Form factor and boundary contribution of amplitude

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ABSTRACT: The boundary contribution of an amplitude in the BCFW recursion relation can be considered as a form factor involving boundary operator and unshifted particles. At the tree-level, we show that by suitable construction of Lagrangian, one can relate the leading order term of boundary operators to some composite operators of $\mathcal{N} = 4$ super-Yang-Mills theory, then the computation of form factors is translated to the computation of amplitudes. We compute the form factors of these composite operators through the computation of corresponding double trace amplitudes.

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[2]The unusual ordering of authors is just to let authors get proper recognition of contributions under outdated practice in China.

Contents

1 Introduction

The ongoing research on the on-shell techniques has gone beyond its primal scattering amplitude domain, to the computation of form factor in recent years. The form factor, sometimes stated as a bridge linking on-shell amplitude and off-shell correlation function, is a quantity containing both on-shell states (ingredients for amplitudes) and gauge invariant operators (ingredients for correlation functions). Its computation can be traced back to the pioneering paper [1] nearly 30 years ago, where the Sudakov form factor of the bilinear scalar operator $\text{Tr}(\phi^2)$ is investigated up to two loops. At present, many revolutionary insights originally designed for the computation of amplitudes,[1] such as MHV vertex expansion [5], BCFW recursion relation [6, 7], color-kinematic duality [8, 9], unitarity cut [10, 11] method (and its generalization to D-dimension [12, 13]), generalized unitarity [14, 15], etc., have played their new roles in evaluating form factors.

These progresses are achieved in various papers. In paper [16], the BCFW recursion relation appears for the first time in the recursive computation of tree-level form factor,

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[1]See reviews, e.g., [2–4].

mainly for the bilinear scalar operator. As a consequence, the solution of recursion relation for split helicity form factor is conquered [17]. Intensive discussion on the recursion relation of form factor is provided later in [18]. A generalization to the form factor of full stress tensor multiplet is discussed in [17] and [19], where in the former one, supersymmetric version of BCFW recursion relation is pointed out to be applicable to super form factor. Shortly after, the color-kinematic duality is implemented in the context of form factor [20], both at tree and loop-level, to generate the integrand of form factor. Most recently, the elegant formulation of amplitudes based on Grassmannian prescription [21] is also extended to tree-level form factors [22]. At loop-level, the form factor is generally computed by unitarity cut method. The generic Maximal-Helicity-Violating (MHV) super form factor as well as some Next-MHV (NMHV) form factor at one-loop are computed in [17, 23–25] with compact results. The Sudakov form factor is computed to three loops in [26–28]. The three-point two-loop form factor of half-BPS operator is achieved in [29], and the general n-point form factor as well as the remainder functions in [30]. The scalar operator with arbitrary number of scalars is discussed in [19, 30, 31]. Beyond the half-BPS operators, form factors of non-protected operators, such as dilatation operator [32], Konishi operator [33], operators in the SU(2) sectors [34], are also under investigation. Furthermore, the soft theorems for the form factor of half-BPS and Konishi operators are studied at tree and one-loop level [35], showing similarity to amplitude case. Carrying on the integrand result of [20], the master integrals for four-loop Sudakov form factor is determined in [36]. An alternative discussion on the master integrals of form factor in massless QCD can be found in [37]. Similar unitarity based studies on Sudakov form factor of three-dimensional ABJM theories are also explored [38–40].

The above mentioned achievements encode the belief that the state-of-art on-shell techniques of amplitude would also be applicable to form factor. Recently, the advances in the computation of boundary contribution have revealed another connection between form factor and amplitude. When talking about the BCFW recursion relation of amplitude, the boundary contribution is generally assumed to be absent. However this assumption is not always true, for example, it fails in the theories involving only scalars and fermions or under the "bad" momentum deformation. Many solutions have been proposed (by auxiliary fields [41, 42], analyzing Feynman diagrams [43–45], studying the zeros [46–48], the factorization limits [49], or using other deformation [50–52]) to deal with the boundary contribution in various situations. Most recently, a new multi-step BCFW recursion relation algorithm [53–55] is proposed to detect the boundary contribution through certain poles step by step. Especially in paper [54], it is pointed out that the boundary contribution possesses similar BCFW recursion relation as amplitudes, and it can be computed recursively from the lower-point boundary contribution. Based on this idea, later in paper [56], the boundary contribution is further interpreted as form factor of certain composite operator named *boundary operator*, while the boundary operator can be extracted from the operator product expansion (OPE) of deformed fields.[2]

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[2]In [57], inspired by the discussion of the Regge limits in string theory [58], the large z behavior of string theory tree amplitudes is analyzed using the OPE of shifted vertex operators. See also [59] for BCFW recursion of string amplitudes.

The idea of boundary operator motives us to connect the computation of form factor to the boundary contribution of amplitudes. Since a given boundary contribution of amplitude can be identified as a form factor of certain boundary operator, we can also interpret a given form factor as the boundary contribution of certain amplitude. In paper [56], the authors showed how to construct the boundary operator starting from a known Lagrangian. We can reverse the logic and ask the question: for a given operator, how can we construct a Lagrangian whose boundary operator under certain momentum deformation is exactly the operator of request? In this paper, we try to answer this question by constructing the Lagrangian for a class of so called composite operators. Once the Lagrangian is ready, we can compute the corresponding amplitude, take appropriate momentum shifting and extract the boundary contribution, which is identical (or proportional) to the form factor of that operator. By this way, the computation of form factor can be considered as a problem of computing the amplitude of certain theory.

This paper is structured as follows. In section 2, we briefly review the BCFW recursion relation and boundary operator. We also list the composite operators of interest, and illustrate how to construct the Lagrangian that generates the boundary operators of request. In section 3, using Sudakov form factor as example, we explain how to compute the form factor through computing the boundary contribution of amplitude, and demonstrate the computation by recursion relation of form factor, amplitude and boundary contribution. We show that these three ways of understanding lead to the same result. In section 4, we compute the form factors of composite operators by constructing corresponding Lagrangian and working out the amplitude of double trace structure. Conclusion and discussion can be found in section 5, while in the appendix, the construction of boundary operator starting from Lagrangian is briefly reviewed for reader's convenience, and the discussion on large z behavior is presented.

2 From boundary contribution to form factor

The BCFW recursion relation [6, 7] provides a new way of studying scattering amplitude in S-matrix framework. Using suitable momentum shifting, for example,

$$\widehat{p}_i = p_i - zq, \quad \widehat{p}_j = p_j + zq \quad \text{while} \quad q^2 = p_i \cdot q = p_j \cdot q = 0, \tag{2.1}$$

one can treat the amplitude as an analytic function $A(z)$ of single complex variable, with poles in finite locations and possible non-vanishing terms in boundary, while the physical amplitude sits at $z = 0$ point. Assuming that under certain momentum shifting, $A(z)$ has no boundary contribution in the contour integration $\frac{1}{2\pi i} \oint \frac{dz}{z} A(z)$, i.e., $A(z) \to 0$ when $z \to \infty$, then the physical amplitude $A(z = 0)$ can be purely determined by the residues of $A(z)$ at finite poles. However, if $A(z)$ does not vanish around the infinity, for example when taking a "bad" momentum shifting or in theories such as $\lambda \phi^4$, the boundary contribution would also appear as a part of physical amplitude. Most people would try to avoid dealing with such theories as well as the "bad" momentum shifting, since the evaluation of boundary contribution is much more complicated than taking the residues of $A(z)$.

Although it is usually unfavored during the direct computation of amplitude, authors in paper [56] found that the boundary contribution is in fact *a form factor involving boundary operator and unshifted particles*,

$$B^{\langle 1|2]} = \langle \Phi(p_3) \cdots \Phi(p_n) | \mathcal{O}^{\langle 1|2]}(0) | 0 \rangle \,, \tag{2.2}$$

where $\Phi(p_i)$ denotes arbitrary on-shell fields, and momenta of $\Phi(p_1), \Phi(p_2)$ have been shifted according to eq. (2.1). The momentum q carried by the boundary operator is $q = -p_1 - p_2 = \sum_{i=3}^n p_i$. Eq. (2.2) is identical to a $(n-2)$-point form factor generated by operator $\mathcal{O}^{\langle 1|2]}$ with off-shell momentum $q^2 \neq 0$. The observation (2.2) provides a new way of computing form factor,

1. Construct the Lagrangian, and compute the corresponding amplitude,

2. Take the appropriate momentum shifting, and pick up the boundary contribution,

3. Read out the form factor from boundary contribution after considering LSZ reduction.

In paper [56], the authors illustrated how to work out the boundary operator $\mathcal{O}^{\langle \Phi_i | \Phi_j]}$ from Lagrangian of a given theory under momentum shifting of two selected external fields. Starting from a Lagrangian, one can eventually obtain a boundary operator. For example, a real massless scalar theory with ϕ^m interaction

$$L = -\frac{1}{2}(\partial\phi)^2 + \frac{\kappa}{m!}\phi^m \,, \tag{2.3}$$

under momentum shifting of two scalars (say ϕ_1 and ϕ_2) will produce a boundary operator

$$\mathcal{O}^{\langle \phi_1 | \phi_2]} = \frac{\kappa}{(m-2)!}\phi^{m-2} \,. \tag{2.4}$$

Hence the boundary contribution of a n-point amplitude $A_n(\phi_1, \ldots, \phi_n)$ in this $\kappa\phi^m$ theory under $\langle \phi_1 | \phi_2]$-shifting is identical to the $(n-2)$-point form factor

$$\mathcal{F}_{\mathcal{O}^{\langle \phi_1 | \phi_2]}, n-2}(\phi_3, \ldots, \phi_n; q) \equiv \frac{\kappa}{(m-2)!} \langle \phi_3 \cdots \phi_n | \phi^{m-2}(0) | 0 \rangle \,. \tag{2.5}$$

However, this form factor is not quite interesting. We are interested in certain kind of operators, such as bilinear half-BPS scalar operator $\mathrm{Tr}(\phi^{AB}\phi^{AB})$ or chiral stress-tensor operator $\mathrm{Tr}(W^{++}W^{++})$ in $\mathcal{N}=4$ super-Yang-Mills (SYM) theory, where W^{++} is a particular projection of the chiral vector multiplet superfield $W^{AB}(x, \theta)$ in SYM. What we want to do is to compute the form factor for a given operator, but not the operators generated from arbitrary Lagrangian. More explicitly, if we want to compute the form factor of operator $\mathcal{O}$, we should first construct a Lagrangian whose boundary operator is identical (or proportional) to $\mathcal{O}$. With such Lagrangian in hand, we can then compute the corresponding amplitude, take the momentum shifting and pick up the boundary contribution. So the problem is how to construct the corresponding Lagrangian.

2.1 The operators of interest

It is obvious that the construction of Lagrangian depends on the operators we want to produce. In this paper, we will study the so called gauge-invariant local composite operators, which are built as traces of product of gauge-covariant fields at a common spacetime point. These fields are taken to be the component fields of $\mathcal{N}=4$ superfield $\Phi^{\mathcal{N}=4}$ [60], given by six real scalars $\phi^I, I = 1, \ldots, 6$(or 3 complex scalars ϕ^{AB}), four fermions $\psi_\alpha^A = \epsilon^{ABCD}\psi_{BCD\alpha}$, four anti-fermions $\bar{\psi}_{A\dot\alpha}$ and the field strength $F_{\mu\nu}$, where $\alpha, \beta, \dot\alpha, \dot\beta = 1, 2$ are spinor indices, $A, B, C, D = 1, 2, 3, 4$ are SU(4) R-symmetric indices, and $\mu, \nu = 0, 1, 2, 3$ are spacetime indices. The field strength can be further split into self-dual and anti-self-dual parts $F_{\alpha\beta}, \bar{F}_{\dot\alpha\dot\beta}$:

$$F_{\alpha\beta\dot\alpha\dot\beta} = F_{\mu\nu}(\sigma^\mu)_{\alpha\dot\alpha}(\sigma^\nu)_{\beta\dot\beta} = \sqrt{2}\epsilon_{\dot\alpha\dot\beta}F_{\alpha\beta} + \sqrt{2}\epsilon_{\alpha\beta}\bar{F}_{\dot\alpha\dot\beta}, \tag{2.6}$$

corresponding to positive gluon and negative gluon respectively.

The number of fields inside the trace is called the length of operator, and the simplest non-trivial ones are the length two operators. There is no limit on the length of operator, for example, the bilinear half-BPS scalar operator $\text{Tr}(\phi^I\phi^J)$ is length two, while we could also have length L scalar operator $\text{Tr}(\phi^{I_1}\cdots\phi^{I_L})$. The operators can also carry spinor indices, such as $\mathcal{O}^{\alpha\beta\dot\alpha\dot\beta} = \text{Tr}(\psi^{A\alpha}\psi^{B\beta}\bar{F}^{\dot\alpha\dot\beta})$ in the $(1,1)$ representation under Lorentz group $\text{SU}(2) \times \text{SU}(2)$.

We will mainly focus on the length two operators. These operators can be classified by their spins and labeled by their representations under $\text{SU}(2) \times \text{SU}(2)$ group. For spin-0 operators in $(0,0)$-representation, we have

$$\mathcal{O}_\text{I}^{[0]} = \text{Tr}(\phi^I\phi^J), \qquad \mathcal{O}_\text{II}^{[0]} = \text{Tr}(\psi^{A\alpha}\psi_\alpha^B), \qquad \mathcal{O}_\text{III}^{[0]} = \text{Tr}(F^{\alpha\beta}F_{\alpha\beta}),$$
$$\bar{\mathcal{O}}_\text{II}^{[0]} = \text{Tr}(\bar{\psi}_A^{\dot\alpha}\bar{\psi}_{B\dot\alpha}), \qquad \bar{\mathcal{O}}_\text{III}^{[0]} = \text{Tr}(\bar{F}^{\dot\alpha\dot\beta}\bar{F}_{\dot\alpha\dot\beta}). \tag{2.7}$$

For spin-$\frac{1}{2}$ operators in $(\frac{1}{2}, 0)$ or $(0, \frac{1}{2})$-representation, we have

$$\mathcal{O}_\text{I}^{[1/2]} = \text{Tr}(\phi^I\psi^{A\alpha}), \qquad \mathcal{O}_\text{II}^{[1/2]} = \text{Tr}(\psi_\beta^A F^{\beta\alpha}),$$
$$\bar{\mathcal{O}}_\text{I}^{[1/2]} = \text{Tr}(\phi^I\bar{\psi}_A^{\dot\alpha}), \qquad \bar{\mathcal{O}}_\text{II}^{[1/2]} = \text{Tr}(\bar{\psi}_{A\dot\beta}\bar{F}^{\dot\beta\dot\alpha}). \tag{2.8}$$

For spin-1 operators in $(1, 0)$ or $(0, 1)$-representation, we have

$$\mathcal{O}_\text{I}^{[1]} = \text{Tr}(\psi^{A\alpha}\psi^{B\beta} + \psi^{A\beta}\psi^{B\alpha}), \qquad \mathcal{O}_\text{II}^{[1]} = \text{Tr}(\phi^I F^{\alpha\beta}),$$
$$\bar{\mathcal{O}}_\text{I}^{[1]} = \text{Tr}(\bar{\psi}_A^{\dot\alpha}\bar{\psi}_B^{\dot\beta} + \bar{\psi}_A^{\dot\beta}\bar{\psi}_B^{\dot\alpha}), \qquad \bar{\mathcal{O}}_\text{II}^{[1]} = \text{Tr}(\phi^I\bar{F}^{\dot\alpha\dot\beta}), \tag{2.9}$$

and in $(\frac{1}{2}, \frac{1}{2})$-representation,

$$\mathcal{O}_\text{III}^{[1]} = \text{Tr}(\psi^{A\alpha}\bar{\psi}_B^{\dot\alpha}). \tag{2.10}$$

For spin-$\frac{3}{2}$ operators in $(1, \frac{1}{2})$ or $(\frac{1}{2}, 1)$-representation, we have

$$\mathcal{O}_\text{I}^{[3/2]} = \text{Tr}(\bar{\psi}_A^{\dot\alpha}F^{\alpha\beta}), \qquad \bar{\mathcal{O}}_\text{I}^{[3/2]} = \text{Tr}(\psi^{A\alpha}\bar{F}^{\dot\alpha\dot\beta}). \tag{2.11}$$

and in $(\frac{3}{2}, 0)$ or $(0, \frac{3}{2})$-representation,

$$\mathcal{O}_\text{II}^{[3/2]} = \text{Tr}(\psi^{A\gamma}F^{\alpha\beta}), \qquad \bar{\mathcal{O}}_\text{II}^{[3/2]} = \text{Tr}(\bar{\psi}_A^{\dot\gamma}\bar{F}^{\dot\alpha\dot\beta}). \tag{2.12}$$

For spin-2 operators in $(1,1)$-representation, we have

$$\mathcal{O}_{\mathrm{I}}^{[2]} = \mathrm{Tr}(F^{\alpha\beta}\bar{F}^{\dot\alpha\dot\beta}).\tag{2.13}$$

For operators of the same class, we can apply similar procedure to construct the Lagrangian. The operators with length larger than two can be similarly written down, and classified according to their spins and representations. For those whose spins are no larger than 2, we can apply the same procedure as is done for length two operators. while if their spins are larger than 2, we need multiple shifts.

Some of above operators are in fact a part of the chiral stress-tensor multiplet operator in $\mathcal{N}=4$ SYM [61, 62], and their form factors are components of $\mathcal{N}=4$ super form factor. However, we have assumed that, all indices of these gauge-covariant fields are general, so above operators are not limited to the chiral part, they are quite general.

2.2 Constructing the Lagrangian

One important property shared by above operators is that they are all traces of fields. Tree-level amplitudes of ordinary gauge theory only possess single trace structure. From the shifting of two external fields, one can not generate boundary operators with trace structures, which can be seen in [56]. The solution is to intentionally add a double trace term in the standard Lagrangian. The added term should be gauge-invariant, and generate the corresponding operator under selected momentum shifting.

For a given operator $\mathcal{O}$ of interest, let us add a double trace term ΔL to the $\mathcal{N}=4$ Lagrangian L_{SYM},

$$L_{\mathcal{O}} = L_{\mathrm{SYM}} + \frac{\kappa}{N}\,\mathrm{Tr}(\Phi^{\alpha'_1}\Phi^{\alpha'_2})\mathcal{O} + \frac{\bar\kappa}{N}\,\mathrm{Tr}(\Phi^{\dagger}_{\alpha'_1}\Phi^{\dagger}_{\alpha'_2})\bar{\mathcal{O}},\tag{2.14}$$

where $\mathrm{SU}(N)$ group is assumed, $\kappa,\bar\kappa$ are coupling constants for the double trace interactions[3](which can be re-scaled to fit the overall factor of final result) and $\Phi^{\alpha'}$, $\Phi^{\dagger}_{\alpha'}$ denotes[4] any type of fields among $\phi^I, \psi^{A\alpha}, \bar\psi^{\dot\alpha}_A, F^{\alpha\beta}, \bar{F}^{\dot\alpha\dot\beta}$. The spinor indices are not explicitly written down for $\Phi, \Phi^{\dagger}$, however we note that they should be contracted with the spinor indices of the operator, so that the added Lagrangian terms are Lorentz invariant. We will show that at the large N limit, momentum shifting of two fields in ΔL indeed generates the boundary operator $\mathcal{O}$.

