Published for SISSA by Springer

Received: March 25, 2014
Revised: May 28, 2014
Accepted: May 28, 2014
Published: June 16, 2014

Scattering theory without large-distance asymptotics

JHEP06(2014)087

Tong Liu,[a] **Wen-Du Li**[a] **and Wu-Sheng Dai**[a,b]

[a] *Department of Physics, Tianjin University, Tianjin 300072, P.R. China*

[b] *LiuHui Center for Applied Mathematics, Nankai University & Tianjin University, Tianjin 300072, P.R. China*

E-mail: liutong@tju.edu.cn, liwendu@tju.edu.cn, daiwusheng@tju.edu.cn

Abstract: In conventional scattering theory, to obtain an explicit result, one imposes a precondition that the distance between target and observer is infinite. With the help of this precondition, one can asymptotically replace the Hankel function and the Bessel function with the sine functions so that one can achieve an explicit result. Nevertheless, after such a treatment, the information of the distance between target and observer is inevitably lost. In this paper, we show that such a precondition is not necessary: without losing any information of distance, one can still obtain an explicit result of a scattering rigorously. In other words, we give an rigorous explicit scattering result which contains the information of distance between target and observer. We show that at a finite distance, a modification factor — the Bessel polynomial — appears in the scattering amplitude, and, consequently, the cross section depends on the distance, the outgoing wave-front surface is no longer a sphere, and, besides the phase shift, there is an additional phase (the argument of the Bessel polynomial) appears in the scattering wave function.

Keywords: Integrable Equations in Physics, Scattering Amplitudes

ArXiv ePrint: 1403.5646

doi:10.1007/JHEP06(2014)087

Contents

1 Introduction

In conventional scattering theory, which is now a standard quantum mechanics textbook content, to seek an explicit result, one imposes a precondition that the distance between target and observer is infinite. As a result, the conventional scattering theory loses all the information of the distance and the result depends only on the angle of emergence. In this paper, we will show that without such a precondition, one can still achieve a rigorous scattering theory which, of course, contains the information of distance that is lost in conventional scattering theory.

The dynamical information of a scattering problem with a spherical potential $V(r)$ are embedded in the radial wave equation,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR_l}{dr}\right)+\left[k^2-\frac{l(l+1)}{r^2}-V(r)\right]R_l=0. \tag{1.1}$$

The scattering boundary condition in conventional scattering theory is taken to be

$$\psi(r,\theta)=e^{ikr\cos\theta}+f(\theta)\frac{e^{ikr}}{r},\quad r\to\infty. \tag{1.2}$$

In conventional scattering theory, in order to achieve an explicit result, two kinds of asymptotic approximations are employed [1].

1) Replace the solution of the free radial equation, i.e., eq. (1.1) with $V(r)=0$, with its asymptotics:

$$R_l(r)=C_l h_l^{(2)}(kr)+D_l h_l^{(1)}(kr) \tag{1.3}$$

$$\overset{r\to\infty}{\sim} A_l\frac{\sin(kr-l\pi/2+\delta_l)}{kr}, \tag{1.4}$$

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where $h_l^{(1)}(z)$ and $h_l^{(2)}(z)$ are the first and second kind spherical Hankel functions, $e^{2i\delta_l} = D_l/C_l$ defines the scattering phase shift δ_l, and $A_l = 2\sqrt{C_l D_l}$.

2) Replace the plane wave expansion in the boundary condition with its asymptotics:

$$e^{ikr\cos\theta} = \sum_{l=0}^{\infty}(2l+1)\, i^l j_l(kr)\, P_l(\cos\theta) \tag{1.5}$$

$$\overset{r\to\infty}{\sim} \sum_{l=0}^{\infty}(2l+1)\, i^l \frac{\sin(kr - l\pi/2)}{kr} P_l(\cos\theta), \tag{1.6}$$

where $j_l(z)$ is the spherical Bessel function.

Technologically speaking, the above two treatments in conventional theory are to replace the spherical Hankel function, $h_l^{(1)}(kr)$ and $h_l^{(2)}(kr)$, and the spherical Bessel function, $j_l(kr)$, with their asymptotics, and, thus, inevitably lead to the loss of information of the distance r.

In this paper, we will show that the above two replacements is not necessary; without these two replacements, we can still obtain a rigorous scattering theory which contains the information of the distance between target and observer.

