

Cut moments and a generalization of DGLAP equations

D. Kotlorz^{a,b} and S.V. Mikhailov^{b,1}

^aOpole University of Technology, Division of Physics,
45-370 Opole, Ozimska 75, Poland

^bBogoliubov Laboratory of Theoretical Physics, JINR,
141980 Dubna, Russia

E-mail: dorota@theor.jinr.ru, mikhs@theor.jinr.ru

ABSTRACT: We elaborate a cut (truncated) Mellin moments (CMM) approach that is constructed to study deep inelastic scattering in lepton-hadron collisions at the natural kinematic constraints. We show that generalized CMM obtained by multiple integrations of the original parton distribution $f(x, \mu^2)$ as well as ones obtained by multiple differentiations of this $f(x, \mu^2)$ also satisfy the DGLAP equations with the correspondingly transformed evolution kernel $P(z)$. Appropriate classes of CMM for the available experimental kinematic range are suggested and analyzed. Similar relations can be obtained for the structure functions $F(x)$, being the Mellin convolution $F = C * f$, where C is the coefficient function of the process.

KEYWORDS: Deep Inelastic Scattering, Renormalization Group, QCD

ARXIV EPRINT: [1404.5172](https://arxiv.org/abs/1404.5172)

¹Corresponding author.

Contents

1	Introduction	1
2	General solution of the DGLAP equation	3
2.1	Evolution equations for original cut Mellin moments	3
2.2	Evolution equations for generalized cut Mellin moments	3
3	Generalized CMM solution of the DGLAP equation at $n_i = \alpha = 1$	5
3.1	The continuation of the gCMM solution in real number $\nu \geq 0$ of integrations	5
3.2	The continuation of the gCMM solutions to $\nu < 0$	6
4	Generalized CMM solution of the DGLAP equation at $n_i = \alpha = 0$	7
5	Conclusion	8
A	Lemma about Mellin convolution	9

1 Introduction

Deep inelastic scattering (DIS) of leptons on hadrons providing unique information about the structure of the hadrons remains nowadays one of the best studied reactions. It tests also the scale evolution of the parton densities (named also distribution functions), one of the most important predictions of perturbative QCD (pQCD). Dependence on the argument x of the parton density $f(x, \mu^2)$ is formed at a hadron scale by nonperturbative forces, while its dependence on factorization/renormalization scale μ can be obtained within pQCD. The evolution of $f(x, \mu^2)$ with μ^2 is governed by the well-known Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) evolution equation [1–4], presented in the space of the Bjorken variable x ($x = Q^2/(2(Pq))$), where q — transferred momentum, $-q^2 = Q^2 > 0$, P — hadron momentum, $P^2 = m^2$).

Alternatively, one can study how to evolve with scale μ^2 ($\mu^2 \sim Q^2$) the Mellin moments of the parton densities, $f(\mu^2, n) \equiv \int_0^1 f(x, \mu^2)x^{n-1}dx$. It seems that these moments provide a natural framework of QCD analysis as they originate from the basic formalism of operator product expansion (OPE). However, the moments $f(\mu^2, n)$ appear as the result of idealization. We need to invent new “real” observables of DIS, named the “cut (truncated) Mellin moments” (CMM) with a goal to overcome kinematic constraints naturally appearing in real DIS experiments. Namely, the small values of the variable x cannot be reached in experiment at bounded below transferred momentum $Q^2 \geq Q_0^2 > 0$ and at not very large $2(Pq) = 2mE \sim$ transferred energy in the laboratory frame. The CMM $f(x_0, \mu^2, n)$ are generalized moments of the parton density $f(x, \mu^2)$ with lower limit of integration $x_0 > 0$,

$f(x, \mu^2) \rightarrow f(x_0, \mu^2, n) \equiv \int_{x_0}^1 f(x, \mu^2) x^{n-1} dx$. In this way, $f(x_0, \mu^2, n)$ in contrast with the standard $f(0, \mu^2, n) \equiv f(\mu^2, n)$ takes into account the kinematic constraint.

The actual requirement to deal with the cut moments appeared, e.g., for JLAB experiment EG1b [5] on the Bjorken integral measured in polarized nonsinglet DIS investigation of parton density $g_1^p - g_1^n$. The values of the standard moment $g_1(0, \mu^2, 1)$ of the parton density g_1 there, obtained by extrapolation of g_1 to the origin, differ 3 ÷ 4 times from the observable $g_1(x_0, \mu^2, 1)$ that is really measured in the experiment (see table 1 in [5]) in a *restricted* region of x . Looks evident that the procedure of extrapolation certainly reduces the accuracy of the results. Otherwise, one would obtain how $g_1(x_0, \mu^2, 1)$ is evolved with μ^2 in contrast with the case of the conserved standard moment $g_1(\mu^2, 1) \sim \Gamma_1^{p-n}(\mu^2)$.