The tree-level amplitudes defined by Lagrangian $L_{\mathcal{O}}$ can have single trace pieces or multiple trace pieces. A full $(n+2)$-point amplitude

$$A_{n+2}^{\mathrm{full}}(\Phi^{\alpha_1 a_1},\ldots,\Phi^{\alpha_n a_n},\Phi^{\alpha_{n+1}a},\Phi^{\alpha_{n+2}b})$$

thus can be decomposed into color-ordered partial amplitudes A as

$$A_{n+2}^{\mathrm{full}} = A_{n+2}(1,2,\ldots,n+2)\,\mathrm{Tr}(t^{a_1}\cdots t^{a_n}t^a t^b) + \cdots\tag{2.15}$$
$$+ \frac{1}{N}A_{k;n+2-k}(1,\ldots,k;k+1,\ldots,n+2)\,\mathrm{Tr}(t^{a_1}\cdots t^{a_k})\,\mathrm{Tr}(t^{a_{k+1}}\cdots t^a t^b) + \cdots$$

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[3]Throughout this paper, we consider operators with various classical scaling dimensions. So in general the corresponding couplings (κ's) have non-vanishing scaling dimensions.

[4]The definition of $\Phi, \Phi^{\dagger}$ can be found in (A.3), and remind that the index here of $\Phi, \Phi^{\dagger}$ is not spinor index but the index of their components, which specifies Φ to be scalar, fermion or gluon.

where A_n denotes n-point single trace amplitude, $A_{k;n-k}$ denotes n-point double trace amplitude. We use i to abbreviate Φ_i, and $\cdots$ stands for all possible permutation terms and other higher order multiple trace pieces. Since the operator $\mathcal{O}$ we want to generate is single trace, the terms with higher multiple trace in $\cdots$ is then irrelevant for our discussion, and also they can be ignored at large N. Now let us contract the color indices a, b, which gives[5]

$$A_{n+2}^{\text{full}} = \frac{N^2 - 1}{N} A_{n+2}(1, 2, \ldots, n+2) \operatorname{Tr}(t^{a_1} \cdots t^{a_n}) + \cdots \tag{2.16}$$
$$+ \frac{N^2 - 1}{N^2} A_{k;n+2-k}(1, \ldots, k; k+1, \ldots, n+2) \operatorname{Tr}(t^{a_1} \cdots t^{a_k}) \operatorname{Tr}(t^{a_{k+1}} \cdots t^{a_n}) + \cdots$$

In this case, the $O(N)$ order terms in (2.16) come from two places, one is the single trace part in (2.15) when t^a and t^b are adjacent, the other is the double trace part in (2.15) whose color factor has the form $\operatorname{Tr}(\cdots) \operatorname{Tr}(t^a t^b)$. So when color indices a, b are contracted, the leading contribution of the full $(n+2)$-point amplitude is

$$A_{n+2}^{\text{full}} = N \operatorname{Tr}(t^{a_1} \cdots t^{a_n}) \mathcal{K}(1, 2, \ldots, n) + \text{possible permutation}\{1, 2, \ldots, n\}, \tag{2.17}$$

where

$$\mathcal{K}(1, \ldots, n) \equiv A_{n+2}(1, \ldots, n, n+1, n+2) + A_{n+2}(1, \ldots, n, n+2, n+1)$$
$$+ A_{n;2}(1, \ldots, n; n+1, n+2). \tag{2.18}$$

The first two terms in $\mathcal{K}$ are the same as the corresponding color-ordered single trace amplitudes, since the other double trace terms in the Lagrangian will not contribute to the $O(N)$ order at tree-level. The third term in $\mathcal{K}$ is double trace amplitude of the trace form $\operatorname{Tr}(\cdots) \operatorname{Tr}(t^a t^b)$, and the Feynman diagrams contributing to this amplitude are those whose Φ_{n+1} and Φ_{n+2} are attached to the same double trace vertex, while the color indices of Φ_{n+1}, Φ_{n+2} are separated from others.

Now let us examine the large z behavior of the amplitude under momentum shifting $\langle \Phi_{n+1}^{\alpha_{n+1}} | \Phi_{n+2}^{\alpha_{n+2}}]$. Since the color indices of two shifted legs are contracted, it is equivalent to consider the large z behavior of $\mathcal{K}(1, 2, \ldots, n)$ under such shifting. Following [56], we find that at the large N limit, the leading interaction part V is given by

$$V^{\alpha\beta} = V_{\text{SYM}}^{\alpha\beta} + N\bar{\kappa} \left(\delta_{\alpha_1'}^{\alpha} \delta_{\alpha_2'}^{\beta} + \delta_{\alpha_2'}^{\alpha} \delta_{\alpha_1'}^{\beta} \right) \bar{\mathcal{O}} + N\kappa (T^{\alpha_1'\alpha} T^{\alpha_2'\beta} + T^{\alpha_2'\alpha} T^{\alpha_1'\beta}) \mathcal{O}, \tag{2.19}$$

where $T^{\alpha\beta}$ is defined through $\Phi^{\alpha} = T^{\alpha\beta} \Phi_{\beta}^{\dagger}$, and $\alpha_1' = \alpha_{n+1}, \alpha_2' = \alpha_{n+2}$, indicating that the shifted fields Φ_{n+1}, Φ_{n+2} are the two fields of $\operatorname{Tr}(\Phi^{\alpha_1'} \Phi^{\alpha_2'})$ in (2.14) with specific field type. In general, the OPE of shifted fields has the form [56]

$$\mathcal{Z}(z) = \epsilon_{\alpha}^{n+1} \epsilon_{\beta}^{n+2} \left[V^{\alpha\beta} - V^{\alpha\beta_1} (D_0^{-1})_{\beta_1\beta_2} V^{\beta_2\beta} + \cdots \right], \tag{2.20}$$

where $\epsilon_{\alpha}^{n+1}, \epsilon_{\beta}^{n+2}$ are external wave functions of Φ_{n+1}, Φ_{n+2}. The terms with $(D_0^{-1})^k$ correspond to Feynman diagrams with k hard propagators. The $\mathcal{Z}(z)$ for $L_{\mathcal{O}}$ contains two parts,

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[5]Remind the identity $(t^a)_{i_1}^{\bar{j}_1} (t^a)_{i_2}^{\bar{j}_2} = \delta_{i_1}^{\bar{j}_2} \delta_{i_2}^{\bar{j}_1} - \frac{1}{N} \delta_{i_1}^{\bar{j}_1} \delta_{i_2}^{\bar{j}_2}$.

one from the single trace and the other from double trace. The single trace amplitudes in $\mathcal{K}$ originate from Feynman diagrams with vertices of $\mathcal{N} = 4$ Lagrangian, thus their $\mathcal{Z}(z)$ can be directly obtained by replacing $V^{\alpha\beta}$ with $V^{\alpha\beta}_{\text{SYM}}$. The double trace amplitudes in $\mathcal{K}$ originate from Feynman diagrams with double trace vertices. Because the two shifted fields Φ_{n+1}, Φ_{n+2} should be attached to the same double trace vertex, in this case the hard propagator will not appear in the corresponding Feynman diagrams. Thus for this part, we only need to keep the first term in (2.20) (more explicitly, the terms with single $\mathcal{O}$ or $\bar{\mathcal{O}}$ in (2.19)). Combined together, we have

$$\mathcal{Z}(z) = \mathcal{Z}_{\text{SYM}}(z) + \epsilon^{n+1}_\alpha \epsilon^{n+2}_\beta N\bar{\kappa}(\delta^\alpha_{\alpha'_1}\delta^\beta_{\alpha'_2} + \delta^\alpha_{\alpha'_2}\delta^\beta_{\alpha'_1})\bar{\mathcal{O}}$$
$$+ \epsilon^{n+1}_\alpha \epsilon^{n+2}_\beta N\kappa(T^{\alpha'_1\alpha}T^{\alpha'_2\beta} + T^{\alpha'_2\alpha}T^{\alpha'_1\beta})\mathcal{O}\,. \qquad (2.21)$$

The summation of α, β runs over all types of fields. For a given momentum shifting $\alpha'_1 = \alpha_{n+1}$, $\alpha'_2 = \alpha_{n+2}$, we can choose the wave function such that $\epsilon^{n+1}_{\alpha_{n+1}}\epsilon^{n+2}_{\alpha_{n+2}} \neq 0$ but all other types of contractions vanish. In this case, the second line of (2.21) contains a factor $(T^{\alpha_{n+1}\alpha_{n+1}}T^{\alpha_{n+2}\alpha_{n+2}} + T^{\alpha_{n+1}\alpha_{n+2}}T^{\alpha_{n+2}\alpha_{n+1}})$. From the definition of $T^{\alpha\beta}$ in (A.4), it is clear that this factor is zero when the two shifted fields are not complex conjugate to each other. So we have,

$$\mathcal{Z}(z) = \mathcal{Z}_{\text{SYM}}(z) + N\bar{\kappa}\epsilon^{n+1}_{\alpha_{n+1}}\epsilon^{n+2}_{\alpha_{n+2}}\bar{\mathcal{O}}\,. \qquad (2.22)$$

However, if the two shifted fields are complex conjugate to each other, then in the definition of Lagrangian (2.14), $\bar{\mathcal{O}}$ is in fact identical to $\mathcal{O}$. This means that there is only one term in ΔL but not two, and consequently there is only the first line in (2.21). After the choice of wave functions, we again get (2.22).

From eq. (2.22), we know that the large z behavior of $L_{\mathcal{O}}$ under $\langle\Phi|\Phi\rangle$-shifting depends on the large z behavior of $\mathcal{N} = 4$ SYM theory as well as the double trace term ΔL. In fact (please refer to appendix B for detailed discussion), for all the shifts we use in this paper,[6] $\mathcal{Z}_{\text{SYM}}(z)$ has lower power in z than the second term in (2.22) at large z. This means that the boundary operator (or the operator defined by the leading z order) is always determined by the second term in (2.22),

$$\mathcal{Z}(z) \sim N\bar{\kappa}\epsilon^{n+1}_{\alpha_{n+1}}\epsilon^{n+2}_{\alpha_{n+2}}\bar{\mathcal{O}}\,. \qquad (2.23)$$

So it produces the desired operator $\bar{\mathcal{O}}$, up to certain possible pre-factor from the external wave functions.

3 Sudakov form factor and more

In this section, we will take the bilinear half-BPS scalar operator $\mathcal{O}_2 \equiv \mathcal{O}^{[0]}_{\text{I}} = \text{Tr}(\phi^I \phi^J)$ as an example to illustrate the idea of computing form factor from boundary contributions. The form factor is defined as

$$\mathcal{F}_{\mathcal{O}_2, n}(s; q) = \int d^4 x e^{-iqx} \langle s| \text{Tr}(\phi^I \phi^J)(x)|0\rangle = \delta^{(4)}(q - \sum_{i=1}^n p_i) \langle s| \text{Tr}(\phi^I \phi^J)(0)|0\rangle\,. \quad (3.1)$$

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[6]Including $\langle\phi^I|\phi^J\rangle$, $\langle\psi^{A\alpha}|\phi^J\rangle$, $\langle\psi^{A\alpha}|\bar{\psi}_{\dot\alpha}\rangle$, $\langle\psi^{A\alpha}|\psi^{B\beta}\rangle$, $\langle\psi^{A\alpha}|F^{\beta\gamma}\rangle$, $\langle\bar{\psi}_{A\dot\alpha}|F^{\beta\gamma}\rangle$ and $\langle\bar{F}^{\dot\alpha\dot\beta}|F^{\gamma\rho}\rangle$.

Here $|s\rangle$ is a n-particle on-shell states, and each state in $|s\rangle$ is on-shell, with a momentum $p_i^2 = 0$, while the operator, carrying momentum $q = \sum_{i=1}^n p_i$, is off-shell. The simplest example is given by taking $|s\rangle = |\phi^I(p_1)\phi^J(p_2)\rangle$, i.e., the Sudakov form factor, and it is simply[7]

$$\langle \phi^I(p_1)\phi^J(p_2)| \operatorname{Tr}(\phi^I\phi^J)(0)|0\rangle = 1\,.$$

A more complicated one is given by taking the on-shell states as two scalars and $(n-2)$ gluons. Depending on the helicities of gluons, it defines the MHV form factor, NMHV form factor and so on.

In order to compute the form factor (3.1) as boundary contribution of certain amplitude under BCFW shifting, we need to relate the operator $\mathcal{O}_2$ with certain boundary operator. This can be done by constructing a new Lagrangian $L_{\mathcal{O}_2}$ by adding an extra double trace term ΔL in the $\mathcal{N} = 4$ Lagrangian as

$$L_{\mathcal{O}_2} = L_{\text{SYM}} - \frac{\kappa}{4N} \operatorname{Tr}(\phi^I\phi^J) \operatorname{Tr}(\phi^K\phi^L)\,, \tag{3.2}$$

where κ is the coupling constant.[8] Since we are dealing with real scalars, there is no need to add the corresponding complex conjugate term. This new term provides a four-scalar vertex, and it equals to $i\kappa$, as shown in figure 1. If we split two scalars into ordinary part and hard part $\phi^{Ia} \to \phi^{Ia} + \phi^{\Lambda Ia}$ and $\phi^{Jb} \to \phi^{Jb} + \phi^{\Lambda Jb}$ (the hard part ϕ^Λ corresponds to the large z part), then the quadratic term $\phi^{\Lambda Ia}\phi^{\Lambda Jb}$ of L_{SYM} part can be read out from the result in appendix B of [56] by setting $A = (A_\mu, \phi^I)$, which is given by

$$2g^2 N\delta^{IJ} \operatorname{Tr}(A \cdot A + \phi \cdot \phi)\,. \tag{3.3}$$

The quadratic term $\phi^{\Lambda Ia}\phi^{\Lambda Jb}$ of ΔL part is simply(at the leading N order)

$$\frac{N}{2}\kappa \operatorname{Tr}(\phi^K\phi^L)\,. \tag{3.4}$$

Thus the boundary operator under two-scalar shifting is

$$\mathcal{O}^{\langle\phi^{Ia}|\phi^{Jb}]} = 2g^2 N\delta^{IJ} \operatorname{Tr}(A \cdot A + \phi \cdot \phi) + \frac{N}{2}\kappa \operatorname{Tr}(\phi^K\phi^L)\,. \tag{3.5}$$

Notice that the traceless part (while $I \neq J$) of boundary operator (3.5) is proportional to the operator $\mathcal{O}_2$. This means that if the two shifted scalars are not the same type of scalar, i.e., $I \neq J$, the corresponding boundary contribution $B^{\langle\phi^{Ia}|\phi^{Jb}]}$ of amplitude defined by Lagrangian $L_{\mathcal{O}_2}$ is identical to the form factor of $\mathcal{O}_2 = \operatorname{Tr}(\phi^K\phi^L)$, up to some over-all factor which can be fixed by hand.

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[7]With coupling constant and delta function of momentum conservation stripped off here and from now on for simplicity.

[8]Notice the operator $\operatorname{Tr}(\phi^I\phi^J)$ can be split into a traceless part and a trace part. The traceless part is the Sudakov operator and the trace part is the Konishi operator, and they are the lowest members of the Stress tensor and Konishi super-multiplets, respectively. In principle, one can work out a supersymmetric version of this subsection, in which super-fields and super BCFW shifts are used instead of fields and the usual BCFW shift, and in the end one could derive the super-form factors of the corresponding operators.
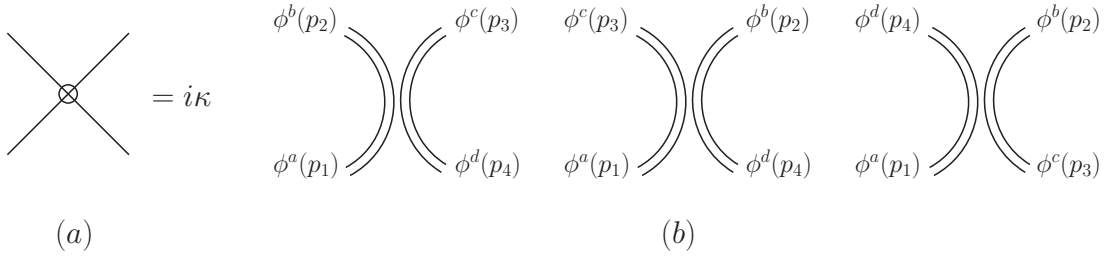
Figure 1. (a)The four-scalar vertex of $\frac{\kappa}{4N}\mathrm{Tr}(\phi^I\phi^J)\,\mathrm{Tr}(\phi^K\phi^L)$ term, (b)The double-line notation of four-scalar vertex, showing the possible trace structures.

More explicitly, let us consider the color-ordered form factor $\langle 1,2,\ldots,n|\mathcal{O}_2|0\rangle$, where i denotes an arbitrary field. It is dressed with a single trace structure $\mathrm{Tr}(t^1t^2\cdots t^n)\mathcal{O}_2$. In the amplitude side, $\mathcal{O}_2$ is generated from the double trace term ΔL, and the corresponding trace structure of color-ordered amplitude is $\mathrm{Tr}(t^1t^2\cdots t^n)\,\mathrm{Tr}(t^{n+1}t^{n+2})$. We denote the amplitude of double trace structure as $A_{n;2}(1,2,\ldots,n;\phi_{n+1},\phi_{n+2})$. It only gets contributions from the Feynman diagrams where ϕ_{n+1},ϕ_{n+2} are attached to the sole four-scalar vertex of ΔL. Then the form factor $\langle 1,2,\ldots,n|\mathcal{O}_2|0\rangle$ is just the boundary contribution of $A_{n;2}(1,2,\ldots,n;\phi_{n+1},\phi_{n+2})$ under BCFW shifting of two scalars ϕ_{n+1},ϕ_{n+2}!

As a simple illustration, let us consider four-point scalar amplitude $A_{2;2}(\phi_1^K,\phi_2^L;\phi_3^I,\phi_4^J)$. In this case, the only possible contributing diagram is a four-scalar vertex defined by ΔL, and we can directly work out as $A_{2;2}(\phi_1,\phi_2;\phi_3,\phi_4)=i\kappa$. After appropriate normalization, it can be set as 1. Since it has no dependence on any external momenta, after momentum shifting

$$|3\rangle \to |3\rangle - z|4\rangle\,,\quad |4\rangle \to |4\rangle + z|3\rangle\,, \tag{3.6}$$

the amplitude still remains the same, while the boundary operator is $\mathrm{Tr}(\phi^K\phi^L)$. There is no pole's term in z, while the zero-th order term in z is $B^{\langle\phi_3^I|\phi_4^J]}(\phi_1^K,\phi_2^L;\phi_{\hat{3}}^I,\phi_{\hat{4}}^J)=1$. Thus we confirm the tree-level Sudakov form factor

$$\langle\phi_1^K,\phi_2^L|\,\mathrm{Tr}(\phi^K\phi^L)|0\rangle = B^{\langle 3|4]}(\phi_1^K,\phi_2^L;\phi_3^I,\phi_4^J)=1\,. \tag{3.7}$$

Now we have three different ways of studying form factor. The first, as stated in [16], form factor obeys a similar BCFW recursion relation as amplitude. This enables us to compute a form factor recursively from lower-point ones. The second, we can compute the corresponding amplitude. Once it is obtained, we can take the BCFW shifting $\langle\phi_{n+1}|\phi_{n+2}]$ and extract the boundary contribution $B^{\langle\phi_{n+1}|\phi_{n+2}]}$, which equals to the corresponding form factor after identification. The third, as stated in [54], the boundary contribution also obeys a similar BCFW recursion relation as amplitude. We can compute boundary contribution recursively from lower-point boundary contributions, and once it is obtained, we can work out the form factor after identification.

In the following subsection, we will take MHV form factor of operator $\mathcal{O}_2$ as an example, to illustrate these three ways of understanding.

