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Neutral bions in the $\mathbb{C}P^{N-1}$ model

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Abstract: We study classical configurations in the $\mathbb{C}P^{N-1}$ model on $\mathbb{R}^1 \times S^1$ with twisted boundary conditions. We focus on specific configurations composed of multiple fractionalized-instantons, termed "neutral bions", which are identified as "perturbative infrared renormalons" by Ünsal and his collaborators. For $\mathbb{Z}_N$ twisted boundary conditions, we consider an explicit ansatz corresponding to topologically trivial configurations containing one fractionalized instanton ($\nu = 1/N$) and one fractionalized anti-instanton ($\nu = -1/N$) at large separations, and exhibit the attractive interaction between the instanton constituents and how they behave at shorter separations. We show that the bosonic interaction potential between the constituents as a function of both the separation and N is consistent with the standard separated-instanton calculus even from short to large separations, which indicates that the ansatz enables us to study bions and the related physics for a wide range of separations. We also propose different bion ansatze in a certain non-$\mathbb{Z}_N$ twisted boundary condition corresponding to the "split" vacuum for $N = 3$ and its extensions for $N \geq 3$. We find that the interaction potential has qualitatively the same asymptotic behavior and N-dependence as those of bions for $\mathbb{Z}_N$ twisted boundary conditions.

Keywords: Field Theories in Lower Dimensions, Solitons Monopoles and Instantons, Wilson, 't Hooft and Polyakov loops, Topological States of Matter

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Contents

1 Introduction

In the recent study on QCD-like theories with spatial compactification (L), fractionalized multi-instanton configurations composed of fractionalized instantons and anti-instantons have been attracting a great deal of attention. It is stressed by Ünsal and his collaborators that these configurations [1–19], which are termed "bions", have two physical significances associated with two types of topologically trivial bion configurations called "magnetic (charged) bions" and "neutral bions", as seen in the following examples: in the weak-coupling regime ($L \ll 1/\Lambda_{\rm QCD}$) in QCD(adj.) on $\mathbb{R}^3 \times S^1$, or in the $\mathrm{U}(1)^{N-1}$ center-symmetric phase [20–29], condensation of magnetic bions (zero topological charge and nonzero magnetic charge) causes the confinement [3–9]. This confinement mechanism may remain responsible for the confinement at strong-coupling regime due to the continuity principle. This argument is also of importance in terms of the recent progress in large-N volume reduction [30–40]. On the other side, neutral bions (zero topological charge and zero magnetic charge) can be identified as the infrared renormalon [10–19, 41–43]. Here imaginary ambiguities arising in bion's amplitude and those arising in non-Borel-summable perturbative series cancel against each other, and it is expected that full semi-classical expansion including perturbative and non-perturbative sectors, which is called "resurgent" expansion [44], leads to unambiguous and self-consistent definition of field theories in the same manner as the Bogomolny-Zinn-Justin (BZJ) prescription in quantum mechanics [45–47]. However, it is not straightforward to verify these arguments in gauge theories directly, since it is difficult to find an explicit ansatz of bion configurations.

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In order to reach deeper understanding on bions and the associated physics, it is of great importance to study examples in the low-dimensional models such as $\mathbb{C}P^{N-1}$ models [12–14], principal chiral models [16, 19] and quantum mechanics [15, 17, 18]. In particular, the $\mathbb{C}P^{N-1}$ model in 1+1 dimensions has been studied for a long time as a toy model of the Yang-Mills theory in 3+1 dimensions [48], because of similarities between them such as dynamical mass gap, asymptotic freedom and the existence of instantons [49]. The $\mathbb{C}P^{N-1}$ model on $\mathbb{R}^1 \times S^1$ with twisted boundary conditions admits fractionalized instantons (domain wall-instantons) as configurations with the minimal topological charge [50–52] (see also refs. [53–55]). In refs. [12, 13], generic arguments on bion configurations were given in the $\mathbb{C}P^{N-1}$ model on $\mathbb{R}^1 \times S^1$ with $\mathbb{Z}_N$ twisted boundary conditions, which is a corresponding situation to $\mathrm{U}(1)^{N-1}$ center-symmetric phase in QCD(adj.), based on the independent instanton description taking account of interactions between far-separated fractionalized instantons and anti-instantons. According to the study, the renormalon ambiguity arising in non-Borel-summable perturbative series is compensated by the amplitude of neutral bions also in the $\mathbb{C}P^{N-1}$ model. This phenomenon, which is called "resurgence", works as follows [12, 13]: the effective interaction action by bosonic exchange between one fractionalized instanton $\mathcal{K}_i$ and one fractionalized anti-instanton $\overline{\mathcal{K}}_j$ is

$$S_{\rm int}(\tau) = -8\xi \frac{\alpha_i \cdot \alpha_j}{g^2} e^{-\xi\tau}\,, \qquad \xi \equiv \frac{2\pi}{N}\,, \tag{1.1}$$

where τ stands for distance (divided by the compact scale L) between two fractionalized instantons. Vectors α_i, α_j are affine co-roots and $\alpha_i \cdot \alpha_j$ is an entry of the extended Cartan matrix. The total bion amplitude including the fermion zero-mode exchange contribution is mainly given by

$$\mathcal{B}_{ij} \propto -e^{-2S_I/N} \int_0^\infty d\tau\, e^{-V_{\rm eff}^{ij}(\tau)}\,, \tag{1.2}$$

with $V_{\rm eff}^{ij}(\tau) = S_{\rm int}(\tau) + 2N_f\xi\tau$ and S_I being the instanton action. N_f stands for fermion flavors. For neutral bion $\alpha_i \cdot \alpha_i > 0$, semiclassical description of independent fractionalized instantons breaks down since the interaction is attractive and instantons are merged in the end. Here, the BZJ-prescription, replacing $g^2 \to -g^2$, works to extract meaningful information from this amplitude. The prescription turns the interaction (spuriously) into a repulsive one and the amplitude becomes well-defined as

$$\mathcal{B}_{ii}(g^2, N_f) \;\to\; \tilde{\mathcal{B}}_{ii}(-g^2, N_f) \;\propto\; (-g^2N/8\pi)^{2N_f}\Gamma(2N_f)e^{-2S_I/N}\,. \tag{1.3}$$

By the use of the analytic continuation in the g^2 complex plane, we can continue back to the original g^2. For $N_f = 0$ case, we then encounter the following imaginary ambiguity in the amplitude as

$$\tilde{\mathcal{B}}_{ii}(g^2, 0) \;\propto\; \left(\log(g^2N/8\pi) - \gamma \pm i\pi\right) e^{-2S_I/N}\,. \tag{1.4}$$

We can rephrase this situation as unstable negative modes of bions give rise to imaginary ambiguities of the amplitude. The imaginary ambiguity has the same magnitude with an opposite sign as the leading-order ambiguity ($\sim \mp i\pi e^{-2S_I/N}$) arising from the non-Borel-summable series expanded around the perturbative vacuum. The ambiguities at

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higher orders ($\mp i\pi e^{-4S_I/N}$, $\mp i\pi e^{-6S_I/N}$, ...) are cancelled by amplitudes of bion molecules (2-bion, 3-bion,...), and the full trans-series expansion around the perturbative and non-perturbative vacua results in unambiguous definition of field theories.

Although this generic argument based on far-separated instantons is clear, it is also worthwhile manifesting and studying an explicit solution or ansatz corresponding to bion configurations, which can be investigated from short to large separation. In ref. [14], the authors found out non-Bogomol'nyi-Prasad-Sommerfield (BPS) solutions in the $\mathbb{C}P^{N-1}$ model on $\mathbb{R}^1 \times S^1$ with a $\mathbb{Z}_N$ twisted boundary condition, and have shown that these solutions can be critical points, around which the resurgent semi-classical expansion is performed. The simplest non-BPS solution that they found is a four-instanton configuration composed of two fractionalized instantons ($\nu = 1/N$) and two fractionalized anti-instantons ($\nu = -1/N$) for $N \geq 3$. (see also [59].) They used the Din-Zakrewski projection method [56–58] generating a tower of non-BPS solutions from a BPS solution. It is known that all possible classical solutions are exhausted by this method at least on $\mathbb{R}^2$ and S^2 [56–58]. This result indicates that if a simple bion configuration containing one instanton and one anti-instanton in the $\mathbb{C}P^{N-1}$ model ($N \geq 2$) exists, it may not be a solution of the equation of motion, but may be some classical configuration which can give significant contributions to path integrals. If it is true, one question arises what ansatz corresponds to such a bion. If such an ansatz exists, the other questions arise how the instanton constituents behave at short separations and whether it is consistent with the amplitude (1.1) obtained in the standard instanton calculus in a far-separated limit. In the present study, we consider and study an ansatz corresponding to bions beyond exact solutions. We also consider more general twisted boundary conditions similar to the "split phase" in QCD(adj.).

The purpose of our work is to study an explicit ansatz corresponding to topologically trivial bion configurations in the $\mathbb{C}P^{N-1}$ on $\mathbb{R}^1 \times S^1$ with several twisted boundary conditions, and show how the instanton constituents behave at an arbitrary separation. For the $\mathbb{Z}_N$ twisted boundary condition, we consider a simple neutral-bion ansatz, which yields configuration involving one fractionalized instanton ($\nu = 1/N$) and one fractionalized anti-instanton ($\nu = -1/N$) in the well-separated limit. By studying separation dependence of the total action, we exhibit the attractive interaction between the instanton constituents and how they are merged in the end, which means that the configuration has a negative mode. By looking into N-dependence of the interaction potential as a function of the separation in comparison with the result in the standard instanton calculus (1.1), we show that our ansatz is consistent with (1.1) even from short to large separations. Our ansatz can be used to study bions and related physics for a wide range of separations. For the non-$\mathbb{Z}_N$ twisted boundary conditions with $N = 3$, which we term a "split" boundary condition, we find out a different fractionalized instanton-anti-instanton ansatz. We again show that the configuration has a negative mode. We extend the ansatz to general $N \geq 3$ cases, and find that the interaction potential between the instantons has qualitatively the same properties as (1.1) up to some factors in the extended versions. This fact indicates universality of resurgence based on neutral bions for general boundary conditions.

In section 2 we introduce $\mathbb{C}P^{N-1}$ models with some notations for calculations. In section 3 we first introduce $\mathbb{Z}_N$ twisted boundary conditions and discuss how fractionalized

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instantons emerge. We then propose a specific ansatz for neutral bions, and discuss the properties. In section 4 we consider non-$\mathbb{Z}_N$ twisted boundary conditions, and discuss bion-like configurations for the cases. Section 5 is devoted to a summary.

