Published for SISSA by Springer

Received: April 10, 2015
Accepted: June 1, 2015
Published: June 22, 2015

Superconformal field theory in three dimensions: correlation functions of conserved currents

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Abstract: For $\mathcal{N}$-extended superconformal field theories in three spacetime dimensions (3D), with $1 \leq \mathcal{N} \leq 3$, we compute the two- and three-point correlation functions of the supercurrent and the flavour current multiplets. We demonstrate that supersymmetry imposes additional restrictions on the correlators of conserved currents as compared with the non-supersymmetric case studied by Osborn and Petkou in hep-th/9307010. It is shown that the three-point function of the supercurrent is determined by a single functional form consistent with the conservation equation and all the symmetry properties. Similarly, the three-point function of the flavour current multiplets is also determined by a single functional form in the $\mathcal{N} = 1$ and $\mathcal{N} = 3$ cases. The specific feature of the $\mathcal{N} = 2$ case is that two independent structures are allowed for the three-point function of flavour current multiplets, but only one of them contributes to the three-point function of the conserved currents contained in these multiplets. Since the supergravity and super-Yang-Mills Ward identities are expected to relate the coefficients of the two- and three-point functions under consideration, the results obtained for 3D superconformal field theory are analogous to those in 2D conformal field theory.

In addition, we present a new supertwistor construction for compactified Minkowski superspace. It is suitable for developing superconformal field theory on 3D spacetimes other than Minkowski space, such as $S^1 \times S^2$ and its universal covering space $\mathbb{R} \times S^2$.

Keywords: Extended Supersymmetry, Superspaces

ArXiv ePrint: 1503.04961

[1]On leave from Tomsk Polytechnic University, 634050 Tomsk, Russia.

doi:10.1007/JHEP06(2015)138

JHEP06(2015)138

Contents

JHEP06(2015)138

1 Introduction

One of the well-known implications of conformal invariance in $d \geq 2$ dimensions [1–4] is that the functional form of the two- and three-point correlation functions of primary fields is fixed up to a finite number of parameters.[1] In general, however, it is a nontrivial technical exercise to determine explicitly the three-point functions of constrained tensor operators, examples of which are the energy momentum-tensor or conserved vector currents. The point is that such operators obey certain differential constraints and some work is required in order to classify those functional contributions to the given three-point function, which are consistent with all the constraints. Building on the theoretical ideas and results that may be traced back as early as the 1970s (see, e.g., [5–10] and references therein), Osborn and Petkou [11] presented the group-theoretic formalism to construct the three-point functions for primary fields of arbitrary spin in d dimensions.[2] They analysed in detail the restrictions on the correlation functions imposed by the conservation equations for the energy-momentum tensor and conserved currents, and the outcomes of their study include the following conclusions: the three-point function of the energy-momentum tensor contains three linearly independent functional forms for $d > 3$, while for $d = 3$ there are two and for $d = 2$ only one.[3] The three-point function of vector currents contains two linearly independent completely antisymmetric functional forms, which is in accord with the result obtained in 1971 by Schreier [2]. In the $d = 4$ case, an additional completely symmetric structure is allowed, which reflects the presence of anomalies [16]. In the same $d = 4$ case, the three parameters describing the three-point function of the energy-momentum tensor were demonstrated [11, 16] to be related to two coefficients in the trace anomaly of a conformal field theory in curved space.

When conformal symmetry is combined with supersymmetry, the story of the two- and three-point functions of conserved currents described in [11, 16] has to be supplemented

[1] In the Euclidean case, these results follow from the following well-known mathematical observation. For any three distinct points p_1, p_2 and p_3 on the d-sphere $S^d = \mathbb{R}^d \bigcup \{\infty\}$, there exists a conformal transformation $g \in \mathsf{SO}_0(d+1,1)$ that maps these points to $\vec{0}$, $(1, 0, \ldots, 0)$ and ∞, respectively. Here S^d is understood to be the conformal compactification of Euclidean space $\mathbb{R}^d$, i.e. the set of all null straight lines through the origin in $\mathbb{R}^{d+1,1}$. The above observation can be rephrased as the statement that there is no conformal invariant of three points.

[2] In four dimensions, the three-point function of the energy-momentum tensor was first derived by Stanev [12].

[3] In three dimensions, conformal invariance also allows parity violating structures for the three-point functions involving either the energy-momentum tensor, flavour currents or higher spin currents [13, 14] (see also [15]).

JHEP06(2015)138

with a sequel, for there appear conceptually new fermionic conserved currents. In the realm of supersymmetric field theories, the energy-momentum tensor is replaced with the supercurrent [17]. The latter is a supermultiplet containing the energy-momentum tensor and the supersymmetry current, along with some additional components such as the R-symmetry current. Thus the supercurrent contains fundamental information about the symmetries of a given supersymmetric field theory.

The supercurrent is the source of supergravity [18–20], in the same way as the energy-momentum tensor is the source of gravity. For every superconformal field theory, the supercurrent is an irreducible multiplet that may be coupled to the Weyl multiplet of conformal supergravity. For non-superconformal theories, the supercurrent is reducible and contains the so-called trace multiplet which includes the trace of the energy-momentum tensor. Different supersymmetric theories may possess different trace multiplets that correspond to different off-shell formulations for supergravity.

For completeness, it is pertinent to recall the structure of (non-)conformal supercurrents in four spacetime dimensions (4D). The $\mathcal{N}=1$ conformal supercurrent [17] is a real vector superfield $J_{\alpha\dot\beta}$ constrained by $\bar D^{\dot\beta} J_{\alpha\dot\beta}=0$ or, equivalently, $D^\alpha J_{\alpha\dot\beta}=0$. The simplest non-conformal supercurrent was given by Ferrara and Zumino [17]; the corresponding conservation equation is $\bar D^{\dot\beta} J_{\alpha\dot\beta} = D_\alpha T$, where the trace multiplet T is chiral (see [21] for a review). The Ferrara-Zumino supercurrent proves to be well defined on a dense set in the space of $\mathcal{N}=1$ supersymmetric field theories. For recent discussions of the most general 4D $\mathcal{N}=1$ non-conformal supercurrents, see [22–25]. The $\mathcal{N}=2$ conformal supercurrent [26, 27] is a real scalar superfield J constrained by $\bar D^i_{\dot\alpha}\bar D^{\dot\alpha j}J=0$ or, equivalently, $D^{\alpha i}D^j_\alpha J=0$, see [28] for more details. Numerous $\mathcal{N}=2$ supersymmetric theories are characterised by the following conservation equation [28, 29]: $\bar D^i_{\dot\alpha}\bar D^{\dot\alpha j}J = \mathrm{i}\,T^{ij} + g^{ij}Y$. Here the trace multiplets $T^{ij}=T^{ji}=\overline{T_{ij}}$ and Y are linear and reduced chiral superfields[4] respectively; $g^{ij}=g^{ji}=\overline{g_{ij}}$ is a constant iso-triplet that might be thought of as an expectation value of the tensor multiplet, one of the two supergravity compensators, see [29] for the details.

Superconformal symmetry imposes additional restrictions on the structure of three-point functions of conserved currents as compared with the non-supersymmetric case studied in [11, 16]. In 4D $\mathcal{N}=1$ superconformal theories, the three-point function of the supercurrent is the sum of two linearly independent functional structures [30] as compared with the three functional forms in the non-supersymmetric case [11]. The corresponding coefficients a and c were shown [30] to be related to those constituting the super-Weyl anomaly in curved superspace studied theoretically in [31] and computed explicitly in [32] (see [21] for a review).[5] The same conclusion holds for the three-point function of the supercurrent in 4D $\mathcal{N}=2$ superconformal field theories [28], while $\mathcal{N}=4$ superconformal symmetry is known to demand $a=c$. As concerns the three-point function of the flavour

[4]The linear superfield is constrained by $D^{(i}_\alpha T^{jk)} = \bar D^{(i}_{\dot\alpha}T^{jk)}=0$, while the reduced chiral superfield obeys the chirality constraint $\bar D^i_{\dot\alpha}Y=0$ and the reality condition $D^{\alpha i}D^j_\alpha Y = \bar D^i_{\dot\alpha}\bar D^{\dot\alpha j}\bar Y$.

[5]In his analysis [30], Osborn used the realisation of superconformal transformations in 4D $\mathcal{N}=1$ Minkowski superspace described in [21]. Earlier works on superconformal transformations in superspace include [33–36].

JHEP06(2015)138

current multiplets, there exist two independent structures in the 4D $\mathcal{N} = 1$ case [30] (as compared with three in the non-supersymmetric case [11, 16]), while $\mathcal{N} = 2$ superconformal symmetry allows only one [28].

The present paper is the first in a series devoted to the correlation functions of conserved currents in $\mathcal{N}$-extended superconformal field theories in three spacetime dimensions. We start with a brief discussion of 3D conformal supercurrents and flavour current multiplets. The 3D $\mathcal{N}$-extended conformal supercurrents have been described in [37] using the conformal superspace formulation for $\mathcal{N}$-extended conformal supergravity given in [38]. For every $\mathcal{N} = 1, 2\ldots$, the supercurrent is a primary real superfield of certain tensor type and dimension, which obeys some conservation equation formulated in terms of covariant derivatives. Denoting by D^I_α the spinor covariant derivatives of $\mathcal{N}$-extended Minkowski superspace $\mathbb{R}^{3|2\mathcal{N}}$, the conformal supercurrents[6] for $\mathcal{N} \leq 4$ are specified by the following properties:

SUSY Type	Supercurrent	Dimension	Conservation Equation
$\mathcal{N} = 1$	$J_{\alpha\beta\gamma}$	5/2	$D^\alpha J_{\alpha\beta\gamma} = 0$
$\mathcal{N} = 2$	$J_{\alpha\beta}$	2	$D^{I\alpha} J_{\alpha\beta} = 0$
$\mathcal{N} = 3$	J_α	3/2	$D^{I\alpha} J_\alpha = 0$
$\mathcal{N} = 4$	J	1	$\Big(D^{I\alpha} D^J_\alpha - \frac{1}{4}\delta^{IJ} D^{L\alpha} D^L_\alpha\Big) J = 0$

(1.1)

For $\mathcal{N} > 4$, the conformal supercurrent is a completely antisymmetric dimension-1 superfield, $J^{IJKL} = J^{[IJKL]}$, subject to the conservation equation

$$D^I_\alpha J^{JKLP} = D^{[I}_\alpha J^{JKLP]} - \frac{4}{\mathcal{N} - 3} D^Q_\alpha J^{Q[JKL}\delta^{P]I} \ . \tag{1.2}$$

The above results follow from the analyses carried out in [37, 38]. Given an $\mathcal{N}$-extended superconformal field theory, it may be coupled to the Weyl multiplet of $\mathcal{N}$-extended conformal supergravity. In curved superspace, the supercurrent J (with its indices suppressed) of the matter model with action $S_{\rm matter}$ is

$$J \propto \frac{\delta S_{\rm matter}}{\delta H} \, , \tag{1.3}$$

with H being an unconstrained prepotential for conformal supergravity. The latter has the following index structure: $H_{\alpha\beta\gamma}$ for $\mathcal{N} = 1$ [41], $H_{\alpha\beta}$ for $\mathcal{N} = 2$ [42, 43], H_α for $\mathcal{N} = 3$ and H for $\mathcal{N} = 4$ [37, 38].[7]

The 3D $\mathcal{N}$-extended flavour current multiplets constitute another family of primary real superfields obeying certain conservation equations. For $\mathcal{N} \leq 3$, they have the following

[6] The 3D $\mathcal{N} = 2$ supercurrents were studied in [39, 40].

[7] Using the harmonic superspace techniques [44], one may derive the $\mathcal{N} = 3$ and $\mathcal{N} = 4$ prepotentials by generalising the 4D $\mathcal{N} = 2$ analysis of [28].

structure:

SUSY Type	Flavour Current	Dimension	Conservation Equation
$\mathcal{N}=1$	L_α	$3/2$	$D^\alpha L_\alpha = 0$
$\mathcal{N}=2$	L	1	$\left(D^{\alpha I} D^J_\alpha - \frac{1}{2}\delta^{IJ} D^{K\alpha} D^K_\alpha\right) L = 0$
$\mathcal{N}=3$	L^I	1	$D^{(I}_\alpha L^{J)} - \frac{1}{3}\delta^{IJ} D^K_\alpha L^K = 0$

(1.4)

In the $\mathcal{N} = 4$ case, there are two inequivalent flavour current multiplets, L^{IJ}_+ and L^{IJ}_-. Each of them is described by a primary antisymmetric dimension-1 superfield, $L^{IJ} = -L^{JI}$, which obeys the conservation equation

$$D^I_\alpha L^{JK} = D^{[I}_\alpha L^{JK]} - \frac{2}{3} D^L_\alpha L^{L[J} \delta^{K]I} . \tag{1.5}$$

What differs between the two flavour current multiplets, L^{IJ}_+ and L^{IJ}_-, is that they are subject to different self-duality constraints

$$\frac{1}{2}\varepsilon^{IJKL} L^{KL}_\pm = \pm L^{IJ}_\pm . \tag{1.6}$$

The above results naturally follow from the known structure of unconstrained prepotentials for the $\mathcal{N}$-extended vector multiplets given in the following publications: [41, 45] for $\mathcal{N} = 1$, [42, 45, 46] for $\mathcal{N} = 2$, [47] for $\mathcal{N} = 3$ and [48, 49] for $\mathcal{N} = 4$.

The general group-theoretic formalism to construct the two- and three-point functions of primary superfields in 3D $\mathcal{N}$-extended Minkowski superspace was developed by Park [50], as a natural extension of earlier 4D [30, 51] and 6D [52] constructions. Instead, we will re-derive the two- and three-point superconformal building blocks, originally given in [50], by making use of the 3D $\mathcal{N}$-extended supertwistor construction of [53]. Such a re-formulation makes it possible to apply the formalism for computing correlation functions on more general (conformally flat) superspaces than the standard Minkowski superspace used in [50].

In this paper we study the correlation functions of conserved current multiplets in superconformal field theories with $\mathcal{N} \leq 3$, while the case $\mathcal{N} > 3$ will be considered elsewhere. The main outcomes of our study are as follows: for $\mathcal{N} \leq 3$, the three-point function of the supercurrent is determined by a single functional form consistent with the conservation equation and all the symmetry properties. The same conclusion holds for the three-point function of flavour current multiplets in the $\mathcal{N} = 1$ and $\mathcal{N} = 3$ cases. As concerns the $\mathcal{N} = 2$ case, two independent structures are allowed for the three-point function of flavour current multiplets, but only one of them contributes to the three-point function of the conserved currents contained in these multiplets. Thus the 3D superconformal story is analogous to that in 2D conformal field theory.

In 3D $\mathcal{N} = 2$ superconformal field theories, of special importance are contact terms of the supercurrent and conserved current multiplets [54, 55]. Such contributions to correlation functions are associated with certain Chern-Simons terms for background fields. In this paper, we will concentrate on studying the correlation functions at distinct points where the contact terms do not contribute.

Before we turn to the technical aspects of this paper, it is worth discussing one more conceptual issue: the symmetry structure of extended supersymmetric field theories from the point of view of "less extended" supersymmetry. Every $\mathcal{N}$-extended superconformal field theory is a special theory with $(\mathcal{N}-1)$-extended superconformal symmetry. It is worth elucidating the structure of $(\mathcal{N}-1)$-extended supermultiplets contained in the $\mathcal{N}$-extended supercurrent or flavour current multiplet. To uncover this, we split the Grassmann coordinates θ^α_I of $\mathcal{N}$-extended Minkowski superspace $\mathbb{M}^{3|2\mathcal{N}}$ onto two subsets: (i) the coordinated $\theta^\alpha_{\hat{I}}$, with $\hat{I}=1,\dots,\mathcal{N}-1$, corresponding to $(\mathcal{N}-1)$-extended Minkowski superspace $\mathbb{M}^{3|2(\mathcal{N}-1)}$; and (ii) two additional coordinates $\theta^\alpha_{\mathcal{N}}$. The corresponding splitting of the spinor derivatives D^I_α is $D^{\hat{I}}_\alpha$ and $D^{\mathcal{N}}_\alpha$. Given a superfield V on $\mathbb{M}^{3|2\mathcal{N}}$, its bar-projection onto $\mathbb{M}^{3|2(\mathcal{N}-1)}$ is defined by $V| := V|_{\theta_{\mathcal{N}}=0}$.

Consider the $\mathcal{N}=2$ case. The spinor covariant derivatives $D^{\hat{I}}_\alpha$ and $D^{\mathcal{N}}_\alpha$ introduced above, now become D_α and D^2_α respectively. The supercurrent $J_{\alpha\beta}$ leads to the following $\mathcal{N}=1$ supermultiplets:

$$S_{\alpha\beta} := J_{\alpha\beta}|\,, \qquad D^\alpha S_{\alpha\beta}=0\;; \tag{1.7a}$$
$$J_{\alpha\beta\gamma} := \mathrm{i}D^2_{(\alpha}J_{\beta\gamma)}|\;; \qquad D^\alpha J_{\alpha\beta\gamma}=0\;. \tag{1.7b}$$

Here $J_{\alpha\beta\gamma}$ is the $\mathcal{N}=1$ supercurrent, while the additional superfield $S_{\alpha\beta}$ contains the $\mathsf{U}(1)$ R-symmetry current (the θ-independent component of $S_{\alpha\beta}$) and the second supersymmetry current (the top component of $S_{\alpha\beta}$).

The $\mathcal{N}=3$ supercurrent J_α leads to the following $\mathcal{N}=2$ supermultiplets:

$$R_\alpha := J_\alpha|\,, \qquad D^{\hat{I}\alpha}R_\alpha=0\;; \tag{1.8a}$$
$$J_{\alpha\beta} := D^3_{(\alpha}J_{\beta)}|\,, \qquad D^{\hat{I}\alpha}J_{\alpha\beta}=0\;. \tag{1.8b}$$

Here $J_{\alpha\beta}$ is the $\mathcal{N}=2$ supercurrent, while R_α contains the third supersymmetry current and two R-symmetry currents corresponding to $\mathsf{SO}(3)/\mathsf{SO}(2)$.

Next, the $\mathcal{N}=4$ supercurrent J contains the following $\mathcal{N}=3$ supermultiplets:

$$S := J|\,, \qquad (D^{\hat{I}\alpha}D^{\hat{J}}_\alpha - \frac{1}{3}\delta^{\hat{I}\hat{J}}D^{\hat{K}\alpha}D^{\hat{K}}_\alpha)S=0\;; \tag{1.9a}$$
$$J_\alpha := \mathrm{i}D^4_\alpha J|\,, \qquad D^{\hat{I}\alpha}J_\alpha=0\;. \tag{1.9b}$$

Here J_α is the $\mathcal{N}=3$ supercurrent, while S contains the fourth supersymmetry current and three R-symmetry currents corresponding to $\mathsf{SO}(4)/\mathsf{SO}(3)$. Upon reduction to $\mathcal{N}=2$ superspace, the scalar S generates two primary $\mathcal{N}=2$ superfields: (i) the scalar $S|_{\theta_2=0}$, which is an $\mathcal{N}=2$ flavour current multiplet; and (ii) the spinor $D^3_\alpha S|_{\theta_2=0}$, which is of the type (1.8a).

Finally, we just mention the $\mathcal{N}\to(\mathcal{N}-1)$ decomposition of flavour current multiplets. The $\mathcal{N}=2$ multiplet L leads to the following $\mathcal{N}=1$ real supermultiplets:

$$S := L|\;; \tag{1.10a}$$
$$L_\alpha := \mathrm{i}D^2_\alpha L|\;; \qquad D^\alpha L_\alpha=0\;. \tag{1.10b}$$

Here L_α is an $\mathcal{N}=1$ flavour current multiplet, and the real scalar S is unconstrained. The $\mathcal{N}=3$ multiplet L^I leads to an $\mathcal{N}=2$ flavour current multiplet L and a chiral scalar.

Therefore, if one studies $\mathcal{N}$-extended superconformal field theories in $(\mathcal{N}-1)$-extended superspace, it is not sufficient to analyse the correlation functions of those currents which correspond to the manifestly realised symmetries.

There is a remarkable difference between superconformal field theories and ordinary conformal ones in diverse dimensions. For the action of the conformal group on (compactified) Minkowski space, there is no conformal invariant of three points. The situation is different in superspace. On (compactified) Minkowski superspace, the superconformal group does not act transitively on the set consisting of triples of distinct superspace points. As a result, there exist *nilpotent* superconformal invariants of three points. Such invariants have been constructed by Park in diverse dimensions [50–52, 56].

This paper is organised as follows. Following [53], in section 2 we review the supertwistor construction of $\mathcal{N}$-extended compactified Minkowski superspace $\overline{\mathbb{M}}^{3|2\mathcal{N}}$. Minkowski superspace $\mathbb{M}^{3|2\mathcal{N}}$ originates as a dense open subset of $\overline{\mathbb{M}}^{3|2\mathcal{N}}$. It is shown that $\overline{\mathbb{M}}^{3|2\mathcal{N}}$ is a homogeneous space for the superconformal group $\mathsf{OSp}(\mathcal{N}|4,\mathbb{R})$, while only the infinitesimal superconformal transformations are well defined on $\mathbb{M}^{3|2\mathcal{N}}$. Section 3 describes a different isomorphic realisation for $\mathsf{OSp}(\mathcal{N}|4,\mathbb{R})$. Using this realisation, we construct a global supermatrix parametrisation of $\overline{\mathbb{M}}^{3|2\mathcal{N}}$ as well as a smooth metric on $\overline{\mathbb{M}}^{3|2\mathcal{N}}$, which only scales under the superconformal transformations. The supertwistor formalism is used in section 4 to derive all building blocks in terms of which the two- and three-point functions of primary superfields are constructed. The general structure of the two- and three-point functions of primary superfields is described in section 5 following [50]. The two- and three-point functions for the supercurrent and flavour current multiplets in superconformal field theories with $\mathcal{N}=1$, $\mathcal{N}=2$ and $\mathcal{N}=3$ are computed in sections 6, 7 and 8 respectively. Concluding comments are given in section 9.

We have also included several technical appendices. Appendix A gives a summary of our 3D notation and conventions. Appendix B is devoted to the correlation functions involving (anti)chiral superfields. Appendix C is concerned with the $\mathcal{N}=2\rightarrow\mathcal{N}=1$ superspace reduction of the three-point functions for the $\mathcal{N}=2$ supercurrent and flavour current multiplets. In appendix D we reduce to components the three-point function for $\mathcal{N}=1$ flavour current multiplets.

2 Supertwistor construction

In this section we describe the supertwistor construction of $\mathcal{N}$-extended compactified Minkowski superspace. Our presentation mostly follows the construction given in [53] and inspired by [57] (see also [58]).

2.1 Supertwistors and the superconformal group

In three spacetime dimensions, the $\mathcal{N}$-extended superconformal group[8] is $\mathsf{OSp}(\mathcal{N}|4;\mathbb{R})$. It naturally acts on the space of *real even* supertwistors and on the space of *real odd* supertwistors.

[8]This supergroup was denoted $\mathsf{OSp}(\mathcal{N}|2,\mathbb{R})$ in [53].

JHEP06(2015)138

An arbitrary supertwistor is a column vector

$$T = (T_A) = \left(\frac{T_{\hat{\alpha}}}{T_I}\right) , \qquad (T_{\hat{\alpha}}) = \begin{pmatrix} f_\alpha \\ g^\beta \end{pmatrix} , \qquad \alpha, \beta = 1, 2 , \quad I = 1, \dots, \mathcal{N} . \tag{2.1}$$

In the case of even supertwistors, $T_{\hat{\alpha}}$ is bosonic and T_I is fermionic. In the case of odd supertwistors, $T_{\hat{\alpha}}$ is fermionic while T_I is bosonic. The even and odd supertwistors are called pure. We introduce the parity function $\varepsilon(T)$ defined as: $\varepsilon(T) = 0$ if T is even, and $\varepsilon(T) = 1$ if T is odd. It is also useful to define

$$\varepsilon_A = \begin{cases} 0 & A = \hat{\alpha} \\ 1 & A = I \end{cases} .$$

Then the components T_A of a pure supertwistor have the following Grassmann parities

$$\varepsilon(T_A) = \varepsilon(T) + \varepsilon_A \pmod 2 . \tag{2.2}$$

A pure supertwistor is said to be real if its components obey the reality condition

$$\overline{T_A} = (-1)^{\varepsilon(T)\varepsilon_A + \varepsilon_A} T_A . \tag{2.3}$$

The space of complex (real) even supertwistors is naturally identified with $\mathbb{C}^{4|\mathcal{N}}$ ($\mathbb{R}^{4|\mathcal{N}}$), while the space of complex (real) odd supertwistors may be identified with $\mathbb{C}^{\mathcal{N}|4}$ ($\mathbb{R}^{\mathcal{N}|4}$).

Introduce a graded antisymmetric supermatrix

$$\mathbb{J} = (\mathbb{J}^{AB}) = \left(\begin{array}{c||c} J & 0 \\ \hline 0 & \mathrm{i}\, \mathbb{1}_{\mathcal{N}} \end{array}\right) , \qquad J = \left(J^{\hat{\alpha}\hat{\beta}}\right) = \begin{pmatrix} 0 & \mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{pmatrix} , \tag{2.4}$$

where $\mathbb{1}_{\mathcal{N}}$ denotes the unit $\mathcal{N} \times \mathcal{N}$ matrix. With the aid of $\mathbb{J}$, we may define a graded symplectic inner product on the space of pure supertwistors by the rule: for arbitrary pure supertwistors T and S, the inner product is

$$\langle T|S\rangle_{\mathbb{J}} := T^{\mathrm{sT}} \mathbb{J}\, S , \tag{2.5}$$

where the row vector T^{sT} is defined by

$$T^{\mathrm{sT}} := \left(T_{\hat{\alpha}}, -(-1)^{\varepsilon(T)} T_I\right) = \left(T_A (-1)^{\varepsilon(T)\varepsilon_A + \varepsilon_A}\right) \tag{2.6}$$

and is called the super-transpose of T. The above inner product has the following symmetry property

$$\langle T_1|T_2\rangle_{\mathbb{J}} = -(-1)^{\varepsilon_1\varepsilon_2} \langle T_2|T_1\rangle_{\mathbb{J}} , \tag{2.7}$$

where ε_i stands for the Grassmann parity of T_i. If T and S are real supertwistors, then applying the complex conjugation gives

$$\overline{\langle T|S\rangle_{\mathbb{J}}} = -\langle S|T\rangle_{\mathbb{J}} . \tag{2.8}$$

JHEP06(2015)138

By definition, the supergroup $\mathsf{OSp}(\mathcal{N}|4;\mathbb{C})$ consists of those even $(4|\mathcal{N})\times(4|\mathcal{N})$ supermatrices

$$g = (g_A{}^B)\,, \qquad \varepsilon(g_A{}^B) = \varepsilon_A + \varepsilon_B\,, \tag{2.9}$$

which preserve the inner product (2.5) under the action

$$T = (T_A) \;\to\; gT = (g_A{}^B T_B)\;. \tag{2.10}$$

Such a transformation maps the space of even (odd) supertwistors onto itself. The condition of invariance of the inner product (2.5) under (2.10) is

$$g^{\rm sT}\,\mathbb{J}\,g = \mathbb{J}\,, \qquad (g^{\rm sT})^A{}_B := (-1)^{\varepsilon_A\varepsilon_B+\varepsilon_B} g_B{}^A\;. \tag{2.11}$$

The subgroup $\mathsf{OSp}(\mathcal{N}|4;\mathbb{R})\subset\mathsf{OSp}(\mathcal{N}|4;\mathbb{C})$ consists of those transformations which preserve the reality condition (2.3),

$$\overline{T_A} = (-1)^{\varepsilon(T)\varepsilon_A+\varepsilon_A}T_A \quad\longrightarrow\quad \overline{(gT)_A} = (-1)^{\varepsilon(T)\varepsilon_A+\varepsilon_A}(gT)_A\;. \tag{2.12a}$$

This is equivalent to

$$\overline{g_A{}^B} = (-1)^{\varepsilon_A\varepsilon_B+\varepsilon_A}g_A{}^B \quad\Longleftrightarrow\quad g^\dagger = g^{\rm sT}\;. \tag{2.12b}$$

In conjunction with (2.11), this reality condition is equivalent to

$$g^\dagger\mathbb{J}\,g = \mathbb{J}\;. \tag{2.12c}$$

A dual supertwistor

$$Z = (Z^A) = \left(Z^{\hat\alpha}, Z^I\right) \tag{2.13}$$

is a row vector that transforms under $\mathsf{OSp}(\mathcal{N}|4;\mathbb{R})$ such that Z^AT_A is invariant for every supertwistor T,

$$Z \;\to\; Z' = Zg^{-1}\,, \qquad g\in\mathsf{OSp}(\mathcal{N}|4;\mathbb{R})\;. \tag{2.14}$$

A dual supertwistor Z is even (odd) if Z^AT_A is a c-number for every even (odd) supertwistor T. Given a pure dual supertwistor Z, its super-transpose $Z^{\rm sT}$ will be defined to be the following column vector

$$(Z^{\rm sT})^A := (-1)^{\varepsilon(Z)\varepsilon_A+\varepsilon(Z)}Z^A\,, \tag{2.15}$$

such that $Z^AT_A = (T^{\rm sT})_A(Z^{\rm sT})^A$.

The superconformal algebra $\mathfrak{osp}(\mathcal{N}|4;\mathbb{R})$ consists of real supermatrices Ω obeying the master equation

$$\Omega^{\rm sT}\mathbb{J} + \mathbb{J}\,\Omega = 0\;. \tag{2.16}$$

JHEP06(2015)138

The general solution of this equation is

$$
\begin{aligned}
\Omega &= \left(\begin{array}{c|c||c}
\lambda-\frac{1}{2}\sigma\mathbb{1}_2 & \check{b} & \sqrt{2}\check{\eta}^{\mathrm{T}} \\
\hline
-\hat{a} & -\lambda^{\mathrm{T}}+\frac{1}{2}\sigma\mathbb{1}_2 & -\sqrt{2}\hat{\epsilon}^{\mathrm{T}} \\
\hline\hline
\mathrm{i}\sqrt{2}\,\hat{\epsilon} & \mathrm{i}\sqrt{2}\,\check{\eta} & \Lambda
\end{array}\right) \\
&\equiv \left(\begin{array}{c|c||c}
\lambda_\alpha{}^\beta-\frac{1}{2}\sigma\delta_\alpha{}^\beta & b_{\alpha\beta} & \sqrt{2}\eta_{\alpha J} \\
\hline
-a^{\alpha\beta} & -\lambda^\alpha{}_\beta+\frac{1}{2}\sigma\delta^\alpha{}_\beta & -\sqrt{2}\epsilon^\alpha{}_J \\
\hline\hline
\mathrm{i}\sqrt{2}\,\epsilon_I{}^\beta & \mathrm{i}\sqrt{2}\,\eta_{I\beta} & \Lambda_{IJ}
\end{array}\right), \\
\lambda_\alpha{}^\alpha &= 0\,, \qquad a^{\alpha\beta}=a^{\beta\alpha}\,, \qquad b_{\alpha\beta}=b_{\beta\alpha}\,, \qquad \Lambda_{IJ}=-\Lambda_{JI}\;.
\end{aligned}
\tag{2.17}
$$

The bosonic parameters $\lambda_\alpha{}^\beta$, σ, $a_{\alpha\beta}$, $b^{\alpha\beta}$ and Λ_{IJ}, as well as the fermionic parameters $\epsilon^\alpha{}_I \equiv \epsilon_I{}^\alpha$ and $\eta_{\alpha I} \equiv \eta_{I\alpha}$ in (2.17) are real.

2.2 Compactified Minkowski superspace

In accordance with [53], the compactified $\mathcal{N}$-extended Minkowski superspace $\overline{\mathbb{M}}^{3|2\mathcal{N}}$ is defined to be the set of all Lagrangian subspaces of $\mathbb{R}^{4|\mathcal{N}}$, the space of real even supertwistors. We recall that a Lagrangian subspace of $\mathbb{R}^{4|\mathcal{N}}$ is a maximal isotropic subspace of $\mathbb{R}^{4|\mathcal{N}}$. By definition, such a subspace is spanned by two even supertwistors T^μ with the properties that (i) the bodies of T^1 and T^2 are linearly independent; (ii) they obey the null condition

$$
\langle T^1|T^2\rangle_{\mathbb{J}} = 0\;;
\tag{2.18}
$$

(iii) they are defined only modulo the equivalence relation

$$
\{T^\mu\} \;\sim\; \{\tilde{T}^\mu\}\,, \qquad \tilde{T}^\mu = T^\nu\,\Xi_\nu{}^\mu\,, \qquad \Xi\in\mathsf{GL}(2,\mathbb{R})\;.
\tag{2.19}
$$

Equivalently, the space $\overline{\mathbb{M}}^{3|2\mathcal{N}}$ can be defined to consist of rank-two supermatrices of the form

$$
\mathcal{P} = \big(T^1\;T^2\big) = \left(\begin{array}{c} F \\ G \\ \hline\hline \mathrm{i}\Upsilon \end{array}\right), \qquad G^{\mathrm{T}}F = F^{\mathrm{T}}G + \mathrm{i}\Upsilon^{\mathrm{T}}\,\Upsilon\,,
\tag{2.20}
$$

which are defined modulo the equivalence relation

$$
\mathcal{P} = \left(\begin{array}{c} F \\ G \\ \hline\hline \mathrm{i}\Upsilon \end{array}\right) \;\sim\; \left(\begin{array}{c} F\,\Xi \\ G\,\Xi \\ \hline\hline \mathrm{i}\Upsilon\,\Xi \end{array}\right) = \mathcal{P}\Xi\,, \qquad \Xi\in\mathsf{GL}(2,\mathbb{R})\;.
\tag{2.21}
$$

Here F and G are 2×2 real bosonic matrices, and Υ is a $\mathcal{N}\times2$ real fermionic matrix. The null condition (2.18) can be rewritten as

$$
\mathcal{P}^{\mathrm{sT}}\mathbb{J}\mathcal{P} = 0\;.
\tag{2.22}
$$

It may be shown that the superconformal group $\mathsf{OSp}(\mathcal{N}|4;\mathbb{R})$ acts transitively on the compactified Minkowski superspace. Thus $\overline{\mathbb{M}}^{3|2\mathcal{N}}$ can be identified with the coset space $\mathsf{OSp}(\mathcal{N}|4;\mathbb{R})/G_{\mathcal{P}_0}$, where $G_{\mathcal{P}_0}$ denotes the isotropy group at a given two-plane $\mathcal{P}_0\in\overline{\mathbb{M}}^{3|2\mathcal{N}}$.

JHEP06(2015)138

2.3 Minkowski superspace

As discussed in [53], Minkowski superspace $\mathbb{M}^{3|2\mathcal{N}}$ is identified with a dense open subset U_F of $\overline{\mathbb{M}}^{3|2\mathcal{N}}$ spanned by supermatrices (2.20) under the condition

$$\det F \neq 0\ . \tag{2.23}$$

Every null two-plane in $\mathbb{M}^{3|2\mathcal{N}}$ may be described by a supermatrix of the form

$$\mathcal{P} \sim \begin{pmatrix} \mathbb{1}_2 \\ -\hat{\boldsymbol{x}} \\ \hline\hline \mathrm{i}\sqrt{2}\hat{\theta} \end{pmatrix} = \begin{pmatrix} \delta_\alpha{}^\beta \\ -\boldsymbol{x}^{\alpha\beta} \\ \hline\hline \mathrm{i}\sqrt{2}\theta_I{}^\beta \end{pmatrix} \equiv \mathcal{P}(z)\,, \tag{2.24}$$

where the real matrix $\hat{\boldsymbol{x}}$ is constrained by

$$\hat{\boldsymbol{x}} - \hat{\boldsymbol{x}}^{\mathrm{T}} = 2\mathrm{i}\hat{\theta}^{\mathrm{T}}\hat{\theta} \quad \Longrightarrow \quad \boldsymbol{x}^{\alpha\beta} = x^{\alpha\beta} + \mathrm{i}\theta_I{}^\alpha\theta_I{}^\beta\,, \quad x^{\alpha\beta} = x^{\beta\alpha}\ . \tag{2.25}$$

The points of $\mathbb{M}^{3|2\mathcal{N}}$ are naturally parametrised by the variables $z^A = (x^a, \theta^\alpha_I)$.