A systematic rigorous result of a scattering with the distance between target and observer is given in section 2. The conclusion and outlook are given in section 3.

2 Rigorous result of scattering without large-distance asymptotics

In this section, a rigorous treatment without large-distance asymptotics for short-range potentials is established. The scattering wave function, scattering amplitude, phase shift, cross section, and a description of the outgoing wave are rigorously obtained.

2.1 Phase shift

In conventional scattering theory, as mentioned above, one replaces the solution of the free radial equation, $R_l(r)$, given by eq. (1.3) with its asymptotics, eq. (1.4), using the asymptotics of the spherical Hankel functions $h_l^{(1)}(kr) \sim \frac{1}{ikr}e^{i(kr-l\pi/2)}$ and $h_l^{(2)}(kr) \sim -\frac{1}{ikr}e^{-i(kr-l\pi/2)}$. Obviously, such a replacement will lose information.

In the following, with $R_l(r)$ given by eq. (1.3), rather than its asymptotics, eq. (1.4), we solve the scattering rigorously.

The first step is to rewrite $R_l(r)$ given by eq. (1.3) as

$$\begin{aligned} R_l(r) &= C_l h_l^{(2)}(kr) + D_l h_l^{(1)}(kr) \\ &= M_l\left(-\frac{1}{ikr}\right)\frac{A_l}{kr}\sin\left[kr - \frac{l\pi}{2} + \delta_l + \Delta_l\left(-\frac{1}{ikr}\right)\right], \end{aligned} \tag{2.1}$$

where $e^{2i\delta_l} = D_l/C_l$ and $M_l(x) = |y_l(x)|$ and $\Delta_l(x) = \arg y_l(x)$ are the modulus and argument of the Bessel polynomial $y_l(x)$, respectively.

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In order to achieve eq. (2.1), we prove the relation

$$C_l h_l^{(2)}(x) + D_l h_l^{(1)}(x) = M_l\left(-\frac{1}{ix}\right)\frac{A_l}{x}\sin\left[x-\frac{l\pi}{2}+\delta_l+\Delta_l\left(-\frac{1}{ix}\right)\right]. \tag{2.2}$$

Proof. The first and second kind spherical Hankel functions, $h_l^{(1)}(x)$ and $h_l^{(2)}(x)$, can be expanded as [2]

$$h_l^{(1)}(x) = e^{ix}\sum_{k=0}^{l}\frac{i^{k-l-1}(l+k)!}{2^k k!(l-k)!x^{k+1}}, \tag{2.3}$$

$$h_l^{(2)}(x) = e^{-ix}\sum_{k=0}^{l}\frac{(-i)^{k-l-1}(l+k)!}{2^k k!(l-k)!x^{k+1}}. \tag{2.4}$$

By the Bessel polynomial [2],

$$y_l(x) = \sum_{k=0}^{l}\frac{(l+k)!}{k!(l-k)!}\left(\frac{x}{2}\right)^k, \tag{2.5}$$

we can rewrite $h_l^{(1)}(x)$ and $h_l^{(2)}(x)$ as

$$\begin{aligned} h_l^{(1)}(x) &= e^{i(x-l\pi/2)}\frac{1}{ix}y_l\left(-\frac{1}{ix}\right), \\ h_l^{(2)}(x) &= -e^{-i(x-l\pi/2)}\frac{1}{ix}y_l\left(\frac{1}{ix}\right). \end{aligned} \tag{2.6}$$

Using eq. (2.6), we have

$$C_l h_l^{(2)}(x) + D_l h_l^{(1)}(x) = C_l\left[-\frac{e^{-i(x-l\pi/2)}}{ix}y_l\left(\frac{1}{ix}\right)+e^{2i\delta_l}\frac{e^{i(x-l\pi/2)}}{ix}y_l\left(-\frac{1}{ix}\right)\right]. \tag{2.7}$$

Writing the Bessel polynomial as $y_l = M_l e^{i\Delta_l}$, we prove the relation (2.2).

The wave function, then, by $\psi(r,\theta) = \sum_{l=0}^{\infty} R_l(r) P_l(\cos\theta)$, can be obtained immediately from eq. (2.1),

$$\psi(r,\theta) = \sum_{l=0}^{\infty} M_l\left(-\frac{1}{ikr}\right)\frac{A_l}{kr}\sin\left[kr-\frac{l\pi}{2}+\delta_l+\Delta_l\left(-\frac{1}{ikr}\right)\right]P_l(\cos\theta). \tag{2.8}$$

When the distance r is finite, the coefficient becomes $M_l A_l$ and the phase becomes $\delta_l + \Delta_l$, where M_l and Δ_l both depend on r. While, in conventional scattering theory, $r \to \infty$, the coefficient is A_l and the phase is δ_l, and they are both independent of r.