The idea of “truncated” Mellin moments of the parton densities in QCD analysis was introduced and developed in the late 1990’s [6–9]. The authors obtained the nondiagonal differential evolution equations, in which the n th truncated moment couples to all higher ones. Later on, diagonal integro-differential DGLAP-type evolution equations for the single and double truncated moments of the parton densities were derived in [10] and [11, 12], respectively. The main finding of the truncated Mellin moments approach (CMMA) is that the n th moment of the parton density obeys the DGLAP equation again, but with a rescaled evolution kernel $P_1(z) = z^n P(z)$ [10]. The CMMA has already been successfully applied, e.g., in spin physics to derive a generalization of the Wandzura-Wilczek relation in terms of the truncated moments and to obtain the evolution equation for the structure function, e.g., g_2 [12, 13]. Truncation of the moments in the upper limit is less important in comparison to the low- x limit because of the rapid decrease of the parton densities as $x \rightarrow 1$; nevertheless, a comprehensive theoretical analysis requires an equal treatment of both truncated limits. The evolution equations for double cut moments and their application to study the quark-hadron duality were also discussed in [14]. A evolution equations for CMM are universal — they are valid in each order of the pQCD expansion and also for the singlet parton distributions [13]. A similar generalization can be obtained for the structure functions $F(x)$, $F = C * f$, where C is the coefficient function of the process, signum $*$ means the Mellin convolution. Indeed, the coefficient functions $C(t)$ rescale in the same way as the evolutionary kernels: $C_1(z) = z^n C(z)$.

In this paper, we present a novel generalization of CMM $f(x, \mu^2, n)$ and the corresponding DGLAP equations for the nonsinglet case. We elaborate an approach to investigate within a unified frame a smoothed parton density obtained by multiple integration of the initial $f(x, \mu^2)$ as well as a sharpened one obtained by a multiple differentiation of this $f(x, \mu^2)$. These generalized solutions in terms of CMM (gCMM) provide a powerful tool to study DIS processes at real kinematic constraints. Moreover, this gives new classes of parton gCMM for DIS description, whose analytical properties are discussed in detail for some interesting partial cases. The main aim of this paper is to present an extended theoretical frame for gCMM and to investigate analytic properties for them. We only mention possible applications of these gCMM for the analysis of experimental data, preparing a specific analysis for an oncoming paper. This paper is organized as follows: in section 2, we formulate the theoretical scheme used in this work. The detailed analysis of a multiple integrated (smoothed) CMM is presented in section 3 by consideration of a methodically

important partial case. We show that in this case the same defined gCMM cover multiple differentiated (sharpen) CMM. Following this line of consideration, in section 4 we propose a gCMM that is appropriate for analysis of DIS sum rules. Our conclusions are drawn in section 5, while important technical details are collected in appendix, where we also discuss a restriction of this generalization.

2 General solution of the DGLAP equation

The goal of this section is to construct new general solutions for the nonsinglet DGLAP equation

$$\mu^2 \frac{d}{d\mu^2} f(z, \mu^2) \equiv \dot{f}(z, \mu^2) = P * f(z) \equiv \int_0^1 P(y) f(x, \mu^2) \delta(z - x \cdot y) dy dx, \quad (2.1)$$

which are inspired by physically motivated CMM. The method of consideration will be illustrated via a brief derivation of the original CMM [10] in subsection 2.1.

2.1 Evolution equations for original cut Mellin moments

Let us apply the integral transformation $\int_{z_1}^1 z^{n-1}(\dots) dz$ to both sides of the DGLAP equation (2.1) taking into account that this transformation can be represented in the form of Mellin convolution (Mc). Introducing the notation for the r.h.s. of (2.1) $\phi(z) \equiv P * f(z)$ and using the Mc notation for the r.h.s. of the transformed equation one obtains (the argument μ^2 is omitted here and below)

$$f(z) \rightarrow f_1(z_1; n) \stackrel{def}{=} \int_{z_1}^1 z^{n-1} f(z) dz, \quad (2.2a)$$

$$\dot{f}_1(z_1; n) = \int_{z_1}^1 z^{n-1} \phi(z) dz = \left(\frac{z_1^n}{t^n} * \phi(z) \right) (z_1) = \left(\frac{z_1^n}{t^n} * (P * f_1) \right) (z_1). \quad (2.2b)$$

Taking into account known properties of Mellin convolution, which are discussed in detail in appendix A (see the chain of eqs. (A.4) and eq. (A.5)), and applying them to the r.h.s. of eq. (2.2b) one gets the evolution equation for CMM f_1

$$\dot{f}_1(z; n) = P_1 * f_1(z), \quad (2.3)$$

where $P_1(y) = P(y) \cdot y^n$. The same relation is evidently derived for the structure function (SF) $F = C * f$ under this integral transform, $F \rightarrow F_1 = C_1 * f_1$, where $C_1(t) = t^n \cdot C(t)$ [10–12].