3.1 MHV case

The n-point color-ordered MHV form factor of operator $\mathcal{O}_2$ is given by

$$\mathcal{F}^{\mathrm{MHV}}_{\mathcal{O}_2,n}(\{g^+\}, \phi_i, \phi_j; q) = -\frac{\langle i\ j\rangle^2}{\langle 1\ 2\rangle\langle 2\ 3\rangle\cdots\langle n\ 1\rangle}\,, \tag{3.8}$$

where $\mathcal{F}^{\mathrm{MHV}}_{\mathcal{O}_2,n}(\{g^+\}, \phi_i, \phi_j; q)$ denotes

$$\mathcal{F}^{\mathrm{MHV}}_{\mathcal{O}_2,n}(g_1^+, \ldots, g_{i-1}^+, \phi_i, g_{i+1}^+, \ldots, g_{j-1}^+, \phi_j, g_{j+1}^+, \ldots, g_n^+; q)\,.$$

BCFW recursion relation of form factor. The result (3.8) has been proven in paper [16][9] by BCFW recursion relation of form factor. As stated therein, after taking BCFW shifting of two momenta p_{i_1}, p_{i_2}, the form factor can be computed as summation of products of lower-point form factor and lower-point amplitude, as long as the large z behavior $\mathcal{F}(z)|_{z\to\infty} \to 0$ is satisfied under such deformation. The n external legs will be split into two parts, with $\widehat{p}_{i_1}, \widehat{p}_{i_2}$ in each part separately. The operator, since it is color-singlet, can be inserted into either part. So it is possible to build up a n-point form factor recursively from three-point amplitudes and three-point form factors. Since this method has already been described in [16], we will not repeat it here.

BCFW recursion relation of amplitude. Instead of computing form factor directly, we can first compute the corresponding $(n+2)$-point amplitude

$$A_{n;2}(g_1^+, \ldots, g_{i-1}^+, \phi_i, g_{i+1}^+, \ldots, g_{j-1}^+, \phi_j, g_{i+1}^+, \ldots, g_n^+; \phi_{n+1}, \phi_{n+2})\,. \tag{3.9}$$

This amplitude can be computed via BCFW recursion relation. If we choose one shifted momentum to be gluon, $A_{n;2}(z)$ will be vanishing when $z \to \infty$, i.e., there is no boundary contribution. So we can take $\langle g^+|\phi]$-shifting in the computation. The four-point amplitude is trivially $A_{2;2}(\phi_1, \phi_2; \phi_3, \phi_4) = 1$. To compute the five-point amplitude $A_{3;2}(\phi_1, \phi_2, g_3^+; \phi_4, \phi_5)$, we can take $\langle g_3^+|\phi_1]$-shifting. There is only one contributing term as shown in figure 2a, which is given by

$$A_{3;2}(\phi_1, \phi_2, g_3^+; \phi_4, \phi_5) = A_{2;2}(\phi_{\widehat{1}}, \phi_{\widehat{P}}; \phi_4, \phi_5)\frac{1}{P_{23}^2}A_3(\phi_{-\widehat{P}}, \phi_2, g_{\widehat{3}}^+)$$

$$= -1 \times \frac{1}{P_{23}^2} \times \frac{[2\ 3][3\ \widehat{P}]}{[\widehat{P}\ 2]} = -\frac{\langle 1\ 2\rangle^2}{\langle 1\ 2\rangle\langle 2\ 3\rangle\langle 3\ 1\rangle}\,, \tag{3.10}$$

where $\widehat{P} = p_2 + p_3 - z|1\rangle|3]$. Similarly, for general amplitude $A_{n;2}$, we can take $\langle g_{i+1}^+|\phi_i]$-shifting.[10] If $j \neq (i+2)$, we need to consider two contributing terms as shown in figure 2b and 2c, while if $j = (i+2)$, we need to consider two contributing terms as shown in figure 2b and 2d. In either case, contribution of diagram 2.b vanishes under $\langle g_{i+1}^+|\phi_i]$-shifting. So

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[9]Note that we have introduced an over-all minus sign in the expression (3.8), so that the Sudakov form factor is defined to be $\mathcal{F}_{\mathcal{O}_2,2}(\phi_1, \phi_2; q) = 1$.

[10]Because of cyclic invariance, we can always do this.

we only need to compute contribution of diagram 2.c or 2.d. Taking $j \neq (i+2)$ as example, we have

$$A_{n;2}(g_1^+, \ldots, \phi_i, \ldots, \phi_j, \ldots, g_n^+; \phi_{n+1}, \phi_{n+2}) \tag{3.11}$$
$$= A_{n-1;2}(g_{i+3}^+, \ldots, \phi_j, \ldots, g_n^+, g_1^+, \ldots, \phi_{\hat{i}}, g_{\hat{P}}^+; \phi_{n+1}, \phi_{n+2}) \frac{1}{P_{i+1,i+2}^2} A_3(g_{-\hat{P}}^-, g_{\widehat{i+1}}^+, g_{i+2}^+).$$

Assuming that

$$A_{n;2}(\{g^+\}, \phi_i, \phi_j; \phi_{n+1}, \phi_{n+2}) = -\frac{\langle i\ j \rangle^2}{\langle 1\ 2 \rangle \langle 2\ 3 \rangle \cdots \langle n\ 1 \rangle} \tag{3.12}$$

is true for $A_{n-1;2}$, then

$$A_{n;2}(g_1^+, \ldots, \phi_i, \ldots, \phi_j, \ldots, g_n^+; \phi_{n+1}, \phi_{n+2}) \tag{3.13}$$
$$= -\frac{\langle i\ j \rangle^2}{\langle 1\ 2 \rangle \cdots \langle i-1, i \rangle \langle i\ \hat{P} \rangle \langle \hat{P}, i+3 \rangle \langle i+3, i+4 \rangle \cdots \langle n\ 1 \rangle} \frac{1}{P_{i+1,i+2}^2} \frac{[i+1, i+2]^3}{[\hat{P}, i+1][i+2, \hat{P}]}$$
$$= -\frac{\langle i\ j \rangle^2}{\langle 1\ 2 \rangle \langle 2\ 3 \rangle \cdots \langle n\ 1 \rangle},$$

where

$$\hat{P} = p_{i+1} + p_{i+2} - z_{i+1,i+2}|i\rangle|i+1], \qquad z_{i+1,i+2} = \frac{\langle i+1, i+2 \rangle}{\langle i, i+2 \rangle}. \tag{3.14}$$

Similar computation shows that for $j \neq i+2$ case, (3.12) is also true for all n. Thus we have proven the result (3.12) by BCFW recursion relation of amplitude.

As discussed, $\langle \phi_{n+1}|\phi_{n+2}]$-shifting generates the boundary operator $\mathcal{O}_2$, and the corresponding boundary contribution is identical to the form factor of operator $\mathcal{O}_2$. Here, $A_{n;2}$ does not depend on momenta p_{n+1}, p_{n+2}, thus

$$B^{\langle \phi_{n+1}|\phi_{n+2}]}(\{g^+\}, \phi_i, \phi_j; \phi_{\widehat{n+1}}, \phi_{\widehat{n+2}}) = -\frac{\langle i\ j \rangle^2}{\langle 1\ 2 \rangle \langle 2\ 3 \rangle \cdots \langle n\ 1 \rangle}, \tag{3.15}$$

and correspondingly

$$\mathcal{F}_{\mathcal{O}_2,n}^{\mathrm{MHV}}(\{g^+\}, \phi_i, \phi_j; q) = B^{\langle \phi_{n+1}|\phi_{n+2}]} = -\frac{\langle i\ j \rangle^2}{\langle 1\ 2 \rangle \langle 2\ 3 \rangle \cdots \langle n\ 1 \rangle}, \tag{3.16}$$

which agrees with the result given by BCFW recursion relation of form factor.

Recursion relation of boundary contribution. We can also compute the boundary contribution directly by BCFW recursion relation without knowing the explicit expression of amplitude, as shown in paper [54]. The boundary contribution of four and five-point amplitudes can be computed directly by Feynman diagrams. For four-point case, there is only one diagram, i.e., four-scalar vertex, as shown in figure 3a, and $B_{2;2}^{\langle \phi_3|\phi_4]}(\phi_1, \phi_2; \phi_{\hat{3}}, \phi_{\hat{4}}) = 1$. For five-point case, under $\langle \phi_4|\phi_5]$-shifting, only those Feynman diagrams whose $\hat{p}_4, \hat{p}_5$ are attached to the same four-scalar vertex contribute to the boundary contribution. There are in total two diagrams as shown in figure 3b, which gives

$$B_{3;2}^{\langle \phi_4|\phi_5]}(\phi_1, \phi_2, g_3^+; \phi_{\hat{4}}, \phi_{\hat{5}}) = -\frac{(p_2 - P_{23})^\mu \epsilon_\mu^+(p_3)}{P_{23}^2} + \frac{(p_1 - P_{13})^\mu \epsilon_\mu^+(p_3)}{P_{13}^2}$$
$$= -\frac{\langle 1\ 2 \rangle^2}{\langle 1\ 2 \rangle \langle 2\ 3 \rangle \langle 3\ 1 \rangle}, \tag{3.17}$$
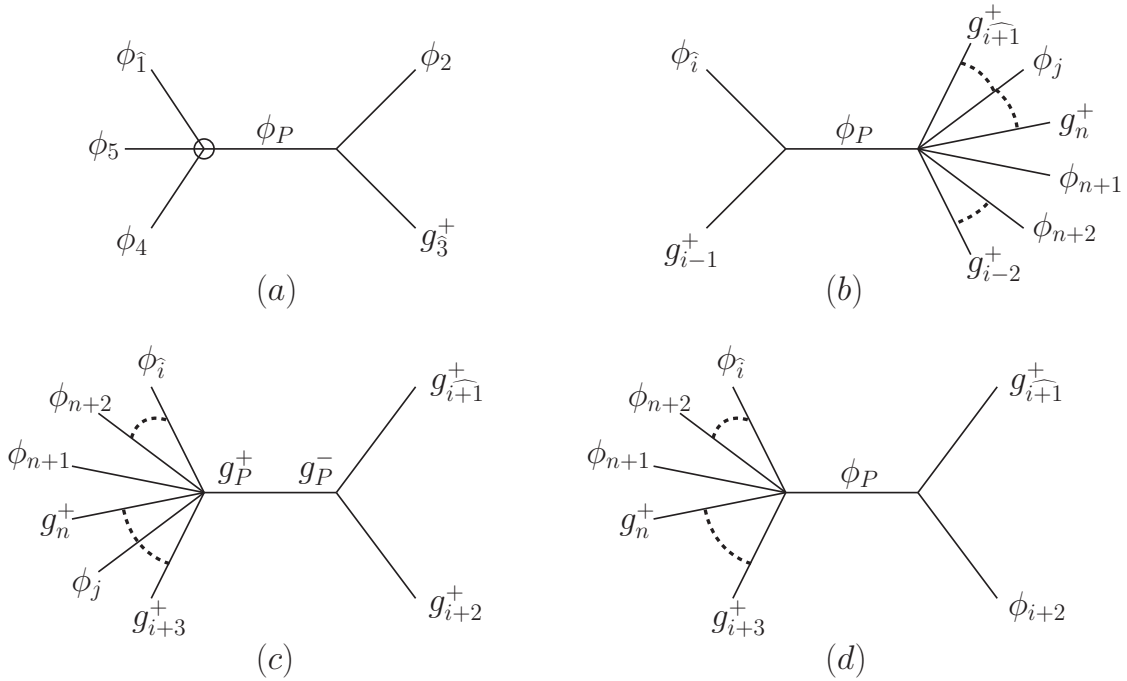
Figure 2. (a) is the contributing diagram for $A_{3;2}(\phi_1, \phi_2, g_3^+; \phi_4, \phi_5)$. (b)(c) are the contributing diagrams for general $A_{n;2}$ when $j \neq i+2$ while (b)(d) are the contributing diagrams for $A_{n;2}$ when $j = i+2$.

where the polarization vector $\epsilon_\mu^\pm(p)$ is defined to be

$$\epsilon_\mu^+(p) = \frac{\langle r|\gamma_\mu|p]}{\sqrt{2}\langle r\ p\rangle}, \quad \epsilon_\mu^-(p) = \frac{\langle p|\gamma_\mu|r]}{\sqrt{2}[p\ r]}, \tag{3.18}$$

with r an arbitrary reference spinor. From these lower-point results, it is not hard to guess that

$$B_{n;2}^{\langle \phi_{n+1}|\phi_{n+2}]}(\{g^+\}, \phi_i, \phi_j; \phi_{\widehat{n+1}}, \phi_{\widehat{n+2}}) = -\frac{\langle i\ j\rangle^2}{\langle 1\ 2\rangle\langle 2\ 3\rangle \cdots \langle n\ 1\rangle}. \tag{3.19}$$

This result can be proven recursively by taking another shifting $\langle i_1|\phi_{n+2}]$ on $B_{n;2}^{\langle \phi_{n+1}|\phi_{n+2}]}$, where p_{i_1} is the momentum other than p_{n+1}, p_{n+2}. If under this second shifting, there is no additional boundary contribution, then $B_{n;2}^{\langle \phi_{n+1}|\phi_{n+2}]}$ can be fully determined by the pole terms under $\langle i_1|\phi_{n+2}]$-shifting. Otherwise we should take a third momentum shifting and so on, until we have detected the complete boundary contribution.

Fortunately, if p_{i_1} is the momentum of gluon, a second shifting $\langle g_{i_1}^+|\phi_{n+2}]$ is sufficient to detect all the contributions [54]. For a general boundary contribution $B_{n;2}^{\langle \phi_{n+1}|\phi_{n+2}]}$, we can take $\langle g_1^+|\phi_{n+2}]$-shifting. It splits the boundary contribution into a sub-amplitude times a lower-point boundary contribution, and only those terms with three-point amplitudes are non-vanishing. Depending on the location of ϕ_i, ϕ_j, the contributing terms are different.
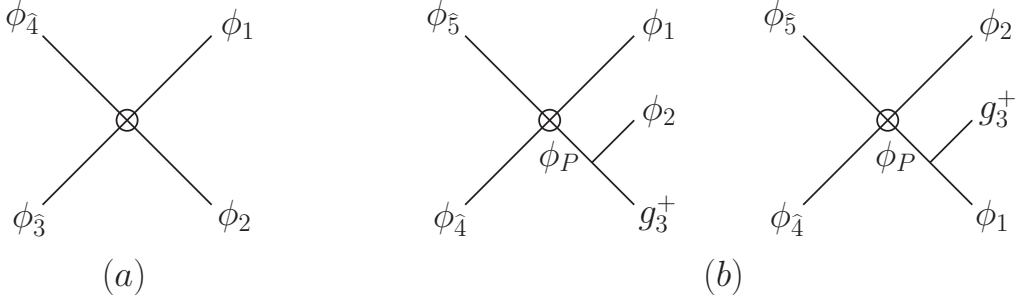
Figure 3. (a)Feynman diagram for boundary contribution $B_{2;2}^{\langle \phi_3 | \phi_4 \rangle}(\phi_1, \phi_2; \phi_{\widehat{3}}, \phi_{\widehat{4}})$, (b)Feynman diagrams for boundary contribution $B_{3;2}^{\langle \phi_4 | \phi_5 \rangle}(\phi_1, \phi_2, g_3^+; \phi_{\widehat{4}}, \phi_{\widehat{5}})$.

Assuming that (3.19) is true for $B_{n-1;2}$, if $i, j \neq 2, n$, we have

$$B_{n;2}^{\langle \phi_{n+1} | \phi_{n+2} \rangle}(\{g^+\}, \phi_i, \phi_j; \phi_{\widehat{n+1}}, \phi_{\widehat{n+2}}) \tag{3.20}$$

$$= A_3(g_n^+, g_{\widehat{1}}^+, g_{\widehat{P}}^-) \frac{1}{P_{1n}^2} B_{n-1;2}^{\langle \phi_{n+1} | \phi_{n+2} \rangle}(g_{-\widehat{P}}^+, g_2^+, \ldots, \phi_i, \ldots, \phi_j, \ldots, g_{n-1}^+; \phi_{\widehat{n+1}}, \phi_{\widehat{\widehat{n+2}}})$$

$$+ A_3(g_{\widehat{1}}^+, g_2^+, g_{\widehat{P}}^-) \frac{1}{P_{12}^2} B_{n-1;2}^{\langle \phi_{n+1} | \phi_{n+2} \rangle}(g_{-\widehat{P}}^+, g_3^+, \ldots, \phi_i, \ldots, \phi_j, \ldots, g_n^+; \phi_{\widehat{n+1}}, \phi_{\widehat{\widehat{n+2}}}) ,$$

while if $i = 2, j \neq n$, we have

$$B_{n;2}^{\langle \phi_{n+1} | \phi_{n+2} \rangle}(\{g^+\}, \phi_2, \phi_j; \phi_{\widehat{n+1}}, \phi_{\widehat{n+2}}) \tag{3.21}$$

$$= A_3(g_n^+, g_{\widehat{1}}^+, g_{\widehat{P}}^-) \frac{1}{P_{1n}^2} B_{n-1;2}^{\langle \phi_{n+1} | \phi_{n+2} \rangle}(g_{-\widehat{P}}^+, \phi_2, g_3^+, \ldots, \phi_j, \ldots, g_{n-1}^+; \phi_{\widehat{n+1}}, \phi_{\widehat{\widehat{n+2}}})$$

$$+ A_3(g_{\widehat{1}}^+, \phi_2, \phi_{\widehat{P}}) \frac{1}{P_{12}^2} B_{n-1;2}^{\langle \phi_{n+1} | \phi_{n+2} \rangle}(\phi_{-\widehat{P}}, g_3^+, \ldots, \phi_j, \ldots, g_n^+; \phi_{\widehat{n+1}}, \phi_{\widehat{\widehat{n+2}}}) ,$$

and if $i = 2, j = n$, we have

$$B_{n;2}^{\langle \phi_{n+1} | \phi_{n+2} \rangle}(\{g^+\}, \phi_2, \phi_n; \phi_{\widehat{n+1, n+2}}) \tag{3.22}$$

$$= A_3(\phi_n, g_{\widehat{1}}^+, \phi_{\widehat{P}}) \frac{1}{P_{1n}^2} B_{n-1;2}^{\langle \phi_{n+1} | \phi_{n+2} \rangle}(\phi_{-\widehat{P}}, \phi_2, g_3^+, \ldots, g_{n-1}^+; \phi_{\widehat{n+1}}, \phi_{\widehat{\widehat{n+2}}})$$

$$+ A_3(g_{\widehat{1}}^+, \phi_2, \phi_{\widehat{P}}) \frac{1}{P_{12}^2} B_{n-1;2}^{\langle \phi_{n+1} | \phi_{n+2} \rangle}(\phi_{-\widehat{P}}, g_3^+, \ldots, g_{n-1}^+, \ldots, \phi_n; \phi_{\widehat{n+1}}, \phi_{\widehat{\widehat{n+2}}}) .$$

All of them lead to the result (3.19), which ends the proof. Again, with the result of boundary contribution, we can work out the corresponding form factor directly.

We have shown that the BCFW recursion relation of form factor, amplitude and boundary contribution lead to the same conclusion. This is not limited to MHV case, since the connection between form factor and boundary contribution of amplitude is universal and does not depend on the external states. In fact, for any form factor with n-particle on-shell states $|s\rangle$, we can instead compute the corresponding amplitude $A_{n;2}(s; \phi_{n+1}, \phi_{n+2})$ defined by Lagrangian $L_{\mathcal{O}_2}$, and extract the boundary contribution under $\langle \phi_{n+1} | \phi_{n+2} \rangle$-shifting. There is no difference between this boundary contribution and form factor of $\mathcal{O}_2$.

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For example, in [17], the authors showed that the split-helicity form factor shares a similar "zigzag diagram" construction as the split-helicity amplitude given in [63]. It is now easy to understand this, since the form factor is equivalent to the boundary contribution of the amplitude, and it naturally inherits the "zigzag" construction with minor modification.