Given a group element $g \in \mathsf{OSp}(\mathcal{N}|4;\mathbb{R})$, its action $\mathcal{P} \to g\mathcal{P}$ on the two-plane $\mathcal{P}(z) \in \mathbb{M}^{3|2\mathcal{N}}$ can be represented as

$$g \begin{pmatrix} \mathbb{1}_2 \\ -\hat{\boldsymbol{x}} \\ \hline\hline \mathrm{i}\sqrt{2}\,\hat{\theta} \end{pmatrix} = \begin{pmatrix} \mathbb{1}_2 \\ -\hat{\boldsymbol{x}}' \\ \hline\hline \mathrm{i}\sqrt{2}\,\hat{\theta}' \end{pmatrix} \Xi(g;z)\,, \qquad \Xi(g;z) \in \mathsf{GL}(2,\mathbb{R}) \tag{2.26}$$

provided the transformed two-plane, $g\mathcal{P}(z)$, belongs to $\mathbb{M}^{3|2\mathcal{N}}$. In general, this property holds only locally, since $\Xi(g;z)$ may become singular for certain group elements g (special conformal transformations) and some spacetime points x.

Let us consider an infinitesimal superconformal transformation, $g = \mathbb{1} + \kappa\Omega$, where κ is an infinitesimal parameter and Ω is given by (2.17). Then from (2.26) we derive

$$\delta\hat{\boldsymbol{x}} = \hat{a} - \lambda^{\mathrm{T}}\hat{\boldsymbol{x}} - \hat{\boldsymbol{x}}\lambda + \sigma\hat{\boldsymbol{x}} + \hat{\boldsymbol{x}}\check{b}\,\hat{\boldsymbol{x}} + 2\mathrm{i}\,\hat{\epsilon}^{\mathrm{T}}\hat{\theta} - 2\mathrm{i}\,\hat{\boldsymbol{x}}\check{\eta}^{\mathrm{T}}\hat{\theta}\,, \tag{2.27a}$$

$$\delta\hat{\theta} = \hat{\epsilon} - \hat{\theta}\lambda + \frac{1}{2}\sigma\hat{\theta} + \Lambda\hat{\theta} + \hat{\theta}\check{b}\hat{\boldsymbol{x}} - \check{\eta}\hat{\boldsymbol{x}} - 2\mathrm{i}\,\hat{\theta}\,\check{\eta}^{\mathrm{T}}\hat{\theta}\ . \tag{2.27b}$$

We see that the matrix elements in (2.17) correspond to a Lorentz transformation ($\lambda_\alpha{}^\beta$), spacetime translation ($a^{\alpha\beta}$), special conformal transformation ($b_{\alpha\beta}$), dilatation (σ), Q-supersymmetry ($\epsilon_I{}^\beta$), S-supersymmetry ($\eta_{I\beta}$) and R-symmetry transformation (Λ_{IJ}).

As pointed out in the previous subsection, $\overline{\mathbb{M}}^{3|2\mathcal{N}}$ can be identified with the homogeneous space $\mathsf{OSp}(\mathcal{N}|4;\mathbb{R})/G_{\mathcal{P}_0}$, where $G_{\mathcal{P}_0}$ denotes the isotropy group at a given two-plane $\mathcal{P}_0 \in \overline{\mathbb{M}}^{3|2\mathcal{N}}$. Consider a special null two-plane $\mathcal{P}_0 \in \overline{\mathbb{M}}^{3|2\mathcal{N}}$ which corresponds to the origin of $\mathbb{M}^{3|2\mathcal{N}}$, that is $\mathcal{P}_0 = \mathcal{P}(z=0)$. Its isotropy group $G_{\mathcal{P}_0}$ is the subgroup of $\mathsf{OSp}(\mathcal{N}|4;\mathbb{R})$ generated by supermatrices (2.17) of the form:

$$\omega = \left(\begin{array}{c|c||c} \lambda - \frac{1}{2}\sigma\mathbb{1}_2 & \check{b} & \sqrt{2}\check{\eta}^{\mathrm{T}} \\ \hline 0 & -\lambda^{\mathrm{T}} + \frac{1}{2}\sigma\mathbb{1}_2 & 0 \\ \hline\hline 0 & \mathrm{i}\sqrt{2}\,\check{\eta} & \Lambda \end{array}\right)\ . \tag{2.28}$$

The isotropy group $G_{\mathcal{P}_0}$ consists of the followings supermatrices

$$\left(\begin{array}{c|c||c} A & A\check{\boldsymbol{b}} & \sqrt{2}A\check{\eta}^{\rm T} \\ \hline 0 & (A^{-1})^{\rm T} & 0 \\ \hline\hline 0 & {\rm i}\sqrt{2}\,R\check{\eta} & R \end{array}\right), \qquad A \in \mathsf{GL}(2,\mathbb{R}), \quad R \in \mathsf{O}(\mathcal{N}), \tag{2.29a}$$

where the 2×2 matrix $\check{\boldsymbol{b}}$ is constrained by

$$\check{\boldsymbol{b}} - \check{\boldsymbol{b}}^{\rm T} = 2{\rm i}\check{\eta}^{\rm T}\check{\eta} \quad \Longrightarrow \quad \boldsymbol{b}_{\alpha\beta} = b_{\alpha\beta} + {\rm i}\eta_{I\alpha}\eta_{I\beta}, \quad b_{\alpha\beta} = b_{\beta\alpha}. \tag{2.29b}$$

As follows from (2.29), $G_{\mathcal{P}_0}$ includes space reflections. Choosing $i = 1$ or $i = 3$, let us consider the following element of $G_{\mathcal{P}_0}$:

$$g_i = \left(\begin{array}{c|c||c} \sigma_i & 0 & 0 \\ \hline 0 & \sigma_i & 0 \\ \hline\hline 0 & 0 & \mathbb{1}_{\mathcal{N}} \end{array}\right), \tag{2.30}$$

with σ_i being the i-th Pauli matrix, $\det \sigma_i = -1$. Associated with this group element is the transformation on $\mathbb{M}^{3|2\mathcal{N}}$

$$P_i: \qquad \hat{x}' = \sigma_i \hat{x} \sigma_i, \qquad \hat{\theta}' = \hat{\theta}\sigma_i, \tag{2.31}$$

which is a reflection about one of the coordinate axes in two-space.

It is also seen from (2.29) that $G_{\mathcal{P}_0}$ includes arbitrary R-symmetry transformations from the group $\mathsf{O}(\mathcal{N})$ and not necessarily from its connected component of the identity, $\mathsf{SO}(\mathcal{N})$, as discussed by [50].

A complement of the subalgebra (2.28) in $\mathfrak{osp}(\mathcal{N}|4;\mathbb{R})$ generates a subgroup of the superconformal group consisting of all supermatrices of the form:

$$s(a,\epsilon) = \left(\begin{array}{c|c||c} \mathbb{1}_2 & 0 & 0 \\ \hline -\hat{\boldsymbol{a}} & \mathbb{1}_2 & -\sqrt{2}\hat{\epsilon}^{\rm T} \\ \hline\hline {\rm i}\sqrt{2}\,\hat{\epsilon} & 0 & \mathbb{1}_{\mathcal{N}} \end{array}\right), \qquad \hat{\boldsymbol{a}} = \hat{a} + {\rm i}\hat{\epsilon}^{\rm T}\hat{\epsilon}. \tag{2.32}$$

Such a supermatrix describes a spacetime translation ($\epsilon = 0$) and a Q-supersymmetry transformation ($a = 0$) when acting on $\mathbb{M}^{3|2\mathcal{N}}$.

The $\mathcal{N}$-extended Minkowski superspace can be also realised as a homogeneous space. The standard realisation is

$$\mathbb{M}^{3|2\mathcal{N}} = \mathfrak{P}(3|\mathcal{N})/\mathsf{SL}(2,\mathbb{R}), \tag{2.33}$$

where $\mathfrak{P}(3|\mathcal{N})$ denotes the $\mathcal{N}$-extended super-Poincaré group and $\mathsf{SL}(2,\mathbb{R})$ the spin group in three spacetime dimensions. Every group element $g \in \mathfrak{P}(3|\mathcal{N})$ can uniquely be represented in the form $g = s(a,\epsilon)\,h(M)$, where

$$h(M) = \left(\begin{array}{c|c||c} M & 0 & 0 \\ \hline 0 & (M^{-1})^{\rm T} & 0 \\ \hline\hline 0 & 0 & \mathbb{1}_{\mathcal{N}} \end{array}\right), \qquad M \in \mathsf{SL}(2,\mathbb{R}). \tag{2.34}$$

Here the supermatrix $h(M)$ describes a Lorentz transformation. The points of $\mathbb{M}^{3|2\mathcal{N}}$ can be parametrised by the following coset representative

$$s(z) := s(x,\theta) = \left(\begin{array}{c|c||c} \mathbb{1}_2 & 0 & 0 \\ \hline -\hat{\boldsymbol{x}} & \mathbb{1}_2 & -\sqrt{2}\hat{\theta}^{\mathrm{T}} \\ \hline\hline \mathrm{i}\sqrt{2}\,\hat{\theta} & 0 & \mathbb{1}_{\mathcal{N}} \end{array}\right) . \tag{2.35}$$

The Q-supersymmetry transformation $s(0,\epsilon)$ acts on $\mathbb{M}^{3|2\mathcal{N}}$ according to the law $s(z) \to s(z') = s(0,\epsilon)s(z)$, and thus

$$x'^{\alpha\beta} = x^{\alpha\beta} + \mathrm{i}(\epsilon_I^\alpha \theta_I^\beta + \epsilon_I^\beta \theta_I^\alpha)\,, \qquad \theta'^\alpha_I = \theta^\alpha_I + \epsilon^\alpha_I\ . \tag{2.36}$$

These results can be rewritten as

$$z'^A = z^A - \mathrm{i}\,\epsilon^\beta_J Q^J_\beta\, z^A\,, \tag{2.37}$$

where we have introduced the supersymmetry generators

$$Q^I_\alpha = \mathrm{i}\,\frac{\partial}{\partial\theta^\alpha_I} + (\gamma^m)_{\alpha\beta}\,\theta^\beta_I\partial_m = \mathrm{i}\,\frac{\partial}{\partial\theta^\alpha_I} + \theta^\beta_I\partial_{\beta\alpha}\ . \tag{2.38}$$

From here we immediately read off the spinor covariant derivatives

$$D^I_\alpha = \frac{\partial}{\partial\theta^\alpha_I} + \mathrm{i}(\gamma^m)_{\alpha\beta}\,\theta^\beta_I\partial_m = \frac{\partial}{\partial\theta^\alpha_I} + \mathrm{i}\theta^\beta_I\partial_{\beta\alpha}\,, \tag{2.39a}$$

which anti-commute with the supercharges, $\{D^I_\alpha, Q^J_\beta\} = 0$, and obey the anti-commutation relations

$$\big\{D^I_\alpha, D^J_\beta\big\} = 2\mathrm{i}\,\delta^{IJ}(\gamma^m)_{\alpha\beta}\,\partial_m\ . \tag{2.39b}$$

We introduce the 3D extension of the Volkov-Akulov supersymmetric one-form [64–66]

$$\hat{e} = \mathrm{d}\hat{\boldsymbol{x}} - 2\mathrm{i}\hat{\theta}^{\mathrm{T}}\mathrm{d}\hat{\theta} = \mathrm{d}\hat{x} + \mathrm{i}\mathrm{d}\hat{\theta}^{\mathrm{T}}\hat{\theta} - \mathrm{i}\hat{\theta}^{\mathrm{T}}\mathrm{d}\hat{\theta}\,, \qquad \hat{e}^{\mathrm{T}} = \hat{e}\ . \tag{2.40}$$

It is obviously invariant under the Q-supersymmetry transformation (2.36).

2.4 Twin Minkowski superspace

The chart U_F, which we have identified with Minkowski superspace, does not cover $\overline{\mathbb{M}}^{3|2\mathcal{N}}$. Another dense open subset U_G of $\overline{\mathbb{M}}^{3|2\mathcal{N}}$ consists of those supermatrices (2.20) which are characterised by

$$\det G \neq 0\ . \tag{2.41}$$

Every null two-plane in U_G may be described by a supermatrix of the form

$$\mathcal{P} \sim \left(\begin{array}{c} \check{\boldsymbol{y}} \\ \mathbb{1}_2 \\ \hline\hline \mathrm{i}\sqrt{2}\check{\rho} \end{array}\right) = \left(\begin{array}{c} \boldsymbol{y}_{\alpha\beta} \\ \delta^\alpha{}_\beta \\ \hline\hline \mathrm{i}\sqrt{2}\rho_{I\beta} \end{array}\right), \tag{2.42}$$

where the real 2×2 matrix $\check{\boldsymbol{y}}$ is constrained by

$$\check{\boldsymbol{y}} - \check{\boldsymbol{y}}^{\mathrm{T}} = 2\mathrm{i}\check{\rho}^{\mathrm{T}}\check{\rho} \quad \Longleftrightarrow \quad \boldsymbol{y}_{\alpha\beta} = y_{\alpha\beta} + \mathrm{i}\rho_{I\alpha}\rho_{I\beta}\,, \quad y_{\alpha\beta} = y_{\beta\alpha}\,. \tag{2.43}$$

One may think of U_G as a twin of U_F obtained by replacing the spacetime translations and Q-supersymmetry transformations with the special conformal boosts and S-supersymmetry transformations, respectively. The two-plane $\mathcal{P}_0$, which is the origin of U_F, is replaced with $\mathcal{P}_\infty$ corresponding to $y_{\alpha\beta} = 0$ and $\rho_{I\alpha} = 0$, the origin of U_G. The two-plane $\mathcal{P}_\infty$ is an infinitely separated point from the viewpoint of U_F. The isotropy group $\mathcal{P}_\infty$, denoted $G_{\mathcal{P}_\infty}$, consists of the following supermatrices

$$\left(\begin{array}{c|c||c} A & 0 & 0 \\ \hline -(A^{-1})^{\mathrm{T}}\hat{\boldsymbol{a}} & (A^{-1})^{\mathrm{T}} & -\sqrt{2}(A^{-1})^{\mathrm{T}}\hat{\theta}^{\mathrm{T}} \\ \hline\hline \mathrm{i}\sqrt{2}\,R\hat{\theta} & 0 & R \end{array}\right), \qquad A \in \mathsf{GL}(2,\mathbb{R})\,, \quad R \in \mathsf{O}(\mathcal{N})\,, \tag{2.44}$$

where $\hat{\boldsymbol{a}}$ is defined in (2.32). The following one-form

$$\check{e}_{\mathrm{M}} = \mathrm{d}\check{\boldsymbol{y}} - 2\mathrm{i}\check{\rho}^{\mathrm{T}}\mathrm{d}\check{\rho}\,, \qquad \check{e}_{\mathrm{M}}^{\mathrm{T}} = \check{e}_{\mathrm{M}} \tag{2.45}$$

is invariant under the S-supersymmetry transformations.

In the intersection of the two charts introduced, $U_F \bigcap U_G$, the transition functions are

$$\check{\boldsymbol{y}} = -\hat{\boldsymbol{x}}^{-1}\,, \qquad \check{\rho} = -\hat{\theta}\hat{\boldsymbol{x}}^{-1}\,. \tag{2.46}$$

The one-forms (2.40) and (2.45) are related to each other by the rule

$$\hat{e} = (\check{\boldsymbol{y}}^{\mathrm{T}})^{-1}\check{e}_{\mathrm{M}}\check{\boldsymbol{y}}^{-1}\,. \tag{2.47}$$

The charts U_F and U_G are mapped onto each other by the superconformal transformation

$$\left(\begin{array}{c||c} -J & 0 \\ \hline\hline 0 & \mathbb{1}_{\mathcal{N}} \end{array}\right) \in \mathsf{OSp}(\mathcal{N}|4;\mathbb{R})\,, \tag{2.48}$$

where the matrix J is defined by (2.4).

The two charts U_F and U_G do not cover the compactified Minkowski superspace. It may be shown [53] that the bosonic body of $\overline{\mathbb{M}}^{3|2\mathcal{N}} \setminus \{U_F \bigcup U_G\}$ is topologically S^1. Additional charts are required if we are interested in the global description of $\overline{\mathbb{M}}^{3|2\mathcal{N}}$. Instead of introducing such additional charts, there is actually a better way out. It turns out that there exists an isomorphic realisation for $\mathsf{OSp}(\mathcal{N}|4;\mathbb{R})$ that is ideally suited for a global description of $\overline{\mathbb{M}}^{3|2\mathcal{N}}$. It will be presented in the next section.

2.5 Alternative definition of compactified Minkowski superspace

We would like to introduce one more refinement of the formalism that will be rather useful for the discussion in next sections. Following [53], we have defined the compactified $\mathcal{N}$-extended Minkowski superspace to be the space of null two-planes (through the origin) in $\mathbb{R}^{4|\mathcal{N}}$. However, every two-plane in $\mathbb{R}^{4|\mathcal{N}}$ is a real two-plane in $\mathbb{C}^{4|\mathcal{N}}$, the space of complex even supertwistors. A two-plane in $\mathbb{C}^{4|\mathcal{N}}$ is described by a supermatrix $\mathcal{P} = (T_1, T_2)$, where the supertwistors T_1 and T_2 constitute a basis of the two-plane. This supermatrix is defined modulo the equivalence relation

$$\mathcal{P} \sim \mathcal{P}\Xi\,, \qquad \Xi \in \mathsf{GL}(2,\mathbb{C})\,. \tag{2.49}$$

The equivalent supermatrices define the same two-plane. A two-plane $\mathcal{P}$ in $\mathbb{C}^{4|\mathcal{N}}$ is said to be real if it possesses a basis $\mathcal{P}_0$ consisting of real supertwistors, which means $\mathcal{P}_0^\dagger = \mathcal{P}_0^{\mathrm{sT}}$. Given an arbitrary basis $\mathcal{P}$ of the real two-plane, it holds that $\mathcal{P} = \mathcal{P}_0\Xi$, for some nonsingular matrix Ξ, and hence $\mathcal{P}^\dagger \sim \mathcal{P}^{\mathrm{sT}}$. We will adopt this new point of view in what follows. It allows us to define the compactified $\mathcal{N}$-extended Minkowski superspace $\overline{\mathbb{M}}^{3|2\mathcal{N}}$ to be the space of all real Lagrangian subspaces of $\mathbb{C}^{4|\mathcal{N}}$, the space of even supertwistors.

3 Pseudo-unitary realisation of $\mathsf{OSp}(\mathcal{N}|4;\mathbb{R})$

In this section we present a different isomorphic realisation for the superconformal group, which allows us to construct (i) a global supermatrix parametrisation of $\overline{\mathbb{M}}^{3|2\mathcal{N}}$; and (ii) a globally defined smooth metric, $\mathrm{d}s^2$, on $\overline{\mathbb{M}}^{3|2\mathcal{N}}$ with the property that $\mathrm{d}s^2$ only scales under the superconformal transformations. The crucial feature of this realisation is that it is suitable for developing superconformal field theory on 3D spacetimes more general than Minkowski space, such as $S^1 \times S^2$ and its universal covering space $\mathbb{R} \times S^2$. Our results in this section are analogous to those for the supersphere $S^{3|4n}$ [59].

3.1 Algebraic aspects

The superconformal group possesses an alternative realisation based on the isomorphism

$$\mathsf{OSp}(\mathcal{N}|4;\mathbb{R}) \cong \mathsf{U}(2,2|\mathcal{N}) \bigcap \mathsf{OSp}(\mathcal{N}|4;\mathbb{C})\,. \tag{3.1}$$

Here the supergroup on the right consists of all even $(4|\mathcal{N}) \times (4|\mathcal{N})$ supermatrices $\underline{g}$ constrained by

$$\underline{g}^\dagger \mathbb{I} \underline{g} = \mathbb{I}\,, \tag{3.2a}$$

$$\underline{g}^{\mathrm{sT}} \mathbb{J}\, \underline{g} = \mathbb{J}\,, \tag{3.2b}$$

where we have introduced

$$\mathbb{I} = \left(\begin{array}{c||c} I & 0 \\ \hline 0 & -\mathbb{1}_{\mathcal{N}} \end{array}\right), \qquad I = \left(\begin{array}{c||c} \mathbb{1}_2 & 0 \\ \hline 0 & -\mathbb{1}_2 \end{array}\right). \tag{3.3}$$

The condition (3.2a) defines the supergroup $\mathsf{U}(2,2|\mathcal{N})$. It should be pointed out that for $\mathcal{N} \neq 4$ the supergroup $\mathsf{SU}(2,2|\mathcal{N})$ is the $\mathcal{N}$-extended superconformal group in four spacetime dimensions, as defined in [28] ($\mathsf{PSU}(2,2|4)$ in the $\mathcal{N}=4$ case). In what follows, the supergroup on the right of (3.1) will be denoted $\mathsf{OSp}(\mathcal{N}|4;\mathbb{R})_U$.

The proof of (3.1) is based on considering the following correspondence

$$g \;\to\; \underline{g} := \mathbb{U} g \mathbb{U}^{-1}\,, \qquad g \in \mathsf{OSp}(\mathcal{N}|4;\mathbb{R})\,, \tag{3.4a}$$

$$T \;\to\; \underline{T} := \mathbb{U} T\,, \tag{3.4b}$$

$$Z \;\to\; \underline{Z} := Z \mathbb{U}^{-1}\,, \tag{3.4c}$$

for every supertwistor T and dual supertwistor Z. Here the supermatrix $\mathbb{U}$ is defined by

$$\mathbb{U} = \left(\begin{array}{c||c} U & 0 \\ \hline\hline 0 & \mathbb{1}_{\mathcal{N}} \end{array}\right) , \qquad U = \frac{1}{\sqrt{2}} \left(\begin{array}{c||c} \mathbb{1}_2 & \mathrm{i}\mathbb{1}_2 \\ \hline\hline \mathrm{i}\mathbb{1}_2 & \mathbb{1}_2 \end{array}\right) = U^{\mathrm{T}} . \tag{3.5}$$

The symmetric 4×4 matrix U is unitary, $U^\dagger U = \overline{U} U = \mathbb{1}_4$, and symplectic, $UJU = J$. Another important property is $UJU^{-1} = -\mathrm{i}I$. These properties have obvious counterparts in terms of $\mathbb{U}$:

$$\mathbb{U}^\dagger \mathbb{U} = \mathbb{1}_{4|\mathcal{N}} , \qquad \mathbb{U}\mathbb{J}\mathbb{U} = \mathbb{J} , \qquad \mathbb{U}\mathbb{J}\mathbb{U}^{-1} = -\mathrm{i}\mathbb{I} . \tag{3.6}$$

Of special importance for us will be the identity

$$\mathbb{U}^{-2}\mathbb{I} = \mathrm{i}\mathbb{J} . \tag{3.7}$$

The above properties of $\mathbb{U}$ imply that $\underline{g}$ defined by (3.4a) obeys the conditions (3.2), and hence $\underline{g} \in \mathsf{OSp}(\mathcal{N}|4;\mathbb{R})_U$, for every $g \in \mathsf{OSp}(\mathcal{N}|4;\mathbb{R})$, and vice versa.

Associated with the supergroup $\mathsf{OSp}(\mathcal{N}|4;\mathbb{R})_U$ are two invariant inner products defined as follows:

$$\langle \underline{T} | \underline{S} \rangle_{\mathbb{J}} := \underline{T}^{\mathrm{sT}} \mathbb{J} \, \underline{S} , \tag{3.8a}$$

$$\langle \underline{T} | \underline{S} \rangle_{\mathbb{I}} := \underline{T}^{\dagger} \mathbb{I} \, \underline{S} , \tag{3.8b}$$

for arbitrary pure supertwistors $\underline{T}$ and $\underline{S}$.

An important feature of the supergroup $\mathsf{OSp}(\mathcal{N}|4;\mathbb{R})_U$ is that one can define an involution $\star$ that acts on the space of supertwistors and commutes with the superconformal transformations. Associated with a pure supertwistor $\underline{T}$ is its star-image, $\star\underline{T}$, defined by

$$\underline{T}^\dagger \mathbb{I} = (\star \underline{T})^{\mathrm{sT}} \mathbb{J} . \tag{3.9}$$

Explicitly the map $\star$ acts as follows:

$$\underline{T} = \left(\begin{array}{c} f \\ g \\ \hline\hline \psi \end{array}\right) \quad \longrightarrow \quad \star\underline{T} = - \left(\begin{array}{c} \bar{g} \\ \bar{f} \\ \hline\hline (-1)^{\varepsilon(\underline{T})} \bar{\psi} \end{array}\right) . \tag{3.10}$$

This shows that $\star(\star\underline{T}) = \underline{T}$, for every supertwistor $\underline{T}$.

3.2 Compactified Minkowski superspace

Let us see how the compactified Minkowski superspace is described within the supergroup realisation introduced above. The null two-plane $\mathcal{P} \in \overline{\mathbb{M}}^{3|2\mathcal{N}}$ turns into $\underline{\mathcal{P}} = \mathbb{U}\mathcal{P}$. Since $\mathcal{P}$ obeys the null condition $\mathcal{P}^{\mathrm{sT}}\mathbb{J}\mathcal{P} = 0$ and is real, $\mathcal{P}^\dagger = \mathcal{P}^{\mathrm{sT}}$, the two-plane $\underline{\mathcal{P}}$ enjoys the two null conditions

$$\underline{\mathcal{P}}^\dagger \mathbb{I} \underline{\mathcal{P}} = 0 , \tag{3.11a}$$

$$\underline{\mathcal{P}}^{\mathrm{sT}} \mathbb{J} \underline{\mathcal{P}} = 0 . \tag{3.11b}$$

JHEP06(2015)138

The reality condition $\mathcal{P}^\dagger = \mathcal{P}^{\rm sT}$ turns into

$$\underline{\mathcal{P}}^\dagger = \underline{\mathcal{P}}^{\rm sT}\mathbb{U}^{-2} . \tag{3.12}$$

It may be seen that this reality condition preserves its form only under the *real* equivalence transformations

$$\underline{\mathcal{P}} \sim \underline{\mathcal{P}}\Xi , \qquad \Xi \in \mathsf{GL}(2,\mathbb{R}) . \tag{3.13}$$

However, making use of the identities (3.6) and (3.7), it may be rewritten in a different but equivalent form

$$\star\underline{\mathcal{P}} \sim \underline{\mathcal{P}} , \tag{3.14}$$

which does not change its form under arbitrary *complex* equivalent transformations

$$\underline{\mathcal{P}} \sim \underline{\mathcal{P}}\Xi , \qquad \Xi \in \mathsf{GL}(2,\mathbb{C}) , \tag{3.15}$$

see subsection 2.5. Thus, the compactified $\mathcal{N}$-extended Minkowski superspace $\overline{\mathbb{M}}^{3|2\mathcal{N}}$ is equivalently defined as the set of all two-planes in the space of even supertwistors $\mathbb{C}^{4|\mathcal{N}}$ which obey (i) the null conditions (3.11); and (ii) the reality condition (3.14).

We can represent

$$\underline{\mathcal{P}} = \begin{pmatrix} \underline{F} \\ \underline{G} \\ \hline\hline \underline{\Lambda} \end{pmatrix} , \tag{3.16}$$

where $\underline{F}$ and $\underline{G}$ are bosonic 2×2 matrices, and the remaining $\mathcal{N}\times 2$ matrix $\underline{\Lambda}$ is fermionic. Then the null condition (3.11a) tells us that

$$\underline{F}^\dagger\underline{F} - \underline{G}^\dagger\underline{G} = \underline{\Lambda}^\dagger\underline{\Lambda} . \tag{3.17}$$

In conjunction with the fact that the supermatrix $\underline{\mathcal{P}}$ has rank two, this condition implies that $\det\underline{F} \neq 0$ and $\det\underline{G} \neq 0$, see [57] for the proof. As a result, making use of the equivalence relation (3.15) allows us to bring every two-plane $\underline{\mathcal{P}} \in \overline{\mathbb{M}}^{3|2\mathcal{N}}$ to the form

$$\underline{\mathcal{P}} \sim \begin{pmatrix} \boldsymbol{h} \\ \mathbb{1}_2 \\ \hline\hline \zeta \end{pmatrix} . \tag{3.18}$$

Now, the null conditions (3.11a) and (3.11b) turn into

$$\boldsymbol{h}^\dagger\boldsymbol{h} - \mathbb{1}_2 = \zeta^\dagger\zeta , \tag{3.19a}$$
$$\boldsymbol{h}^{\rm T} - \boldsymbol{h} = {\rm i}\zeta^{\rm T}\zeta . \tag{3.19b}$$

Moreover, the reality condition (3.14) gives

$$\bar{\boldsymbol{h}} = \boldsymbol{h}^{-1} , \tag{3.20a}$$
$$\bar{\zeta} = -{\rm i}\zeta\boldsymbol{h}^{-1} . \tag{3.20b}$$

The relations (3.18)–(3.20) provide a global supermatrix parametrisation of $\overline{\mathbb{M}}^{3|2\mathcal{N}}$.

Consider the bosonic body $\overline{\mathbb{M}}^3$ of compactified Minkowski superspace $\overline{\mathbb{M}}^{3|2\mathcal{N}}$, which is obtained by switching off the Grassmann variables ζ and is described by a 2×2 matrix h defined by $h := \boldsymbol{h}|_{\zeta=0}$. As follows from (3.19), its properties are $h^\dagger h = \mathbb{1}_2$ and $h^{\rm T} = h$. The general solution of these constraints is

$$h = {\rm e}^{{\rm i}\varphi}\Big(a\mathbb{1}_2 + {\rm i}b\sigma_1 + {\rm i}c\sigma_3\Big)\,, \qquad a^2+b^2+c^2 = 1\,, \tag{3.21}$$

for real parameters φ and a, b, c parametrising, respectively, S^1 and S^2. We also have

$${\rm e}^{{\rm i}\varphi}\Big(a\mathbb{1}_2 + {\rm i}b\sigma_1 + {\rm i}c\sigma_3\Big) = {\rm e}^{{\rm i}(\varphi+\pi)}\Big(-a\mathbb{1}_2 - {\rm i}b\sigma_1 - {\rm i}c\sigma_3\Big)\,,$$

and thus compactified Minkowski space $\overline{\mathbb{M}}^3$ is $(S^1\times S^2)/\mathbb{Z}_2$.

3.3 Superconformal metric

As shown in the previous subsection, every null two-plane $\underline{\mathcal{P}} \in \overline{\mathbb{M}}^{3|2\mathcal{N}}$ is uniquely represented in the form (3.18) for some matrices $\boldsymbol{h}$ and ζ constrained according to (3.19) and (3.20). This means that, given a group element $\underline{g} \in \mathsf{OSp}(\mathcal{N}|4;\mathbb{R})_U$, it acts on $\underline{\mathcal{P}}$ as

$$\underline{g}\begin{pmatrix}\boldsymbol{h}\\ \mathbb{1}_2\\ \hline\hline \zeta\end{pmatrix} = \begin{pmatrix}\boldsymbol{h}'\\ \mathbb{1}_2\\ \hline\hline \zeta'\end{pmatrix}\varphi(\underline{g},\boldsymbol{h},\zeta)\,, \qquad \varphi(\underline{g},\boldsymbol{h},\zeta) \in \mathsf{GL}(2,\mathbb{C})\,, \tag{3.22}$$

for some nonsingular matrix $\varphi(\underline{g},\boldsymbol{h},\zeta)$. Explicitly, if we represent $\underline{g}$ in the block form

$$\underline{g} = \left(\begin{array}{c|c||c} A & B & \gamma\\ \hline C & D & \delta\\ \hline\hline \lambda & \rho & R\end{array}\right) \in \mathsf{OSp}(\mathcal{N}|4;\mathbb{R})_U\,, \tag{3.23}$$

then $\boldsymbol{h}'$ and ζ' are seen to be fractional linear functions of $\boldsymbol{h}$ and ζ,

$$\boldsymbol{h}' = (A\boldsymbol{h}+B+\gamma\zeta)(C\boldsymbol{h}+D+\delta\zeta)^{-1}\,, \tag{3.24a}$$
$$\zeta' = (\lambda\boldsymbol{h}+\rho+R\zeta)(C\boldsymbol{h}+D+\delta\zeta)^{-1}\,, \tag{3.24b}$$

and $\varphi(\underline{g},\boldsymbol{h},\zeta) = C\boldsymbol{h}+D+\delta\zeta$. By construction, $\varphi(\underline{g},\boldsymbol{h},\zeta)$ is nonsingular for every group element $\underline{g}\in\mathsf{OSp}(\mathcal{N}|4;\mathbb{R})_U$.

Cartan's one-form

$$\mathcal{E} := \underline{\mathcal{P}}^\dagger\mathbb{I}{\rm d}\underline{\mathcal{P}} = \boldsymbol{h}^\dagger{\rm d}\boldsymbol{h} - \zeta^\dagger{\rm d}\zeta\,, \qquad \mathcal{E}^\dagger = -\mathcal{E} \tag{3.25}$$

takes its values in the superalgebra $\mathfrak{osp}(\mathcal{N}|4;\mathbb{R})_U$ and possesses the superconformal transformation law

$$\mathcal{E} \;\to\; \mathcal{E}' = (\varphi^\dagger)^{-1}\mathcal{E}\varphi^{-1}\,, \tag{3.26}$$

where the shorthand notation $\varphi = \varphi(\underline{g},\boldsymbol{h},\zeta)$ has been used. We can introduce a superinterval

$${\rm d}s^2 := \frac{1}{4}\det\mathcal{E}\,, \tag{3.27}$$

which is a globally defined tensor field over $\overline{\mathbb{M}}^{3|2\mathcal{N}}$. It follows from (3.26) that $\mathrm{d}s^2$ only scales under the superconformal transformations,

$$\mathrm{d}s^2 \ \to \ \mathrm{d}s^2 |\det \varphi|^{-2} \ . \tag{3.28}$$

In the Minkowski chart, it may be seen that the variables $\boldsymbol{h}$ and ζ are expressed in terms of the superspace coordinates as

$$\boldsymbol{h} = -\mathrm{i}(\mathbb{1}_2 - \mathrm{i}\hat{\boldsymbol{x}})(\mathbb{1}_2 + \mathrm{i}\hat{\boldsymbol{x}})^{-1} , \qquad \zeta = 2\theta(\mathbb{1}_2 + \mathrm{i}\hat{\boldsymbol{x}})^{-1} \ . \tag{3.29}$$

A direct calculation of $\mathcal{E}$ gives the following expression:

$$\mathcal{E} = -2\mathrm{i}(\mathbb{1}_2 - \mathrm{i}\hat{\boldsymbol{x}}^{\mathrm{T}})^{-1}\hat{e}\,(\mathbb{1}_2 + \mathrm{i}\hat{\boldsymbol{x}})^{-1} , \tag{3.30}$$

where $\hat{e}$ is the supersymmetric one-form (2.40).

3.4 Pseudo inversion

Consider a particular superconformal transformation

$$\underline{\mathcal{F}} = \left(\begin{array}{c|c||c} \sigma_2 & 0 & 0 \\ \hline 0 & -\sigma_2 & 0 \\ \hline\hline 0 & 0 & \mathbb{1}_{\mathcal{N}} \end{array}\right) \in \mathsf{OSp}(\mathcal{N}|4;\mathbb{R})_U \ , \qquad \underline{\mathcal{F}}^2 = \mathbb{1}_{4|\mathcal{N}} \ , \tag{3.31}$$

where σ_2 is the second Pauli matrix. It acts on $\overline{\mathbb{M}}^{3|2\mathcal{N}}$ as follows

$$\boldsymbol{h}' = -\sigma_2 \boldsymbol{h} \sigma_2 \ , \qquad \zeta' = -\zeta\sigma_2 \ . \tag{3.32}$$

In the real realisation of the superconformal group, the supermatrix (3.31) turns into

$$\mathcal{F} = \mathbb{U}^{-1}\underline{\mathcal{F}}\mathbb{U} = \left(\begin{array}{c|c||c} 0 & -\varepsilon & 0 \\ \hline -\varepsilon^{-1} & 0 & 0 \\ \hline\hline 0 & 0 & \mathbb{1}_{\mathcal{N}} \end{array}\right) \in \mathsf{OSp}(\mathcal{N}|4;\mathbb{R}) \ , \qquad \varepsilon := (\varepsilon_{\alpha\beta}) = -\mathrm{i}\sigma_2 \ . \tag{3.33}$$

Its action on the two-plane $\mathcal{P}(z)$ defined by (2.24) is

$$\mathcal{F}\mathcal{P}(z) = \left(\begin{array}{c} \mathbb{1}_2 \\ \varepsilon^{-1}\hat{\boldsymbol{x}}^{-1}\varepsilon^{-1} \\ \hline\hline \mathrm{i}\sqrt{2}\hat{\theta}\hat{\boldsymbol{x}}^{-1}\varepsilon^{-1} \end{array}\right) \Xi(z) = \mathcal{P}(z')\Xi(z) \ , \qquad \Xi(z) := \varepsilon\hat{\boldsymbol{x}} \ . \tag{3.34}$$

This leads to

$$\hat{\boldsymbol{x}}' = -\check{\boldsymbol{x}}^{-1} \ , \qquad \hat{\theta}' = \check{\theta}\check{\boldsymbol{x}}^{-1} \tag{3.35a}$$

or, equivalently,

$$\check{\boldsymbol{x}}' = -\hat{\boldsymbol{x}}^{-1} \ , \qquad \check{\theta}' = -\hat{\theta}\hat{\boldsymbol{x}}^{-1} \ . \tag{3.35b}$$

JHEP06(2015)138

Let $g = \mathbb{1} + \kappa\Omega$ be an infinitesimal superconformal transformation, where κ is an infinitesimal parameter and Ω is an arbitrary element of the superconformal algebra $\mathfrak{osp}(\mathcal{N}|4;\mathbb{R})$ given by (2.17). In Minkowski superspace, its action is given by eq. (2.27). It is an instructive exercise to check that the transformation $\mathcal{F}(\mathbb{1} + \kappa\Omega)\mathcal{F}$ generates the following infinitesimal transformation:

$$\delta\check{\boldsymbol{x}} = \check{b} + \lambda\check{\boldsymbol{x}} + \check{\boldsymbol{x}}\lambda^{\mathrm{T}} - \sigma\check{\boldsymbol{x}} + \check{\boldsymbol{x}}\hat{a}\,\check{\boldsymbol{x}} + 2\mathrm{i}\,\check{\eta}^{\mathrm{T}}\check{\theta} + 2\mathrm{i}\,\check{\boldsymbol{x}}\hat{\epsilon}^{\mathrm{T}}\check{\theta}\,, \tag{3.36a}$$

$$\delta\check{\theta} = \check{\eta} + \check{\theta}\lambda^{\mathrm{T}} - \frac{1}{2}\sigma\check{\theta} + \Lambda\check{\theta} + \check{\theta}\hat{a}\check{\boldsymbol{x}} + \hat{\epsilon}\check{\boldsymbol{x}} + 2\mathrm{i}\,\check{\theta}\,\hat{\epsilon}^{\mathrm{T}}\check{\theta}\;. \tag{3.36b}$$

As compared with $g = \mathbb{1}+\kappa\Omega$, the transformation $\mathcal{F}(\mathbb{1}+\kappa\Omega)\mathcal{F}$ swaps the spacetime translations and special conformal boosts, as well as the Q-supersymmetry and S-supersymmetry transformations. It also changes the sign of the scale parameter σ.