2.2 Evolution equations for generalized cut Mellin moments

Here we present a generalization of the results shown in the previous section, obtained for multiintegration of the original function. Namely, if $f(x, \mu^2)$ is a solution of the nonsinglet DGLAP equation with the kernel $P(y)$: then the k -integrated function $f_k(z, \mu^2; \{n\}_k)$ (the argument μ^2 and the index k is omitted below for simplicity)

$$\begin{aligned} f(z; \{n\}_k) &\equiv f(z; n_1, n_2, \dots, n_k) \\ &= \int_z^1 z_k^{n_k-1} dz_k \int_{z_k}^1 z_{k-1}^{n_{k-1}-1} dz_{k-1} \dots \int_{z_2}^1 z_1^{n_1-1} f(z_1, \mu^2) dz_1, \end{aligned} \quad (2.4)$$

which is a generalization of CMM and is also the solution of the DGLAP equation

$$\dot{f}(z; \{n\}_k) = \mathcal{P} * f(z; \{n\}_k) \equiv \int_0^1 \mathcal{P}(y) f(x; \{n\}_k) \delta(z - x \cdot y) dy dx \quad (2.5a)$$

with the kernel

$$\mathcal{P}(y) = P(y) \cdot y^{n_1+n_2+\dots+n_k}. \quad (2.5b)$$

Proof. One should apply k times the transformation (2.2) (or apply Lemma in appendix A.) to both sides of DGLAP eq. (2.1). At the same transformation of SF $F = C * f$ one obtains the new SF \mathcal{F} and the new coefficient function \mathcal{C} ,

$$F, C \rightarrow \mathcal{F} = \mathcal{C} * f(z; \{n\}_k), \quad \mathcal{C} = C(t) \cdot t^{n_1+n_2+\dots+n_k}. \quad (2.6)$$

The generalized evolution equation similar to eq. (2.5) can be obtained also for \mathcal{F} . For this purpose, let us recall that the corresponding original equation reads [15],

$$\dot{F}(z; \mu^2) = (K * F)(z); \quad (2.7a)$$

$$K = P + \beta(a_s) (\partial_{a_s} C) * C^{-1}, \quad (2.7b)$$

where K is the modified kernel, while β is the QCD β -function, $\mu^2 \frac{d}{d\mu^2} a_s(\mu^2) = \beta(a_s)$. Again, applying the integration in eq. (2.2) k times to eq. (2.7) one arrives at the evolution equation for \mathcal{F}

$$\dot{\mathcal{F}}(z; \{n\}_k) = \mathcal{K} * \mathcal{F}(z; \{n\}_k) \quad (2.8)$$

with the kernel $\mathcal{K}(y) = K(y) \cdot y^{n_1+n_2+\dots+n_k}$.

The general solution (2.4) is the source of various new partial solutions and also already known results, e.g., from (2.5) at $k = 1$ follows the original equation from [10], namely,

$$\dot{f}_1(z; n_1) = \int_0^1 [P(y)y^{n_1}] f_1(x; n_1) \delta(z - x \cdot y) dx dy. \quad (2.9)$$

If one puts $z = 0$ in (2.9) it reduces to

$$\dot{f}_1(0; n_1) = \left(\int_0^1 P(y)y^{n_1-1} dy \right) \cdot f_1(0; n_1) \equiv -\gamma(n_1) \cdot f_1(0; n_1),$$

which is the renormalization group equation for the standard moments $f(\mu^2, n_1) \equiv f_1(0; n_1)$ with the corresponding nonsinglet anomalous dimension $\gamma(n_1) = - \int_0^1 P(y)y^{n_1-1} dy$.

A simple way to explore the properties of gCMM in eq. (2.4) is to admit a single weight for all integrations in the r.h.s. of (2.4), namely, $n_i = \alpha$. Hence, after the integration one obtains

$$f(z; \{\alpha\}_k) = \int_z^1 \left[\frac{t^\alpha - z^\alpha}{\alpha} \right]^{k-1} \frac{f(t)}{\Gamma(k)} dt, \quad (2.10a)$$

$$\mathcal{P}(y) = P(y) \cdot y^{\alpha k}; \quad \mathcal{C}(t) = C(t) \cdot t^{\alpha k}. \quad (2.10b)$$

Based on eqs. (2.10), (2.5) different interesting gCMM can be constructed. The results of two simplest cases, $\alpha = 1, 0$, will be considered below.