The tree amplitude $A_{n;2}(1, \ldots, n; n+1, n+2)$ associated with the double trace structure is cyclically invariant inside legs $\{1, 2, \ldots, n\}$ and $\{n+1, n+2\}$, so no surprisingly, the color-ordered form factor is also cyclically invariant on its n legs. Since the trace structure $\mathrm{Tr}(t^{n+1} t^{n+2})$ is completely isolated from the other color structure, while the later one is constructed only from structure constant f^{abc}. Thus for amplitudes $A_{n;2}$, we also have Kleiss-Kuijf (KK) relation [64] among permutation of legs $\{1, 2, \ldots, n\}$ as

$$A_{n;2}(1, \{\alpha\}, n, \{\beta\}; \phi_{n+1}, \phi_{n+2}) = (-)^{n_\beta} \sum_{\sigma \in OP\{\alpha\} \cup \{\beta^T\}} A_{n;2}(1, \sigma, n; \phi_{n+1}, \phi_{n+2}), \quad (3.23)$$

where n_β is the length of set β, β^T is the reverse of set β, and OP is the ordered permutation, containing all the possible permutations between two sets while keeping each set ordered. This relation can be similarly extended to form factors. Especially for operator $\mathcal{O}_2$, we can relate all form factors to those with two adjacent scalars,

$$\mathcal{F}_{\mathcal{O}_2,n}(\phi_1, \{\alpha\}, \phi_n, \{\beta\}; q) = (-)^{n_\beta} \sum_{\sigma \in OP\{\alpha\} \cup \{\beta^T\}} \mathcal{F}_{\mathcal{O}_2,n}(\phi_n, \phi_1, \sigma; q). \quad (3.24)$$

3.2 Form factor of operator $\mathcal{O}_k \equiv \mathrm{Tr}(\phi^{M_1} \phi^{M_2} \cdots \phi^{M_k})$

Let us further consider a more general operator $\mathcal{O}_k \equiv \mathrm{Tr}(\phi^{M_1} \phi^{M_2} \cdots \phi^{M_k})$ and the form factor $\mathcal{F}_{\mathcal{O}_k,n}(s; q) = \langle s | \mathcal{O}_k(0) | 0 \rangle$. In order to generate the operator $\mathcal{O}_k$ under certain BCFW shifting, we need to add an additional Lagrangian term

$$\Delta L = \frac{\kappa}{(2k)N} \mathrm{Tr}(\phi^I \phi^J) \mathrm{Tr}(\phi^{M_1} \phi^{M_2} \cdots \phi^{M_k}) \quad (3.25)$$

to construct a new Lagrangian $L_{\mathcal{O}_k} = L_{\mathrm{SYM}} + \Delta L$. Then the boundary contribution of corresponding amplitude $A_{n;2}(s; \phi_{n+1}, \phi_{n+2})$ under $\langle \phi_{n+1} | \phi_{n+2}]$-shifting is identical to the form factor $\mathcal{F}_{\mathcal{O}_k,n}(s; q)$.

To see that the boundary operator $\mathcal{O}^{\langle \phi^{I_a} | \phi^{J_b}]}$ is indeed the operator $\mathcal{O}_k$, we can firstly compute the variation of Lagrangian $L_{\mathcal{O}_k}$ from left with respect to ϕ^{I_a}, and then the variation of $\frac{\delta L_{\mathcal{O}_k}}{\delta \phi^{I_a}}$ from right with respect to ϕ^{J_b}, which we shall denote as $\frac{\overleftarrow{\delta}}{\delta \phi^{J_b}}$ to avoid ambiguities. The variation of L_{SYM} part is given in (3.3), while for ΔL part, we have

$$\begin{aligned}
\frac{\delta \Delta L}{\delta \phi^{I_a}} &= \frac{\kappa}{kN} \mathrm{Tr}(\phi^J t^a) \mathrm{Tr}(\phi^{M_1} \phi^{M_2} \cdots \phi^{M_k}) \\
&+ \frac{\kappa}{2N} \mathrm{Tr}(\phi^{N_1} \phi^{N_2}) \mathrm{Tr}(t^a \phi^{M_1} \phi^{M_2} \cdots \phi^{M_{k-1}}),
\end{aligned} \quad (3.26)$$

and

$$\begin{aligned}
\frac{\overleftarrow{\delta}}{\delta \phi^{J_b}} \left(\frac{\delta \Delta L}{\delta \phi^{I_a}} \right) &= \frac{N^2 - 1}{2kN} \kappa \, \mathrm{Tr}(\phi^{M_1} \phi^{M_2} \cdots \phi^{M_k}) + \frac{\kappa}{2N} \phi^{Ma} \mathrm{Tr}(t^a \phi^{M_1} \phi^{M_2} \cdots \phi^{M_{k-1}}) \\
&+ \sum_i \frac{\kappa}{2N} \mathrm{Tr}(\phi^{N_1} \phi^{N_2}) \mathrm{Tr}(t^a \phi^{M_1} \cdots \phi^{M_i} t^a \phi^{M_{i+1}} \cdots \phi^{M_{k-2}}).
\end{aligned}$$

The first term contains $O(N)$ order result, with a single trace proportional to $\text{Tr}(\phi^k)$, while the second term is $O(\frac{1}{N})$ order, and the third term is also $O(\frac{1}{N})$ order with even triple trace structure. Thus at the leading N order, the boundary operator of $L_{\mathcal{O}_k}$ is

$$\mathcal{O}^{\langle\phi^{Ia}|\phi^{Jb}]} = 2g^2 N \delta^{IJ} \text{Tr}(A \cdot A + \phi^K \phi^K) + \frac{N}{2k}\kappa \, \text{Tr}(\phi^{M_1}\phi^{M_2}\cdots\phi^{M_k}). \qquad (3.27)$$

Similar to the $\mathcal{O}_2$ case, the traceless part of (3.27) is proportional to the operator $\mathcal{O}_k$.

The ΔL term introduces a $(k+2)$-scalar vertex, besides this it has no difference to $\mathcal{O}_2$ case. We can compute the amplitude $A_{n;2}(s; \phi_{n+1}, \phi_{n+2})$, take $\langle\phi_{n+1}|\phi_{n+2}]$-shifting and extract the boundary contribution. Then transforming it to form factor is almost trivial. For instance, $A_{k;2}(\phi_1, \ldots, \phi_k; \phi_{k+1}, \phi_{k+2}) = 1$, thus $\mathcal{F}_{\mathcal{O}_k,k}(\phi_1, \ldots, \phi_k; q) = 1$. It is also easy to conclude that, since the Feynman diagrams of amplitude

$$A_{n;2}(\phi_1, \cdots, \phi_k, g^+_{k+1}, \ldots, g^+_n; \phi_{n+1}, \phi_{n+2})$$

defined by $L_{\mathcal{O}_k}$ have one-to-one mapping to the Feynman diagrams of amplitude

$$A_{n-(k-2);2}(\phi_1, \phi_k, g^+_{k+1}, \ldots, g^+_n; \phi_{n+1}, \phi_{n+2})$$

defined by $L_{\mathcal{O}_2}$ by just replacing the $(k+2)$-scalar vertex with four-scalar vertex, we have

$$A^{\mathcal{O}_k}_{n;2}(\phi_1, \ldots, \phi_k, g^+_{k+1}, \ldots, g^+_n; \phi_{n+1}, \phi_{n+2}) = A^{\mathcal{O}_2}_{n-(k-2);2}(\phi_1, \phi_k, g^+_{k+1}, \ldots, g^+_n; \phi_{n+1}, \phi_{n+2})$$
$$= -\frac{\langle 1\ k\rangle}{\langle k, k+1\rangle\langle k+1, k+2\rangle\cdots\langle n\ 1\rangle}. \qquad (3.28)$$

Thus we get

$$\mathcal{F}_{\mathcal{O}_k;n}(\phi_1, \ldots, \phi_k, g^+_{k+1}, \ldots, g^+_n; q) = -\frac{\langle 1\ k\rangle}{\langle k, k+1\rangle\langle k+1, k+2\rangle\cdots\langle n\ 1\rangle}. \qquad (3.29)$$

4 Form factor of composite operators

Now we move to the computation of form factors for the composite operators introduced in section 2.1. For convenience we will use complex scalars $\phi^{AB}, \bar{\phi}_{AB}$ instead of real scalars ϕ^I in this section. We will explain the construction of Lagrangian which generates the corresponding operators, and compute the MHV form factors through amplitudes of double trace structure.

4.1 The spin-0 operators

There are three operators

$$\mathcal{O}^{[0]}_{\text{I}} = \text{Tr}(\phi^{AB}\phi^{CD}), \quad \mathcal{O}^{[0]}_{\text{II}} = \text{Tr}(\psi^{A\gamma}\psi^B_\gamma), \quad \mathcal{O}^{[0]}_{\text{III}} = \text{Tr}(F^{\alpha\beta}F_{\alpha\beta}), \qquad (4.1)$$

with their complex conjugate partners $\bar{\mathcal{O}}^{[0]}_{\text{I}}$, $\bar{\mathcal{O}}^{[0]}_{\text{II}}$ and $\bar{\mathcal{O}}^{[0]}_{\text{III}}$. For these operators, in order to construct Lorentz invariant double trace Lagrangian terms ΔL, we need to product them with another spin-0 trace term. Since shifting a gluon is always more complicated than

shifting a fermion, and shifting a fermion is more complicated than shifting a scalar, we would like to choose the spin-0 trace term as trace of two scalars, as already shown in operator $\mathcal{O}_2$ case.

For operator $\mathcal{O}_{\text{II}}^{[0]}$, we could construct the Lagrangian as

$$L_{\mathcal{O}_{\text{II}}^{[0]}} = L_{\text{SYM}} + \frac{\kappa}{N} \text{Tr}(\phi^{A'B'}\phi^{C'D'}) \text{Tr}(\psi^{A\gamma}\psi_\gamma^B) + \frac{\bar{\kappa}}{N} \text{Tr}(\bar{\phi}_{A'B'}\bar{\phi}_{C'D'}) \text{Tr}(\bar{\psi}_A^{\dot\gamma}\bar{\psi}_{B\dot\gamma}). \quad (4.2)$$

The momentum shifting of two scalars ϕ_{n+1}, ϕ_{n+2} will generate the boundary operator $\mathcal{O}^{\langle \phi_{n+1}|\phi_{n+2}]} = \text{Tr}(\bar{\psi}_A^{\dot\gamma}\bar{\psi}_{B\dot\gamma})$, while the shifting of two scalars $\bar{\phi}_{n+1}, \bar{\phi}_{n+2}$ will generate the boundary operator $\mathcal{O}^{\langle \bar{\phi}_{n+1}|\bar{\phi}_{n+2}]} = \text{Tr}(\psi^{A\gamma}\psi_\gamma^B)$. Thus the form factor

$$\mathcal{F}_{\mathcal{O}_{\text{II}}^{[0]},n}(s;q) = \langle s|\mathcal{O}_{\text{II}}^{[0]}|0\rangle$$

is identical to the boundary contribution of amplitude $A_{n;2}(s; \bar{\phi}_{n+1}, \bar{\phi}_{n+2})$ defined by $L_{\mathcal{O}_{\text{II}}^{[0]}}$ under $\langle \bar{\phi}_{n+1}|\bar{\phi}_{n+2}]$-shifting. This amplitude can be computed by Feynman diagrams or BCFW recursion relation method.

The ΔL Lagrangian term introduces ϕ-ϕ-ψ-ψ and $\bar{\phi}$-$\bar{\phi}$-$\bar{\psi}$-$\bar{\psi}$ vertices in the Feynman diagrams, and it defines the four-point amplitude $A_{2;2}(\bar{\psi}_1, \bar{\psi}_2; \bar{\phi}_3, \bar{\phi}_4) = \langle 1\,2 \rangle$ as well as $A_{2;2}(\psi_1, \psi_2; \phi_3, \phi_4) = [1\,2]$. Thus it is immediately know that the boundary contribution $B^{\langle \bar{\phi}_3|\bar{\phi}_4]}(\bar{\psi}_1, \bar{\psi}_2; \bar{\phi}_{\hat{3}}, \bar{\phi}_{\hat{4}}) = \langle 1\,2 \rangle$, and the form factor $\mathcal{F}_{\mathcal{O}_{\text{II}}^{[0]},n}(\bar{\psi}_1, \bar{\psi}_2; q) = \langle 1\,2 \rangle$. We can also compute the five-point amplitude $A_{3;2}(\bar{\psi}_1, \bar{\psi}_2, g_3^+; \bar{\phi}_4, \bar{\phi}_5)$, and the contributing Feynman diagrams are similar to figure 3b but now we have $\bar{\psi}_1, \bar{\psi}_2$ instead of $\bar{\phi}_1, \bar{\phi}_2$. It is given by

$$A_{3;2}(\bar{\psi}_1, \bar{\psi}_2, g_3^+; \bar{\phi}_4, \bar{\phi}_5) = \frac{\langle 1|P_{23}|\gamma^\mu|2\rangle}{s_{23}} \epsilon_\mu^+(p_3) + \frac{\langle 2|P_{13}|\gamma^\mu|1\rangle}{s_{13}} \epsilon_\mu^+(p_3) = -\frac{\langle 1\,2\rangle^2}{\langle 2\,3\rangle\langle 3\,1\rangle}. \quad (4.3)$$

Generalizing this result to $(n+2)$-point double trace amplitude, we have

$$A_{n;2}(\{g^+\}, \bar{\psi}_i, \bar{\psi}_j; \bar{\phi}_{n+1}, \bar{\phi}_{n+2}) = -\frac{\langle i\,j\rangle^3}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\cdots\langle n\,1\rangle}. \quad (4.4)$$

It is easy to verify above result by BCFW recursion relation of amplitude, for example, by taking $\langle g_1^+|\bar{\psi}_i]$-shifting. Similar to the $\mathcal{O}_2$ case, only those terms with three-point sub-amplitudes can have non-vanishing contributions, and after substituting the explicit results for A_3 and $A_{n-1;2}$, we arrive at the result (4.4). The boundary contribution of amplitude (4.4) under $\langle \bar{\phi}_{n+1}|\bar{\phi}_{n+2}]$-shifting keeps the same as $A_{n;2}$ itself, thus consequently we get the form factor

$$\mathcal{F}_{\mathcal{O}_{\text{II}}^{[0]},n}(\{g^+\}, \bar{\psi}_i, \bar{\psi}_j; q) = -\frac{\langle i\,j\rangle^3}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,4\rangle\cdots\langle n\,1\rangle}. \quad (4.5)$$

It is also interesting to consider another special n-point external states, i.e., two fermions with $(n-2)$ gluons of negative helicities. For five-point amplitude $A_{3;2}(\bar{\psi}_1, \bar{\psi}_2, g_3^-; \bar{\phi}_4, \bar{\phi}_5)$, the contributing Feynman diagrams can be obtained by replacing g_3^+ as g_3^- in amplitude $A_{3;2}(\bar{\psi}_1, \bar{\psi}_2, g_3^+; \bar{\phi}_4, \bar{\phi}_5)$, so we have

$$\begin{aligned} A_{3;2}(\bar{\psi}_1, \bar{\psi}_2, g_3^-; \bar{\phi}_4, \bar{\phi}_5) &= \frac{\langle 1|P_{23}|\gamma^\mu|2\rangle}{s_{23}} \epsilon_\mu^-(p_3) + \frac{\langle 2|P_{13}|\gamma^\mu|1\rangle}{s_{13}} \epsilon_\mu^-(p_3) \\ &= \frac{(p_4 + p_5)^2[1\,2]}{[1\,2][2\,3][3\,1]}. \end{aligned} \quad (4.6)$$

More generally, we have

$$A_{n;2}(\{g^-\}, \bar{\psi}_i, \bar{\psi}_j; \bar{\phi}_{n+1}, \bar{\phi}_{n+2}) = \frac{(p_{n+1} + p_{n+2})^2 [i\,j]}{[1\,2][2\,3]\cdots[n\,1]}. \tag{4.7}$$

This result can be proven recursively by BCFW recursion relation. Assuming eq. (4.7) is valid for $A_{n-1;2}$, then taking $\langle \bar{\phi}_{n+2} | g_n]$-shifting, we get two contributing terms[11] for $A_{n;2}$. The first term is

$$A_3(g_{\hat{n}}^-, g_1^-, g_{\hat{P}_{1n}}^+) \frac{1}{P_{1n}^2} A_{n-1;2}(g_{-\hat{P}_{1n}}^-, g_2^-, \ldots, \bar{\psi}_i, \ldots, \bar{\psi}_j, \ldots, g_{n-1}^-; \bar{\phi}_{n+1}, \bar{\phi}_{\widehat{n+2}})$$

$$= \frac{[i\,j](p_{n+1} + p_{n+2})^2}{[1\,2][2\,3]\cdots[n-1,n][n\,1]} \frac{[n+2,1][n,n-1]}{[n-1,1][n+2,n]}$$

$$+ \frac{[i\,j]}{[1\,2][2\,3]\cdots[n-1,n][n\,1]} \frac{\langle n+1,n\rangle[n+2,n+1]}{[n-1,1][n+2,n]} [n\,1][n,n-1], \tag{4.8}$$

while the second term is

$$A_3(g_{n-1}^-, g_{\hat{n}}^-, g_{\hat{P}_{n-1,n}}^+) \frac{1}{P_{n-1,n}^2} A_{n-1;2}(g_{-\hat{P}_{n-1,n}}^-, g_1^-, \ldots, \bar{\psi}_i, \ldots, \bar{\psi}_j, \ldots, g_{n-2}^-; \bar{\phi}_{n+1}, \bar{\phi}_{\widehat{n+2}})$$

$$= \frac{[i\,j](p_{n+1} + p_{n+2})^2}{[1\,2][2\,3]\cdots[n-1,n][n\,1]} \frac{[n-1,n+2][n,1]}{[n-1,1][n+2,n]}$$

$$+ \frac{[i\,j]}{[1\,2][2\,3]\cdots[n-1,n][n\,1]} \frac{\langle n+1,n\rangle[n+2,n+1]}{[n-1,1][n+2,n]} [n\,1][n-1,n]. \tag{4.9}$$

Summing above two contributions, we get the desired eq. (4.7).

Note that $q = -p_{n+1} - p_{n+2}$ shows up in result (4.7), which is the momentum carried by the operator in form factor. The $\langle \bar{\phi}_{n+1} | \bar{\phi}_{n+2}]$-shifting assures that $\hat{p}_{n+1} + \hat{p}_{n+2} = p_{n+1} + p_{n+2}$, thus we get the form factor

$$\mathcal{F}_{\mathcal{O}_{\mathrm{II}}^{[0]}, n}(\{g^-\}, \bar{\psi}_i, \bar{\psi}_j; q) = \frac{q^2 [i\,j]}{[1\,2][2\,3]\cdots[n\,1]}. \tag{4.10}$$

For operator $\mathcal{O}_{\mathrm{III}}^{[0]}$, we can also construct the Lagrangian as

$$L_{\mathcal{O}_{\mathrm{III}}^{[0]}} = L_{\mathrm{SYM}} + \frac{\kappa}{N} \mathrm{Tr}(\phi^{A'B'} \phi^{C'D'}) \mathrm{Tr}(F^{\alpha\beta} F_{\alpha\beta}) + \frac{\bar{\kappa}}{N} \mathrm{Tr}(\bar{\phi}_{A'B'} \bar{\phi}_{C'D'}) \mathrm{Tr}(\bar{F}^{\dot{\alpha}\dot{\beta}} \bar{F}_{\dot{\alpha}\dot{\beta}}). \tag{4.11}$$

As usual, the $\langle \bar{\phi}_{n+1} | \bar{\phi}_{n+2}]$-shifting generates the boundary operator $\mathcal{O}^{\langle \bar{\phi}_{n+1} | \bar{\phi}_{n+2}]} = \mathrm{Tr}(F^{\alpha\beta} F_{\alpha\beta})$, while the ΔL double trace Lagrangian term introduces four, five and six-point vertices in the Feynman diagrams. For computational convenience, let us take the following definition of self-dual $F_{\mu\nu}^+$ and anti-self-dual $F_{\mu\nu}^-$ field strengthes

$$F_{\mu\nu}^{\pm} = \frac{1}{2} F_{\mu\nu} \pm \frac{1}{4i} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} \quad \text{and} \quad \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\pm\rho\sigma} = \pm F_{\mu\nu}^{\pm}, \tag{4.12}$$

and rewrite the Lagrangian as

$$L_{\mathcal{O}_{\mathrm{III}}^{[0]}} = L_{\mathrm{SYM}} + \kappa \mathrm{Tr}(\phi^{A'B'} \phi^{C'D'}) \mathrm{Tr}(F^{+\mu\nu} F_{\mu\nu}^+) + \bar{\kappa} \mathrm{Tr}(\bar{\phi}_{A'B'} \bar{\phi}_{C'D'}) \mathrm{Tr}(F^{-\mu\nu} F_{\mu\nu}^-).$$

---

[11]We assumed that $i, j \neq 1, n-1$, otherwise the two contributing terms are slightly different. However they lead to the same conclusion.