The transformation $\mathcal{F}$ has properties analogous to those of the superinversion (i.e., a supersymmetric extension of the conformal inversion), see e.g. [21] for the 4D case. However, the restriction of $\mathcal{F}$ to compactified Minkowski space is a transformation that belongs to the connected component of the identity of the conformal group, and thus it differs from the 3D conformal inversion

$$\hat{x}' = \check{x}^{-1}\,, \tag{3.37}$$

which belongs to the other component of the conformal group. This is why it is appropriate to call $\mathcal{F}$ "pseudo inversion." The transformation (3.35) was called "superinversion" in [50]. Our consideration shows that this terminology is somewhat misleading. An extension of conformal inversion (3.37) is unclear to us.

3.5 Fibre bundles over compactified Minkowski superspace

Fibre bundles over $\overline{\mathbb{M}}^{3|2\mathcal{N}}$, such as compactified harmonic/projective superspaces in three spacetime dimensions [53], can be obtained by generalising the construction of subsection 3.2 to include odd supertwistors.[9] Odd supertwistors are destined to parametrise fibres over $\overline{\mathbb{M}}^{3|2\mathcal{N}}$. In the unitary realisation of the superconformal group, given an odd supertwistor $\underline{\Sigma}$, it is defined by the following two conditions:

- it is orthogonal to the even supertwistors $\underline{T}^\mu$ which form a basis of the null two-plane $\underline{\mathcal{P}} \in \overline{\mathbb{M}}^{3|2\mathcal{N}}$, with respect to the inner products (3.8a) and (3.8b),

$$\langle \underline{T}^\mu|\underline{\Sigma}\rangle_{\mathbb{J}} = 0\,, \qquad \langle \underline{T}^\mu|\underline{\Sigma}\rangle_{\mathbb{I}} = 0\;; \tag{3.38}$$

- it is defined modulo the equivalence relation

$$\underline{\Sigma} \sim \underline{\Sigma} + \underline{T}^\mu a_\mu\,, \tag{3.39}$$

  for arbitrary a-numbers a_μ (i.e. odd elements of the Grassmann algebra).

[9] Our approach here is inspired by the construction of compactified harmonic/projective superspaces with Lorentzian signature given in [53, 57–59]. These papers built on earlier works [60–62].

If the null two-plane $\underline{\mathcal{P}}$ is chosen in the form (3.18), then imposing the first null condition (3.38) and making use of the equivalence relation (3.39), the odd supertwistor may be brought to the form

$$\underline{\Sigma} = \begin{pmatrix} -\mathrm{i}\zeta^{\mathrm{T}} v \\ 0 \\ \hline\hline v \end{pmatrix} , \tag{3.40}$$

with $v = (v_I)$ being a bosonic $\mathcal{N}$-vector. Here we have used the reality conditions (3.20). It is important to point out that the second null condition (3.38) also leads to the same explicit expression (3.40) for $\underline{\Sigma}$. Thus the space of odd supertwistors at $\underline{\mathcal{P}}$ may be identified with $\mathbb{C}^{\mathcal{N}}$.

As simple examples of fibre bundles over $\overline{\mathbb{M}}^{3|2\mathcal{N}}$, we can consider odd supertwistor Grassmannians $\mathfrak{G}(m, \mathcal{N})$, where m may take values from 1 to $\mathcal{N}$. By definition, the points of $\mathfrak{G}(m, \mathcal{N})$ are described by m odd supertwistors $\underline{\Sigma}^{\underline{i}}$, with $\underline{i} = 1, \ldots, m$, such that (i) the bodies of $\underline{\Sigma}^{\underline{i}}$ are linearly independent; and (ii) the supertwistors $\underline{\Sigma}^{\underline{i}}$ are defined modulo the equivalence relation

$$\underline{\Sigma}^{\underline{i}} \sim \underline{\Sigma}^{\underline{j}} \mathscr{D}_{\underline{j}}{}^{\underline{i}} , \qquad \mathscr{D} \in \mathsf{GL}(m, \mathbb{C}) . \tag{3.41}$$

In this paper, we are mostly interested in the $m = \mathcal{N}$ case, for which $\underline{\Sigma}^{\underline{i}}$ may be chosen in the form:

$$\underline{\mathcal{O}} = (\underline{\Sigma}^1, \ldots, \underline{\Sigma}^{\mathcal{N}}) = \begin{pmatrix} -\mathrm{i}\zeta^{\mathrm{T}} \\ 0 \\ \hline\hline \mathbb{1}_{\mathcal{N}} \end{pmatrix} . \tag{3.42}$$

In the remainder of the paper, we use the real realisation of the superconformal group described in section 2.1. In this realisation, only one of the two null conditions (3.38) remains,

$$\langle T^\mu | \Sigma \rangle_{\mathbb{J}} = 0 . \tag{3.43}$$

We recall that for every point z in Minkowski superspace, the supermatrix $\mathcal{P} = (T^1, T^2)$ can be chosen as

$$\mathcal{P}(z) = \begin{pmatrix} \mathbb{1}_2 \\ -\hat{\boldsymbol{x}} \\ \hline\hline \mathrm{i}\sqrt{2}\hat{\theta} \end{pmatrix} , \qquad \hat{\boldsymbol{x}} = \hat{x} + \mathrm{i}\hat{\theta}^{\mathrm{T}}\hat{\theta} . \tag{3.44}$$

Therefore, instead of the representation (3.40), now every odd supertwistor Σ from the fibre at $\mathcal{P}(z)$ can be brought to the form:

$$\Sigma = \begin{pmatrix} 0 \\ -\sqrt{2}\hat{\theta}^{\mathrm{T}} v \\ \hline\hline v \end{pmatrix} . \tag{3.45}$$

Finally, the expression (3.42) turns into

$$\mathcal{O}(z) = (\Sigma^1, \dots, \Sigma^{\mathcal{N}}) = \begin{pmatrix} 0 \\ -\sqrt{2}\hat{\theta}^{\mathrm{T}} \\ \hline\hline \mathbb{1}_{\mathcal{N}} \end{pmatrix} . \tag{3.46}$$

4 Two-point and three-point building blocks

Here we derive those two- and three-point functions of superspace coordinates which are building blocks for the correlation functions of primary superfields. An alternative derivation was given by Park [50].

As is seen from (3.44) and (3.46), the coset representative $s(z)$, defined by (2.35), is built from the supermatrices corresponding to the even two-plane $\mathcal{P}(z)$ and the odd $\mathcal{N}$-plane $\mathcal{O}(z)$ according to the rule

$$s(z) = \Big(\mathcal{P}(z),\ \mathcal{P}_\infty,\ \mathcal{O}(z)\Big) , \qquad \mathcal{P}_\infty = \begin{pmatrix} 0 \\ \mathbb{1}_2 \\ \hline\hline 0 \end{pmatrix} . \tag{4.1}$$

Here $\mathcal{P}_\infty$ denotes the null two-plane corresponding to the origin of the chart $U_G \subset \overline{\mathbb{M}}^{3|2\mathcal{N}}$, see subsection 2.4. The two-plane $\mathcal{P}_\infty$ is an infinitely separated point from the viewpoint of an observer living in Minkowski superspace.

4.1 Infinitesimal superconformal transformations

For our subsequent analysis, it is advantageous to recast the superconformal transformation laws of $\mathcal{P}(z)$ and $\mathcal{O}(z)$ in terms of the coset representative (4.1). Before discussing the transformation of $s(z)$, we first point out that the infinitesimal transformation $z^A \to z^A + \delta z^A$, with δz^A given by (2.27), can be rewritten as

$$\delta z^A = \xi z^A \quad \Longleftrightarrow \quad \delta x^a = \xi^a + \mathrm{i}\, \xi^\alpha_I (\gamma^a)_{\alpha\beta} \theta^\beta_I , \qquad \delta\theta^\alpha_I = \xi^\alpha_I , \tag{4.2}$$

where ξ denotes a first-order differential operator

$$\xi = \xi^A(z) D_A = \xi^a \partial_a + \xi^\alpha_I D^I_\alpha = -\frac{1}{2}\xi^{\alpha\beta}\partial_{\alpha\beta} + \xi^\alpha_I D^I_\alpha . \tag{4.3}$$

The components of this operator are as follows:

$$\begin{aligned} \xi^{\alpha\beta} &= a^{\alpha\beta} - \lambda^\alpha{}_\gamma x^{\gamma\beta} - x^{\alpha\gamma}\lambda_\gamma{}^\beta + \sigma x^{\alpha\beta} + 4\mathrm{i}\epsilon^{(\alpha}_I \theta^{\beta)}_I + 2\mathrm{i}\Lambda_{IJ}\theta^\alpha_J\theta^\beta_I + x^{\alpha\gamma}x^{\beta\delta}b_{\gamma\delta} \\ &\quad + \mathrm{i}b^{(\alpha}_\delta x^{\beta)\delta}\theta^2 - \frac{1}{4}b^{\alpha\beta}\theta^2\theta^2 - 4\mathrm{i}\eta_{I\gamma}x^{\gamma(\alpha}\theta^{\beta)}_I + 2\eta^{(\alpha}_I\theta^{\beta)}_I\theta^2 , \end{aligned} \tag{4.4a}$$

$$\xi^\alpha_I = \epsilon^\alpha_I - \lambda^\alpha{}_\beta\theta^\beta_I + \frac{1}{2}\sigma\theta^\alpha_I + \Lambda_{IJ}\theta^\alpha_J + b_{\beta\gamma}\boldsymbol{x}^{\beta\alpha}\theta^\gamma_I + \eta_{J\beta}(2\mathrm{i}\theta^\beta_I\theta^\alpha_J - \delta_{IJ}\boldsymbol{x}^{\beta\alpha}) . \tag{4.4b}$$

Following the terminology of [21], the supervector field ξ is called a conformal Killing supervector field. It may equivalently be defined [53] as the most general solution to the equation $[\xi, D^I_\alpha] \propto D^J_\beta$, and therefore

$$[\xi, D^I_\alpha] = -(D^I_\alpha\xi^\beta_J)D^J_\beta = \lambda_\alpha{}^\beta(z)D^I_\beta + \Lambda^{IJ}(z)D^J_\alpha - \frac{1}{2}\sigma(z)D^I_\alpha . \tag{4.5}$$

JHEP06(2015)138

Here the coefficient functions in the right-hand side read

$$\lambda_{\alpha\beta}(z) = -\frac{1}{\mathcal{N}} D^I_{(\alpha}\xi^I_{\beta)}\,, \quad \Lambda^{IJ}(z) = -2D^{[I}_\alpha \xi^{J]\alpha}\,, \quad \sigma(z) = \frac{1}{\mathcal{N}} D^I_\alpha \xi^\alpha_I = \frac{1}{3}\partial_a \xi^a\,, \tag{4.6}$$

and may be thought of as the parameters of local Lorentz, R-symmetry and scale transformations, respectively. The explicit calculation of these parameters gives

$$\lambda^{\alpha\beta}(z) = \lambda^{\alpha\beta} - x^{\gamma(\alpha}b_\gamma^{\beta)} - \frac{\mathrm{i}}{2} b^{\alpha\beta}\theta^\gamma_I \theta_{I\gamma} + 2\mathrm{i}\eta_I^{(\alpha}\theta_I^{\beta)}\,, \tag{4.7a}$$

$$\Lambda_{IJ}(z) = \Lambda_{IJ} + 4\mathrm{i}\eta^\alpha_{[I}\theta_{J]\alpha} + 2\mathrm{i}b_{\alpha\beta}\theta^\alpha_I\theta^\beta_J\,, \tag{4.7b}$$

$$\sigma(z) = \sigma + b_{\alpha\beta}x^{\beta\alpha} + 2\mathrm{i}\theta^\alpha_I \eta_{I\alpha}\,. \tag{4.7c}$$

Under the infinitesimal superconformal transformation associated with Ω, eq. (2.17), the even two-plane $\mathcal{P}(z)$ and the odd $\mathcal{N}$-plane $\mathcal{O}(z)$ vary as $\xi\mathcal{P}(z) = \mathcal{P}(z+\delta z) - \mathcal{P}(z)$ and $\xi\mathcal{O}(z) = \mathcal{O}(z+\delta z) - \mathcal{O}(z)$, respectively. These variations are computed by the rule[10]

$$\Omega\mathcal{P}(z) = \xi\mathcal{P}(z) + \mathcal{P}(z)\Big(\lambda(z) - \frac{1}{2}\sigma(z)\mathbb{1}_2\Big)\,, \tag{4.8a}$$

$$\Omega\mathcal{O}(z) = \xi\mathcal{O}(z) + \mathcal{O}(z)\Lambda(z) + \mathcal{P}(z)\sqrt{2}\check{\eta}^{\mathrm{T}}(z)\,, \tag{4.8b}$$

where we have introduced the z-dependent S-supersymmetry parameter

$$\eta_{I\alpha}(z) := \eta_{I\alpha} - b_{\alpha\beta}\theta^\beta_I = -\frac{\mathrm{i}}{2}D^I_\alpha \sigma(z)\,. \tag{4.9}$$

As concerns the coset representative (4.1), it follows from first principles that

$$\Omega s(z) = \xi s(z) + s(z)\omega(z)\,, \tag{4.10}$$

for some supermatrix $\omega(z)$ belonging to the isotropy subalgebra (2.28). Making use of (4.8), the explicit form of $\omega(z)$ is

$$\omega(z) = \left(\begin{array}{c|c||c} \lambda_\alpha{}^\beta(z) - \frac{1}{2}\delta_\alpha{}^\beta\sigma(z) & b_{\alpha\beta} & \sqrt{2}\eta_{I\alpha}(z) \\ \hline 0 & -\lambda^\alpha{}_\beta(z) + \frac{1}{2}\delta^\alpha{}_\beta\sigma(z) & 0 \\ \hline\hline 0 & \mathrm{i}\sqrt{2}\eta_{I\beta}(z) & \Lambda_{IJ}(z) \end{array}\right)\,. \tag{4.11}$$

4.2 Two-point functions

By construction, it holds that $\mathcal{P}^{\mathrm{sT}}(z)\mathbb{J}\mathcal{P}(z) = 0$ and $\mathcal{P}^{\mathrm{sT}}(z)\mathbb{J}\mathcal{O}(z) = 0$. Consider two different superspace points z_1 and z_2. Then we can define two-point functions

$$\mathcal{P}^{\mathrm{sT}}(z_1)\mathbb{J}\mathcal{P}(z_2) = \hat{\boldsymbol{x}}_1^{\mathrm{T}} - \hat{\boldsymbol{x}}_2 + 2\mathrm{i}\hat{\theta}_1^{\mathrm{T}}\hat{\theta}_2 \equiv \hat{\boldsymbol{x}}_{12}\,, \qquad \hat{\boldsymbol{x}}_{21} = -\hat{\boldsymbol{x}}_{12}^{\mathrm{T}} \tag{4.12a}$$

$$\mathcal{P}^{\mathrm{sT}}(z_1)\mathbb{J}\mathcal{O}(z_2) = \sqrt{2}(\hat{\theta}_1 - \hat{\theta}_2)^{\mathrm{T}} \equiv \sqrt{2}\hat{\theta}_{12}^{\mathrm{T}}\,, \qquad \hat{\theta}_{21} = -\hat{\theta}_{12}\,. \tag{4.12b}$$

Making use of (2.25), $\hat{\boldsymbol{x}}_{12}$ and $\hat{\theta}_{12}$ can be rewritten with explicit spinor indices as follows:

$$\boldsymbol{x}_{12}^{\alpha\beta} = (x_1 - x_2)^{\alpha\beta} + 2\mathrm{i}\theta^{(\alpha}_{1I}\theta^{\beta)}_{2I} - \mathrm{i}\theta^\alpha_{12I}\theta^\beta_{12I}\,, \tag{4.13a}$$

$$\theta^\alpha_{12I} = (\theta_1 - \theta_2)^\alpha_I\,. \tag{4.13b}$$

[10] The index structure of the matrices $\lambda(z)$, $\Lambda(z)$ and $\check{\eta}(z)$ in (4.8) is the same as in (2.17).

JHEP06(2015)138

According to (4.8), the above two-point functions transform semi covariantly under the superconformal group

$$\widetilde{\delta}\boldsymbol{x}_{12}^{\alpha\beta} = \left(\frac{1}{2}\delta^{\alpha}{}_{\gamma}\sigma(z_1) - \lambda^{\alpha}{}_{\gamma}(z_1)\right)\boldsymbol{x}_{12}^{\gamma\beta} + \boldsymbol{x}_{12}^{\alpha\gamma}\left(\frac{1}{2}\delta_{\gamma}{}^{\beta}\sigma(z_2) - \lambda_{\gamma}{}^{\beta}(z_2)\right), \tag{4.14a}$$

$$\widetilde{\delta}\theta_{12I}^{\alpha} = \left(\frac{1}{2}\delta^{\alpha}{}_{\beta}\sigma(z_1) - \lambda^{\alpha}{}_{\beta}(z_1)\right)\theta_{12I}^{\beta} - \boldsymbol{x}_{12}^{\alpha\beta}\eta_{I\beta}(z_2) + \Lambda_{IJ}(z_2)\theta_{12J}^{\alpha}\,. \tag{4.14b}$$

Here the variation $\widetilde{\delta}$ is defined by its action on an n-point function $\Phi(z_1, \dots, z_n)$ to be

$$\widetilde{\delta}\Phi(z_1, \dots, z_n) = \sum_{i=1}^{n}\xi_{z_i}\Phi(z_1, \dots, z_n)\,. \tag{4.15}$$

We note that the definitions (4.12) can be recast in terms of the coset representative (4.1) by introducing the following two-point supermatrix [50]

$$\mathcal{S}(z_1, z_2) := (s(z_1))^{\mathrm{sT}}\mathbb{J}s(z_2)\,. \tag{4.16}$$

Using the transformation law of $s(z)$, eq. (4.10), we read off the superconformal transformation of $\mathcal{S}(z_1, z_2)$

$$\widetilde{\delta}\mathcal{S}(z_1, z_2) = -\omega(z_1)^{\mathrm{sT}}\mathcal{S}(z_1, z_2) - \mathcal{S}(z_1, z_2)\omega(z_2)\,. \tag{4.17}$$

Let us introduce the following objects

$$\boldsymbol{x}_{12}{}^2 := -\frac{1}{2}\mathrm{tr}(\hat{\boldsymbol{x}}_{12}\check{\boldsymbol{x}}_{12}^{\mathrm{T}}) = -\frac{1}{2}\boldsymbol{x}_{12}^{\alpha\beta}\boldsymbol{x}_{12\alpha\beta}\,, \tag{4.18a}$$

$$\underline{\hat{\boldsymbol{x}}}_{12} := \frac{\hat{\boldsymbol{x}}_{12}}{\sqrt{-\boldsymbol{x}_{12}{}^2}}\,, \qquad (\varepsilon\underline{\hat{\boldsymbol{x}}}_{12})^2 = \mathbb{1}_2\,, \tag{4.18b}$$

with $\varepsilon = (\varepsilon_{\alpha\beta})$. Using (4.14) it is easy to check that $\boldsymbol{x}_{12}{}^2$ transforms only under local scale transformations while $\underline{\hat{\boldsymbol{x}}}_{12}$ varies only with the local Lorentz parameters

$$\widetilde{\delta}\boldsymbol{x}_{12}{}^2 = (\sigma(z_1) + \sigma(z_2))\,\boldsymbol{x}_{12}{}^2\,, \tag{4.19a}$$

$$\widetilde{\delta}\underline{\boldsymbol{x}}_{12}^{\alpha\beta} = -\lambda^{\alpha}{}_{\gamma}(z_1)\,\underline{\boldsymbol{x}}_{12}^{\gamma\beta} - \underline{\boldsymbol{x}}_{12}^{\alpha\gamma}\,\lambda_{\gamma}{}^{\beta}(z_2)\,. \tag{4.19b}$$

Thus, they will naturally appear as two-point building blocks in the correlation functions of primary superfields to be studied in the next sections.

Since the two-point function $\boldsymbol{x}_{12}^{\alpha\beta}$ has the symmetry property

$$\boldsymbol{x}_{21}^{\alpha\beta} = -\boldsymbol{x}_{12}^{\beta\alpha}\,, \tag{4.20}$$

it can be divided into the symmetric and antisymmetric parts

$$\boldsymbol{x}_{12}^{\alpha\beta} = x_{12}^{\alpha\beta} + \frac{\mathrm{i}}{2}\varepsilon^{\alpha\beta}\theta_{12}{}^2\,, \qquad \theta_{12}{}^2 \equiv \theta_{12I}^{\alpha}\theta_{12I\alpha}\,. \tag{4.21}$$

The symmetric part is nothing but the bosonic component of the standard two-point superspace interval

$$x_{12}^{\alpha\beta} = (x_1 - x_2)^{\alpha\beta} + 2\mathrm{i}\theta_{1I}^{(\alpha}\theta_{2I}^{\beta)}\,. \tag{4.22}$$

JHEP06(2015)138

We stress that both $x_{12}^{\alpha\beta}$ and $\boldsymbol{x}_{12}^{\alpha\beta}$ are invariant under supersymmetry while only the latter transforms covariantly under the superconformal group according to (4.14).

To introduce one more important building block, we point out that the pseudo inversion $\mathcal{F}$ acts on $\mathcal{P}(z)$ by the rule (3.34). One may also work out the action of $\mathcal{F}$ on $\mathcal{O}(z)$. These results allow us to compute the action of $\mathcal{F}$ on the coset representative (2.35)

$$\mathcal{F}s(z) = s(z')h(\mathcal{F};z)\,, \tag{4.23}$$

where $h(\mathcal{F};z)$ is a supermatrix from the isotropy group $G_{\mathcal{P}_0}$. The latter supermatrix is of the type (2.29) with the following block matrix elements:

$$A = \varepsilon\hat{\boldsymbol{x}}\,, \qquad \check{\boldsymbol{b}} = -\hat{\boldsymbol{x}}^{-1}\,, \qquad \check{\eta} = \hat{\theta}(\boldsymbol{x}^{\mathrm{T}})^{-1}\,, \tag{4.24a}$$

$$R = \mathbb{1}_{\mathcal{N}} - 2\mathrm{i}\hat{\theta}\hat{\boldsymbol{x}}^{-1}\hat{\theta}^{\mathrm{T}}\;. \tag{4.24b}$$

The $\mathcal{N}\times\mathcal{N}$ matrix R is orthogonal, $R^{\mathrm{T}}R = \mathbb{1}_{\mathcal{N}}$, and unimodular, $\det R = 1$.

Let us denote the two-point analog of the matrix (4.24b) as u_{12}. It is defined by

$$u_{12} = \mathbb{1}_{\mathcal{N}} + 2\mathrm{i}\hat{\theta}_{12}\hat{\boldsymbol{x}}_{12}^{-1}\hat{\theta}_{12}^{\mathrm{T}} \tag{4.25}$$

and has the properties

$$u_{12}^{\mathrm{T}}u_{12} = \mathbb{1}_{\mathcal{N}}\,, \qquad \det u_{12} = 1\;. \tag{4.26}$$

The sign difference in the right-hand sides of (4.24b) and (4.25) follows from the fact that $\hat{\boldsymbol{x}}_{12}$ has the symmetry property

$$\hat{\boldsymbol{x}}_{12} - \hat{\boldsymbol{x}}_{12}^{\mathrm{T}} = -2\mathrm{i}\hat{\theta}_{12}^{\mathrm{T}}\hat{\theta}_{12}\,, \tag{4.27}$$

which differs by sign from (2.25).

Here the inverse matrix $\hat{\boldsymbol{x}}_{12}^{-1}$ is expressed in terms of $\check{\boldsymbol{x}}_{12}$ as

$$\hat{\boldsymbol{x}}_{12}^{-1} = -\frac{\check{\boldsymbol{x}}_{12}^{\mathrm{T}}}{\boldsymbol{x}_{12}{}^{2}}\;. \tag{4.28}$$

With the use of (4.14) one can check that this matrix transforms as

$$\widetilde{\delta} u_{12}^{IJ} = \Lambda^{IK}(z_1)u_{12}^{KJ} - u_{12}^{IK}\Lambda^{KJ}(z_2) \tag{4.29a}$$

or, equivalently,

$$-(\xi_{z_1} + \xi_{z_2})u_{12}^{IJ} + \Lambda^{IK}(z_1)u_{12}^{KJ} - u_{12}^{IK}\Lambda^{KJ}(z_2) = 0\;. \tag{4.29b}$$

This shows that u_{12}^{IJ} is an invariant tensor two-point function of the superconformal group (compare with the transformation law (5.1) describing a primary superfield). Therefore this object will naturally appear in correlation functions of primary superfields with $\mathsf{O}(\mathcal{N})$ indices.

4.3 Three-point functions

Given three superspace points z_1, z_2 and z_3, we construct the following three-point functions

$$\check{\boldsymbol{X}}_1 = -\hat{\boldsymbol{x}}_{21}^{-1}\hat{\boldsymbol{x}}_{23}\hat{\boldsymbol{x}}_{13}^{-1}\,, \qquad \check{\Theta}_1 = \hat{\boldsymbol{x}}_{21}^{-1}\hat{\theta}_{12} - \hat{\boldsymbol{x}}_{31}^{-1}\hat{\theta}_{13}\,, \tag{4.30a}$$

$$\check{\boldsymbol{X}}_2 = -\hat{\boldsymbol{x}}_{32}^{-1}\hat{\boldsymbol{x}}_{31}\hat{\boldsymbol{x}}_{21}^{-1}\,, \qquad \check{\Theta}_2 = \hat{\boldsymbol{x}}_{32}^{-1}\hat{\theta}_{23} - \hat{\boldsymbol{x}}_{12}^{-1}\hat{\theta}_{21}\,, \tag{4.30b}$$

$$\check{\boldsymbol{X}}_3 = -\hat{\boldsymbol{x}}_{13}^{-1}\hat{\boldsymbol{x}}_{12}\hat{\boldsymbol{x}}_{32}^{-1}\,, \qquad \check{\Theta}_3 = \hat{\boldsymbol{x}}_{13}^{-1}\hat{\theta}_{31} - \hat{\boldsymbol{x}}_{23}^{-1}\hat{\theta}_{32}\,. \tag{4.30c}$$

The structures in (4.30b) and (4.30c) follow from (4.30a) by applying cyclic permutations of superspace points. This is why it suffices to study the properties of (4.30a).

With the use of (4.14) one can check that $\check{\boldsymbol{X}}_1$ and $\check{\Theta}_1$ transform as tensors at the superspace point z_1 and scalars at z_2 and z_3,

$$\tilde{\delta}\boldsymbol{X}_{1\alpha\beta} = \lambda_\alpha{}^\gamma(z_1)\boldsymbol{X}_{1\gamma\beta} + \boldsymbol{X}_{1\alpha\gamma}\lambda^\gamma{}_\beta(z_1) - \sigma(z_1)\boldsymbol{X}_{1\alpha\beta}\,, \tag{4.31a}$$

$$\tilde{\delta}\Theta_{1\alpha I} = \left(\lambda_\alpha{}^\beta(z_1) - \frac{1}{2}\delta_\alpha{}^\beta\sigma(z_1)\right)\Theta_{1\beta I} + \Lambda_{IJ}(z_1)\Theta_{1J\alpha}\,. \tag{4.31b}$$

Thus, they turn out to be essential building blocks for correlation functions of primary superfields.

Let us consider the squares of the structures in (4.30a)

$$\boldsymbol{X}_1{}^2 := -\frac{1}{2}\mathrm{tr}(\hat{\boldsymbol{X}}_1\check{\boldsymbol{X}}_1^{\mathrm{T}}) = \frac{\boldsymbol{x}_{23}{}^2}{\boldsymbol{x}_{12}{}^2\boldsymbol{x}_{13}{}^2}\,, \qquad \Theta_1^2 := \Theta_{1I}^\alpha\Theta_{1I\alpha}\,. \tag{4.32}$$

The variations of these objects involve only the parameter of local scale transformation

$$\tilde{\delta}\boldsymbol{X}_1{}^2 = -2\sigma(z_1)\boldsymbol{X}_1{}^2\,, \qquad \tilde{\delta}\Theta_1^2 = -\sigma(z_1)\Theta_1^2\,. \tag{4.33}$$

As a consequence, the combination [50]

$$\frac{\Theta_1^2}{\sqrt{\boldsymbol{X}_1{}^2}} \tag{4.34}$$

is a superconformal invariant and the superconformal symmetry can fix the form of correlation functions only up to this combination.

The two-point function (4.13a) has the following distributive property

$$\boldsymbol{x}_{23}^{\alpha\beta} = \boldsymbol{x}_{21}^{\alpha\beta} + \boldsymbol{x}_{13}^{\alpha\beta} - 2\mathrm{i}\theta_{21}^{I\alpha}\theta_{13}^{I\beta}\,. \tag{4.35}$$

As a consequence, the three-point functions (4.30a) obey

$$\varepsilon^{\alpha\beta}\boldsymbol{X}_{1\alpha\beta} = \mathrm{i}\Theta_1^2\,. \tag{4.36}$$

Hence, similar to the two-point function (4.21), the decomposition of the three-point function $\boldsymbol{X}_1^{\alpha\beta}$ into symmetric and antisymmetric parts reads

$$\boldsymbol{X}_{1\alpha\beta} = X_{1\alpha\beta} - \frac{\mathrm{i}}{2}\varepsilon_{\alpha\beta}\Theta_1^2\,, \qquad X_{1\alpha\beta} = X_{1\beta\alpha}\,. \tag{4.37}$$

Given the symmetric object $X_{1\alpha\beta}$ we construct a vector by the standard rule, $X_{1m} = -\frac{1}{2}\gamma_m^{\alpha\beta} X_{1\alpha\beta}$, and introduce analogs of the covariant spinor derivative (2.39) and the corresponding supercharge operator

$$\mathcal{D}^I_{(1)\alpha} = \frac{\partial}{\partial \Theta^\alpha_{1I}} + \mathrm{i}\gamma^m_{\alpha\beta}\Theta_1^{I\beta}\frac{\partial}{\partial X_1^m}\,, \qquad \mathcal{Q}^I_{(1)\alpha} = \mathrm{i}\frac{\partial}{\partial \Theta^\alpha_{1I}} + \gamma^m_{\alpha\beta}\Theta_1^{I\beta}\frac{\partial}{\partial X_1^m}\,. \tag{4.38}$$

They obey standard anticommutation relations

$$\{\mathcal{D}^I_\alpha, \mathcal{D}^J_\beta\} = \{\mathcal{Q}^I_\alpha, \mathcal{Q}^J_\beta\} = 2\mathrm{i}\delta^{IJ}\gamma^m_{\alpha\beta}\frac{\partial}{\partial X^m}\,, \tag{4.39}$$

where we omit the subscript (1) labeling the superspace point.

There are various identities that involve the three-point functions $\boldsymbol{X}_{i\,\alpha\beta}$ and $\Theta_{i\,\alpha}$ at different points, e.g.,

$$\boldsymbol{x}_{13}^{\alpha\alpha'}\boldsymbol{X}_{3\alpha'\beta'}\boldsymbol{x}_{31}^{\beta'\beta} = -(\boldsymbol{X}_1^{-1})^{\beta\alpha} = \frac{\boldsymbol{X}_1^{\alpha\beta}}{\boldsymbol{X}_1{}^2}\,, \qquad \Theta^I_{1\gamma}\boldsymbol{x}_{13}^{\gamma\delta}\boldsymbol{X}_{3\delta\beta} = u_{13}^{IJ}\Theta^J_{3\beta}\,. \tag{4.40}$$

These relations allow us to prove the following properties of three-point functions (4.30c)

$$\begin{aligned} D^I_{(1)\alpha}\Theta^J_{3\beta} &= -\boldsymbol{x}^{-1}_{13\beta\alpha}u_{13}^{IJ}\,, \qquad & D^I_{(1)\gamma}\boldsymbol{X}_{3\alpha\beta} &= 2\mathrm{i}\boldsymbol{x}^{-1}_{13\alpha\gamma}u_{13}^{IJ}\Theta^J_{3\beta}\,,\\ D^I_{(2)\alpha}\Theta^J_{3\beta} &= \boldsymbol{x}^{-1}_{23\beta\alpha}u_{23}^{IJ}\,, \qquad & D^I_{(2)\gamma}\boldsymbol{X}_{3\alpha\beta} &= 2\mathrm{i}\boldsymbol{x}^{-1}_{23\beta\gamma}u_{23}^{IJ}\Theta^J_{3\alpha}\,. \end{aligned} \tag{4.41}$$

Here $D^I_{(i)\alpha}$ is the conventional covariant spinor derivative (2.39) which acts on the superspace coordinates at the point z_i.

Given a function $f(\boldsymbol{X}_3, \Theta_3)$ depending on the objects (4.30c) one can prove the following differential identities

$$D^I_{(1)\gamma}f(\boldsymbol{X}_3,\Theta_3) = \boldsymbol{x}^{-1}_{13\alpha\gamma}u_{13}^{IJ}\mathcal{D}^{J\alpha}_{(3)}f(\boldsymbol{X}_3,\Theta_3)\,, \tag{4.42a}$$

$$D^I_{(2)\gamma}f(\boldsymbol{X}_3,\Theta_3) = \mathrm{i}\boldsymbol{x}^{-1}_{23\alpha\gamma}u_{23}^{IJ}\mathcal{Q}^{J\alpha}_{(3)}f(\boldsymbol{X}_3,\Theta_3)\,. \tag{4.42b}$$

Note that on the left of these identities there are standard covariant spinor derivatives (2.39) while on the right there are generalized derivative and supercharge given in (4.38). The above properties (4.42) will be important in the next sections.

Using the relations (4.40) one can also check that the object (4.34) is invariant under permutations of superspace points,

$$\frac{\Theta_1^2}{\sqrt{\boldsymbol{X}_1{}^2}} = \frac{\Theta_2^2}{\sqrt{\boldsymbol{X}_2{}^2}} = \frac{\Theta_3^2}{\sqrt{\boldsymbol{X}_3{}^2}}\,. \tag{4.43}$$

Finally, we introduce the following three-point functions

$$U_1^{IJ} = u_{12}^{IK}u_{23}^{KL}u_{31}^{LJ}\,, \qquad U_2^{IJ} = u_{23}^{IK}u_{31}^{KL}u_{12}^{LJ}\,, \qquad U_3^{IJ} = u_{31}^{IK}u_{12}^{KL}u_{23}^{LJ}\,, \tag{4.44}$$

which have simple transformation properties. One may see that U_1^{IJ} transforms as an $\mathsf{O}(\mathcal{N})$ tensor at superspace point z_1

$$\tilde{\delta}U_1^{IJ} = \Lambda^{IK}(z_1)U_1^{KJ} - U_1^{IK}\Lambda^{KJ}(z_1)\,. \tag{4.45}$$

JHEP06(2015)138

By construction, the matrix U_1 is orthogonal, $U_1^{\rm T} U_1 = \mathbb{1}_{\mathcal{N}}$, and unimodular, $\det U_1 = 1$. It can be expressed in terms of the three-point functions (4.30c)

$$U_1^{IJ} = \delta^{IJ} + 2{\rm i}\Theta^I_{1\alpha}(\boldsymbol{X}_1^{-1})^{\alpha\beta}\Theta^J_{1\beta} = \delta^{IJ} - 2{\rm i}\frac{\Theta^I_{1\alpha}\boldsymbol{X}_1^{\beta\alpha}\Theta^J_{1\beta}}{\boldsymbol{X}_1{}^2} . \tag{4.46}$$

Analogous results hold for U_2 and U_3.