3 Generalized CMM solution of the DGLAP equation at $n_i = \alpha = 1$

In this section we, consider the partial solution of eq. (2.10) with $n_1 = n$ and all others $n_k = \alpha = 1, k > 1$,

$$\begin{aligned} z^n f(z) \rightarrow f(z; \{n, 1\}_k) &\equiv f(z; n, 1, \dots, 1) = \int_z^1 dz_k \int_{z_k}^1 dz_{k-1} \dots \int_{z_2}^1 z_1^n f(z_1) dz_1 \\ &= \int_z^1 \frac{(t-z)^{(k-1)}}{\Gamma(k)} t^n f(t) dt, \end{aligned} \quad (3.1a)$$

$$\text{with the kernel } \mathcal{P}(y) = P(y) \cdot y^{n+k}. \quad (3.1b)$$

$$f(z; \{n, 1\}_k) \rightarrow z^n f(z) = \left(-\frac{d}{dz}\right)^k f(z; \{n, 1\}_k) \quad - \text{ inverse operator} \quad (3.1c)$$

In comparison to eq. (2.10), in gCMM (3.1) and below we use for further convenience additional weight t^n at $f(t)$. Our aim here is to analyze solution (3.1) and to extend the range of definition of the integer parameter k to any real value $\nu, k \rightarrow \nu$. We shall show that the generalization of (3.1a), (3.1b) for $\nu \rightarrow -k$ naturally leads to a solution with few times differentiated initial parton density $f(z)$. This result will give us an important methodical lesson for further consideration.

3.1 The continuation of the gCMM solution in real number $\nu \geq 0$ of integrations

Let us consider the kernel $\frac{(t-z)^{(k-1)}}{\Gamma(k)}$ of the integrand in (3.1a) that accumulates all the dependence on the k . This kernel enables us to generalize $f(z; \{\rho, 1\}_k)$ to any real index $k \rightarrow \nu, f(z; \{\rho, 1\}_\nu)$ can be defined as

$$\frac{(t-z)^{(k-1)}}{\Gamma(k)} \rightarrow \frac{(t-z)^{(\nu-1)}}{\Gamma(\nu)} \quad (3.2a)$$

$$f(z; \{\rho, 1\}_k) \rightarrow f(z; \{\rho, 1\}_\nu) \stackrel{def}{=} \int_z^1 \frac{(t-z)^{(\nu-1)}}{\Gamma(\nu)} t^\rho f(t) dt \quad (3.2b)$$

with the DGLAP kernel

$$\mathcal{P}_\nu(y) = P(y) \cdot y^{\rho+\nu}.$$

Using definition (3.2b) $f(z; \{\rho, 1\}_\nu)$ can be analytically extended at the point $\nu = 0$. To show this, let term $t^\rho f(t) = \varphi_\rho(t)$ for shortcut notation and put $\nu = \varepsilon \rightarrow 0$, then

$$\begin{aligned} f(z; \{\rho, 1\}_0) &= \lim_{\varepsilon \rightarrow 0} [f(z; \{\rho, 1\}_\varepsilon)] \\ &= \lim_{\varepsilon \rightarrow 0} \left[\int_z^1 \frac{(t-z)^{(\varepsilon-1)}}{\Gamma(\varepsilon)} \varphi_\rho(t) dt = \frac{1}{\Gamma(\varepsilon)} \int_z^1 \frac{\varphi_\rho(t) - \varphi_\rho(z)}{(t-z)^{1-\varepsilon}} dt + \frac{\varphi_\rho(z)}{\Gamma(\varepsilon)} \int_z^1 \frac{dt}{(t-z)^{1-\varepsilon}} \right. \\ &\quad \left. = \varepsilon \int_z^1 \frac{\varphi_\rho(t) - \varphi_\rho(z)}{(t-z)} dt + \varphi_\rho(z) (1 + O(\varepsilon)) \right] \stackrel{\varepsilon \rightarrow 0}{=} \varphi_\rho(z). \end{aligned} \quad (3.3)$$

3.2 The continuation of the gCMM solutions to $\nu < 0$

Successively integrating by part the r.h.s. of (3.2b) the $f(z; \{\rho, 1\}_\nu)$ can be analytically extended to negative ν . E.g., integrating by part the def. (3.2) for $\nu \geq 0$ one obtains

$$f(z; \{\rho, 1\}_\nu) = \frac{\bar{z}^\nu}{\Gamma(\nu + 1)} \varphi_\rho(1) - \int_z^1 \varphi'_\rho(t) \frac{(t - z)^\nu}{\Gamma(\nu + 1)} dt. \quad (3.4)$$