The off-shell Feynman rules for the four-point vertices defined by the corresponding terms inside $\mathrm{Tr}(\phi\phi)\,\mathrm{Tr}(F^+F^+)$ or $\mathrm{Tr}(\bar\phi\bar\phi)\,\mathrm{Tr}(F^-F^-)$ of ΔL are given by

$$M^{\pm}_{\mu\nu} = (p_{i_1}\cdot p_{i_2})\eta_{\mu\nu} - p_{i_1\nu}p_{i_2\mu} \pm \frac{1}{i}\epsilon_{\mu\nu\rho\sigma}p^{\rho}_{i_1}p^{\sigma}_{i_1}\,, \tag{4.13}$$

where p_{i_1}, p_{i_2} are the momenta of two gluons. In fact, $M^+_{\mu\nu}$ can only attach gluons with positive helicities while $M^-_{\mu\nu}$ can only attach gluons with negative helicities, since

$$\epsilon^{+\mu}_1 M^+_{\mu\nu} = \frac{[1|\gamma_\nu|p_2|1]}{\sqrt{2}}\,,\quad \epsilon^{-\mu}_1 M^+_{\mu\nu} = 0 \ \text{ and }\ \epsilon^{+\mu}_1 M^-_{\mu\nu} = 0\,,\quad \epsilon^{-\mu}_1 M^-_{\mu\nu} = \frac{\langle 1|\gamma_\nu|p_2|1\rangle}{\sqrt{2}}\,.$$

And the four-point amplitudes defined by these vertices are given by

$$A_{2;2}(g^-_1, g^-_2; \bar\phi_3, \bar\phi_4) = \epsilon^{-\mu}_1 M^-_{\mu\nu}\epsilon^{-\nu}_2 = \langle 1\ 2\rangle^2\,,\quad A_{2;2}(g^+_1, g^+_2; \phi_3, \phi_4) = \epsilon^{+\mu}_1 M^+_{\mu\nu}\epsilon^{+\nu}_2 = [1\ 2]^2\,.$$

In order to compute the five-point amplitude $A_{3;2}(g^-_1, g^-_2, g^+_3; \bar\phi_4, \bar\phi_5)$, we also need the Feynman rule for five-point vertex defined by the corresponding terms inside $\mathrm{Tr}(\phi\phi)\,\mathrm{Tr}(F^+F^+)$ or $\mathrm{Tr}(\bar\phi\bar\phi)\,\mathrm{Tr}(F^-F^-)$, which is given by

$$V^{abc}_{\mu\nu\rho} = \frac{ig}{2}f^{abc}\Big((p_1-p_2)_\rho\eta_{\mu\nu} + (p_2-p_3)_\mu\eta_{\nu\rho} + (p_3-p_1)_\nu\eta_{\rho\mu} + i\kappa(p_1+p_2+p_3)^\sigma\epsilon_{\mu\nu\sigma\rho}\Big)\,. \tag{4.14}$$

There are in total three contributing Feynman diagrams, as shown in figure 4. We need to sum up all of three results. The first diagram gives

$$\begin{aligned}
(a) &= \frac{\langle 1|P_{23}|\gamma_\mu|1\rangle}{P^2_{23}}\Big((\epsilon^+_3\cdot\epsilon^-_2)p^\mu_2 - (P_{23}\cdot\epsilon_3)\epsilon^{-\mu}_2 + (p_3\cdot\epsilon^-_2)\epsilon^{+\mu}_3\Big)\\
&= \frac{\langle r_3\ 2\rangle\langle 1\ 2\rangle^2}{\langle 2\ 3\rangle\langle 3\ r_3\rangle} + \frac{\langle 1\ 2\rangle\langle r_3\ 1\rangle[r_2\ 3]}{\langle r_3\ 3\rangle[2\ r_2]}\,,
\end{aligned} \tag{4.15}$$

where r_1, r_2, r_3 are reference momenta of $\epsilon^-_\mu(p_1), \epsilon^-_\mu(p_2), \epsilon^+_\mu(p_3)$ (abbreviate as $\epsilon^-_1, \epsilon^-_2, \epsilon^+_3$) respectively. The second diagram gives

$$\begin{aligned}
(b) &= \frac{\langle 2|P_{13}|\gamma_\mu|2\rangle}{P^2_{13}}\Big(-(P_{13}\cdot\epsilon^-_1)\epsilon^{+\mu}_3 + (p_1\cdot\epsilon^+_3)\epsilon^{-\mu}_1 + (\epsilon^-_1\epsilon^+_3)p^\mu_3\Big)\\
&= \frac{\langle 1\ r_3\rangle\langle 1\ 2\rangle^2}{\langle 3\ 1\rangle\langle r_3\ 3\rangle} + \frac{\langle 1\ 2\rangle\langle r_3\ 2\rangle[r_1\ 3]}{\langle r_3\ 3\rangle[1\ r_1]}\,.
\end{aligned} \tag{4.16}$$

The third diagram 4.c is defined by the five-point vertex (4.14), while the result of first three terms in the bracket of (4.14) is

$$\begin{aligned}
(c.1) &= \frac{1}{2}\big((p_2-p_1)\cdot\epsilon^+_3\big)(\epsilon^-_1\cdot\epsilon^-_2) + \big((p_1-p_3)\cdot\epsilon^-_2\big)(\epsilon^+_3\cdot\epsilon^-_1) + \big((p_3-p_2)\cdot\epsilon^-_1\big)(\epsilon^-_2\cdot\epsilon^+_3)\\
&= \frac{1}{2}\Big(\frac{[r_1\ 3]\langle 1\ 2\rangle\langle 2\ r_3\rangle}{\langle r_3\ 3\rangle[1\ r_1]} - \frac{[3\ r_2]\langle 1\ 2\rangle\langle 1\ r_3\rangle}{\langle r_3\ 3\rangle[2\ r_2]} + \frac{[r_1\ 3][r_2\ 3]\langle 2\ 1\rangle}{[1\ r_1][2\ r_2]}\Big)\,.
\end{aligned} \tag{4.17}$$

Using

$$i\epsilon_{\mu\nu\rho\sigma}p^\mu_1 p^\nu_2 p^\rho_3 p^\sigma_4 = \langle 1\ 2\rangle[2\ 3]\langle 3\ 4\rangle[4\ 1] - [1\ 2]\langle 2\ 3\rangle[3\ 4]\langle 4\ 1\rangle\,,$$
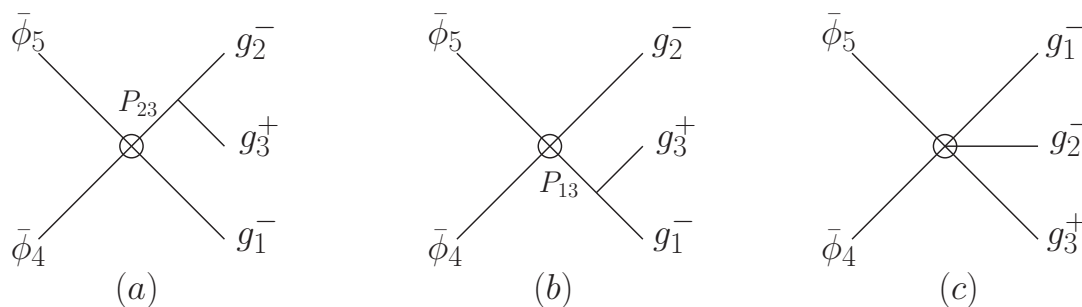
Figure 4. Feynman diagrams for $A_{3;2}(g_1^-, g_2^-, g_3^+; \bar{\phi}_4, \bar{\phi}_5)$ defined by $L_{\mathcal{O}_{\text{III}}^{[0]}}$. All external particles are out-going.

the last term in the bracket of (4.14) can be computed as

$$
\begin{aligned}
(c.2) &= \frac{1}{2} i \epsilon_{\mu\nu\sigma\rho} \epsilon_1^{-\mu} \epsilon_2^{-\nu} (p_1 + p_2 + p_3)^\sigma \epsilon_3^{+\rho} \\
&= \frac{1}{2} \left(\frac{\langle 1\,2 \rangle \langle 2\,r_3 \rangle [r_1\,3]}{[1\,r_1] \langle r_3\,3 \rangle} - \frac{\langle 1\,2 \rangle \langle 1\,r_3 \rangle [3\,r_2]}{[2\,r_2] \langle r_3\,3 \rangle} + \frac{\langle 1\,2 \rangle [r_2\,3][r_1\,3]}{[1\,r_1][2\,r_2]} \right).
\end{aligned}
\tag{4.18}
$$

Summing above contributions, we get

$$
A_{3;2}(g_1^-, g_2^-, g_3^+; \bar{\phi}_4, \bar{\phi}_5) = -\frac{\langle 1\,2 \rangle^4}{\langle 1\,2 \rangle \langle 2\,3 \rangle \langle 3\,1 \rangle}.
\tag{4.19}
$$

More generally, we have

$$
A_{n;2}(\{g^+\}, g_i^-, g_j^-; \bar{\phi}_{n+1}, \bar{\phi}_{n+2}) = -\frac{\langle i\,j \rangle^4}{\langle 1\,2 \rangle \langle 2\,3 \rangle \cdots \langle n\,1 \rangle},
\tag{4.20}
$$

which can be trivially proven by BCFW recursion relation. This expression is exactly the same as the pure-gluon n-point MHV amplitude of Yang-Mills theory. By taking $\langle \bar{\phi}_{n+1} | \bar{\phi}_{n+2}]$-shifting, we can get the form factor as

$$
\mathcal{F}_{\mathcal{O}_{\text{III}}^{[0]}, n}(\{g^+\}, g_i^-, g_j^-; q) = -\frac{\langle i\,j \rangle^4}{\langle 1\,2 \rangle \langle 2\,3 \rangle \cdots \langle n\,1 \rangle}.
\tag{4.21}
$$

Again, let us consider another configuration of external states, i.e., n gluons with negative helicities and two scalars. Computation of $A_{3;2}(g_1^-, g_2^-, g_3^-; \bar{\phi}_4, \bar{\phi}_5)$ is almost the same as $A_{3;2}(g_1^-, g_2^-, g_3^+; \bar{\phi}_4, \bar{\phi}_5)$, and we only need to replace ϵ_3^+ by ϵ_3^-. Direct computation shows that, contributions of all three diagrams lead to

$$
\frac{s_{12}^2 + s_{13}^2 + s_{23}^2 + 2 s_{12} s_{13} + 2 s_{12} s_{23} + 2 s_{13} s_{23}}{[1\,2][2\,3][3\,1]} = \frac{((p_4 + p_5)^2)^2}{[1\,2][2\,3][3\,1]}.
\tag{4.22}
$$

This result can be generalized to $A_{n;2}$ as

$$
A_{n;2}(g_1^-, g_2^-, \ldots, g_n^-; \bar{\phi}_{n+1}, \bar{\phi}_{n+2}) = \frac{((p_{n+1} + p_{n+2})^2)^2}{[1\,2][2\,3] \cdots [n\,1]},
\tag{4.23}
$$

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and can be proven recursively by BCFW recursion relation. In fact, assuming eq. (4.23) is true for $A_{n-1;2}$ and taking $\langle g_n^- | g_1^-]$-shifting, there is only one non-vanishing term in BCFW expansion, which gives

$$A_3(g_{\hat{1}}^-, g_2^-, g_{\hat{P}_{12}}^+) \frac{1}{P_{12}^2} A_{n-1;2}(g_{-\hat{P}_{12}}^-, g_3^-, \ldots, g_{\hat{n}}^-; \bar{\phi}_{n+1}, \bar{\phi}_{n+2}) = \frac{((p_{n+1} + p_{n+2})^2)^2}{[1\ 2][2\ 3]\cdots[n\ 1]}. \quad (4.24)$$

So the corresponding form factor is

$$\mathcal{F}_{\mathcal{O}_{\mathrm{III}}^{[0]},n}(g_1^-, g_2^-, \ldots, g_n^-; q) = \frac{(q^2)^2}{[1\ 2][2\ 3]\cdots[n\ 1]}. \quad (4.25)$$

4.2   The spin-$\frac{1}{2}$ operators

For operators

$$\mathcal{O}_{\mathrm{I}}^{[1/2]} = \mathrm{Tr}(\phi^{AB}\psi^{C\alpha}), \qquad \mathcal{O}_{\mathrm{II}}^{[1/2]} = \mathrm{Tr}(\psi_{\hat{\beta}}^A F^{\beta\alpha}), \quad (4.26)$$

and their complex conjugates $\bar{\mathcal{O}}_{\mathrm{I}}^{[1/2]}, \bar{\mathcal{O}}_{\mathrm{II}}^{[1/2]}$, we need to product them with another spin-$\frac{1}{2}$ trace term, which can be chosen as trace of product of scalar and fermion.

For operator $\mathcal{O}_{\mathrm{I}}^{[1/2]}$, we can construct the Lagrangian as

$$L_{\mathcal{O}_{\mathrm{I}}^{[1/2]}} = L_{\mathrm{SYM}} + \frac{\kappa}{N} \mathrm{Tr}(\phi^{A'B'}\psi^{C'\alpha}) \mathrm{Tr}(\phi^{AB}\psi_\alpha^C) + \frac{\bar{\kappa}}{N} \mathrm{Tr}(\bar{\phi}_{A'B'}\bar{\psi}_{C'}^{\dot{\alpha}}) \mathrm{Tr}(\bar{\phi}_{AB}\bar{\psi}_{C\dot{\alpha}}). \quad (4.27)$$

In order to generate operator $\mathcal{O}_{\mathrm{I}}^{[1/2]}$, we should shift $\bar{\phi}_{n+1}, \bar{\psi}_{n+2}$. However, there are two ways of shifting, and their large z behaviors are different. If we consider $\langle \bar{\phi}_{n+1} | \bar{\psi}_{n+2}]$-shifting, the leading term in z is $O(z^0)$, and the boundary operator after considering the LSZ reduction is

$$\mathcal{O}^{\langle \bar{\phi}_{n+1} | \bar{\psi}_{n+2}]} = \lambda_{n+2,\alpha} \mathrm{Tr}(\phi\psi^\alpha), \quad (4.28)$$

hence it has a $\lambda_{n+2,\alpha}$ factor difference with $\mathcal{O}_{\mathrm{I}}^{[1/2]}$. If we consider $\langle \bar{\psi}_{n+2} | \bar{\phi}_{n+1}]$-shifting, the leading term in z is $O(z)$ order. The boundary operator associated with the $O(z^0)$ term is quite complicated, but in the $O(z)$ order, we have

$$\mathcal{O}_z^{\langle \bar{\psi}_{n+2} | \bar{\phi}_{n+1}]} = -\lambda_{n+1,\alpha} \mathrm{Tr}(\phi\psi^\alpha). \quad (4.29)$$

These two ways of shifting would give the same result for form factor of $\mathcal{O}_{\mathrm{I}}^{[1/2]}$. However, it is better to take the shifting where the leading z term has lower rank, preferably $O(z^0)$ order, since the computation would be simpler.

The ΔL term introduces ϕ-ψ-ϕ-ψ and $\bar{\phi}$-$\bar{\psi}$-$\bar{\phi}$-$\bar{\psi}$ vertices in the Feynman diagrams. It is easy to know from Feynman diagram computation that $A_{2;2}(\bar{\phi}_1, \bar{\psi}_2; \bar{\phi}_3, \bar{\psi}_4) = \langle 4\ 2 \rangle$, and

$$A_{3;2}(\bar{\phi}_1, \bar{\psi}_2, g_3^+; \bar{\phi}_4, \bar{\psi}_5) = \frac{\langle 5 | P_{23} | \gamma_\mu | 2 \rangle}{s_{23}} \epsilon_3^{+\mu} - \frac{\langle 5\ 2 \rangle}{s_{13}} (p_1 - P_{13})_\mu \epsilon_3^{+\mu}$$

$$= \frac{\langle 1\ 2 \rangle^2 \langle 2\ 5 \rangle}{\langle 1\ 2 \rangle \langle 2\ 3 \rangle \langle 3\ 1 \rangle}. \quad (4.30)$$

This result can be generalized to

$$A_{n;2}(\{g^+\}, \bar{\phi}_i, \bar{\psi}_j; \bar{\phi}_{n+1}, \bar{\psi}_{n+2}) = \frac{\langle i\ j \rangle^2 \langle j, n+2 \rangle}{\langle 1\ 2 \rangle \langle 2\ 3 \rangle \cdots \langle n\ 1 \rangle}, \quad (4.31)$$

and similarly be proven by BCFW recursion relation. Note that this amplitude depends on p_{n+2} (more strictly speaking, λ_{n+2}^α) but not p_{n+1}, if we take $\langle\bar\phi_{n+1}|\bar\psi_{n+2}]$-shifting, the boundary contribution equals to the amplitude itself. Thus subtracting the factor[12] $\lambda_{n+2,\alpha}$, we obtain the form factor of operator $\mathcal{O}_{\mathrm{I}}^{[1/2]}$ as

$$\mathcal{F}^\alpha_{\mathcal{O}_{\mathrm{I}}^{[1/2]},n}(\{g^+\},\bar\phi_i,\bar\psi_j;q) = \frac{\langle i\ j\rangle^2\lambda_j^\alpha}{\langle 1\ 2\rangle\langle 2\ 3\rangle\cdots\langle n\ 1\rangle}\,. \tag{4.32}$$

If we instead take $\langle\bar\psi_{n+2}|\bar\phi_{n+1}]$-shifting, the boundary contribution of amplitude $A_{n;2}$ is

$$B_{n;2}^{\langle\bar\psi_{n+2}|\bar\phi_{n+1}]}(\{g^+\},\bar\phi_i,\bar\psi_j,\bar\phi_{n+1},\bar\psi_{n+2}) = \frac{\langle i\ j\rangle^2(\langle j,n+2\rangle - z\langle j,n+1\rangle)}{\langle 1\ 2\rangle\langle 2\ 3\rangle\cdots\langle n\ 1\rangle}\,. \tag{4.33}$$

The coefficient of z in above result is identical to the form factor of $\mathcal{O}_z^{\langle\bar\psi_{n+2}|\bar\phi_{n+1}]}$, and in order to get the form factor of $\mathcal{O}_{\mathrm{I}}^{[1/2]}$, we should subtract $-\lambda_{n+1,\alpha}$. The final result is again (4.32).

