Published for SISSA by Springer

Received: *February 19, 2018*
Revised: *April 6, 2018*
Accepted: *April 16, 2018*
Published: *May 17, 2018*

Pure spinor superspace action for $D = 6$, $N = 1$ super-Yang-Mills theory

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Abstract: A Batalin-Vilkovisky action for $D = 6$, $N = 1$ super-Yang-Mills theory, including coupling to hypermultiplets, is given. The formalism involves pure spinor superfields. The geometric properties of the $D = 6$, $N = 1$ pure spinors (which differ from Cartan pure spinors) are examined. Unlike the situation for maximally supersymmetric models, the fields and antifields (including ghosts) of the vector multiplet reside in separate superfields. The formalism provides an off-shell superspace formulation for matter hypermultiplets, which in a traditional treatment are on-shell.

Keywords: Extended Supersymmetry, Superspaces

ArXiv ePrint: 1712.02284

https://doi.org/10.1007/JHEP05(2018)115

Contents

1 Introduction

Pure spinor superfields (see ref. [1] and references therein) have been used in the construction of actions for maximally supersymmetric theories [2–13]. It is there that the formalism, originating in superstring theory [14–17] and in the deformation theory for maximally supersymmetric super-Yang-Mills theory (SYM) and supergravity [18–25], has its greatest power. The superspace constraints, turned into a relation of the form "$Q\Psi + \ldots = 0$", where Q is nilpotent, become the equations of motion in a Batalin-Vilkovisky (BV) framework. Not only does this allow for a solution to the long-standing problem of off-shell formulation of maximally supersymmetric theories, the actions are also typically of a very simple kind. Generically, they turn out to be of finite and low order in the fields, even when the component field dynamics is non-polynomial.

Surprisingly little work has been done on pure spinor superfields for models with less than maximal supersymmetry. A classical description of $D = 6$, $N = 1$ super-Yang-Mills theory was given in ref. [26]. It was based on minimal pure spinor variables, which precludes the treatment of important issues like integration. It is the aim of the present work to take this construction to the level of an action principle. Such a formulation may, after gauge fixing, be used for quantum calculations, to be compared e.g. to the ones performed in harmonic superspace [27].

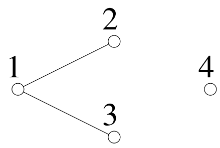

Figure 1. Labelling of the Dynkin diagram for $D_3 \times A_1$.

2 $D = 6$, $N = 1$ pure spinors

2.1 Minimal pure spinor variables

The important property for pure spinors in relation to supersymmetry is the constraint

$$(\lambda\gamma^a\lambda) = 0\,. \tag{2.1}$$

When the anticommutator of two fermionic covariant derivatives contains the torsion $T_{\alpha\beta}{}^a = 2\gamma^a_{\alpha\beta}$, this ensures that the BRST operator

$$q = \lambda^\alpha D_\alpha \tag{2.2}$$

is nilpotent, and (physical) fields may be defined as belonging to some cohomology of q. The pure spinor λ carries ghost number one.

The $D = 6$, $N = 1$ spinors transform under Spin$(1, 5) \times$ SU(2), the latter being the R-symmetry group. For Minkowski signature, this allows for (pseudo-)real 8-dimensional chiral spinor representations in the form of so called SU(2)-Majorana spinors. A convenient way to represent them is as two-component spinors with quaternionic entries. One then uses the isomorphism SL$(2;\mathbb{H}) \approx$ Spin$(1, 5)$, and the R-symmetry SU(2) acts by quaternionic multiplication with elements of unit norm from the right. We will use this language only occasionally, but instead work with matrices $(\gamma^a)_{\alpha\beta}$ or $(\gamma^a)^{\alpha\beta}$, $a = 1,\ldots,6$, acting on the respective chiral spinors, and $(\sigma_i)^\alpha{}_\beta$ or $(\sigma_i)_\alpha{}^\beta$, $i = 1, 2, 3$. In the quaternionic language, the latter are identified with right multiplication by $-e_i$, the imaginary quaternionic units. They satisfy $\sigma_i\sigma_j = -\delta_{ij} + \epsilon_{ijk}\sigma_k$. Some more spinor identities are collected in appendix A. The numbering for Dynkin labels is that of figure 1, where an upper spinor index is represented by (001)(1).