It should be emphasized that δ_l here is the same as that in conventional scattering theory. This is because δ_l is determined only by the coefficient C_l and D_l and $y_l\left(-\frac{1}{ikr}\right) \overset{r\to\infty}{=} 1$. Thus when $r \to \infty$, C_l, D_l, and, accordingly, δ_l remains unchanged.

The modification factors, Δ_l and M_l, are independent of potentials. When $r \to \infty$, $M_l(r\to\infty) = 1$ and $\Delta_l(r\to\infty) = 0$. □

2.2 Asymptotic boundary condition

The outgoing wave is no longer a spherical wave when the observer stands at a finite distance from the target, other than that in large-distance asymptotics. The outgoing wave now becomes a surface of revolution around the incident direction, determined by the potential and the observation distance. Because the outgoing waves are different at different distances, there is no uniform expression of the asymptotic boundary condition like eq. (1.2). Here, we express the boundary condition as

$$\psi(r,\theta)=e^{ikr\cos\theta}+f(r,\theta)\frac{e^{ikr}}{r}, \tag{2.9}$$

where $f(r,\theta)$ depends not only on θ but also on r.

When the distance r is finite, however, the differential scattering cross section is no longer the square modulus of $f(r,\theta)$. Only when $r\to\infty$, $f(\infty,\theta)=f(\theta)$ and the differential cross section reduces to $|f(\theta)|^2$.

To calculate $f(r,\theta)$, as that in conventional scattering theory, we expand the incoming plane wave $e^{ikr\cos\theta}$ by the eigenfunction of the angular momentum. Now, we prove that the expansion of $e^{ikr\cos\theta}$, eq. (1.5), can be exactly rewritten as

$$\begin{aligned}e^{ikr\cos\theta}&=\sum_{l=0}^{\infty}(2l+1)\,i^l j_l(kr)\,P_l(\cos\theta)\\&=\sum_{l=0}^{\infty}(2l+1)\,i^l M_l\left(-\frac{1}{ikr}\right)\frac{1}{kr}\sin\left[kr-\frac{l\pi}{2}+\Delta_l\left(-\frac{1}{ikr}\right)\right]P_l(\cos\theta).\end{aligned} \tag{2.10}$$

Proof. A plane wave can be expanded as [3]

$$e^{ikr\cos\theta}=\sum_{l=0}^{\infty}(2l+1)\,i^l j_l(kr)\,P_l(\cos\theta). \tag{2.11}$$

By the relations $h_l^{(1)}(x)=j_l(x)+in_l(x)$ and $h_l^{(2)}(x)=j_l(x)-in_l(x)$, the spherical Bessel function $j_l(x)$ can be rewritten as $j_l(x)=\frac{1}{2}\left[h_l^{(1)}(x)+h_l^{(2)}(x)\right]$, where $n_l(x)$ is the spherical Neumann function [2]. By eq. (2.6), we have

$$j_l(kr)=M_l\left(-\frac{1}{ikr}\right)\frac{1}{kr}\sin\left[kr-\frac{l\pi}{2}+\Delta_l\left(-\frac{1}{ikr}\right)\right]. \tag{2.12}$$

Substituting this result into eq. (2.11) proves eq. (2.10).

The plane wave expansion (2.10) is exact, rather than the asymptotic one, eq. (1.6), used in conventional scattering theory. In conventional scattering theory, the spherical Bessel function $j_l(kr)$ given by eq. (2.12) is replaced by its asymptotics: $j_l(kr)\sim\frac{1}{kr}\sin(kr-l\pi/2)$, i.e., M_l and Δ_l are asymptotically taken to be $M_l\left(-\frac{1}{ikr}\right)\sim1$ and $\Delta_l\left(-\frac{1}{ikr}\right)\sim0$; as a result, the information embedded in M_l and Δ_l is lost.

