \\ g_{rr} &= \frac{\ell^2}{r^2} + \mathcal{O}(r^{-5}), \\ g_{\varphi\varphi} &= r^2 + f_{\varphi\varphi} + \mathcal{O}(r^{-1}), \\ g_{r\varphi} &= \mathcal{O}(r^{-3}), \end{aligned} \quad (6.1)$$

with

$$\mu^{\pm} = \mathcal{N} \ell^{-1} \pm \mathcal{N}^{\varphi}. \quad (6.2)$$

Therefore, in an ADM decomposition, the lapse and the shift asymptotically behave as

$$N^\perp = \frac{r}{\ell} \mathcal{N} + \mathcal{O}(r^{-1}) , \quad (6.3)$$

$$N^r = -r \mathcal{N}^{\varphi'} + \mathcal{O}(r^{-1}) , \quad (6.4)$$

$$N^\varphi = \mathcal{N}^\varphi + \mathcal{O}(r^{-2}) , \quad (6.5)$$

and consequently, by virtue of (6.2), they become determined at the boundary according to eq. (2.4).

The functions $f_{\varphi\varphi}$, $f_{t\varphi}$, and f_{tt} are given by

$$\begin{aligned} f_{\varphi\varphi} &= \frac{\ell^2}{4} (\mathcal{L}_+ + \mathcal{L}_-) , \\ f_{t\varphi} &= -\frac{\ell^2}{2} \mathcal{N}^{\varphi''} + f_{\varphi\varphi} \mathcal{N}^\varphi + \frac{\ell}{4} (\mathcal{L}_+ - \mathcal{L}_-) \mathcal{N} , \\ f_{tt} &= \left(\frac{1}{\ell^2} \mathcal{N}^2 - \mathcal{N}^{\varphi 2} \right) f_{\varphi\varphi} + 2 f_{t\varphi} \mathcal{N}^\varphi + \ell^2 \mathcal{N}^{\varphi' 2} - \mathcal{N} \mathcal{N}'' . \end{aligned}$$

The asymptotic form of the metric (6.1) then implies that the Einstein equations with negative cosmological constant in vacuum are fulfilled provided

$$\dot{\mathcal{L}}_\pm = \pm \mathcal{D}^\pm \mu^\pm , \quad (6.6)$$

in full agreement with (1.4).

The asymptotic form of the metric (6.1) is mapped into itself under asymptotic Killing vectors ξ^μ that fulfill $\delta_\xi g_{\mu\nu} = \mathcal{L}_\xi g_{\mu\nu}$, whose components are given by

$$\xi^t = \frac{\ell}{2\mathcal{N}} \left[\varepsilon^+ + \varepsilon^- + \frac{\ell^2}{2\mathcal{N}r^2} \left(\mathcal{N} (\varepsilon^+ + \varepsilon^-)'' - \mathcal{N}'' (\varepsilon^+ + \varepsilon^-) \right) \right] + \mathcal{O}(r^{-4}) , \quad (6.7)$$

$$\begin{aligned} \xi^r &= -\frac{1}{2\mathcal{N}} \left[(\varepsilon^+ - \varepsilon^-)' \mathcal{N} - \ell \mathcal{N}^{\varphi'} (\varepsilon^+ + \varepsilon^-) \right] r \\ &\quad + \frac{\ell^3 \mathcal{N}^{\varphi'}}{4\mathcal{N}r} \left[(\varepsilon^+ + \varepsilon^-)'' - (\varepsilon^+ + \varepsilon^-) \frac{\mathcal{N}''}{\mathcal{N}} \right] \frac{1}{r} + \mathcal{O}(r^{-2}) , \end{aligned} \quad (6.8)$$

$$\begin{aligned} \xi^\varphi &= \frac{1}{2\mathcal{N}} \left[(\varepsilon^+ - \varepsilon^-) \mathcal{N} - \ell (\varepsilon^+ + \varepsilon^-) \mathcal{N}^\varphi \right] - \frac{\ell^2}{2\mathcal{N}r^2} \left[(\varepsilon^+ + \varepsilon^-)'' \mathcal{N} + \ell (\varepsilon^+ - \varepsilon^-)'' \mathcal{N}^\varphi \right. \\ &\quad \left. - \frac{\ell}{\mathcal{N}} (\varepsilon^+ + \varepsilon^-) (\mathcal{N} \mathcal{N}^{\varphi''} + \mathcal{N}'' \mathcal{N}^\varphi) \right] + \mathcal{O}(r^{-4}) , \end{aligned} \quad (6.9)$$