2 $\mathbb{C}P^{N-1}$ model

Let $\omega(x)$ be an N-component vector of complex scalar fields, and $n(x)$ be a normalized complex N-component vector composed from ω: $n(x) \equiv \omega(x)/|\omega(x)|$ with $|\omega| = \sqrt{\omega^\dagger \omega}$. Then, the action and topological charge representing $\pi_2(\mathbb{C}P^{N-1}) \simeq \mathbb{Z}$ of the $\mathbb{C}P^{N-1}$ model in Euclidean two dimensions are given by (see, e.g., refs. [56–58])

$$S = \frac{1}{g^2}\int d^2x (D_\mu n)^\dagger (D_\mu n)\,, \tag{2.1}$$

$$Q = \int d^2x\, i\epsilon_{\mu\nu}(D_\mu n)^\dagger (D_\nu n) = \int d^2x \epsilon_{\mu\nu}\partial_\mu A_\nu\,, \tag{2.2}$$

respectively, where $d^2x \equiv dx_1 dx_2$ and $\mu,\nu = 1,2$. Here, we have defined the covariant derivative by $D_\mu = \partial_\mu - iA_\mu$ with a composite gauge field $A_\mu(x) \equiv -in^\dagger \partial_\mu n$.

The action S and topological charge Q can be expressed in terms of the projection operator $\mathbf{P} \equiv nn^\dagger = \frac{\omega\omega^\dagger}{\omega^\dagger\omega}$ and using the complex coordinate $z \equiv x_1 + ix_2$,

$$S = \frac{2}{g^2}\int d^2x \mathrm{Tr}\left[\partial_z \mathbf{P}\partial_{\bar z}\mathbf{P}\right]\,, \tag{2.3}$$

$$Q = 2\int d^2x \mathrm{Tr}\left[\mathbf{P}(\partial_{\bar z}\mathbf{P}\partial_z\mathbf{P} - \partial_z\mathbf{P}\partial_{\bar z}\mathbf{P})\right]\,. \tag{2.4}$$

All through this paper, we focus the geometry $\mathbb{R}^1 \times S^1$ and configurations on it satisfying periodicity in the x_2 direction with compactification scale L. For all the configurations considered in the present paper, the action density and the topological charge density are reduced to be functions of x_1 after the integration over x_2:

$$S = \frac{1}{g^2\pi}\int dx_1 s(x_1) = \frac{1}{g^2\pi}\int dx_1 \mathrm{Tr}\left[\partial_z \mathbf{P}\partial_{\bar z}\mathbf{P}\right]\,, \tag{2.5}$$

$$Q = \frac{1}{\pi}\int dx_1 q(x_1) = \frac{1}{\pi}\int dx_1 \mathrm{Tr}\left[\mathbf{P}(\partial_{\bar z}\mathbf{P}\partial_z\mathbf{P} - \partial_z\mathbf{P}\partial_{\bar z}\mathbf{P})\right]\,, \tag{2.6}$$

where we have defined the action density $s(x_1)$ and the charge density $q(x_1)$ depending only on x_1. We note that the action and charge with the factor $1/(2\pi)$ yield integers or multiples of $1/N$ after x_1 integration. In this paper, we omit the coupling $1/g^2$ for simplicity.

The $\mathbb{C}P^1$ model is equivalent to the $O(3)$ nonlinear sigma model, described by three real scalar fields $\mathbf{m}(x) = (m^1(x), m^2(x), m^3(x))^T$ with a constraint $\mathbf{m}(x)^2 = 1$. More explicitly,

$$\begin{aligned}\mathbf{m}(x) &= n^\dagger(x)\vec\sigma n(x) = \frac{\omega^\dagger(x)\vec\sigma\omega(x)}{\omega^\dagger(x)\omega(x)} \\ &= \frac{(\omega^{*1}\omega^2 + \omega^{*2}\omega^1, -i\omega^{*1}\omega^2 + i\omega^{*2}\omega^1, |\omega^1|^2 - |\omega^2|^2)}{\omega^\dagger(x)\omega(x)},\end{aligned} \tag{2.7}$$

with the Pauli matrices $\vec\sigma$. Then, the action is

$$S = \frac{1}{g^2}\int d^2x \partial_\mu \mathbf{m}\cdot\partial_\mu\mathbf{m}. \tag{2.8}$$

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3 Fractionalized instantons and neutral-bion configuration in $\mathbb{Z}_N$ twisted boundary conditions

3.1 $\mathbb{Z}_N$ twisted boundary conditions

In the present section, we propose a neutral bion ansatz for a $\mathbb{Z}_N$ twisted boundary condition in the $\mathbb{C}P^{N-1}$ model on $\mathbb{R}^1 \times S^1$. $\mathbb{Z}_N$ twisted boundary conditions in a compactified direction is expressed as [12–14]

$$\omega(x_1, x_2 + L) = \Omega\,\omega(x_1, x_2)\,, \qquad \Omega = \text{diag.}\left[1, e^{2\pi i/N}, e^{4\pi i/N}, \cdots, e^{2(N-1)\pi i/N}\right]. \tag{3.1}$$

In SU(N) gauge theories with adjoint quarks, this $\mathbb{Z}_N$ twisted boundary condition corresponds to the vacuum with the gauge symmetry breaking SU(N) $\to$ U(1)$^{N-1}$, where Wilson-loop holonomy in the compactified direction is given by

$$\langle A_2 \rangle = (0, 2\pi/N, \cdots, 2(N-1)\pi/N)\,, \qquad \text{for} \quad N \geq 3\,, \tag{3.2}$$

and

$$\langle A_2 \rangle = (-\pi/2, \pi/2)\,, \qquad \text{for} \quad N = 2\,, \tag{3.3}$$

where A_2 is the gauge field in the compactified direction. (See also [60–63] for topics related to $\mathbb{Z}_N$ twisted boundary conditions.) We here omit permutation copies. We note that the gauge field defined in the $\mathbb{C}P^{N-1}$ model (2.2) also has the same Wilson-loop holonomy for $\mathbb{Z}_N$ twisted boundary condition. Difference between (3.1) and (3.3) for $N = 2$ is just superficial and unphysical, since two different ansatz of $\omega(x)$ with an overall boundary condition factor $e^{-i\pi/2}$ result in the same projection field $\mathbf{P}(x)$ as we will show later. Fractionalized instantons (domain wall-instantons) carry the minimum topological charges in the $\mathbb{C}P^{N-1}$ model on $\mathbb{R}^1 \times S^1$ with a twisted boundary condition [50–52]. For simplicity, we begin with the $\mathbb{C}P^1$ model and generalize the argument to the $\mathbb{C}P^{N-1}$ model subsequently. From next subsection we make all the dimensionful quantities and parameters dimensionless by using the compact scale L ($L \to 1$) unless we have a special reason to recover it.

3.2 Fractionalized instantons

In this subsection, we illustrate fractionalized instantons in the $\mathbb{C}P^1$ model satisfying a $\mathbb{Z}_2$ twisted boundary condition (3.1) as

$$\omega(x_1, x_2 + 1) = \text{diag.}[1, e^{\pi i}]\omega(x_1, x_2) = \text{diag.}[1, -1]\omega(x_1, x_2), \tag{3.4}$$

$$\begin{aligned}(m^1(x_1, x_2 + 1), m^2(x_1, x_2 + 1), m^3(x_1, x_2 + 1)) \\ = (-m^1(x_1, x_2), -m^2(x_1, x_2), m^3(x_1, x_2 + 1))\,,\end{aligned} \tag{3.5}$$

on $\mathbb{R}^1 \times S^1$ with the unexplicit unit compactification scale L.[1] Here, we have used the relation (2.8) for the second equation.

[1] Fractionalized instantons can also exist even in $\mathbb{R}^2$ when the target space has a sigularity, e.g. [64], and/or a U(1) isometry is gauged with a potential term, e.g. [65].

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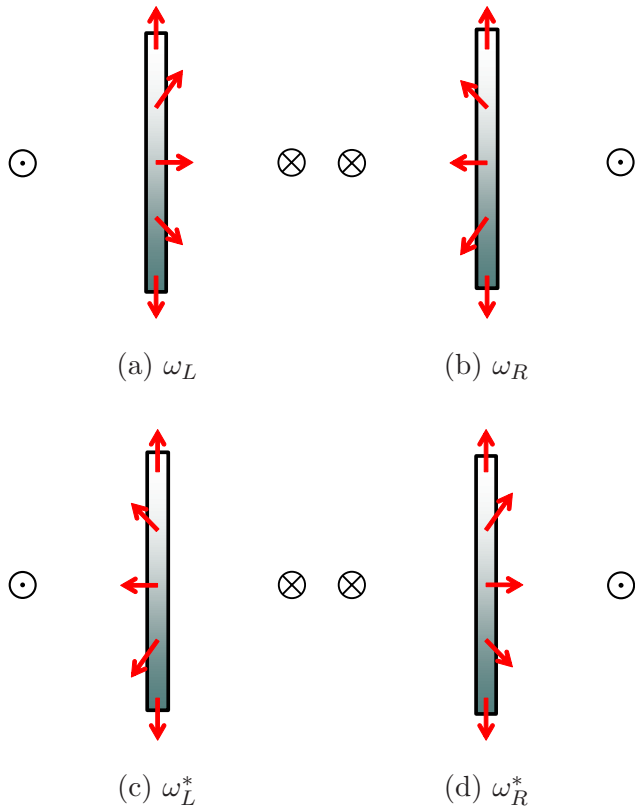

(a) ω_L (b) ω_R

(c) ω_L^* (d) ω_R^*

Figure 1. Fractionalized instantons in the $\mathbb{C}P^1$ model with the $\mathbb{Z}_2$ twisted boundary condition, corresponding to (a) ω_L, (b) ω_R, (c) ω_L^*, and (d) ω_R^* (in which we have taken the phase modulus to be $\theta = -\pi/2$). The horizontal and vertical directions are x_1 and x_2, respectively. The symbols $\odot$, $\otimes$, $\leftarrow$, $\rightarrow$, $\uparrow$ and $\downarrow$ denote $\mathbf{m} = (0,0,1), (0,0,-1), (-1,0,0), (1,0,0), (0,1,0)$ and $(0,-1,0)$, respectively. The shaded regions imply domain walls with $m^3 \sim 0$. The $\uparrow$ and $\downarrow$ at the boundaries at $x_2 = +1$ and $x_2 = 0$ are identified by the twisted boundary condition. The domain wall charges are (a) $+1$, (b) -1, (c) $+1$, (d) -1, and the instanton charges Q are (a) $+1/2$, (b) $+1/2$, (c) $-1/2$, (d) $-1/2$.