The r.h.s. of eq. (3.4) at $\nu = 0$,

$$f(z; \{\rho, 1\}_0) = \varphi_\rho(1) - \int_z^1 \varphi'_\rho(t) dt = \varphi_\rho(z), \quad (3.5)$$

coincides with the r.h.s. of (3.3). By means of further integrations eq. (3.4) can be extended to the left for $\nu > -1$. Then, each of the integrations by part shifts to the left on 1 domain of analyticity in ν ; finally, one arrives at the proposition:

$f(z; \{\rho, 1\}_\nu)$ can be extended into strip $\nu > -K - 1$ at any integer $K \geq 0$ in the form

$$f(z; \{\rho, 1\}_\nu) = \sum_{m=1}^{K+1} \frac{\bar{z}^{\nu+m-1}}{\Gamma(\nu+m)} (-)^{m-1} \varphi_\rho^{(m-1)}(1) - (-)^K \int_z^1 \frac{(t-z)^{\nu+K}}{\Gamma(\nu+K)} \varphi_\rho^{(K+1)}(t) dt, \quad (3.6a)$$

$$\mathcal{P}_\nu(y) = P(y) \cdot y^{\rho+\nu} \quad \text{at } \rho + \nu \geq 0; \quad \varphi_\rho^{(m)}(x) \equiv \frac{d^m}{dx^m} \varphi_\rho(x). \quad (3.6b)$$

From eq. (3.6a) follows the expression for $f(z; \{\rho, 1\}_{-K})$ at integer negative $\nu = -K$ (within the strip of extension $\nu > -K - 1$). Put $\nu = -K + \varepsilon$ and taking the limit $\varepsilon \rightarrow 0$ in the r.h.s. of (3.6a) one arrives at

$$\lim_{\varepsilon \rightarrow 0} f(z; \{\rho, 1\}_{-K+\varepsilon}) = f(z; \{\rho, 1\}_{-K}) = (-)^K \varphi_\rho^{(K)}(z). \quad (3.7)$$

A few partial results for $f(x; \{\rho, 1\}_\nu)$ are shown in table 1.

An interesting case of solution provides the condition $\rho = \nu$ (*number of derivatives*). For any $\nu \geq 0$, $f(x; \{\nu, 1\}_{-\nu})$ from equation (3.6) evolves following to the DGLAP equation (2.1) with the same initial kernel $\mathcal{P}_\nu = P$ (see (3.6b)) that has no ν dependence. The solutions $\varphi_1^{(k)}(x) = (xf(x))^{(k)}$, were considered in [16, 17].

We conclude that the partial solution joints different results, previously presented in the literature [10–12, 16, 17], in a unit frame: ones are related with the “truncated integration” of $f(x)$ in [10–12], while the other is related with differentiated $f(x)$ in [16, 17]. The applicable domain of $f(x; \{\rho, 1\}_\nu)$ in ν can be extended to the real axis, following eq. (3.6a).

$\nu =$	2	1	0	-1	-2
$f(x; \{\rho, 1\}_\nu) =$	$\int_x^1 dz \int_z^1 \varphi_\rho(t) dt$	$\int_x^1 \varphi_\rho(t) dt$	$\varphi_\rho(x)$	$-\varphi'_\rho(x)$	$\varphi''_\rho(x)$
$\mathcal{P}_\nu =$	$P(y)y^{\rho+2}$	$P(y)y^{\rho+1}$	$P(y)y^\rho$	$P(y)y^{\rho-1}$	$P(y)y^{\rho-2}$

Table 1. Partial results for CMM $f(x; \{\rho, 1\}_\nu)$ (see description in section 3.2).

4 Generalized CMM solution of the DGLAP equation at $n_i = \alpha = 0$

Here we construct a gCMM that would be appropriate for analysis of the DIS sum rules. To this end, it is convenient to take $f(z; \{n\}_k)$ with a single DGLAP kernel \mathcal{P} that is independent of k . Following the condition eq. (2.10b), let us consider the case of eq. (2.10) with $n_1 = n$, and all $n_k = \alpha = 0$, $k > 1$, then $\lim_{\alpha \rightarrow 0} \left[\frac{t^\alpha - z^\alpha}{\alpha} \right]^{k-1} = \ln^{(k-1)}(t/z)$,

$$z^n f(z) \rightarrow f(z; \{n, 0\}_k) \equiv f(z; \underbrace{n, 0, \dots, 0}_k) = \int_z^1 \frac{\ln^{(k-1)}(t/z)}{\Gamma(k)} t^\rho f(t) \frac{dt}{t}, \quad (4.1a)$$

$$\text{with the kernel } \mathcal{P}(y) = P(y)y^n, \quad (4.1b)$$

This y^n -factor in the kernel appears due to the weight z^n in $\varphi_n(x) = x^n f(x)$.

