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On Yangian and long representations of the centrally extended $\mathfrak{su}(2|2)$ superalgebra

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ABSTRACT: The centrally extended $\mathfrak{su}(2|2)$ superalgebra is an asymptotic symmetry of the light-cone string sigma model on $\text{AdS}_5 \times S^5$. We consider an evaluation representation of the conventional Yangian built over a particular 16-dimensional long representation of the centrally extended $\mathfrak{su}(2|2)$. Interestingly, we find that S-matrices compatible with this evaluation representation do not exist. On the other hand, by requiring centrally extended $\mathfrak{su}(2|2)$ invariance and explicitly solving the Yang-Baxter equation, we find a scattering matrix for long-short representations of the Lie superalgebra. We notice that this S-matrix is invariant under a different representation of non-evaluation type, induced from the tensor product of short representations. Our findings concern the conventional Yangian only, and are not applied to possible algebraic extensions of the latter.

KEYWORDS: AdS-CFT Correspondence, Exact S-Matrix, Extended Supersymmetry, Quantum Groups

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1 Introduction

An important progress towards the solution of the finite-size spectral problem of the planar AdS/CFT system has been recently made. On the one hand, a generalized Lüscher approach for treating the leading wrapping effects was developed [1–13] and successfully confronted against direct field-theoretic computations [14, 15]. On the other hand, the Thermodynamic Bethe Ansatz based on the mirror theory [16] was advanced as a tool to capture the exact string spectrum in both the 't Hooft coupling constant λ and the size L of the system [17–30]. In many respects the success of this research is based on the existence of an asymptotic symmetry, which consists of (two copies of) the $\mathfrak{su}(2|2)$ superalgebra [31–33] centrally extended by two central charges. In particular, this superalgebra and the associated Yangian [34] have been used to explicitly determine the S-matrices describing scattering of fundamental and bound-state particles of the light-cone string sigma model [35, 36], which is important for setting up the Thermodynamic Bethe Ansatz approach.

Albeit nice, some of the recent developments were based on certain assumptions and clever guesses, which provides us with further motivation to better understand the representation theory of the centrally extended $\mathfrak{su}(2|2)$ superalgebra, as well as its implications. So far mainly short (atypical) representations have been the subject of interest in the context

of the string sigma model. This is because these representations correspond to bound states of fundamental particles [37] and, together with the latter, they constitute the asymptotic spectrum of the sigma model. On the other hand, long (typical) representations naturally enter in the construction of the large L asymptotic solution of the TBA equations via the so-called Y-functions [21]. It is therefore interesting to look for at the scattering theory involving long representations. Another independent aspect where long representations should play a role concerns the issue of the universal R-matrix. If such a quantity exists as an abstract element in $\mathcal{A} \otimes \mathcal{A}$, where $\mathcal{A}$ is a Hopf algebra, then it can be evaluated in any two representation of $\mathcal{A}$. In the case of (the Yangian of) centrally-extended $\mathfrak{su}(2|2)$, these must include the long ones. Therefore, there should exist a concrete (matrix) realization for an intertwiner of symmetry generators in the tensor product of the corresponding (long) representations (the S-matrix).

In this paper, we will make a first step towards understanding the scattering problem involving long representations of the centrally extended $\mathfrak{su}(2|2)$. We start with building such long representations by applying an outer $\mathfrak{sl}(2)$ automorphism to the representations of the unextended $\mathfrak{su}(2|2)$ superalgebra [38]. These representations, in turn, can be obtained from those constructed for $\mathfrak{gl}(2|2)$ by Gould and Zhang [39], see also [40, 41]. They are parameterized by a continuous parameter $q \in \mathbb{C}$, which is the value of the unique central charge (the Hamiltonian) in a given representation. An outer $\mathfrak{sl}(2)$ automorphism acting on $\mathfrak{su}(2|2)$ can be used to generate two extra central charges, which depend on additional parameters P and g . Here P is identified with the (generically complex) particle momentum, while g with the coupling constant. We will be interested in the lowest (16-dimensional) long representation, for which we will obtain an explicit realization in terms of 16×16 matrices depending on q, P and g . As usual, special values of q correspond to the shortening conditions. In particular, $q = 1$ corresponds to an indecomposable formed out of two short 8-dimensional representations.

Given an explicit realization of the long 16-dimensional representation, we construct the corresponding evaluation representation for the Yangian introduced by Beisert [34]. The defining relations are given in appendices B and C. We will refer to this Yangian, exclusively built upon the centrally extended $\mathfrak{su}(2|2)$ superalgebra, as the *conventional* Yangian. Whenever the term ‘Yangian’ will be used throughout the paper, it will always be understood as conventional, even if we will not mention it explicitly.

We will then try to find an S-matrix which scatters the long evaluation with the long evaluation or the long evaluation with a short four-dimensional representation. Namely, we try to find an S-matrix which acts as the following intertwiner:

$$\Delta^{\text{op}}(\mathbb{J})\mathbb{S} = \mathbb{S}\Delta(\mathbb{J}), \quad \Delta^{\text{op}}(\hat{\mathbb{J}})\mathbb{S} = \mathbb{S}\Delta(\hat{\mathbb{J}}),$$

where $\mathbb{J}$ is a generator of centrally extended $\mathfrak{su}(2|2)$, $\hat{\mathbb{J}}$ is the corresponding Yangian generator in the evaluation representation, and Δ and Δ^{op} are the coproduct and its opposite (see section 2 for the precise definitions). We recall that this construction proved to work well for the fundamental or bound-state, i.e. short, representations, and it lead to the complete determination of the corresponding bound state scattering matrix [35, 42]. However, where

one of the representations involved is long evaluation, we find that the S-matrix satisfying the invariance conditions above *does not exist*.

The origin of this problem can be clearly seen in Drinfeld's second realization [43],¹ where it can be traced back to non-co-commutativity of the higher Yangian central charges $\mathbb{C}_n$ with $n \geq 2$. Co-commutativity means that the coproduct map coincides with its opposite, namely $\Delta = \Delta^{\text{op}}$. When a coproduct will be called co-commutative without further specification, it means it is such when it acts on all the generators of the algebra in question. We will often use the terminology co-commutativity of the generator X as a shortcut for co-commutativity of the coproduct when acting on generator X , namely $\Delta(X) = \Delta^{\text{op}}(X)$. The term quasi-co-commutative referred to a coproduct means that the coproduct map is equal to its opposite up to conjugation with an invertible element $\mathbb{S}$, namely $\Delta^{\text{op}} \mathbb{S} = \mathbb{S} \Delta$. This fact can happen in some representations, and not in others. If it happens in all representations, and the element $\mathbb{S}$ can be determined in a representation-independent way as an abstract object, we will call it the universal R-matrix. If only some representations allow the coproduct to be related to its opposite by conjugation with a invertible matrix $\mathbb{S}$, we will say that those representations admit an S-matrix, but there is no universal R-matrix.

Since the coproducts of the Yangian central charges only involve central elements (see formula (B.3)), co-commutativity of the central charges in a specific representation is a necessary condition for the existence of an S-matrix in that representation. In fact, one has

$$\Delta^{\text{op}}(\mathbb{C}_n) \mathbb{S} = \mathbb{S} \Delta^{\text{op}}(\mathbb{C}_n) = \mathbb{S} \Delta(\mathbb{C}_n) \quad (1.1)$$

from which invertibility of $\mathbb{S}$ implies $\Delta^{\text{op}}(\mathbb{C}_n) = \Delta(\mathbb{C}_n)$. Finding at least one representation of the Yangian where this necessary condition is not satisfied² implies that the corresponding universal R-matrix does not exist.

Even if the Yangian evaluation representation does not admit an S-matrix, one can still look for centrally extended $\mathfrak{su}(2|2)$ -invariant solutions of the Yang-Baxter equation. Indeed, we find two such solutions. It is important to understand if there is still some extended symmetry they correspond to. To answer this question, we recall that, generically, a product of two short representations gives an *irreducible* long representation [38]. In particular,

$$V_{4d}(p_1) \otimes V_{4d}(p_2) \approx V_{16d}(P, q).$$

Here, $V_{4d}(p)$ is a fundamental 4-dimensional representation which depends on the particle momentum and the coupling constant. Analogously, $V_{16d}(P, q)$ is a long 16-dimensional representation described by the momentum P , the coupling constant g and the parameter q . We find an explicit relation between the pairs (p_1, p_2) and (P, q) at fixed g , in

¹Given the existence of an invertible map between the generators of Drinfeld's second [43] and first [34] realization of the Yangian, we will use either realizations according to the needs considering them as completely equivalent. A rigorous proof of this equivalence has however never been derived.

²For instance, if one takes the coproduct of $\mathbb{C}_2$ in a *long* $\otimes$ *long* representation, and subtracts from it its opposite, one obtains an expression which expands semiclassically as

$$\Delta(\mathbb{C}_2) - \Delta^{\text{op}}(\mathbb{C}_2) = \frac{x_1 x_2 q_1 q_2 (q_2^2 - q_1^2)}{2g(x_1^2 - 1)(x_2^2 - 1)} + \mathcal{O}(g^{-2}),$$

with x a classical rapidity.

particular $P = p_1 + p_2$. Obviously, for a given p_1 and p_2 there is a unique long representation. However, any long representation can be written as a tensor product of two short representations in two different ways.

The observed relationship between long and short representations suggests that the S-matrix $\mathbb{S}_{\text{LS}}$, which scatters a long representation with a short one, can simply be composed as a product of two S-matrices $\mathbb{S}_{13}$ and $\mathbb{S}_{23}$ describing the scattering of the corresponding short representations, i.e.

$$\mathbb{S}_{\text{LS}}(P, q; p_3) = \mathbb{S}_{13}(p_1, p_3) \mathbb{S}_{23}(p_2, p_3).$$

In this formula, the tensor product of two short representations in the spaces 1 and 2 with momenta p_1 and p_2 gives a long representation (P, q) , which scatters with a short representation in the third space with momentum p_3 . We then verify that the two S-matrices we found by solving the Yang-Baxter equations indeed coincide with the product of two “short” S-matrices. The fact that we find two matrices is explained by the double-covering relationship between (p_1, p_2) and (P, q) . This finding also shows that the Yangian symmetry can be induced on long representation from the one defined on the short ones, and this tensor product representation automatically admits an S-matrix. It naturally does that in both branches of the double-covering. Importantly, this (double-branched) tensor product representation of the Yangian is *not* isomorphic to the long evaluation representation we were discussing before, even though the two short representations composing it are the short evaluation representations of the Yangian described in [34, 43]. This is also clear from the fact that the long evaluation representation discussed before *does not* admit an S-matrix.

The existence of *two* solutions for $\mathbb{S}_{\text{LS}}$, corresponding to two Yangian representations induced from short representations, is an unexpected feature which we do not have a good explanation for. Both S-matrices come with the canonical normalization and, therefore, they cannot be related to each other by any multiplicative factor (an extra dressing phase). They are not related by a similarity transformation either. We note, however, that at the special value $q = 1$ where the long multiplet becomes reducible, the two matrices $\mathbb{S}_{\text{LS}}$ become of the form (the block structure refers to the split into the 8-dimensional sub- and factor representations one finds at $q = 1$, as we will discuss in the paper)

$$\begin{pmatrix} \alpha A & B + \alpha C \\ 0 & D \end{pmatrix}, \quad (1.2)$$

where only the scalar coefficient α is different for the two solutions. Here, D corresponds to the factor representation (symmetric), and coincides with (the inverse of) the known symmetric bound-state S-matrix $\mathbb{S}_{AB}$ [44]. This is in agreement with the fact that there is a unique bound-state S-matrix.

The paper is organized as follows. In the next section, starting from a construction of long representations, we discuss the Yangian and prove the non-existence of a universal R-matrix. This no-go theorem applies to the conventional Yangian only, and it may

not hold when considering algebraic extensions of the latter.³ In section 3 we find the “long-short” S-matrix by solving the corresponding Yang-Baxter equation. In section 4 we present an alternative construction of long representations and the associated S-matrix *via* tensor product of short ones. In appendices A–C we provide several computational details. Finally, in appendix D we discuss some aspects of the Hirota equations related to the long representations we construct in the paper. Most of the corresponding discussion should be known to experts, and we include it only for completeness.

2 Long representations and Yangian

We start with discussing the representation theory of $\mathfrak{sl}(2|2)$ and its generalization to the centrally extended case. We provide a matrix realization of the simplest 16-dimensional long multiplet. Then we discuss the evaluation representation of the corresponding Yangian algebra based on this long multiplet and show the absence of a universal R-matrix.