Using the complex coordinate $z = x_1 + ix_2$ on $\mathbb{R}^1 \times S^1$, fractionalized instanton solutions are given by

$$\begin{aligned} \omega_L &= \left(1, \lambda e^{i\theta} e^{+\pi z}\right)^T, \quad \omega_R = \left(1, \lambda e^{i\theta} e^{-\pi z}\right)^T, \\ \omega_L^* &= \left(1, \lambda e^{i\theta} e^{+\pi \bar{z}}\right)^T, \quad \omega_R^* = \left(1, \lambda e^{i\theta} e^{-\pi \bar{z}}\right)^T, \end{aligned} \tag{3.6}$$

with real constants λ and θ which are moduli. The configurations ω_L and ω_R are BPS which are holomorphic and depend on z, while their complex conjugate ω_L^* and ω_R^* are anti-BPS which are anti-holomorphic and depend on $\bar{z}$ only. Figure 1 shows configurations in $\mathbf{m}(x)$ of these solutions. The configuration ω_L (ω_L^*) goes to $n = (1,0)$ ($\mathbf{m} = (0,0,1)$) denoted by $\odot$ at $x_1 \to -\infty$ and to $n = (0,1)$ ($\mathbf{m} = (0,0,-1)$) denoted by $\otimes$ at $x_1 \to +\infty$. The configuration ω_R (ω_R^*) goes to $n = (0,1)$ ($\mathbf{m} = (0,0,-1)$) at $x_1 \to -\infty$ and to $\omega = (1,0)$ ($\mathbf{m} = (0,0,+1)$) at $x_1 \to +\infty$. The configurations ω_L (ω_L^*) and ω_R (ω_R^*) can be regarded as a domain wall and anti-domain wall, respectively. A domain wall at each constant x_2

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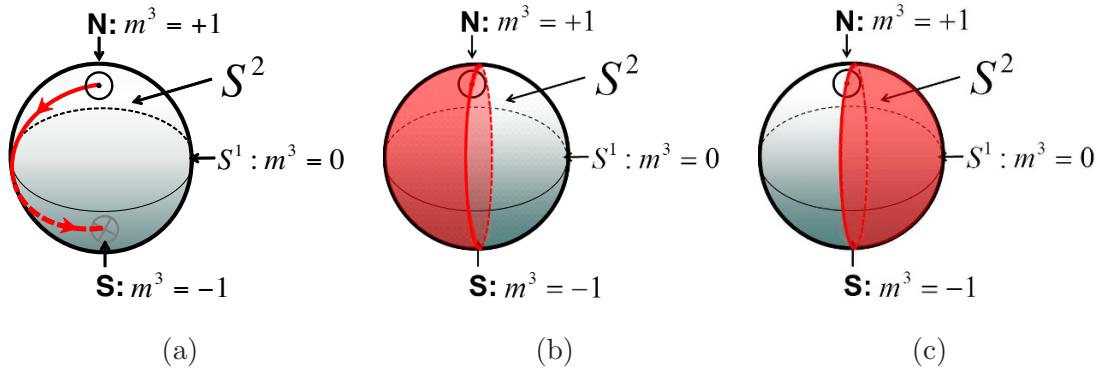

Figure 2. (a) Domain wall, and (b) and (c) fractionalized instantons in the target space S^2. (b) corresponds to the configurations ω_R and ω_L^* while (c) corresponds to the configurations ω_L and ω_R^*.

slice corresponds to a path connecting the north pole $\mathbf{m} = (0, 0, +1)$ and the south pole $\mathbf{m} = (0, 0, -1)$ in the target space, as illustrated in figure 2(a). A U(1) modulus is localized on these domain walls characterizing which point on the equator in the target space a domain wall passes through [66, 67]. This U(1) modulus is twisted along the domain wall to satisfy the boundary condition at $x_2 = 0$ and $x_2 = 1$. When one changes a constant x_2 slice from $x_2 = 0$ to $x_2 = 1$, a path in the target space changes with sweeping a half of the sphere as the target space, as shown in figure 2(b) and (c). Therefore, these configurations give maps from the space $\mathbb{R} \times S^1$ to a half of the target space. BPS configurations ω_L and ω_R carry a half of the unit instanton charge, $Q = 1/2$, while anti-BPS configurations ω_L^* and ω_R^* carry $Q = -1/2$. This fact also can be understood by noting that the U(1) modulus is twisted half along the domain wall [68, 69, 77, 78].

Fractionalized instantons can exist in the $\mathbb{C}P^{N-1}$ model too. The configuration (3.6) of the $\mathbb{C}P^1$ model can be generalized into the N-vector ω for the $\mathbb{C}P^{N-1}$ model with the $\mathbb{Z}_N$ twisted boundary condition in eq. (3.1) as

$$\omega_L = \left(0, \cdots, 0, 1, \lambda e^{i\theta} e^{+2\pi z/N}, 0, \cdots\right)^T, \quad \omega_R = \left(0, \cdots, 0, 1, \lambda e^{i\theta} e^{-2\pi z/N}, 0, \cdots, 0\right)^T. \tag{3.7}$$

3.3 Neutral bions

A neutral bion configuration is a composite of a fractionalized instanton and fractionalized anti-intanton with the total instanton charge canceled out. Let us discuss the $\mathbb{C}P^1$ model first. From the solutions in eq. (3.6) and their complex conjugates, it is reasonable to consider the following ansatz for the $\mathbb{C}P^1$ model satisfying a $\mathbb{Z}_2$ twisted boundary condition (3.1) as

$$\omega = \left(1 + \lambda_2 e^{i\theta_2} e^{\pi(z+\bar{z})},\ \lambda_1 e^{i\theta_1} e^{\pi z}\right)^T, \tag{3.8}$$

constructed from fractionalized instantons ω_L and ω_R^* in eq. (3.6). As we mentioned, the ansatz $\omega = e^{-\pi z/2}\left(1 + \lambda_2 e^{i\theta_2} e^{\pi(z+\bar{z})},\ \lambda_1 e^{i\theta_1} e^{\pi z}\right)^T$ also gives the same $\mathbf{P}(x)$, thus these are equivalent. $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $0 \leq \theta_1, \theta_2 < 2\pi$ are all real parameters characterizing the configuration associated with this ansatz, as λ_1^2/λ_2 and λ_2 govern a relative separation and

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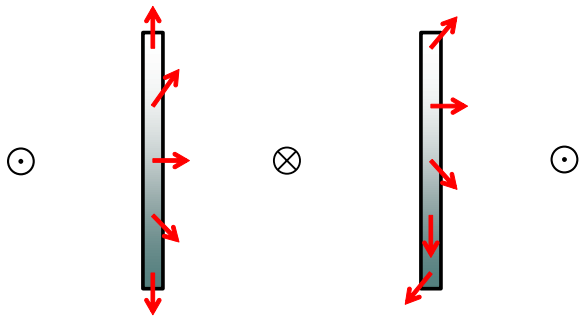

Figure 3. Neutral bion. This is a composite of fractionalized instantons ω_L and ω_R^*, where we have introduced a relative phase. The notation is the same as figure 1.

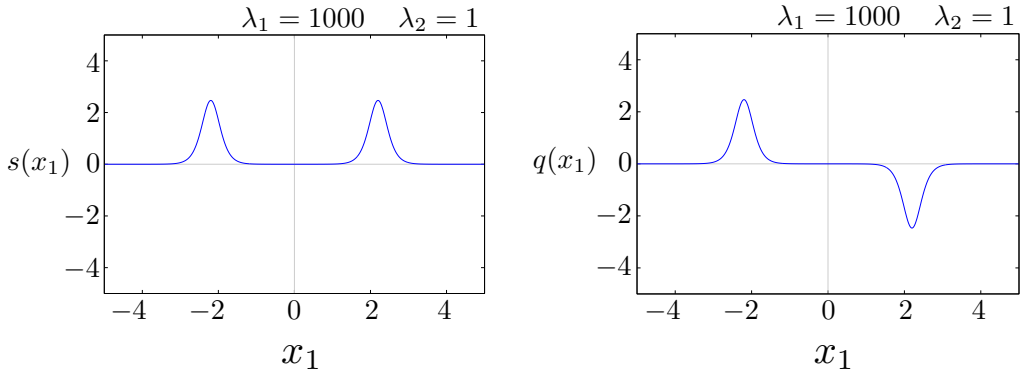

Figure 4. Action density $s(x_1)$ and topological charge density $q(x_1)$ for the configuration of eq. (3.8) for $\lambda_1 = 1000$, $\lambda_2 = 1$ and $\theta_2 = 0$. The distance between the peaks of two fractionalized instantons is given by ~ 4.3976, which is consistent with the separation $(1/\pi)\log(1000^2)$ obtained from eq. (3.21).

a center location between the instanton constituents respectively. We have no parameter characterizing the size of fractionalized instantons in the present ansatz. For $\lambda_1^2 \gg \lambda_2$, this configuration is composed of two components, a BPS fractionalized instanton ($S = 1/2$, $Q = 1/2$) and a BPS fractionalized anti-instanton ($S = 1/2$, $Q = -1/2$), which are separately located as shown in figure 3. Figure 4 shows the action and topological charge densities of this configuration.

The superposition ansatz such as ours has been studied long for Yang-Mills instantons and $\mathbb{C}P^{N-1}$ instantons on S^2 and $\mathbb{R}^2$ (see [70–72] for example). On the other hand, for these theories, a multiple-type ansatz has been also investigated [73–76], in relation to the study on "zindons". However, due to the fixed twisted boundary condition, it is not straightforward to construct an ansatz for the present case, with keeping, non-triviality of configurations, finite energy and the boundary conditions. The twisted boundary condition strongly restricts patterns of ansatz. This is why we begin with the simple ansatz (3.8). It is notable that the action density and topological charge density are independent of θ_1.

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The operators $\mathbf{P}$, $\partial_z\mathbf{P}$, $\partial_{\bar{z}}\mathbf{P}$, $\partial_z\mathbf{P}\partial_{\bar{z}}\mathbf{P}$ and $\partial_{\bar{z}}\mathbf{P}\partial_z\mathbf{P}$ have the following form as

$$\begin{pmatrix} a & b\,e^{-i\theta_1} \\ c\,e^{+i\theta_1} & d \end{pmatrix}, \tag{3.9}$$

where a, b, c and d are some functions of z and $\bar{z}$ including λ_1, λ_2 and θ_2 as parameters. Then, it is obvious that both $s(x_1) \sim \mathrm{Tr}[\partial_z\mathbf{P}\partial_{\bar{z}}\mathbf{P}]$ and $q(x_1) = \mathrm{Tr}[\mathbf{P}(\partial_{\bar{z}}\mathbf{P}\partial_z\mathbf{P} - \partial_z\mathbf{P}\partial_{\bar{z}}\mathbf{P})]$ have no θ_1 dependence. It means that θ_1 corresponds to a bosonic zero mode, which does not cost the configuration energy. On the other hand, the configuration depends on θ_2. For now we assume $\theta_2 = 0$, and will consider $\theta_2 \neq 0$ cases later.

The total action and the net topological charge in the large-separation limit are given by

$$S = 1, \quad Q = 0\,, \tag{3.10}$$

respectively. We note that the topological charge is zero for any values of separation, and this configuration corresponds to a topologically trivial vacuum.

Generalization of this configuration into the $\mathbb{C}P^{N-1}$ model is straightforward as

$$\omega = \left(0, \cdots, 0, 1 + \lambda_2 e^{i\theta_2} e^{2\pi(z+\bar{z})/N},\ \lambda_1 e^{i\theta_1} e^{2\pi z/N}, 0, \cdots, 0\right)^T. \tag{3.11}$$

The corresponding configuration again has no θ_1 dependence. For $\lambda_1^2 \gg \lambda_2$, this configuration corresponds to a $1/N$ instanton ($S = 1/N$, $Q = 1/N$) and a $1/N$ anti-instanton ($S = 1/N$, $Q = -1/N$) at large separations. The total action and the net topological charge in this large-separation limit are given by

$$S = 2/N, \quad Q = 0\,, \tag{3.12}$$

respectively.