The matrices U_2 and U_3 are related to U_1 as

$$U_2^{IJ} = u_{21}^{IK} U_1^{KL} u_{12}^{LJ} , \qquad U_3^{IJ} = u_{31}^{IK} U_1^{KL} u_{13}^{LJ} . \tag{4.47}$$

These properties will be useful in checking the invariance under permutations of superspace points of correlation functions of superfields with $\mathsf{O}(\mathcal{N})$ indices.

5 Correlation functions of primary superfields

Consider a superfield $\Phi^{\mathcal{I}}_{\mathcal{A}}(z)$ that transforms in a representation T of the Lorentz group with respect to its index $\mathcal{A}$ and in a representation D of the R-symmetry group $\mathsf{O}(\mathcal{N})$ with respect to the index $\mathcal{I}$. Such a superfield is called primary of dimension q if its superconformal transformation law is

$$\delta\Phi^{\mathcal{I}}_{\mathcal{A}} = -\xi\Phi^{\mathcal{I}}_{\mathcal{A}} - q\sigma(z)\Phi^{\mathcal{I}}_{\mathcal{A}} + \lambda^{\alpha\beta}(z)(M_{\alpha\beta})_{\mathcal{A}}{}^{\mathcal{B}}\Phi^{\mathcal{I}}_{\mathcal{B}} + \Lambda_{IJ}(z)(R^{IJ})^{\mathcal{I}}{}_{\mathcal{J}}\Phi^{\mathcal{J}}_{\mathcal{A}} . \tag{5.1}$$

Here ξ is the conformal Killing supervector (4.3), and the z-dependent parameters $\sigma(z)$, $\lambda^{\alpha\beta}(z)$ and $\Lambda_{IJ}(z)$ associated with ξ are given in (4.7). The matrices $M_{\alpha\beta}$ and R^{IJ} are the Lorentz and $\mathsf{O}(\mathcal{N})$ generators, respectively.

In the non-supersymmetric case, the formalism to construct the correlation functions of primary fields in conformal field theories in diverse dimensions was developed in [11] (see also [16]). In four dimensions, this approach was generalised to $\mathcal{N} = 1$ superconformal field theories formulated in superspace in [30] (see also [56]) as well as to higher $\mathcal{N}$ [51]. The correlation functions of primary superfields in three and six dimensions were studied in [50] and [52], respectively. Here we briefly review the 3D formalism of [50] as it will be employed further for constructing correlation functions of conserved current multiplets in 3D superconformal field theories.

The two-point correlation function of the primary superfield $\Phi^{\mathcal{I}}_{\mathcal{A}}$ and its conjugate $\bar{\Phi}^{\mathcal{A}}_{\mathcal{I}}$ is fixed by the superconformal symmetry up to a single coefficient c and has the form

$$\langle \Phi^{\mathcal{I}}_{\mathcal{A}}(z_1)\bar{\Phi}^{\mathcal{B}}_{\mathcal{J}}(z_2)\rangle = c\frac{T_{\mathcal{A}}{}^{\mathcal{B}}(\varepsilon\hat{\boldsymbol{x}}_{12})D^{\mathcal{I}}{}_{\mathcal{J}}(u_{12})}{(\boldsymbol{x}_{12}{}^2)^q} \tag{5.2}$$

provided the representations T and D are irreducible. The two-point functions $\boldsymbol{x}_{12}{}^2$, $\hat{\boldsymbol{x}}_{12}$ and u_{12} are defined in eqs. (4.18) and (4.25), respectively, and $\varepsilon = (\varepsilon_{\alpha\beta})$. The denominator in (5.2) is fixed by the dimension of Φ.

Let Φ, Ψ and Π be primary superfields (with indices suppressed) of dimensions q_1, q_2 and q_3, respectively. The three-point correlation function for these superfields can be found with the use of the ansatz

$$\langle \Phi^{\mathcal{I}_1}_{\mathcal{A}_1}(z_1)\Psi^{\mathcal{I}_2}_{\mathcal{A}_2}(z_2)\Pi^{\mathcal{I}_3}_{\mathcal{A}_3}(z_3)\rangle = \frac{T^{(1)}{}_{\mathcal{A}_1}{}^{\mathcal{B}_1}(\varepsilon\hat{\underline{\boldsymbol{x}}}_{13})T^{(2)}{}_{\mathcal{A}_2}{}^{\mathcal{B}_2}(\varepsilon\hat{\underline{\boldsymbol{x}}}_{23})D^{(1)\mathcal{I}_1}{}_{\mathcal{J}_1}(u_{13})D^{(2)\mathcal{I}_2}{}_{\mathcal{J}_2}(u_{23})}{(\boldsymbol{x}_{13}{}^2)^{q_1}(\boldsymbol{x}_{23}{}^2)^{q_2}} \times H^{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}_{\mathcal{B}_1\mathcal{B}_2\mathcal{A}_3}(\boldsymbol{X}_3,\Theta_3,U_3)\,, \tag{5.3}$$

where $H^{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}_{\mathcal{B}_1\mathcal{B}_2\mathcal{A}_3}$ is a tensor constructed in terms of the three-point functions (4.30) and (4.44). The functional form of this tensor is highly constrained by the following conditions:

(i) It should obey the scaling property

$$H^{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}_{\mathcal{B}_1\mathcal{B}_2\mathcal{A}_3}(\lambda^2\boldsymbol{X},\lambda\Theta,U) = (\lambda^2)^{q_3-q_2-q_1}H^{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}_{\mathcal{B}_1\mathcal{B}_2\mathcal{A}_3}(\boldsymbol{X},\Theta,U)\,, \qquad \forall\lambda\in\mathbb{R}\setminus\{0\} \tag{5.4}$$

in order for the correlation function to have the correct transformation law under the superconformal group.

(ii) When some of the superfields Φ, Ψ and Π obey differential equations such as the conservation conditions of conserved current multiplets, the tensor $H^{\mathcal{J}_1\mathcal{J}_2\mathcal{I}_3}_{\mathcal{B}_1\mathcal{B}_2\mathcal{A}_3}$ is constrained by certain differential equations as well. In deriving such equations the identities (4.42) may be useful.

(iii) When two of the superfields Φ, Ψ and Π (or all of them) coincide, the tensor H should obey certain constraints originating from the symmetry under permutations of superspace points, e.g.

$$\langle \Phi^{\mathcal{A}}_{\mathcal{I}}(z_1)\Phi^{\mathcal{B}}_{\mathcal{J}}(z_2)\Pi^{\mathcal{C}}_{\mathcal{K}}(z_3)\rangle = (-1)^{\epsilon(\Phi)}\langle \Phi^{\mathcal{B}}_{\mathcal{J}}(z_2)\Phi^{\mathcal{A}}_{\mathcal{I}}(z_1)\Pi^{\mathcal{C}}_{\mathcal{K}}(z_3)\rangle\,, \tag{5.5}$$

where $\epsilon(\Phi)$ is the Grassmann parity of $\Phi^{\mathcal{A}}_{\mathcal{I}}$.

These constraints fix the functional form of the tensor H (and, hence, the three-point correlation function) up to a few arbitrary constants.

The procedure described reduces the problem of computing thee-point correlation functions to deriving the single function H subject to the above mentioned constraints. In the next sections we will apply this procedure to compute the two- and three-point correlation functions of the supercurrents and the flavour current multilpets in superconformal field theories with $1 \le \mathcal{N} \le 3$.

6 Correlators in $\mathcal{N} = 1$ superconformal field theory

To start with, we give an example of a classically $\mathcal{N} = 1$ superconformal field theory. It is described by n primary real scalar superfields $\vec{\varphi}$ of dimension 1/2 with action

$$S = \int \mathrm{d}^3x\mathrm{d}^2\theta\left\{\frac{1}{2}D^\alpha\vec{\varphi}\cdot D_\alpha\vec{\varphi} + \frac{\mathrm{i}}{4}\lambda(\vec{\varphi}\cdot\vec{\varphi})^2\right\}, \tag{6.1}$$

with λ a coupling constant. This action is invariant under the superconformal transformation

$$\delta\vec{\varphi} = -\xi\vec{\varphi} - \frac{1}{2}\sigma(z)\vec{\varphi}\ . \tag{6.2}$$

The supercurrent of this model [67] is

$$J_{\alpha\beta\gamma} = \mathrm{i}\Big(\vec{\varphi}\cdot D_{(\alpha}\partial_{\beta\gamma)}\vec{\varphi} - 3D_{(\alpha}\vec{\varphi}\cdot\partial_{\beta\gamma)}\vec{\varphi}\Big)\ . \tag{6.3}$$

The flavour current multiplet reads

$$J_\alpha^{\bar{a}} = \mathrm{i}\Big(\vec{\varphi}\cdot\Sigma^{\bar{a}}D_\alpha\vec{\varphi} - D_\alpha\vec{\varphi}\cdot\Sigma^{\bar{a}}\vec{\varphi}\Big)\,, \tag{6.4}$$

where $\Sigma^{\bar{a}}$ denotes the generator of the flavour $\mathsf{O}(n)$ group. One may check that the currents (6.3) and (6.4) transform as primary superfields under the superconformal group and obey the corresponding conservation laws given in (1.1) and (1.4) on the equations of motion for $\vec{\varphi}$.

A natural generalisation of (6.1) is the most general off-shell 3D $\mathcal{N}=1$ superconformal sigma model given in [53].[11]

6.1 $\mathcal{N}=1$ flavour current multiplets

In $\mathcal{N}=1$ supersymmetric field theory, the flavour current multiplet is described by a primary real spinor superfield L_α of dimension 3/2 (with its flavour index suppressed) which transforms under superconformal group as

$$\delta L_\alpha = -\xi L_\alpha - \frac{3}{2}\sigma(z)L_\alpha + \lambda_\alpha{}^\beta(z)L_\beta \tag{6.5}$$

and obeys the conservation equation

$$D^\alpha L_\alpha = 0\ . \tag{6.6}$$

Let us assume that the superconformal field theory under study has several flavour current multiplets $L^{\bar{a}}_\alpha$, with $\bar{a}$ the flavour index. According to the general formula (5.2), the two-point function of such operators is fixed up to one real coefficient $a_{\mathcal{N}=1}$,

$$\langle L^{\bar{a}}_\alpha(z_1)L^{\bar{b}}_\beta(z_2)\rangle = \mathrm{i}a_{\mathcal{N}=1}\frac{\delta^{\bar{a}\bar{b}}\boldsymbol{x}_{12\alpha\beta}}{(\boldsymbol{x}_{12}{}^2)^2}\,, \tag{6.7}$$

assuming that the flavour group is simple. With the relation (4.20) it is easy to see that (6.7) obeys the right symmetry property under the permutation of superspace points, $\langle L^{\bar{a}}_\alpha(z_1)L^{\bar{b}}_\beta(z_2)\rangle = -\langle L^{\bar{b}}_\beta(z_2)L^{\bar{a}}_\alpha(z_1)\rangle$. Next, using the explicit expression for $\boldsymbol{x}_{12\alpha\beta}$ given in (4.13a), one may check that (6.7) respects the conservation condition (6.6)

$$D^\alpha_{(1)}\langle L^{\bar{a}}_\alpha(z_1)L^{\bar{b}}_\beta(z_2)\rangle = 0\,, \qquad z_1\neq z_2\ . \tag{6.8}$$

Consider now the three-point correlation function $\langle L^{\bar{a}}_\alpha(z_1)L^{\bar{b}}_\beta(z_2)L^{\bar{c}}_\gamma(z_3)\rangle$. Since the superspace coordinates do not carry any flavour group indices, the dependence of the correlation function on $\bar{a}$, $\bar{b}$ and $\bar{c}$ should factorise in the form of an invariant tensor of the flavour group, which is completely antisymmetric, $f^{\bar{a}\bar{b}\bar{c}} = f^{[\bar{a}\bar{b}\bar{c}]}$, or completely symmetric,

[11] On-shell superconformal sigma models in three dimensions were proposed in [68–70].

$d^{\bar{a}\bar{b}\bar{c}} = d^{(\bar{a}\bar{b}\bar{c})}$. These tensors are defined in terms of the generators $\Sigma^{\bar{a}}$ of the flavour group as follows

$$[\Sigma^{\bar{a}}, \Sigma^{\bar{b}}] = \mathrm{i} f^{\bar{a}\bar{b}\bar{c}} \Sigma^{\bar{c}}, \qquad d^{\bar{a}\bar{b}\bar{c}} = \frac{1}{2} \mathrm{tr}(\{\Sigma^{\bar{a}}, \Sigma^{\bar{b}}\} \Sigma^{\bar{c}}) . \tag{6.9}$$

In principle, the correlator may be a sum of two terms, one of which is proportional to $f^{\bar{a}\bar{b}\bar{c}}$ and the other to $d^{\bar{a}\bar{b}\bar{c}}$. In four dimensions, contributions with $d^{\bar{a}\bar{b}\bar{c}}$ arise as a consequence of anomalies. In three dimensions, gauge theories are anomaly-free. Therefore, it is natural to expect that the part of $\langle L^{\bar{a}}_\alpha(z_1) L^{\bar{b}}_\beta(z_2) L^{\bar{c}}_\gamma(z_3)\rangle$ with $d^{\bar{a}\bar{b}\bar{c}}$ should vanish as it was observed in the non-supersymmetric case studied in [11]. Nevertheless, here we start by considering the most general expression for the correlation function including contributions of both types, with $f^{\bar{a}\bar{b}\bar{c}}$ and $d^{\bar{a}\bar{b}\bar{c}}$, and then show that the latter vanishes upon imposing all the relevant constrains.

According to the general formula (5.3), we have to look for the three-point correlator in the form

$$\langle L^{\bar{a}}_\alpha(z_1) L^{\bar{b}}_\beta(z_2) L^{\bar{c}}_\gamma(z_3)\rangle = \frac{\boldsymbol{x}_{13\alpha\alpha'} \boldsymbol{x}_{23\beta\beta'}}{(\boldsymbol{x}_{13}{}^2)^2 (\boldsymbol{x}_{23}{}^2)^2} \Big(f^{\bar{a}\bar{b}\bar{c}} H_{(f)}^{\alpha'\beta'}{}_\gamma(\boldsymbol{X}_3, \Theta_3) + d^{\bar{a}\bar{b}\bar{c}} H_{(d)}^{\alpha'\beta'}{}_\gamma(\boldsymbol{X}_3, \Theta_3) \Big) , \tag{6.10}$$

where the tensors $H^{\alpha\beta\gamma}_{(f,d)}$ should obey the following scaling property:

$$H^{\alpha\beta\gamma}_{(f,d)}(\lambda^2 \boldsymbol{X}, \lambda\Theta) = \lambda^{-3} H^{\alpha\beta\gamma}_{(f,d)}(\boldsymbol{X}, \Theta) . \tag{6.11}$$

Recall that the superfield $L^{\bar{a}}_\alpha$ is Grassmann odd. Hence, the correlator (6.10) changes its sign when we interchange any pair of superfields in it, e.g.

$$\langle L^{\bar{b}}_\beta(z_2) L^{\bar{a}}_\alpha(z_1) L^{\bar{c}}_\gamma(z_3)\rangle = -\langle L^{\bar{a}}_\alpha(z_1) L^{\bar{b}}_\beta(z_2) L^{\bar{c}}_\gamma(z_3)\rangle . \tag{6.12}$$

This equation imposes the following constraint on the tensors $H^{\alpha\beta\gamma}_{(f,d)}$:

$$H^{\beta\alpha\gamma}_{(f)}(-\boldsymbol{X}^{\mathrm{T}}, -\Theta) = H^{\alpha\beta\gamma}_{(f)}(\boldsymbol{X}, \Theta) , \qquad H^{\beta\alpha\gamma}_{(d)}(-\boldsymbol{X}^{\mathrm{T}}, -\Theta) = -H^{\alpha\beta\gamma}_{(d)}(\boldsymbol{X}, \Theta) . \tag{6.13}$$

The most general expressions for these tensors subject to the constraints (6.11) and (6.13) read

$$H^{\alpha\beta\gamma}_{(f)} = \mathrm{i} \sum_n c_n H^{\alpha\beta\gamma}_{(f)n} , \qquad H^{\alpha\beta\gamma}_{(d)} = \mathrm{i} \sum_n d_n H^{\alpha\beta\gamma}_{(d)n} , \tag{6.14}$$

where c_n and d_n are some real coefficients and

$$H^{\alpha\beta\gamma}_{(f)1} = \frac{\varepsilon^{\alpha\beta}\Theta^\gamma}{\boldsymbol{X}^2} , \qquad H^{\alpha\beta\gamma}_{(f)2} = \frac{\boldsymbol{X}^{\alpha\beta}\Theta^\gamma}{\boldsymbol{X}^3} , \qquad H^{\alpha\beta\gamma}_{(f)3} = \frac{\varepsilon^{\beta\gamma}\boldsymbol{X}^{\alpha\mu}\Theta_\mu + \varepsilon^{\alpha\gamma}\boldsymbol{X}^{\beta\mu}\Theta_\mu}{\boldsymbol{X}^3} ; \tag{6.15a}$$

$$H^{\alpha\beta\gamma}_{(d)1} = \frac{\boldsymbol{X}^{\alpha\beta}\boldsymbol{X}^{\gamma\delta}\Theta_\delta}{\boldsymbol{X}^4} , \qquad H^{\alpha\beta\gamma}_{(d)2} = \frac{\varepsilon^{\alpha\gamma}\Theta^\beta + \varepsilon^{\beta\gamma}\Theta^\alpha}{\boldsymbol{X}^2} , \qquad H^{\alpha\beta\gamma}_{(d)3} = \frac{\varepsilon^{\alpha\beta}\boldsymbol{X}^{\gamma\delta}\Theta_\delta}{\boldsymbol{X}^3} . \tag{6.15b}$$

Recall that we use the notation in which $\boldsymbol{X}^2 = -\frac{1}{2}\boldsymbol{X}^{\alpha\beta}\boldsymbol{X}_{\alpha\beta}$ and $\boldsymbol{X}^k \equiv (\boldsymbol{X}^2)^{k/2}$.

Note that there is no need to add one more addmisible structure $\frac{1}{\boldsymbol{X}^3}(\boldsymbol{X}^{\alpha\gamma}\Theta^\beta + \boldsymbol{X}^{\beta\gamma}\Theta^\alpha)$ to the list (6.15a), since it is linearly dependent of the others,

$$\boldsymbol{X}^{\alpha\gamma}\Theta^\beta + \boldsymbol{X}^{\beta\gamma}\Theta^\alpha = 2\boldsymbol{X}^{\alpha\beta}\Theta^\gamma + \varepsilon^{\beta\gamma}\boldsymbol{X}^{\alpha\mu}\Theta_\mu + \varepsilon^{\alpha\gamma}\boldsymbol{X}^{\beta\mu}\Theta_\mu . \tag{6.16}$$

JHEP06(2015)138

To fix the values of the coefficients c_n and d_n in (6.14) we have to take into account the conservation condition (6.6),

$$D^{\alpha}_{(1)}\langle L^{\bar a}_{\alpha}(z_1)L^{\bar b}_{\beta}(z_2)L^{\bar c}_{\gamma}(z_3)\rangle = 0\ . \tag{6.17}$$

Making use of (4.42a), this equation imposes the following constraint on the tensors $H^{\alpha\beta\gamma}_{(f,d)}$:

$$\mathcal{D}_\alpha H^{\alpha\beta\gamma}_{(f,d)} = 0\ . \tag{6.18}$$

Here $\mathcal{D}_\alpha$ is the generalized covariant spinor derivative defined in (4.38). The equation (6.18) leads to the following constraints on the coefficients c_n and d_n:

$$c_1 = 0\,,\quad c_2 + c_3 = 0\ ;\qquad d_1 = 3d_2\,,\quad d_3 = 0\ . \tag{6.19}$$

To find further constrains on the coefficients, we recall that the correlation function changes its sign if we swap any two superfields in it, e.g.,

$$\langle L^{\bar a}_{\alpha}(z_1)L^{\bar b}_{\beta}(z_2)L^{\bar c}_{\gamma}(z_3)\rangle = -\langle L^{\bar c}_{\gamma}(z_3)L^{\bar b}_{\beta}(z_2)L^{\bar a}_{\alpha}(z_1)\rangle\ . \tag{6.20}$$

Using the identities (4.40), we find the following corollaries of (6.20):

$$H^{\alpha\beta\gamma}_{(f,d)}(\boldsymbol{X}_3,\Theta_3) = \pm\frac{\boldsymbol{X}_3^{\rho\beta}\boldsymbol{x}_{32\rho\rho'}\boldsymbol{x}_{31}{}^{\gamma}{}_{\gamma'}(\boldsymbol{x}_{13}^{-1})^{\alpha\alpha'}H^{\gamma'\rho'}_{(f,d)\alpha'}(-\boldsymbol{X}_1^{\mathrm{T}},-\Theta_1)}{\boldsymbol{X}_3{}^4\boldsymbol{x}_{13}{}^4}\,, \tag{6.21}$$

where the right-hand side should be taken with the plus sign for $H_{(f)}$ and with minus for $H_{(d)}$. The constraints (6.21) are satisfied under the conditions

$$c_1 = 0\,,\quad c_2 + c_3 = 0\ ;\qquad d_1 = -d_2\,,\quad d_3 = 0\ . \tag{6.22}$$

Thus, from (6.19) and (6.22) we see that all d-coefficients vanish, $d_n = 0$, and therefore $H^{\alpha\beta\gamma}_{(d)} = 0$. Furthermore, only one independent coefficient remains among c_n, which we denote by $b_{\mathcal{N}=1} = c_2 = -c_3$. Our final expression for the correlator (6.10) is

$$\langle L^{\bar a}_{\alpha}(z_1)L^{\bar b}_{\beta}(z_2)L^{\bar c}_{\gamma}(z_3)\rangle = f^{\bar a\bar b\bar c}\frac{\boldsymbol{x}_{13\alpha\alpha'}\boldsymbol{x}_{23\beta\beta'}}{(\boldsymbol{x}_{13}{}^2)^2(\boldsymbol{x}_{23}{}^2)^2}H^{\alpha'\beta'}_{(f)}{}_{\gamma}(\boldsymbol{X}_3,\Theta_3)\,, \tag{6.23a}$$

$$H^{\alpha\beta\gamma}_{(f)}(\boldsymbol{X},\Theta) = \frac{\mathrm{i}b_{\mathcal{N}=1}}{\boldsymbol{X}^3}(\boldsymbol{X}^{\alpha\beta}\Theta^{\gamma} - \varepsilon^{\beta\gamma}\boldsymbol{X}^{\alpha\delta}\Theta_\delta - \varepsilon^{\alpha\gamma}\boldsymbol{X}^{\beta\delta}\Theta_\delta)\ . \tag{6.23b}$$

The superfield operator $L^{\bar a}_{\alpha}(z)$ contains an ordinary conserved flavour current $L^{\bar a}_m(x)$ as its linear in θ component,

$$L^{\bar a}_{\alpha\beta} = D_\alpha L^{\bar a}_{\beta}|\,,\qquad L^{\bar a}_{\alpha\beta} = \gamma^m_{\alpha\beta}L^{\bar a}_m\,,\qquad \partial^m L^{\bar a}_m = 0\,, \tag{6.24}$$

where $|$ indicates that we have to set $\theta = 0$. From (6.23) we can extract the three-point function $\langle L^{\bar a}_{\alpha\alpha'}(x_1)L^{\bar b}_{\beta\beta'}(x_2)L^{\bar c}_{\gamma\gamma'}(x_3)\rangle$ by the rule:

$$\langle L^{\bar a}_{\alpha\alpha'}(x_1)L^{\bar b}_{\beta\beta'}(x_2)L^{\bar c}_{\gamma\gamma'}(x_3)\rangle = -D_{(1)\alpha}D_{(2)\beta}D_{(3)\gamma}\langle L^{\bar a}_{\alpha'}(z_1)L^{\bar b}_{\beta'}(z_2)L^{\bar c}_{\gamma'}(z_3)\rangle|\ . \tag{6.25}$$

It is instructive to compare the flavour current correlation function (6.23) with the corresponding non-supersymmetric expression found in [11]. After a straightforward but lengthy calculation (see appendix D for the technical details) we find

$$\langle L^{\bar a}_m(x_1) L^{\bar b}_n(x_2) L^{\bar c}_k(x_3)\rangle = \frac{f^{\bar a\bar b\bar c}}{{x_{12}}^2 {x_{23}}^2 {x_{13}}^2} I_{mm'}(x_{13}) I_{nn'}(x_{23}) t^{m'n'}{}_k(X_3)\ . \tag{6.26}$$

Here we have defined

$$I_{mn}(x) = \eta_{mn} - \frac{2x_m x_n}{x^2}\ , \tag{6.27a}$$

$$X_3^m = \frac{x_{13}^m}{{x_{13}}^2} - \frac{x_{23}^m}{{x_{23}}^2}\ , \tag{6.27b}$$

$$t_{mnk}(X) = b_1 \frac{X_m X_n X_k}{X^3} + b_2 \frac{X_m \eta_{nk} + X_n \eta_{mk} - X_k \eta_{mn}}{X}\ . \tag{6.27c}$$

According to (D.17), the coefficients b_1 and b_2 are given in terms of $b_{\mathcal{N}=1}$ as follows

$$b_1 = b_2 = 3 b_{\mathcal{N}=1}\ . \tag{6.28}$$

This is the same result as the one obtained in [11] except for the fact that the two coefficients b_1 and b_2, which were completely independent in the non-supersymmetric case, are now equal to each other due to supersymmetry.

As pointed out in [13], in 3D conformal field theories an additional *parity violating*[12] structure can arise in the three-point correlator of flavour currents,

$$\tilde t_{mnk}(X) = b_3 \frac{-\varepsilon_{nkp} X_m X^p + \varepsilon_{mkp} X_n X^p + \varepsilon_{mnp} X_k X^p}{X^2}\ . \tag{6.29}$$

However, this structure does not appear upon the $\mathcal{N}=1 \to \mathcal{N}=0$ reduction of our result (6.23) and, hence, it is not consistent with supersymmetry. The same conclusion holds in all cases considered below in this paper. Specifically, the correlators of both flavour current multiplets and supercurrents contain only parity even contributions.[13]

6.2 $\mathcal{N}=1$ supercurrent

The $\mathcal{N}=1$ supercurrent is described by a primary symmetric third-rank spinor $J_{\alpha\beta\gamma} = J_{(\alpha\beta\gamma)}$ of dimension 5/2, which obeys the conservation law

$$D^\alpha J_{\alpha\beta\gamma} = 0\ . \tag{6.30}$$

This conservation equation is invariant under the superconformal transformation of $J_{\alpha\beta\gamma}$, which is

$$\delta J_{\alpha\beta\gamma} = -\xi J_{\alpha\beta\gamma} - \frac{5}{2}\sigma(z) J_{\alpha\beta\gamma} + 3\lambda^\delta{}_{(\alpha}(z) J_{\beta\gamma)\delta}\ . \tag{6.31}$$

[12] The parity transformation in question is $x^m \to -x^m$. The correlator of three flavour currents acquires a minus sign under this transformation.

[13] One way to check whether a given contribution is even or odd under parity is to reduce it to the $\mathcal{N}=0$ case to see whether or not it contains an ε_{mnp} tensor. This is easy to see from the general structure of the supersymmetric result without performing the reduction in detail. We will not discuss details of the reduction of our results to $\mathcal{N}=0$ in other sections of this paper.

JHEP06(2015)138

According to the general discussion in section 5, the two-point function of the supercurrent is given by

$$\langle J_{\alpha\beta\gamma}(z_1) J^{\alpha'\beta'\gamma'}(z_2)\rangle = \mathrm{i}c_{\mathcal{N}=1} \frac{\boldsymbol{x}_{12\alpha}{}^{(\alpha'} \boldsymbol{x}_{12\beta}{}^{\beta'} \boldsymbol{x}_{12\gamma}{}^{\gamma')}}{(\boldsymbol{x}_{12}{}^2)^4} \ . \tag{6.32}$$

It is easy to show that the two-point function (6.32) has the right symmetry property under the change of superspace points, $\langle J_{\alpha\beta\gamma}(z_1) J_{\alpha'\beta'\gamma'}(z_2)\rangle = \langle J_{\alpha'\beta'\gamma'}(z_2) J_{\alpha\beta\gamma}(z_1)\rangle$, and satisfies

$$D^\alpha_{(1)} \langle J_{\alpha\beta\gamma}(z_1) J_{\alpha'\beta'\gamma'}(z_2)\rangle = 0\,, \qquad z_1 \neq z_2 \ . \tag{6.33}$$

Similarly, we can write the most general form for the three-point function that is consistent with the superconformal symmetry. Let us denote by $\mathcal{A} = (\alpha, \alpha', \alpha'')$ a symmetric combination of the three spinor indices, $J_\mathcal{A} \equiv J_{\alpha\alpha'\alpha''}$. Then

$$\langle J_\mathcal{A}(z_1) J_\mathcal{B}(z_2) J_\mathcal{C}(z_3)\rangle = \frac{T_\mathcal{A}{}^\mathcal{R}(\boldsymbol{x}_{13}) T_\mathcal{B}{}^\mathcal{S}(\boldsymbol{x}_{23})}{(\boldsymbol{x}_{13}{}^2)^4 (\boldsymbol{x}_{23}{}^2)^4} H_{\mathcal{RSC}}(\boldsymbol{X}_3, \Theta_3) \ . \tag{6.34}$$

Explicitly,

$$T_\mathcal{A}{}^\mathcal{R}(\boldsymbol{x}) = \boldsymbol{x}_{(\alpha}{}^{\rho} \boldsymbol{x}_{\alpha'}{}^{\rho'} \boldsymbol{x}_{\alpha'')}{}^{\rho''} \tag{6.35}$$

and the function[14] $H_{\mathcal{ABC}}(\boldsymbol{X}, \Theta) \equiv H_{(\alpha\alpha'\alpha''),(\beta\beta'\beta''),(\gamma\gamma'\gamma'')}(\boldsymbol{X}, \Theta)$ should satisfy the scaling property

$$H_{\mathcal{ABC}}(\lambda^2 \boldsymbol{X}, \lambda\Theta) = \lambda^{-5} H_{\mathcal{ABC}}(\boldsymbol{X}, \Theta) \ . \tag{6.36}$$

If we exchange the first and the second superspace points $z_1 \leftrightarrow z_2$ it follows that $\boldsymbol{X}_3 \to -\boldsymbol{X}_3^{\mathrm{T}}$, $\Theta_3 \to -\Theta_3$. Since the supercurrent $J_{\alpha\beta\gamma}$ is Grassmann odd the correlation function (6.34) has to change the sign under $z_1 \leftrightarrow z_2$, $\mathcal{A} \leftrightarrow \mathcal{B}$. It implies that the function $H_{\mathcal{ABC}}(\boldsymbol{X}, \Theta)$ has to satisfy

$$H_{\mathcal{ABC}}(-\boldsymbol{X}^{\mathrm{T}}, -\Theta) = -H_{\mathcal{BAC}}(\boldsymbol{X}, \Theta) \ . \tag{6.37}$$

The three-point function (6.34) now has the right symmetry property under $z_1 \leftrightarrow z_2$, but it does not necessarily has the right symmetry under $z_1 \leftrightarrow z_3$ and $z_2 \leftrightarrow z_3$. Additionally, the function $H_{\mathcal{ABC}}(\boldsymbol{X}, \Theta)$ is constrained by the conservation law (6.30). Upon the use of the identity (4.42a) the latter is translated to

$$\mathcal{D}_\alpha H^{\alpha\beta\gamma,\alpha'\beta'\gamma',\alpha''\beta''\gamma''}(\boldsymbol{X}, \Theta) = 0 \ . \tag{6.38}$$

Now our aim is to find the most general solution for H. The standard approach used in 4D superconformal field theories with $\mathcal{N} = 1$ [30] and $\mathcal{N} = 2$ [28] is based on writing the most general ansatz in terms of $\boldsymbol{X}$ and Θ consistent with the symmetries and the scaling property (6.36) and constrain it by the conservation law (6.38). However, because of a large number of tensorial indices it appears to be inefficient as such an ansatz would require to analyse quite a considerable number of possible terms. Hence, we will take a slightly indirect route.

[14] Here and below we sometimes use a comma to separate various groups of indices.

JHEP06(2015)138

First, let us trade a pair of spinor indices of H for a vector index in each triple. That is, we write

$$H^{\alpha\alpha'\alpha'',\beta\beta'\beta'',\gamma\gamma'\gamma''} = (\gamma_m)^{\alpha'\alpha''}(\gamma_n)^{\beta'\beta''}(\gamma_k)^{\gamma'\gamma''}H^{\alpha m,\beta n,\gamma k}\,. \tag{6.39}$$

Note that eq. (6.39) is not quite correct as it stands because the left-hand side is fully symmetric in each triple while the right hand side is symmetric only in (α',α''), (β',β'') and (γ',γ''). For eq. (6.39) to make sense, we have to make sure that the antisymmetric part in (α,α'), (β,β') and (γ,γ') vanishes on the right hand side. That is, we have to impose the following conditions on $H^{\alpha m,\beta n,\gamma k}$

$$(\gamma_m)_{\alpha\delta}H^{\alpha m,\beta n,\gamma k} = 0\,, \quad (\gamma_n)_{\beta\delta}H^{\alpha m,\beta n,\gamma k} = 0\,, \quad (\gamma_k)_{\gamma\delta}H^{\alpha m,\beta n,\gamma k} = 0\,. \tag{6.40}$$

From (6.38) we still have the conservation law

$$\mathcal{D}_\alpha H^{\alpha m,\beta n,\gamma k} = 0\,. \tag{6.41}$$

Since H is Grassmann odd and since

$$\Theta^\alpha\Theta^\beta\Theta^\gamma = 0\,, \tag{6.42}$$

it follows that H must contain only linear Θ-terms. Then eq. (6.41) is equivalent to two independent equations

$$\partial_\alpha H^{\alpha m,\beta n,\gamma k} = 0\,, \tag{6.43a}$$
$$\Theta^\delta(\gamma^t)_{\alpha\delta}\partial_t H^{\alpha m,\beta n,\gamma k} = 0\,. \tag{6.43b}$$

Let us decompose $H^{\alpha m,\beta n,\gamma k}$ into symmetric and antisymmetric parts in the first and second pair of indices

$$H^{\alpha m,\beta n,\gamma k} = H^{(\alpha m,\beta n),\gamma k} + H^{[\alpha m,\beta n],\gamma k}\,. \tag{6.44}$$

In our subsequent analysis, it is more convenient to view H as a function of X^m rather than of $\boldsymbol{X}^{\alpha\beta}$. Then it is easy to see from eqs. (6.37) that $H^{(\alpha m,\beta n),\gamma k}$ has to be an even function of X^m while $H^{[\alpha m,\beta n],\gamma k}$ has to be an odd function. Since even and odd functions cannot mix in the conservation law (6.43a), (6.43b) $H^{(\alpha m,\beta n),\gamma k}$ and $H^{[\alpha m,\beta n],\gamma k}$ must satisfy (6.43a), (6.43b) separately. This means that we can consider $H^{(\alpha m,\beta n),\gamma k}$ and $H^{[\alpha m,\beta n],\gamma k}$ independently.