2.1 Constructing long representations

The paper [39] explicitly constructs all finite-dimensional irreducible representations of $\mathfrak{gl}(2|2)$ in an oscillator basis. Generators of $\mathfrak{gl}(2|2)$ are denoted by E_{ij} , with commutation relations

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - (-)^{(d[i]+d[j])(d[k]+d[l])} \delta_{il} E_{kj}. \quad (2.1)$$

Indices i, j, k, l run from 1 to 4, and the fermionic grading is assigned as $d[1] = d[2] = 0$, $d[3] = d[4] = 1$. The quadratic Casimir of this algebra is $C_2 = \sum_{i,j=1}^4 (-)^{d[j]} E_{ij} E_{ji}$. Finite dimensional irreps are labelled by two half-integers $j_1, j_2 = 0, \frac{1}{2}, \dots$, and two complex numbers q and y . These numbers correspond to the values taken by appropriate generators on the highest weight state $|\omega\rangle$ of the representation, defined by the following conditions:

$$\begin{aligned} H_1 |\omega\rangle &= (E_{11} - E_{22}) |\omega\rangle = 2j_1 |\omega\rangle, & H_2 |\omega\rangle &= (E_{33} - E_{44}) |\omega\rangle = 2j_2 |\omega\rangle, \\ I |\omega\rangle &= \sum_{i=1}^4 E_{ii} |\omega\rangle = 2q |\omega\rangle, & N |\omega\rangle &= \sum_{i=1}^4 (-)^{[i]} E_{ii} |\omega\rangle = 2y |\omega\rangle, & E_{i < j} |\omega\rangle &= 0. \end{aligned} \quad (2.2)$$

The generator N never appears on the right hand side of the commutation relations, therefore it is defined up to the addition of a central element βI , with β a constant.⁴ This also means that we can consistently mod out the generator N , and obtain $\mathfrak{sl}(2|2)$ as a subalgebra of the original $\mathfrak{gl}(2|2)$ algebra.⁵ In order to construct representations of the

³In fact, certain extensions may result in further relations one has to impose on the generators. These extra relations may not be satisfied by the evaluation representation, therefore ruling it out from the list of irreps. We thank Niklas Beisert for a discussion about this point.

⁴We decided to drop the term βI since it will not affect our discussion.

⁵Further modding out of the center I produces the simple Lie superalgebra $\mathfrak{psl}(2|2)$. Its representations can be understood as that of $\mathfrak{sl}(2|2)$ for which $q = 0$. Correspondingly, $(2|2)$ has long irreps of dimension $16(2j_1 + 1)(2j_2 + 1)$ with $j_1 \neq j_2$ and short irreps with $j_1 = j_2 = j$ of dimension $16j(j + 1) + 2$. For a discussion of the tensor product decomposition of $\mathfrak{psl}(2|2)$, see [45].

centrally-extended $\mathfrak{su}(2|2)$ Lie superalgebra,⁶ we then first mod out N , and subsequently perform an $\mathfrak{sl}(2)$ rotation by means of the outer automorphism of $\mathfrak{su}(2|2)$ [38].

As usual for superalgebras, irreps are divided into typical (long), which have generic values of the labels j_1, j_2, q , and atypical (short), for which special relations are satisfied by these labels. Short representations occur here for $\pm q = j_1 - j_2$ and $\pm q = j_1 + j_2 + 1$. When these relations are satisfied, the dimension of the representation is smaller than what it would generically be for the same values of j_1, j_2 , but q arbitrary (that is, the dimension is in these cases smaller than $16(2j_1 + 1)(2j_2 + 1)$). One notices also that, when starting from a long irrep and reaching these special values by continuous variation of the parameter q , one generically ends up into a reducible but indecomposable representation.

We can identify the values of the labels which will produce the representations we are particularly interested in in this paper. First of all, the fundamental 4-dimensional short representation [31] corresponds to $j_1 = \frac{1}{2}, j_2 = 0$ (or, equivalently, $j_1 = 0, j_2 = \frac{1}{2}$) and $q = \frac{1}{2}$ ($q = -\frac{1}{2}$). More generally, the bound state (symmetric short) representations [37, 38, 44, 46–48] are given by $j_2 = 0, q = j_1$, with $j_1 = \frac{1}{2}, 1, \dots$ and bound state number $M \equiv s = 2j_1$. In addition, there are the antisymmetric short representations given by $j_1 = 0, q = 1 + j_2$, with $j_2 = 0, \frac{1}{2}, \dots$ and bound state number $M \equiv a = 2(j_2 + 1)$. Both symmetric and antisymmetric representations have dimension $4M$. We see that symmetric and antisymmetric representations are associated with the different shortening conditions $\pm q = j_1 - j_2$ and $\pm q = 1 + j_1 + j_2$.

Second, we consider the simplest long representation of dimension 16. In terms of the $\mathfrak{gl}(2|2)$ labels introduced above, this is the 16-dimensional long representation characterized by $j_1 = j_2 = 0$, and arbitrary q . It is instructive to see how it branches under the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ algebra. We denote as $[l_1, l_2]$ the subset of states which furnish a representation of $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ with angular momentum l_1 w.r.t the first $\mathfrak{su}(2)$, and l_2 w.r.t the second $\mathfrak{su}(2)$, respectively. The branching rule is

$$(2, 2) \rightarrow 2 \times [0, 0] \oplus 2 \times \left[\frac{1}{2}, \frac{1}{2} \right] \oplus [1, 0] \oplus [0, 1]. \quad (2.3)$$

One can straightforwardly verify that the total dimension adds up to 16, since $[l_1, l_2]$ has dimension $(2l_1 + 1) \times (2l_2 + 1)$.

For generic values of q , the corresponding long representations have no interpretation in terms of Young tableaux. However, when q is a certain integer, such an interpretation becomes possible. Consider rectangular Young tableaux, with one side made of 2 boxes, and the other side made of arbitrarily many boxes. These are long representations, denoted by $(2, s)$ and $(a, 2)$ according to the length (in boxes) of their sides. Together with the short irreps, denoted accordingly as $(1, s)$ (symmetric) and $(a, 1)$ (antisymmetric), they span all the admissible rectangular representations. In fact, every allowed representation has to have its associated Young tableaux fit into the so-called “fat hook” [49], which has branches of width equal to two boxes. All representations $(2, s)$ (respectively, $(a, 2)$) with

⁶The reality condition on the algebra will be imposed later, and will not affect the present discussion.

$s \geq 2$ (respectively, $a \geq 2$) have dimension⁷ 16, central charge $q = s$ and Dynkin labels $[0, q, 0]$. For both long and short representations that have an interpretation in terms of a rectangular Young tableaux, the charge q is simply given by the number of boxes in the tableaux multiplied by the charge $q = 1/2$ of the fundamental representation.

As a first step of our study, we have explicitly constructed the oscillator representation by using the formulas of [39], and derived from it the 16×16 matrix realization of the algebra generators. We have done this before acting with the outer automorphism, in such a way that the subsequent $\mathfrak{sl}(2)$ rotation provides an explicit matrix representation of centrally-extended $\mathfrak{su}(2|2)$. This explicit realization is reported in appendix A. Below we discuss some of the salient features of this realization.

The way the outer automorphism is implemented is by mapping the $\mathfrak{gl}(2|2)$ non-diagonal generators into new generators as follows:

$$\begin{aligned}\mathbb{L}_a^b &= E_{ab} \quad \forall a \neq b, & \mathbb{R}_\alpha^\beta &= E_{\alpha\beta} \quad \forall \alpha \neq \beta, \\ \mathbb{Q}_\alpha^a &= a E_{\alpha a} + b \epsilon_{\alpha\beta} \epsilon^{ab} E_{b\beta}, \\ \mathbb{G}_a^\alpha &= c \epsilon_{ab} \epsilon^{\alpha\beta} E_{\beta b} + d E_{a\alpha},\end{aligned}\tag{2.4}$$

subject to the constraint

$$ad - bc = 1.\tag{2.5}$$

Diagonal generators are automatically obtained by commuting positive and negative roots. In particular, from the explicit matrix realization one obtains the following values of the central charges:

$$\mathbb{H} = 2q(ad + bc) \mathbb{1}, \quad \mathbb{C} = 2q ab \mathbb{1}, \quad \mathbb{C}^\dagger = 2q cd \mathbb{1},\tag{2.6}$$

($\mathbb{1}$ is the 16-dimensional identity matrix), satisfying the condition

$$\frac{\mathbb{H}^2}{4} - \mathbb{C}\mathbb{C}^\dagger = q^2 \mathbb{1}.\tag{2.7}$$

When $q^2 = 1$, this becomes a shortening condition. In fact, for $q = 1$ the 16-dimensional representation becomes reducible but indecomposable. Its subrepresentation [45] is a short anti-symmetric 8-dimensional representation. Formula (2.7) above, however, tells us that we can conveniently think of q as a *generalized* bound state number, since for short representations $2q$ would be replaced by the bound state number M in the analogous formula for the central charges. This is particularly useful, since it allows us to parameterize the labels a, b, c, d in terms of the familiar bound state variables⁸ $x^\pm$, just replacing the bound state number M by $2q$. The explicit parameterization is given by

$$\begin{aligned}a &= \sqrt{\frac{g}{4q}} \eta, & b &= -\sqrt{\frac{g}{4q}} \frac{i}{\eta} \left(1 - \frac{x^+}{x^-}\right), \\ c &= -\sqrt{\frac{g}{4q}} \frac{\eta}{x^+}, & d &= \sqrt{\frac{g}{4q}} \frac{x^+}{i\eta} \left(1 - \frac{x^-}{x^+}\right),\end{aligned}\tag{2.8}$$

⁷Formulas for computing the dimension of representations of superalgebras from their Young tableaux can be found in [50].

⁸We use the conventions of [35].

where

$$\eta = e^{\frac{ip}{4}} \sqrt{i(x^- - x^+)} \quad (2.9)$$

and

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{4iq}{g}. \quad (2.10)$$

As in the case of short representations, there exist a uniformizing torus with variable z and periods depending on q [51]. The choice (2.9) for η is historically preferred in the string theory analysis [16, 35, 44, 52], and will actually ensure our S-matrix to be symmetric. Finally, we point out that positive and negative values of q correspond to positive and negative energy representations, respectively.

2.2 Hopf algebra and Yangian

Having in mind the derivation of an S-matrix in the above described long representation, we equip the symmetry algebra with the deformed Hopf-algebra coproduct⁸ [53, 54]

$$\begin{aligned} \Delta(\mathbb{J}) &= \mathbb{J} \otimes \mathbb{U}^{[[\mathbb{J}]]} + \mathbb{1} \otimes \mathbb{J}, \\ \Delta(\mathbb{U}) &= \mathbb{U} \otimes \mathbb{U}, \end{aligned} \quad (2.11)$$

where $\mathbb{J}$ is any generator of centrally-extended $\mathfrak{su}(2|2)$, $[[\mathbb{J}]] = 0$ for the bosonic $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ generators and for the energy generator $\mathbb{H}$, $[[\mathbb{J}]] = 1$ (resp., -1) for the $\mathbb{Q}$ (resp., $\mathbb{G}$) supercharges, and $[[\mathbb{J}]] = 2$ (resp., -2) for the central charge $\mathbb{C}$ (resp., $\mathbb{C}^\dagger$). The value of $\mathbb{U}$ is determined by the consistency requirement that the coproduct is co-commutative on the center, which is a necessary condition for the existence of an S-matrix $\mathbb{S}$ satisfying⁹

$$\Delta^{\text{op}}(\mathbb{J}) \mathbb{S} = \mathbb{S} \Delta(\mathbb{J}) \quad \forall \mathbb{J}. \quad (2.12)$$

This produces the algebraic condition

$$\mathbb{U}^2 = \kappa \mathbb{C} + \mathbb{1} \quad (2.13)$$

for some representation-independent constant κ . With our choice of parametrization (2.8), κ gets re-expressed *via* the coupling constant g as $\kappa = \frac{2}{ig}$, and we obtain the familiar relation

$$\mathbb{U} = \sqrt{\frac{x^+}{x^-}} \mathbb{1} = e^{i\frac{p}{2}} \mathbb{1}. \quad (2.14)$$

The advantage of the choice (2.8) is that the above relations are valid as they stand both for long and short representations. This will be particularly useful, since we plan to project the coproduct (2.11) into a long representation in the first space, and a short one in the second space. This is so because we will be primarily interested in the S-matrix scattering long representations against short ones.

⁹ $\mathbb{S}$ acts as $\mathbb{S} : V_1 \otimes V_2 \rightarrow V_1 \otimes V_2$ on representation modules.