with $\varepsilon^\pm = \varepsilon^\pm(t, \varphi)$, provided $\mathcal{L}_\pm$ and $\mu^\pm$ transform precisely according to eqs. (2.5) and (2.6), respectively. Note that consistency of (2.6) implies that the time derivative of $\varepsilon^\pm$ fulfills (2.7), which means that these parameters in general depend on $\mathcal{L}_\pm$ and their derivatives.

The variation of the global charges associated to the asymptotic symmetries can then be obtained in the canonical approach [1], and they are found to agree with (2.8).

The general solution of the field equations that fulfills our boundary conditions (6.1) is described by spacetime metrics that in an ADM decomposition read,

$$ds^2 = - \left(N^\perp \right)^2 dt^2 + g_{ij} (N^i dt + dx^i) (N^j dt + dx^j) , \quad (6.10)$$

where the spacelike geometry is given by

$$dl^2 = g_{ij}dx^i dx^j = \frac{\ell^2}{r^2} \left[dr^2 + \ell^2 \left(\frac{r^2}{\ell^2} + \frac{1}{4}\mathcal{L}_+ \right) \left(\frac{r^2}{\ell^2} + \frac{1}{4}\mathcal{L}_- \right) d\varphi^2 \right], \quad (6.11)$$

with the following shift and lapse functions

$$N^r = -r\mathcal{N}^{\varphi'}, \quad N^\varphi = \mathcal{N}^\varphi + \frac{\left(\frac{r^2}{\ell^2}\mathcal{N} + \frac{1}{4}\mathcal{N}'' \right) (\mathcal{L}_+ - \mathcal{L}_-) - 2 \left(\frac{r^2}{\ell^2} + \frac{1}{8}(\mathcal{L}_+ + \mathcal{L}_-) \right) \mathcal{N}^{\varphi''}}{4\ell \left(\frac{r^2}{\ell^2} + \frac{1}{4}\mathcal{L}_+ \right) \left(\frac{r^2}{\ell^2} + \frac{1}{4}\mathcal{L}_- \right)}, \quad (6.12)$$

$$N^\perp = \frac{\ell \left[\left(\frac{r^4}{\ell^4} - \frac{1}{16}\mathcal{L}_+\mathcal{L}_- \right) \mathcal{N} + \frac{1}{2} \left(\frac{r^2}{\ell^2} + \frac{1}{8}(\mathcal{L}_+ + \mathcal{L}_-) \right) \mathcal{N}'' - \frac{\ell}{16}(\mathcal{L}_+ - \mathcal{L}_-) \mathcal{N}^{\varphi''} \right]}{r \sqrt{\left(\frac{r^2}{\ell^2} + \frac{1}{4}\mathcal{L}_+ \right) \left(\frac{r^2}{\ell^2} + \frac{1}{4}\mathcal{L}_- \right)}}, \quad (6.13)$$

respectively, where $\mathcal{L}_\pm$ satisfy eq. (6.6).

Therefore, for the choice of boundary conditions that is labelled by a nonnegative integer k in (3.14), the class of spacetimes described by (6.10) solves the Einstein equations with negative cosmological constant in vacuum as long as the functions $\mathcal{L}_\pm$ fulfill the field equations of left and right movers for the k -th element of the KdV hierarchy (6.6).

In this case, the line element (6.10) turns out to be manifestly invariant under the anisotropic (Lifshitz) scaling transformation given by (3.18), provided the radial coordinate scales as $r \rightarrow \lambda^{-1}r$.