The boundary condition, eq. (2.9), then, by eq. (2.10), can be expressed as

$$\begin{aligned}\psi(r,\theta)=&\sum_{l=0}^{\infty}(2l+1)\,i^l M_l\left(-\frac{1}{ikr}\right)\frac{1}{kr}\sin\left[kr-\frac{l\pi}{2}+\Delta_l\left(-\frac{1}{ikr}\right)\right]P_l(\cos\theta)\\&+f(r,\theta)\frac{e^{ikr}}{r}.\end{aligned} \tag{2.13}$$

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2.3 Scattering wave function

The scattering wave function can be calculated by imposing the boundary condition (2.13) on the asymptotic wave function (2.8).

Observing the outgoing part of the wave function (2.13), $f(r,\theta)\, e^{ikr}/r$, we can see that the leading contribution of $f(r,\theta)$ must only be a zero power of r, or else the outgoing wave is not a spherical wave when $r \to \infty$. Thus, we can expand $f(r,\theta)$ by the Bessel polynomial, which is complete and orthogonal [4], as

$$f(r,\theta) = \sum_{l=0}^{\infty} g_l(\theta)\, y_l\left(-\frac{1}{ikr}\right). \tag{2.14}$$

The reason why only $y_l\left(-\frac{1}{ikr}\right)$ appears in the expansion (2.14) is that only the flux corresponding to $y_l\left(-\frac{1}{ikr}\right) e^{ikr}/r$ is an outgoing spherical wave; or, in other words, the requirement that the scattering wave must be an outgoing wave rules out the terms including $y_l^*\left(-\frac{1}{ikr}\right) = y_l\left(\frac{1}{ikr}\right)$.

Equating eqs. (2.8) and (2.13), using the expansion (2.14), $\sin\theta = \left(e^{i\theta} - e^{-i\theta}\right)/(2i)$, and the orthogonality, and noting that $M_l e^{i\Delta_l} = y_l$, we arrive at

$$\begin{aligned} \frac{1}{2ik}\left[(2l+1) - A_l e^{i(-l\pi/2+\delta_l)}\right] P_l(\cos\theta) + g_l(\theta) &= 0, \\ (2l+1)\, e^{il\pi} - A_l e^{-i(-l\pi/2+\delta_l)} &= 0. \end{aligned} \tag{2.15}$$

Solving these two equations gives

$$A_l = (2l+1)\, e^{il\pi} e^{i(-l\pi/2+\delta_l)}, \tag{2.16}$$

$$g_l(\theta) = -\frac{1}{2ik}(2l+1)\left(1 - e^{i2\delta_l}\right) P_l(\cos\theta). \tag{2.17}$$

Then, we arrive at

$$f(r,\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\left(e^{2i\delta_l} - 1\right) P_l(\cos\theta)\, y_l\left(-\frac{1}{ikr}\right). \tag{2.18}$$

When taking the limit $r \to \infty$, the modification factor — the Bessel polynomial $y_l\left(-\frac{1}{ikr}\right)$ — tends to 1, and $f(r,\theta)$ recovers the scattering amplitude in conventional scattering theory: $f^{\infty}(\theta) = f(\infty,\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\left(e^{2i\delta_l} - 1\right) P_l(\cos\theta)$.

The leading is a p-wave modification, because the s-wave modification is $y_0(x) = 1$.

2.4 Outgoing wave-front surface

In conventional scattering theory, the observer is at $r \to \infty$ and the outgoing wave is a spherical wave. When the observer is at a finite distance r, the outgoing wave-front surface, however, is a surface of revolution around the incident direction, since for a spherical potential the outgoing wave must be cylindrically symmetric.

The outgoing wave-front surface is determined by the outgoing flux $\mathbf{j}^{\rm sc}$ which serves as its surface normal vector. The outgoing flux is $\mathbf{j}^{\rm sc} = \mathbf{j} - \mathbf{j}^{\rm in}$, where $\mathbf{j} = \frac{\hbar}{m}\,\mathrm{Im}\left(\psi^*\nabla\psi\right)$

and $\mathbf{j}^{\rm in}=\frac{\hbar}{m}\,{\rm Im}\left(\psi^{\rm in *}\nabla\psi^{\rm in}\right)$. Here we write the wave function (2.9) as $\psi = \psi^{\rm in} + \psi^{\rm sc}$ with $\psi^{\rm in} = e^{ikr\cos\theta}$ and $\psi^{\rm sc} = f\left(r,\theta\right)e^{ikr}/r$.