In order to have a complete realization of the Hopf algebra, one needs to remember the antipode map $\mathcal{S}$ [44, 51, 54], and specify the charge conjugation matrix $\mathcal{C}$ that implements the antipode in the long representations.¹⁰ One has in particular

$$\mathcal{S}(\mathbb{J}) = -\mathbb{U}^{-[[\mathbb{J}]]} \mathbb{J} = \mathcal{C} \bar{\mathbb{J}}^{\text{st}} \mathcal{C}^{-1}, \quad (2.15)$$

where $\bar{\mathbb{J}}$ is the *antiparticle* representation associated to the representation we choose on the l.h.s. of (2.15). One finds that the charge conjugation matrix in the 16-dimensional long representation is given by

$$\mathcal{C} = \begin{pmatrix} \boxed{-1} & & & & & & \\ & \boxed{\begin{smallmatrix} & & i \\ i & -i & \end{smallmatrix}} & & & & \\ & & \boxed{\begin{smallmatrix} & -1 \\ -1 & 1 \end{smallmatrix}} & & & & \\ & & & \boxed{\begin{smallmatrix} & -1 \\ -1 & 1 \end{smallmatrix}} & & & \\ & & & & \boxed{\begin{smallmatrix} & & i \\ i & -i & \end{smallmatrix}} & & \\ & & & & & \boxed{-1} \end{pmatrix} \quad (2.16)$$

The blocks in (2.16) refer to the branching rule (2.3), with the ordering of states given in section 3.1. The antiparticle representation $\bar{\mathbb{J}}$ is still defined by sending $p \rightarrow -p$ (together with changing sign to the eigenvalue of the energy generator $\mathbb{H}$), which means

$$x^{\pm} \rightarrow \frac{1}{x^{\pm}}, \quad (2.17)$$

exactly as in the case of short representations. Once again, on the uniformizing torus (cf. comment to (2.8)), applying the particle to anti-particle transformation four times gives the identity, which corresponds to the $\mathbb{Z}_4$ graded Lie-algebra structure of centrally extended $\mathfrak{su}(2|2)$.

The next step is to study the Yangian in this representation. One can prove that the defining commutation relations of Drinfeld's first realization of the Yangian given in [34] (see appendix B) are satisfied (by the generators and their coproducts) if we assume the evaluation representation¹¹

$$\widehat{\mathbb{J}} = u \mathbb{J}, \quad (2.18)$$

where the spectral parameter u assumes the familiar form

$$u = \frac{g}{4i}(x^+ + x^-) \left(1 + \frac{1}{x^+ x^-} \right). \quad (2.19)$$

¹⁰The counit and all other bialgebra structures are straightforwardly implemented, and do not present any novel features.

¹¹We use the conventions of [35].

The Yangian coproducts are given by the same formulas used in [35], and the above value of u is determined by requiring co-commutativity of the Yangian central charges $\hat{\mathbb{C}}, \hat{\mathbb{C}}^\dagger$. We report the details in appendix B for convenience of the reader.

Drinfeld's second realization is also obtained by applying a similar (Drinfeld's) map as in [43].¹² This ensures the fulfilment of the Serre relations (see also [55]). All defining relations in [43] are satisfied (see appendix C), although the representation one obtains after Drinfeld map is not any longer of a simple evaluation-type, but more complicated. In fact, level- n simple roots $\mathbb{J}_n$ are not obtained from level-zero ones *via* multiplication by a (possibly shifted) spectral parameter to the power n . Nevertheless, the representation we obtain for Drinfeld's second realization of the Yangian is consistent, and the coproducts obtained after Drinfeld's map respect all commutation and Serre relations.¹³ We give details of this realization in appendix C.

However, surprisingly, it turns out that the Yangian in this representation, both for coproducts projected into *long* $\otimes$ *short* and for *long* $\otimes$ *long* representations, does *not* admit an S-matrix. This is easily seen by considering the Yangian central charges $\mathbb{C}_n, \mathbb{C}_n^\dagger$. While for $n = 0, 1$, their coproducts are co-commutative, this is not so for $n \geq 2$. Only for the special case $q^2 = 1$ the Yangian central charges appear to be co-commutative also for $n = 2$ and higher.¹⁴ Nevertheless, even for the special case $q^2 = 1$, the Yangian still does not seem to admit an S-matrix in this representation. One way to see it is by noticing that the equation

$$\Delta^{\text{op}}(\hat{\mathbb{J}}) \mathbb{S} = \mathbb{S} \Delta(\hat{\mathbb{J}}), \quad (2.20)$$

when applied to certain combinations of generators and on particular states (for instance, of highest weight w.r.t. to the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ splitting (2.3)), leads to a contradiction when the explicit matrix realization is used. This means that such an S-matrix does not exist for this representation of the Yangian, which also implies that a universal R-matrix for the Yangian [34] does not exist.

However, as we will discuss in section 4, a different Yangian representation, for which an S-matrix does indeed exist, can be induced on the space of long representations. This Yangian representation is obtained *via* the decomposition of long representations into short ones, and is therefore built upon the Yangian representations that have been already built on short representations. This induced representation is quite different from the one described above (cf. (2.18)), and, in particular, it is not related to (2.18) *via* any similarity transformation combined with redefinition of the spectral parameters.

A remark is in order. In principle, it should be possible to deduce non-co-commutativity of the higher central charges directly from the corresponding formulas for the coproducts written in terms of algebra generators, without referring to a specific representation. These formulas should also imply that the non-co-commutative part must disappear for representations which satisfy the shortening conditions. However, the ab-

¹²We have checked that the map we use in this paper (see appendix C) also works for the fundamental representation equally well, and is, in this sense, universal. This map might be related to the one used in [43] by redefinitions of the generators in the various realizations.

¹³Antipode and charge conjugation are also perfectly consistent with Drinfeld's second realization.

¹⁴For these special values of q we actually checked co-commutativity only up to $n = 4$.

	f_1	f_2	f_3	f_4	f_5	f_6	f_7	f_8	f_9	f_{10}	f_{11}	f_{12}	f_{13}	f_{14}	f_{15}	f_{16}
$\mathbb{L}_1^1$	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	1	0	-1	0	0	0	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0
$\mathbb{R}_3^3$	0	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0	0	0	1	0	-1	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	0

	e_1	e_2	e_3	e_4
$\mathbb{L}_1^1$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
$\mathbb{R}_3^3$	0	0	$\frac{1}{2}$	$-\frac{1}{2}$

Table 1. The $\mathfrak{su}(2)_{\mathbb{L}} \oplus \mathfrak{su}(2)_{\mathbb{R}}$ weights of the basis vectors for the long (f_i) and short (e_i) representation, respectively.

stract formulation of the coproducts is quite cumbersome, and we find it more illuminating to exhibit a concrete representation for which the higher central charges show non-commutativity, cf. footnote 2.

3 The long-short S-matrix

Let us start by deriving all S-matrices satisfying the relation (2.12) exclusively for the (level-zero) algebra generators. Such an S-matrix describes the scattering of an excitation in the long representation with momentum P and parameter q , against a fundamental particle with momentum p :

$$\mathbb{S} : V_{16d}(P, q) \otimes V_{4d}(p) \longrightarrow V_{16d}(P, q) \otimes V_{4d}(p). \quad (3.1)$$

This S-matrix should relate the Hopf algebra structure to the opposite Hopf algebra one. This means that $\mathbb{S}$ should satisfy (2.12).

3.1 Kinematic structure

It is useful to apply a procedure similar to the one performed in [35], this time without the explicit help of Yangian symmetry. We again begin by using the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ invariance (with trivial coproduct) to divide the space $V_{16d}(P, q) \otimes V_{4d}(p)$ into blocks with definite $\mathfrak{su}(2)_{\mathbb{L}} \oplus \mathfrak{su}(2)_{\mathbb{R}}$ weights $(a; b)$. Let us denote the basis-vectors of the long representation by f_i and the basis for the short representation by e_i , respectively. They correspond to 16-dimensional (resp., 4-dimensional) vectors with all zeroes, except in position i , where there is a 1. The $\mathfrak{su}(2)_{\mathbb{L}} \oplus \mathfrak{su}(2)_{\mathbb{R}}$ weights are explicitly given in table 1.

From this table it is now easy to read off the blocks corresponding to weights $(a; b)$. Explicitly, we find four one-dimensional blocks

$$V_{(\frac{3}{2}; 0)} = \{f_6 \otimes e_1\}, \quad V_{(-\frac{3}{2}; 0)} = \{f_8 \otimes e_2\}, \quad (3.2)$$

$$V_{(0; \frac{3}{2})} = \{f_9 \otimes e_3\}, \quad V_{(0; -\frac{3}{2})} = \{f_{11} \otimes e_4\}, \quad (3.3)$$

corresponding to vectors that have maximum/minimum weight. These subspaces are related in the following way: $(\Delta \mathbb{L}_2^1)^3 V_{(\frac{3}{2}; 0)} = V_{(-\frac{3}{2}; 0)}$ and $(\Delta \mathbb{R}_4^3)^3 V_{(0; \frac{3}{2})} = V_{(0; -\frac{3}{2})}$.

Figure 1. The relations between the different subspaces. The arrows with tildes denote isomorphic subspaces, which therefore have the same S-matrix block.

Next, we have eight three-dimensional blocks

$$V_{(1;\frac{1}{2})} = \{f_2 \otimes e_1, f_6 \otimes e_3, f_{12} \otimes e_1\}, \quad V_{(-1;\frac{1}{2})} = \{f_3 \otimes e_2, f_8 \otimes e_3, f_{13} \otimes e_2\}, \quad (3.4)$$

$$V_{(1;-\frac{1}{2})} = \{f_4 \otimes e_1, f_6 \otimes e_4, f_{14} \otimes e_1\}, \quad V_{(-1;-\frac{1}{2})} = \{f_5 \otimes e_2, f_8 \otimes e_4, f_{15} \otimes e_2\}, \quad (3.5)$$

and

$$V_{(\frac{1}{2};1)} = \{f_2 \otimes e_3, f_9 \otimes e_1, f_{12} \otimes e_3\}, \quad V_{(\frac{1}{2};-1)} = \{f_4 \otimes e_4, f_{11} \otimes e_1, f_{14} \otimes e_4\}, \quad (3.6)$$

$$V_{(-\frac{1}{2};1)} = \{f_3 \otimes e_3, f_9 \otimes e_2, f_{13} \otimes e_3\}, \quad V_{(-\frac{1}{2};-1)} = \{f_5 \otimes e_4, f_{11} \otimes e_2, f_{15} \otimes e_4\}, \quad (3.7)$$

Both sets of subspaces are again related *via* the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ generators as is indicated in figure 1.

Finally, there are four nine-dimensional blocks

$$\begin{aligned} V_{(\frac{1}{2};0)} &= \{f_1 \otimes e_1, f_2 \otimes e_4, f_4 \otimes e_3, f_6 \otimes e_2, f_7 \otimes e_1, f_{10} \otimes e_1, f_{12} \otimes e_4, f_{14} \otimes e_3, f_{16} \otimes e_1\}, \\ V_{(-\frac{1}{2};0)} &= \{f_1 \otimes e_2, f_3 \otimes e_4, f_5 \otimes e_3, f_7 \otimes e_2, f_8 \otimes e_1, f_{10} \otimes e_2, f_{13} \otimes e_4, f_{15} \otimes e_3, f_{16} \otimes e_2\}, \\ V_{(0;\frac{1}{2})} &= \{f_1 \otimes e_3, f_2 \otimes e_2, f_9 \otimes e_1, f_7 \otimes e_3, f_9 \otimes e_4, f_{10} \otimes e_3, f_{12} \otimes e_2, f_{13} \otimes e_1, f_{16} \otimes e_3\}, \\ V_{(0;-\frac{1}{2})} &= \{f_1 \otimes e_4, f_4 \otimes e_2, f_5 \otimes e_2, f_7 \otimes e_4, f_{10} \otimes e_4, f_{11} \otimes e_3, f_{14} \otimes e_2, f_{15} \otimes e_1, f_{16} \otimes e_4\}. \end{aligned} \quad (3.8)$$

Because of the relations between the different subspaces one only has to find the action of the S-matrix on the following subspaces

$$V_{(\frac{3}{2};0)}, \quad V_{(0;\frac{3}{2})}, \quad V_{(1;\frac{1}{2})}, \quad V_{(\frac{1}{2};1)}, \quad V_{(\frac{1}{2};0)}, \quad V_{(0;\frac{1}{2})}. \quad (3.9)$$

Let us conveniently denote the vectors in these spaces by

$$V_{(a;b)} = \{|a;b\rangle_i\}_{i=1, \dim V_{(a;b)}}. \quad (3.10)$$

The final step consists in introducing a (opposite) coproduct basis that allows for a quick derivation of the S-matrix. It turns out that we must use as building blocks both $|\frac{3}{2}; 0\rangle$ and $|0; \frac{3}{2}\rangle$. We find for the aforementioned three-dimensional subspaces

$$V_{(1; \frac{1}{2})} = \left\{ \Delta Q_3^1 |\frac{3}{2}; 0\rangle, \Delta G_2^4 |\frac{3}{2}; 0\rangle, \Delta Q_4^2 \Delta G_1^3 |0; \frac{3}{2}\rangle \right\}, \quad (3.11)$$

$$V_{(\frac{1}{2}; 1)} = \left\{ \Delta Q_4^2 |0; \frac{3}{2}\rangle, \Delta G_1^3 |0; \frac{3}{2}\rangle, \Delta Q_3^1 \Delta G_2^4 |\frac{3}{2}; 0\rangle \right\}. \quad (3.12)$$

For the relevant nine-dimensional spaces we find

$$V_{(\frac{1}{2}; 0)} = \left\{ \Delta R_4^3 |\frac{1}{2}; 1\rangle_i, \Delta Q_4^1 |1; \frac{1}{2}\rangle_i, \Delta G_2^3 |1; \frac{1}{2}\rangle_i \right\}, \quad (3.13)$$

$$V_{(0; \frac{1}{2})} = \left\{ \Delta L_2^1 |1; \frac{1}{2}\rangle_i, \Delta Q_4^1 |\frac{1}{2}; 1\rangle_i, \Delta G_2^3 |\frac{1}{2}; 1\rangle_i \right\}, \quad (3.14)$$

where $i = 1, 2, 3$.