For operator $\mathcal{O}_{\mathrm{II}}^{[1/2]}$, we can construct the Lagrangian as

$$L_{\mathcal{O}_{\mathrm{II}}^{[1/2]}} = L_{\mathrm{SYM}} + \frac{\kappa}{N}\operatorname{Tr}(\phi^{A'B'}\psi_\alpha^{C'})\operatorname{Tr}(\psi_\beta^A F^{\beta\alpha}) + \frac{\bar\kappa}{N}\operatorname{Tr}(\bar\phi_{AB}\bar\psi_{C\dot\alpha})\operatorname{Tr}(\bar\psi_{A\dot\beta}\bar F^{\dot\beta\dot\alpha})\,. \tag{4.34}$$

Here we choose $\langle\bar\phi_{n+1}|\bar\psi_{n+2}]$-shifting so that the leading term in z is $O(z^0)$ order. The corresponding boundary operator is

$$\mathcal{O}^{\langle\bar\phi_{n+1}|\bar\psi_{n+2}]} = \lambda_{n+2,\alpha}\operatorname{Tr}(\psi_\beta F^{\beta\alpha})\,. \tag{4.35}$$

The ΔL term introduces four-point (scalar-fermion-fermion-gluon) and five-point (scalar-fermion-fermion-gluon-gluon) vertices. The four-point amplitude defined by the four-point vertex is given by

$$A_{2;2}(\bar\psi_1,g_2^-;\bar\phi_3,\bar\psi_4) = \frac{\langle 1|2|\gamma_\mu|4\rangle + \langle 4|2|\gamma_\mu|1\rangle}{2}\epsilon_2^{-\mu} = \langle 1\ 2\rangle\langle 4\ 2\rangle\,. \tag{4.36}$$

The five-point amplitude $A_{3;2}(\bar\psi_1,g_2^-,g_3^+;\bar\phi_4,\bar\psi_5)$ can be computed from three Feynman diagrams as shown in figure 5. The first diagram gives

$$(a) = \frac{1}{2}\Big(-\frac{\langle 1|P_{23}|\gamma_\mu|5\rangle}{s_{23}} - \frac{\langle 5|P_{23}|\gamma_\mu|1\rangle}{s_{23}}\Big)\Big(-(P_{23}\cdot\epsilon_3^+)\epsilon_2^{-\mu} + (p_3\cdot\epsilon_2^{-\mu})\epsilon_3^{+\mu} + (\epsilon_3^+\cdot\epsilon_2^-)p_2^\mu\Big)$$
$$= \frac{\langle 2\ r_3\rangle\langle 1\ 2\rangle\langle 2\ 5\rangle}{\langle 2\ 3\rangle\langle r_3\ 3\rangle} - \frac{1}{2}\frac{\langle 1\ 2\rangle[r_2\ 3]\langle r_3\ 5\rangle}{\langle r_3\ 3\rangle[2\ r_2]} - \frac{1}{2}\frac{\langle 5\ 2\rangle[r_2\ 3]\langle r_3\ 1\rangle}{\langle r_3\ 3\rangle[2\ r_2]}\,, \tag{4.37}$$

and the second diagram gives

$$(b) = -\frac{\langle 5\ 2\rangle\langle 2|P_{13}|\gamma_\mu|1\rangle}{s_{13}}\epsilon_3^{+\mu} = \frac{\langle 2\ 5\rangle\langle 1\ 2\rangle\langle r_3\ 1\rangle}{\langle 1\ 3\rangle\langle r_3\ 3\rangle}\,, \tag{4.38}$$

while the third diagram gives

$$(c) = \frac{\langle 1|\gamma_\mu\gamma_\nu|5\rangle + \langle 5|\gamma_\mu\gamma_\nu|1\rangle}{2}\epsilon_2^{-\mu}\epsilon_3^\nu = \frac{1}{2}\frac{\langle 1\ 2\rangle[r_2\ 3]\langle r_3\ 5\rangle}{\langle r_3\ 3\rangle[2\ r_2]} + \frac{1}{2}\frac{\langle 5\ 2\rangle[r_2\ 3]\langle r_3\ 1\rangle}{\langle r_3\ 3\rangle[2\ r_2]}\,. \tag{4.39}$$

---

[12]We take the convention that $\langle i\ j\rangle = \epsilon_{\alpha\beta}\lambda_i^\alpha\lambda_j^\beta = \lambda_i^\alpha\lambda_{j\alpha}$, $[i\ j] = \epsilon^{\dot\alpha\dot\beta}\tilde\lambda_{i\dot\alpha}\tilde\lambda_{j\dot\beta} = \tilde\lambda_{i\dot\alpha}\tilde\lambda_j^{\dot\alpha}$.
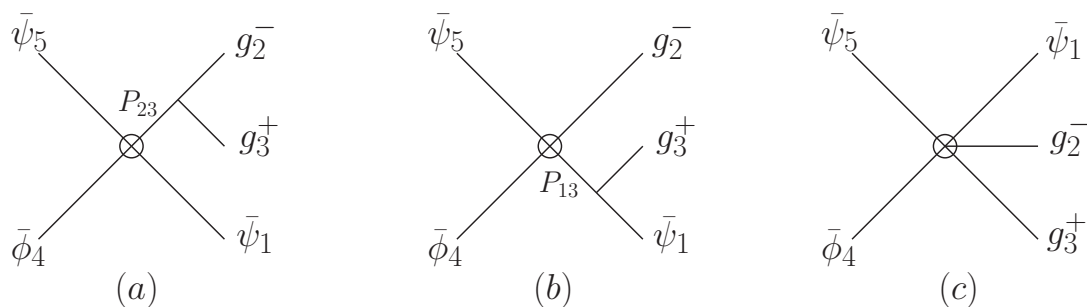
Figure 5. Feynman diagrams for $A_{3;2}(\bar\psi_1, g_2^-, g_3^+; \bar\phi_4, \bar\psi_5)$ defined by $L_{\mathcal{O}_{\mathrm{II}}^{[1/2]}}$. All external particles are out-going.

Summing above contributions, we get

$$A_{3;2}(\bar\psi_1, g_2^-, g_3^+; \bar\phi_4, \bar\psi_5) = \frac{\langle 1\ 2\rangle^3 \langle 2\ 5\rangle}{\langle 1\ 2\rangle\langle 2\ 3\rangle\langle 3\ 1\rangle}\,. \tag{4.40}$$

Then it is simple to generalize it to

$$A_{n;2}(\{g^+\}, \bar\psi_i, g_j^-; \bar\phi_{n+1}, \bar\psi_{n+2}) = \frac{\langle i\ j\rangle^3 \langle j, n+2\rangle}{\langle 1\ 2\rangle\langle 2\ 3\rangle\dots\langle n\ 1\rangle}\,, \tag{4.41}$$

which can be proven by BCFW recursion relation. Taking $\langle \bar\phi_{n+1}|\bar\psi_{n+2}]$-shifting and Subtracting $\lambda_{n+2,\alpha}$, we get the form factor

$$\mathcal{F}^\alpha_{\mathcal{O}_{\mathrm{II}}^{[1/2]},n}(\{g^+\}, \bar\psi_i, g_j^-; q) = \frac{\langle i\ j\rangle^3 \lambda_j^\alpha}{\langle 1\ 2\rangle\langle 2\ 3\rangle\dots\langle n\ 1\rangle}\,. \tag{4.42}$$

4.3 The spin-1 operators

There are three spin-1 operators

$$\mathcal{O}_{\mathrm{I}}^{[1]} = \mathrm{Tr}(\psi^{A\alpha}\psi^{B\beta} + \psi^{A\beta}\psi^{B\alpha})\,, \quad \mathcal{O}_{\mathrm{II}}^{[1]} = \mathrm{Tr}(\phi^{AB}F^{\alpha\beta})\,, \quad \mathcal{O}_{\mathrm{III}}^{[1]} = \mathrm{Tr}(\psi^{A\alpha}\bar\psi_B^{\dot\alpha})\,, \tag{4.43}$$

and their complex conjugates. In order to construct the Lagrangian, we need to product them with spin-1 trace term. Since a computation involving $F^{\alpha\beta}$ is always harder than those involving fermion and scalar, it is better to choose the trace of two fermions.

For operator $\mathcal{O}_{\mathrm{I}}^{[1]}$, we can construct the Lagrangian as

$$L_{\mathcal{O}_{\mathrm{I}}^{[1]}} = L_{\mathrm{SYM}} + \left(\frac{\kappa}{N}\mathrm{Tr}(\psi_\alpha^{A'}\psi_\beta^{B'} + \psi_\beta^{A'}\psi_\alpha^{B'})\,\mathrm{Tr}(\psi^{A\alpha}\psi^{B\beta} + \psi^{A\beta}\psi^{B\alpha}) + c.c.\right)\,. \tag{4.44}$$

Here in order to generate operator $\mathcal{O}_{\mathrm{I}}^{[1]}$, we should shift two fermions $\bar\psi_{n+1}, \bar\psi_{n+2}$. Taking $\langle \bar\psi_{n+1}|\bar\psi_{n+2}]$-shifting and considering the LSZ reduction, we find that the leading term in z is $O(z)$ order, and the corresponding boundary operator is

$$\mathcal{O}_z^{\langle \bar\psi_{n+1}|\bar\psi_{n+2}]} = -2\lambda_{n+2,\alpha}\lambda_{n+2,\beta}\,\mathrm{Tr}(\psi^{A\alpha}\psi^{B\beta} + \psi^{A\beta}\psi^{B\alpha})\,. \tag{4.45}$$

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Thus we also need to take the $O(z)$ order term in the boundary contribution of amplitude $A_{n;2}$ under $\langle \bar{\psi}_{n+1} | \bar{\psi}_{n+2}]$-shifting.

The ΔL Lagrangian term introduces four-fermion vertex, which defines the four-point amplitude $A_{2;2}(\bar{\psi}_1, \bar{\psi}_2; \bar{\psi}_3, \bar{\psi}_4) = \langle 3\ 1 \rangle \langle 2\ 4 \rangle + \langle 4\ 1 \rangle \langle 2\ 3 \rangle$. For five-point amplitude $A_{3;2}(\bar{\psi}_1, \bar{\psi}_2, g_3^+; \bar{\psi}_4, \bar{\psi}_5)$, there are two contributing Feynman diagrams, and the first diagram gives

$$
\begin{aligned}
(a) &= -\langle 5\ 2 \rangle \frac{\langle 1 | \gamma_\mu | P_{13} | 4 \rangle}{s_{13}} \epsilon_3^{+\mu} - \langle 4\ 2 \rangle \frac{\langle 1 | \gamma_\mu | P_{13} | 5 \rangle}{s_{13}} \epsilon_3^{+\mu} \\
&= -\frac{\langle 5\ 2 \rangle \langle 4\ 1 \rangle \langle 1\ r_3 \rangle}{\langle 3\ 1 \rangle \langle r_3\ 3 \rangle} - \frac{\langle 4\ 2 \rangle \langle 5\ 1 \rangle \langle 1\ r_3 \rangle}{\langle 3\ 1 \rangle \langle r_3\ 3 \rangle},
\end{aligned}
\tag{4.46}
$$

while the second gives

$$
\begin{aligned}
(b) &= \langle 5\ 1 \rangle \frac{\langle 2 | \gamma_\mu | P_{23} | 4 \rangle}{s_{23}} \epsilon_3^{+\mu} + \langle 4\ 1 \rangle \frac{\langle 2 | \gamma_\mu | P_{23} | 5 \rangle}{s_{23}} \epsilon_3^{+\mu} \\
&= \frac{\langle 5\ 1 \rangle \langle 4\ 2 \rangle \langle 2\ r_3 \rangle}{\langle 3\ 2 \rangle \langle r_3\ 3 \rangle} + \frac{\langle 4\ 1 \rangle \langle 5\ 2 \rangle \langle 2\ r_3 \rangle}{\langle 3\ 2 \rangle \langle r_3\ 3 \rangle}.
\end{aligned}
\tag{4.47}
$$

Thus

$$
A_{3;2}(\bar{\psi}_1, \bar{\psi}_2, g_3^+; \bar{\psi}_4, \bar{\psi}_5) = \frac{\langle 1\ 2 \rangle^2}{\langle 1\ 2 \rangle \langle 2\ 3 \rangle \langle 3\ 1 \rangle} (\langle 4\ 1 \rangle \langle 2\ 5 \rangle + \langle 5\ 1 \rangle \langle 2\ 4 \rangle).
\tag{4.48}
$$

By BCFW recursion relation, we also have

$$
\begin{aligned}
&A_{n;2}(\{g^+\}, \bar{\psi}_i, \bar{\psi}_j; \bar{\psi}_{n+1}, \bar{\psi}_{n+2}) \\
&= \frac{\langle i\ j \rangle^2}{\langle 1\ 2 \rangle \langle 2\ 3 \rangle \cdots \langle n\ 1 \rangle} (\langle n+1, i \rangle \langle j, n+2 \rangle + \langle n+2, i \rangle \langle j, n+1 \rangle).
\end{aligned}
\tag{4.49}
$$

Notice that this amplitude depends on both $\lambda_{n+1}^\alpha, \lambda_{n+2}^\alpha$, thus the $O(z)$ term is unavoidable when shifting two fermions. The boundary contribution under $\langle \bar{\psi}_{n+1} | \bar{\psi}_{n+2}]$-shifting is

$$
\begin{aligned}
&B_{n;2}^{\langle \bar{\psi}_{n+1} | \bar{\psi}_{n+2}]}(\{g^+\}, \bar{\psi}_i, \bar{\psi}_j; \bar{\psi}_{\widehat{n+1}}, \bar{\psi}_{\widehat{n+2}}) \\
&= -2z \frac{\langle i\ j \rangle^2}{\langle 1\ 2 \rangle \langle 2\ 3 \rangle \cdots \langle n\ 1 \rangle} \langle n+2, i \rangle \langle j, n+2 \rangle \\
&\quad + \frac{\langle i\ j \rangle^2}{\langle 1\ 2 \rangle \langle 2\ 3 \rangle \cdots \langle n\ 1 \rangle} (\langle n+1, i \rangle \langle j, n+2 \rangle + \langle n+2, i \rangle \langle j, n+1 \rangle).
\end{aligned}
\tag{4.50}
$$

Taking the $O(z)$ contribution and subtracting the factor $-2\lambda_{n+2,\alpha} \lambda_{n+2,\beta}$, we get the form factor

$$
\mathcal{F}_{\mathcal{O}_1^{[1]}, n}^{\alpha\beta}(\{g^+\}, \bar{\psi}_i, \bar{\psi}_j; q) = \frac{\langle i\ j \rangle^2}{\langle 1\ 2 \rangle \langle 2\ 3 \rangle \cdots \langle n\ 1 \rangle} \left(\frac{\lambda_i^\alpha \lambda_j^\beta + \lambda_j^\alpha \lambda_i^\beta}{2} \right),
\tag{4.51}
$$

where we have symmetrized the indices α, β.

Similar construction can be applied to the operator $\mathcal{O}_{II}^{[1]}$, where we have

$$
L_{\mathcal{O}_{II}^{[1]}} = L_{\text{SYM}} + \left(\frac{\kappa}{N} \text{Tr}(\psi_\alpha^{A'} \psi_\beta^{B'} + \psi_\beta^{A'} \psi_\alpha^{B'}) \text{Tr}(\phi^{AB} F^{\alpha\beta}) + c.c. \right).
\tag{4.52}
$$

The leading term in z under $\langle\bar\psi_{n+1}|\bar\psi_{n+2}]$-shifting is $O(z)$ order, and the boundary operator is

$$\mathcal{O}_z^{\langle\bar\psi_{n+1}|\bar\psi_{n+2}]} = -\lambda_{n+2,\alpha}\lambda_{n+2,\beta}\,\mathrm{Tr}(\phi^{AB}F^{\alpha\beta})\,. \tag{4.53}$$

The ΔL Lagrangian term introduces four-point (fermion-fermion-scalar-gluon) vertex and five-point (fermion-fermion-scalar-gluon-gluon) vertex. The four-point vertex defines four-point amplitude $A_{2;2}(\bar\phi_1,g_2^-;\bar\psi_3,\bar\psi_4) = -\frac{1}{2}(\langle3|2|\gamma_\mu|4] + \langle4|2|\gamma_\mu|3])\epsilon_2^{-\mu} = \langle2\,3\rangle\langle2\,4\rangle$, while for five-point amplitude $A_{3;2}(\bar\phi_1,g_2^-,g_3^+;\bar\psi_4,\bar\psi_5)$, we need to consider three Feynman diagrams, as shown in figure 6. The first diagram gives

$$(a) = \frac{\langle2\,4\rangle\langle2\,5\rangle}{s_{13}}(p_1 + P_{13})_\mu\epsilon_3^{+\mu} = \frac{\langle2\,4\rangle\langle2\,5\rangle\langle r_3\,1\rangle}{\langle3\,1\rangle\langle r_3\,3\rangle}\,, \tag{4.54}$$

the second diagram gives

$$\begin{aligned}
(b) &= \frac{1}{2}\Big(-\frac{\langle4|P_{23}|\gamma_\mu|5]}{s_{23}} - \frac{\langle5|P_{23}|\gamma_\mu|4]}{s_{23}}\Big)\Big(-(P_{23}\cdot\epsilon_3^+)\epsilon_2^{-\mu} + (p_3\cdot\epsilon_2^-)\epsilon_3^\mu + (\epsilon_3^+\cdot\epsilon_2^-)p_2^\mu\Big)\\
&= \frac{\langle r_3\,2\rangle\langle2\,4\rangle\langle2\,5\rangle}{\langle2\,3\rangle\langle r_3\,3\rangle} + \frac{1}{2}\frac{[r_2\,3]\langle r_3\,4\rangle\langle2\,5\rangle}{\langle r_3\,3\rangle[2\,r_2]} + \frac{1}{2}\frac{[r_2\,3]\langle r_3\,5\rangle\langle2\,4\rangle}{\langle r_3\,3\rangle[2\,r_2]}\,,
\end{aligned} \tag{4.55}$$

and the third diagram gives

$$(c) = \frac{\langle4|\gamma_\mu\gamma_\nu|5] + \langle5|\gamma_\mu\gamma_\nu|4]}{2}\epsilon_2^{-\mu}\epsilon_3^{+\nu} = \frac{1}{2}\frac{\langle4\,2\rangle[r_2\,3]\langle r_3\,5\rangle}{\langle r_3\,3\rangle[2\,r_2]} + \frac{1}{2}\frac{\langle5\,2\rangle[r_2\,3]\langle r_3\,4\rangle}{\langle r_3\,3\rangle[2\,r_2]}\,. \tag{4.56}$$

Summing above contributions, we get

$$A_{3;2}(\bar\phi_1,g_2^-,g_3^+;\bar\psi_4,\bar\psi_5) = \frac{\langle1\,2\rangle^2}{\langle1\,2\rangle\langle2\,3\rangle\langle3\,1\rangle}\langle4\,2\rangle\langle2\,5\rangle\,. \tag{4.57}$$

Generalizing above result to $(n+2)$-point amplitude, we have

$$A_{n;2}(\{g^+\},\bar\phi_i,g_j^-;\bar\psi_{n+1},\bar\psi_{n+2}) = \frac{\langle i\,j\rangle^2}{\langle1\,2\rangle\langle2\,3\rangle\cdots\langle n\,1\rangle}\langle n+1,j\rangle\langle j,n+2\rangle\,, \tag{4.58}$$

which can be trivially proven by BCFW recursion relation. We are only interested in the $O(z)$ term of the boundary contribution under $\langle\bar\psi_{n+1}|\bar\psi_{n+2}]$-shifting, which is

$$B_{n;2}^{\langle\bar\psi_{n+1}|\bar\psi_{n+2}]}(\{g^+\},\bar\phi_i,g_j^-;\widehat{\bar\psi_{n+1}},\widehat{\bar\psi_{n+2}}) = -z\frac{\langle i\,j\rangle^2\langle n+2,j\rangle\langle j,n+2\rangle}{\langle1\,2\rangle\langle2\,3\rangle\cdots\langle n\,1\rangle} + O(z^0)\,. \tag{4.59}$$

After subtracting the factor $-\lambda_{n+2,\alpha}\lambda_{n+2,\beta}$, we get

$$\mathcal{F}_{\mathcal{O}_{\mathrm{II}}^{[0]},n}^{\alpha\beta}(\{g^+\},\bar\phi_i,g_j^-;q) = \frac{\langle i\,j\rangle^2}{\langle1\,2\rangle\langle2\,3\rangle\cdots\langle n\,1\rangle}(-\lambda_j^\alpha\lambda_j^\beta)\,. \tag{4.60}$$

Now let us turn to the operator $\mathcal{O}_{\mathrm{III}}^{[1]}$, and construct the Lagrangian as

$$L_{\mathcal{O}_{\mathrm{III}}^{[1]}} = L_{\mathrm{SYM}} + \frac{\kappa}{N}\,\mathrm{Tr}(\psi_\alpha^{A'}\bar\psi_{B'\dot\alpha})\,\mathrm{Tr}(\psi^{A\alpha}\bar\psi_B^{\dot\alpha})\,. \tag{4.61}$$
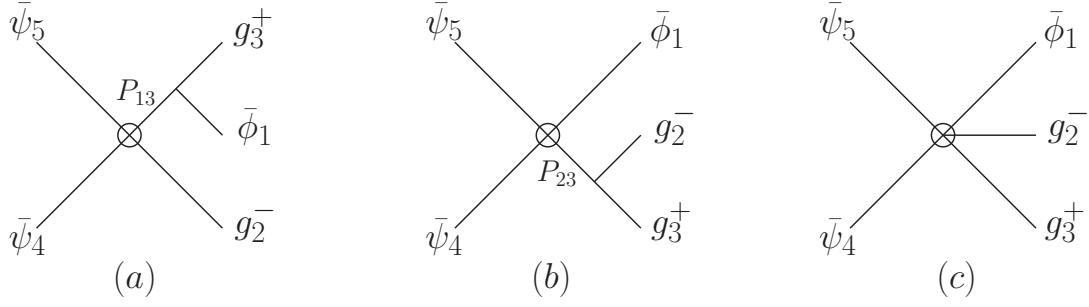
Figure 6. Feynman diagrams for $A_{3;2}(\bar{\phi}_1, g_2^-, g_3^+; \bar{\psi}_4, \bar{\psi}_5)$ defined by $L_{\mathcal{O}_{\mathrm{II}}^{[1]}}$. All external particles are out-going.