The outgoing wave-front surface is a surface of revolution. Its generatrix, $r = r\left(\theta\right)$, with $\mathbf{j}^{\rm sc}$ as the normal vector, is determined by

$$\frac{1}{r\left(\theta\right)}\frac{dr\left(\theta\right)}{d\theta} = -\frac{j_\theta^{\rm sc}}{j_r^{\rm sc}} = -\tan\gamma^{\rm sc}, \tag{2.19}$$

where $\gamma^{\rm sc}$ is the intersection angle between $\mathbf{j}^{\rm sc}$ and the radial vector.

The equation of the generatrix, eq. (2.19), is a differential equation. The integration constant can be chosen as $r\left(0\right) = R$, where R is the intersection between the outgoing wave-front surface on which the observer stands and the target along the z-axis. Then the solution of eq. (2.19) can be formally written as $r = r\left(\theta, R\right)$.

Moreover, the Gaussian curvature of the outgoing wave-front surface reads

$$K\left(\theta\right) = \frac{1}{r^2}\cos^2\gamma^{\rm sc}\left(1+\frac{d\gamma^{\rm sc}}{d\theta}\right)\left(1-\tan\gamma^{\rm sc}\cot\theta\right). \tag{2.20}$$

When $r\to\infty$, $\gamma^{\rm sc}\to 0$, and then $K = 1/r^2$ reduces to a curvature of a sphere.

2.5 Differential scattering cross section

The differential scattering section is $d\sigma = \mathbf{j}^{\rm sc}\cdot d\mathbf{S}/j^{\rm in}$. The scattering flux $\mathbf{j}^{\rm sc}$, other than that in conventional scattering theory, is not along the radial direction. Thus,

$$d\sigma = \frac{\mathbf{j}^{\rm sc}\cdot d\mathbf{S}}{j^{\rm in}} = \frac{j^{\rm sc}}{j^{\rm in}}\frac{r^2 d\Omega}{\cos\gamma^{\rm sc}} = \left(1+\tan^2\gamma^{\rm sc}\right)\frac{j_r^{\rm sc}}{j^{\rm in}}r^2 d\Omega, \tag{2.21}$$

where $j^{\rm sc} = \sqrt{j_r^{sc2}+j_\theta^{sc2}}$ and $\tan\gamma^{\rm sc} = j_\theta^{\rm sc}/j_r^{\rm sc}$. A straightforward calculation gives

$$\frac{d\sigma}{d\Omega} = \left[\left|f\left(r,\theta\right)\right|^2 + \eta\left(r,\theta\right)\right]\left(1+\tan^2\gamma^{\rm sc}\right), \tag{2.22}$$

where

$$\eta\left(r,\theta\right) = \frac{1}{k}\,{\rm Im}\left\{f^*\frac{\partial f}{\partial r} + e^{ikr(1-\cos\theta)}\left\{\left[ikr(1+\cos\theta)-1\right]f + r\frac{\partial f}{\partial r}\right\}\right\}. \tag{2.23}$$

2.6 Total scattering cross section

For simplicity, we only consider the leading contribution of the total scattering cross section, $\sigma_t\left(R\right) = 2\pi\int_0^\pi\left|f\left(R,\theta\right)\right|^2\sin\theta d\theta$, in which the outgoing wave-front surface is approximately a sphere of radius R.

The total cross section then reads

$$\sigma_t\left(R\right) = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}\left(2l+1\right)\sin^2\delta_l\left|y_l\left(-\frac{1}{ikR}\right)\right|^2. \tag{2.24}$$

In comparison with conventional scattering theory, a modification factor $\left|y_l\left(-\frac{1}{ikR}\right)\right|^2$ appears.

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2.7 Condition on potentials

In this section, we discuss the condition on the potential $V(r)$ that preserves the validity of the solution of the radial wave equation without the large-distance asymptotics.