3.2 S-matrix

From the coproduct basis it is easily seen that the S-matrix will be fixed upon specifying its action on $|\frac{3}{2}; 0\rangle$ and $|0; \frac{3}{2}\rangle$. Since these vectors both form a one-dimensional block, they are mapped onto themselves by the S-matrix. We will normalize the S-matrix as follows:

$$\mathbb{S} |\frac{3}{2}; 0\rangle = |\frac{3}{2}; 0\rangle, \quad \mathbb{S} |0; \frac{3}{2}\rangle = \mathcal{X} |0; \frac{3}{2}\rangle. \quad (3.15)$$

Let us start considering the action of the S-matrix on the three-dimensional block $V_{(1; \frac{1}{2})}$. We first define

$$q_{\pm} \equiv \sqrt{q \pm 1}. \quad (3.16)$$

The basis transformation that relates the standard basis to the coproduct basis and the opposite coproduct basis can be written in terms of the following matrix:

$$\Lambda_{3d, tot}(i, j, \kappa) = \begin{pmatrix} b_i q_- & d_i q_- & \kappa (b_j d_i - b_i d_j) q_+ \\ a_j & c_j & -\kappa q_+ q_- \\ -a_i q_+ & -c_i q_+ & \kappa (b_j c_i - a_i d_j) q_- \end{pmatrix}, \quad (3.17)$$

More precisely one finds that the basis transformations Λ_{3d} and Λ_{3d}^{op} are given by

$$\Lambda_{3d} = \Lambda_{3d, tot}(1, 2, 1), \quad \Lambda_{3d}^{\text{op}} = \Lambda_{3d, tot}(3, 4, \mathcal{X}). \quad (3.18)$$

We use the coefficients (B.4), (B.5) that explicitly include the braiding factors. By construction, the action of the S-matrix is now given by

$$\mathbb{S} |1; \frac{1}{2}\rangle_i = \sum_{j=1}^3 \mathcal{X}_i^j |1; \frac{1}{2}\rangle_i, \quad (3.19)$$

with

$$\mathcal{Y} = \Lambda_{3d}^{\text{op}} \Lambda_{3d}^{-1}. \quad (3.20)$$

The other three-dimensional space $V_{(\frac{1}{2};1)}$ has transformation matrix

$$\bar{\Lambda}_{3d,tot}(i,j,\kappa) = \begin{pmatrix} \kappa b_i q_+ & \kappa d_i q_+ & (a_j d_i - b_i c_j) q_- \\ \kappa b_j & \kappa d_j & -q_+ q_- \\ \kappa a_i q_- & \kappa c_i q_- & (a_i c_j - a_j c_i) q_+ \end{pmatrix}. \quad (3.21)$$

One again finds

$$\bar{\Lambda}_{3d} = \bar{\Lambda}_{3d,tot}(1,2,1), \quad \bar{\Lambda}_{3d}^{\text{op}} = \bar{\Lambda}_{3d,tot}(3,4,\mathcal{X}). \quad (3.22)$$

This in turn leads to

$$\mathbb{S}|\frac{1}{2};1\rangle_i = \sum_{j=1}^3 \tilde{\mathcal{Y}}_i^j |\frac{1}{2};1\rangle_i, \quad (3.23)$$

with

$$\left(\tilde{\mathcal{Y}}_i^j\right) = \bar{\Lambda}_{3d}^{\text{op}} (\bar{\Lambda}_{3d})^{-1}. \quad (3.24)$$

The S-matrices in the other three-dimensional blocks are also described by the above expressions. From figure 1 we see that they are isomorphic via the $\mathfrak{su}(2)$ operators. They are related in a straightforward way; the specified maps map basis vectors to basis vectors, e.g.

$$\Delta R_4^3 |1;\frac{1}{2}\rangle_i = |1;-\frac{1}{2}\rangle_i. \quad (3.25)$$

This leads to

$$\mathbb{S}|\pm 1;\pm\frac{1}{2}\rangle_i = \sum_{j=1}^3 \mathcal{Y}_i^j |\pm 1;\pm\frac{1}{2}\rangle_i, \quad \mathbb{S}|\pm\frac{1}{2};\pm 1\rangle_i = \sum_{j=1}^3 \tilde{\mathcal{Y}}_i^j |\pm\frac{1}{2};\pm 1\rangle_i. \quad (3.26)$$

For the nine-dimensional blocks we again have two distinct cases. We start with $V_{(\frac{1}{2};0)}$ and write

$$\mathbb{S}|\frac{1}{2};0\rangle_i = \sum_{j=1}^9 \mathcal{Z}_i^j |\frac{1}{2};0\rangle_i. \quad (3.27)$$

Since we have expressed the coproduct basis in terms of the three-dimensional subspaces, we find, in analogy with the previous discussion,

$$(\mathcal{Z}) = \Lambda_{9d}^{\text{op}} \text{diag}(\bar{\mathcal{Y}}, \mathcal{Y}, \mathcal{Y}) \Lambda_{9d}^{-1}. \quad (3.28)$$

The matrices Λ_{9d} and Λ_{9d}^{op} are given by

$$\Lambda_{9d} = \begin{pmatrix} 0 & 0 & 0 & b_1\sqrt{q} & 0 & 0 & -d_1\sqrt{q} & 0 & 0 \\ 1 & 0 & 0 & -a_2 & 0 & 0 & c_2 & 0 & 0 \\ 1 & 0 & 0 & 0 & b_1q_- & 0 & 0 & -d_1q_- & 0 \\ 0 & 0 & 0 & 0 & -b_2 & 0 & 0 & d_2 & 0 \\ 0 & 0 & 0 & -\frac{a_1q_-}{\sqrt{2}} & 0 & \frac{b_1q_+}{\sqrt{2}} & \frac{c_1q_-}{\sqrt{2}} & 0 & -\frac{d_1q_+}{\sqrt{2}} \\ 0 & \sqrt{2} & 0 & \frac{a_1q_+}{\sqrt{2}} & 0 & \frac{b_1q_-}{\sqrt{2}} & -\frac{c_1q_+}{\sqrt{2}} & 0 & -\frac{d_1q_-}{\sqrt{2}} \\ 0 & 0 & 1 & 0 & 0 & -a_2 & 0 & 0 & c_2 \\ 0 & 0 & 1 & 0 & -a_1q_+ & 0 & 0 & c_1q_+ & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_1\sqrt{q} & 0 & 0 & c_1\sqrt{q} \end{pmatrix}, \quad \Lambda_{9d}^{\text{op}} = \Lambda_{9d}|_{(1 \leftrightarrow 3, 2 \leftrightarrow 4)}. \quad (3.29)$$

On the other hand, we have

$$\mathbb{S}|0; \frac{1}{2}\rangle_i = \sum_{j=1}^9 \mathcal{Z}_i^j |0; \frac{1}{2}\rangle_i. \quad (3.30)$$

Here we find

$$(\bar{\mathcal{Z}}) = \bar{\Lambda}_{9d}^{\text{op}} \text{diag}(\mathcal{Y}, \bar{\mathcal{Y}}, \bar{\mathcal{Y}})(\bar{\Lambda}_{9d})^{-1}. \quad (3.31)$$

The matrices $\bar{\Lambda}_{9d}$ and $\bar{\Lambda}_{9d}^{\text{op}}$ are given by

$$\bar{\Lambda}_{9d} = \begin{pmatrix} 0 & 0 & 0 & b_1\sqrt{q} & 0 & 0 & -d_1\sqrt{q} & 0 & 0 \\ 1 & 0 & 0 & b_2 & 0 & 0 & -d_2 & 0 & 0 \\ 1 & 0 & 0 & 0 & -b_1q_+ & 0 & 0 & d_1q_+ & 0 \\ 0 & \sqrt{2} & 0 & -\frac{a_1q_-}{\sqrt{2}} & 0 & \frac{b_1q_+}{\sqrt{2}} & \frac{c_1q_-}{\sqrt{2}} & 0 & -\frac{d_1q_+}{\sqrt{2}} \\ 0 & 0 & 0 & 0 & a_2 & 0 & 0 & -c_2 & 0 \\ 0 & 0 & 0 & \frac{a_1q_+}{\sqrt{2}} & 0 & \frac{b_1q_-}{\sqrt{2}} & -\frac{c_1q_+}{\sqrt{2}} & 0 & -\frac{d_1q_-}{\sqrt{2}} \\ 0 & 0 & 1 & 0 & 0 & b_2 & 0 & 0 & -d_2 \\ 0 & 0 & 1 & 0 & -a_1q_- & 0 & 0 & c_1q_- & 0 \\ 0 & 0 & 0 & 0 & 0 & -a_1\sqrt{q} & 0 & 0 & c_1\sqrt{q} \end{pmatrix}, \quad \bar{\Lambda}_{9d}^{\text{op}} = \bar{\Lambda}_{9d}|_{(1 \leftrightarrow 3, 2 \leftrightarrow 4)}. \quad (3.32)$$

Again one can use the $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ generators to relate these two nine-dimensional S-matrices to the two remaining ones. However, the relation is slightly less straightforward. In particular we find

$$\mathbb{S}|0; -\frac{1}{2}\rangle_i = \sum_{j=1}^9 (\mathcal{Z}')_i^j |0; -\frac{1}{2}\rangle_i, \quad (\mathcal{Z}') = L(\mathcal{Z})L^{-1}, \quad (3.33)$$

$$\mathbb{S}|-\frac{1}{2}; 0\rangle_i = \sum_{j=1}^9 (\bar{\mathcal{Z}}')_i^j |-\frac{1}{2}; 0\rangle_i, \quad (\bar{\mathcal{Z}}') = R(\bar{\mathcal{Z}})R^{-1}. \quad (3.34)$$

where

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \quad (3.35)$$

It is easily checked that this S-matrix is symmetric and is indeed invariant under the full centrally extended $\mathfrak{su}(2|2)$ algebra. By construction, one can also see that the S-matrix is automatically obtained in the factorized form $\mathbb{S} = F_{21} F_{12}^{-1}$ (Drinfeld twist) [56].

3.3 Yang-Baxter equation

From the previous section we saw that invariance under the symmetry algebra is not enough to fix the S-matrix completely. We still have a free parameter $\mathcal{X}$. This parameter can be fixed by imposing that the S-matrix solves the Yang-Baxter equation:

$$\mathbb{S}_{12}(P, p_2)\mathbb{S}_{13}(P, p_3)\mathbb{S}_{23}(p_2, p_3) = \mathbb{S}_{23}(p_2, p_3)\mathbb{S}_{13}(P, p_3)\mathbb{S}_{12}(P, p_2). \quad (3.36)$$

By considering the scattering processes

$$f_6 \otimes e_1 \otimes e_3 \rightarrow f_2 \otimes e_1 \otimes e_1, \quad f_6 \otimes e_1 \otimes e_2 \rightarrow f_6 \otimes e_3 \otimes e_4 \quad (3.37)$$

we obtain two quadratic equations for $\mathcal{X}$ of the form

$$A + B\mathcal{X}(P, p_2) + C\mathcal{X}(P, p_3) + D\mathcal{X}(P, p_2)\mathcal{X}(P, p_3) = 0, \quad (3.38)$$

where A, B, C, D are functions of P, p_2, p_3 . It is easily seen that there are *two* different solutions to these equations. This means that we find *two* S-matrices, and they are not related by a similarity transformation. The solutions for $\mathcal{X}$ appear however rather complicated and we refrain from giving their explicit expressions. It can be checked that *both* solutions satisfy the following relations

$$\textit{Unitarity:} \quad \mathbb{S}_{12}\mathbb{S}_{21} = \mathbb{1}.$$

$$\textit{Hermiticity:} \quad \mathbb{S}_{12}(z_L, z)\mathbb{S}_{12}(z_L^*, z^*)^\dagger = \mathbb{1}.$$

$$\textit{CPT Invariance:} \quad \mathbb{S}_{12} = \mathbb{S}_{12}^t.$$

$$\textit{Yang-Baxter:} \quad \mathbb{S}_{12}\mathbb{S}_{13}\mathbb{S}_{23} = \mathbb{S}_{23}\mathbb{S}_{13}\mathbb{S}_{12}.$$

This completes our derivation of the S-matrices based on the $\mathfrak{su}(2|2)$ symmetry.