Moreover, the gCMM $f(z; \{1, 0\}_k)$ holds the same value of the normalization $\langle f \rangle = \int_0^1 f(x) dx$ as one has for the parton density $f(x)$. Indeed, gCMM can be represented as

$$f(z; \{n, 0\}_k) = \int_z^1 z_k^{-1} dz_k \int_{z_k}^1 z_{k-1}^{-1} dz_{k-1} \dots \int_{z_2}^1 \varphi_n z_1^{-1} dz_1 \quad (4.2)$$

$$\equiv (1 * 1 * \dots * \varphi_n)(z), \quad (4.3)$$

therefore, $\langle f(z; \{n, 0\}_k) \rangle = \prod_{i=1}^k 1 \cdot \langle \varphi_n \rangle |_{n=1} = \langle f \rangle$.

Following the previous proposition in eq. (3.2), we extend the solution (4.1a) to any real $\nu > 0$ (and ρ)

$$f(z; \{\rho, 0\}_\nu) \stackrel{\text{def}}{=} \int_z^1 \frac{\ln^{(\nu-1)}(t/z)}{\Gamma(\nu)} \varphi_\rho(t) \frac{dt}{t}, \quad (4.4a)$$

$$\text{with DGLAP kernel } \mathcal{P}(y) = P(y) \cdot y^\rho. \quad (4.4b)$$

The contribution to $f(z; \{\rho, 0\}_\nu)$ is reinforced at the right end $t = 1$ in eq. (4.4a) by powers of logs. This reinforcement becomes especially useful for the case when the experimental data are better known at larger x and, in contrast, ones are unreliable or worse known at lower x .

An important property of gCMM (4.4a) is the independence of the corresponding kernel (4.4b) of the parameter ν . Therefore, collecting these solutions $f(z; \{\rho, 0\}_\nu)$ with the different weights we can obtain the new solution at the same kernel. In other words, the integrands $\ln^{(\nu-1)}(t/z)/\Gamma(\nu)$ at different ν can be considered as ‘‘bricks’’ for any new gCMM constructions that evolve following the same DGLAP equation.

The next step of extension can be done like one in eq. (3.6) in the previous section: $f(z; \{\rho, 0\}_\nu)$ can be extended into strip $\nu > -K - 1$ at any integer K in the form

$$f(z; \{\rho, 0\}_\nu) = \sum_{m=1}^{K+1} \frac{l^{\nu+m-1}}{\Gamma(\nu+m)} (-)^{(m-1)} \varphi_\rho^{(m-1)}[l] - (-)^K \int_0^l \frac{y^{\nu+K}}{\Gamma(\nu+K+1)} \varphi_\rho^{(K+1)}[y] dy, \quad (4.5a)$$

$$\mathcal{P}(t) = P(t) \cdot t^\rho \text{ at } \rho \geq 0, \quad (4.5b)$$

where $l = \ln(1/z)$ and $y = \ln(t/z)$ is a new appropriate variable. It is convenient to invent the new notation for φ_ρ , $\varphi_\rho[y] \equiv \varphi_\rho(ze^y)$ that depends on the variable y . It is instructive to obtain from (4.5a) a partial case at $\nu \rightarrow -1$ (for the strip at $K = 1$)

$$\lim_{\nu \rightarrow -1} f(z; \{\rho, 0\}_\nu) = f(z; \{\rho, 0\}_{-1}) = -z \frac{d}{dz} \varphi_\rho(z).$$

In general, for any integer $n > 0 : -n > -K - 1$ one can obtain

$$f(z; \{\rho, 0\}_{-n}) = \left(-z \frac{d}{dz}\right)^n \varphi_\rho(z), \tag{4.6}$$

the result is similar to one in eq. (3.7) and also to that one obtained in [16, 17]. While for any fractional $\nu > 0 : -\nu > -K - 1$, $f(x; \{\rho, 0\}_{-\nu})$ from equation (4.5a) can be considered as generalized derivatives $(-x \frac{d}{dx})^\nu \varphi_\rho(x)$.