The radial wave equation, eq. (1.1), can be rewritten as $\frac{d^2u_l(r)}{dr^2}+\left[k^2-\frac{l(l+1)}{r^2}-V(r)\right]u_l(r)=0$ by introducing $R_l(r)=u_l(r)/r$. At the distance that the influence of $V(r)$ can be ignored, as pointed above, the solution of the radial wave equation with $V(r)=0$ is $u_l(r)=y_l\left(\mp\frac{1}{ikr}\right)e^{\pm ikr}$. In the region that the potential cannot be ignored, we express $u_l(r)$ as

$$u_l(r)=e^{h(r)}y_l\left(\mp\frac{1}{ikr}\right)e^{\pm ikr} \tag{2.25}$$

$$=[1+h(r)+\cdots]\left[1\mp\frac{l(l+1)}{2ik}\frac{1}{r}+\cdots\right]e^{\pm ikr}. \tag{2.26}$$

When taking the large-distance asymptotics, $y_l\left(\mp\frac{1}{ikr}\right)\big|_{r\to\infty}=1$ and $u_l(r)$ returns to the large-distance asymptotics: $u_l(r)=e^{h(r)}e^{\pm ikr}=[1+h(r)+\cdots]e^{\pm ikr}$. This requires $h(r)|_{r\to\infty}\to 0$. Without the large-distance asymptotics, however, since $u_l(r)$ must tend to $y_l\left(\mp\frac{1}{ikr}\right)e^{\pm ikr}$ as r increases, $h(r)$ has to decrease more rapidly than $y_l\left(\mp\frac{1}{ikr}\right)$; or, $h(r)$ tends to zero before the vanishing of $y_l\left(\mp\frac{1}{ikr}\right)$. It can be directly seen by observing eq. (2.26) that this requirement imposes a condition on $h(r)$: without the large-distance asymptotics, $h(r)$ must decrease more rapidly than $\frac{1}{r}$, i.e.,

$$h(r)=\frac{\alpha}{r^{1+\epsilon}}, \tag{2.27}$$

where $\epsilon>0$. While, as a comparison, in the case of large-distance asymptotics, since $y_l\left(\mp\frac{1}{ikr}\right)=1$, we only needs

$$h_{asym}(r)=\frac{\alpha}{r^{\epsilon}}, \tag{2.28}$$

Next, we discuss the condition on $h(r)$ will impose what a restriction on the potential $V(r)$.

The equation determining $h(r)$ can be constructed by substituting eq. (2.25) into the radial wave equation:

$$h''(r)+\left[h'(r)\right]^2\pm 2ikh'(r)+2h'(r)\frac{\frac{d}{dr}y_l\left(\mp\frac{1}{ikr}\right)}{y_l\left(\mp\frac{1}{ikr}\right)}$$

$$\pm 2ik\frac{\frac{d}{dr}y_l\left(\mp\frac{1}{ikr}\right)}{y_l\left(\mp\frac{1}{ikr}\right)}+\frac{\frac{d^2}{dr^2}y_l\left(\mp\frac{1}{ikr}\right)}{y_l\left(\mp\frac{1}{ikr}\right)}=\frac{l(l+1)}{r^2}+V(r). \tag{2.29}$$

For a large r, we have

$$\frac{\frac{d}{dr}y_l\left(\mp\frac{1}{ikr}\right)}{y_l\left(\mp\frac{1}{ikr}\right)}=\pm\frac{l(l+1)}{2ik}\frac{1}{r^2}+\cdots, \tag{2.30}$$

$$\frac{\frac{d^2}{dr^2}y_l\left(\mp\frac{1}{ikr}\right)}{y_l\left(\mp\frac{1}{ikr}\right)}=\mp\frac{l(l+1)}{ik}\frac{1}{r^3}+\cdots. \tag{2.31}$$

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Then by eqs. (2.29), (2.30), (2.31), and condition (2.27), we obtain a condition on the potential:

$$V(r) \sim \frac{1}{r^{2+\epsilon}}. \tag{2.32}$$

As a comparison, in the case of large-distance asymptotics, by condition (2.28) and eq. (2.29) with $y_l \to 1$, we obtain the condition for the potential in the conventional scattering theory [5]:

$$V(r) \sim \frac{1}{r^{1+\epsilon}}. \tag{2.33}$$

When $k = 0$, however, there is something different. The condition on the potential for $k = 0$ becomes stronger than that for $k \neq 0$.