4 Long representations via tensor product of short ones

The scope of this section is to prove that the two S-matrices we have just derived, relying only on the Lie superalgebra symmetry and the Yang-Baxter equation, can both be obtained from the tensor product of two short evaluation representations of the Yangian. In principle from the previous analysis we could have found *more* solutions than those related to short evaluation representations, but we will show that this is not the case.

Consider the tensor product of two short representations labelled by momentum (p_1, p_2) ,

$$V(p_1) \otimes V(p_2). \quad (4.1)$$

This vector space naturally carries a representation of centrally extended $\mathfrak{su}(2|2)$ via the (opposite) coproduct, i.e. for any generator $\mathbb{J}$ we have

$$\mathbb{J}_{V(p_1) \otimes V(p_2)} = \Delta\mathbb{J}. \quad (4.2)$$

It is easily seen by considering the central charges on this space that we are dealing with a long representation. To be precise, we find

$$(2q)^2 = \Delta \mathbb{H}^2 - 4\Delta \mathbb{C} \Delta \mathbb{C}^\dagger = [E(p_1) + E(p_2)]^2 - E(p_1 + p_2)^2 + 1, \quad (4.3)$$

where the energy $E(p)$ is

$$E(p)^2 = 1 + 4g^2 \sin^2 \frac{p}{2}. \quad (4.4)$$

The momentum of the long representation is found to be

$$P = p_1 + p_2. \quad (4.5)$$

One has therefore

$$V(p_1) \otimes V(p_2) \cong V(P, q) \quad (4.6)$$

with

$$P = p_1 + p_2, \quad q = \frac{E(p_1) + E(p_2)}{\sqrt{[E(p_1) + E(p_2)]^2}} \frac{\sqrt{[E(p_1) + E(p_2)]^2 - E(p_1 + p_2)^2 + 1}}{2}. \quad (4.7)$$

The dispersion relation (4.4) has two branches, corresponding to particles and anti-particles. Fixing momentum p and choosing a branch specifies the fundamental representation completely. Then, the tensor product of two such representations is identified with a unique 16-dimensional long representation with momentum P and the central charge q specified above.

Consider now the inverse problem, i.e. suppose we are given a long representation (P, q) and we want to factorize it into the tensor product of two fundamental representations. It is convenient to label the representation space corresponding to particles as V_+ and the one corresponding to anti-particles as V_- . Thus, the carrier space of the long representation can be identified with one of the following four spaces:

- I. $V_+ \otimes V_+$,
- II. $V_+ \otimes V_-$,
- III. $V_- \otimes V_+$,
- IV. $V_- \otimes V_-$.

In the emerging solutions of the factorization problem the momenta p_1 and p_2 can be ordered, and we always assume that the ordering is such that $p_1 \prec p_2$.¹⁵ Assuming for simplicity that q is real, we find that to a long representation $V(P, q)$ one can associate two solutions of the factorization problem. For instance, for q positive the two solutions are both associated with the case I, or one of the solutions is from I and the second is

¹⁵The details of the ordering are irrelevant, since its only function is to choose a unique representative between the couple (p_1, p_2) and its permuted couple (p_2, p_1) .

from II. Analogous situation takes place for q negative. Thus, any long representation can be written as a tensor product of two *different* short representations. Actually, this observation was reflected earlier in the fact that we found two independent long-short S-matrices that solve the Yang-Baxter equation.

Instead of particle momenta p_i and P one can use the corresponding rapidity variables u_i and u . The equation for q and the momentum conservation take the form

$$-\left[x\left(u_1 - \frac{i}{g}\right) + x\left(u_1 + \frac{i}{g}\right) + x\left(u_2 - \frac{i}{g}\right) + x\left(u_2 + \frac{i}{g}\right) + \frac{2i}{g}\right]^2 + \frac{x\left(u + \frac{2i}{g}q\right)}{x\left(u - \frac{2i}{g}q\right)} + \frac{x\left(u - \frac{2i}{g}q\right)}{x\left(u + \frac{2i}{g}q\right)} = 4q^2 + 2, \quad (4.8)$$

and

$$\frac{x\left(u_1 + \frac{i}{g}\right) x\left(u_2 + \frac{i}{g}\right)}{x\left(u_1 - \frac{i}{g}\right) x\left(u_2 - \frac{i}{g}\right)} = \frac{x\left(u + \frac{2i}{g}q\right)}{x\left(u - \frac{2i}{g}q\right)}, \quad (4.9)$$

where $x(u) = \frac{u}{2} \left(1 + \sqrt{1 - \frac{4}{u^2}}\right)$ maps the u -plane on the kinematic region of the string theory [16]. The energy of the long representation is given by

$$E(P) = igx\left(u - \frac{2i}{g}q\right) - igx\left(u + \frac{2i}{g}q\right) - 2q = \text{sign}(q) \sqrt{(2q)^2 + 4g^2 \sin^2 \frac{P}{2}}. \quad (4.10)$$

In general, given u_1 and u_2 , the variable q will appear as a complicated function $q \equiv q(u_1, u_2, g)$. However, there are two special cases, where q is a constant independent of u_i and g . Indeed, for $u_1 = u \pm \frac{i}{g}$ and $u_2 = u \mp \frac{i}{g}$ one gets $q = 1$. Analogously, for $u_1 = -u \pm \frac{i}{g}$ and $u_2 = -u \mp \frac{i}{g}$ one finds $q = -1$. These values of q correspond to the shortening conditions, for which the long multiplet becomes reducible but indecomposable. Imposing the ordering $u_1 \prec u_2$ we get *e.g.* for $q = 1$ only one solution. This is an artifact of our parametrization $x(u)$ in eqs. (4.8) and (4.9). As is known, the u -plane covers through the map $x(u)$ only the string region on the z -torus [16]. To find the other solution, one has to change the map $x(u)$ for the one which covers the mirror regions on the z -torus. For $q = 1$ both solutions are from $V_+ \otimes V_+$, which is also the case for q close to one. However, when q deviates from $q = 1$ sufficiently enough, two solutions can occur in $V_+ \otimes V_+$ and $V_+ \otimes V_-$, respectively.

We further note that one can explicitly find the similarity transformation that relates the long algebra generators to the ones that arise from the coproduct. It is convenient to first express the coefficients a_L, b_L, c_L, d_L parameterizing long representations via a_1 , etc. that describe the short representations (again we use the coefficients that already include braiding factors, (B.4), (B.5))

$$a_L = \frac{a_1 d_1 + a_2 d_2 + q - 1}{2q d_L}, \quad b_L = \frac{d_L (a_1 d_1 + a_2 d_2 - q - 1)}{c_1 d_1 + c_2 d_2}, \quad c_L = \frac{c_1 d_3 + c_2 d_2}{2q d_L}. \quad (4.11)$$

In terms of these coefficients we find that the algebra generators are related via a similarity transformation

$$\mathbb{J}_L = V_\Delta \Delta \mathbb{J} V_\Delta^{-1}, \quad (4.12)$$

with

$$V_{\Delta} = \begin{pmatrix} 0 & -v_1 & 0 & 0 & v_1 & 0 & 0 & 0 & 0 & 0 & 0 & -v_2 & 0 & 0 & v_2 & 0 \\ 0 & 0 & v_3 & 0 & 0 & 0 & 0 & 0 & v_4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v_3 & 0 & 0 & v_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_3 & 0 & 0 & 0 & 0 & v_4 & 0 & 0 \\ \sqrt{2}v_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & v_5 & 0 & 0 & v_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2}v_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}v_6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_6 & 0 & 0 & v_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}v_6 \\ 0 & 0 & v_7 & 0 & 0 & 0 & 0 & 0 & v_8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v_7 & 0 & 0 & v_8 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & v_7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_7 & 0 & 0 & 0 & 0 & 0 & v_8 & 0 & 0 \\ 0 & -v_9 & 0 & 0 & v_9 & 0 & 0 & 0 & 0 & 0 & 0 & -v_{10} & 0 & 0 & v_{10} & 0 \end{pmatrix}, \quad (4.13)$$

where the coefficients are given by

$$\begin{aligned} v_1 &= -\frac{(d_L a_1 - b_L c_1) v_4}{\sqrt{q}}, & v_2 &= \frac{(d_L b_2 - b_L d_2) v_4}{\sqrt{q}}, \\ v_3 &= -\frac{(d_L a_1 - b_L c_1) v_4}{d_L a_2 - b_L c_2}, & v_5 &= -\frac{q_- (d_L a_1 - b_L c_1) v_4}{\sqrt{2} (d_L a_2 - b_L c_2) (d_L b_2 - b_L d_2)}, \\ v_6 &= -\frac{q_+ v_4}{\sqrt{2} (d_L a_2 - b_L c_2)}, & v_7 &= \frac{q_+ q_- (d_L a_1 - b_L c_1) v_4}{2 (d_L a_2 - b_L c_2)^2 (d_L b_2 - b_L d_2)}, \\ v_8 &= \frac{q_+ q_- v_4}{2 (d_L a_2 - b_L c_2) (d_L b_2 - b_L d_2)}, & v_9 &= -\frac{q_+ q_- (c_L a_1 - a_L c_1) v_4}{2 \sqrt{q} (d_L a_2 - b_L c_2) (d_L b_2 - b_L d_2)}, \\ v_{10} &= \frac{q_+ q_- (c_L b_2 - a_L d_2) v_4}{2 \sqrt{q} (d_L a_2 - b_L c_2) (d_L b_2 - b_L d_2)}. \end{aligned} \quad (4.14)$$

The coproduct on three short representation is given by $(\Delta \otimes \mathbb{1})\Delta$. It is easily seen that

$$\begin{aligned} \mathbb{S}_{13}\mathbb{S}_{23}(\Delta \otimes \mathbb{1})\Delta\mathbb{J} &= \mathbb{S}_{13}\mathbb{S}_{23}(\Delta\mathbb{J} \otimes \mathbb{U}^{[\mathbb{J}]} + \mathbb{1}_L \otimes \mathbb{J}) \\ &= \mathbb{S}_{13}\mathbb{S}_{23}(\mathbb{J} \otimes \mathbb{U}^{[\mathbb{J}]} \otimes \mathbb{U}^{[\mathbb{J}]} + \mathbb{1} \otimes \mathbb{J} \otimes \mathbb{U}^{[\mathbb{J}]} + \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{J}) \\ &= (\mathbb{J} \otimes \mathbb{U}^{[\mathbb{J}]} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{J} \otimes \mathbb{1} + \mathbb{U}^{[\mathbb{J}]} \otimes \mathbb{U}^{[\mathbb{J}]} \otimes \mathbb{J})\mathbb{S}_{13}\mathbb{S}_{23} \\ &= (\Delta\mathbb{J} \otimes \mathbb{1} + \mathbb{U}_L^{[\mathbb{J}]} \otimes \mathbb{J})\mathbb{S}_{13}\mathbb{S}_{23}. \end{aligned} \quad (4.15)$$

Thus we see that $\mathbb{S}_{13}\mathbb{S}_{23}$ intertwines the coproduct on the tensor product of a long and a short representation. By the above similarity transformation, this means that we can interpret $\mathbb{S}$ as being built up out of fundamental S-matrices.

$$\mathbb{S} = V_{\Delta} \otimes \mathbb{1} \mathbb{S}_{13}\mathbb{S}_{23} V_{\Delta}^{-1} \otimes \mathbb{1}. \quad (4.16)$$

The two different choices of short representations that give rise to the long representation then indeed gives two different solutions for $\mathbb{S}$. They exactly coincide with the ones that are found from the Yang-Baxter equation.

As we discussed in section 2.2, the fact that the S-matrix in short representation possesses Yangian symmetries (in evaluation representations) automatically induces, *via* the above mentioned tensor product procedure, a Yangian representation associated to the long representation. The generators are simply given by

$$\widehat{\mathbb{J}}_{V(p_1) \otimes V(p_2)} = \Delta(\widehat{\mathbb{J}}). \quad (4.17)$$

Figure 2. The various representations, their $q \rightarrow 1$ limits, and the block structure of the corresponding S-matrix in this limit. We denoted with V_{ZG} the representation obtained from [39]. As a consequence of the upper-triangular structure, the bottom-right block of the limiting S-matrices satisfies the YBE by itself.

Δ is projected into short $\otimes$ short Yangian representations, the latter being characterized by the known (‘short’) spectral parameters u_1 and u_2 (on the first and second factor of the tensor product, respectively). These short spectral parameters are linked to the parameters of the two corresponding short representations as in (2.19). When using the formulas in appendices B and C for the Yangian generators and their coproducts, taking into account (4.2) and (4.17), one can check the perfect consistency with all the relations in both Drinfeld’s first and second realization, *and* one finds of course that the Yangian in this representation admits an S-matrix (in particular, the higher central charges in Drinfeld’s second realization are co-commutative, and no contradiction with the existence of an S-matrix is found when acting on specific states, cf. section 2.2). However, The Yangian representation obtained in this way is not isomorphic to the Yangian evaluation representation discussed in section 2.2. This is consistent with the fact that the evaluation representation of section 2.2 does not admit an S-matrix, while the tensor product of two short representations does.