5 Conclusion

We propose a generalization of the standard parton density $f(x)$ by means of the extended cut Mellin moments. These CMM are appropriate for direct study of real observables in DIS at the natural kinematic constraints of experiments. We found that functions obtained by multiple integration of the original parton density $f(x, \mu^2)$ as well as ones obtained by multiple differentiations of this $f(x, \mu^2)$ satisfy the same corresponding DGLAP equations. Moreover, this gives new classes of parton generalized CMM for DIS descriptions where both the previous cases are considered within a unified frame. Some interesting partial cases can be especially useful when the experimental data are better known at large x and are less known at low x . We showed also similar relations for the structure functions $F(x)$, being the Mellin convolution $F = C * f$, where C is the coefficient function of the process. In other words, the presented here CMM approach provides novel gCMM functions, satisfying the same DGLAP equations, which reinforce the experimentally available x -region. This seems to be of great importance in QCD analysis of a wide class of high-energy processes.

Our goal here was to present an extended theoretical frame for gCMM. We only mentioned possible applications of these gCMM while the specific analysis of the experimental data is reserved for forthcoming papers.

Acknowledgments

We are grateful to O. Teryaev for the fruitful discussions of the considered subject which inspired this work. DK would like to thank BLTP community for their warm hospitality and cozy atmosphere. This work is supported by the Bogoliubov-Infeld Program, Grant No 01-3-1113-2014/2018. SVM acknowledges support from the Russian Foundation for Fundamental Research (Grant No. 14-01-00647a).

A Lemma about Mellin convolution

Let us consider the r.h.s. of the DGLAP equation $\phi(z)$

$$\mu^2 \frac{d}{d\mu^2} f(z, \mu^2) = \phi(z) \equiv (P * f)(z), \quad (\text{A.1})$$

and define the map for any function ϕ , $\phi \rightarrow \phi_1$ by means of the Mellin convolution

$$\phi_1(z_1) = \int_{z_1}^1 \omega(z) \phi(z) dz = \left(\omega \left(\frac{z_1}{t} \right) \frac{z_1}{t} * \phi \right) (z_1). \quad (\text{A.2})$$

For the partial case of the monomial smearing function $\omega(z) = z^{n-1}$ one can write

$$\phi_1(z_1) = \int_{z_1}^1 z^{n-1} \phi(z) dz = \left(\frac{z_1^n}{t^n} * \phi \right) (z_1), \quad (\text{A.3a})$$

and for k -times recursion, used in (2.5),

$$\phi_{k+1}(z_{k+1}) = \int_{z_{k+1}}^1 z^{n_k-1} \phi_k(z) dz. \quad (\text{A.3b})$$

What is the r.h.s. of eq. (A.3a)? Substituting (A.1) in the r.h.s. of (A.3a) and using the commutative and associative properties of Mellin convolution one has

$$\begin{aligned} \phi_1(z_1) &= \left(\frac{z_1^n}{t^n} * (P * f) \right) (z_1) = \left((P * f) * \frac{z_1^n}{t^n} \right) (z_1) = \left(P * \left(f * \frac{z_1^n}{t^n} \right) \right) (z_1) \\ &= \left(z_1^n P * \frac{1}{z^n} \left(f * \frac{z^n}{t^n} \right) (z) \right) (z_1) = \left(z_1^n P * \frac{1}{z^n} f_1(z; n) \right) (z_1) \end{aligned} \quad (\text{A.4})$$

The variable z , explicitly invented in the intrinsic convolution in the l.h.s. of the last equation, is the argument of this intrinsic convolution, i.e., $(f * 1/t^n)(z)$. In the r.h.s. of (A.4) there appears CMM $f_1(z_1; n) = \int_{z_1}^1 z^{n-1} f(z) dz$ following eq. (A.1). As the last step one can obtain the transformed kernel P_1 in the convolution

$$\phi_1(z_1) = \left(z_1^n P * \frac{1}{z^n} f_1(z; n) \right) (z_1) = (P_{1(n)} * f_1)(z_1), \quad (\text{A.5a})$$

$$f_1(z_1, n) = \int_{z_1}^1 z^{n-1} f(z) dz; \quad P_{1(n)}(t) = P(t)t^n, \quad (\text{A.5b})$$

where the last equation for P_1 followed from the definition of the Mellin convolution in (2.1).

If P is in turn $P = A * B$, then

$$P_1(t) = (A_1 * B_1)(t), \text{ where } A_1(\alpha) = \alpha^n A(\alpha), \quad B_1(\beta) = \beta^n B(\beta). \quad (\text{A.6})$$

Let us emphasize here that the chain of conclusions from Eq. (A.1), Eq. (A.3a) to Eq. (A.5) is possible only for monomial weight $\omega(z) = z^{n-1}$ in the definition of transform in Eq. (A.2). For another form of the weight ω we do not obtain a covariant form of the r.h.s. of eq. (A.1). Really, substituting the linear combination of the monomials in $w = az^{n-1} + bz^{m-1}$ in eq. (A.2) one can obtain $\varphi_1(z_1)$ for the l.h.s. of eq. (A.1), while for the r.h.s. one will arrive at the decomposition

$$a (P_{1(n)} * f_1(z; n))(z_1) + b (P_{1(m)} * f_1(z; m))(z_1) \quad (\text{A.7})$$

that can not be presented as a single convolution with φ_1 .