As x_1 varies from $-\infty$ to ∞, the normalized complex vector $n(x_1)$ takes the following three different values, which we denote as n_1, n_2, n_3,

$$n_1 = (1, 0, \cdots, 0)^T \ \rightarrow\ n_2 = (0, 1, \cdots, 0)^T \ \rightarrow\ n_3 = (1, 0, \cdots, 0)^T, \tag{3.13}$$

for $\lambda_1^2 > \lambda_2$. The above three domains are divided by two critical points corresponding to the locations of the two kinks. As shown in [12, 13], the two affine co-roots α_i and α_j, which correspond to the two kinks (fractionalized instantons) in figure 4 are given by

$$\alpha_i = n_2 - n_1\,, \tag{3.14}$$

$$\alpha_j = -(n_3 - n_2)\,, \tag{3.15}$$

which satisfies

$$n(x_1 = \infty) = n(x_1 = -\infty) + \alpha_i - \alpha_j\,. \tag{3.16}$$

In the present case, α_i and α_j are identical, which we define as $\alpha_i = \alpha_j \equiv \alpha$. It is given by

$$\alpha \ = \ (0, 1, \cdots, 0)^T \ - \ (1, 0, \cdots, 0)^T \ = \ (-1, 1, \cdots, 0)^T. \tag{3.17}$$

We note that $\alpha_i \cdot \alpha_j = \alpha \cdot \alpha > 0$ for this case.

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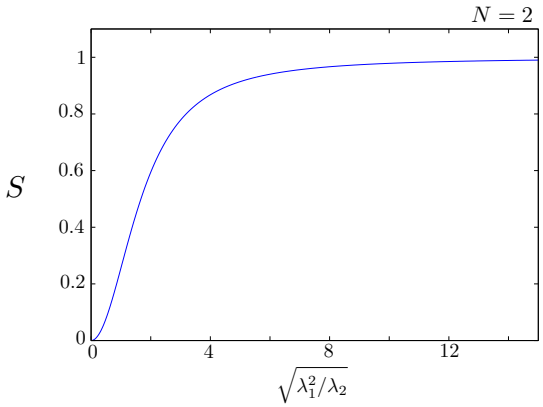

Figure 5. The $\sqrt{\lambda_1^2/\lambda_2}$ dependence of the total action S with $\theta_2 = 0$ for (3.8). The action is independent of λ_2 for λ_1^2/λ_2 fixed. The configuration is changed from $S = 1$ to $S = 0$, due to the attractive force.

The explicit form of the action density $s(x_1)$ for general N is given by

$$\begin{aligned} s(x_1) =& \frac{4\pi^2}{N^2\left(1+(\lambda_1^2+2\lambda_2\cos\theta_2)e^{4\pi x_1/N}+\lambda_2^2 e^{8\pi x_1/N}\right)^4}\times \\ & \Big[2(\lambda_1^2 e^{4\pi x_1/N}-\lambda_1^2\lambda_2^2 e^{12\pi x_1/N})^2 \\ & +(\lambda_1 e^{2\pi x_1/N}+2\lambda_1\lambda_2 e^{i\theta_2}e^{6\pi x_1/N}+\lambda_1\lambda_2 e^{i\theta_2}(\lambda_1^2+\lambda_2 e^{i\theta_2})e^{10\pi x_1/N})\times \\ & \quad (\lambda_1 e^{2\pi x_1/N}+2\lambda_1\lambda_2 e^{-i\theta_2}e^{6\pi x_1/N}+\lambda_1\lambda_2 e^{-i\theta_2}(\lambda_1^2+\lambda_2 e^{-i\theta_2})e^{10\pi x_1/N}) \\ & +(\lambda_1(\lambda_1^2+\lambda_2 e^{i\theta_2})e^{6\pi x_1/N}+2\lambda_1\lambda_2^2 e^{10\pi x_1/N}+\lambda_1\lambda_2^3 e^{-i\theta_2}e^{14\pi x_1/N})\times \\ & \quad (\lambda_1(\lambda_1^2+\lambda_2 e^{-i\theta_2})e^{6\pi x_1/N}+2\lambda_1\lambda_2^2 e^{10\pi x_1/N}+\lambda_1\lambda_2^3 e^{i\theta_2}e^{14\pi x_1/N})\Big]\,. \end{aligned} \tag{3.18}$$

Figure 5 depicts the $\sqrt{\lambda_1^2/\lambda_2}$ dependence of the total action S with $\theta_2 = 0$ for $N = 2$.

We will from now look into the attractive interaction between the two fractionalized instantons. In order to understand precise separation dependence of action and interaction force, we need to know the exact separation between the two components of fractionalized instantons in our configuration. The positions τ_1 and τ_2 of fractionalized instantons and fractionalized anti-instantons in the x_1-coordinate are given by the balance conditions

$$1 = \lambda_1 e^{2\pi\tau_1/N} \quad \to \quad \tau_1 = \frac{N}{2\pi}\log\left(\frac{1}{\lambda_1}\right)\,, \tag{3.19}$$

$$\lambda_2 e^{4\pi\tau_2/N} = \lambda_1 e^{2\pi\tau_2/N} \quad \to \quad \tau_2 = \frac{N}{2\pi}\log\left(\frac{\lambda_1}{\lambda_2}\right)\,, \tag{3.20}$$

respectively. Then, the separation τ between them is given by

$$\tau = \tau_2 - \tau_1 = \frac{N}{2\pi}\log\left(\frac{\lambda_1^2}{\lambda_2}\right)\,. \tag{3.21}$$

For $\tau \geq 0$, τ can be interpreted as separation between the fractionalized-instanton components. The definition of separation depends on N for this configuration. As noted in the

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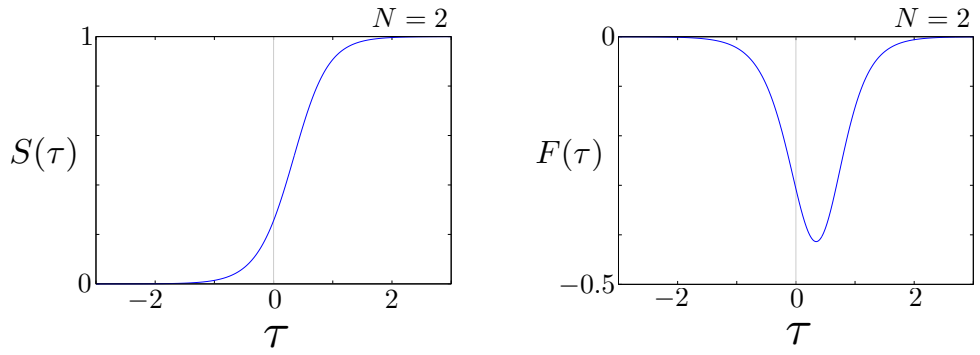

Figure 6. The $\tau = (1/\pi)\log\lambda_1^2/\lambda_2$ dependence of the total action S and the force $F = -\frac{dS}{d\tau}$ with $\theta_2 = 0$ for (3.8). For $\tau \geq 0$, we can interpret τ as separation between the instanton constituents. The configuration is changed from $S = 1$ to $S = 0$, due to the attractive force. The configuration for $\tau \gtrsim 1$ corresponds to neutral bions.

caption in figure 4, this definition of separation precisely describes the distance between the locations of two fractionalized instantons.

Figure 6 depicts the separation τ dependence of the total action S and the static force $F = -\frac{dS}{d\tau}$ with $\lambda_2 = 1$ fixed for $N = 2$. It indicates that the total action monotonically decreases as τ gets smaller, and the interaction force is negative for wide τ range. It clearly shows that the fractionalized-instanton constituents exert an attractive force. To be precise, as will be shown later, the interaction force is exponentially suppressed for large separation $\tau \gg 1$ or the merged limit $\tau \ll 0$ ($\lambda_1^2/\lambda_2 \ll 1$). It indicates that our ansatz yields intermediate configurations between two (approximate) solutions, a two-separated fractionalized-instanton solution ($S = 1$, $Q = 0$) and a trivial perturbative vacuum ($S = 0$, $Q = 0$). Our analysis is easily generalized to $\lambda_2 \neq 1$, where we find that the total action is independent of λ_2 if λ_1^2/λ_2 or τ are fixed. From this analysis, we see that the location of the center of mass λ_2 corresponds to a bosonic zero mode while λ_1^2/λ_2 to a negative mode.

The two constituents are getting closer and finally are merged by the attractive force, as shown in figure 7. The resultant configuration at $\tau = -\infty$ ($\lambda_1^2/\lambda_2 = 0$) is given by

$$\omega(\tau = -\infty) \to \left(1 + \lambda_2 e^{\pi(z+\bar{z})},\ 0\right)^T , \tag{3.22}$$

for $N = 2$, and

$$\omega(\tau = -\infty) \to \left(0, \cdots, 0, 1 + \lambda_2 e^{i\theta_2} e^{2\pi(z+\bar{z})/N},\ 0, 0, \cdots, 0\right)^T . \tag{3.23}$$

for general N, with the quantum number

$$S = 0, \quad Q = 0\,, \tag{3.24}$$

which is identical to a trivial vacuum.

Here we discuss the characteristic size of neutral bions. As shown in [12, 13], the size of "charged" bions $N_f > 0$ is clearly determined since the instanton constituents are bound

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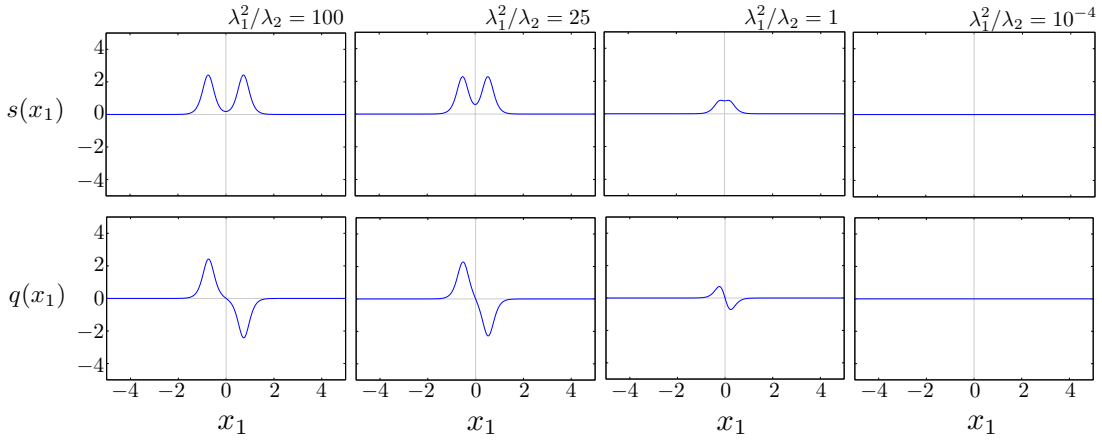

Figure 7. Action density $s(x_1)$ (up) and charge density $q(x_1)$ (down) of the configuration of eq. (3.8) for $\lambda_1^2/\lambda_2 = 100, 25, 1, 10^{-4}$ ($\tau = 1.47, 1.02, 0, -2.93$) for $\lambda_2 = 1$ and $\theta_2 = 0$. The configurations for $\lambda_1^2/\lambda_2 = 100, 25$ correspond to neutral bions.

due to the balance of bosonic repulsive and fermionic attractive forces. This calculation can be extended to the case of $N_f = 0$, and it gives the length scale of charged bions as $\tau^* = \frac{N}{2\pi} \log \frac{8\pi}{g^2 N}$, which reads $\tau^* = 0.5 \sim 1.5$ for $N = O(1)$ with $g^2 = 1$. On the other hand, the calculation of the neutral bion size is not straightforward since its bosonic interaction is attractive and it requires analytic continuation of g^2 to negative values and its returning back to positive values. However, taking into account the fact that the actions of neutral and charged bions in the amplitudes are common except for the imaginary part, we speculate that neutral bions have a similar size or length scale to that of the charged bions $\tau^* \sim 1$. In the rest of the paper, we assume that the neutral bions arise from the separation scale $\tau \gtrsim 1$. We thus regard our ansatz as the neutral bions only for the separation $\tau > 1$. We will discuss whether this assumption is appropriate or not in terms of BZJ-prescription in section 5.