The symmetry properties of spinor bilinears are:

$$\begin{array}{rll} \text{symmetric:} & (\gamma^a)_{\alpha\beta} & (\gamma^{abc}\sigma_i)_{\alpha\beta} \\ \text{antisymmetric:} & (\gamma^a\sigma_i)_{\alpha\beta} & (\gamma^{abc})_{\alpha\beta} \end{array} \tag{2.3}$$

A bosonic spinor λ^α in $(\mathbf{4}, \mathbf{2}) = (001)(1)$, subject to the pure spinor constraint (2.1), will only yield the single representation $(00n)(n)$ in its n'th power. Counting the dimensions of these representations immediately gives the partition function for the pure spinor (cf. refs. [17, 28–31])

$$Z(t) = \sum_{n=0}^{\infty}(n+1)\binom{n+3}{3} = \frac{1+3t}{(1-t)^5} = \frac{1-6t^2+8t^3-3t^4}{(1-t)^8}\,. \tag{2.4}$$

A refined partition function, counting the actual representation content at each level, is given by

$$\mathcal{Z}(t)=\sum_{n=0}^{\infty}(00n)(n)t^n=\mathcal{Z}_0(t)\otimes\left[(000)(0)-(100)(0)t^2+(010)(1)t^3-(000)(2)t^4\right]\,, \quad (2.5)$$

where $\mathcal{Z}_0$ is the partition function for an unconstrained spinor,

$$\mathcal{Z}_0(t)=\sum_{n=0}^{\infty}\otimes_s^n(001)(1)t^n\,. \quad (2.6)$$

As usual, the second factor in eq. (2.5) encodes the zero mode cohomology of the BRST operator q, which will be described in section 3.

An attempt to solve the pure spinor constraint immediately shows that complex pure spinors are needed. The manifold of pure spinors is a 5-dimensional complex manifold. The dimensionality is reflected in the power of the denominator of eq. (2.4). If one considers a complex spinor as a bifundamental λ^{Aa} of $\mathrm{SU}(4)\times\mathrm{SU}(2)$, the pure spinor constraint takes the form $\epsilon_{ab}\lambda^{Aa}\lambda^{Bb}=0$. Obviously, any spinor of the form $\lambda=\big(\ell^A,0\big)$ is pure,[1] and all solutions can be obtained from this solution by transformations in $\mathrm{SU}(4)\times\mathrm{SU}(2)$. This tells us that the space of pure spinors is $\mathbb{C}^4\times\mathbb{C}P^1$.

The conjugate variable to λ^α, $\omega_\alpha=\frac{\partial}{\partial\lambda^\alpha}$ is not well defined, since it does not preserve the pure spinor constraint. However, the operators

$$N=(\lambda\omega)\,,\quad N^{ab}=(\lambda\gamma^{ab}\omega)\,,\quad N^i=(\lambda\sigma^i\omega) \quad (2.7)$$

are well defined.

2.2 Non-minimal variables and integration

For several reasons, it is necessary to include non-minimal variables [16], a bosonic variable $\bar\lambda_\alpha$ and the "fermionic" $r_\alpha=d\bar\lambda_\alpha$. One reason is the construction of a non-degenerate integration measure, another, as we will see, is the need for operators with negative ghost number. The BRST operator is modified to

$$Q=q+\bar\partial=\lambda^\alpha D_\alpha+d\bar\lambda_\alpha\frac{\partial}{\partial\bar\lambda_\alpha}\,. \quad (2.8)$$

$\bar\lambda$ can be considered as the complex conjugate of λ. It is pure, and differentiation gives $(\bar\lambda\gamma^a d\bar\lambda)=0$.

If superfields are functions of the non-minimal variables x^a, θ^α, λ^α, $\bar\lambda_\alpha$ and $d\bar\lambda_\alpha$, they are forms with antiholomorphic indices on complex pure spinor space. A tentative integration can then be taken as

$$\int[dZ]\phi\sim\int d^6x\,d^8\theta\int\Omega\wedge\phi\,, \quad (2.9)$$

if it is possible to find a holomorphic 5-form Ω.

[1]This amounts to the statement that any Spin(6) spinor is pure, in the sense of Cartan.

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From the description of pure spinor space as $\mathbb{C}^4 \times \mathbb{C}P^1$, it is clear that there is not only one, but three holomorphic 5-forms, which can be written as $d^4y\, z^p dz$, $p = 0, 1, 2$, where y parametrises $\mathbb{C}^4$ and z $\mathbb{C}P^1$. They transform as a triplet under R-symmetry. We will in fact use the full triplet, and have a "triplet integration". It will become clear, when actions are formed in section 4, that this is necessary in order to maintain covariance, and absorb transformations of diverse fields.