For $k = 0$, the expansions (2.30) and (2.31) become

$$\frac{\frac{d}{dr} y_l\left(\mp\frac{1}{ikr}\right)}{y_l\left(\mp\frac{1}{ikr}\right)} = -\frac{l}{r} + \cdots, \tag{2.34}$$

$$\frac{\frac{d^2}{dr^2} y_l\left(\mp\frac{1}{ikr}\right)}{y_l\left(\mp\frac{1}{ikr}\right)} = \frac{l(l+1)}{r^2} + \cdots. \tag{2.35}$$

Substituting expansions (2.34) and (2.35) into (2.29) and taking $k = 0$ give

$$h''(r) + \left[h'(r)\right]^2 + 2h'(r)\left(-\frac{l}{r}\right) = V(r). \tag{2.36}$$

By condition (2.27) and eq. (2.36), we obtain a condition on the potential for $k = 0$:

$$V(r) \sim \frac{1}{r^{3+\epsilon}}. \tag{2.37}$$

As a comparison, similarly, in the case of large-distance asymptotics with $k = 0$, by condition (2.28) and eq. (2.29) with $y_l \to 1$ and $k = 0$, we have

$$h''_{asym}(r) + \left[h'_{asym}(r)\right]^2 = \frac{l(l+1)}{r^2} + V(r), \tag{2.38}$$

and then we obtain the condition for the potential in the conventional scattering theory:

$$V(r) \sim \frac{1}{r^{2+\epsilon}}. \tag{2.39}$$

It is worth to note that for $l \neq 0$, we need only $V(r) \sim 1/r^2$, but for $l = 0$, we need the somewhat stronger condition (2.39).

From the above discussion we learn that the condition on the potential in the scattering theory without large-distance asymptotics is stronger than that in the scattering theory with large-distance asymptotics. Without large-distance asymptotics, the influence of the potential must be small enough at a finite distance r_0; when $r > r_0$, the influence due to the potential can be safely neglected. In the region of $r > r_0$, the solution is determined by the free radial wave equation.

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Now we estimate the magnitude of r_0. From eq. (2.26), we have

$$u_l(r) = \left[1 + h(r) \pm \frac{l(l+1)}{2ik}\frac{1}{r} + \cdots\right] e^{\pm ikr}. \tag{2.40}$$

As analyzed above, $h(r)$, which reflects the influence of the potential, must decrease more rapidly than $\left|\frac{l(l+1)}{2ik}\frac{1}{r}\right|$. When $r < r_0$ the influence of the potential cannot be neglected and when $r > r_0$ the influence of the potential can be neglected, so r_0 can be estimated by

$$|h(r_0)| = \left|\frac{l(l+1)}{2ik}\frac{1}{r_0}\right|. \tag{2.41}$$

Substituting the condition on $h(r)$, eq. (2.27), and the condition on $V(r)$, eq. (2.32), into eq. (2.29) gives $\alpha \sim \beta/[2k(1+\epsilon)]$. Then substituting eq. (2.27) and α into eq. (2.41) gives

$$r_0 \sim \left[\frac{\beta}{(1+\epsilon)}\frac{1}{l(l+1)}\right]^{1/\epsilon}. \tag{2.42}$$

Note that the case of $l = 0$ does not contribute to the correction from the solution without large-distance asymptotics.

In principle, different potentials correspond to different r_0. The range r_0 given here is indeed an upper limit of the range of the influence of the potential without large-distance asymptotics, since the potential considered here is given by condition (2.27).

3 Conclusions and outlook

We show that one can obtain a rigorous scattering theory without the precondition $r \to \infty$. A rigorous scattering theory contains the information of the distance between target and observer is presented. The conventional scattering theory can be recovered by setting $r \to \infty$.

In comparison with conventional scattering theory, there is an additional factor — the l-th Bessel polynomial — appears in the l-th partial-wave contribution. The leading modification is p-wave.

Quantum scattering theory plays an important role in many physical area and is intensively studied. Nevertheless, all studies are based on conventional scattering theory. Based on our result, we can further consider many scattering-related problems. For example, at low temperatures, the thermal wavelength has the same order of magnitude as the interparticle spacing, so the scattering in a BEC transition [6, 7] and in a transport of spin-polarized fermions [8, 9] may need to take the effect of the distance into account. The scattering spectrum method is important in quantum field theory [10–13]; a scattering spectrum method without asymptotics can also be discussed. Moreover, the relation between scattering spectrum method and heat kernel method, which is given by ref. [14] based on refs. [15, 16], can also be improved by the exact result of the scattering theory without infinite-distance asymptotics. Moreover, a related inverse scattering problems can also be systematically studied, and the result can be applied to, e.g., the interference pattern of Bose-Einstein condensates [17] and the Aharonov-Bohm effect [18].

JHEP06(2014)087

Acknowledgments

We are very indebted to Dr. G. Zeitrauman for his encouragement. This work is supported in part by NSF of China under Grant No. 11075115.

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