We now investigate interaction part of the action for this configuration, to compare our concrete ansatz to the far-separated instanton argument (1.1) in refs. [12, 13]. The interaction part of the action density is written as the action density $s(x_1)$ minus the one fractionalized-instanton density and one fractionalized-anti-instanton density $s_{\nu=1/N}(x_1) + s_{\nu=-1/N}(x_1)$,

$$s_{\rm int}(x_1) = s(x_1) - \left(s_{\nu=1/N}(x_1) + s_{\nu=-1/N}(x_1)\right). \tag{3.25}$$

The integrated interaction action is then given by

$$S_{\rm int}(N,\tau) \;=\; \frac{1}{\pi}\int dx\, s_{\rm int}(x_1)\,. \tag{3.26}$$

In figure 8, we plot the logarithm of the total interaction action $S_{\rm int}(N,\tau)$ as a function of τ for $N = 2, 3, 4$. For $\tau \gtrsim 1$ region, $\log(-S_{\rm int}(N,\tau))$ can be well approximated by analytic lines,

$$\log\left[-S_{\rm int}(N,\tau)\right] \;\sim\; -\xi(N)\,\tau \;+\; C(N)\,, \qquad (\tau \gtrsim 1)\,, \tag{3.27}$$

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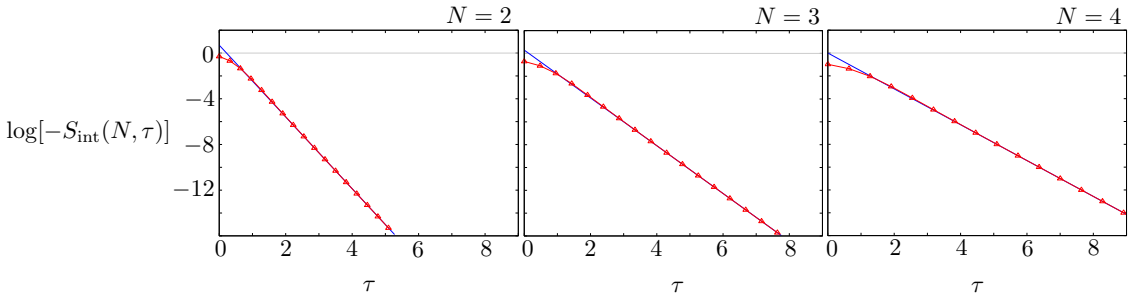

Figure 8. Plot of $\log(-S_{\rm int}(N,\tau))$ as a function of τ for $N=2$ (left), $N=3$ (center) and $N=4$ (right) for (3.11) (red curves with triangle points). For $\tau > 1$, the curve is almost equivalent to $-(2\pi/N)\tau + C(N)$ (blue curves).

where $\xi(N)$ is a slope and $C(N)$ is a y-intercept. In figure 8 we simultaneously depict these analytic lines for the three cases. The slopes ξ of the approximate lines read $\xi \sim \pi$ for $N=2$, $\xi \sim 2\pi/3$ for $N=3$ and $\xi \sim \pi/2$ for $N=4$, which indicates that the slope ξ can be generally expressed as

$$\xi(N) \sim \frac{2\pi}{N}\,. \tag{3.28}$$

Therefore we observe that the interaction action can be written as the following form for $\tau \gtrsim 1$ region,

$$S_{\rm int}(N,\tau) \sim -e^{C}\, e^{-\xi\tau}\,, \qquad \xi = \frac{2\pi}{N}\,, \qquad (\tau \gtrsim 1)\,. \tag{3.29}$$

This ξ is equivalent to the (dimensionless) lowest Kaluza-Klein spectrum Lm_{LKK}, which is given as $Lm_{LKK} = |q_i - q_j| = 2\pi/N$, where q_i and q_j are two nonzero components of Wilson-loop holonomy in (3.2), (3.3).

We next determine the N dependence of the y-intercept $C(N)$. In figure 9 we plot exponential of the intercept $\exp[C(N)]$ as a function of N for $N = 2,3,4,5,6,7$. We find out that this dependence is well approximated by $\exp[C(N)] \sim 4/N$, and depict it simultaneously in the figure. This result shows that the interaction action for $\tau \gtrsim 1$ can be written as

$$S_{\rm int}(N,\tau) \sim -\frac{4}{N}\, e^{-\xi\tau}\,, \qquad \xi = \frac{2\pi}{N}\,, \qquad (\tau \gtrsim 1)\,. \tag{3.30}$$

It means, for a wide range of separations $\tau \gtrsim 1$, the interaction part of the action for our configuration is consistent with the neutral bion action (1.1) obtained from the far-separated instanton calculation up to 2π factor, which we introduced for convenience, as

$$S_{\rm int}(N,\tau) = -8\xi\,(\alpha_i \cdot \alpha_j)\, e^{-\xi\tau} = -\frac{8\pi}{N} e^{-\xi\tau}\,, \qquad \xi = \frac{2\pi}{N}\,, \tag{3.31}$$

with $\alpha_i \cdot \alpha_j = \alpha \cdot \alpha = 1$ for our ansatz following the Lie algebra notation in [12, 13].

We have shown that our ansatz (3.8) gives a configuration consistent to (1.1) except in the merged region $\tau < 1$. It means that (3.8) is a good ansatz describing the neutral bion, and can be identified as an infrared renormalon since the imaginary part of its amplitude obtained through the BZJ-prescription ($g^2 \to -g^2$) and analytic continuation cancels the

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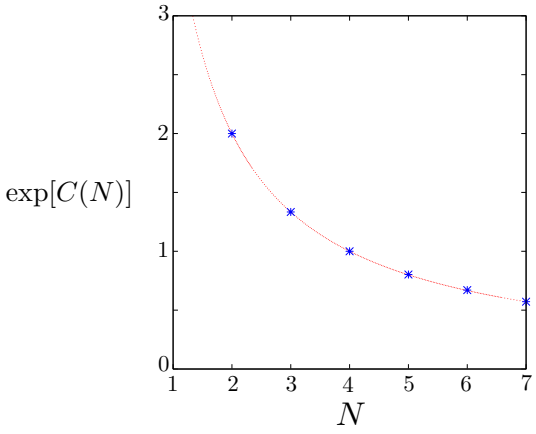

Figure 9. The coefficient of the interaction action $\exp[C(N)]$ in eq. (3.29) as a function of N for $N = 2, 3, 4, 5, 6, 7$ for the Ansatz (3.11) (blue points). The coefficient can be approximated by $4/N$ (a red curve).

notorious ambiguity arising in the Borel re-summation of the perturbative series. By use of the present ansatz, we can study properties of bions and the related physics, not only at large separation $\tau \gg 1$, but also at short separation $\tau \gtrsim 1$.

We here discuss a physical meaning of our ansatz for the merged region $\tau < 1$. The coincidence of the interactive actions for $\tau \gtrsim 1$ (figure 8) implies that the neutral bion scale can be determined by such a coincidence point, or $\tau \sim 1$ for this case, which is consistent with the charged bion scale. On the other hand, for the region $\tau < 1$, the configurations (see the right two columns in figure 7) are regarded as those around the perturbative vacuum rather than the bion saddle. It means that, in the semi-classical calculation, these configurations correspond to fluctuations around the perturbative saddle, but not around the bion saddle point. It is also notable that our ansatz connects these two different saddle points continuously by a single parameter. We will briefly discuss how to classify the parameter regions into the two sectors in section 5.

We here make a comment on cases for $\theta_2 \neq 0$. For $0 < \theta_2 < \pi/2$, the interaction force is qualitatively the same as the case for $\theta_2 = 0$, or attractive. For $\pi/2 \leq \theta_2 \leq \pi$, things change: the potential barrier emerges around $\sqrt{\lambda_1^2/\lambda_2} = 1$ ($\tau = 0$), and the height becomes infinite for $\theta_2 = \pi$ as shown in figure 10. Of course, this does not mean that the interaction is repulsive since θ_2 is also a dynamical field variable and should relax eventually to $\theta_2 = 0$ in order to minimize the total action. The result indicates that θ_2 corresponds to a positive mode.

4 Bions with non-$\mathbb{Z}_N$ twisted boundary conditions

The configuration discussed in the previous section is specific to the $\mathbb{Z}_N$ twisted boundary condition. In this section we consider different boundary conditions and bion-like configurations. We here begin with the $\mathbb{C}P^2$ model and extend the result to the $\mathbb{C}P^{N-1}$.

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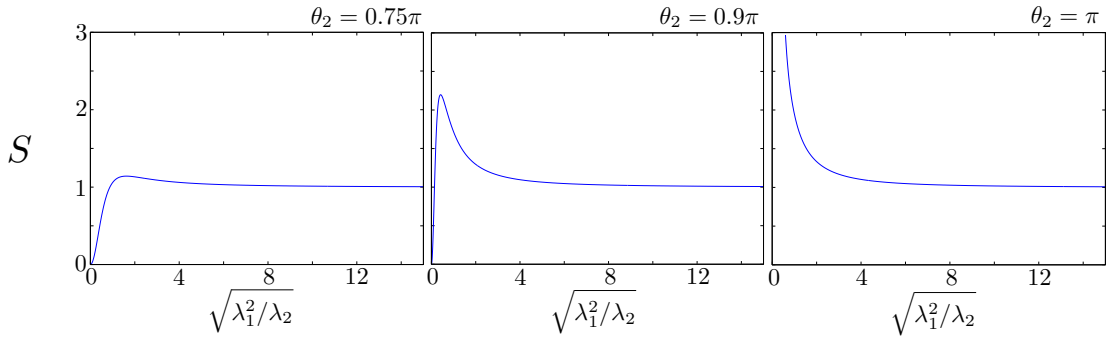

Figure 10. The $\sqrt{\lambda_1^2/\lambda_2}$ dependence of the total action S for the configuration in eq. (3.8) for $\theta_2 = 0.75\pi, 0.9\pi, \pi$.