Let us add one more remark on the situation corresponding to $q = 1$. In this case, both the similarity transformation (A.5) that connects the Gould-Zhang representation [39] to the unitary one, and the one connecting the unitary representation to a tensor product of short ones (*via* coproduct), namely (4.13), are singular. In fact, by sending $q \rightarrow 1$ in both the Gould-Zhang representation and in the tensor product of short ones, one gets a reducible but indecomposable representation, and the limit of the S-matrix is not block-diagonal in the subrepresentation and factor representation spaces. Instead, by sending $q \rightarrow 1$ in the unitary representation, one ends up into a decomposable representation (which is the only way it can be represented by hermitean matrices). The reducible components are the sym-

metric and antisymmetric bound-state representations, and the S-matrix trivially factorizes in the two spaces, each block becoming equal to the corresponding bound-state S-matrices. The relative unknown coefficient is not fixed, and this S-matrix trivially satisfies the Yang-Baxter equation blockwise, and has usual blockwise bound-state Yangian invariance in the evaluation representation. As we said, this unitary representation at $q = 1$ is not isomorphic to a tensor (or more precisely, to the co-) product of short fundamental representations, and it turns out to furnish a way of extracting the subrepresentation and factor representation from the indecomposable one we are dealing with. We have summarized the situation for convenience in figure 2.

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A Explicit parameterization

We list in this appendix the generators of centrally extended $\mathfrak{su}(2|2)$ in the long representation. We only report explicitly the simple roots for a distinguished Dynkin diagram, the remainder of the algebra being generated *via* commutation relations. We present the roots in a unitary representation. To achieve this, we perform a similarity transformation on the generator constructed directly from the oscillator basis of [39], in order to obtain hermitean matrices. First, the bosonic $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ roots are given by

$$\mathbb{L}_2^1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{L}_1^2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (\text{A.1})$$

and

$$\mathbb{R}_4^3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbb{R}_3^4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.2})$$

Next, we show two fermionic roots as an example:

$$\mathbb{Q}_3^1 = \begin{pmatrix} 0 & 0 & 0 & -b\sqrt{q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & bq_- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a\sqrt{q} & 0 & 0 & 0 & 0 & 0 & \frac{bq_-}{\sqrt{2}} & 0 & \frac{bq_+}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & bq_+ & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{aq_-}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{bq_+}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & aq_- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -bq_+ & 0 \\ 0 & aq_+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & bq_- & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{aq_+}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{bq_-}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -aq_+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{aq_+}{\sqrt{2}} & 0 & \frac{aq_-}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -b\sqrt{q} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & aq_- & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a\sqrt{q} & 0 & 0 & 0 \end{pmatrix}, \quad (\text{A.3})$$

$$\mathbb{G}_2^4 = \begin{pmatrix} 0 & 0 & 0 & -d\sqrt{q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & dq_- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ c\sqrt{q} & 0 & 0 & 0 & 0 & 0 & \frac{dq_-}{\sqrt{2}} & 0 & \frac{dq_+}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & dq_+ & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{cq_-}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{dq_+}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & cq_- & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -dq_+ & 0 & 0 \\ 0 & cq_+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & dq_- & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{cq_+}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{dq_-}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -cq_+ & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{cq_+}{\sqrt{2}} & 0 & \frac{cq_-}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & -d\sqrt{q} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & cq_- & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & c\sqrt{q} & 0 & 0 & 0 \end{pmatrix}. \quad (\text{A.4})$$

Finally, the similarity transformation that relates the unitary representation to the one from [39] is given by

$$\mathcal{V} = \text{diag}\left(\sqrt{q^3 - q}, q_+q_-, q_+q_-, q_+q_-, q_+q_-, 2q_+, \sqrt{2}q_+, 2q_+, 2q_-, \sqrt{2}q_-, 2q_-, 1, 1, 1, 1, \frac{1}{\sqrt{q}}\right) \quad (\text{A.5})$$

where $q_{\pm}$ have been defined in (3.16). We notice that this transformation is singular for $q^2 = 1$, where the representation becomes reducible but indecomposable.

B Yangians and coproducts: Drinfeld's first realization

The double Yangian [57] $DY(\mathfrak{g})$ of a (simple) Lie algebra $\mathfrak{g}$ is a deformation of the universal enveloping algebra $U(\mathfrak{g}[u, u^{-1}])$ of the loop algebra $\mathfrak{g}[u, u^{-1}]$. The Yangian is obtained by adding to the Lie algebra a set of partner generators $\hat{\mathbb{J}}_n^A$, $n \in \mathbb{Z}$ satisfying the commutation relations

$$[\mathbb{J}^A, \hat{\mathbb{J}}^B] = F_C^{AB} \hat{\mathbb{J}}^C, \quad (\text{B.1})$$

where F_C^{AB} are the structure constants of $\mathfrak{g}$. The centrally-extended $\mathfrak{su}(2|2)$ Yangian has the following coproduct¹⁶ [34, 53, 54]:

$$\Delta(\hat{\mathbb{J}}_n^A) = \hat{\mathbb{J}}_n^A \otimes \mathbb{1} + \mathbb{U}^{[A]} \otimes \hat{\mathbb{J}}_n^A + \hbar \sum_{m=0}^{n-1} F_{BC}^A \mathbb{J}_{n-1-m}^B \mathbb{U}^{[C]} \otimes \mathbb{J}_m^C + \mathcal{O}(\hbar^2), \quad (\text{B.2})$$

where $\mathbb{U}$, $[[A]]$ are given in section 2.2.

The evaluation representation we have been discussing in section 2.2 is obtained as $\hat{\mathbb{J}}^A = u \mathbb{J}^A$ [34]. In this representation the coproduct structure is fixed in terms of the coproducts of $\mathbb{J}, \hat{\mathbb{J}}$. As in the case of short representations, it is possible to absorb the factors arising due to the presence of $\mathbb{U}$ into a non-local redefinition of the representation labels a_i, b_i, c_i, d_i . We will give here the formulas for the Yangian coproducts and for these redefined labels.

$$\begin{aligned} \Delta(\hat{\mathbb{L}}_b^a) &= \hat{\mathbb{L}}_{1;b}^a + \hat{\mathbb{L}}_{2;b}^a + \frac{1}{2} \mathbb{L}_{1;c}^c \mathbb{L}_{2;c}^a - \frac{1}{2} \mathbb{L}_{1;c}^a \mathbb{L}_{2;b}^c - \frac{1}{2} \mathbb{G}_{1;b}^\gamma \mathbb{Q}_{2;\gamma}^a - \frac{1}{2} \mathbb{Q}_{1;\gamma}^a \mathbb{G}_{2;b}^\gamma \\ &\quad + \frac{1}{4} \delta_b^a \mathbb{G}_{1;c}^\gamma \mathbb{Q}_{2;\gamma}^c + \frac{1}{4} \delta_b^a \mathbb{Q}_{1;\gamma}^c \mathbb{G}_{2;c}^\gamma, \\ \Delta(\hat{\mathbb{R}}_\beta^\alpha) &= \hat{\mathbb{R}}_{1;\beta}^\alpha + \hat{\mathbb{R}}_{2;\beta}^\alpha - \frac{1}{2} \mathbb{R}_{1;\beta}^\gamma \mathbb{R}_{2;\gamma}^\alpha + \frac{1}{2} \mathbb{R}_{1;\gamma}^\alpha \mathbb{R}_{2;\beta}^\gamma + \frac{1}{2} \mathbb{G}_{1;c}^\alpha \mathbb{Q}_{2;\beta}^c + \frac{1}{2} \mathbb{Q}_{1;\beta}^c \mathbb{G}_{2;c}^\alpha \\ &\quad - \frac{1}{4} \delta_\beta^\alpha \mathbb{G}_{1;c}^\gamma \mathbb{Q}_{2;\gamma}^c - \frac{1}{4} \delta_\beta^\alpha \mathbb{Q}_{1;\gamma}^c \mathbb{G}_{2;c}^\gamma, \\ \Delta(\hat{\mathbb{Q}}_\beta^a) &= \hat{\mathbb{Q}}_{1;\beta}^a + \hat{\mathbb{Q}}_{2;\beta}^a - \frac{1}{2} \mathbb{R}_{1;\beta}^\gamma \mathbb{Q}_{2;\gamma}^a + \frac{1}{2} \mathbb{Q}_{1;\gamma}^a \mathbb{R}_{2;\beta}^\gamma - \frac{1}{2} \mathbb{L}_{1;c}^a \mathbb{Q}_{2;\beta}^c + \frac{1}{2} \mathbb{Q}_{1;\beta}^c \mathbb{L}_{2;c}^a \\ &\quad - \frac{1}{4} \mathbb{H}_1 \mathbb{Q}_{2;\beta}^a + \frac{1}{4} \mathbb{Q}_{1;\beta}^a \mathbb{H}_2 + \frac{1}{2} \epsilon_{\beta\gamma} \epsilon^{ad} \mathbb{C}_1 \mathbb{G}_{2;d}^\gamma - \frac{1}{2} \epsilon_{\beta\gamma} \epsilon^{ad} \mathbb{G}_{1;d}^\gamma \mathbb{C}_2, \\ \Delta(\hat{\mathbb{G}}_b^\alpha) &= \hat{\mathbb{G}}_{1;b}^\alpha + \hat{\mathbb{G}}_{2;b}^\alpha + \frac{1}{2} \mathbb{L}_{1;c}^c \mathbb{G}_{2;c}^\alpha - \frac{1}{2} \mathbb{G}_{1;c}^\alpha \mathbb{L}_{2;b}^c + \frac{1}{2} \mathbb{R}_{1;\gamma}^\alpha \mathbb{G}_{2;b}^\gamma - \frac{1}{2} \mathbb{G}_{1;b}^\gamma \mathbb{R}_{2;\gamma}^\alpha \\ &\quad + \frac{1}{4} \mathbb{H}_1 \mathbb{G}_{2;b}^\alpha - \frac{1}{4} \mathbb{G}_{1;b}^\alpha \mathbb{H}_2 - \frac{1}{2} \epsilon_{bc} \epsilon^{\alpha\gamma} \mathbb{C}_1^\dagger \mathbb{Q}_{2;\gamma}^c + \frac{1}{2} \epsilon_{bc} \epsilon^{\alpha\gamma} \mathbb{Q}_{1;\gamma}^c \mathbb{C}_2^\dagger, \\ \Delta(\hat{\mathbb{H}}) &= \hat{\mathbb{H}}_1 + \hat{\mathbb{H}}_2 + \mathbb{C}_1 \mathbb{C}_2^\dagger - \mathbb{C}_1^\dagger \mathbb{C}_2, \\ \Delta(\hat{\mathbb{C}}) &= \hat{\mathbb{C}}_1 + \hat{\mathbb{C}}_2 - \frac{1}{2} \mathbb{H}_1 \mathbb{C}_2 + \frac{1}{2} \mathbb{C}_1 \mathbb{H}_2, \\ \Delta(\hat{\mathbb{C}}^\dagger) &= \hat{\mathbb{C}}_1^\dagger + \hat{\mathbb{C}}_2^\dagger + \frac{1}{2} \mathbb{H}_1 \mathbb{C}_2^\dagger - \frac{1}{2} \mathbb{C}_1^\dagger \mathbb{H}_2. \end{aligned} \quad (\text{B.3})$$

We have used in the above formulas the shorthand notation $\mathbb{J}_1 \mathbb{Y}_2 = \mathbb{J} \otimes \mathbb{Y}$. In case of long representation in space 1 and short representation in space 2 of the tensor product, the

¹⁶Here, $\hbar = \frac{1}{g}$, and it can be reabsorbed in the definition of the algebra generators, which is the convention we use in the paper. We display $\hbar$ in this particular formula just to show how the ‘tail’ of the Yangian coproduct organizes itself. The terms we omit from (B.2) are also completely determined by the knowledge of the level zero and level one coproducts, in a recursive fashion.

labels used in Δ are given by:

$$\begin{aligned}
 a_1 &= \sqrt{\frac{g}{4q}} \eta_1, & b_1 &= -ie^{ip_2} \sqrt{\frac{g}{4q}} \frac{1}{\eta_1} \left(\frac{x_1^+}{x_1^-} - 1 \right), \\
 c_1 &= -e^{-ip_2} \sqrt{\frac{g}{4q}} \frac{\eta_1}{x_1^+}, & d_1 &= i \sqrt{\frac{g}{4q}} \frac{x_1^+}{\eta_1} \left(\frac{x_1^-}{x_1^+} - 1 \right), \\
 \eta_1 &= e^{i\frac{p_1}{4}} e^{i\frac{p_2}{2}} \sqrt{ix_1^- - ix_1^+}, \\
 a_2 &= \sqrt{\frac{g}{2}} \eta_2, & b_2 &= -i \sqrt{\frac{g}{2}} \frac{1}{\eta_2} \left(\frac{x_2^+}{x_2^-} - 1 \right), \\
 c_2 &= -\sqrt{\frac{g}{2}} \frac{\eta_2}{x_2^+}, & d_2 &= i \sqrt{\frac{g}{2}} \frac{x_2^+}{\eta_2} \left(\frac{x_2^-}{x_2^+} - 1 \right), \\
 \eta_2 &= e^{i\frac{p_2}{4}} \sqrt{ix_2^- - ix_2^+}.
 \end{aligned} \tag{B.4}$$