For our purposes, and a closer correspondence with the cohomology of section 3, we will write down an expression for the holomorphic 5-forms Ω_i in a fully covariant way. They are

$$\Omega_i = (\lambda\bar{\lambda})^{-1}(\bar{\lambda}\sigma^j d\lambda)(d\lambda\gamma^a\sigma_j d\lambda)(d\lambda\gamma_a\sigma_i d\lambda)\,. \tag{2.10}$$

Although $\bar{\lambda}$ is used to form a covariant expression, it can be checked that $\bar{\partial}\Omega = 0$. In addition, the forms satisfy

$$(\sigma^i\lambda)^\alpha\Omega_i = 0\,. \tag{2.11}$$

Except for the presence of a triple of holomorphic top-forms instead of single one, this mirrors closely the construction for $D = 10$ pure spinor superfields. As we will see in the following section, the integration measure is directly connected to the highest cohomology of a scalar pure spinor superfield, which is the present case will be the triplet of auxiliary fields H_i in the super-Yang-Mills multiplet.

The geometry corresponding to the integration at hand, with a volume form $\mathrm{Vol} = \Omega_i \wedge \bar{\Omega}_i$, is not the one inherited by embedding pure spinor space in flat spinor space. The latter one would scale like $d\lambda^5 d\bar{\lambda}^5$, while the actual volume scales like $\mathrm{Vol} \sim \lambda^{-1}\bar{\lambda}^{-1}d\lambda^5 d\bar{\lambda}^5$. This is quite similar to the 10-dimensional situation [9]. As usual, integrals have to be regularised by a factor $\exp\{Q,\chi\}$. A convenient choice is $\chi = -(\bar{\lambda}\theta)$, giving

$$e^{\{Q,\chi\}} = e^{-(\lambda\bar{\lambda})-(d\bar{\lambda}\theta)}\,, \tag{2.12}$$

which both regulates the integral over pure spinor space at infinity and saturates the fermionic integral.

Conider the behaviour of an integral at $\lambda = 0$. Define $\varrho = \sqrt{\lambda\bar{\lambda}}$. The radial integration contains $\int d\varrho\varrho^9$. The holomorphic top form behaves as $\Omega_i \sim \lambda^{-1}$. Take an integrand $A_i \sim \lambda(\lambda\bar{\lambda})^p$. Then, the radial integral behaves as $\int d\varrho\varrho^{9+2p}$, and converges at $\varrho = 0$ if $p > -5$. This is minus the complex dimension of pure spinor space, which is a generic feature.

As always in pure spinor superfield theory, the fields must be regular enough at the singular point $\lambda = 0$ of pure spinor space. Too non-singular behaviour corresponds to non-normalisable modes. This is also desirable, since inclusion of too singular functions destroys the cohomology in the non-minimal picture. This is a universal feature, which is shared by the present model. We refer too ref. [16] for details. If e.g. gauge variations of the fields, in the form of BRST variations or shift symmetries (to be discussed later) are considered, they must obey the corresponding regularity condition.

3 Cohomology and supermultiplets

In this section we will construct pure spinor superfields containing the off-shell SYM multiplet and its current multiplet, and the on-shell hypermultiplet.

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gh#=	1	0	-1	-2
dim = 0	(000)(0)			
$\frac{1}{2}$	•	•		
1	•	(100)(0)	•	
$\frac{3}{2}$	•	(001)(1)	•	•
2	•	(000)(2)	•	•
$\frac{5}{2}$	•	•	•	•
3	•	•	•	•

Table 1. The zero-mode cohomology of a scalar superfield.

3.1 The vector multiplet

The standard superspace treatment of supersymmetric gauge theory formulates SYM as gauge theory on superspace. A connection 1-form A is decomposed as $A = E^a A_a(x,\theta) + E^\alpha A_\alpha(x,\theta)$. The dimension 1 part of the field strength $F_{\alpha\beta}$ is set to zero. This contains two parts: a vector $(\gamma^a)^{\alpha\beta}F_{\alpha\beta}$ and a triplet of selfdual 3-forms $(\gamma^{abc}\sigma_i)^{\alpha\beta}F_{\alpha\beta}$ in (020)(2). As usual, setting the vector to 0 is the conventional constraint, expressing the superfield A_a, and thereby the entire field content, in terms of the superfield A_α.

One can now work with A_α alone. Consider a scalar pure spinor superfield $\Psi(x,\theta,\lambda)$ of ghost number 1. Its expansion in λ contains the physical fields as $\lambda^\alpha A_\alpha$. The (linearised) constraint on F in (020)(2) now arises as the condition $q\Psi = 0$. In addition, a transformation $\delta\Psi = q\Lambda$ gives a gauge transformation, and physical fields, modulo gauge transformations, arise as cohomology of q. It is well known that the relation $F_{\alpha\beta} = 0$ does not imply the field equations, but leaves the SYM fields off-shell. Calculating the zero-mode cohomology indeed gives the SYM multiplet, including the triplet H_i of auxiliary fields, as shown in table 1. In this and the following tables, the representations and quantum numbers (dimension, ghost number) of the component fields are listed.