We first consider the following twisted boundary condition for $\mathbb{C}P^2$

$$\omega(x_1, x_2 + L) = \Omega\,\omega(x_1, x_2)\,, \qquad \Omega = \text{diag.}\left[e^{\pi i}, e^{\pi i}, 1\right] = \text{diag.}\left[1, 1, e^{-\pi i}\right] e^{\pi i}\,. \tag{4.1}$$

This boundary condition corresponds to the vacuum

$$\langle A_2 \rangle = (\pi, \pi, 0)\,, \tag{4.2}$$

where we omit permutation copies. In gauge theory, this boundary condition is realized by special Wilson-loop holonomy in the exotic gauge-broken phase in SU(3) gauge theory with adjoint quarks [20, 21], where the gauge symmetry is broken as $\text{SU}(3) \to \text{SU}(2) \times \text{U}(1)$. In this vacuum two elements of the Wilson-loop holonomy have the same value, but the other has a different one. This phase is called "split phase", thus we term the above boundary condition a "split" twisted boundary condition. Although how a neutral bion works in the split vacuum has not yet been elucidated, it is worth investigating bion-like configurations in this vacuum.

4.1 Bions for the split twisted boundary condition

We first consider a configuration in $\mathbb{C}P^2$ satisfying the split twisted boundary condition (4.1),

$$\omega = \left(1,\ \lambda_2 e^{i\theta_2} e^{\pi(z+\bar{z})},\ \lambda_1 e^{i\theta_1} e^{\pi z}\right)^T\,, \tag{4.3}$$

This is an ansatz beyond the simple superposition ansatz. For $\lambda_1^2 \gg \lambda_2$, this configuration is composed of a BPS fractionalized instanton ($S = 1/2$, $Q = 1/2$) and a BPS fractionalized anti-instanton ($S = 1/2$, $Q = -1/2$) in figure 11. The total action and the net topological charge in a far-separated limit are given by

$$S = 1, \quad Q = 0\,. \tag{4.4}$$

We note the action density and topological charge density are independent of the parameters θ_1 and θ_2. Figure 12 depicts $\sqrt{\lambda_1^2/\lambda_2}$ dependence of the total action S for (4.3). The total action is independent of λ_2 with λ_1^2/λ_2 fixed.

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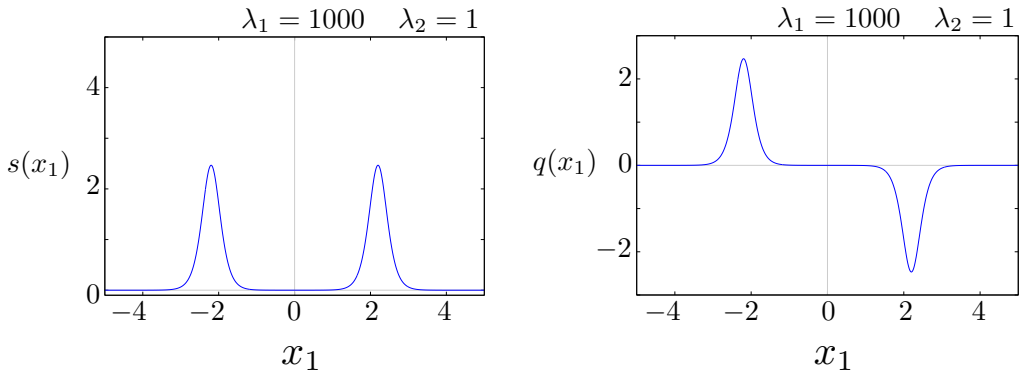

Figure 11. Action density $s(x_1)$ and topological charge density $q(x_1)$ for the configuration in eq. (4.3) for $\lambda_1 = 1000$ and $\lambda_2 = 1$. The distance between two fractionalized instantons is ~ 4.3976, which is consistent with the $(1/\pi)\log(1000^2)$ in eq. (4.6).

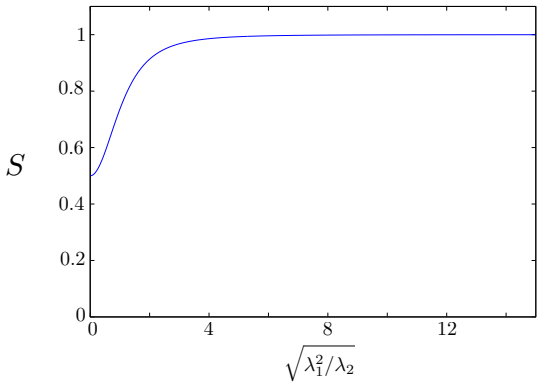

Figure 12. The $\sqrt{\lambda_1^2/\lambda_2}$ dependence of the total action S for the configuration in eq. (4.3). It is independent of λ_2 (corresponding to the center of two fractionalized instantons) with λ_1^2/λ_2 fixed. The configuration is changed from $S = 1$ to $S = 1/2$, due to the attractive force.

For this ansatz, the normalized complex vector $n(x_1)$ takes the following three different values as x_1 varies from $-\infty$ to ∞,

$$n_1 = (1,0,0)^T \quad \to \quad n_2 = (0,0,1)^T \quad \to \quad n_3 = (0,1,0)^T \,, \tag{4.5}$$

for $\lambda_1^2 > \lambda_2$. For this case, the entry of extended Cartan matrix is again positive $\alpha_i \cdot \alpha_j > 0$.

The separation τ is given by

$$\tau = \frac{1}{\pi} \log\left(\frac{\lambda_1^2}{\lambda_2}\right) , \tag{4.6}$$

which is obtained from the balance conditions [50–52, 79–83],

$$1 = \lambda_1 e^{\pi\tau_1} \quad \to \quad \tau_1 = \frac{1}{\pi} \log\left(\frac{1}{\lambda_1}\right) , \tag{4.7}$$

$$\lambda_2 e^{2\pi\tau_2} = \lambda_1 e^{\pi\tau_2} \quad \to \quad \tau_2 = \frac{1}{\pi} \log\left(\frac{\lambda_1}{\lambda_2}\right) , \tag{4.8}$$

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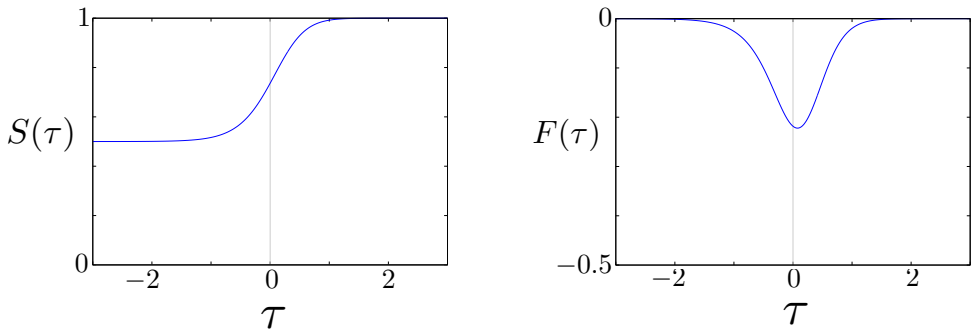

Figure 13. The $\tau = (1/\pi)\log\lambda_1^2/\lambda_2$ dependence of the total action S and the force $F = -\frac{dS}{d\tau}$ for the configuration in eq. (4.3). For $\tau \geq 0$, we can interpret τ as the separation between the instanton constituents. The configuration is changed from $S = 1$ to $S = 1/2$ with $Q = 0$ conserved, due to the attractive force. The configuration for $\tau \gtrsim 1$ corresponds to neutral bions.

with $\tau = \tau_2 - \tau_1$. For $\tau \geq 0$, τ stands for a separation between the fractionalized-instanton constituents. In figure 13 we depicts τ dependence of the total action S and the static force $F = -\frac{dS}{d\tau}$ for (4.3). The result indicates that the force is negative for $-\infty < \tau < \infty$, and the fractionalized instanton constituents have an attractive force. As with the configuration in the previous section, the two fractionalized instantons are merged by the attractive force, and finally resulting in the configuration,

$$\omega(\tau = -\infty) \;\to\; \left(1,\ \lambda_2 e^{i\theta_2} e^{\pi(z+\bar{z})},\ 0\right)^T , \tag{4.9}$$

at $\tau = -\infty$ ($\lambda_1^2/\lambda_2 = 0$) with

$$S = 1/2, \quad Q = 0\,. \tag{4.10}$$

From this analysis, we see that λ_2, θ_1 and θ_2 correspond to bosonic zero modes while λ_1^2/λ_2 to a negative mode.

The interaction part of the action takes the same form as that in the previous section $S_{\rm int}(\tau) = \frac{1}{\pi}\int dx s_{\rm int}(x_1)$. In figure 14, we plot the logarithm of the interaction action $S_{\rm int}(\tau)$ as a function of τ for the present case (4.3). For $\tau \gtrsim 2$ region, $\log(-S_{\rm int}(\tau))$ is approximated by

$$\log\left[-S_{\rm int}(\tau)\right] \;\sim\; -6\,\tau \;+\; 1.9657\,, \qquad (\tau \gtrsim 2)\,. \tag{4.11}$$

Therefore, the interaction action can be written as the following form for $\tau \gtrsim 2$ region,

$$S_{\rm int}(\tau) \;\sim\; -7.14\, e^{-6\tau}\,. \qquad (\tau \gtrsim 2)\,. \tag{4.12}$$

This is qualitatively consistent with (1.1), while the coefficients are different from $\xi(3)$ and $C(3)$ in $\mathbb{Z}_3$ twisted boundary conditions. Although how the coefficients are fixed needs to be uncovered, we at least argue that neutral bion-type configurations exist also for the split boundary condition with $N = 3$, which are responsible for the cancellation of infrared renormalon ambiguity.

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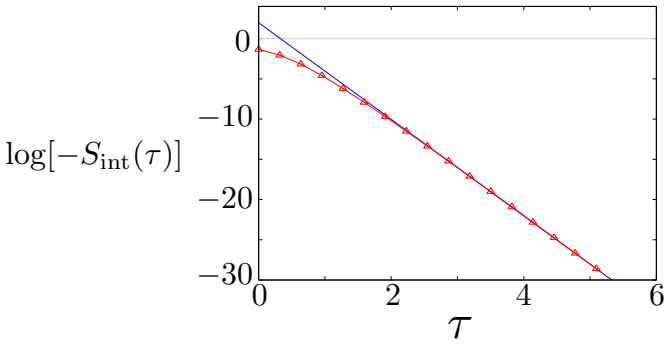

Figure 14. Plot of $\log(-S_{\rm int}(\tau))$ as a function of τ for (4.3) (a red curve with triangle points). For $\tau > 2$, the curve is almost equivalent to $-6\tau + 1.97$ (a blue curve).