First, we will consider the case of $H^{(\alpha m,\beta n),\gamma k}$. Due to its symmetry properties, it is the sum of four possible terms:

1. $H_1^{(\alpha m,\beta n),\gamma k} = \varepsilon^{\alpha\beta}\Theta^\gamma A^{[mn],k}\,,$

2. $H_2^{(\alpha m,\beta n),\gamma k} = \varepsilon^{\alpha\beta}(\gamma_r)^\gamma{}_\delta\Theta^\delta B^{[mn],k,r}\,,$

3. $H_3^{(\alpha m,\beta n),\gamma k} = (\gamma_p)^{\alpha\beta}\Theta^\gamma C^{(mn),k,p}\,,$

4. $H_4^{(\alpha m,\beta n),\gamma k} = (\gamma_p)^{\alpha\beta}(\gamma_r)^\gamma{}_\delta\Theta^\delta D^{(mn),k,p,r}\,.$

JHEP06(2015)138

Here we use the fact that every symmetric in (α,β) matrix is proportional to a gamma-matrix. We also indicated that the matrices A and B are antisymmetric in (m,n) and C and D are symmetric. The tensors A, B, C, D depend on X^m and are symmetric under $X^m \to -X^m$. Now we will impose the conditions (6.40) as well as the conservation law (6.43a), (6.43b). To begin with, we will impose $\partial_\alpha H^{(\alpha m,\beta n),\gamma k}=0$. Then it is easy to see that

$$\begin{aligned} A^{[mn],k} &= 0\,, & B^{[mn],k,r} &= 0\,, \\ \eta_{pr} D^{(mn),k,p,r} &= 0\,, & \varepsilon_{rpq} D^{(mn),k,p,r} + \eta_{qq'} C^{(mn),k,q'} &= 0\,. \end{aligned} \tag{6.45}$$

Hence, $H_1^{(\alpha m,\beta n),\gamma k} = H_2^{(\alpha m,\beta n),\gamma k} = 0$.

Upon imposing (6.40) we obtain

$$\begin{aligned} & C^{(mn),k,p} = C^{(mnp),k}\,, \qquad D^{(mn),k,p,r} = D^{(mnp),(kr)} + \frac{1}{2}\varepsilon^{krq}\eta_{qq'}C^{(mnp),q'}\,, \\ & \eta_{mn}C^{(mnp),k} = 0\,, \qquad \eta_{mn}D^{(mnp),(kr)} = 0\,, \qquad \eta_{kr}D^{(mnp),(kr)} = 0\,. \end{aligned} \tag{6.46}$$

That is we find two symmetric traceless tensors $C^{(mnp),k}$ and $D^{(mnp),(kr)}$. Substituting now (6.46) into (6.45) we find that $C^{(mnp),k}$ and $D^{(mnp),(kr)}$ are related to each other as follows

$$\eta_{pr}D^{(mnp),(kr)} + \frac{1}{2}\varepsilon^{kpq}\eta_{pp'}\eta_{qq'}C^{(mnp'),q'} = 0\,, \tag{6.47a}$$

$$\varepsilon^{rpq}\eta_{rr'}\eta_{pp'}D^{(mnp'),(kr')} + C^{(mnq),k} + \frac{1}{2}C^{(mnk),q} - \frac{1}{2}\eta^{qk}\eta_{pt}C^{(mnp),t} = 0\,. \tag{6.47b}$$

Quite remarkably, eqs. (6.47) allow us to fully solve for $D^{(mnp),(kr)}$ in terms of $C^{(mnp),k}$. In order to do this we will decompose $C^{(mnp),k}$ and $D^{(mnp),(kr)}$ into irreducible components. To understand which irreducible components are relevant it is convenient to trade each vector index for a pair of spinor ones. Since $C^{(mnp),k}$ and $D^{(mnp),(kr)}$ are symmetric and traceless they become equivalent to symmetric tensors $C^{(\alpha_1\ldots\alpha_6),(\beta_1\beta_2)}$ and $D^{(\alpha_1\ldots\alpha_6),(\beta_1\ldots\beta_4)}$. Hence, C contains irreducible components (that is, *totally* symmetric tensors) of rank 8, 6, 4 and 2, whereas D contains irreducible components of rank 10, 8, 6, 4 and 2 (note that neither C nor D contains the rank 0 representation since the number of α and β indices is different). Now let us recall that all irreducible components must be even functions of $X^{\alpha\beta}$. This means that irreducible tensors of rank 10, 6 and 2 must vanish since they contain an odd number of $X^{\alpha\beta}$. Therefore, in both C and D only irreducible components of rank 8 and rank 4 can contribute. Going back to the vector indices, let us denote the irreducible components of C as $C_1^{(mnpk)}$ and $C_2^{(mn)}$ and the irreducible components of D as $D_1^{(mnpk)}$ and $D_2^{(mn)}$. By construction, all these tensors are symmetric and traceless. It is not hard to construct explicit decompositions of $C^{(mnp),k}$ and $D^{(mnp),(kr)}$ into the irreducible components. The decomposition of $C^{(mnp),k}$ reads

$$\begin{aligned} C^{(mnp),k} = {} & C_1^{(mnpk)} + \eta^{pk}C_2^{(mn)} + \eta^{nk}C_2^{(mp)} + \eta^{mk}C_2^{(np)} \\ & + \eta^{mn}C_3^{(pk)} + \eta^{mp}C_3^{(nk)} + \eta^{np}C_3^{(mk)}\,. \end{aligned} \tag{6.48}$$

Here we have taken into account that $C^{(mnp),k}$ is symmetric in (m,n,p). Recalling now that it is also traceless (see eq. (6.46)), $\eta_{mn}C^{(mnp),k}=0$, gives

$$C_3^{(mn)} = -\frac{2}{5}C_2^{(mn)} \,. \tag{6.49}$$

Similarly, we have the following decomposition of $D^{(mnp),(kr)}$:

$$\begin{aligned} D^{(mnp),(kr)} &= \varepsilon^{mks}\eta_{ss'}T^{(np),r,s'} + \varepsilon^{nks}\eta_{ss'}T^{(mp),r,s'} + \varepsilon^{pks}\eta_{ss'}T^{(mn),r,s'} \\ &\quad + \varepsilon^{mrs}\eta_{ss'}T^{(np),k,s'} + \varepsilon^{nrs}\eta_{ss'}T^{(mp),k,s'} + \varepsilon^{prs}\eta_{ss'}T^{(mn),k,s'} \,, \end{aligned} \tag{6.50}$$

where $T^{(np),r,s}$ is given in terms of $D_1^{(nprs)}$ and $D_2^{(np)}$ by

$$T^{(np),r,s} = D_1^{(nprs)} + \eta^{nr}D_2^{(ps)} + \eta^{pr}D_2^{(ns)} + \eta^{np}D_3^{(rs)} \,. \tag{6.51}$$

Recalling that $D^{(mnp),(kr)}$ is traceless in each group of indices relates

$$D_3^{(mn)} = -\frac{2}{5}D_2^{(mn)} \,. \tag{6.52}$$

Let us point out that symmetry allows us to add in (6.51) terms of the form $\eta^{ns}D_4^{(pr)} + \eta^{ps}D_4^{(nr)} + \eta^{rs}D_5^{(np)}$ with some symmetric traceless tensors $D_4^{(np)}$ and $D_5^{(np)}$. However, it is straightforward to show that such terms will cancel when we substitute them in (6.50) and, hence, they can be ignored. Substituting the irreducible decompositions (6.48), (6.49), (6.50), (6.51), (6.52) into (6.37) yields the solution

$$D_1^{(nprs)} = -\frac{3}{10}C_1^{(nprs)} \,, \quad D_2^{(np)} = -\frac{1}{8}C_2^{(np)} \,. \tag{6.53}$$

Thus, the tensor D is fully determined in terms of C.

Finally, let us consider the equation (6.43b) which involves the derivative with respect to X. It is possible to show using (6.47) that (6.43b) is equivalent to a pair of simple equations

$$\partial_m C^{(mnp),k} = 0 \,, \quad \partial_m D^{(mnp),(kr)} = 0 \,. \tag{6.54}$$

Now we are ready to construct an explicit solution. It is enough to consider $C^{(mnp),k}$ since $D^{(mnp),(kr)}$ is fully expressed in terms of it. Using the symmetry in (m,n,p), the scaling property (6.36) and the fact that it is an even function of X^m we have the following most general ansatz

$$\begin{aligned} C^{(mnp),k} &= \frac{a}{X^3}\Big[\eta^{mn}\eta^{pk} + \eta^{mk}\eta^{np} + \eta^{mp}\eta^{nk}\Big] + \frac{b}{X^5}\Big[\eta^{mn}X^pX^k + \eta^{mp}X^nX^k + \eta^{np}X^mX^k\Big] \\ &\quad + \frac{c}{X^5}\Big[\eta^{pk}X^mX^n + \eta^{nk}X^mX^p + \eta^{mk}X^nX^p\Big] + \frac{d}{X^7}X^mX^nX^pX^k \,. \end{aligned} \tag{6.55}$$

Here we adopt the vector notation $X^2 = X_mX^m$ and a, b, c, d are some coefficients. Imposing $\eta_{mn}C^{(mnp),k}=0$ gives $c=-5a$, $d=10a-5b$. Imposing $\partial_m C^{(mnp),k}=0$ gives $b=3a$, $d=-10b-5c$. Thus, we obtain that there is only one independent coefficient which we choose to be a and the remaining three coefficients are given by

$$b = 3a \,, \quad c = d = -5a \,. \tag{6.56}$$

At this step it is convenient to give particular values to a, b, c, d, say

$$a = 1\,, \quad b = 3\,, \quad c = d = -5\,. \tag{6.57}$$

Then, the free parameter, which we denote as $d_{\mathcal{N}=1}$, will show up as an overall coefficient in the final answer for the correlation function presented below.

It is now straightforward to compute $T^{(np),r,s}$ (and, hence, $D^{(mnp),(kr)}$). Using eqs. (6.51), (6.52), (6.53) and the explicit form of $C^{(mnp),k}$ in (6.55), (6.56) we obtain[15]

$$T^{(np),r,s} = \frac{1}{2}\Big[\frac{\eta^{nr}X^pX^s + \eta^{pr}X^nX^s - \eta^{np}X^rX^s}{X^5} + \frac{3X^nX^pX^rX^s}{X^7}\Big]\,. \tag{6.58}$$

As the last step, one can check that with $T^{(np),r,s}$ given by (6.58) the differential constraint $\partial_m D^{(mnp),(kr)} = 0$ from eq. (6.54) is indeed satisfied. Thus, we have shown that $H^{(\alpha m,\beta n),\gamma k}$ is fixed by symmetries and by the conservation law up to an overall coefficient $d_{\mathcal{N}=1}$.

In a similar manner we can consider the antisymmetric part $H^{[\alpha m,\beta n],\gamma k}$. Fortunately, the consideration is much simpler. It is not hard to show following the same logic as above that already imposing $\partial_\alpha H^{[\alpha m,\beta n],\gamma k} = 0$ and eq. (6.40) sets $H^{[\alpha m,\beta n],\gamma k} = 0$.

To summarise, we have shown that the three-point function of supercurrents in $\mathcal{N} = 1$ superconformal theories is fixed up to one overall coefficient $d_{\mathcal{N}=1}$. The explicit form of the function $H^{\alpha m,\beta n,\gamma k}$ is given by

$$\begin{aligned} H^{\alpha m,\beta n,\gamma k}(X,\Theta) = \mathrm{i}d_{\mathcal{N}=1}\Big[&(\gamma_p)^{\alpha\beta}\Theta^\gamma C^{(mnp),k} + \frac{1}{2}(\gamma_p)^{\alpha\beta}(\gamma_r)^\gamma{}_\delta\Theta^\delta\varepsilon^{krq}\eta_{qq'}C^{(mnp),q'} \\ &+(\gamma_p)^{\alpha\beta}(\gamma_r)^\gamma{}_\delta\Theta^\delta D^{(mnp),(kr)}\Big]\,. \end{aligned} \tag{6.59}$$

The tensors $C^{(mnp),k}$ and $D^{(mnp),(kr)}$ are given by (6.55), (6.57) and (6.50), (6.58), respectively.

Obviously, the correlation function (6.34) changes its sign under permutation of the superspace points z_1 and z_3 with the simultaneous swap of indices $\mathcal{A}$ and $\mathcal{C}$

$$\langle J_{\mathcal{A}}(z_1)J_{\mathcal{B}}(z_2)J_{\mathcal{C}}(z_3)\rangle = -\langle J_{\mathcal{C}}(z_3)J_{\mathcal{B}}(z_2)J_{\mathcal{A}}(z_1)\rangle\,. \tag{6.60}$$

As a consequence, the tensor H should obey the following equation

$$\begin{aligned} H_{\alpha_1\alpha_2\alpha_3,\beta_1\beta_2\beta_3,\gamma_1\gamma_2\gamma_3}(\boldsymbol{X}_3,\Theta_3) = &\frac{1}{\boldsymbol{X}_3{}^8\boldsymbol{x}_{13}{}^8}(\boldsymbol{x}_{13}^{-1})_{\alpha_1}{}^{\alpha_1'}(\boldsymbol{x}_{13}^{-1})_{\alpha_2}{}^{\alpha_2'}(\boldsymbol{x}_{13}^{-1})_{\alpha_3}{}^{\alpha_3'}\boldsymbol{x}_{13}{}^{\gamma_1'}{}_{\gamma_1}\boldsymbol{x}_{13}{}^{\gamma_2'}{}_{\gamma_2} \\ &\times\boldsymbol{x}_{13}{}^{\gamma_3'}{}_{\gamma_3}\boldsymbol{x}_{13}^{\beta_1'\delta_1}\boldsymbol{X}_{3\delta_1\beta_1}\boldsymbol{x}_{13}^{\beta_2'\delta_2}\boldsymbol{X}_{3\delta_2\beta_2}\boldsymbol{x}_{13}^{\beta_3'\delta_3}\boldsymbol{X}_{3\delta_3\beta_3} \\ &\times H_{\gamma_1'\gamma_2'\gamma_3',\beta_1'\beta_2'\beta_3',\alpha_1'\alpha_2'\alpha_3'}(-\boldsymbol{X}_1^{\mathrm{T}},-\Theta_1)\,. \end{aligned} \tag{6.61}$$

However, it appears to be very difficult to check that the tensor (6.59) obeys this equation because of its complicated structure. Alternatively, in appendix C we demonstrate that the expression (6.59) can be derived as a result of reduction of the $\mathcal{N} = 2$ supercurrent correlation function which will be computed in subsection 7.2. This will prove that (6.59) obeys the required property (6.61).

[15] An explicit calculation of $T^{(np),r,s}$ also gives additional terms containing η^{ns}, η^{ps} or η^{rs}. However, all such terms will cancel when we substitute them into the expression for $D^{(mnp),(kr)}$ in (6.50) and, hence, they can be ignored. It is analogous to the cancellation discussed below (6.52).

JHEP06(2015)138

7 Correlators in $\mathcal{N}=2$ superconformal field theory

We start with an example of a classically $\mathcal{N}=2$ superconformal field theory. It is described by n primary chiral scalars Φ (viewed as a column vector) of dimension $1/2$, $\bar{D}_\alpha \Phi = 0$, and their conjugate antichiral superfields $\Phi^\dagger$ with action[16]

$$S = \int d^3x d^2\theta d^2\bar{\theta}\, \Phi^\dagger \Phi + \left\{ \lambda \int d^3x d^2\theta\, (\Phi^{\mathrm{T}} \Phi)^2 + \text{c.c.} \right\} \ . \tag{7.1}$$

Here λ is a dimensionless coupling constant. The supercurrent of this model is [39, 40]

$$J_{\alpha\beta} = 2\bar{D}_{(\alpha} \Phi^\dagger D_{\beta)} \Phi + \frac{1}{2}[D_{(\alpha}, \bar{D}_{\beta)}](\bar{\Phi}^\dagger \Phi) \ . \tag{7.2}$$

The action is obviously $\mathsf{O}(n)$ invariant. The corresponding flavour current multiplet is

$$L^{\bar{a}} = \Phi^\dagger \Sigma^{\bar{a}} \Phi \, , \tag{7.3}$$

with $\Sigma^{\bar{a}}$ being the generator of the flavour $\mathsf{O}(n)$ group. It is not difficult to check that on-shell the currents (7.2) and (7.3) obey the $\mathcal{N}=2$ conservation equations in (1.1) and (1.4), respectively. In the free case, $\lambda = 0$, the action is $\mathsf{U}(n)$ invariant; the corresponding flavour current multiplet is given by (7.3), in which $\Sigma^{\bar{a}}$ now stands for the generator of the $\mathsf{U}(n)$ group. The free model is trivially superconformal at the quantum level.

A natural generalisation of (7.1) is the most general off-shell 3D $\mathcal{N}=2$ superconformal sigma model given in [53].[17]

7.1 $\mathcal{N}=2$ flavour current multiplets

The $\mathcal{N}=2$ flavour current is described by a primary scalar L of dimension 1, which means that its superconformal transformation is

$$\delta L = -\xi L - \sigma(z) L \ . \tag{7.4}$$

This transformation law is uniquely fixed by requiring the conservation equation

$$\left(D^{\alpha(I} D^{J)}_\alpha - \frac{1}{2}\delta^{IJ} D^{\alpha K} D^K_\alpha \right) L = 0 \tag{7.5}$$

to be superconformal.

As in the $\mathcal{N}=1$ case, we assume that the $\mathcal{N}=2$ superconformal field theory under study has a set of flavour current multiplets $L^{\bar{a}}$ associated with a simple flavour group. Since

[16]For the action (7.1) and the associated conserved current multiplets (7.2) and (7.3), we have employed the complex basis for the superspace Grassmann coordinates introduced in appendix B. In the remainder of this section, the real basis for the superspace Grassmann coordinates will be used.

[17]For target spaces with $\mathsf{U}(1)$ isometries, 3D supersymmetric sigma models may be formulated in terms of Abelian vector multiplets described in terms of gauge invariant field strengths. In the $\mathcal{N}=2$ case, the field strength of a vector multiplet is a real linear superfield. The $\mathcal{N}=2$ superconformal sigma models formulated using real linear superfields were studied in [40, 71].

JHEP06(2015)138

the superfields $L^{\bar a}$ carry neither spinor nor R-symmetry indices, their two-point correlation function is simply

$$\langle L^{\bar a}(z_1)L^{\bar b}(z_2)\rangle = a_{\mathcal{N}=2}\frac{\delta^{\bar a\bar b}}{\boldsymbol{x}_{12}{}^2}\,, \tag{7.6}$$

where $a_{\mathcal{N}=2}$ is a free coefficient. It is straightforward to check that this correlator is symmetric, $\langle L^{\bar a}(z_1)L^{\bar b}(z_2)\rangle = \langle L^{\bar b}(z_2)L^{\bar a}(z_1)\rangle$, and respects the conservation equation (7.5),

$$\left(D^{\alpha(I}_{(1)}D^{J)}_{(1)\alpha} - \frac{1}{2}\delta^{IJ}D^{\alpha K}_{(1)}D^{K}_{(1)\alpha}\right)\langle L^{\bar a}(z_1)L^{\bar b}(z_2)\rangle = 0\,, \qquad z_1 \neq z_2\,. \tag{7.7}$$

Our next goal is to work out the most general expression for the three-point function $\langle L^{\bar a}(z_1)L^{\bar b}(z_2)L^{\bar c}(z_3)\rangle$ compatible with all the physical requirements. According to (5.3), we have to make the ansatz

$$\langle L^{\bar a}(z_1)L^{\bar b}(z_2)L^{\bar c}(z_3)\rangle = \frac{1}{\boldsymbol{x}_{13}{}^2\boldsymbol{x}_{23}{}^2}\Big[f^{\bar a\bar b\bar c}H_{(f)}(\boldsymbol{X}_3,\Theta_3) + d^{\bar a\bar b\bar c}H_{(d)}(\boldsymbol{X}_3,\Theta_3)\Big]\,, \tag{7.8}$$

where $f^{\bar a\bar b\bar c}$ and $d^{\bar a\bar b\bar c}$ are antisymmetric and symmetric invariant tensors, respectively. Both functions $H_{(f)}$ and $H_{(d)}$ should have the same scaling property

$$H_{(f,d)}(\lambda^2\boldsymbol{X},\lambda\Theta) = \lambda^{-2}H_{(f,d)}(\boldsymbol{X},\Theta) \tag{7.9}$$

and obey the conservation condition

$$\left(\mathcal{D}^{\alpha(I}\mathcal{D}^{J)}_{\alpha} - \frac{1}{2}\delta^{IJ}\mathcal{D}^{\alpha K}\mathcal{D}^{K}_{\alpha}\right)H_{(f,d)} = 0\,. \tag{7.10}$$

The latter constraint is obtained from (7.5) with the use of (4.42a).

The correlation function (7.8) is invariant under exchange of the superspace points z_1 and z_2 and the flavour indices $\bar a$ and $\bar b$. As a consequence, the functions $H_{(f)}$ and $H_{(d)}$ are constrained by

$$H_{(f)}(-\boldsymbol{X}^{\mathrm{T}},-\Theta) = -H_{(f)}(\boldsymbol{X},\Theta)\,, \qquad H_{(d)}(-\boldsymbol{X}^{\mathrm{T}},-\Theta) = H_{(d)}(\boldsymbol{X},\Theta)\,. \tag{7.11}$$

The general solutions of the equations (7.9), (7.10) and (7.11) prove to be

$$H_{(f)}(\boldsymbol{X},\Theta) = b_{\mathcal{N}=2}\frac{\mathrm{i}\varepsilon_{IJ}\Theta^I_\alpha X^{\alpha\beta}\Theta^J_\beta}{X^3}\,, \qquad H_{(d)}(\boldsymbol{X},\Theta) = \tilde b_{\mathcal{N}=2}\frac{1}{X}\,. \tag{7.12}$$

Here $b_{\mathcal{N}=2}$ and $\tilde b_{\mathcal{N}=2}$ are two real coefficients. One can also check that the functions $H_{(f)}$ and $H_{(d)}$ obey the equations

$$H_{(f)}(-\boldsymbol{X}_1^{\mathrm{T}},-\Theta_1) = -\boldsymbol{x}_{13}{}^2\boldsymbol{X}_3{}^2H_{(f)}(\boldsymbol{X}_3,\Theta_3)\,, \tag{7.13a}$$
$$H_{(d)}(-\boldsymbol{X}_1^{\mathrm{T}},-\Theta_1) = \boldsymbol{x}_{13}{}^2\boldsymbol{X}_3{}^2H_{(d)}(\boldsymbol{X}_3,\Theta_3)\,, \tag{7.13b}$$

which are corollaries of the following symmetry property

$$\langle L^{\bar a}(z_1)L^{\bar b}(z_2)L^{\bar c}(z_3)\rangle = \langle L^{\bar c}(z_3)L^{\bar b}(z_2)L^{\bar a}(z_1)\rangle\,. \tag{7.14}$$

Finally we point out that the functions (7.12) can be rewritten in terms of the covariant object $\boldsymbol{X}_{\alpha\beta}$ with the use of (4.37)

$$H_{(f)}(\boldsymbol{X},\Theta) = b_{\mathcal{N}=2}\frac{\mathrm{i}\varepsilon_{IJ}\Theta^{\alpha I}\Theta^{J\beta}\boldsymbol{X}_{\alpha\beta}}{\boldsymbol{X}^3}\,, \tag{7.15a}$$

$$H_{(d)}(\boldsymbol{X},\Theta) = \tilde{b}_{\mathcal{N}=2}\frac{1}{\boldsymbol{X}}\left(1+\frac{1}{8}\frac{\Theta^4}{\boldsymbol{X}^2}\right)\,. \tag{7.15b}$$

In verifying eq. (7.15a), the $\mathcal{N}=2$ identity $\varepsilon^{IJ}\Theta^I_\alpha\Theta^J_\beta\Theta^2=0$ may be useful. We point out that the expression in parentheses in (7.15b) involves the square of the superconformal invariant (4.34).

It should be stressed that the appearance of the d-term in the flavour current correlation function (7.8) is a novel feature which distinguishes the $\mathcal{N}=2$ superconformal field theories from the $\mathcal{N}=1$ ones considered in section 6.1 and from non-supersymmetric ones studied in [11]. In contrast to the four-dimensional theories, in three dimensions this part of the correlation function cannot be considered as an anomaly induced contribution. To understand the role of this part of the correlation function it would be interesting to consider some examples of $\mathcal{N}=2$ theories in which this contribution is non-trivial.[18] We leave this issue for further studies.

7.2 $\mathcal{N}=2$ supercurrent

The $\mathcal{N}=2$ supercurrent is described by a primary symmetric second-rank spinor $J_{\alpha\beta}=J_{(\alpha\beta)}$ of dimension 2, hence its superconformal transformation is

$$\delta J_{\alpha\beta} = -\xi J_{\alpha\beta} - 2\sigma(z)J_{\alpha\beta} + 2\lambda_{(\alpha}{}^{\gamma}(z)J_{\beta)\gamma}\,. \tag{7.16}$$

This transformation law is uniquely fixed by the condition that the supercurrent conservation equation

$$D^\alpha_I J_{\alpha\beta} = 0 \tag{7.17}$$

is superconformal.

According to the general prescription (5.2), the two-point function for the supercurrent is given by

$$\langle J_{\alpha\beta}(z_1)J^{\alpha'\beta'}(z_2)\rangle = c_{\mathcal{N}=2}\frac{\boldsymbol{x}_{12\alpha}{}^{(\alpha'}\boldsymbol{x}_{12\beta}{}^{\beta')}}{(\boldsymbol{x}_{12}{}^2)^3}\,, \tag{7.18}$$

where $c_{\mathcal{N}=2}$ is a real coefficient. It is not difficult to see that this correlator is symmetric, $\langle J_{\alpha\beta}(z_1)J_{\alpha'\beta'}(z_2)\rangle = \langle J_{\alpha'\beta'}(z_2)J_{\alpha\beta}(z_1)\rangle$, and respects the conservation equation (7.17),

$$D^{I\alpha}_{(1)}\langle J_{\alpha\beta}(z_1)J_{\alpha'\beta'}(z_2)\rangle = 0\,, \qquad z_1\neq z_2\,. \tag{7.19}$$

The most general expression for the three-point function for the supercurrent is

$$\langle J_{\alpha\alpha'}(z_1)J_{\beta\beta'}(z_2)J_{\gamma\gamma'}(z_3)\rangle = \frac{\boldsymbol{x}_{13\alpha\rho}\boldsymbol{x}_{13\alpha'\rho'}\boldsymbol{x}_{23\beta\sigma}\boldsymbol{x}_{23\beta'\sigma'}}{(\boldsymbol{x}_{13}{}^2)^3(\boldsymbol{x}_{23}{}^2)^3}H^{\rho\rho',\sigma\sigma'}{}_{\gamma\gamma'}(\boldsymbol{X}_3,\Theta_3)\,, \tag{7.20}$$

[18] It may be shown that both f- and d-terms are generated in the free model (7.1) with $\lambda=0$.

where, by construction, the function $H^{\rho\rho',\sigma\sigma'}{}_{\gamma\gamma'}(\boldsymbol{X},\Theta)$ obeys the symmetry property

$$H^{\rho\rho',\sigma\sigma'}{}_{\gamma\gamma'}(\boldsymbol{X},\Theta) = H^{(\rho\rho'),(\sigma\sigma')}{}_{(\gamma\gamma')}(\boldsymbol{X},\Theta) \ . \tag{7.21}$$

Since both the supercurrent $J_{\alpha\beta}$ and $H^{\alpha\alpha',\beta\beta',\gamma\gamma'}$ are Grassmann even, the three-point function (7.21) has to be symmetric under the exchange $z_1 \leftrightarrow z_2$, $\alpha,\alpha' \leftrightarrow \beta,\beta'$. Hence, $H^{\alpha\alpha',\beta\beta',\gamma\gamma'}(\boldsymbol{X},\Theta)$ satisfies the following symmetry property

$$H^{\alpha\alpha',\beta\beta',\gamma\gamma'}(-\boldsymbol{X}^{\mathrm{T}},-\Theta) = H^{\beta\beta',\alpha\alpha',\gamma\gamma'}(\boldsymbol{X},\Theta) \ . \tag{7.22}$$

In addition, $H^{\alpha\alpha',\beta\beta',\gamma\gamma'}$ is characterised by the scaling property

$$H^{\alpha\alpha',\beta\beta',\gamma\gamma'}(\lambda^2\boldsymbol{X},\lambda\Theta) = \lambda^{-4}H^{\alpha\alpha',\beta\beta',\gamma\gamma'}(\boldsymbol{X},\Theta) \ . \tag{7.23}$$

With the use of eq. (4.42a), the supercurrent conservation condition (7.17) is translated to the following equation for $H^{\alpha\alpha',\beta\beta',\gamma\gamma'}$:

$$\mathcal{D}^I_\alpha H^{\alpha\alpha',\beta\beta',\gamma\gamma'}(\boldsymbol{X},\Theta) = 0 \ . \tag{7.24}$$

Just like in the problem of the three-point correlator for the $\mathcal{N}=1$ supercurrent considered in section 6.2, it is convenient to trade each pair of spinor indices for vector ones,

$$H^{\alpha\alpha',\beta\beta',\gamma\gamma'} = (\gamma_m)^{\alpha\alpha'}(\gamma_n)^{\beta\beta'}(\gamma_p)^{\gamma\gamma'}H^{mnp} \ , \tag{7.25}$$

where $H^{mnp}(\boldsymbol{X},\Theta)$ satisfies the same scaling property as (7.23) as well as

$$H^{mnp}(-\boldsymbol{X}^{\mathrm{T}},-\Theta) = H^{nmp}(\boldsymbol{X},\Theta) \tag{7.26}$$

and

$$(\gamma^m)^{\alpha\beta}\mathcal{D}^I_\alpha H_{mnp}(\boldsymbol{X},\Theta) = 0 \ . \tag{7.27}$$

Unlike the $\mathcal{N}=1$ case, now it is not hard to list all possible structures consistent with the symmetry (7.26) and the scaling property (7.23). This makes the analysis considerably simpler than in the previous section. Just like in the $\mathcal{N}=1$ case, it is more convenient to view H as function of X^m rather than $\boldsymbol{X}^{\alpha\beta}$. Then the building blocks which can appear in H are

$$\begin{aligned} &\eta_{mn}\,, \quad \varepsilon_{mnp}\,, \quad X^m = -\frac{1}{2}(\gamma^m)_{\alpha\beta}X^{\alpha\beta}\,, \qquad && X = \sqrt{X_m X^m}\,, \\ &\qquad\qquad (\Theta\Theta)_m = -\frac{\mathrm{i}}{2}(\gamma_m)^{\alpha\beta}\Theta_{\alpha I}\Theta_{\beta J}\varepsilon^{IJ}\,, && \Theta^2 = \Theta^\alpha_I\Theta_{\alpha I} \ . \end{aligned} \tag{7.28}$$

Note that there is the following $\mathcal{N}=2$ identity

$$(\Theta\Theta)_m\Theta^2 = 0 \ . \tag{7.29}$$

Taking into account the symmetry property (7.26) and (7.23) we get the following general expression for H:

$$H_{mnp} = \sum_i A_i H_{i,mnp} + \sum_i B_i \mathcal{H}_{i,mnp} + \sum_i C_i \mathbf{H}_{i,mnp} \ , \tag{7.30}$$

where $A_i,\ B_i,\ C_i$ are some coefficients and the tensors $H_{i,mnp}$, $\mathcal{H}_{i,mnp}$, $\mathbf{H}_{i,mnp}$ are explicitly given by

$$\begin{aligned}
H_{1,mnp} &= \frac{\eta_{mn}(\Theta\Theta)_p}{X^3}\,, \qquad\qquad H_{2,mnp} = \frac{X_m X_n(\Theta\Theta)_p}{X^5}\,,\\
H_{3,mnp} &= \frac{X_m X_p(\Theta\Theta)_n + X_n X_p(\Theta\Theta)_m}{X^5}\,,\\
H_{4,mnp} &= \frac{\eta_{mp}(\Theta\Theta)_n + \eta_{np}(\Theta\Theta)_m}{X^3}\,,\\
H_{5,mnp} &= \frac{X_m X_n X_p X^q(\Theta\Theta)_q}{X^7}\,, \quad H_{6,mnp} = \frac{\eta_{mn}X_p X^q(\Theta\Theta)_q}{X^5}\,,\\
H_{7,mnp} &= \frac{\eta_{mp}X_n X^q(\Theta\Theta)_q + \eta_{np}X_m X^q(\Theta\Theta)_q}{X^5}\,;
\end{aligned} \tag{7.31}$$

$$\begin{aligned}
\mathcal{H}_{1,mnp} &= \frac{\varepsilon_{mnr}X^r(\Theta\Theta)_p}{X^4}\,, \qquad \mathcal{H}_{2,mnp} = \frac{\varepsilon_{mnr}X_p(\Theta\Theta)^r}{X^4}\,,\\
\mathcal{H}_{3,mnp} &= \frac{\varepsilon_{mnp}X^r(\Theta\Theta)_r}{X^4}\,, \qquad \mathcal{H}_{4,mnp} = \frac{\varepsilon_{mnr}X_p X^r X^q(\Theta\Theta)_q}{X^6}\,;
\end{aligned} \tag{7.32}$$

$$\begin{aligned}
\mathbf{H}_{1,mnp} &= \frac{\eta_{np}X_m - \eta_{mp}X_n}{X^3}\,, \qquad \mathbf{H}_{2,mnp} = \frac{\eta_{np}X_m - \eta_{mp}X_n}{X^4}\Theta^2\,,\\
\mathbf{H}_{3,mnp} &= \frac{\eta_{np}X_m - \eta_{mp}X_n}{X^5}\Theta^4\,.
\end{aligned} \tag{7.33}$$

Note that $H_{i,mnp} = H_{i,(mn)p}$, $\mathcal{H}_{i,mnp} = \mathcal{H}_{i,[mn]p}$, $\mathbf{H}_{i,mnp} = \mathbf{H}_{i,[mn]p}$.

It is easy to realize that $H_{i,mnp}$, $\mathcal{H}_{i,mnp}$ and $\mathbf{H}_{i,mnp}$ do not mix in the equation (7.27) and, hence, they must satisfy the conservation law independently. Let us now substitute (7.31), (7.32) and (7.33) in (7.27). This equation will lead to two types of terms: terms linear in Θ and terms proportional to Θ^3. Clearly, these terms must vanish separately. Let us first consider the terms linear in Θ. Using the identities

$$\begin{aligned}
(\gamma^m)_\beta^\alpha \mathcal{D}_\alpha^I X^n &= -\mathrm{i}\eta^{mn}\Theta_\beta^I - \mathrm{i}\Theta_\alpha^I\varepsilon^{mnp}(\gamma_p)_\beta^\alpha\,,\\
(\gamma^m)_\beta^\alpha \mathcal{D}_\alpha^I \Theta^2 &= 2(\gamma^m)_\beta^\alpha\Theta_\alpha^I\,,\\
(\gamma_m)_\beta^\alpha \mathcal{D}_\alpha^I(\Theta\Theta)_n &= \mathrm{i}\varepsilon_{IJ}\eta_{mn}\Theta_\beta^J + \mathrm{i}\varepsilon_{IJ}\varepsilon_{mnp}(\gamma^p)_\beta^\alpha\Theta_\alpha^J
\end{aligned} \tag{7.34}$$

it is straightforward to show that

$$B_1 = B_2 = B_3 = B_4 = 0\,, \quad C_1 = C_2 = C_3 = 0\,. \tag{7.35}$$

Thus, $\mathcal{H}_{i,mnp}$ and $\mathbf{H}_{i,mnp}$ can be ignored and we can concentrate only on $H_{i,mnp}$ in eq. (7.31). Substituting (7.31) in (7.27) and considering only the terms linear in Θ gives the following constraints on the coefficients A_i:

$$\begin{aligned}
A_1 - A_4 = 0\,, \quad A_3 - A_6 = 0\,, \quad A_2 - A_7 = 0\,,\\
A_1 + 4A_4 + A_7 = 0\,, \quad A_2 + 4A_3 + A_5 + A_6 + A_7 = 0\,.
\end{aligned} \tag{7.36}$$

Similarly, concentrating on the terms cubic in Θ, after straightforward but lengthy calculations we obtain the following system:

$$\begin{aligned}
3A_1 + A_2 + A_3 + 6A_4 + A_6 + 2A_7 = 0\,, \qquad 3A_1 + A_2 - A_3 - 3A_4 + A_6 - A_7 = 0\,,\\
5A_3 + 3A_5 = 0\,, \quad 3A_1 + A_2 + 2A_3 + 3A_4 + A_5 + A_6 + A_7 = 0\,.
\end{aligned} \tag{7.37}$$

To derive the system of equations (7.37), it is important to make use of the following $\mathcal{N}=2$ identity

$$\Theta_{\alpha I}(\Theta\Theta)_m = \frac{\mathrm{i}}{2}(\gamma_m)_{\alpha\beta}\Theta^{\beta J}\Theta^2\varepsilon_{IJ}\,, \tag{7.38}$$

which can easily be obtained by differentiating (7.29).

The systems (7.36) and (7.37) turn out to be consistent and can be solved in terms of one independent coefficient which we choose to be $A_1 \equiv A$:

$$A_2 = A_5 = A_7 = -5A\,, \quad A_3 = A_6 = 3A\,, \quad A_4 = A\,. \tag{7.39}$$

Thus, the three-point function of the supercurrent is fixed up to a single coefficient A.