The leading term in z under $\langle\bar{\psi}_{n+2}|\psi_{n+1}]$-shifting is $O(z^2)$ order, while the leading term in z under $\langle\psi_{n+1}|\bar{\psi}_{n+2}]$-shifting is $O(z^0)$ order. In the later case, the boundary operator is

$$\mathcal{O}^{\langle\psi_{n+1}|\bar{\psi}_{n+2}]} = \widetilde{\lambda}_{n+1,\dot{\alpha}}\lambda_{n+2,\alpha}\,\mathrm{Tr}(\psi^{A\alpha}\bar{\psi}_B^{\dot{\alpha}})\,. \tag{4.62}$$

The four-point amplitude $A_{2;2}(\psi_1, \bar{\psi}_2; \psi_3, \bar{\psi}_4) = [1\,3]\langle 2\,4\rangle$, while the five-point amplitude

$$
\begin{aligned}
A_{3;2}(\psi_1, \bar{\psi}_2, g_3^+; \psi_4, \bar{\psi}_5) &= \langle 2\,5\rangle\frac{[1|\gamma_\mu|P_{13}|4\rangle}{s_{13}}\epsilon_3^{+\mu} - [1\,4]\frac{\langle 2|\gamma_\mu|P_{23}|5]}{s_{23}}\epsilon_3^{+\mu}\\
&= \frac{\langle 1\,2\rangle\langle 2\,5\rangle\langle 2|1+3|4]}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}\,.
\end{aligned}
\tag{4.63}
$$

Note that $\langle 2|1+3|4] = \langle 2|1+2+3|4] = \langle 2|q|4]$, where $q = -p_4 - p_5$, we can generalize above result to $(n+2)$-point as

$$A_{n;2}(\{g^+\}, \psi_i, \bar{\psi}_j; \psi_{n+1}, \bar{\psi}_{n+2}) = \frac{\langle i\,j\rangle\langle j, n+2\rangle\langle j|q|n+1]}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}\,, \tag{4.64}$$

where $q = -p_{n+1} - p_{n+2}$. Let us verify eq. (4.64) by induction method. Assuming eq. (4.64) is valid for $A_{n-1;2}$, and taking $\langle g_1^+|g_n^+]$-shifting, we get two contributing terms[13] from BCFW expansion. One is

$$A_{n-1;2}(g_{\hat{1}}^+, \ldots, \psi_i, \ldots, \bar{\psi}_j, \ldots, g_{n-2}^+, g_{\hat{P}_{n-1,n}}^+; \psi_{n+1}, \bar{\psi}_{n+2})\frac{1}{P_{n-1,n}^2}A_3(g_{-\hat{P}_{n-1,n}}^-, g_{n-1}^+, g_{\hat{n}}^+)\,.$$

Since $\hat{P}_{n-1,n}^2 = \langle n-1, n\rangle[\hat{n}, n-1] = 0$, so $A_3(g_{\hat{P}_{n-1,n}}^-, g_{n-1}^+, g_{\hat{n}}^+) \sim [n-1, \hat{n}]^3 \to 0$, and this term vanishes. The other contributing term is

$$A_3(g_{\hat{1}}^+, g_2^+, g_{\hat{P}_{12}}^-)\frac{1}{P_{12}^2}A_{n-1;2}(g_{-\hat{P}_{12}}^+, g_3^+, \ldots, \psi_i, \ldots, \bar{\psi}_j, \ldots, g_{\hat{n}}^+; \psi_{n+1}, \bar{\psi}_{n+2})\,. \tag{4.65}$$

By inserting the explicit expressions of A_3 and $A_{n-1;2}$, we arrive at eq. (4.64).

---

[13]We have assumed that $i, j \neq 2, n-1$, otherwise the contributing terms are slightly different. But the conclusion is the same.

Under $\langle\psi_{n+1}|\bar{\psi}_{n+2}]$-shifting, the boundary contribution is

$$B_{n;2}^{\langle\psi_{n+1}|\bar{\psi}_{n+2}]}(\{g^+\},\psi_i,\bar{\psi}_j;\psi_{\widehat{n+1}},\bar{\psi}_{\widehat{n+2}}) = \frac{\langle i\ j\rangle\langle j,n+2\rangle\langle j|q|n+1]}{\langle 1\ 2\rangle\langle 2\ 3\rangle\cdots\langle n\ 1\rangle}. \tag{4.66}$$

So subtracting the factor $\lambda_{n+2,\alpha}\widetilde{\lambda}_{n+1,\dot{\alpha}}$, we get the form factor

$$\mathcal{F}_{\mathcal{O}_{\mathrm{III}}^{[1]},n}^{\alpha\dot{\alpha}}(\{g^+\},\psi_i,\bar{\psi}_j;q) = \frac{\langle i\ j\rangle}{\langle 1\ 2\rangle\langle 2\ 3\rangle\cdots\langle n\ 1\rangle}\lambda_j^\alpha(\lambda_{j\beta}q^{\beta\dot{\alpha}}). \tag{4.67}$$

4.4 The spin-$\frac{3}{2}$ operators

There are two operators

$$\mathcal{O}_{\mathrm{I}}^{[3/2]} = \mathrm{Tr}(\bar{\psi}^{\dot{\alpha}}F^{\alpha\beta}), \quad \mathcal{O}_{\mathrm{II}}^{[3/2]} = \mathrm{Tr}(\psi^\gamma F^{\alpha\beta}) \tag{4.68}$$

with their complex conjugate partners. We need to product them with spin-$\frac{3}{2}$ trace term to construct ΔL.

For the operator $\mathcal{O}_{\mathrm{I}}^{[3/2]}$, we can construct the Lagrangian as

$$L_{\mathcal{O}_{\mathrm{I}}^{[3/2]}} = L_{\mathrm{SYM}} + \frac{\kappa}{N}\mathrm{Tr}(\bar{\psi}_{\dot{\alpha}}F_{\alpha\beta})\mathrm{Tr}(\bar{\psi}^{\dot{\alpha}}F^{\alpha\beta}) + \frac{\bar{\kappa}}{N}\mathrm{Tr}(\psi_\alpha\bar{F}_{\dot{\alpha}\dot{\beta}})\mathrm{Tr}(\psi^\alpha\bar{F}^{\dot{\alpha}\dot{\beta}}). \tag{4.69}$$

It introduces new four-point vertices $\bar{\psi}$-g^+-$\bar{\psi}$-g^+ and ψ-g^--ψ-g^-, as well as five, six-point vertices.

From Feynman diagrams, we can directly compute $A_{2;2}(\psi_1,g_2^-;\psi_3,g_4^-) = \langle 2\ 4\rangle^2[1\ 3]$, while for the five-point amplitude $A_{3;2}(\psi_1,g_2^-,g_3^+;\psi_4,g_5^-)$, we need to compute three Feynman diagrams, which are given by

$$(a) = \langle 2\ 5\rangle^2\frac{[1|\gamma_\mu|P_{13}|4\rangle}{s_{13}}\epsilon_3^{+\mu} = \frac{\langle 2\ 5\rangle^2[3\ 4]}{\langle 3\ 1\rangle} + \frac{\langle 2\ 5\rangle^2[1\ 4]\langle r_3\ 1\rangle}{\langle 3\ 1\rangle\langle r_3\ 3\rangle}, \tag{4.70}$$

$$\begin{aligned}(b) &= -[1\ 4]\frac{\langle 5|P_{23}|\gamma_\mu|5\rangle}{s_{23}}\Big(-(P_{23}\cdot\epsilon_3^+)\epsilon_2^{-\mu} + (p_3\cdot\epsilon_2^-)\epsilon_3^{+\mu} + (\epsilon_3^+\cdot\epsilon_2^-)p_2^\mu\Big)\\ &= \frac{[1\ 4]\langle 2\ 5\rangle^2\langle r_3\ 2\rangle}{\langle 2\ 3\rangle\langle r_3\ 3\rangle} + \frac{[1\ 4]\langle 2\ 5\rangle[r_2\ 3]\langle r_3\ 5\rangle}{\langle r_3\ 3\rangle[2\ r_2]},\end{aligned} \tag{4.71}$$

and

$$(c) = [1\ 4]\langle 5|\gamma_\mu\gamma_\nu|5\rangle\epsilon_2^{-\mu}\epsilon_3^{+\nu} = \frac{[1\ 4]\langle 5\ 2\rangle[r_2\ 3]\langle r_3\ 5\rangle}{\langle r_3\ 3\rangle\langle 2\ r_2\rangle}. \tag{4.72}$$

So the final result is

$$A_{3;2}(\psi_1,g_2^-,g_3^+;\psi_4,g_5^-) = \frac{\langle 1\ 2\rangle\langle 2\ 5\rangle^2\langle 2|q|4]}{\langle 1\ 2\rangle\langle 2\ 3\rangle\langle 3\ 1\rangle}, \tag{4.73}$$

where $q = -p_4 - p_5$. This result can be generalized to

$$A_{n;2}(\{g^+\},\psi_i,g_j^-;\psi_{n+1},g_{n+2}^-) = \frac{\langle i\ j\rangle\langle j,n+2\rangle^2\langle j|q|n+1]}{\langle 1\ 2\rangle\langle 2\ 3\rangle\cdots\langle n\ 1\rangle}, \tag{4.74}$$

where $q = -p_{n+1} - p_{n+2}$, and proven by BCFW recursion relation as done for the $\mathcal{O}_{\mathrm{III}}^{[1]}$ case.

If taking $\langle g_{n+2}^-|\psi_{n+1}]$-shifting, the leading z term in the boundary operator would be $O(z^3)$ order. We can however choose $\langle\psi_{n+1}|g_{n+2}^-]$-shifting, under which there is only $O(z^0)$ term in the boundary operator,

$$\mathcal{O}^{\langle\psi_{n+1}|g_{n+2}^-]} = \widetilde{\lambda}_{n+1,\dot{\alpha}}\lambda_{n+2,\alpha}\lambda_{n+2,\beta}\operatorname{Tr}(\bar{\psi}^{\dot{\alpha}}F^{\alpha\beta})\,. \tag{4.75}$$

The boundary contribution of amplitude $A_{n;2}$ under $\langle\psi_{n+1}|g_{n+2}^-]$-shifting equals to $A_{n;2}$ itself, thus after subtracting factor $\widetilde{\lambda}_{n+1,\dot{\alpha}}\lambda_{n+2,\alpha}\lambda_{n+2,\beta}$, we get the form factor

$$\mathcal{F}_{\mathcal{O}_{\mathrm{I}}^{[3/2]},n}^{\dot{\alpha}\,\alpha\beta}(\{g^+\},\psi_i,g_j^-;q) = \frac{\langle i\,j\rangle}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}\lambda_j^\alpha\lambda_j^\beta(\lambda_{j\gamma}q^{\gamma\dot{\alpha}})\,. \tag{4.76}$$

Discussion on the operator $\mathcal{O}_{\mathrm{II}}^{[3/2]}$ is almost the same as operator $\mathcal{O}_{\mathrm{I}}^{[3/2]}$, while we only need to change $\psi\to\bar{\psi}$. We can construct the Lagrangian as

$$L_{\mathcal{O}_{\mathrm{II}}^{[3/2]}} = L_{\mathrm{SYM}} + \frac{\kappa}{N}\operatorname{Tr}(\psi_\gamma F_{\alpha\beta})\operatorname{Tr}(\psi^\gamma F^{\alpha\beta}) + \frac{\bar{\kappa}}{N}\operatorname{Tr}(\bar{\psi}_{\dot{\gamma}}\bar{F}_{\dot{\alpha}\dot{\beta}})\operatorname{Tr}(\bar{\psi}^{\dot{\gamma}}\bar{F}^{\dot{\alpha}\dot{\beta}})\,. \tag{4.77}$$

In order to generate the operator $\operatorname{Tr}(\psi^\gamma F^{\alpha\beta})$, we need to shift $\bar{\psi}_{n+1},g_{n+2}^-$. Under $\langle g_{n+2}^-|\bar{\psi}_{n+1}]$-shifting, the leading term in z is $O(z^2)$ order, and the corresponding boundary operator is

$$\mathcal{O}_{z^2}^{\langle g_{n+2}^-|\bar{\psi}_{n+1}]} = \lambda_{n+1,\alpha}\lambda_{n+1,\beta}\lambda_{n+1,\gamma}\operatorname{Tr}(\psi^\gamma F^{\alpha\beta})\,. \tag{4.78}$$

We can also take $\langle\bar{\psi}_{n+1}|g_{n+2}^-]$-shifting, and the corresponding boundary operator is $O(z)$ order,

$$\mathcal{O}_z^{\langle\bar{\psi}_{n+1}|g_{n+2}^-]} = \lambda_{n+2,\gamma}\lambda_{n+2,\alpha}\lambda_{n+2,\beta}\operatorname{Tr}(\psi^\gamma F^{\alpha\beta})\,. \tag{4.79}$$

Computation of double trace amplitudes defined by $L_{\mathcal{O}_{\mathrm{II}}^{[3/2]}}$ is similar to those defined by $L_{\mathcal{O}_{\mathrm{I}}^{[3/2]}}$, and we immediately get $A_{2;2}(\bar{\psi}_1,g_2^-;\bar{\psi}_3,g_4^-) = \langle 2\,4\rangle^2\langle 3\,1\rangle$, and

$$A_{3;2}(\bar{\psi}_1,g_2^-,g_3^+;\bar{\psi}_4,g_5^-) = \frac{\langle 1\,2\rangle^2\langle 2\,5\rangle^2\langle 1\,4\rangle}{\langle 1\,2\rangle\langle 2\,3\rangle\langle 3\,1\rangle}\,. \tag{4.80}$$

For general $(n+2)$-point amplitude, we have

$$A_{n;2}(\{g^+\},\bar{\psi}_i,g_j^-;\bar{\psi}_{n+1},g_{n+2}^-) = \frac{\langle i\,j\rangle^2\langle j,n+2\rangle^2\langle i,n+1\rangle}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}\,. \tag{4.81}$$

We can either take $\langle g_{n+2}^-|\bar{\psi}_{n+1}]$-shifting or $\langle\bar{\psi}_{n+1}|g_{n+2}^-]$-shifting to compute the form factor of $\mathcal{O}_{\mathrm{II}}^{[3/2]}$. For example, under $\langle g_{n+2}^-|\bar{\psi}_{n+1}]$-shifting, we pick up the $O(z^2)$ term of boundary contribution, which is

$$z^2\frac{\langle i\,j\rangle^2\langle j,n+1\rangle^2\langle i,n+1\rangle}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}\,,$$

subtract the factor $\lambda_{n+1,\alpha}\lambda_{n+1,\beta}\lambda_{n+1,\gamma}$, and finally get the form factor,

$$\mathcal{F}_{\mathcal{O}_{\mathrm{II}}^{[3/2]},n}^{\alpha\beta\gamma}(\{g^+\},\bar{\psi}_i,g_j^-;\bar{\psi}_{n+1},g_{n+2}^-) = \frac{\langle i\,j\rangle^2}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}\lambda_j^\alpha\lambda_j^\beta\lambda_i^\gamma\,. \tag{4.82}$$

4.5 The spin-2 operator

For the spin-2 operator

$$\mathcal{O}_{\mathrm{I}}^{[2]} = \mathrm{Tr}(F^{\alpha\beta}\bar{F}^{\dot{\alpha}\dot{\beta}}), \tag{4.83}$$

we can construct the Lagrangian as

$$L_{\mathcal{O}_{\mathrm{I}}^{[2]}} = L_{\mathrm{SYM}} + \frac{\kappa}{N}\,\mathrm{Tr}(F_{\alpha\beta}\bar{F}_{\dot{\alpha}\dot{\beta}})\,\mathrm{Tr}(F^{\alpha\beta}\bar{F}^{\dot{\alpha}\dot{\beta}}). \tag{4.84}$$

The ΔL Lagrangian term introduces four to eight-point gluon vertices in Feynman diagrams. It is easy to know that the four-point amplitude $A_{2;2}(g_1^-, g_2^+; g_3^-, g_4^+) = \langle 1\,3\rangle^2[2\,4]^2$. The general $(n+2)$-point amplitude is given by

$$A_{n;2}(\{g^+\}, g_i^-; g_{n+1}^-, g_{n+2}^+) = -\frac{\langle i|q|n+2]^2\langle i, n+1\rangle^2}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}, \tag{4.85}$$

where $q = -p_{n+1} - p_{n+2}$. Let us verify this result by BCFW recursion relation. Assuming eq. (4.85) is valid for $A_{n-1;2}$, and taking $\langle g_{n-1}^+|g_n^+]$-shifting, we get two contributing terms[14] in BCFW expansion. The first term is

$$A_{n-1;2}(g_2^+, \ldots, g_{i-1}^+, g_i^-, g_{i+1}^+, \ldots, g_{\widehat{n-1}}^+, g_{\widehat{P}_{1n}}^+; g_{n+1}^-, g_{n+2}^+)\frac{1}{P_{1n}^2}A_3(g_{-\widehat{P}_{1n}}^-, g_{\widehat{n}}^+, g_1^+), \tag{4.86}$$

and this one vanishes, since the on-shell condition of propagator $\widehat{P}_{1n}^2 = \langle 1\,n\rangle[\widehat{n}\,1] = 0$ implies $A_3(g_{-\widehat{P}_{1n}}^-, g_{\widehat{n}}^+, g_1^+) \sim [\widehat{n}\,1]^3 \to 0$. The other term is

$$A_3(g_{n-2}^+, g_{\widehat{n-1}}^+, g_{\widehat{P}}^-)\frac{1}{P_{n-2,n-1}^2}A_{n-1;2}(g_{-\widehat{P}}^+, g_{\widehat{n}}^+, g_1^+, \ldots, g_{i-1}^+, g_i^-, g_{i+1}^+, \ldots, g_{n-3}^+; g_{n+1}^-, g_{n+2}^+).$$

After inserting the explicit expressions for A_3 and $A_{n-1;2}$, we arrive at the result (4.85).