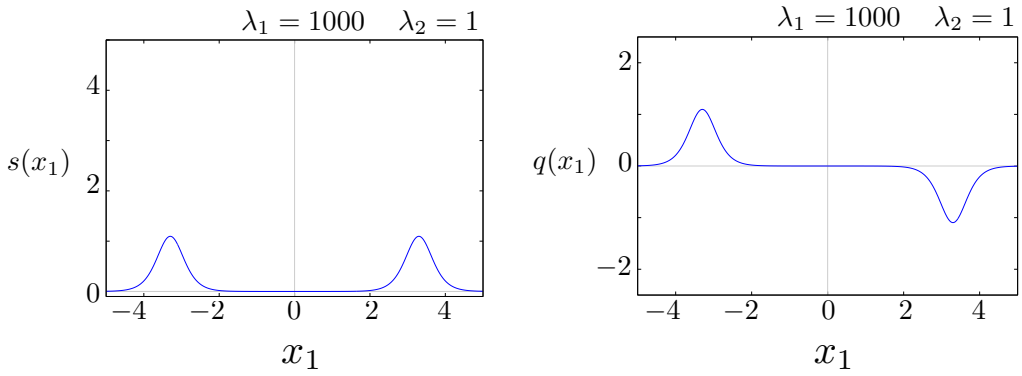

Figure 15. Action density $s(x_1)$ and topological charge density $q(x_1)$ for the configuration of eq. (4.13) for $\lambda_1 = 1000$ and $\lambda_2 = 1$. The distance between two fractionalized instantons is ~ 6.596, which is consistent with the $(3/2\pi)\log(1000^2)$ in eq. (4.18).

4.2 Bions in extended split boundary conditions

As an extension of the split boundary condition, we consider the following ansatz of the $\mathbb{C}P^2$ model on $\mathbb{R}^1 \times S^1$,

$$\omega = \left(1,\ \lambda_2 e^{i\theta_2} e^{\frac{2\pi}{3}(z+\bar{z})},\ \lambda_1 e^{i\theta_1} e^{\frac{2\pi}{3}z}\right)^T . \tag{4.13}$$

In this case we no longer regard the boundary condition as the Wilson-loop holonomy in the split vacuum, rather one specific twisted boundary condition, with $\Omega = (1, 1, e^{2\pi i/3})$. We will investigate the ansatz from pure-theoretical interest.

For $\lambda_1^2 \gg \lambda_2$, this configuration is composed of two constituents, a BPS fractionalized instanton ($S = 1/3$, $Q = 1/3$) and a BPS fractionalized anti-instanton ($S = 1/3$, $Q = -1/3$), which are separately located as shown in figure 15. The total action and the net topological charge in this limit are given by

$$S = 2/3, \quad Q = 0\,, \tag{4.14}$$

respectively. We note that the topological charge is zero and conserved.

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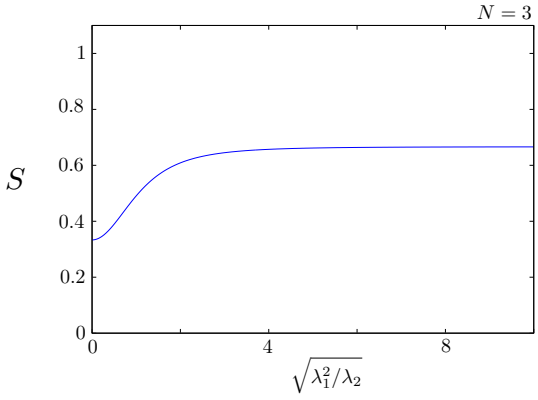

Figure 16. The $\sqrt{\lambda_1^2/\lambda_2}$ dependence of the total action S for the configuration in eq. (4.13). It is independent of the center location λ_2 with λ_1^2/λ_2 fixed. The configuration is changed from $S = 2/3$ to $S = 1/3$ with $Q = 0$ conserved, due to the attractive force.

We generalize this configuration to the $\mathbb{C}P^{N-1}$ model,

$$\omega = \left(1,\ \lambda_2 e^{i\theta_2} e^{\frac{2\pi}{N}(z+\bar{z})},\ \lambda_1 e^{i\theta_1} e^{\frac{2\pi}{N}z},\ \dots\ , 0\right)^T . \tag{4.15}$$

For $\lambda_1^2 \gg \lambda_2$, this configuration corresponds to a $1/N$ instanton ($S = 1/N$, $Q = 1/N$) and a $1/N$ anti-instanton ($S = 1/N$, $Q = -1/N$). The total action and the net topological charge in this large-separation limit are given by

$$S = 2/N, \quad Q = 0\,, \tag{4.16}$$

respectively. The explicit form of the action density $s(x_1)$ for N is given by

$$\begin{aligned} s(x_1) =& \frac{4\pi^2}{N^2(1+\lambda_1^2 e^{4\pi x_1/N} + \lambda_2^2 e^{8\pi x_1/N})^4} \times \\ & \Big[\lambda_1^2 e^{4\pi x_1/N}(1+\lambda_1^2 e^{4\pi x_1/N})^2 + 2\lambda_2^2 e^{8\pi x_1/N}(1+\lambda_1^2 e^{8\pi x_1/N})^2 \\ & + \lambda_1^2\lambda_2^2 e^{12\pi x_1/N}(7 + 6\lambda_1^2 e^{4\pi x_1/N} + 7\lambda_2^2 e^{8\pi x_1/N} + 2\lambda_1^2\lambda_2^2 e^{12\pi x_1/N} \\ & + \lambda_1^4 e^{8\pi x_1/N} + \lambda_2^2 e^{16\pi x_1/N})\Big] . \end{aligned} \tag{4.17}$$

Figure 16 depicts the $\sqrt{\lambda_1^2/\lambda_2}$ dependence of the total action S for $N = 3$. We note that the total action is independent of the central location λ_2 for λ_1^2/λ_2 fixed.

The separation τ is given by

$$\tau = \frac{N}{2\pi} \log\left(\frac{\lambda_1^2}{\lambda_2}\right) , \tag{4.18}$$

which is obtained from the two balance conditions [50–52, 79–83],

$$1 = \lambda_1 e^{2\pi\tau_1/N} \quad \to \quad \tau_1 = \frac{N}{2\pi} \log\left(\frac{1}{\lambda_1}\right) , \tag{4.19}$$

$$\lambda_2 e^{4\pi\tau_2/N} = \lambda_1 e^{2\pi\tau_2/N} \quad \to \quad \tau_2 = \frac{N}{2\pi} \log\left(\frac{\lambda_1}{\lambda_2}\right) , \tag{4.20}$$

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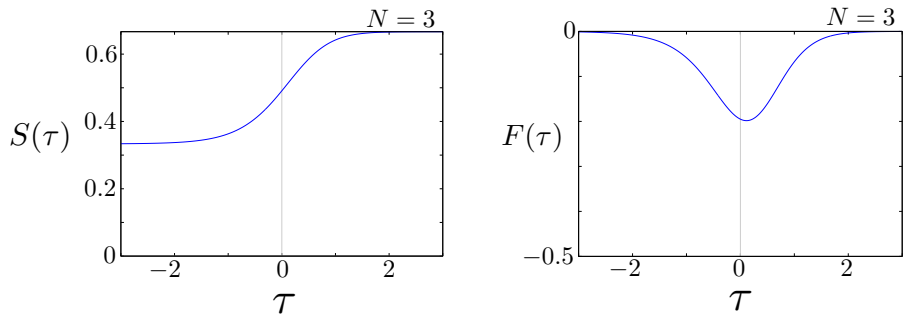

Figure 17. The $\tau = (3/2\pi)\log\lambda_1^2/\lambda_2$ dependence of the total action S and the force $F = -\frac{dS}{d\tau}$ for the configuration in eq. (4.13). For $\tau \geq 0$, we can interpret τ as the separation between the instanton constituents. The configuration is changed from $S = 2/3$ to $S = 1/3$ with $Q = 0$ conserved, due to the attractive force. The configuration for $\tau \gtrsim 1$ corresponds to neutral bions.

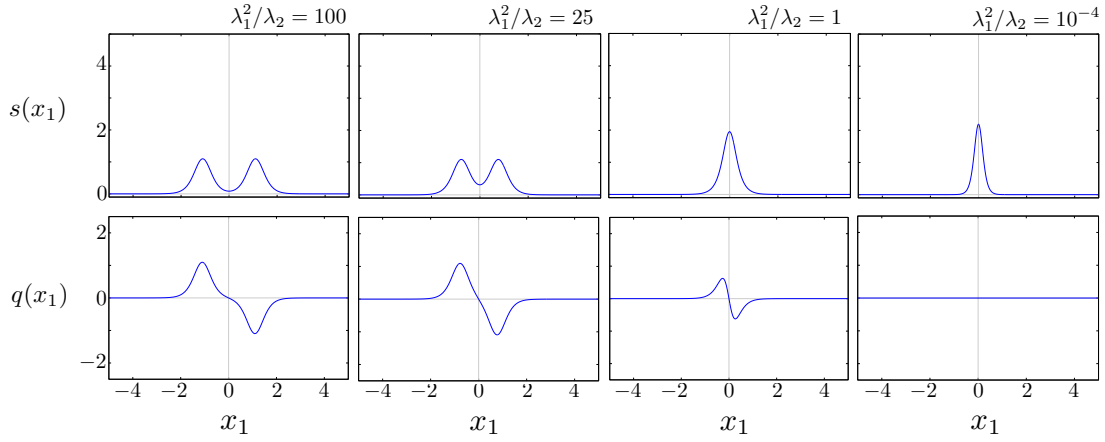

Figure 18. Action density $s(x_1)$ (up) and charge density $q(x_1)$ (down) of the configuration of eq. (4.13) ($N = 3$) for $\lambda_1^2/\lambda_2 = 100, 25, 1, 10^{-4}$ ($\tau = 2.20, 1.54, 0, -4.40$) with $\lambda_2 = 1$ fixed. The configurations for $\lambda_1^2/\lambda_2 = 100, 25$ correspond to neutral bions.

with $\tau = \tau_2 - \tau_1$. In figure 17 we depict the τ dependence of the total action S and the static force $F = -\frac{dS}{d\tau}$ for $N = 3$. The result clearly shows that the fractionalized instanton constituents have the attractive force.

The two constituents are merged by the attractive force, as shown in figure 18. For $N = 3$, the configuration results in the following form

$$\omega(\tau = -\infty) \;\to\; \left(1,\ \lambda_2 e^{i\theta_2} e^{\frac{2\pi}{3}(z+\bar{z})},\ 0\right)^T , \tag{4.21}$$

at $\tau = -\infty$ ($\lambda_1^2/\lambda_2 = 0$) with

$$S = 1/3, \quad Q = 0\,. \tag{4.22}$$

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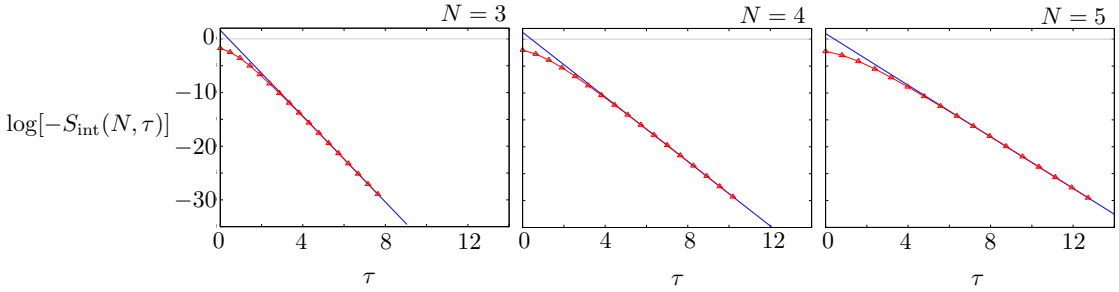

Figure 19. Plot of $\log(-S_{\rm int}(N,\tau))$ as a function of τ for $N=3$ (left), $N=4$ (center) and $N=5$ (right) for the configuration in eq. (4.15) (red curves with triangle points). For sufficiently large τ, the curve is approximated by $-(12/N)\tau + C(N)$ (blue curves).