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] V.N. Gribov and L.N. Lipatov, *Deep inelastic ep scattering in perturbation theory*, *Sov. J. Nucl. Phys.* **15** (1972) 438 [[INSPIRE](#)].
- [2] V.N. Gribov and L.N. Lipatov, *e^+e^- pair annihilation and deep inelastic ep scattering in perturbation theory*, *Sov. J. Nucl. Phys.* **15** (1972) 675 [[INSPIRE](#)].
- [3] Y.L. Dokshitzer, *Calculation of the structure functions for deep inelastic scattering and e^+e^- annihilation by perturbation theory in quantum chromodynamics*, *Sov. Phys. JETP* **46** (1977) 641 [[INSPIRE](#)].
- [4] G. Altarelli and G. Parisi, *Asymptotic freedom in parton language*, *Nucl. Phys.* **B 126** (1977) 298 [[INSPIRE](#)].
- [5] A. Deur et al., *Experimental study of isovector spin sum rules*, *Phys. Rev.* **D 78** (2008) 032001 [[arXiv:0802.3198](#)] [[INSPIRE](#)].
- [6] S. Forte and L. Magnea, *Truncated moments of parton distributions*, *Phys. Lett.* **B 448** (1999) 295 [[hep-ph/9812479](#)] [[INSPIRE](#)].
- [7] S. Forte, L. Magnea, A. Piccione and G. Ridolfi, *Evolution of truncated moments of singlet parton distributions*, *Nucl. Phys.* **B 594** (2001) 46 [[hep-ph/0006273](#)] [[INSPIRE](#)].
- [8] A. Piccione, *Solving the Altarelli-Parisi equations with truncated moments*, *Phys. Lett.* **B 518** (2001) 207 [[hep-ph/0107108](#)] [[INSPIRE](#)].
- [9] S. Forte, J.I. Latorre, L. Magnea and A. Piccione, *Determination of α_s from scaling violations of truncated moments of structure functions*, *Nucl. Phys.* **B 643** (2002) 477 [[hep-ph/0205286](#)] [[INSPIRE](#)].
- [10] D. Kotlorz and A. Kotlorz, *Evolution equations for truncated moments of the parton distributions*, *Phys. Lett.* **B 644** (2007) 284 [[hep-ph/0610282](#)] [[INSPIRE](#)].
- [11] D. Kotlorz and A. Kotlorz, *Evolution equations of the truncated moments of the parton densities. A possible application*, *Acta Phys. Polon.* **B 40** (2009) 1661 [[arXiv:0906.0879](#)] [[INSPIRE](#)].
- [12] D. Kotlorz and A. Kotlorz, *Truncated Mellin moments: useful relations and implications for the spin structure function g_2* , *Acta Phys. Polon.* **B 42** (2011) 1231 [[arXiv:1106.3753](#)] [[INSPIRE](#)].
- [13] D. Kotlorz and A. Kotlorz, *Evolution of the truncated Mellin moments of the parton distributions in QCD analysis*, *Phys. Part. Nucl.* **45** (2014) 357 [[arXiv:1405.5315](#)] [[INSPIRE](#)].
- [14] A. Psaker, W. Melnitchouk, M.E. Christy and C. Keppel, *Quark-hadron duality and truncated moments of nucleon structure functions*, *Phys. Rev.* **C 78** (2008) 025206 [[arXiv:0803.2055](#)] [[INSPIRE](#)].
- [15] E.G. Floratos, R. Lacaze and C. Kounnas, *Space and timelike cut vertices in QCD beyond the leading order. 1. The singlet sector*, *Phys. Lett.* **B 98** (1981) 285 [[INSPIRE](#)].

- [16] O. Teryaev, *QCD evolution and density matrix positivity*, in the proceedings of the *XXX PNPI International Winter School*, February 26–March 5, Repino, Russia (2005).
- [17] X. Artru, M. Elchikh, J.-M. Richard, J. Soffer and O.V. Teryaev, *Spin observables and spin structure functions: inequalities and dynamics*, *Phys. Rept.* **470** (2009) 1 [[arXiv:0802.0164](#)] [[INSPIRE](#)].