Since the three-point function has only one overall coefficient, our result should possess the right symmetry properties under the exchange $z_1 \leftrightarrow z_3$, $z_2 \leftrightarrow z_3$. However, since the final result is rather simple and contains only a few terms listed in (7.31), the symmetry under, say, the $z_1 \leftrightarrow z_3$ exchange is not hard to verify. The invariance of the three-point function

$$\langle J_{\alpha\alpha'}(z_1)J_{\beta\beta'}(z_2)J_{\gamma\gamma'}(z_3)\rangle = \langle J_{\gamma\gamma'}(z_3)J_{\beta\beta'}(z_2)J_{\alpha\alpha'}(z_1)\rangle \tag{7.40}$$

implies the following equation[19] on the tensor H

$$\begin{aligned} H^{\rho\rho',\sigma,\sigma'}{}_{\tau\tau'}(-\boldsymbol{X}_1^{\mathrm{T}},-\Theta_1) \,=\, & \boldsymbol{X}_3{}^4\boldsymbol{X}_1^{\sigma\lambda}\boldsymbol{X}_1^{\sigma'\lambda'}\boldsymbol{x}_{13\lambda\alpha}\boldsymbol{x}_{13\lambda'\alpha'}\boldsymbol{x}_{13}^{\rho\gamma}\boldsymbol{x}_{13}^{\rho'\gamma'}\boldsymbol{x}_{13\tau\beta}\boldsymbol{x}_{13\tau'\beta'} \\ & \times H^{\beta\beta,\alpha,\alpha'}{}_{\gamma\gamma'}(\boldsymbol{X}_3,\Theta_3)\,. \end{aligned} \tag{7.41}$$

Using the formulae (4.40) we can relate $\boldsymbol{X}_3$ with $\boldsymbol{X}_1$ and Θ_3 with Θ_1

$$\boldsymbol{X}_{1\alpha\beta} = -\frac{\boldsymbol{x}_{13\alpha\alpha'}\boldsymbol{x}_{13\beta\beta'}\boldsymbol{X}_3^{\alpha'\beta'}}{\boldsymbol{x}_{13}{}^4\boldsymbol{X}_3{}^2}\,, \quad \Theta^I_{1\alpha} = \frac{u_{13}^{IJ}\boldsymbol{x}_{13\alpha\beta}\boldsymbol{X}_3^{\beta\gamma}\Theta_{3\gamma J}}{\boldsymbol{x}_{13}{}^2\boldsymbol{X}_3{}^2}\,. \tag{7.42}$$

It is now straightforward to substitute eq. (7.31) into (7.41) and verify that it is indeed fulfilled if the coefficients A_i satisfy (7.39). More precisely, eq. (7.41) constrains the coefficients A_i as follows

$$A_1 - A_4 = 0\,, \quad 2A_1 + A_2 + A_3 = 0\,, \quad 2A_1 + A_6 + A_7 = 0\,. \tag{7.43}$$

The system of equations (7.43) is weaker than the system (7.36), (7.37) and is contained there. That is why the conservation law alone fully constrains the coefficients.

To conclude this section, we rewrite explicitly the final result for the tensor H in terms of the objects (4.30):

$$\begin{aligned} H^{\alpha\alpha',\beta\beta',\gamma\gamma'} = \mathrm{i}d_{\mathcal{N}=2}\Big\{ & \frac{2}{\boldsymbol{X}^3}\Big[\varepsilon^{\alpha(\beta}\varepsilon^{\beta')\alpha'}\Theta_I^\gamma\Theta_J^{\gamma'} + \varepsilon^{\alpha(\gamma}\varepsilon^{\gamma')\alpha'}\Theta_I^\beta\Theta_J^{\beta'} + \varepsilon^{\beta(\gamma}\varepsilon^{\gamma')\beta'}\Theta_I^\alpha\Theta_J^{\alpha'}\Big]\varepsilon^{IJ} \\ & + \frac{1}{\boldsymbol{X}^5}\Big[3\boldsymbol{X}^{\alpha\alpha'}\boldsymbol{X}^{\gamma\gamma'}\Theta_I^\beta\Theta_J^{\beta'} + 3\boldsymbol{X}^{\beta\beta'}\boldsymbol{X}^{\gamma\gamma'}\Theta_I^\alpha\Theta_J^{\alpha'} - 5\boldsymbol{X}^{\alpha\alpha'}\boldsymbol{X}^{\beta\beta'}\Theta_I^\gamma\Theta_J^{\gamma'}\Big]\varepsilon^{IJ} \\ & + \frac{1}{\boldsymbol{X}^5}\Big[5\varepsilon^{\alpha(\gamma}\varepsilon^{\gamma')\alpha'}\boldsymbol{X}^{\beta\beta'} + 5\varepsilon^{\beta(\gamma}\varepsilon^{\gamma')\beta'}\boldsymbol{X}^{\alpha\alpha'} - 3\varepsilon^{\alpha(\beta}\varepsilon^{\beta')\alpha'}\boldsymbol{X}^{\gamma\gamma'}\Big]\boldsymbol{X}^{\delta\delta'}\Theta_\delta^I\Theta_{\delta'}^J\varepsilon_{IJ} \\ & + \frac{5}{2}\frac{1}{\boldsymbol{X}^7}\boldsymbol{X}^{\alpha\alpha'}\boldsymbol{X}^{\beta\beta'}\boldsymbol{X}^{\gamma\gamma'}\boldsymbol{X}^{\delta\delta'}\Theta_\delta^I\Theta_{\delta'}^J\varepsilon_{IJ}\Big\}\,. \end{aligned} \tag{7.44}$$

Here we have denoted the overall coefficient by $d_{\mathcal{N}=2}$.

[19] Due to the identity (7.29) it is trivial to rewrite (7.31) in terms of $\boldsymbol{X}$ rather than X.

JHEP06(2015)138

8 Correlators in $\mathcal{N}=3$ superconformal field theory

The off-shell $\mathcal{N}=3$ superconformal sigma model in three dimensions proposed in [53] is a nontrivial example of classically $\mathcal{N}=3$ superconformal theories. Its formulation is based on the projective superspace techniques [72–74] (see [75] for a review). An alternative approach to describe off-shell $\mathcal{N}=3$ hypermultiplets in three dimensions [76] is provided by the harmonic superspace formalism [44, 77], see, e.g., [78] for the formulation of the ABJM models [79] in $\mathcal{N}=3$ harmonic superspace. In the present paper, we will not discuss the harmonic and the projective superspace formulations for the off-shell hypermultiplet as it goes beyond our goals. Here we will simply provide examples of $\mathcal{N}=3$ supercurrent and flavour current multiplets, and for this it suffices to consider a free on-shell massless $\mathcal{N}=3$ hypermultiplet.[20] It is described by a primary superfield q^i, and its conjugate $\bar{q}_i$, subject to the equation of motion [76]

$$D_\alpha^{(ij}q^{k)}=0\,, \tag{8.1}$$

which is the 3D $\mathcal{N}=3$ analogue of the famous 4D $\mathcal{N}=2$ hypermultiplet constraints due to Sohnius [80]. Here D_α^{ij} is obtained from D_α^I by replacing its isovector index with a pair of isospinor ones by the general rule [53]

$$Z^I \;\to\; Z_i{}^j := \frac{\mathrm{i}}{\sqrt{2}}(\vec{Z}\cdot\vec{\sigma})_i{}^j \equiv Z^I(\tau^I)_i{}^j\,, \qquad Z_i{}^i=0\,, \tag{8.2}$$

with $\vec{\sigma}$ being the Pauli matrices. The hypermultiplet q^i transforms in the defining representation of $\mathsf{SU}(2)$, which is the double cover of the R-symmetry group $\mathsf{SO}(3)$. The $\mathsf{SU}(2)$ indices are raised and lowered with the antisymmetric tensors ε^{ij} and ε_{ij}, $\varepsilon^{12}=-\varepsilon_{12}=1$.

Let us consider a system of n free on-shell hypermultiplets. It is described by a column n-vector $\boldsymbol{q}^i$ constrained by (8.1) and its conjugate $\boldsymbol{q}^\dagger_i$. The supercurrent J_α and a flavour current multiplet $L^{\bar a}_{ij}$ are given by

$$\begin{aligned} J_\alpha &= \mathrm{i}\,\boldsymbol{q}^\dagger_i \overleftrightarrow{D}{}^{ij}_\alpha\, \boldsymbol{q}_j\,, &(8.3\text{a})\\ L^{\bar a}_{ij} &= \mathrm{i}\,\boldsymbol{q}^\dagger_{(i}\Sigma^{\bar a}\boldsymbol{q}_{j)}\,, &(8.3\text{b})\end{aligned}$$

where $\Sigma^{\bar a}$ is a flavour group generator. With the use of (8.1) it is possible to check that the operators (8.3) obey the $\mathcal{N}=3$ conservation equations given in (1.1) and (1.4). In the $\mathsf{SU}(2)$ notation, these equations read

$$\begin{aligned} D^{ij\,\alpha}J_\alpha &= 0\,, &(8.4\text{a})\\ D_\alpha^{(ij}L^{kl)} &= 0\,. &(8.4\text{b})\end{aligned}$$

We now turn to studying the correlation functions of the supercurrent and flavour current multiplets in quantum $\mathcal{N}=3$ superconformal models.

[20] Although the harmonic and the projective formulations for the free hypermultiplet differ off the mass shell, they lead to the same on-shell superfield.

8.1 $\mathcal{N}=3$ flavour current multiplets

The $\mathcal{N}=3$ flavour current multiplet is described by a primary real isovector L^I of dimension 1, which transforms under the superconformal group as

$$\delta L^I = -\xi L^I - \sigma(z)L^I + \Lambda^{IJ}(z)L^J \tag{8.5}$$

and obeys the conservation equation

$$D_\alpha^{(I}L^{J)} - \frac{1}{3}\delta^{IJ}D_\alpha^K L^K = 0 \ . \tag{8.6}$$

Similar to the $\mathcal{N}=1$ and $\mathcal{N}=2$ cases, we assume that the $\mathcal{N}=3$ superconformal field theory under study has a set of flavour current multiplets $L^{I\bar{a}}$ associated with a simple flavour group. According to (5.2), the two-point correlator for these multiplets is

$$\langle L^{I\bar{a}}(z_1)L^{J\bar{b}}(z_2)\rangle = a_{\mathcal{N}=3}\frac{\delta^{\bar{a}\bar{b}}u_{12}^{IJ}}{\boldsymbol{x}_{12}{}^2} \,, \tag{8.7}$$

with some coefficient $a_{\mathcal{N}=3}$. It may be checked that this two-function is symmetric, $\langle L^{I\bar{a}}(z_1)L^{J\bar{b}}(z_2)\rangle = \langle L^{J\bar{b}}(z_2)L^{I\bar{a}}(z_1)\rangle$, and respects the conservation law (8.6),

$$D_{(1)\alpha}^{(K}\langle L^{I)\bar{a}}(z_1)L^{J\bar{b}}(z_2)\rangle - \frac{1}{3}\delta^{KI}D_{(1)\alpha}^{L}\langle L^{L\bar{a}}(z_1)L^{J\bar{b}}(z_2)\rangle = 0\,, \qquad z_1 \neq z_2 \ . \tag{8.8}$$

For the three-point function $\langle L^{I\bar{a}}(z_1)L^{J\bar{b}}(z_2)L^{K\bar{c}}(z_3)\rangle$, we follow (5.3) and make the ansatz

$$\langle L^{I\bar{a}}(z_1)L^{J\bar{b}}(z_2)L^{K\bar{c}}(z_3)\rangle = \frac{u_{13}^{II'}u_{23}^{JJ'}}{\boldsymbol{x}_{13}{}^2\boldsymbol{x}_{23}{}^2}\Big(f^{\bar{a}\bar{b}\bar{c}}H_{(f)}^{I'J'K}(\boldsymbol{X}_3,\Theta_3) + d^{\bar{a}\bar{b}\bar{c}}H_{(d)}^{I'J'K}(\boldsymbol{X}_3,\Theta_3)\Big)\,, \tag{8.9}$$

where $f^{\bar{a}\bar{b}\bar{c}}$ and $d^{\bar{a}\bar{b}\bar{c}}$ are antisymmetric and symmetric invariant tensors of the flavour group. The functions $H_{(f,d)}^{IJK}$ should obey the following scaling property

$$H_{(f,d)}^{IJK}(\lambda^2\boldsymbol{X},\lambda\Theta) = \lambda^{-2}H_{(f,d)}^{IJK}(\boldsymbol{X},\Theta) \ . \tag{8.10}$$

The three-point function under consideration has to possess the symmetry property

$$\langle L^{I\bar{a}}(z_1)L^{J\bar{b}}(z_2)L^{K\bar{c}}(z_3)\rangle = \langle L^{J\bar{b}}(z_2)L^{I\bar{a}}(z_1)L^{K\bar{c}}(z_3)\rangle\,, \tag{8.11}$$

which implies the following constraints for $H_{(f,d)}$

$$H_{(f)}^{IJK}(\boldsymbol{X},\Theta) = -H_{(f)}^{JIK}(-\boldsymbol{X}^{\mathrm{T}},-\Theta)\,, \quad H_{(d)}^{IJK}(\boldsymbol{X},\Theta) = H_{(d)}^{JIK}(-\boldsymbol{X}^{\mathrm{T}},-\Theta) \ . \tag{8.12}$$

The most general solution of the equations (8.10) and (8.12) can be written as

$$H_{(d)}^{IJK} = \sum_n b_n H_{(d)n}^{IJK}\,, \qquad H_{(f)}^{IJK} = \sum_n c_n H_{(f)n}^{IJK} + \sum_n d_n \mathcal{H}_{(f)n}^{IJK}\,, \tag{8.13}$$

JHEP06(2015)138

where b_n, c_n and d_n are some coefficients and the tensors $H^{IJK}_{(d)n}$, $H^{IJK}_{(f)n}$ and $\mathcal{H}^{IJK}_{(f)n}$ are

$$\begin{aligned} H^{IJK}_{(d)1} &= \frac{\varepsilon_{IJL}A_{LK}}{X^3}\,, & H^{IJK}_{(d)2} &= \frac{\varepsilon^{IKL}B^{JL}+\varepsilon^{JKL}B^{IL}}{X^2}\,, \\ H^{IJK}_{(d)3} &= \frac{\varepsilon^{IJL}A_{LK}\Theta^2}{X^4}\,, & H^{IJK}_{(d)4} &= \frac{\varepsilon^{IKL}B^{JL}+\varepsilon^{JKL}B^{IL}}{X^3}\Theta^2\,; \end{aligned} \tag{8.14}$$

$$\begin{aligned} H^{IJK}_{(f)1} &= \frac{\varepsilon^{IJK}}{X}\,, & H^{IJK}_{(f)2} &= -\frac{1}{2}\frac{A^{IL}\varepsilon^{LJK}+A^{JL}\varepsilon^{LIK}}{X^3}\,, \\ H^{IJK}_{(f)3} &= -\frac{1}{2}\frac{\delta^{IJ}\varepsilon^{KMN}A^{MN}}{X^3}\,, & H^{IJK}_{(f)4} &= \frac{1}{4}\frac{A^{IJ}\varepsilon^{KMN}A^{MN}}{X^5}\,, \\ H^{IJK}_{(f)5} &= -\frac{1}{2}\frac{\varepsilon^{IJL}B^{KL}\Theta^2}{X^3}\,, & H^{IJK}_{(f)6} &= -\frac{1}{2}\frac{\varepsilon^{IJK}\Theta^4}{X^3}\,; \end{aligned} \tag{8.15}$$

$$\begin{aligned} \mathcal{H}^{IJK}_{(f)1} &= \frac{\varepsilon^{IJK}\Theta^2}{X^2}\,, & \mathcal{H}^{IJK}_{(f)2} &= \frac{\varepsilon^{IJL}B^{LK}}{X^2}\,, \\ \mathcal{H}^{IJK}_{(f)3} &= -\frac{1}{2}\frac{\varepsilon^{IJK}\Theta^6}{X^4}\,, & \mathcal{H}^{IJK}_{(f)4} &= -\frac{1}{2}\frac{\delta^{IJ}\varepsilon^{KMN}A^{MN}\Theta^2}{X^4}\,, \\ \mathcal{H}^{IJK}_{(f)5} &= -\frac{1}{2}\frac{\delta^{IK}\varepsilon^{JMN}A^{MN}+\delta^{JK}\varepsilon^{IMN}A^{MN}}{X^4}\Theta^2\,, \\ \mathcal{H}^{IJK}_{(f)6} &= -\frac{1}{2}\frac{B^{IJ}\varepsilon^{KMN}A^{MN}}{X^4}\,. \end{aligned} \tag{8.16}$$

Here we have introduced

$$A^{IJ} := \mathrm{i}\Theta^{I\alpha}X_{\alpha\beta}\Theta^{J\beta} = -A^{JI}\,, \qquad B^{IJ} := \Theta^{I\alpha}\Theta^J_\alpha = B^{JI}\,. \tag{8.17}$$

In principle, the set (8.16) could be extended by one more term

$$\mathcal{H}^{IJK}_{(f)7} = -\frac{1}{2}\frac{B^{IK}\varepsilon^{JMN}A^{MN}+B^{JK}\varepsilon^{IMN}A^{MN}}{X^4}\,, \tag{8.18}$$

which obeys both equations (8.10) and (8.12). However this term is linearly dependent of the others,

$$\mathcal{H}^{IJK}_{(f)7} = \mathcal{H}^{IJK}_{(f)4} + \mathcal{H}^{IJK}_{(f)5} - \mathcal{H}^{IJK}_{(f)6}\,, \tag{8.19}$$

and therefore the list (8.16) is complete.

The tensor $H^{IJK}_{(f)}$ in (8.13) is determined by two sectors with functions $H^{IJK}_{(f)n}$ and $\mathcal{H}^{IJK}_{(f)n}$. As will be seen further, these pieces should independently obey the constraints imposed by the conservation condition and symmetry of the correlation function under the permutation of superspace points.

With the aid of the identity (4.42a), the supercurrent conservation law (8.6) leads to the following constraint on $H^{IJK}_{(f,d)}$

$$\mathcal{D}^{(I}_\alpha H^{J)KL}_{(f,d)} - \frac{1}{3}\delta^{IJ}\mathcal{D}^M_\alpha H^{MKL}_{(f,d)} = 0\,. \tag{8.20}$$

Computing the derivatives of the tensors (8.15) and (8.16) and substituting them in the equation (8.20) we find the following constraints on the coefficients c_n and d_n:

$$\text{all } b_i = 0 \; ; \tag{8.21a}$$
$$c_2 = 2c_1 \,, \qquad c_3 = c_1 \,, \qquad c_5 = -2c_6 \,, \qquad c_4 = -4c_6 \; ; \tag{8.21b}$$
$$d_1 = d_2 = d_3 = d_4 = 0 \,, \quad 2d_5 + d_6 = 0 \; . \tag{8.21c}$$

In deriving these equations we have used the following $\mathcal{N} = 3$ identities:

$$\Theta^{I\alpha}\Theta^{J\beta}\Theta^{K\gamma}\Theta^{L\delta}\varepsilon_{JKL} = -\frac{1}{2}\varepsilon^{\alpha\beta}\Theta^2\Theta^{J\gamma}\Theta^{K\delta}\varepsilon_{IJK} - \frac{1}{2}\varepsilon^{\alpha\gamma}\Theta^2\Theta^{J\beta}\Theta^{K\delta}\varepsilon_{IJK} - \frac{1}{2}\varepsilon^{\alpha\delta}\Theta^2\Theta^{J\beta}\Theta^{K\gamma}\varepsilon_{IJK} \,, \tag{8.22a}$$
$$\Theta^{I\alpha}\Theta^{J}_{\alpha}\Theta^{K\beta}\Theta^{L}_{\gamma}\varepsilon_{JKL} = 2\Theta^2\Theta^{J\beta}\Theta^{K}_{\gamma}\varepsilon_{IJK} \,, \tag{8.22b}$$

which are differential consequences of the more general $\mathcal{N} = 3$ identity

$$\Theta^2\varepsilon_{IJK}\Theta^I_\alpha\Theta^J_\beta\Theta^K_\gamma = 0 \; . \tag{8.23}$$

As is seen from (8.21a), the part of the correlation function with the symmetric tensor $d^{\bar{a}\bar{b}\bar{c}}$ vanishes since $H^{IJK}_{(d)} = 0$. In the rest of this section we concentrate on the derivation of the part of the flavour current correlator with the antisymmetric tensor $f^{\bar{a}\bar{b}\bar{c}}$.

The three-point correlation function has to possess the symmetry property

$$\langle L^{I\bar{a}}(z_1)L^{J\bar{b}}(z_2)L^{K\bar{c}}(z_3)\rangle = \langle L^{K\bar{c}}(z_3)L^{J\bar{b}}(z_2)L^{I\bar{a}}(z_1)\rangle \; . \tag{8.24}$$

It imposes the following constraint on the tensor $H^{IJK}_{(f)}$

$$H^{IJK}_{(f)}(-\boldsymbol{X}_1^{\mathrm{T}}, -\Theta_1) = -\boldsymbol{x}_{13}{}^2\boldsymbol{X}_3{}^2 u^{II'}_{13}u^{JL}_{13}U^{LJ'}_3 u^{KK'}_{13} H^{K'J'I'}_{(f)}(\boldsymbol{X}_3, \Theta_3) \,, \tag{8.25}$$

as a consequence of (4.47). This equation gives additional relations among coefficients c_i and d_i, which are:

$$c_2 = 2c_1 - c_5 + \frac{1}{2}c_4 \,, \quad c_3 = c_1 - \frac{3}{2}c_5 + \frac{3}{4}c_4 \,, \quad c_6 = -\frac{5}{12}c_1 \; ;$$
$$d_2 = d_6 = 0 \,, \qquad d_5 = d_4 + 2d_1 \; . \tag{8.26}$$

Comparing these equations with (8.21b) and (8.21c) we see that all coefficients d_i vanish while all c_i can be expressed in terms of $c_1 \equiv b_{\mathcal{N}=3}$,

$$c_2 = c_3 = b_{\mathcal{N}=3} \,, \qquad c_4 = 3c_5 = -4c_6 = \frac{5}{3}b_{\mathcal{N}=3} \,, \qquad d_i = 0 \; . \tag{8.27}$$

Taking into account these relations, we can rewrite the resulting expression for the tensor $H^{IJK}_{(f)}$ (8.13) in terms of the matrix (4.44). Our final result for the three-point function is

$$\langle L^{I\bar{a}}(z_1)L^{J\bar{b}}(z_2)L^{K\bar{c}}(z_3)\rangle = f^{\bar{a}\bar{b}\bar{c}}\frac{u^{II'}_{13}u^{JJ'}_{23}}{\boldsymbol{x}_{13}{}^2\boldsymbol{x}_{23}{}^2}H^{I'J'K}_{(f)}(\boldsymbol{X}_3, \Theta_3) \tag{8.28a}$$

where

$$
\begin{aligned}
H^{IJK}_{(f)} = \frac{b_{\mathcal{N}=3}}{\boldsymbol{X}}\Big[& \varepsilon^{IJK} - U^{LJ}\varepsilon^{LIK} + U^{IL}\varepsilon^{LJK} \\
& -\frac{1}{16}(\delta^{IJ}\varepsilon^{KMN}U^{MN} + \varepsilon^{IMN}U^{MN}U^{KJ} + \varepsilon^{JMN}U^{MN}U^{IK}) \\
& +\frac{5}{16}(U^{IJ}\varepsilon^{KMN}U^{MN} + \delta^{IK}\varepsilon^{JMN}U^{MN} + \delta^{JK}\varepsilon^{IMN}U^{MN})\Big] \ .
\end{aligned} \tag{8.28b}
$$

Here we have used the following relation between the matrix U^{IJ} given by (4.46) and the composites in (8.17)

$$
U^{IJ} = \delta^{IJ} - 2\frac{A^{IJ}}{\boldsymbol{X}^2} + \frac{B^{IJ}\Theta^2}{\boldsymbol{X}^2} \ . \tag{8.29}
$$

The three-point function (8.28) looks more complicated than its 4D $\mathcal{N}=2$ counterpart [28]. The reason for that is the isovector notation used for the R-symmetry indices. Switching to the isospinor notation, following the prescription (8.2), should simplify the structure of the correlation function.

8.2 $\mathcal{N}=3$ supercurrent

The $\mathcal{N}=3$ supercurrent is described by a primary real spinor J_α of dimension 3/2, which is characterised by the superconformal transformation law

$$
\delta J_\alpha = -\xi J_\alpha - \frac{3}{2}\sigma(z)J_\alpha + \lambda_\alpha{}^\beta(z)J_\beta \tag{8.30}
$$

and obeys the conservation equation

$$
D^{I\alpha}J_\alpha = 0 \ . \tag{8.31}
$$

According to (5.2), the two-point correlation function for the supercurrent reads

$$
\langle J_\alpha(z_1)J_\beta(z_2)\rangle = \mathrm{i}c_{\mathcal{N}=3}\frac{\boldsymbol{x}_{12\alpha\beta}}{(\boldsymbol{x}_{12}{}^2)^2} \ , \tag{8.32}
$$

with $c_{\mathcal{N}=3}$ a parameter. It is antisymmetric, $\langle J_\alpha(z_1)J_\beta(z_2)\rangle = -\langle J_\beta(z_2)J_\alpha(z_1)\rangle$, and respects the conservation equation (8.31),

$$
D^{I\alpha}_{(1)}\langle J_\alpha(z_1)J_\beta(z_2)\rangle = 0 \ , \qquad z_1 \neq z_2 \ . \tag{8.33}
$$

In accordance with (5.3), the three-point correlator for the supercurrent has the form

$$
\langle J_\alpha(z_1)J_\beta(z_2)J_\gamma(z_3)\rangle = \frac{\boldsymbol{x}_{13\alpha\alpha'}\boldsymbol{x}_{23\beta\beta'}}{(\boldsymbol{x}_{13}{}^2)^2(\boldsymbol{x}_{23}{}^2)^2}H^{\alpha'\beta'}{}_\gamma(\boldsymbol{X}_3,\Theta_3) \ , \tag{8.34}
$$

where H should have the following scaling property

$$
H^{\alpha\beta\gamma}(\lambda^2\boldsymbol{X},\lambda\Theta) = \lambda^{-3}H^{\alpha\beta\gamma}(\boldsymbol{X},\Theta) \ . \tag{8.35}
$$

Due to

$$
\langle J_\beta(z_2)J_\alpha(z_1)J_\gamma(z_3)\rangle = -\langle J_\alpha(z_1)J_\beta(z_2)J_\gamma(z_3)\rangle \ , \tag{8.36}
$$

the tensor H should obey the following symmetry property

$$H^{\beta\alpha\gamma}(-\boldsymbol{X}^{\rm T},-\Theta)=-H^{\alpha\beta\gamma}(\boldsymbol{X},\Theta)\ . \tag{8.37}$$

The most general form for H compatible with the relations (8.35) and (8.37) is

$$H^{\alpha\beta\gamma}(\boldsymbol{X},\Theta)=\sum c_i H_i^{\alpha\beta\gamma}(\boldsymbol{X},\Theta)\,, \tag{8.38}$$

where c_i are some coefficients and

$$\begin{aligned}
H_1^{\alpha\beta}{}_\gamma(\boldsymbol{X},\Theta)&=\frac{\boldsymbol{X}^{\alpha\alpha'}\boldsymbol{X}^{\beta'\beta}\Theta^I_{\alpha'}\Theta^J_{\beta'}\Theta^K_\gamma\varepsilon_{IJK}}{\boldsymbol{X}^5}\,,\\
H_2^{\alpha\beta}{}_\gamma(\boldsymbol{X},\Theta)&=\frac{\boldsymbol{X}^{\beta\alpha}\boldsymbol{X}^{\mu\nu}\Theta^I_\mu\Theta^J_\nu\Theta^K_\gamma\varepsilon_{IJK}}{\boldsymbol{X}^5}\,,\\
H_3^{\alpha\beta}{}_\gamma(\boldsymbol{X},\Theta)&=\frac{(\delta^\beta_\gamma\boldsymbol{X}^{\alpha\rho}+\delta^\alpha_\gamma\boldsymbol{X}^{\rho\beta})\boldsymbol{X}^{\mu\nu}\Theta^I_\mu\Theta^J_\nu\Theta^K_\rho\varepsilon_{IJK}}{\boldsymbol{X}^5}\,,\\
H_4^{\alpha\beta}{}_\gamma(\boldsymbol{X},\Theta)&=\frac{\boldsymbol{X}^{\beta'\beta}\Theta^{I\alpha}\Theta^J_{\beta'}\Theta^K_\gamma}{\boldsymbol{X}^4}\varepsilon_{IJK}-\frac{\boldsymbol{X}^{\alpha\alpha'}\Theta^I_{\alpha'}\Theta^{J\beta}\Theta^K_\gamma}{\boldsymbol{X}^4}\varepsilon_{IJK}\ .
\end{aligned} \tag{8.39}$$

Here we have listed all linearly independent structures. Note that owing to the identity (8.23) there are no terms of order $O(\Theta^5)$.

Now we have to impose the constraint

$$\mathcal{D}^I_\alpha H^{\alpha\beta\gamma}=0\,, \tag{8.40}$$

which follows from the conservation law (8.31). In deriving (8.40), the identity (4.42a) has been used. At order $O(\Theta^2)$ the equation (8.40) gives

$$-c_1+2c_2=0\,,\quad c_2-c_3=0\,,\quad c_4=0\,, \tag{8.41}$$

while collecting the terms of order $O(\Theta^4)$ we find

$$-c_1+2c_2=0\ . \tag{8.42}$$

In the derivation of these equations we have used the identities (8.22b). The general solution of (8.41), (8.42) reads

$$c_4=0\,,\quad c_2=c_3=d_{\mathcal{N}=3}\,,\quad c_1=2d_{\mathcal{N}=3}\,, \tag{8.43}$$

where $d_{\mathcal{N}=3}$ is a single free coefficient.

As a result, the tensor $H^{\alpha\beta\gamma}$ has the following explicit form

$$\begin{aligned}
H^{\alpha\beta}{}_\gamma(\boldsymbol{X},\Theta)=\frac{d_{\mathcal{N}=3}}{\boldsymbol{X}^5}\Big[&(\delta^\beta_\gamma\boldsymbol{X}^{\alpha\rho}+\delta^\alpha_\gamma\boldsymbol{X}^{\rho\beta})\boldsymbol{X}^{\mu\nu}\Theta^I_\mu\Theta^J_\nu\Theta^K_\rho\varepsilon_{IJK}\\
&+\boldsymbol{X}^{\beta\alpha}\boldsymbol{X}^{\mu\nu}\Theta^I_\mu\Theta^J_\nu\Theta^K_\gamma\varepsilon_{IJK}+2\boldsymbol{X}^{\alpha\mu}\boldsymbol{X}^{\nu\beta}\Theta^I_\mu\Theta^J_\nu\Theta^K_\gamma\varepsilon_{IJK}\Big]\ .
\end{aligned} \tag{8.44}$$

One can also check that this expression obeys the equation

$$H^{\mu\nu}{}_\alpha(-\boldsymbol{X}_1^{\rm T},-\Theta_1)=\boldsymbol{X}_3{}^2\boldsymbol{x}_{13}^{\mu\mu'}\boldsymbol{x}_{13\alpha\alpha'}\boldsymbol{x}_{13}^{\nu\nu'}\boldsymbol{X}_{3\nu'\rho}H^{\alpha'\rho}{}_{\mu'}(\boldsymbol{X}_3,\Theta_3)\,, \tag{8.45}$$

which is a consequence of the symmetry property

$$\langle J_\gamma(z_3)J_\beta(z_2)J_\alpha(z_1)\rangle=-\langle J_\alpha(z_1)J_\beta(z_2)J_\gamma(z_3)\rangle\ . \tag{8.46}$$

JHEP06(2015)138

9 Concluding remarks

In this paper, we have demonstrated that for three-dimensional $\mathcal{N}$-extended superconformal field theories with $1 \leq \mathcal{N} \leq 3$ each of the two-point and three-point functions for the supercurrent is fixed by the symmetries and by the conservation law up to a single overall coefficient. In particular, our results imply that each of the two- and three-point functions for the stress-energy tensor in 3D superconformal theories are fixed up to one coefficient. It is natural to expect that the coefficients in the two- and three-point functions for the supercurrent are related to each other through a Ward identity just like in 4D (super)conformal theories [11, 30]. Although the required Ward identities may be derived using the known prepotential formulations for 3D $\mathcal{N} = 1$ supergravity [41] and 3D $\mathcal{N} = 2$ supergravity [42, 43], we postpone the study of such a relation for future work. The fact that such correlation functions are constrained up to an overall coefficient makes 3D superconformal theories similar to the well-studied case of 2D conformal field theory.

We have also proved that the three-point function of the flavour current multiplets is determined by a single functional form in the $\mathcal{N} = 1$ and $\mathcal{N} = 3$ cases. The specific feature of the $\mathcal{N} = 2$ case is that two independent structures are allowed for the three-point function of the flavour current multiplets, but only one of them contributes to the three-point function of the conserved currents contained in these multiplets. This is explicitly demonstrated in appendix C.

As was shown in [13, 14], 3D non-supersymmetric conformal theories can have certain *odd parity* contributions to three-point functions of the stress-energy tensors and flavour currents. They do not exist for an arbitrary number of space-time dimensions but are a specific 3D (and, perhaps, in general, an odd-dimensional) feature. Such terms do not arise in free conformal field theories but can appear in interacting Chern-Simons theories coupled to parity violating matter. Some general constructions of the parity violating terms in $\mathcal{N} = 1$ superconformal theories were later discussed in [81]. There it was shown that in some examples correlators of conserved currents can also contain parity odd contributions. In our approach we did not distinguish whether various allowed structures are even or odd under parity. For $\mathcal{N} = 1, 2, 3$ we always assumed the most general ansatz. Hence, our analysis demonstrates that odd parity contributions do not appear in the supersymmetric cases for both the flavour current and supercurrent correlators.[21] This is explicitly proved in appendix D for the $\mathcal{N} = 1$ flavour current multiplets.

It is an interesting problem to generalise the present results to the superconformal theories with $\mathcal{N} \geq 4$ supersymmetry. This is also postponed for future work.

We hope that the techniques developed in our paper will be useful in the context of generalised higher spin superconformal theories formulated on hyper-superspaces, see [82] and references therein.

[21] It is not difficult to show that our results are in complete agreement with [81]. For example, the correlator of three $\mathcal{N} = 1$ supercurrents corresponds to $\langle J_{3/2} J_{3/2} J_{3/2} \rangle$ in [81]. This correlator admits only one parity odd structure respecting the proper symmetry under the permutation of the three points but this structure is inconsistent with the conservation law.

JHEP06(2015)138

Recently, the so-called superembedding formalism in four dimensions [83–85], which was originally introduced by Siegel [86, 87] and fully elaborated in [58] under the name bi-supertwistor formalism, has been applied to compute correlation functions of multiplets containing conserved currents in 4D $\mathcal{N} = 1$ superconformal theories [85, 88, 89].[22] The 3D $\mathcal{N}$-extended bi-supertwistor formalism was presented in [58]. It would be of interest to apply this approach for an alternative computation of the correlation functions of the supercurrent and flavour current multiplets derived in our paper. The results given in section 3 might be useful for that.

Acknowledgments

We are grateful to Joseph Novak for comments on the manuscript. This work is supported in part by the ARC DP project DP140103925. The work of E.I.B. was also supported by the ARC Future Fellowship FT120100466.

A 3D notation and conventions

We mostly follow the notation and conventions adopted in [53]. In particular, the Minkowski metric is $\eta_{mn} = \mathrm{diag}(-1, 1, 1)$.

The spinor indices are raised and lowered using the $\mathsf{SL}(2,\mathbb{R})$ invariant tensors

$$\varepsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \varepsilon^{\alpha\gamma}\varepsilon_{\gamma\beta} = \delta_\beta^\alpha \tag{A.1}$$

by the standard rule:

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta\,, \qquad \psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta\,. \tag{A.2}$$

We employ real gamma-matrices, $\gamma_m := \big((\gamma_m)_\alpha{}^\beta\big)$, which are expressed in terms of the Pauli matrices as $\gamma_0 = -\mathrm{i}\sigma_2$, $\gamma_1 = \sigma_3$, $\gamma_2 = -\sigma_1$. They obey the algebra

$$\gamma_m\gamma_n = \eta_{mn}\mathbb{1} + \varepsilon_{mnp}\gamma^p\,, \tag{A.3}$$

where the Levi-Civita tensor is normalised as $\varepsilon^{012} = -\varepsilon_{012} = 1$. The completeness relation for the gamma-matrices reads

$$(\gamma^m)_{\alpha\beta}(\gamma_m)^{\rho\sigma} = -(\delta_\alpha^\rho\delta_\beta^\sigma + \delta_\alpha^\sigma\delta_\beta^\rho)\,. \tag{A.4}$$

Here $(\gamma_m)^{\alpha\beta}$ and $(\gamma_m)_{\alpha\beta}$ are obtained from $\gamma_m = (\gamma_m)_\alpha{}^\beta$ by the rules (A.2).