Accordingly, the labels used in Δ^{op} are given by:

$$\begin{aligned}
 a_3 &= \sqrt{\frac{g}{4q}} \eta_1^{\text{op}}, & b_3 &= -i \sqrt{\frac{g}{4q}} \frac{1}{\eta_1^{\text{op}}} \left(\frac{x_1^+}{x_1^-} - 1 \right), \\
 c_3 &= -\sqrt{\frac{g}{4q}} \frac{\eta_1^{\text{op}}}{x_1^+}, & d_3 &= i \sqrt{\frac{g}{4q}} \frac{x_1^+}{\eta_1^{\text{op}}} \left(\frac{x_1^-}{x_1^+} - 1 \right), \\
 \eta_1^{\text{op}} &= e^{i\frac{p_1}{4}} \sqrt{ix_1^- - ix_1^+}, \\
 a_4 &= \sqrt{\frac{g}{2}} \eta_2^{\text{op}}, & b_4 &= -ie^{ip_1} \sqrt{\frac{g}{2}} \frac{1}{\eta_2^{\text{op}}} \left(\frac{x_2^+}{x_2^-} - 1 \right), \\
 c_4 &= -e^{-ip_1} \sqrt{\frac{g}{2}} \frac{\eta_2^{\text{op}}}{x_2^+}, & d_4 &= i \sqrt{\frac{g}{2}} \frac{x_2^+}{\eta_2^{\text{op}}} \left(\frac{x_2^-}{x_2^+} - 1 \right), \\
 \eta_2^{\text{op}} &= e^{i\frac{p_2}{4}} e^{i\frac{p_1}{2}} \sqrt{ix_2^- - ix_2^+}.
 \end{aligned} \tag{B.5}$$

The non-trivial braiding factors are all hidden in the parameters of the four representations involved.

C Yangians and coproducts: Drinfeld's second realization

The second realization of the Yangian [58] is given in terms of Chevalley-Serre type generators and relations. The formulas for the centrally-extended $\mathfrak{su}(2|2)$ case have been given in [43]. They are expressed in terms of Cartan generators $\kappa_{i,m}$ and fermionic simple roots $\xi_{i,m}^\pm$, $i = 1, 2, 3$, $m = 0, 1, 2, \dots$, subject to the following relations:

$$\begin{aligned}
 [\kappa_{i,m}, \kappa_{j,n}] &= 0, & [\kappa_{i,0}, \xi_{j,m}^+] &= a_{ij} \xi_{j,m}^+, \\
 [\kappa_{i,0}, \xi_{j,m}^-] &= -a_{ij} \xi_{j,m}^-, & \{\xi_{i,m}^+, \xi_{j,n}^-\} &= \delta_{i,j} \kappa_{j,n+m}, \\
 [\kappa_{i,m+1}, \xi_{j,n}^+] &- [\kappa_{i,m}, \xi_{j,n+1}^+] &= \frac{1}{2} a_{ij} \{\kappa_{i,m}, \xi_{j,n}^+\}, \\
 [\kappa_{i,m+1}, \xi_{j,n}^-] &- [\kappa_{i,m}, \xi_{j,n+1}^-] &= -\frac{1}{2} a_{ij} \{\kappa_{i,m}, \xi_{j,n}^-\},
 \end{aligned}$$

$$\begin{aligned}\{\xi_{i,m+1}^+, \xi_{j,n}^+\} - \{\xi_{i,m}^+, \xi_{j,n+1}^+\} &= \frac{1}{2}a_{ij}[\xi_{i,m}^+, \xi_{j,n}^+], \\ \{\xi_{i,m+1}^-, \xi_{j,n}^-\} - \{\xi_{i,m}^-, \xi_{j,n+1}^-\} &= -\frac{1}{2}a_{ij}[\xi_{i,m}^-, \xi_{j,n}^-],\end{aligned}\quad (\text{C.1})$$

$$\begin{aligned}i \neq j, \quad n_{ij} = 1 + |a_{ij}|, \quad \text{Sym}_{\{k\}}[\xi_{i,k_1}^+, [\xi_{i,k_2}^+, \dots \{\xi_{i,k_{n_{ij}}}^+, \xi_{j,l}^+\} \dots]] &= 0, \\ i \neq j, \quad n_{ij} = 1 + |a_{ij}|, \quad \text{Sym}_{\{k\}}[\xi_{i,k_1}^-, [\xi_{i,k_2}^-, \dots \{\xi_{i,k_{n_{ij}}}^-, \xi_{j,l}^-\} \dots]] &= 0, \\ \text{except for } \{\xi_{2,n}^+, \xi_{3,m}^+\} = \mathbb{C}_{n+m}, \quad \{\xi_{2,n}^-, \xi_{3,m}^-\} = \mathbb{C}_{n+m}^\dagger, &\end{aligned}\quad (\text{C.2})$$

where the symmetric Cartan matrix a_{ij} has all zeroes except for $a_{12} = a_{21} = 1$ and $a_{13} = a_{31} = -1$. We call the index n of the generators in this realization the *level*. The Dynkin diagram corresponds to the following Chevalley-Serre basis, composed of Cartan generators $\mathfrak{H}_i$, and positive (negative) simple roots $\mathfrak{E}_i^+$ ($\mathfrak{E}_i^-$, respectively) [43]

$$\mathfrak{E}_1^+ = \mathbb{G}_2^4, \quad \mathfrak{E}_1^- = \mathbb{Q}_4^2, \quad \mathfrak{H}_1 = -\mathbb{L}_1^1 - \mathbb{R}_3^3 + \frac{1}{2}\mathbb{H}, \quad (\text{C.3})$$

$$\mathfrak{E}_2^+ = i\mathbb{Q}_4^1, \quad \mathfrak{E}_2^- = i\mathbb{G}_1^4, \quad \mathfrak{H}_2 = -\mathbb{L}_1^1 + \mathbb{R}_3^3 - \frac{1}{2}\mathbb{H}, \quad (\text{C.4})$$

$$\mathfrak{E}_3^+ = i\mathbb{Q}_3^2, \quad \mathfrak{E}_3^- = i\mathbb{G}_2^3, \quad \mathfrak{H}_3 = \mathbb{L}_1^1 - \mathbb{R}_3^3 - \frac{1}{2}\mathbb{H}. \quad (\text{C.5})$$

The isomorphism (Drinfeld's map) between the first and the second realization is given as follows:

$$\begin{aligned}\kappa_{i,0} &= \mathfrak{H}_i, & \xi_{i,0}^+ &= \mathfrak{E}_i, & \xi_{i,0}^- &= \mathfrak{F}_i, \\ \kappa_{i,1} &= \widehat{\mathfrak{H}}_i - v_i, & \xi_{i,1}^+ &= \widehat{\mathfrak{E}}_i - w_i, & \xi_{i,1}^- &= \widehat{\mathfrak{F}}_i - z_i,\end{aligned}\quad (\text{C.6})$$

where $\widehat{\mathfrak{H}}_i, \widehat{\mathfrak{E}}_i, \widehat{\mathfrak{F}}_i$ are the Yangian partners of $\mathfrak{H}_i, \mathfrak{E}_i, \mathfrak{F}_i$ in the first realization, and the special elements are given by

$$\begin{aligned}v_1 &= -\frac{1}{2}\kappa_{1,0}^2 + \frac{1}{4}\mathbb{R}_3^4\mathbb{R}_3^3 + \frac{1}{4}\mathbb{R}_4^4\mathbb{R}_3^3 + \frac{3}{4}\mathbb{L}_1^2\mathbb{L}_2^1 - \frac{1}{4}\mathbb{L}_2^1\mathbb{L}_1^2 - \frac{1}{4}\mathbb{Q}_3^2\mathbb{G}_2^3 - \frac{1}{4}\mathbb{Q}_4^1\mathbb{G}_1^4 - \frac{3}{4}\mathbb{G}_1^4\mathbb{Q}_4^1 \\ &\quad + \frac{1}{4}\mathbb{G}_2^3\mathbb{Q}_3^2 + \frac{1}{2}\mathbb{C}\mathbb{C}^\dagger, \\ v_2 &= -\frac{1}{2}\kappa_{2,0}^2 - \mathbb{R}_3^4\mathbb{R}_4^3 + \frac{1}{2}\mathbb{R}_4^4\mathbb{R}_3^3 + \frac{1}{2}\mathbb{L}_2^1\mathbb{L}_1^2 + \mathbb{Q}_3^1\mathbb{G}_1^3 + \frac{1}{2}\mathbb{G}_1^3\mathbb{Q}_3^1 - \frac{1}{2}\mathbb{G}_2^4\mathbb{Q}_4^2 - \frac{1}{2}\mathbb{C}\mathbb{C}^\dagger, \\ v_3 &= -\frac{1}{2}\kappa_{3,0}^2 + \frac{1}{2}\mathbb{R}_3^4\mathbb{R}_4^3 - \frac{1}{2}\mathbb{L}_1^2\mathbb{L}_2^1 + \frac{1}{2}\mathbb{G}_1^3\mathbb{Q}_3^1 + \frac{1}{2}\mathbb{G}_2^4\mathbb{Q}_4^2 - \frac{1}{2}\mathbb{C}\mathbb{C}^\dagger, \\ w_1 &= -\frac{1}{4}(\xi_{1,0}^+\kappa_{1,0} + \kappa_{1,0}\xi_{1,0}^+) + \frac{3}{4}\mathbb{G}_1^4\mathbb{L}_2^1 - \frac{1}{4}\mathbb{L}_2^1\mathbb{G}_1^4 + \frac{1}{4}\mathbb{G}_2^3\mathbb{R}_3^4 + \frac{1}{4}\mathbb{R}_3^4\mathbb{G}_2^3 + \frac{1}{2}\mathbb{Q}_3^1\mathbb{C}^\dagger, \\ w_2 &= -\frac{1}{4}(\xi_{2,0}^+\kappa_{2,0} + \kappa_{2,0}\xi_{2,0}^+) + \frac{3i}{4}\mathbb{Q}_3^1\mathbb{R}_4^3 - \frac{i}{4}\mathbb{L}_2^1\mathbb{Q}_4^2 - \frac{i}{4}\mathbb{Q}_4^2\mathbb{L}_2^1 - \frac{i}{4}\mathbb{R}_4^3\mathbb{Q}_3^1 - \frac{i}{2}\mathbb{G}_2^3\mathbb{C}, \\ w_3 &= -\frac{1}{4}(\xi_{3,0}^+\kappa_{3,0} + \kappa_{3,0}\xi_{3,0}^+) - \frac{i}{4}\mathbb{Q}_3^1\mathbb{L}_1^2 + \frac{3i}{4}\mathbb{L}_1^2\mathbb{Q}_3^1 - \frac{i}{4}\mathbb{Q}_4^2\mathbb{R}_3^4 - \frac{i}{4}\mathbb{R}_3^4\mathbb{Q}_4^2 - \frac{i}{2}\mathbb{G}_1^4\mathbb{C}, \\ z_1 &= -\frac{1}{4}(\xi_{1,0}^-\kappa_{1,0} + \kappa_{1,0}\xi_{1,0}^-) - \frac{1}{4}\mathbb{Q}_4^1\mathbb{L}_1^2 + \frac{3}{4}\mathbb{L}_1^2\mathbb{Q}_4^1 + \frac{1}{4}\mathbb{Q}_3^2\mathbb{R}_4^3 + \frac{1}{4}\mathbb{R}_4^3\mathbb{Q}_3^2 + \frac{1}{2}\mathbb{G}_1^3\mathbb{C}, \\ z_2 &= -\frac{1}{4}(\xi_{2,0}^-\kappa_{2,0} + \kappa_{2,0}\xi_{2,0}^-) - \frac{i}{4}\mathbb{G}_1^3\mathbb{R}_4^3 + \frac{3i}{4}\mathbb{R}_3^4\mathbb{G}_1^3 - \frac{i}{4}\mathbb{G}_2^4\mathbb{L}_1^2 - \frac{i}{4}\mathbb{L}_1^2\mathbb{G}_2^4 - \frac{i}{2}\mathbb{Q}_3^2\mathbb{C}^\dagger, \\ z_3 &= -\frac{1}{4}(\xi_{3,0}^-\kappa_{3,0} + \kappa_{3,0}\xi_{3,0}^-) - \frac{i}{4}\mathbb{G}_2^4\mathbb{R}_4^3 - \frac{i}{4}\mathbb{R}_4^3\mathbb{G}_2^4 + \frac{3i}{4}\mathbb{G}_1^3\mathbb{L}_2^1 - \frac{i}{4}\mathbb{L}_2^1\mathbb{G}_1^3 - \frac{i}{2}\mathbb{Q}_4^1\mathbb{C}^\dagger.\end{aligned}$$

By knowing level-zero and level-one generators, one can recursively construct all higher-level generators by repeated use of the relations (C.1). We have performed extensive checks of the consistency of the (long) representation we find after Drinfeld's map with all relations (C.1). The explicit form of these generators is not particularly illuminating and we omit to report it here. The only interesting point is that it is not of a simple evaluation type, but rather more complicated. The Cartan generators at level one, for instance, are not represented by diagonal matrices, still with all relations (C.1) being satisfied.