For general N, the resultant configuration at $\tau=-\infty$ ($\lambda_1^2/\lambda_2=0$) is given by

$$\omega(\tau=-\infty) \;\to\; \left(1,\ \lambda_2 e^{i\theta_2} e^{\frac{2\pi}{N}(z+\bar{z})}\ 0,\ \ldots\ 0\right)^T , \tag{4.23}$$

with

$$S = 1/N, \quad Q = 0\,. \tag{4.24}$$

We now investigate the interaction part of the action for this configuration. The interaction action is given by

$$S_{\rm int}(N,\tau) = \frac{1}{\pi}\int dx s_{\rm int}(x_1)\,, \tag{4.25}$$

$$s_{\rm int}(x_1) = s(x_1) - \left(s_{\nu=1/N}(x_1) + s_{\nu=-1/N}(x_1)\right). \tag{4.26}$$

In figure 19, we plot the logarithm of the total interaction action $S_{\rm int}(N,\tau)$ as a function of τ for $N=2,3,4$. For sufficiently large separation $\tau \gtrsim 4$, $\log(-S_{\rm int})$ can be approximated by analytic lines,

$$\log\left[-S_{\rm int}(N,\tau)\right] \;\sim\; -\xi(N)\,\tau \;+\; C(N)\,, \qquad (\tau \gtrsim 4)\,, \tag{4.27}$$

where $\xi(N)$ is a slope and $C(N)$ is an y-intercept. For this case, the slope is expressed as

$$\xi(N) \sim \frac{12}{N}\,. \tag{4.28}$$

The interaction action can be written as the following form for large τ region,

$$S_{\rm int}(N,\tau) \;\sim\; -e^{C}\, e^{-\xi\tau}\,, \qquad \xi = \frac{12}{N}\,, \qquad (\tau \gtrsim 4)\,. \tag{4.29}$$

We next determine the N dependence of $C(N)$. In figure 20 we plot $\exp[C(N)]$ for $N=3,4,5,6,7,8$. We find out that it is approximated by $\exp[C(N)] \sim 14.3/N$, and depict it simultaneously in the figure. This result indicates that the interaction action (potential) for large-separation region can be written as

$$S_{\rm int}(N,\tau) \sim -\frac{14.3}{N}\, e^{-\xi\tau}\,, \qquad \xi = \frac{12}{N}\,, \qquad (\tau \gtrsim 4)\,, \tag{4.30}$$

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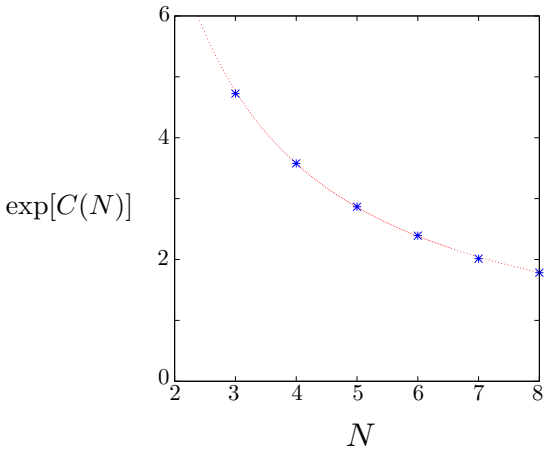

Figure 20. Plot of $\exp[C(N)]$ (a coefficient of the interaction potential) for $N = 3, 4, 5, 6, 7, 8$ for the configuration in eq. (4.15) (blue points). The plot is approximated by $14.3/N$ (a red curve).

which implies that the interaction part of the action at large τ region is expressed as

$$S_{\rm int}(\tau) \propto -\xi e^{-\xi\tau} \,. \tag{4.31}$$

This asymptotic form of the interaction potential is qualitatively consistent to (1.1). For our special boundary conditions in the present subsection, which can no longer be identified as Wilson-loop holonomy, it is not straightforward to understand the meaning of the values of ξ and e^C. However, as with the case for the $\mathbb{Z}_N$ twisted boundary condition, it is true that the prescription ($g^2 \to -g^2$) and analytic continuation lead to the ambiguity in the imaginary part of the amplitude for this case too. It implies that the resurgence procedure based on neutral bions universally works for general boundary conditions and vacua in field theories.

By calculating the renormalon ambiguity in the Borel re-summation of the perturbative series for the present non-$\mathbb{Z}_N$ boundary conditions, we can check if the two ambiguities are cancelled against each other. In the future work, we will investigate whether resurgence procedure based on bions or bion-like configuratons still works for non-$\mathbb{Z}_N$ vacuum such as split phases and its extensions.

5 Summary and discussion

In this paper, we have revisited topologically trivial configurations in the $\mathbb{C}P^{N-1}$ model on $\mathbb{R}^1 \times S^1$ with twisted boundary conditions, to study properties of bions composed of multiple fractionalized-instantons. In the $\mathbb{C}P^{N-1}$ model with center-symmetric and non-center-symmetric twisted boundary conditions, we have considered an explicit ansatz of a configuration containing one fractionalized instanton ($\nu = 1/N$) and one fractionalized anti-instanton ($\nu = -1/N$), which has an attractive force. We have shown that the separation-dependence and N-dependence of the interaction potential of the ansatz agree with the results of the far-separated instanton calculus [12, 13], even at small values of the separation.

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In section 3, we have considered a simple neutral-bion ansatz for the $\mathbb{Z}_N$ twisted boundary condition, which represents a molecule of one fractionalized instanton ($\nu = 1/N$) and one fractionalized anti-instanton ($\nu = -1/N$). From the separation dependence of the total action we show that the interaction between the instanton constituents are attractive, thus the configuration has a negative mode. The separation dependence and N-dependence of the interaction potential between the instanton constituents is compared with the result in the standard far-separated instanton calculus in eq. (1.1), we show that our ansatz is consistent with eq. (1.1) even from short to large separations. This result indicates that our ansatz well describes the neutral bion related to renormalon ambiguity, which can be used from short ($\tau \gtrsim 1$) to long ($\tau \gg 1$) separations.

In section 4, we have proposed bion-like ansatze in non-$\mathbb{Z}_N$ twisted boundary conditions including the one corresponding to the split vacuum in QCD(adj.) and its extensions for $N \geq 3$. We have shown that the interaction between the constituents is again attractive. In this case, we have found that the separation and N dependences of the interaction potential at large separation is qualitatively consistent to the result for $\mathbb{Z}_N$ twisted boundary conditions [12, 13] up to a numerical coefficient. It implies that the bion resurgence procedure universally works for a wide range of boundary conditions and vacua in field theories.

Our ansatz in the $\mathbb{C}P^{N-1}$ model corresponding to bion configurations can be a good starting point for studying properties of bions and related physics explicitly. Indeed, by using our ansatz, we can study physics related to neutral bions, not only at large separations $\tau \gg 1$, but also at short separations $\tau \gtrsim 1$, which cannot be reached by the far-separated instanton approach.

We here discuss how to determine which of the saddle points (perturbative or bion) configurations around $\tau \sim 1$ should be classified into. We so far have no systematic way to classify the parameters to the two associated regions by looking into our ansatz itself. On the other hand, from the viewpoint of the BZJ-prescription, the length scale where the imaginary ambiguity in the amplitude for our ansatz gets close to the ambiguity in the perturbative Borel-sum calculation should be regarded as the neutral bion scale. In our calculation, this scale is about $\tau \sim 1$ as shown in figure 8, which is consistent with the charged bion scale too. Thus, for now, what we can do for this purpose is just to sort out the configurations by the separation of the instanton constituents based on the plausible bion size $\tau \sim 1$, as performed in this paper. Exact classification of the parameter regions for the two saddles should be pursued in the future study.

As a future work, we consider to study charged bion configurations, whose instanton constituents have a repulsive interaction in a bosonic sector and also have an attractive interaction due to the fermion zero mode exchange. Due to the balance between the attractive and repulsive interactions, the size of charged bions will be clearly determined and there will be no complicated problem on the size of bions for this case. In the Ünsal's argument [3, 4], this configuration has great significance in weak-coupling-regime confinement via "bion condensation". While understanding of phase diagram in the L-m_{adj} plane is required to elucidate its relation to confinement in pure Yang-Mills or QCD theories, it should be also worth investigating a concrete configuration contributing confinement in a toy model.

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One straightforward extension will be bions in the Grassmanian sigma model with the target space $\mathrm{SU}(N)/[\mathrm{SU}(N-M)\times\mathrm{SU}(M)\times\mathrm{U}(1)]$. Domain walls in the Grassmannian sigma model were constructed in refs. [79–82, 86, 87]. Fractionalized instantons and bions can be composed from these solutions with twisting U(1) moduli. While the $\mathbb{C}P^1$ model with the twisted boundary condition has the Wilson-loop holonomy of a U(1) gauge field, the Grassmanian sigma model with the twisted boundary condition can have that of a non-Abelian gauge field. We will see that the Grassmanian sigma model admits charged bions in addition to neutral bions. The D-brane configurations in refs. [51, 52] will turn out to be very useful for analyzing this model.

One path to connect our results of bions in the $\mathbb{C}P^{N-1}$ model to QCD may be to consider a non-Abelian vortex [84–87] in Yang-Mills theory in the Higgs vacuum. U(N) Yang-Mills theory coupled with suitable number of Higgs matter fields in the fundamental representation admits a non-Abelian vortex, whose effective theory can be described by the $\mathbb{C}P^{N-1}$ model. In this case, the Yang-Mills instantons and monopoles become $\mathbb{C}P^{N-1}$ instantons and domain walls, respectively, when trapped inside a vortex [50, 83, 88–91]. Therefore, when the vortex world-sheet is wrapped around S^1 with a Wilson-loop holonomy, bions (instanton-monopoles) in Yang-Mills theory can exist inside the vortex as the $\mathbb{C}P^{N-1}$ bions (instanton-domain walls). By taking a un-Higgsing limit, the vortex disappears, and therefore we expect that they remain as Yang-Mills bions.

The same relation holds between quark matter in high density QCD and the $\mathbb{C}P^2$ model on a non-Abelian vortex [92, 93] (see ref. [94] for a review). This may give a hint to understand a quark-hadron duality between the confining phase at low density and the Higgs phase at high density, through a non-Abelian vortex [95].

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