In the particular case of constant $\mathcal{L}_\pm$, the metric (6.10) reduces to

$$ds^2 = \ell^2 \left[\frac{dr^2}{r^2} + \frac{\mathcal{L}_+}{4} (d\tilde{x}^+)^2 + \frac{\mathcal{L}_-}{4} (d\tilde{x}^-)^2 - \left(\frac{r^2}{\ell^2} + \frac{\ell^2 \mathcal{L}_+ \mathcal{L}_-}{16r^2} \right) d\tilde{x}^+ d\tilde{x}^- \right], \quad (6.14)$$

with

$$d\tilde{x}^\pm = \mu^\pm dt \pm d\varphi, \quad (6.15)$$

which corresponds to the BTZ black hole where the lapse and the shift are determined by $\mu^\pm = \mathcal{L}_\pm^k$. The left and right temperatures in (4.5) can then be readily found by requiring the Euclidean metric to be smooth at the horizon.

For a generic choice of boundary conditions, the lapse and the shift can be obtained from $\mu^\pm = \frac{\delta H^\pm}{\delta \mathcal{L}_\pm}$, and the Euclidean metric becomes regular for $\mu^\pm = \frac{2\pi}{\sqrt{\mathcal{L}_\pm}}$.

7 Discussion

We have shown that the dynamics of left and right movers that evolve according to the field equations of k -th element of KdV hierarchy can be fully geometrized. Indeed, the general solution of the three-dimensional Einstein equations with negative cosmological constant with our boundary conditions in (3.14) is described by spacetime geometries of the form (6.10), where $\mathcal{L}_\pm(t, \varphi)$ fulfill the field equations (3.15). Consequently, in this framework, the parameters that characterize the k -th KdV equation acquire a gravitational meaning.

Interestingly, different phenomena that have been observed in KdV can then be interpreted in the context of gravitation and vice versa.

In particular, let us consider the simplest solutions of the k -th KdV equations with $k \neq 0$, being described by constants $\mathcal{L}_\pm$, which possesses fixed values of left and right energies, determined by $H_{(k)}^\pm$. One can then generate a generic solution by acting on the simplest one with an arbitrary linear combination of the remaining generators $H_{(j)}^\pm$ with $j \neq k$. As it is well known in the case $k = 1$, see e.g. [24], the evolution in time of this generic configuration will settle down to describe a superposition of left/right solitons (cnoidal waves) with right/left dispersive waves, with the same energy of the original configuration, since $\{H_{(k)}^\pm, H_{(j)}^\pm\} = 0$.

The gravitational interpretation of this phenomenon is the following: the simplest configuration with constants $\mathcal{L}_\pm$ describes a BTZ black hole. The generic solution is then obtained by acting on the BTZ black hole with a generic element of the asymptotic symmetry group globally. Since the asymptotic symmetry generators commute with the Hamiltonian, this operation turns out to be a “soft boost” [25]. One then obtains an inequivalent gravitational configuration that can be regarded as a black hole with “soft gravitons” on it, in the sense of Hawking, Perry and Strominger [26]. This is because the new configuration is generically endowed with nontrivial global charges $H_{(j)}^\pm$, so that it is not related with the original one by a pure gauge transformation. Hence, since these global charges commute with the energy, they can be properly cast as “soft hair”. Nonetheless, one should check whether the generic configuration could properly describe a black hole, since a priori there is no guarantee that it possesses a regular horizon in the standard sense.

Besides, and quite remarkably, our choice of boundary conditions in (3.14) describes constant curvature spacetimes that are locally AdS with anisotropic Lifshitz scaling with dynamical exponent $z = 2k + 1$. This opens the possibility of studying nonrelativistic holography along the lines of [12, 17, 27, 28], but without the need of bulk geometries described by asymptotically Lifshitz spacetimes. In other words, Lifshitz scaling does not necessarily requires the use of Lifshitz spacetimes. Indeed, the BTZ black hole entropy with our boundary conditions, given by (4.4), is successfully reproduced from the asymptotic growth of the number of states of a field theory that describes left and right movers with Lifshitz scaling with the same dynamical exponent z in (5.8). In the non-chiral case, the latter formula reduces to the one proposed in [18], which also precisely reproduces the entropy of different classes of asymptotically Lifshitz black holes [18, 29, 30].⁵ A different link between asymptotically Lifshitz black holes and a generalization of the KdV equation has been pointed out in [39].