The above reported Drinfeld's map is also used to derive the Yangian coproducts in Drinfeld's second realization by knowing the coproducts in Drinfeld's first realization (see previous section) and using the homomorphism property $\Delta(ab) = \Delta(a)\Delta(b)$. Same consistency we have found for coproducts and other Hopf algebra structures.

D A remark on long representations and Hirota equations

The large L asymptotic solution for the Y-system (see the Introduction) is most conveniently written in terms of certain transfer-matrices associated with the underlying symmetry group of the model [59]. In the context of the string sigma model the corresponding asymptotic solution was presented in [21]. In this solution the corresponding Y-functions are re-expressed in terms of suitable T-functions $T_{a,s}$. The latter must obey the so-called Hirota equations

$$T_{a,s}^+(u) T_{a,s}^-(u) = T_{a+1,s}(u) T_{a-1,s}(u) + T_{a,s+1}(u) T_{a,s-1}(u), \quad (\text{D.1})$$

where $f^\pm(u) = f(u \pm \frac{i}{g})$. These equations are formally solved by the Bazhanov-Reshetikhin (BR) determinant formula [60]

$$\begin{aligned} T_{a,s}(u) &= \det_{1 \leq i, j \leq s} T_{a+i-j, 1}(u + \frac{i}{g}(s+1-i-j)) = \\ &= \det_{1 \leq i, j \leq a} T_{1, s+i-j}(u + \frac{i}{g}(a+1-i-j)), \end{aligned} \quad (\text{D.2})$$

which expresses all $T_{a,s}$ either in terms of $T_{1,s}$ or $T_{a,1}$. In the large L -limit the T-function $T_{a,s}$ is supposed to coincide with (the eigenvalues of) the transfer matrix evaluated in the rectangular representation (a, s) of the centrally extended $\mathfrak{sl}(2|2)$. For the case without central extension, this fact has been proved in [61]. Here we will be concerned with centrally-extended $\mathfrak{sl}(2|2)$. Rather than developing a general theory, we will construct explicitly for one simple example the corresponding transfer matrices, *i.e. without* appealing to the BR formula, and show that the Hirota equations are indeed satisfied.

Our construction also allows one to better understand the role of long representations giving rise to a generic transfer matrix T_L . Namely, long representations for which the central charge q satisfies the shortening condition become reducible but indecomposable. Using the relationship between long and short representations, we show that the transfer matrix T_L admit a factorization into a tensor product of the transfer-matrices corresponding to short representations. We carry out this construction for our simplest 16-dimensional long representation with $q = \pm 1$. For these values, the corresponding transfer matrix $T_L(u)$

admits a factorisation

$$T_L^{16|1}(u) = T_{1,1}^{4|\frac{1}{2}}\left(u + \frac{i}{g}\right) T_{1,1}^{4|\frac{1}{2}}\left(u - \frac{i}{g}\right). \quad (\text{D.3})$$

Obviously, this is the left hand side of (D.1). Here we use the notation $T^{\dim|q}$ to indicate the dimension and the charge q of the corresponding representation. To write down the right hand side, we recall that the Hirota equations are invariant under a certain gauge symmetry. This symmetry can be used to set $T_{0,s} = 1$ for all s . Since in the large L limit $T_{a,0} = 1$ for all a , the Hirota equation takes the form¹⁷

$$T_L^{16|1}(u) = T_{1,1}^{4|\frac{1}{2}}\left(u + \frac{i}{g}\right) T_{1,1}^{4|\frac{1}{2}}\left(u - \frac{i}{g}\right) = T_{2,1}^{8|1}(u) + T_{1,2}^{8|1}(u). \quad (\text{D.4})$$

Here $T_{2,1}$ and $T_{1,2}$ are the transfer matrices corresponding to short 8-dimensional anti-symmetric and symmetric representations, respectively. These are precisely those which appear as the subrepresentation and the factor representation of the 16-dimensional long multiplet with $q = 1$. Obviously, eq. (D.4) represents the fusion mechanism.

All transfer matrix eigenvalues $T_{1,s}$ has been obtained with the help of the Algebraic Bethe Ansatz technique in [36]. Alternatively, they can be found with the help of the quantum characteristic function [38]. Here, we first need the eigenvalues which correspond to the $\mathfrak{su}(2)$ sector. They are given by

$$T_{\mathfrak{su}(2)}(u|\vec{v}) = 1 + \prod_{i=1}^{K^1} \frac{(x_i^- - x_i^-)(1 - x_i^- x_i^+) x^+}{(x_i^+ - x_i^-)(1 - x_i^+ x_i^+) x^-} \quad (\text{D.5})$$

$$- 2 \sum_{k=0}^{s-1} \prod_{i=1}^{K^1} \frac{x_i^+ - x_i^+}{x_i^+ - x_i^-} \sqrt{\frac{x_i^-}{x_i^+}} \left[1 - \frac{\frac{2ik}{g}}{u - v_i + \frac{i}{g}(s-1)} \right] + \sum_{m=\pm} \sum_{k=1}^{s-1} \prod_{i=1}^{K^1} \lambda_m(u, v_i, k).$$

This transfer matrix is associated with the canonically normalized S-matrix which is equal to unity on the $\mathfrak{su}(2)$ vacuum. We recall that the fundamental representation can be realized on the space of two bosonic variables w^1 and w^2 , and two fermionic variables θ^3 and θ^4 [44]. The $\mathfrak{su}(2)$ vacuum state is composed of a chain of w^1 's, i.e. $(w^1)^{\otimes K^1}$, where K^1 is the number of excited particles with rapidities v_i and kinematic variables $x_i^\pm = x(v_i \pm \frac{i}{g})$. In the formula (D.5) the quantities $x^\pm$ are the kinematic variables corresponding to the auxiliary bound-state particle with rapidity u : $x^\pm = x(u \pm \frac{i}{g}s)$. Finally, the quantities $\lambda_\pm$ are given by

$$\lambda_\pm(u, v_i, k) = \frac{1}{2} \left[1 - \frac{(x_i^- x^+ - 1)(x^+ - x_i^+)}{(x_i^- - x^+)(x^+ x_i^+ - 1)} + \frac{2ik}{g} \frac{x^+(x_i^- + x_i^+)}{(x_i^- - x^+)(x^+ x_i^+ - 1)} \right. \\ \left. \pm \frac{ix^+(x_i^- - x_i^+)}{(x_i^- - x^+)(x^+ x_i^+ - 1)} \sqrt{4 - \left(u - \frac{i(2k-a)}{g}\right)^2} \right]. \quad (\text{D.6})$$

¹⁷Notice that this can also be interpreted in the light of the discussion at the end of section 4. The transfer matrix is insensitive to a similarity transformation in the auxiliary space, therefore the unitary representation should really give the same result as the tensor (or co-) product of short ones. In fact, in the unitary representation the above splitting in the limit of $q \rightarrow 1$ is an obvious consequence of decomposability into the bound-state irreducible components.

By construction, we can identify $T_{1,s} \equiv T_{\mathfrak{su}(2)}(u | \vec{v})$.

On the other hand, the eigenvalue of the transfer matrix on the $\mathfrak{sl}(2)$ vacuum, i.e. on a fermionic state $(\theta^3)^{\otimes K^1}$, takes the form [36]

$$T_{\mathfrak{sl}(2)}(v | \vec{u}) = d(a, u, K^1) \left[(1+a) \prod_{i=1}^{K^1} \frac{x^- - x_i^-}{x^+ - x_i^-} + (a-1) \prod_{i=1}^{K^1} \frac{x^- - x_i^+}{x^+ - x_i^-} \frac{x_i^- - \frac{1}{x^+}}{x_i^+ - \frac{1}{x^+}} \right. \\ \left. - a \prod_{i=1}^{K^1} \frac{x^- - x_i^+}{x^+ - x_i^-} \sqrt{\frac{x_i^-}{x_i^+}} - a \prod_{i=1}^{K^1} \frac{x^- - x_i^-}{x^+ - x_i^-} \frac{x_i^- - \frac{1}{x^+}}{x_i^+ - \frac{1}{x^+}} \sqrt{\frac{x_i^+}{x_i^-}} \right], \quad (\text{D.7})$$

where we include the following normalization factor

$$d(a, u, K^1) = (-1)^a \prod_{i=1}^{K^1} \frac{x^+ - x_i^-}{x^- - x_i^+} \left(\frac{x_i^+}{x_i^-} \right)^{\frac{a}{2}} \prod_{n=1}^{a-1} \frac{x(u + \frac{2n-a}{g}i) - x_i^-}{x(u - \frac{2n-a}{g}i) - x_i^+}. \quad (\text{D.8})$$

The matrix $T_{a,1}$ in an anti-symmetric irrep is obtained from $T_{\mathfrak{sl}(2)}(v | \vec{u})$ through the replacement $T_{a,1} \equiv T_{\mathfrak{sl}(2)}|_{x^\pm \rightarrow x^\mp, x_i^\pm \rightarrow x_i^\mp}$.

Now we discuss the factorization of the transfer matrix which has an auxiliary space corresponding to the long 16-dimensional irrep with the central charge $q = 1$. Recall that the transfer matrix is defined as

$$T_L(u | \vec{v}) = \text{str}_0 \prod_{i>0}^{\leftarrow} \mathbb{S}_{0i}(u, v_i). \quad (\text{D.9})$$

We take the auxiliary space 0 to be the one corresponding to the long representation. This transfer matrix acts on the tensor product

$$V(v_1) \otimes \dots \otimes V(v_K). \quad (\text{D.10})$$

It is convenient to identify the long representation $V_0(u, q)$ as the tensor product of two short $V_a(u_a(u, q)) \otimes V_b(u_b(u, q))$. Under this identification we have $\mathbb{S}_{0i} = \mathbb{S}_{ai}\mathbb{S}_{bi}$ and this allows for the determination of the transfer matrix

$$T_L(u | \vec{v}) = \text{str}_{V_a \otimes V_b} \left(\prod_{i>0}^{\leftarrow} \mathbb{S}_{ai} \right) \left(\prod_{i>0}^{\leftarrow} \mathbb{S}_{bi} \right) = \text{str}_{V_a} \left(\prod_{i>0}^{\leftarrow} \mathbb{S}_{ai} \right) \text{str}_{V_b} \left(\prod_{i>0}^{\leftarrow} \mathbb{S}_{bi} \right) \quad (\text{D.11})$$

in terms of short representations. In the last formula the factorization property of the supertrace operation has been used. Obviously, the right hand side of the (D.11) coincides with the product $T_{1,1}(u_a | \vec{v}) T_{1,1}(u_b | \vec{v})$. This factorization happens for any q . For $q = 1$ it takes the form (D.3). Having proved eq. (D.3), one can substitute in (D.4) the expressions for $T_{1,2}$ and $T_{2,1}$ discussed above, and verify that the left and the right hand sides agree with each other.

Higher Hirota equations have an analogous origin. For instance, one has

$$T_L^{64|2} = T_{1,2}^{8|1} \left(u + \frac{i}{g} \right) T_{1,2}^{8|1} \left(u - \frac{i}{g} \right) = T_{2,2}^{16|2}(u) + T_{1,1}^{4|\frac{1}{2}}(u) T_{1,3}^{12|\frac{3}{2}}, \quad (\text{D.12})$$

$$T_L^{256|4} = T_{2,2}^{16|2} \left(u + \frac{i}{g} \right) T_{2,2}^{16|2} \left(u - \frac{i}{g} \right) = T_{1,2}^{8|1}(u) T_{3,2}^{16|3}(u) + T_{2,1}^{8|1}(u) T_{2,3}^{16|3}(u). \quad (\text{D.13})$$

On the left hand side we indicate the long representations which for generic q are irreducible and can be written as the tensor product of lower dimensional irreps. They do not have a description in terms of the Young tableaux and for special values of q become reducible but indecomposable. All the representations appearing on the right hand side of the Hirota equations, like, for instance $T_{2,2}^{16|2}$, have an associated Young tableaux.

Concrete transfer matrices in long representations can be obtained by using the S-matrix we have constructed in this paper. When trying to check the above with this concrete realization, however, one has to take into account an extra degree of freedom corresponding to the normalization of the T-functions. This normalization comes on top of the one chosen for the S-matrix, which we fix in the paper to be the canonical normalization. Although we have not studied the normalization issue in detail, we have checked in several cases that solving for transfer matrices of long representations from some of the Hirota equations and plugging them in the remaining equations leads to the consistency conditions on the transfer matrices of short representations which are indeed satisfied.

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