Given a three-vector x_m, it can be equivalently described by a symmetric second-rank spinor $x_{\alpha\beta}$ defined as

$$x_{\alpha\beta} := (\gamma^m)_{\alpha\beta}x_m = x_{\beta\alpha}\,, \qquad x_m = -\frac{1}{2}(\gamma_m)^{\alpha\beta}x_{\alpha\beta}\,. \tag{A.5}$$

[22] So far, the thee-point function for the $\mathcal{N} = 1$ supercurrent originally computed by Osborn [30] has not been re-derived within the superembedding approach.

The same convention is adopted for the spacetime derivatives,

$$\partial_{\alpha\beta} = (\gamma^m)_{\alpha\beta}\partial_m \,, \qquad \partial_m = -\frac{1}{2}(\gamma_m)^{\alpha\beta}\partial_{\alpha\beta} \tag{A.6}$$

such that

$$\partial_m x^n = \delta_m^n \,, \qquad \partial_{\alpha\beta}x^{\gamma\delta} = -(\delta_\alpha^\gamma\delta_\beta^\delta + \delta_\alpha^\delta\delta_\beta^\gamma) \;. \tag{A.7}$$

Note also that the square of a vector in terms of spinor indices reads

$$x^2 = x^m x_m = -\frac{1}{2}x^{\alpha\beta}x_{\alpha\beta} \;. \tag{A.8}$$

B $\mathcal{N} = 2$ correlation functions in chiral basis

The case $\mathcal{N} = 2$ is special since the R-symmetry group $\mathsf{SO}(2)$ is isomorphic to $\mathsf{U}(1)$, and one can define a chiral subspace of the full superspace on which the superconformal group $\mathsf{OSp}(2|4;\mathbb{R})$ acts by holomorphic transformations. This appendix is devoted to a brief discussion of the correlation functions involving (anti)chiral superfields.

B.1 (Anti)chiral two-point functions

Instead of the real Grassmann coordinates $\theta^{I\alpha} = (\theta^{1\alpha}, \theta^{2\alpha})$, we introduce new complex variables,

$$\theta^\alpha = \frac{1}{\sqrt{2}}(\theta^{1\alpha} + \mathrm{i}\theta^{2\alpha}) \,, \qquad \bar\theta^\alpha = \frac{1}{\sqrt{2}}(\theta^{1\alpha} - \mathrm{i}\theta^{2\alpha}) \,, \tag{B.1}$$

which have definite $\mathsf{U}(1)$ charges with respect to the R-symmetry group. The corresponding spinor covariant derivatives

$$D_\alpha = \frac{1}{\sqrt{2}}(D_\alpha^1 - \mathrm{i}D_\alpha^2) \,, \qquad \bar D_\alpha = -\frac{1}{\sqrt{2}}(D_\alpha^1 + \mathrm{i}D_\alpha^2) \tag{B.2}$$

obey the anti-commutation relations

$$\{D_\alpha, D_\beta\} = 0 \,, \quad \{\bar D_\alpha, \bar D_\beta\} = 0 \,, \quad \{D_\alpha, \bar D_\beta\} = -2\mathrm{i}\partial_{\alpha\beta} \,, \tag{B.3}$$

which guarantee the existence of a chiral subspace of the full superspace. The crucial features of the chiral subspace are that (i) it is invariant under the $\mathcal{N} = 2$ super-Poincaré group; and (ii) its bosonic y^a and fermionic θ^α coordinates are annihilated by the operators $\bar D_\gamma$. Its bosonic coordinate is

$$y^{\alpha\beta} = x^{\alpha\beta} + 2\mathrm{i}\bar\theta^{(\alpha}\theta^{\beta)} \;. \tag{B.4}$$

The superconformal transformation law of the real superspace coordinates, eq. (2.27), implies that the superconformal group acts by holomorphic transformations on the chiral subspace. The superconformal variations of the chiral coordinates $y^{\alpha\beta}$ and θ^α are

$$\begin{aligned}\delta y^{\alpha\beta} &= a^{\alpha\beta} + 4\mathrm{i}\bar\epsilon^{(\alpha}\theta^{\beta)} - \lambda^\alpha{}_\gamma y^{\gamma\beta} - y^{\alpha\gamma}\lambda_\gamma{}^\beta + \sigma y^{\alpha\beta} + y^{\alpha\gamma}y^{\beta\delta}b_{\gamma\delta} \\ &\qquad + \mathrm{i}y^{\alpha\gamma}\theta_\gamma\bar\eta^\beta + \mathrm{i}y^{\alpha\gamma}\theta^\beta\bar\eta_\gamma + 2\mathrm{i}y^{\beta\gamma}\theta^\alpha\bar\eta_\gamma + \mathrm{i}y^{\alpha\beta}\theta^\gamma\bar\eta_\gamma \,, \end{aligned} \tag{B.5a}$$

$$\delta\theta^\alpha = \epsilon^\alpha - \theta^\beta\lambda_\beta{}^\alpha + \frac{1}{2}\sigma\theta^\alpha - \mathrm{i}\Lambda\theta^\alpha + y^{\alpha\gamma}b_{\gamma\beta}\theta^\beta - y^{\alpha\beta}\eta_\beta + \mathrm{i}\theta^2\bar\eta^\alpha \;. \tag{B.5b}$$

The parameter Λ of $\mathsf{U}(1)$ R-symmetry is related to the $\mathsf{SO}(2)$ parameters Λ_{IJ} as $\Lambda_{IJ} = \varepsilon_{IJ}\Lambda$, where ε_{IJ} is the antisymmetric tensor.

Let us consider the following Q-supersymmetric two-point function

$$y_{12}^{\alpha\beta} = (x_1 - x_2)^{\alpha\beta} - 2\mathrm{i}\theta_1^{(\alpha}\bar{\theta}_{12}^{\beta)} + 2\mathrm{i}\theta_{12}^{(\alpha}\bar{\theta}_2^{\beta)} , \tag{B.6}$$

which is chiral in its first superspace argument and antichiral in the other,

$$\bar{D}_{(1)\gamma}y_{12}^{\alpha\beta} = D_{(2)\gamma}y_{12}^{\alpha\beta} = 0 . \tag{B.7}$$

It is related to the two-point function (4.13a) as

$$y_{12}^{\alpha\beta} = \boldsymbol{x}_{12}^{\alpha\beta} + 2\mathrm{i}\bar{\theta}_{12}^{\alpha}\theta_{12}^{\beta} . \tag{B.8}$$

For its square we have

$${y_{12}}^2 = {\boldsymbol{x}_{12}}^2 + \mathrm{i}\theta_{12}^{I\alpha}(\boldsymbol{x}_{12})_{\alpha\beta}\theta_{12}^{I\beta} + \varepsilon_{IJ}\theta_{12}^{I\alpha}(\boldsymbol{x}_{12})_{\alpha\beta}\theta_{12}^{J\beta} . \tag{B.9}$$

Using this formula, we get the conjugation rule for ${y_{12}}^2$:

$$\overline{{y_{12}}^2} \equiv {\bar{y}_{12}}^2 = {y_{21}}^2 . \tag{B.10}$$

One also finds the following useful identity for the product of ${y_{12}}^2$ and ${\bar{y}_{12}}^2$

$${y_{12}}^2{\bar{y}_{12}}^2 = ({\boldsymbol{x}_{12}}^2)^2 . \tag{B.11}$$

Now, consider the orthogonal matrix u_{12}^{IJ} given by (4.25) and transforming by the rule (4.29). In this transformation, the matrix $\Lambda_{IJ}(z)$ is antisymmetric, and thus has one independent component which we denote by $\Lambda(z)$,

$$\Lambda_{IJ}(z) = \varepsilon_{IJ}\Lambda(z) . \tag{B.12}$$

Given the matrix u_{12}^{IJ}, we construct a complex scalar two-point function v_{12}

$$v_{12} = \frac{1}{2}(u_{12}^{II} + \mathrm{i}\varepsilon_{IJ}u_{12}^{IJ}) , \tag{B.13}$$

which transforms as

$$\widetilde{\delta}v_{12} = \mathrm{i}(\Lambda(z_1) - \Lambda(z_2))v_{12} . \tag{B.14}$$

Using the explicit expressions for the two-point functions (4.13a) and (4.25) it can be shown that (B.9) and (B.13) are related to each other as

$${y_{12}}^2 = {\boldsymbol{x}_{12}}^2 v_{12} . \tag{B.15}$$

Aapplying (4.19) and (B.14), we find the transformation of (B.9) to be

$$\widetilde{\delta}{y_{12}}^2 = (\boldsymbol{\sigma}(z_1) + \boldsymbol{\sigma}(z_2)){y_{12}}^2 , \tag{B.16}$$

where the chiral superfield $\boldsymbol{\sigma}$ includes parameters of local scale and $\mathsf{U}(1)$ transformations [53]

$$\boldsymbol{\sigma}(z) = \sigma(z) + \mathrm{i}\Lambda(z) , \qquad \bar{D}_\alpha\boldsymbol{\sigma}(z) = 0 . \tag{B.17}$$

Thus (B.15) is a natural (anti)chiral generalisation of ${\boldsymbol{x}_{12}}^2$ which can serve as a building block for correlation functions involving (anti)chiral superfields.

B.2 $\mathcal{N}=2$ correlation functions with (anti)chiral superfields

In this section, we consider some simple correlation functions which involve chiral and antichiral primary superfields. First, we will consider a two-point correlator with chiral and antichiral superfields, and then we will derive the general expression for the three-point correlation function with chiral, antichiral and linear superfields.

Let Φ be a chiral superfield of dimension q with no spinor indices. Its superconformal transformation reads

$$\delta\Phi(z) = -\xi\Phi(z) - q\boldsymbol{\sigma}(z)\Phi(z)\,, \tag{B.18}$$

where $\boldsymbol{\sigma}$ is given by (B.17). This transformation preserves chirality since $\boldsymbol{\sigma}$ is chiral. Using the two-point function (B.9) it is straightforward to construct the two-point correlator of the chiral superfield Φ and its conjugate $\bar{\Phi}$,

$$\langle\Phi(z_1)\bar{\Phi}(z_2)\rangle = \frac{c}{({y_{12}}^2)^q}\,, \tag{B.19}$$

where c is an arbitrary coefficient. Owing to (3.38) this expression automatically possesses correct chirality properties with respect to both arguments and has the right transformation rule because of (B.16).

As an example of a three-point function, we consider the correlator of a linear superfield $G^{\bar{a}}$, a chiral superfield $\Phi^{\bar{a}}$ and an antichiral one $\bar{\Phi}^{\bar{a}}$. Here the index $\bar{a}$ can be considered as a flavour group index. In this case these superfields can be identified with $\mathcal{N}=2$ superfield components of the $\mathcal{N}=3$ flavour current studied in section 8.1. Assuming that all these three superfields have dimension one, we look for the correlation function with the use of the standard ansatz

$$\langle G^{\bar{a}}(z_1)\Phi^{\bar{b}}(z_2)\bar{\Phi}^{\bar{c}}(z_3)\rangle = \frac{1}{{\boldsymbol{x}_{13}}^2{y_{23}}^2}[f^{\bar{a}\bar{b}\bar{c}}H_{(f)}(\boldsymbol{X}_3,\Theta_3) + d^{\bar{a}\bar{b}\bar{c}}H_{(d)}(\boldsymbol{X}_3,\Theta_3)]\,, \tag{B.20}$$

where the functions $H_{(f,d)}$ should have the following homogeneity property

$$H_{(f,d)}(\lambda^2\boldsymbol{X},\lambda\Theta) = \lambda^{-1}H_{(f,d)}(\boldsymbol{X},\Theta)\;. \tag{B.21}$$

Using the identity (4.42a), the linearity of the superfield $G^{\bar{a}}$, $D^2G^{\bar{a}} = \bar{D}^2G^{\bar{a}} = 0$, turns into the following equations for the functions $H_{(f,d)}$

$$\mathcal{D}^2H_{(f,d)} = \bar{\mathcal{D}}^2H_{(f,d)} = 0\,, \tag{B.22}$$

where

$$\mathcal{D}_\alpha = \frac{\partial}{\partial\Theta^\alpha} + \mathrm{i}\bar{\Theta}^\beta\frac{\partial}{\partial X^{\alpha\beta}}\,, \qquad \bar{\mathcal{D}}_\alpha = -\frac{\partial}{\partial\bar{\Theta}^\alpha} - \mathrm{i}\Theta^\beta\frac{\partial}{\partial X^{\alpha\beta}}\;. \tag{B.23}$$

The objects Θ_α and $\bar{\Theta}_\alpha$ are expressed in terms of Θ^I_α by the rule (B.1). One can check that the equations (B.23) being rewritten in terms of the derivatives $\mathcal{D}^I_\alpha$ are equivalent to (7.10). Therefore the solution of (B.22) is

$$H_{(f,d)}(\boldsymbol{X},\Theta) = \frac{\mathrm{i}c_{(f,d)1}}{X} + c_{(f,d)2}\frac{\mathrm{i}\varepsilon_{IJ}\Theta^I_\alpha X^{\alpha\beta}\Theta^J_\beta}{X^3}\,, \tag{B.24}$$

where $c_{(f,d)1}$ and $c_{(f,d)2}$ are some complex coefficients.

Obviously, the correlation function (B.20) obeys the reality condition

$$\langle G^{\bar{a}}(z_1)\Phi^{\bar{b}}(z_2)\bar{\Phi}^{\bar{c}}(z_3)\rangle^* = \langle G^{\bar{a}}(z_1)\Phi^{\bar{c}}(z_3)\bar{\Phi}^{\bar{b}}(z_2)\rangle \ . \tag{B.25}$$

The latter leads to the constraints for the functions $H_{(f,d)}$:

$$H_{(f)}(-\boldsymbol{X}_2^{\rm T}, -\Theta_2) = -\frac{\boldsymbol{x}_{12}{}^2}{\boldsymbol{x}_{13}{}^2}\bar{H}_{(f)}(\boldsymbol{X}_3, \Theta_3)\,, \quad H_{(d)}(-\boldsymbol{X}_2^{\rm T}, -\Theta_2) = \frac{\boldsymbol{x}_{12}{}^2}{\boldsymbol{x}_{13}{}^2}\bar{H}_{(d)}(\boldsymbol{X}_3, \Theta_3)\,. \tag{B.26}$$

This equation implies the following reality properties of the constants $c_{(f,d)1}$ and $c_{(f,d)2}$ in (7.12)

$$\bar{c}_{(f)1} = c_{(f)1}\,, \quad \bar{c}_{(f)2} = c_{(f)2}\,, \quad \bar{c}_{(d)1} = -c_{(d)1}\,, \quad \bar{c}_{(d)2} = -c_{(d)2}\ . \tag{B.27}$$

Finally, we have to take into account the chirality of the correlation function with respect to the second argument

$$\bar{D}_{(2)\alpha}H_{(f,d)}(\boldsymbol{X}_3, \Theta_3) = 0\ . \tag{B.28}$$

With the use of the identity (4.42b) the latter equation gives the following constraint to the functions $H_{(f,d)}$

$$\bar{\mathcal{Q}}_\alpha H_{(f,d)}(\boldsymbol{X}, \Theta) = 0\,, \qquad \bar{\mathcal{Q}}_\alpha = -{\rm i}\frac{\partial}{\partial\Theta^\alpha} - \gamma^m_{\alpha\beta}\Theta^\beta\frac{\partial}{\partial X^m}\ . \tag{B.29}$$

This equation is satisfied if the coefficients $c_{(f,d)1}$ and $c_{(f,d)2}$ in (7.12) are related to each other as

$$c_{(f,d)1} = 2c_{(f,d)2} \equiv 2c_{(f,d)}\ . \tag{B.30}$$

As a result, each of the functions $H_{(f)}$ and $H_{(d)}$ has one free coefficient

$$H_{(f,d)} = {\rm i}c_{(f,d)}\left(\frac{2}{X} + \frac{\varepsilon_{IJ}\Theta^I_\alpha X^{\alpha\beta}\Theta^J_\beta}{X^3}\right) = {\rm i}c_{(f,d)}\left(\frac{2}{\boldsymbol{X}} + \frac{\Theta^4}{4\boldsymbol{X}^3} + \frac{\varepsilon_{IJ}\Theta^I_\alpha \boldsymbol{X}^{\alpha\beta}\Theta^J_\beta}{\boldsymbol{X}^3}\right)\ . \tag{B.31}$$

Here we used the relation (4.37) to represent the function H in terms of covariant objects (4.30). Note that in (B.31) the coefficient $c_{(f)}$ is real while $c_{(d)}$ is imaginary.

C $\mathcal{N} = 2 \to \mathcal{N} = 1$ superspace reduction

This appendix is devoted to the $\mathcal{N} = 2 \to \mathcal{N} = 1$ superspace reduction of the three-point functions for the $\mathcal{N} = 2$ supercurrent and flavour current multiplets.

C.1 The supercurrent correlation function

As discussed in section 1, every $\mathcal{N} = 2$ superconformal field theory is a special $\mathcal{N} = 1$ superconformal field theory. The $\mathcal{N} = 1$ supercurrent $J_{\alpha\beta\gamma}$ for this theory is related to its $\mathcal{N} = 2$ supercurrent $J_{\alpha\beta}$ by the first equation in (1.7b). As a consequence, the $\mathcal{N} = 1$ supercurrent correlation function (6.34) appears as a result of the $\mathcal{N} = 2 \to \mathcal{N} = 1$ reduction from (7.20)

$$\langle J_{\alpha\alpha'\alpha''}(z_1)J_{\beta\beta'\beta''}(z_2)J_{\gamma\gamma'\gamma''}(z_3)\rangle = -{\rm i}D^{\mathbf{2}}_{(1)\alpha}D^{\mathbf{2}}_{(2)\beta}D^{\mathbf{2}}_{(3)\gamma}\langle J_{\alpha'\alpha''}(z_1)J_{\beta'\beta''}(z_2)J_{\gamma'\gamma''}(z_3)\rangle|\,. \tag{C.1}$$

Recall that here the symbol $|$ means that we have to set $\theta^{\mathbf{2}}_\alpha$ to zero after computing the derivatives. In this section we denote the values of the $\mathsf{SO}(2)$ indices $I = \mathbf{1}, \mathbf{2}$ with boldface font to distinguish them from indices of other types.

According to (6.34), the $\mathcal{N} = 1$ supercurrent correlation function is expressed in terms of the tensor $H^{\alpha\alpha'\alpha'',\beta\beta'\beta'',\gamma\gamma'\gamma''}$ (or $H^{\alpha m,\beta n,\gamma k}$ if we trade the pairs of the spinor indices into the vector ones by the rule (6.39)) which was found in the form (6.59) with the tensors C and D given by (6.55), (6.57) and (6.50), (6.58), respectively. In its turn, the $\mathcal{N} = 2$ supercurrent correlation function is represented by the tensor $H^{\alpha\alpha',\beta\beta',\gamma\gamma'}$ given explicitly by (7.44). In this section we will show that the former can be derived from the latter by means of the equation (C.1).

To start with, we point out that the expression (7.44) for the tensor $H^{\alpha\alpha',\beta\beta',\gamma\gamma'}$ with spinor indices converted into Lorentz ones can be rewritten as

$$H^{mnk} = -\mathrm{i} d_{\mathcal{N}=2}\, \gamma_p^{\mu\nu} \Theta^{\mathbf{1}}_\mu \Theta^{\mathbf{2}}_\nu C^{mnp,k}\,, \tag{C.2}$$

where the tensor $C^{mnp,k}$ has the form

$$\begin{aligned} C^{mnp,k} = & \frac{1}{X^3}(\eta^{mn}\eta^{kp} + \eta^{mk}\eta^{np} + \eta^{nk}\eta^{mp}) + \frac{3}{X^5}(X^m X^k \eta^{np} + X^n X^k \eta^{mp} + X^p X^k \eta^{mn}) \\ & - \frac{5}{X^5}(X^m X^n \eta^{pk} + X^n X^p \eta^{mk} + X^m X^p \eta^{nk}) - \frac{5}{X^7} X^m X^n X^p X^k\,. \end{aligned} \tag{C.3}$$

In the formula (C.2) the dependence on the Grassmann variables is only through the factor $\gamma_p^{\mu\nu}\Theta^{\mathbf{1}}_\mu\Theta^{\mathbf{2}}_\nu$ while the rest is described by the tensor (C.3) which is a function of X. It is interesting to note that (C.3) coincides with the similar tensor in eqs. (6.55), (6.57) which was encountered in section 6.2. As was already shown there, this tensor is symmetric and traceless over the first three indices

$$C^{mnp,k} = C^{(mnp),k}\,, \qquad \eta_{mn} C^{mnp,k} = 0\,, \tag{C.4}$$

and obeys the differential equation

$$\partial_m C^{mnp,k} = 0\,, \tag{C.5}$$

where $\partial_m = \frac{\partial}{\partial X^m}$. We also showed in section 6.2 that the equations (C.4) and (C.5) define the form of the tensor (C.3) uniquely, up to an overall coefficient.

Now we substitute the expression (7.20) for the $\mathcal{N} = 2$ supercurrent correlation functions into (C.1) and represent it in the following form

$$\begin{aligned} & \langle J_{\alpha\alpha'\alpha''}(z_1) J_{\beta\beta'\beta''}(z_2) J_{\gamma\gamma'\gamma''}(z_3)\rangle \\ & = -\mathrm{i} D^{\mathbf{2}}_{(1)\alpha} D^{\mathbf{2}}_{(2)\beta} D^{\mathbf{2}}_{(3)\gamma} \frac{\boldsymbol{x}_{13\alpha'\rho'} \boldsymbol{x}_{13\alpha''\rho''} \boldsymbol{x}_{23\beta'\sigma'} \boldsymbol{x}_{23\beta''\sigma''}}{\boldsymbol{x}_{13}{}^6 \boldsymbol{x}_{23}{}^6} H^{\rho'\rho'',\sigma'\sigma''}{}_{\gamma'\gamma''}(X_3, \Theta_3)| \\ & = A + B\,, \end{aligned} \tag{C.6}$$

where in these two terms A and B the derivatives are distributed as follows

$$\begin{aligned} A = & \,\mathrm{i} \frac{\boldsymbol{x}_{13\alpha'\rho'} \boldsymbol{x}_{13\alpha''\rho''}}{\boldsymbol{x}_{13}{}^6} \left(D^{\mathbf{2}}_{(3)\gamma} D^{\mathbf{2}}_{(2)\beta} \frac{\boldsymbol{x}_{23\beta'\sigma'} \boldsymbol{x}_{23\beta''\sigma''}}{\boldsymbol{x}_{23}{}^6} \right) D^{\mathbf{2}}_{(1)\alpha} H^{\rho'\rho'',\sigma'\sigma''}{}_{\gamma'\gamma''}| \\ & - \mathrm{i} \frac{\boldsymbol{x}_{23\beta'\sigma'} \boldsymbol{x}_{23\beta''\sigma''}}{\boldsymbol{x}_{23}{}^6} \left(D^{\mathbf{2}}_{(3)\gamma} D^{\mathbf{2}}_{(1)\alpha} \frac{\boldsymbol{x}_{13\alpha'\rho'} \boldsymbol{x}_{13\alpha''\rho''}}{\boldsymbol{x}_{13}{}^6} \right) D^{\mathbf{2}}_{(2)\beta} H^{\rho'\rho'',\sigma'\sigma''}{}_{\gamma'\gamma''}|\,, \end{aligned} \tag{C.7}$$

$$B = \mathrm{i} \frac{\boldsymbol{x}_{13\alpha'\rho'} \boldsymbol{x}_{13\alpha''\rho''} \boldsymbol{x}_{23\beta'\sigma'} \boldsymbol{x}_{23\beta''\sigma''}}{\boldsymbol{x}_{13}{}^6 \boldsymbol{x}_{23}{}^6} D^{\mathbf{2}}_{(3)\gamma} D^{\mathbf{2}}_{(2)\beta} D^{\mathbf{2}}_{(1)\alpha} H^{\rho'\rho'',\sigma'\sigma''}{}_{\gamma'\gamma''}|\,. \tag{C.8}$$

It is easy to see that the terms with the covariant spinor derivatives distributed in other ways vanish since the expressions like $D^{\mathbf{2}}_{(1)\alpha}\boldsymbol{x}_{13\beta\beta'} = -2\mathrm{i}\theta^{\mathbf{2}}_{13\beta'}\varepsilon_{\beta\alpha}$ die in the $|$-projection. We will analyse the A and B sectors separately.

We begin by considering the A term. Using the explicit expression for $\boldsymbol{x}$, eq. (4.13a), we find

$$D^{\mathbf{2}}_{(3)\gamma}D^{\mathbf{2}}_{(1)\alpha}\frac{\boldsymbol{x}_{13\alpha'\rho'}\boldsymbol{x}_{13\alpha''\rho''}}{\boldsymbol{x}_{13}{}^6}\Big| = 2\mathrm{i}\frac{\varepsilon_{\rho'\gamma}\varepsilon_{\alpha'\alpha}\boldsymbol{x}_{13\alpha''\rho''}+\varepsilon_{\rho''\gamma}\varepsilon_{\alpha''\alpha}\boldsymbol{x}_{13\alpha'\rho'}}{\boldsymbol{x}_{13}{}^6}+6\mathrm{i}\frac{\boldsymbol{x}_{13\alpha\gamma}\boldsymbol{x}_{13\alpha'\rho'}\boldsymbol{x}_{13\alpha''\rho''}}{\boldsymbol{x}_{13}{}^8}\,,$$
$$D^{\mathbf{2}}_{(3)\gamma}D^{\mathbf{2}}_{(2)\beta}\frac{\boldsymbol{x}_{23\beta'\sigma'}\boldsymbol{x}_{23\beta''\sigma''}}{\boldsymbol{x}^6_{23}}\Big| = 2\mathrm{i}\frac{\varepsilon_{\sigma'\gamma}\varepsilon_{\beta'\beta}\boldsymbol{x}_{13\beta''\sigma''}+\varepsilon_{\sigma''\gamma}\varepsilon_{\beta''\beta}\boldsymbol{x}_{23\beta'\sigma'}}{\boldsymbol{x}_{23}{}^6}+6\mathrm{i}\frac{\boldsymbol{x}_{23\beta\gamma}\boldsymbol{x}_{23\beta'\sigma'}\boldsymbol{x}_{13\beta''\sigma''}}{\boldsymbol{x}_{23}{}^8}\,. \tag{C.9}$$

Next, with the use of (4.42) we get

$$D^{\mathbf{2}}_{(1)\alpha}H^{mnk}\big| = (\boldsymbol{x}^{-1}_{13})_{\rho\alpha}\mathcal{D}^{\mathbf{2}\rho}H^{mnk}\big| = -\frac{\boldsymbol{x}_{13\alpha\rho}}{\boldsymbol{x}_{13}{}^2}\mathcal{D}^{\mathbf{2}\rho}H^{mnk}\big|\,,$$
$$D^{\mathbf{2}}_{(2)\beta}H^{mnk}\big| = \mathrm{i}(\boldsymbol{x}^{-1}_{23})_{\rho\beta}\mathcal{Q}^{\mathbf{2}\rho}H^{mnk}\big| = \frac{\boldsymbol{x}_{23\beta\rho}}{\boldsymbol{x}_{23}{}^2}\mathcal{D}^{\mathbf{2}\rho}H^{mnk}\big|\,. \tag{C.10}$$

Now we substitute (C.9) and (C.10) into (C.7) and apply simple identities like

$$\varepsilon_{\sigma'\gamma}\varepsilon_{\beta'\beta} = \frac{1}{\boldsymbol{x}_{23}{}^2}(\boldsymbol{x}_{23\beta\sigma'}\boldsymbol{x}_{23\beta'\gamma}-\boldsymbol{x}_{23\beta\gamma}\boldsymbol{x}_{23\beta'\sigma'}) \tag{C.11}$$

to represent the A sector of the supercurrent correlation function in the form

$$A = \frac{\boldsymbol{x}_{13\alpha'\rho'}\boldsymbol{x}_{13\alpha''\rho''}\boldsymbol{x}_{13\alpha\rho}}{\boldsymbol{x}_{13}{}^8}\frac{\boldsymbol{x}_{23\beta\sigma}\boldsymbol{x}_{23\beta'\sigma'}\boldsymbol{x}_{23\beta''\sigma''}}{\boldsymbol{x}_{23}{}^8}H_{(A)}^{\rho\rho'\rho'',\sigma\sigma'\sigma''}{}_{\gamma\gamma'\gamma''}(X_3,\Theta_3)\,, \tag{C.12}$$

where

$$H_{(A)}^{\alpha\alpha'\alpha'',\beta\beta'\beta''}{}_{\gamma\gamma'\gamma''} = 6\delta^{(\underline{\beta}}_{\gamma}\mathcal{D}^{\mathbf{2}\alpha}H^{\alpha'\alpha'',\underline{\beta'}\,\underline{\beta''})}{}_{\gamma'\gamma''} + 6\delta^{(\underline{\alpha}}_{\gamma}\mathcal{D}^{\mathbf{2}\beta}H^{\underline{\alpha'}\,\underline{\alpha''}),\beta'\beta''}{}_{\gamma'\gamma''}\big|\,. \tag{C.13}$$

Here the symmetrisation involves only the underlined indices.

We stress that the covariant spinor derivatives $\mathcal{D}^{\mathbf{2}}_{\alpha}$ in the expression (C.13) act only on the Grassmann variable $\Theta^{\mathbf{2}}_{\nu}$ and do not hit the X-dependent tensor $C^{mnp,k}$ since the action of the covariant spinor derivative on any combination of X^m is proportional to $\Theta^{\mathbf{2}}$ which dies in the $|$-projection. Hence, after converting pairs of spinor indices into vector ones and using identities with three-dimensional gamma-matrices (A.3), (A.4), the expression (C.13) can be rewritten in the form

$$H_{(A)}^{\alpha m,\beta n,\gamma k} = \mathrm{i}d_{\mathcal{N}=2}\big[-6(\gamma_p)^{\alpha\beta}\Theta^{\gamma}C^{mnp,k} - (\gamma^p)^{\alpha\beta}(\gamma^r)^{\gamma\delta}\Theta_{\delta}\eta_{pp'}\eta_{qq'}\eta_{rr'}(4\varepsilon^{p'q'r'}C^{mnq,k}$$
$$+\varepsilon^{np'q'}C^{mqr',k}+\varepsilon^{nq'r'}C^{mqp',k}+\varepsilon^{mq'p'}C^{qnr',k}+\varepsilon^{mq'r'}C^{qnp',k})\big]\,, \tag{C.14}$$

where $\Theta_{\delta} \equiv \Theta^{\mathbf{1}}_{\delta}$. The first term here coincides (up to the factor -6) with the corresponding term in (6.59). To match the other terms we need to consider also contributions to $H^{\alpha m,\beta n,\gamma k}$ from the B part given by (C.8).

In the B sector of the correlation function we need to compute three covariant spinor derivatives of the tensor (C.2),

$$D^{\mathbf{2}}_{(3)\gamma}D^{\mathbf{2}}_{(2)\beta}D^{\mathbf{2}}_{(1)\alpha}H^{mnk}\big| = \mathrm{i}(\boldsymbol{x}^{-1}_{23})^{\sigma}{}_{\beta}(\boldsymbol{x}^{-1}_{13})^{\rho}{}_{\alpha}D^{\mathbf{2}}_{(3)\gamma}[\mathcal{Q}^{\mathbf{2}}_{\sigma}\mathcal{D}^{\mathbf{2}}_{\rho}+u^{\mathbf{21}}_{23}\mathcal{Q}^{\mathbf{1}}_{\sigma}\mathcal{D}^{\mathbf{2}}_{\rho}+u^{\mathbf{21}}_{13}\mathcal{Q}^{\mathbf{2}}_{\sigma}\mathcal{D}^{\mathbf{1}}_{\rho}]H^{mnk}\big|\,. \tag{C.15}$$

JHEP06(2015)138

In this expression, $u^{\mathbf{21}}_{23}$ and $u^{\mathbf{21}}_{13}$ are components of the matrix (4.25) which appear in (C.15) owing to the identities (4.42). The factor $(\boldsymbol{x}_{23}^{-1})^{\sigma}{}_{\beta}(\boldsymbol{x}_{13}^{-1})^{\rho}{}_{\alpha}$ in the right-hand side of (C.15) is the right one which is required to form the expression (6.34). Now we have to analyse the remaining piece of this expression.