Note that in order to obtain the asymptotic growth of the number of states in (5.8), neither the asymptotic symmetries nor central charges were required. Indeed, the role of the central charge in our case is played by the left and right energies $\Delta_0^\pm [z^{-1}] = -E_0^\pm [z^{-1}]$ of AdS spacetime, given by (4.2) with $z \rightarrow z^{-1}$. In this context an interesting remark is worth to be mentioned. Note that for our boundary conditions with odd values of k , $E_0^\pm [z]$ turns out to be positive. However, in these cases, when one performs the Euclidean continuation, it is found that the Euclidean BTZ with our boundary conditions is diffeomorphic to thermal AdS as in [40, 41], but with reversed orientation, which is a consequence of the fact that the lapse of AdS spacetime reverses its sign. Therefore, left and right energies of

⁵Different generalizations of the Cardy formula have also been found for alternative scaling laws in three-dimensional spacetimes [31–38].

(Euclidean) thermal AdS possess an opposite sign as compared with the Lorentzian ones $E_0^\pm[z]$ of AdS spacetime for odd k .

It would also be interesting to explore different possible choices of boundary conditions for which the Lagrange multipliers depend on the dynamical fields in different ways as compared with the ones considered here. For instance, this is the case for the set of boundary conditions that has been recently proposed in [25], in which the Lagrange multipliers $\mu^\pm$ depend non-locally on $\mathcal{L}_\pm$. Indeed, up to the conventional normalization factor in (3.9), one verifies that the recursion relation allows to recover the Brown-Henneaux boundary conditions starting from the ones in [25], because the former stand for the kernel of ∂_φ , while the latter corresponds to the kernel of the operator $\mathcal{D}$. Consequently, the boundary conditions in [25] are associated with an anisotropic scaling with $z = 0$, that is consistent with labelling this set as a member of an extended hierarchy with $k = -1/2$. Such kind of extensions of the KdV hierarchy have been studied in e.g. [42–45].

A more conservative choice of boundary conditions corresponds to deform the Hamiltonian of the k -th KdV hierarchy $H_{(k)}^\pm$ by a linear combination of the remaining generators of the asymptotic symmetries, so that $H^\pm = H_{(k)}^\pm + \sum_{j \neq k} \xi_{(j)}^\pm H_{(j)}^\pm$, with $\xi_{(j)}^\pm$ constants. It is clear that in this case the asymptotic symmetries remain the same, but nonetheless, the Lifshitz scaling symmetry is lost. Note that in the context of the AdS/CFT correspondence, this is interpreted as a multitrace deformation of the dual theory [46–48]. The particular case of $k = 0$ then corresponds to a multitrace deformation of the Brown-Henneaux boundary conditions, which has been recently considered in [49], in the context of generalized Gibbs ensembles for which all of the additional charges $H_{(j)}^\pm$ possess nonvanishing chemical potentials $\xi_{(j)}^\pm$.

Our analysis of generic choices of boundary conditions can be readily generalized to the case of higher spin gravity in three-spacetime dimensions. Some particular examples have already been reported in [50–52].

If one follows the lines explained in section 2, one finds that the Lagrange multipliers are of the form $\mu_i = \mu_i(\mathcal{W}_s)$, where $\mathcal{W}_s$, with $i, s = 2, 3, \dots, N$, stand for the spin- s charges which appear in the component a_φ of the $sl(N, \mathbb{R})$ gauge fields, while μ_i enter through a_t . Note that a_φ, a_t form a Lax pair. Hence, for instance, our proposal for the boundary conditions in section 3.3, in the case of $sl(3, \mathbb{R})$ corresponds to choosing the Lagrange multipliers associated to the spin-2 and spin-3 charges, given by $\mathcal{L}$ and $\mathcal{W}$, respectively, according to $\mu_{\mathcal{L}} = R_{(k)}$ and $\mu_{\mathcal{W}} = S_{(k)}$, where the doublet $(R_{(k)}, S_{(k)})$ stands for the generalized Gelfand-Dikii polynomials associated to the Boussinesq hierarchy, see e.g., [53–55]. The case of $k = 0$ then reduces to the set of boundary conditions proposed in [8, 9].

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