The leading z term of boundary operator under $\langle g_{n+1}^-|g_{n+2}^+]$-shifting is $O(z^4)$ order. Instead, we would like to take $\langle g_{n+2}^+|g_{n+1}^-]$-shifting, under which the boundary operator is $O(z^0)$ order. After considering LSZ reduction, we have

$$\mathcal{O}^{\langle g_{n+2}^+|g_{n+1}^-]} = \widetilde{\lambda}_{n+2,\dot{\alpha}}\widetilde{\lambda}_{n+2,\dot{\beta}}\lambda_{n+1,\alpha}\lambda_{n+1,\beta}\,\mathrm{Tr}(F^{\alpha\beta}\bar{F}^{\dot{\alpha}\dot{\beta}}). \tag{4.87}$$

Hence by picking up the boundary contribution of amplitude $A_{n;2}$ under $\langle g_{n+2}^+|g_{n+1}^-]$-shifting, and subtracting factor $\widetilde{\lambda}_{n+2,\dot{\alpha}}\widetilde{\lambda}_{n+2,\dot{\beta}}\lambda_{n+1,\alpha}\lambda_{n+1,\beta}$, we get the form factor

$$\mathcal{F}_{\mathcal{O}_{\mathrm{I}}^{[2]},n}^{\dot{\alpha}\dot{\beta}\,\alpha\beta}(\{g^+\}, g_i^-; q) = -\frac{(\lambda_{i\gamma_1}q^{\gamma_1\dot{\alpha}})(\lambda_{i\gamma_2}q^{\gamma_2\dot{\beta}})\lambda_i^\alpha\lambda_i^\beta}{\langle 1\,2\rangle\langle 2\,3\rangle\cdots\langle n\,1\rangle}. \tag{4.88}$$

5 Summary and discussion

The boundary operator is initially introduced as a formal technique to study the boundary contribution of amplitude when doing BCFW recursion relation in paper [56]. It defines a form factor, and practically this off-shell quantity is difficult to compute. In this paper, we

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[14]We have assumed $i \neq 1, n-2$, which can always be true by cyclic invariance of the external legs.

take the reversed way to study the form factor from boundary contribution of amplitude of certain theory. We show that by suitable construction of Lagrangian, it is possible to generate boundary operators which are identical (or proportional) to the given operators of interest. This means that the form factor of given operator can be extracted from the boundary contribution of corresponding amplitude defined by that Lagrangian. We demonstrate this procedure for a class of composite operators by computing amplitudes of double trace structure and reading out the form factors from corresponding boundary contribution. Thus the computation of form factor becomes a problem of computing the scattering amplitude.

We have considered a class of composite operators, which are traces of product of two component fields from $\mathcal{N} = 4$ SYM, and the sum of spins of those two fields is no larger than two. In fact, the construction of Lagrangian has no difference for other operators with length (the number of fields inside the trace) larger than two, provided the sum of their spins is no larger than two. This is because we can always product them with a length-two trace term to make a Lorentz-invariant Lagrangian term, and deform the two fields in the extra trace term to produce the required boundary operators. However, if the operator has spin larger than two, in order to make a Lorentz-invariant Lagrangian term, the length of extra trace term should be larger than two. Then deformation of two fields in the extra trace term is not sufficient to produce the desired boundary operators, and we need multi-step deformation. It would be interesting to investigate how this multi-step deformation works out. It would also be interesting to find out how to apply this story to other kind of operators such as stress-tensor multiplet or amplitude with off-shell currents.

Note that in this paper we only considered local operators, but the method could as well be used to compute the form factors of non-local operators like Wilson lines and Wilson loops. In fact, the $\mathcal{O}(z^3)$ order operator of the $\langle g^- | g^+]$ shift in Yang-Mills theory can be interpreted as a Wilson line in the direction of the q,

$$\mathcal{O}_{z^3}^{\langle g^-(k_1) | g^+(k_2)]} = 2igz^3 \; k_1 \cdot k_2 \; q \cdot A \; S_q \,, \tag{5.1}$$

where S_q is the Wilson line

$$S_q(x) = \mathbf{P} \exp \left[ig \int_{-\infty}^{0} ds q \cdot A(x + sq) \right] . \tag{5.2}$$

All the discussions considered in this paper are at tree-level. While it is argued [56] that the boundary operator is generalizable to loop-level since the OPE can be defined therein, it is interesting to see if similar connection between form factor and amplitude also exists at loop-level or not. For this purpose, it would be better to study the loop corrections to the boundary operators, which is under investigation.

Acknowledgments

A Brief review on constructing the boundary operator

For reader's convenience we briefly review the results of paper [56] in this appendix. Please refer to that paper for more details.

The whole idea is to consider the OPE expansion in momentum space in the large z limits, and work out the expansion coefficients of each z order. Denoting the two shifted fields as $\Phi_1^\Lambda \equiv \Phi^\Lambda(p_1 + zq)$ and $\Phi_n^\Lambda \equiv \Phi^\Lambda(p_n - zq)$, one found that the z-dependence can be computed from

$$\mathcal{Z}(z) = -i \int D\Phi^\Lambda \exp\left(iS_2^\Lambda[\Phi^\Lambda, \Phi]\right)\Phi_1^\Lambda \Phi_n^\Lambda, \tag{A.1}$$

where $S_2^\Lambda[\Phi^\Lambda, \Phi]$ is the quadratic term of Φ^Λ in action S after field splitting $\Phi \to \Phi + \Phi^\Lambda$ (soft part and hard part). This can be interpreted as the OPE of Φ_1^Λ and Φ_n^Λ. Expanding $\mathcal{Z}(z)$ around $z = \infty$ yields

$$\mathcal{Z}(z) = \cdots + \frac{1}{z}\mathcal{O}_{z^{-1}} + \mathcal{O}^{\langle\phi_1|\Phi_n]} + z\mathcal{O}_z^{\langle\Phi_1|\Phi_n]} + \cdots. \tag{A.2}$$

In order to construct the boundary operator for given z order, one should compute $\mathcal{Z}(z)$, i.e., evaluate the integral (A.1). Since S_2^Λ only contains terms quadratic in Φ^Λ, integral (A.1) can be evaluated exactly. Assume a theory has M real fields ψ^I and N complex fields ϕ^A, compactly expressed as

$$\Phi^\alpha = \begin{pmatrix} \varphi^I \\ \phi^A \\ \bar{\phi}^A \end{pmatrix}, \quad H^\alpha = \begin{pmatrix} \widehat{\varphi}^I \\ \widehat{\phi}^A \\ \widehat{\bar{\phi}}^A \end{pmatrix}, \tag{A.3}$$

where we have combined hard fields into H^α. The complex conjugates of Φ^α is $\Phi_\alpha^\dagger = \begin{pmatrix} \varphi^I & \bar{\phi}_A & \phi^A \end{pmatrix}$, and be related to Φ^α as $\Phi^\alpha = T^{\alpha\beta}\Phi_\beta^\dagger$ through matrix

$$T^{\alpha\beta} = \begin{pmatrix} I_M & 0 & 0 \\ 0 & 0 & I_N \\ 0 & I_N & 0 \end{pmatrix}. \tag{A.4}$$

With these notations, the quadratic term in the Lagrangian is

$$L_2^\Lambda = \frac{1}{2}H_\alpha^\dagger \mathcal{D}^\alpha_\beta H^\beta, \quad \mathcal{D}^\alpha_\beta = \frac{\delta^2}{\delta\Phi_\alpha^\dagger \delta\Phi^\beta}L. \tag{A.5}$$

Following the standard procedure of computing generating functions, one can get

$$\mathcal{Z}(z) = \mathcal{Z}^\Lambda[\Phi](\mathcal{D}^{-1})^{\alpha\beta}(x, y; \Phi). \tag{A.6}$$

$\mathcal{D}(\Phi)$ is a function of Φ, and in general can be decomposed into a free part D_0 and an interaction part V as $\mathcal{D}^\alpha_\beta(\Phi) = (D_0)^\alpha_\beta + V^\alpha_\beta(\Phi)$. The $\mathcal{Z}^\Lambda(\Phi)$ can be dropped at tree-level. After some evaluation including LSZ reduction for fields $H(p_1 + zq)$, $H(p_n - zq)$, the remaining part yields

$$\mathcal{Z}(z) = \epsilon_{\alpha_1}^1 \epsilon_{\alpha_n}^n \left[V^{\alpha_1\alpha_n} - V^{\alpha_1\beta_1}(D_0^{-1})_{\beta_1\beta_2}V^{\beta_2\alpha_2} + \cdots\right]. \tag{A.7}$$

Then we can read out the z-dependence from above result.

B Discussion on the large z behavior

Let us start from the Lagrangian of $\mathcal{N} = 4$ SYM in component fields,

$$L = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \frac{1}{2}\mathcal{D}_\mu\phi^{Ia}\mathcal{D}^\mu\phi^{Ia} - i\bar{\psi}^a_A\bar{\sigma}^\mu\mathcal{D}_\mu\psi^{Aa}$$
$$+ \frac{ig}{2}f^{abc}\left(\bar{T}^I_{AB}\phi^{Ia}\psi^{Ab}\psi^{Bc} + T^{IBA}\phi^{Ia}\bar{\psi}^b_A\bar{\psi}^c_B\right) - \frac{g^2}{4}f^{abe}f^{cde}\phi^{Ia}\phi^{Jb}\phi^{Ic}\phi^{Jd}, \qquad \text{(B.1)}$$

where T^{IAB} is the transformation matrix between SO(6) and SU(4) representations of scalar fields $\phi^{AB} = \frac{1}{\sqrt{2}}\phi^I T^{IAB}$. The gauge fixing term is

$$L_{gf} = -\frac{1}{2}(\mathcal{D}_\mu A^{\Lambda\mu a} + gf^{abc}\phi^{Ib}\phi^{\Lambda Ic})^2. \qquad \text{(B.2)}$$

In order to get the quadratic terms of shifted hard fields, we need to compute the second order variation of L. Since

$$\frac{\delta L}{\delta A^a_\mu} = -\mathcal{D}_\nu F^{\mu\nu a} - gf^{abc}\phi^{Ib}D^\mu\phi^{Ic} - igf^{abc}\bar{\psi}^c_A\bar{\sigma}^\mu\psi^{Ab},$$

$$\frac{\delta L}{\delta\phi^{Ia}} = \mathcal{D}^2\phi^{Ia} + \frac{ig}{2}f^{abc}\left(\bar{T}^I_{AB}\psi^{Ab}\psi^{Bc} + T^{IBA}\bar{\psi}^b_A\bar{\psi}^c_B\right) - g^2f^{abe}f^{cde}\phi^{Jb}\phi^{Ic}\phi^{Jd},$$

$$\frac{\delta L}{\delta\bar{\psi}^a_A} = -i\bar{\sigma}^\mu D_\mu\psi^{Aa} + igf^{abc}T^{IBA}\phi^{Ic}\bar{\psi}^b_B,$$

$$\frac{\delta L}{\delta\psi^{Aa}} = i\sigma^\mu D_\mu\bar{\psi}^a_A + igf^{abc}\bar{T}^I_{AB}\phi^{Ic}\psi^{Bb},$$

we have

$$D = \begin{pmatrix} \frac{\delta}{\delta A^a_\mu} \\ \frac{\delta}{\delta\phi^{Ia}} \\ \frac{\delta}{\delta\bar{\psi}^a_A} \\ \frac{\delta}{\delta\psi^{Aa}} \end{pmatrix} L \left(\frac{\overleftarrow{\delta}}{\delta A^b_\nu} \quad \frac{\overleftarrow{\delta}}{\delta\phi^{Jb}} \quad \frac{\overleftarrow{\delta}}{\delta\bar{\psi}^b_B} \quad \frac{\overleftarrow{\delta}}{\delta\psi^{Bb}} \right)$$

$$= \begin{pmatrix} D_{11} & -2gf^{abc}\mathcal{D}^\mu\phi^{Jc} & igf^{abc}\psi^{Bc}\sigma^\mu & igf^{abc}\bar{\psi}^c_B\bar{\sigma}^\mu \\ 2gf^{abc}\mathcal{D}^\nu\phi^{Ic} & D_{22} & -igf^{abc}T^{IBA}\bar{\psi}^c_A & -igf^{abc}\bar{T}^I_{AB}\psi^{Ac} \\ -igf^{abc}\bar{\sigma}^\nu\psi^{Ac} & -igf^{abc}T^{JBA}\bar{\psi}^c_B & igf^{abc}T^{IBA}\phi^{Ic} & -i\delta^A_B\bar{\sigma}^\mu\mathcal{D}^{ab}_\mu \\ igf^{abc}\sigma^\nu\bar{\psi}^c_A & -igf^{abc}\bar{T}^J_{AB}\psi^{Bc} & i\delta^B_A\sigma^\mu\mathcal{D}^{ab}_\mu & igf^{abc}\bar{T}^I_{AB}\phi^{Ic} \end{pmatrix}, \qquad \text{(B.3)}$$

where

$$D_{11} = \eta^{\mu\nu}\left[(\mathcal{D}^2)^{ab} - g^2f^{ace}f^{bde}\phi^{Kc}\phi^{Kd}\right] - 2gf^{abc}F^{\mu\nu c}, \qquad \text{(B.4)}$$

$$D_{22} = \delta^{IJ}\left[(\mathcal{D}^2)^{ab} - g^2f^{ace}f^{bde}\phi^{Kc}\phi^{Kd}\right] - 2g^2f^{abc}f^{cde}\phi^{Id}\phi^{Je}, \qquad \text{(B.5)}$$

and $\mathcal{D}^{-ab} = \delta^{ab}\partial^- - gf^{abc}A^{-c}$. The operator D can be decomposed into two parts, the interaction part $V(z) = V + zX$, where

$$X = \begin{pmatrix} 2igf^{abc}\eta^{\mu\nu}A^{-c} & 2igf^{abc}q^\mu\phi^{Jc} & 0 & 0 \\ -2igf^{abc}q^\nu\phi^{Ic} & 2igf^{abc}\delta^{IJ}A^{-c} & 0 & 0 \\ 0 & 0 & 0 & -\delta^{ab}\delta^A_B q_\mu\bar{\sigma}^\mu \\ 0 & 0 & \delta^{ab}\delta^B_A q_\mu\sigma^\mu & 0 \end{pmatrix}, \qquad \text{(B.6)}$$

and the free field part

$$D_0 = \delta^{ab} \begin{pmatrix} \eta^{\mu\nu}\partial^2 & 0 & 0 & 0 \\ 0 & \delta^{IJ}\partial^2 & 0 & 0 \\ 0 & 0 & 0 & -i\delta^A_B\bar{\sigma}^\mu\partial_\mu \\ 0 & 0 & i\delta^B_A\sigma^\mu\partial_\mu & 0 \end{pmatrix}, \tag{B.7}$$

where we can write $D_0^{-1} = d_0 + \frac{d_1}{z} + \frac{d_2}{z^2} + \mathcal{O}(\frac{1}{z^3})$. Defining

$$\delta_0 = \begin{pmatrix} \eta_{\mu\nu} & 0 & 0 & 0 \\ 0 & \delta^{IJ} & 0 & 0 \\ 0 & 0 & 0 & -i\delta^B_A\bar{\sigma}^\mu\partial_\mu \\ 0 & 0 & i\delta^A_B\sigma^\mu\partial_\mu & 0 \end{pmatrix}, \quad \delta_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta^B_A\bar{\sigma}^\mu q_\mu \\ 0 & 0 & \delta^A_B\sigma^\mu q_\mu & 0 \end{pmatrix}, \quad \tag{B.8}$$

we have

$$D_0^{-1} = \delta^{ab}(\partial^2)^{-1}\delta_0, \qquad\qquad d_0 = \delta^{ab}\frac{i\delta_1}{2\partial^-},$$

$$d_1 = \delta^{ab}[\frac{\partial^2\delta_1}{4(\partial^-)^2} + \frac{i\delta_0}{2\partial^-}], \qquad d_2 = \delta^{ab}[-\frac{i(\partial^2)^2\delta_1}{8(\partial^-)^3} + \frac{\partial^2\delta_0}{4(\partial^-)^2}]. \tag{B.9}$$

It is very crucial to have $d_0 X = X d_0 = 0$, then the expansion

$$V(z)\big(1 + D_0^{-1}V(z)\big)^{-1} = (V + zX)\left(1 + (d_0 + \frac{d_1}{z} + \frac{d_2}{z^2} + \cdots)(V + zX)\right)^{-1}$$

$$= zX(1 + d_1 X)^{-1} + \mathcal{O}(z^0). \tag{B.10}$$

Now let us first consider $\langle g_1^-|g_n^+]$-shifting, and determine the leading order of $\mathcal{Z}^{\langle g_1^-|g_n^+]}(z)$. The helicity vectors of $g_{\hat{1}}^-, g_{\hat{n}}^+$ both introduce a factor of z, while $zX(1+d_1X)^{-1}$ introduce another factor of z. Notice that both d_1 and X are block-diagonal, which means fermion operators will not appear. It implies that, in this order, the $\mathcal{Z}^{\langle g_1^-|g_n^+]}(z)$ of $\mathcal{N} = 4$ SYM is the same as its bosonic sub-theory, which is a 4-dimensional reduction of 10-dimensional Yang-Mills theory. According to [56],

$$\mathcal{Z}^{\langle g_1^{-a}|g_n^{+b}]}(z) = -2iz^3 g f^{abc}(p_1 \cdot p_n)q_\mu A^{\mu c} + O(z^2). \tag{B.11}$$

The leading order is z^3. If the color indices of two shifted fields are contracted, then the first term vanishes due to $f^{abc} = 0$ when $a = b$, and the leading order becomes z^2, while we know in section 4.5 that the leading order of double trace term under such shifting is z^4.

In paper [65], it was proved that the large z behavior of $\langle \Phi^{U_1 a}|\Phi^{U_2 b}]$-shifting is

$$\mathcal{Z}^{\langle \Phi^{U_1 a}|\Phi^{U_2 b}]}(z) = O(z^{|U_1/U_n|-1}). \tag{B.12}$$

We would like to refine their result as

$$\mathcal{Z}^{\langle \Phi^{U_1 a}|\Phi^{U_2 b}]}(z) = z^{|U_1/U_n|-1} f^{abc}\mathcal{L}^c_{\langle \Phi^{U_1 a}|\Phi^{U_2 b}]} + O(z^{|U_1/U_n|-2}), \tag{B.13}$$

where $\mathcal{L}^c$ is an arbitrary operator, and there is always a f^{abc} associated with the leading order term. We already proved (B.13) for $\langle g^{-a}|g^{+b}]$-shifting. Since all states in $\mathcal{N} = 4$ SYM are related by SUSY, (B.13) also holds for any shifting, and the proof will be complete parallel to section 7.1 of [65]. This means that after contracting the indices a, b, the first term vanishes, and the large z behaves even better than expected.

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