Using the explicit expressions (4.38) for $\mathcal{Q}^{\mathbf{2}}_{\sigma}$ and $\mathcal{D}^{\mathbf{2}}_{\rho}$ the first term in the right-hand side of (C.15) can be rewritten as

$$\begin{aligned}
&D^{\mathbf{2}}_{(3)\gamma}\mathcal{Q}^{\mathbf{2}}_{\sigma}\mathcal{D}^{\mathbf{2}}_{\rho}H^{mnk}| \\
&= D^{\mathbf{2}}_{(3)\gamma}\big[\Theta^{\mathbf{2}\mu}\frac{\partial}{\partial X^{\sigma\mu}}\frac{\partial}{\partial\Theta^{\mathbf{2}\rho}}+\Theta^{\mathbf{2}\mu}\frac{\partial}{\partial X^{\rho\mu}}\frac{\partial}{\partial\Theta^{\mathbf{2}\sigma}}-\Theta^{\mathbf{2}\mu}\frac{\partial}{\partial X^{\rho\sigma}}\frac{\partial}{\partial\Theta^{\mathbf{2}\mu}}\big]H^{mnk}| \\
&= [(\boldsymbol{x}_{13}^{-1})^{\mu}{}_{\gamma}-(\boldsymbol{x}_{23}^{-1})^{\mu}{}_{\gamma}]\left(\frac{\partial}{\partial X^{\sigma\mu}}\frac{\partial}{\partial\Theta^{\mathbf{2}\rho}}+\frac{\partial}{\partial X^{\rho\mu}}\frac{\partial}{\partial\Theta^{\mathbf{2}\sigma}}-\frac{\partial}{\partial X^{\rho\sigma}}\frac{\partial}{\partial\Theta^{\mathbf{2}\mu}}\right)H^{mnk}|\ . \qquad \text{(C.16)}
\end{aligned}$$

To get the last line we used the fact that in the |-projection only those terms survive in which the derivative $D^{\mathbf{2}}_{(3)\gamma}$ acts on $\Theta^{\mathbf{2}\mu}$ and produces the factor $[(\boldsymbol{x}_{13}^{-1})^{\mu}{}_{\gamma}-(\boldsymbol{x}_{23}^{-1})^{\mu}{}_{\gamma}]$. However, the latter structure is non-covariant in the sense that it cannot be expressed solely in terms of $\boldsymbol{X}_{\alpha\beta}$ and Θ^{I}_{α}. Indeed, using the identity (4.35) and the definition of $\boldsymbol{X}_{3\alpha\beta}$ (4.30), we represent the factor $[(\boldsymbol{x}_{13}^{-1})^{\mu}{}_{\gamma}-(\boldsymbol{x}_{23}^{-1})^{\mu}{}_{\gamma}]$ in (C.16) as

$$(\boldsymbol{x}_{13}^{-1})_{\alpha\beta}-(\boldsymbol{x}_{23}^{-1})_{\alpha\beta}=-\boldsymbol{X}_{3\alpha\beta}+\mathrm{i}\frac{\varepsilon_{\alpha\beta}}{\boldsymbol{x}_{23}{}^{2}}\theta^{2}_{23}+2\mathrm{i}(\boldsymbol{x}_{13}^{-1})_{\alpha\mu}\theta^{\mu}_{13}\theta^{\nu}_{32}(\boldsymbol{x}_{32}^{-1})_{\nu\beta}\ . \qquad \text{(C.17)}$$

The last two terms here are non-covariant. Therefore, they must cancel against similar terms coming from the last two terms in (C.15)

$$\begin{aligned}
D^{\mathbf{2}}_{(3)\gamma}[u^{\mathbf{21}}_{23}\mathcal{Q}^{\mathbf{1}}_{\sigma}\mathcal{D}^{\mathbf{2}}_{\rho}+u^{\mathbf{21}}_{13}\mathcal{Q}^{\mathbf{2}}_{\sigma}\mathcal{D}^{\mathbf{1}}_{\rho}]H^{mnk}| &= 2\Theta^{\mathbf{1}}_{\gamma}\frac{\partial}{\partial\Theta^{\mathbf{1}\sigma}}\frac{\partial}{\partial\Theta^{\mathbf{2}\rho}}H^{mnk} \\
&-2\mathrm{i}[(\boldsymbol{x}_{23}^{-1})_{\gamma\mu}\theta^{\mu}_{23}\Theta^{\mathbf{1}\nu}\frac{\partial}{\partial X^{\sigma\nu}}\frac{\partial}{\partial\Theta^{\mathbf{2}\rho}}+(\boldsymbol{x}_{13}^{-1})_{\gamma\mu}\theta^{\mu}_{13}\Theta^{\mathbf{1}\nu}\frac{\partial}{\partial X^{\rho\nu}}\frac{\partial}{\partial\Theta^{\mathbf{2}\sigma}}]H^{mnk}|\ . \qquad \text{(C.18)}
\end{aligned}$$

To prove the cancellation of non-covariant terms we have to use the fact the tensor H is linear in $\Theta^{\mathbf{1}}$ and can be represented in the form $H=\Theta^{\mathbf{1}}_{\kappa}h^{\kappa}$, for some h^{κ}. Here we suppress all indices of the tensor H as they do not play role in this consideration. Then, using the explicit expression for $\Theta^{\mathbf{1}}_{\alpha}$ (4.30), we observe that the non-covariant terms have the same structure and cancel against each other

$$\begin{aligned}
\mathrm{i}(\boldsymbol{x}^{-1}_{13\mu\gamma}-\boldsymbol{x}^{-1}_{23\mu\gamma})\Theta^{\mathbf{1}}_{\kappa}h^{\kappa} &= -\mathrm{i}\boldsymbol{X}_{3\mu\gamma}H+\left(\frac{\theta^{2}_{23}}{\boldsymbol{x}^{2}_{23}}\boldsymbol{x}^{-1}_{13\gamma\kappa}\theta^{\kappa}_{13}+\frac{\theta^{2}_{13}}{\boldsymbol{x}^{2}_{13}}\boldsymbol{x}^{-1}_{23\gamma\kappa}\theta^{\kappa}_{23}\right)h_{\mu}\,, \\
2\boldsymbol{x}^{-1}_{23\gamma\mu}\theta^{\mu}_{23}\Theta^{\mathbf{1}\nu}\Theta^{\mathbf{1}}_{\kappa}h^{\kappa} &= -\left(\frac{\theta^{2}_{13}}{\boldsymbol{x}^{2}_{13}}\boldsymbol{x}^{-1}_{23\gamma\mu}\theta^{\mu}_{23}+\frac{\theta^{2}_{23}}{\boldsymbol{x}^{2}_{23}}\boldsymbol{x}^{-1}_{13\gamma\mu}\theta^{\mu}_{13}\right)h^{\nu}\,, \\
2\boldsymbol{x}^{-1}_{13\gamma\mu}\theta^{\mu}_{13}\Theta^{\mathbf{1}\nu}\Theta^{\mathbf{1}}_{\kappa}h^{\kappa} &= -\left(\frac{\theta^{2}_{13}}{\boldsymbol{x}^{2}_{13}}\boldsymbol{x}^{-1}_{23\gamma\mu}\theta^{\mu}_{23}+\frac{\theta^{2}_{23}}{\boldsymbol{x}^{2}_{23}}\boldsymbol{x}^{-1}_{13\gamma\mu}\theta^{\mu}_{13}\right)h^{\nu}\ . \qquad \text{(C.19)}
\end{aligned}$$

Thus in the expression (C.15) only covariant terms remain

$$\begin{aligned}
D^{\mathbf{2}}_{(3)\gamma}D^{\mathbf{2}}_{(2)\beta}D^{\mathbf{2}}_{(1)\alpha}H^{mnk}| &= \mathrm{i}(\boldsymbol{x}_{23}^{-1})^{\sigma}{}_{\beta}(\boldsymbol{x}_{13}^{-1})^{\rho}{}_{\alpha}\Big(2\Theta^{\mathbf{1}}_{\gamma}\frac{\partial}{\partial\Theta^{\mathbf{1}\sigma}}\frac{\partial}{\partial\Theta^{\mathbf{2}\rho}}+X^{\mu}_{\gamma}\frac{\partial}{\partial X^{\rho\sigma}}\frac{\partial}{\partial\Theta^{\mathbf{2}\mu}} \\
&\qquad -X^{\mu}_{\gamma}\frac{\partial}{\partial X^{\sigma\mu}}\frac{\partial}{\partial\Theta^{\mathbf{2}\rho}}-X^{\mu}_{\gamma}\frac{\partial}{\partial X^{\rho\mu}}\frac{\partial}{\partial\Theta^{\mathbf{2}\sigma}}\Big)H^{mnk}|\ . \qquad \text{(C.20)}
\end{aligned}$$

To summarise, the B part of the correlation function (C.8) can be represented in the form

$$B = \frac{\boldsymbol{x}_{13\alpha'\rho'}\boldsymbol{x}_{13\alpha''\rho''}\boldsymbol{x}_{13\alpha\rho}}{\boldsymbol{x}_{13}{}^8}\frac{\boldsymbol{x}_{23\beta\sigma}\boldsymbol{x}_{23\beta'\sigma'}\boldsymbol{x}_{23\beta''\sigma''}}{\boldsymbol{x}_{23}{}^8} H_{(B)}^{\rho\rho'\rho'',\sigma\sigma'\sigma''}{}_{\gamma\gamma'\gamma''}(X_3,\Theta_3)\,, \tag{C.21}$$

where the tensor $H_{(B)}$, after converting the pairs of spinor indices into the vector ones, is expressed in terms of the derivatives of (C.2) as follows

$$\begin{aligned} H_{(B)}^{\alpha m,\beta n}{}_{\gamma k} = \Big(&-2\Theta^{\mathbf{1}}_\gamma \frac{\partial}{\partial\Theta^{\mathbf{1}}_\beta}\frac{\partial}{\partial\Theta^{\mathbf{2}}_\alpha} + X_{\mu\gamma}\frac{\partial}{\partial X_{\beta\mu}}\frac{\partial}{\partial\Theta^{\mathbf{2}}_\alpha} \\ &+X_{\mu\gamma}\frac{\partial}{\partial X_{\alpha\mu}}\frac{\partial}{\partial\Theta^{\mathbf{2}}_\beta} - X_{\mu\gamma}\frac{\partial}{\partial X_{\alpha\beta}}\frac{\partial}{\partial\Theta^{\mathbf{2}}_\mu}\Big) H^{mn}{}_k| \,. \end{aligned} \tag{C.22}$$

Now we substitute here the tensor H^{mnk} in the form (C.2) and compute the derivatives over the Grassmann variables. As a result, with the use of identities with three-dimensional gamma-matrices (A.3), (A.4), we find

$$\begin{aligned} H_{(B)}^{\alpha m,\beta n,\gamma k} = \mathrm{i}d_{\mathcal{N}=2}\big[&(\gamma^p)^{\alpha\beta}\Theta^\gamma C^{mnp,k} - 3(\gamma^p)^{\alpha\beta}(\gamma^r)^{\gamma\rho}\Theta_\rho\varepsilon_{qpr}C^{mnq,k} \\ &-(\gamma^p)^{\alpha\beta}(\gamma^t)^{\gamma\rho}\Theta_\rho\varepsilon_{lst}X^l\partial^s C^{mnp,k} - (\gamma^p)^{\alpha\beta}(\gamma_s)^{\gamma\rho}\Theta_\rho\varepsilon_{qpl}X^l\partial^s C^{mnq,k} \\ &+(\gamma^p)^{\alpha\beta}(\gamma_l)^{\gamma\rho}\Theta_\rho\varepsilon_{qps}X^l\partial^s C^{mnq,k} + (\gamma^p)^{\alpha\beta}(\gamma^r)^{\gamma\rho}\Theta_\rho\varepsilon_{qsr}X^q\partial_p C^{mns,k}\big]. \end{aligned} \tag{C.23}$$

In deriving this expression we have also used the simple relation

$$X^l\partial_l C^{mnp,k} = -3C^{mnp,k}\,, \tag{C.24}$$

which reflects the fact that the tensor C is homogeneous of degree -3 with respect to X_m.

The final result of computing the correlation function of $\mathcal{N}=1$ supercurrent is given by the sum of the tensors (C.14) and (C.23). It can be represented in the form similar to (6.59):

$$H^{\alpha m,\beta n,\gamma k} = -5\mathrm{i}\,d_{\mathcal{N}=2}\left(\gamma_p^{\alpha\beta}\Theta^\gamma C^{mnp,k} + \gamma_p^{\alpha\beta}\gamma_r^{\gamma\delta}\Theta_\delta D^{(mn),k,p,r}\right)\,, \tag{C.25}$$

where

$$\begin{aligned} D^{(mn),k,p,r} = \frac{1}{5}\big(&\varepsilon^{pqr}\eta_{qq'}C^{mnq',k} + \varepsilon^{nqk}\eta_{qq'}C^{mq'r,t} + \varepsilon^{nqr}\eta_{qq'}C^{mq'p,k} + \varepsilon^{mqp}\eta_{qq'}C^{q'nr,k} \\ &+\varepsilon^{mqr}\eta_{qq'}C^{q'np,k} + \varepsilon^{lsr}X_l\partial_s C^{mnp,k} + \varepsilon^{qpl}\eta_{qq'}X_l\partial^r C^{mnq',k} \\ &-\varepsilon^{qpl}\eta_{qq'}X^r\partial_l C^{mnq',k} - \varepsilon^{lqr}\eta_{qq'}X_q\partial^p C^{mnq',k}\big)\,. \end{aligned} \tag{C.26}$$

Our final task is to match the tensor (C.26) with the last two terms in (6.59). A straightforward comparison is rather complicated and we will give an indirect proof. For this we will show that (C.26) satisfies all the same equations as $D^{(mnp),k,r}$ from section 6.2.

First, one can show that (C.26) is symmetric and traceless in (m,n,p) (though this symmetry is not manifest)

$$D^{(mn),k,p,r} = D^{(mnp),k,r}\,, \quad \eta_{mn}D^{(mnp),k,r} = 0\,. \tag{C.27}$$

Now we split $D^{(mnp),k,r}$ in (C.26) into the symmetric and antisymmetric parts in (k,r). Using the explicit form of the tensor $C^{mnp,k}$ in (C.3) one can show that

$$\varepsilon^{krq}\eta_{kk'}\eta_{rr'}D^{(mnp),k',r'} = -C^{mnp,q}\,, \quad \eta_{kr}D^{(mnp),k,r} = 0\,. \tag{C.28}$$

This implies that $D^{(mnp),k,r}$ can be written as follows

$$D^{(mnp),k,r} = D^{(mnp),(kr)} + \frac{1}{2}\varepsilon^{krq}\eta_{qq'}C^{mnp,q'}\,, \quad \eta_{kr}D^{(mnp),(kr)} = 0\,. \tag{C.29}$$

We see that the antisymmetric part in eq. (C.29) precisely agrees with that of the tensor $D^{(mnp),k,r}$ in section 6.2, see eq. (6.46). We are now left to match the symmetric part $D^{(mnp),(kr)}$.

To continue, we contract (C.26) with η_{pr} and ε_{prq} to obtain

$$\eta_{pr}D^{(mnp),k,r} = 0\,, \quad \varepsilon_{rpq}D^{(mnp),k,r} + \eta_{qq'}C^{mnq',k} = 0\,, \tag{C.30}$$

which, using (C.29), imply (6.47). Finally, using eqs. (C.3), (C.4), (C.5) we find that

$$\partial_m D^{(mnp),k,r} = 0\,, \quad \partial_m D^{(mnp),(kr)} = 0\,. \tag{C.31}$$

As a result, we found that $D^{(mnp),(kr)}$ satisfies exactly the same equations as in section 6.2. On the other hand, we have shown in section 6.2 that these equations allow us to fully solve for D in terms of C and such a solution is unique. Since the tensor C in (C.2) coincides with the one from section 6.2 we conclude that (C.26) is the same as $D^{(mnp),k,r}$ found in section 6.2. This completes our proof.

As a byproduct of the above computations, we find that the coefficients in the three-point functions of $\mathcal{N}=1$ and $\mathcal{N}=2$ supercurrents derived in the sections 6.2 and 7.2 are related to each other as

$$d_{\mathcal{N}=1} = -5d_{\mathcal{N}=2}\,. \tag{C.32}$$

C.2 The flavour current correlation function

The $\mathcal{N}=2 \to \mathcal{N}=1$ superspace reduction of the flavour current correlation functions given by (7.8) and (6.10) goes the same way as in the previous section. Therefore here we mention only the essential details of this derivation.

Recall that the $\mathcal{N}=1$ flavour current multiplet L_α appears as a component of the $\mathcal{N}=2$ flavour current superfield L as in eq. (1.10b). Hence, the corresponding relation for the correlation functions reads

$$\langle L^{\bar a}_\alpha(z_1)L^{\bar b}_\beta(z_2)L^{\bar c}_\gamma(z_3)\rangle = -\mathrm{i}D^{\mathbf{2}}_{(1)\alpha}D^{\mathbf{2}}_{(2)\beta}D^{\mathbf{2}}_{(3)\gamma}\langle L^{\bar a}(z_1)L^{\bar b}(z_2)L^{\bar c}(z_3)\rangle|\,, \tag{C.33}$$

where $|$ means that we set $\theta^{\mathbf{2}}_\alpha = 0$ at each superspace point. Recall that the $\mathcal{N}=2$ flavour current correlation function (7.8) consists of two parts which include tensors $f^{\bar a\bar b\bar c}$ and $d^{\bar a\bar b\bar c}$ and both functions $H_{(f)}$ and $H_{(d)}$ are non-trivial, see (7.15). One could expect that the corresponding $\mathcal{N}=1$ correlator appearing in (C.33) may include both such parts. However, as we will show further, the part with the symmetric tensor $d^{\bar a\bar b\bar c}$ vanishes upon this reduction and does not contribute to the $\mathcal{N}=1$ flavour current correlator.

Substituting (7.8) into (C.33) we represent the latter as a sum of the two pieces

$$\langle L_\alpha(z_1)L_\beta(z_2)L_\gamma(z_3)\rangle = f^{\bar{a}\bar{b}\bar{c}}(A_{(f)}+B_{(f)}) + d^{\bar{a}\bar{b}\bar{c}}(A_{(d)}+B_{(d)})\,, \tag{C.34}$$

where

$$\begin{aligned} A_{(f,d)} &= \frac{\mathrm{i}}{\boldsymbol{x}_{13}{}^2}\left(D^{\mathbf{2}}_{(3)\gamma}D^{\mathbf{2}}_{(2)\beta}\frac{1}{\boldsymbol{x}_{23}{}^2}\right)D^{\mathbf{2}}_{(1)\alpha}H_{(f,d)}(X_3,\Theta_3) \\ &\quad -\frac{\mathrm{i}}{\boldsymbol{x}_{23}{}^2}\left(D^{\mathbf{2}}_{(3)\gamma}D^{\mathbf{2}}_{(1)\alpha}\frac{1}{\boldsymbol{x}_{13}{}^2}\right)D^{\mathbf{2}}_{(2)\beta}H_{(f,d)}(X_3,\Theta_3)|\,, \\ B_{(f,d)} &= \frac{\mathrm{i}}{\boldsymbol{x}_{13}{}^2\boldsymbol{x}_{23}{}^2}D^{\mathbf{2}}_{(3)\gamma}D^{\mathbf{2}}_{(2)\beta}D^{\mathbf{2}}_{(1)\alpha}H_{(f,d)}(X_3,\Theta_3)|\;. \end{aligned} \tag{C.35}$$

The functions $H_{(f,d)}$ are given in (7.12).

In the A part, we apply the following equations:

$$D^{\mathbf{2}}_{(3)\gamma}D^{\mathbf{2}}_{(2)\beta}\frac{1}{\boldsymbol{x}_{23}{}^2}| = 2\mathrm{i}\frac{\boldsymbol{x}_{23\beta\gamma}}{\boldsymbol{x}_{23}{}^4}\,, \qquad D^{\mathbf{2}}_{(3)\gamma}D^{\mathbf{2}}_{(1)\alpha}\frac{1}{\boldsymbol{x}_{13}{}^2}| = 2\mathrm{i}\frac{\boldsymbol{x}_{23\alpha\gamma}}{\boldsymbol{x}_{12}{}^4}\;. \tag{C.36}$$

Then using analogs of the equations (C.10), we represent the A sector in the form

$$A_{(f,d)} = \frac{\boldsymbol{x}_{13\alpha\alpha'}\boldsymbol{x}_{23\beta\beta'}}{\boldsymbol{x}_{13}{}^4\boldsymbol{x}_{23}{}^4}H^{\alpha'\beta'}_{(A,f,d)\gamma}(X_3,\Theta_3)\,, \tag{C.37}$$

where

$$H^{\alpha\beta\gamma}_{(A,f)} = 2\varepsilon^{\gamma\beta}\mathcal{D}^{\mathbf{2}\alpha}H_{(f)} + 2\varepsilon^{\gamma\alpha}\mathcal{D}^{\mathbf{2}\beta}H_{(f)}| = b_{\mathcal{N}=2}\frac{4\mathrm{i}}{X^3}(\varepsilon^{\gamma\beta}X^{\alpha\rho}\Theta_\rho + \varepsilon^{\gamma\alpha}X^{\beta\rho}\Theta_\rho)\,, \tag{C.38a}$$

$$H^{\alpha\beta\gamma}_{(A,d)} = 2\varepsilon^{\gamma\beta}\mathcal{D}^{\mathbf{2}\alpha}H_{(d)} + 2\varepsilon^{\gamma\alpha}\mathcal{D}^{\mathbf{2}\beta}H_{(d)}| = 0\;. \tag{C.38b}$$

Now we consider the part B in (C.35). Computation of this piece goes similarly to the analysis given in the previous section. Indeed, the equations (C.15)–(C.20) remain exactly the same with the only modification that we have to discard the indices m,n,k in the tensor H. Thus, we can immediately write down the analog of (C.21):

$$B_{(f,d)} = \frac{\boldsymbol{x}_{13\alpha\alpha'}\boldsymbol{x}_{23\beta\beta'}}{\boldsymbol{x}_{13}{}^4\boldsymbol{x}_{23}{}^4}H^{\alpha'\beta'}_{(B,f,d)\gamma}(X_3,\Theta_3)\,, \tag{C.39}$$

where

$$\begin{aligned} H^{\alpha\beta}_{(B,f,d)\gamma} &= \Bigg(-2\Theta^{\mathbf{1}}_\gamma\frac{\partial}{\partial\Theta^{\mathbf{1}}_\beta}\frac{\partial}{\partial\Theta^{\mathbf{2}}_\alpha} + X_{\mu\gamma}\frac{\partial}{\partial X_{\beta\mu}}\frac{\partial}{\partial\Theta^{\mathbf{2}}_\alpha} + X_{\mu\gamma}\frac{\partial}{\partial X_{\alpha\mu}}\frac{\partial}{\partial\Theta^{\mathbf{2}}_\beta} \\ &\quad -X_{\mu\gamma}\frac{\partial}{\partial X_{\alpha\beta}}\frac{\partial}{\partial\Theta^{\mathbf{2}}_\mu}\Bigg)H_{(f,d)}|\;. \end{aligned} \tag{C.40}$$

Substituting the function (7.12) into (C.40) and computing the derivatives we find

$$\begin{aligned} H^{\alpha\beta\gamma}_{(B,f)} &= b_{\mathcal{N}=2}\frac{\mathrm{i}}{X^3}(10X^{\alpha\beta}\Theta^\gamma - 4X^{\alpha\gamma}\Theta^\beta - 4X^{\beta\gamma}\Theta^\alpha \\ &\quad -6\varepsilon^{\gamma\alpha}X^{\beta\rho}\Theta_\rho - 6\varepsilon^{\gamma\beta}X^{\alpha\rho}\Theta_\rho)\,, \end{aligned} \tag{C.41a}$$

$$H^{\alpha\beta\gamma}_{(B,d)} = 0\;. \tag{C.41b}$$

The equations (C.38b) and (C.41b) show that the $\mathcal{N} = 1$ flavour current correlation function does not receive contributions with the symmetric tensor $d^{\bar{a}\bar{b}\bar{c}}$,

$$H_{(d)}^{\alpha\beta\gamma} = H_{(A,d)}^{\alpha\beta\gamma} + H_{(B,d)}^{\alpha\beta\gamma} = 0 \ . \tag{C.42}$$

The other part with the antisymmetric tensor $f^{\bar{a}\bar{b}\bar{c}}$ is non-trivial. It is given by the sum of the expressions (C.38a) and (C.41a)

$$\begin{aligned} H_{(f)}^{\alpha\beta\gamma} &= H_{(A,f)}^{\alpha\beta\gamma} + H_{(B,f)}^{\alpha\beta\gamma} \\ &= b_{\mathcal{N}=2}\frac{2\mathrm{i}}{X^3}(5X^{\alpha\beta}\Theta^{\gamma} - 2X^{\alpha\gamma}\Theta^{\beta} - 2X^{\beta\gamma}\Theta^{\alpha} - \varepsilon^{\gamma\alpha}X^{\beta\rho}\Theta_{\rho} - \varepsilon^{\gamma\beta}X^{\alpha\rho}\Theta_{\rho}) \ . \end{aligned} \tag{C.43}$$

Finally, applying the identity (6.16), the tensor (C.43) can be brought to the form

$$\begin{aligned} H_{(f)}^{\alpha\beta\gamma}(X,\Theta) &= b_{\mathcal{N}=2}\frac{2\mathrm{i}}{X^3}(X^{\alpha\beta}\Theta^{\gamma} - \varepsilon^{\alpha\gamma}X^{\beta\rho}\Theta_{\rho} - \varepsilon^{\beta\gamma}X^{\alpha\rho}\Theta_{\rho}) \\ &= b_{\mathcal{N}=2}\frac{2\mathrm{i}}{\boldsymbol{X}^3}(\boldsymbol{X}^{\alpha\beta}\Theta^{\gamma} - \varepsilon^{\alpha\gamma}\boldsymbol{X}^{\beta\rho}\Theta_{\rho} - \varepsilon^{\beta\gamma}\boldsymbol{X}^{\alpha\rho}\Theta_{\rho}) \ . \end{aligned} \tag{C.44}$$

Comparing the last expression with (6.23) we conclude that the coefficients $b_{\mathcal{N}=1}$ and $b_{\mathcal{N}=2}$ are related to each other as

$$b_{\mathcal{N}=1} = 2b_{\mathcal{N}=2} \ . \tag{C.45}$$

D Component reduction

The correlation functions of the energy-momentum tensor and flavour currents originate as components in the θ-expansion of the correlation functions for the supercurrent and flavour current multiplets, respectively. In this section, we consider a particular example in which we demonstrate how to derive the correlation function of the flavour current from the corresponding superfield correlator obtained in section 6.1.

We start with the $\mathcal{N} = 1$ flavour current correlator in the form (6.10). Substituting the latter into (6.25), we represent the correlation function as a sum of two pieces

$$\langle L^{\bar{a}}_{\alpha\alpha'}(x_1)L^{\bar{b}}_{\beta\beta'}(x_2)L^{\bar{c}}_{\gamma\gamma'}(x_3)\rangle = A + B \,, \tag{D.1}$$

where

$$\begin{aligned} A &= f^{\bar{a}\bar{b}\bar{c}}\frac{\boldsymbol{x}_{13\alpha'\alpha''}}{\boldsymbol{x}_{13}{}^4}\left(D_{(3)\gamma}D_{(2)\beta}\frac{\boldsymbol{x}_{23\beta'\beta''}}{\boldsymbol{x}_{23}{}^4}\right)D_{(1)\alpha}H^{\alpha''\beta''}{}_{\gamma'}(X_3,\Theta_3) \\ &\quad -f^{\bar{a}\bar{b}\bar{c}}\frac{\boldsymbol{x}_{23\beta'\beta''}}{\boldsymbol{x}_{23}{}^4}\left(D_{(3)\gamma}D_{(1)\alpha}\frac{\boldsymbol{x}_{13\alpha'\alpha''}}{\boldsymbol{x}_{13}{}^4}\right)D_{(2)\beta}H^{\alpha''\beta''}{}_{\gamma'}(X_3,\Theta_3)| \,, \end{aligned} \tag{D.2}$$

$$B = f^{\bar{a}\bar{b}\bar{c}}\frac{\boldsymbol{x}_{13\alpha'\alpha''}\boldsymbol{x}_{23\beta'\beta''}}{\boldsymbol{x}_{13}{}^4\boldsymbol{x}_{23}{}^4}D_{(3)\gamma}D_{(2)\beta}D_{(1)\alpha}H^{\alpha''\beta''}{}_{\gamma'}(X_3,\Theta_3)| \ . \tag{D.3}$$

In the A sector, we compute the derivatives of the objects $\boldsymbol{x}_{13}$ and $\boldsymbol{x}_{23}$ using their explicit form (4.13a),

$$\begin{aligned} D_{(3)\gamma}D_{(1)\alpha}\frac{\boldsymbol{x}_{13\alpha'\alpha''}}{\boldsymbol{x}_{13}{}^4}| &= \frac{2\mathrm{i}}{\boldsymbol{x}_{13}{}^6}(\boldsymbol{x}_{13\alpha\alpha''}\boldsymbol{x}_{13\alpha'\gamma} + \boldsymbol{x}_{13\alpha\gamma}\boldsymbol{x}_{13\alpha'\alpha''}) \,, \\ D_{(3)\gamma}D_{(2)\beta}\frac{\boldsymbol{x}_{23\beta'\beta''}}{\boldsymbol{x}_{23}{}^4}| &= \frac{2\mathrm{i}}{\boldsymbol{x}_{23}{}^6}(\boldsymbol{x}_{23\beta\beta''}\boldsymbol{x}_{23\beta'\gamma} + \boldsymbol{x}_{23\beta\gamma}\boldsymbol{x}_{23\beta'\beta''}) \ . \end{aligned} \tag{D.4}$$

JHEP06(2015)138

With the use of (4.42), the derivatives of the tensor H in (D.2) can be written as

$$D_{(1)\alpha} H^{\alpha''\beta''}{}_{\gamma'}| = -\frac{\boldsymbol{x}_{13\alpha\rho}}{\boldsymbol{x}_{13}{}^2} \mathcal{D}^\rho H^{\alpha''\beta''}{}_{\gamma'}| , \quad D_{(2)\beta} H^{\alpha''\beta''}{}_{\gamma'}| = \frac{\boldsymbol{x}_{23\beta\rho}}{\boldsymbol{x}_{23}{}^2} \mathcal{D}^\rho H^{\alpha''\beta''}{}_{\gamma'}| . \tag{D.5}$$

Substituting (D.4) and (D.5) into (D.2) we represent the A part in the form

$$A = \frac{\boldsymbol{x}_{13\alpha\rho}\boldsymbol{x}_{13\alpha'\rho'}\boldsymbol{x}_{23\beta\sigma}\boldsymbol{x}_{23\beta'\sigma'}}{\boldsymbol{x}_{13}{}^6 \boldsymbol{x}_{23}{}^6} f^{\bar a \bar b \bar c} H_{(A)}^{\rho\rho',\sigma\sigma'}{}_{\gamma\gamma'}(X_3) , \tag{D.6}$$

where

$$H_{(A)}^{\rho\rho',\sigma\sigma'}{}_{\gamma\gamma'} = -4\mathrm{i}\delta_\gamma^{(\underline{\sigma}} \mathcal{D}^\rho H^{\rho'\underline{\sigma}')}{}_{\gamma'} - 4\mathrm{i}\delta_\gamma^{(\underline{\rho}} \mathcal{D}^\sigma H^{\underline{\rho}')\sigma'}{}_{\gamma'}| . \tag{D.7}$$

The symmetrisation here involves only the underlined indices.

Consider the B part given by (D.3). Similarly as in eqs. (C.15) and (C.16), we find

$$\begin{aligned} D_{(3)\gamma} D_{(2)\beta} D_{(1)\alpha} H^{\alpha'\beta'\gamma'} &= \mathrm{i}(\boldsymbol{x}_{23}^{-1})^\sigma{}_\beta (\boldsymbol{x}_{13}^{-1})^\rho{}_\alpha D_{(3)\gamma} \mathcal{Q}_\sigma \mathcal{D}_\rho H^{\alpha'\beta'\gamma'} \\ &= -\mathrm{i}(\boldsymbol{x}_{23}^{-1})^\sigma{}_\beta (\boldsymbol{x}_{13}^{-1})^\rho{}_\alpha X^\mu_\gamma \left(\frac{\partial}{\partial X^{\sigma\mu}} \frac{\partial}{\partial \Theta^\rho} + \frac{\partial}{\partial X^{\rho\mu}} \frac{\partial}{\partial \Theta^\sigma} - \frac{\partial}{\partial X^{\rho\sigma}} \frac{\partial}{\partial \Theta^\mu} \right) H^{\alpha'\beta'\gamma'}| . \end{aligned} \tag{D.8}$$

Here we used an analog of the identity (C.17) in which all Grassmann variables vanish. Substituting (D.8) into (D.3) we represent the B part of the correlation function in the form

$$B = \frac{\boldsymbol{x}_{13\alpha\rho}\boldsymbol{x}_{13\alpha'\rho'}\boldsymbol{x}_{23\beta\sigma}\boldsymbol{x}_{23\beta'\sigma'}}{\boldsymbol{x}_{13}{}^6 \boldsymbol{x}_{23}{}^6} f^{\bar a \bar b \bar c} H_{(B)}^{\rho\rho',\sigma\sigma'}{}_{\gamma\gamma'}(X_3) , \tag{D.9}$$

where

$$H_{(B)\rho\rho',\sigma\sigma',\gamma\gamma'} = -\mathrm{i} X^\mu_\gamma \left(\frac{\partial}{\partial X^{\sigma\mu}} \frac{\partial}{\partial \Theta^\rho} + \frac{\partial}{\partial X^{\rho\mu}} \frac{\partial}{\partial \Theta^\sigma} - \frac{\partial}{\partial X^{\rho\sigma}} \frac{\partial}{\partial \Theta^\mu} \right) H_{\rho'\sigma'\gamma'}| . \tag{D.10}$$

Now we substitute the tensor (6.23) into (D.7) and (D.10), and after computing derivatives, we find

$$\begin{aligned} H_{\rho\rho',\sigma\sigma',\gamma\gamma'} &= H_{(A)\rho\rho',\sigma\sigma',\gamma\gamma'} + H_{(B)\rho\rho',\sigma\sigma',\gamma\gamma'} \\ &= \frac{3b_{\mathcal{N}=1}}{X^5} X_{\rho\sigma} X_{\rho'\sigma'} X_{\gamma\gamma'} - \frac{b_{\mathcal{N}=1}}{X^3} \big[X_{\rho'\sigma'} (\varepsilon_{\gamma\sigma}\varepsilon_{\rho\gamma'} + \varepsilon_{\gamma\rho}\varepsilon_{\sigma\gamma'}) \\ &\quad -5X_{\rho\sigma}(\varepsilon_{\gamma\sigma'}\varepsilon_{\rho'\gamma'} + \varepsilon_{\gamma\rho'}\varepsilon_{\sigma'\gamma'}) + X_{\rho\rho'}(\varepsilon_{\gamma\sigma}\varepsilon_{\sigma'\gamma'} - 2\varepsilon_{\gamma\sigma'}\varepsilon_{\sigma\gamma'}) \\ &\quad +X_{\sigma\sigma'}(\varepsilon_{\gamma\rho}\varepsilon_{\rho'\gamma'} - 2\varepsilon_{\gamma\rho'}\varepsilon_{\rho\gamma'}) + X_{\rho\sigma'}(\varepsilon_{\gamma\sigma}\varepsilon_{\rho'\gamma'} - 2\varepsilon_{\gamma\rho'}\varepsilon_{\sigma\gamma'}) \\ &\quad +X_{\sigma\rho'}(\varepsilon_{\gamma\rho}\varepsilon_{\sigma'\gamma'} - 2\varepsilon_{\gamma\sigma'}\varepsilon_{\rho\gamma'}) - X_{\gamma\gamma'}(\varepsilon_{\rho\rho'}\varepsilon_{\sigma\sigma'} + \varepsilon_{\rho\sigma'}\varepsilon_{\sigma\rho'}) \\ &\quad +2X_{\rho\gamma}(\varepsilon_{\sigma\sigma'}\varepsilon_{\rho'\sigma'} + \varepsilon_{\sigma\rho'}\varepsilon_{\sigma'\gamma'}) + 2X_{\sigma\gamma}(\varepsilon_{\rho\rho'}\varepsilon_{\sigma'\gamma'} + \varepsilon_{\rho\sigma'}\varepsilon_{\rho'\gamma'}) \\ &\quad +X_{\sigma'\gamma}(\varepsilon_{\rho\rho'}\varepsilon_{\sigma\gamma'} + \varepsilon_{\sigma\rho'}\varepsilon_{\rho\gamma'}) + X_{\gamma\rho'}(\varepsilon_{\sigma\sigma'}\varepsilon_{\rho\gamma'} + \varepsilon_{\sigma\gamma'}\varepsilon_{\rho\sigma'}) \big] . \end{aligned} \tag{D.11}$$

This tensor defines the flavour current three-point correlation function,

$$\langle L^{\bar a}_{\alpha\alpha'}(x_1) L^{\bar b}_{\beta\beta'}(x_2) L^{\bar c}_{\gamma\gamma'}(x_3) \rangle = \frac{\boldsymbol{x}_{13\alpha\rho}\boldsymbol{x}_{13\alpha'\rho'}\boldsymbol{x}_{23\beta\sigma}\boldsymbol{x}_{23\beta'\sigma'}}{\boldsymbol{x}_{13}{}^6 \boldsymbol{x}_{23}{}^6} f^{\bar a \bar b \bar c} H^{\rho\rho',\sigma\sigma'}{}_{\gamma\gamma'}(X_3) . \tag{D.12}$$

It is instructive to convert the pairs of spinor indices into vector ones in the correlation function (D.12) to compare it with the corresponding expression obtained in [11]. Using the identity[23]

$$-\frac{1}{2}\gamma_m^{\alpha\alpha'}\gamma_n^{\beta\beta'}\frac{\boldsymbol{x}_{13\alpha\beta}\boldsymbol{x}_{13\alpha'\beta'}}{\boldsymbol{x}_{13}{}^2} = \eta_{mn} - 2\frac{x_{13m}x_{13n}}{x_{13}{}^2} \equiv I_{mn}(x_{13})\,, \tag{D.13}$$

we find

$$\begin{aligned}\langle L_m^{\bar a}(x_1)L_n^{\bar b}(x_2)L_k^{\bar c}(x_3)\rangle &= -\frac{1}{8}\gamma_m^{\alpha\alpha'}\gamma_n^{\beta\beta'}\gamma_k^{\gamma\gamma'}\langle L_{\alpha\alpha'}^{\bar a}(x_1)L_{\beta\beta'}^{\bar b}(x_2)L_{\gamma\gamma'}^{\bar c}(x_3)\rangle \\ &= \frac{I_{mm'}(x_{13})I_{nn'}(x_{23})}{x_{13}{}^4x_{23}{}^4}f^{\bar a\bar b\bar c}H^{m'n'}{}_k(X_3)\,,\end{aligned} \tag{D.14}$$

where

$$\begin{aligned}H_{mnk}(X) &= -\frac{1}{8}\gamma_m^{\rho\rho'}\gamma_n^{\sigma\sigma'}\gamma_k^{\gamma\gamma'}H_{\rho\rho',\sigma\sigma',\gamma\gamma'}(X) \\ &= 3d_{\mathcal{N}=1}\left(\frac{X_mX_nX_k}{X^5} + \frac{\eta_{nk}X_m - \eta_{mk}X_n - \eta_{mn}X_k}{X^3}\right)\,.\end{aligned} \tag{D.15}$$

Finally, using the identity $X_3{}^2 = \frac{x_{12}{}^2}{x_{23}{}^2x_{13}{}^2}$ we represent the denominator in (D.14) in a symmetric form with respect to the indices labelling spacetime points

$$\langle L_m^{\bar a}(x_1)L_n^{\bar b}(x_2)L_k^{\bar c}(x_3)\rangle = \frac{I_{mm'}(x_{13})I_{nn'}(x_{23})}{x_{12}{}^2x_{13}{}^2x_{23}{}^2}f^{\bar a\bar b\bar c}t^{m'n'}{}_k(X_3)\,, \tag{D.16}$$

where

$$t_{mnk}(X) = X^2H_{mnk}(X) = 3d_{\mathcal{N}=1}\left(\frac{X_mX_nX_k}{X^3} + \frac{\eta_{nk}X_m - \eta_{mk}X_n - \eta_{mn}X_k}{X}\right)\,. \tag{D.17}$$

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JHEP06(2015)138