Unlike the situation in $D = 10$, where the SYM multiplet is an on-shell multiplet, there is no cohomology at negative ghost numbers, which also means that there is no room for differential constraints (equations of motion) on the physical fields. The equations of motion do not follow from $Q\Psi = 0$. Instead we will need some relation that effectively implies the vanishing of the auxiliary fields. This will amount to finding an operator $\hat{H}_i$ of ghost number -1 and dimension 2, the rôle of which is to map the auxiliary fields to the "beginning" of the superfield, and postulate $\hat{H}_i\Psi = 0$. Such an operator will be constructed in section 4.

3.2 The current (antifield) multiplet

The scalar superfield of the previous subsection contains the ghost and the physical off-shell SYM multiplet. In order to write a Batalin-Vilkovisky action (section 4), also the antifields for the fields and ghost are needed. They will come in a field that is conjugate to Ψ in the

gh#=	-1	-2	-3	-4
dim=2	(000)(2)			
$\frac{5}{2}$	(001)(1)	•		
3	(100)(1)	•	•	
$\frac{7}{2}$	•	•	•	•
4	•	(000)(0)	•	•
$\frac{9}{2}$	•	•	•	•
5	•	•	•	•

Table 2. The zero-mode cohomology of the triplet antifield.

BV sense. This differs from the situation in maximally supersymmetric SYM, where the scalar superfield is self-conjugate, and $Q\Psi = 0$ gives the equations of motion.

The antifield should have the auxiliary fields H_i as its lowest component, and must therefore itself be a triplet Ψ_i^* with ghost number -1 and dimension 2. In order for a non-scalar superfield to carry a cohomology which is not a product of its representation and the scalar cohomology, it has to be subject to some condition. This has been encountered in a number of situations [2, 3, 5, 6, 8, 13], and was named "shift" symmetry in ref. [8]. The appropriate condition is to consider the equivalence class

$$\Psi_i^* \approx \Psi_i^* + (\lambda\sigma_i\zeta) \tag{3.1}$$

for all possible spinor superfields ζ. This will have consequences for the cohomology. An immediate one is that the zero-mode cohomology will contain $(\theta\sigma_i\chi^*)$, where χ^* is the antifield for the physical spinor (acting with Q gives precisely a shift as in eq. (3.1)). A complete calculation of the zero-mode cohomology yields table 2, and the correct structure as the mirror of the fields in table 1 is reproduced.

It now becomes clear that the operator $\hat{H}_i$, needed to put the vector multiplet on shell, should be an operator that maps the scalar field Ψ to a triplet field of the type described in the present subsection.

We also note that the shift symmetry can be implemented in some action, if the triplet integration and the antifield are used together; an expression $\int [dZ]_i \Psi_i^* \ldots$ will automatically imply it, since, as noted in section 2.2 (eq. (2.11)), $[dZ]_i(\sigma_i\lambda)^\alpha \ldots = 0$.

3.3 The hypermultiplet

Finally, we give the superfield corresponding to the hypermultiplet. There are no ghosts, so the superfield should have as its lowest component the scalars of dimension 1 and ghost number 0. The four scalars transform as $(2,2)$ under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, where the second factor is an additional SU(2) R-symmetry leaving the vector multiplet inert. It is convenient to collect them in a quaternion ϕ, where the "old" $\mathrm{SU}(2)_L$ acts by left multiplication and the new one by right multiplication by unit quaternions. We thus introduce a superfield

gh#=	0	-1	-2	-3
dim = 1	(000)(1)(1)			
$\frac{3}{2}$	(001)(0)(1)	•		
2	•	•	•	
$\frac{5}{2}$	•	(010)(0)(1)	•	•
3	•	(000)(1)(1)	•	•
$\frac{7}{2}$	•	•	•	•
4	•	•	•	•

Table 3. The zero-mode cohomology of the hypermultiplet field.

$\Phi \in \mathbb{H}$ with dimension 1 and ghost number 0. It enjoys a shift symmetry

$$\Phi \approx \Phi + \lambda^{\dagger}\rho\,, \tag{3.2}$$

where now λ is written in the quaternionic 2-component notation described in section 2.1. The parameter ρ in the shift term is a spinor transforming under the new R-symmetry from the right, but inert under the old one. It implies the occurrence of such a spinor in the zero-mode cohomology. The complete zero-mode cohomology is displayed in table 3.

The field is self-conjugate, in that it contains both the fields of the hypermultiplet and their antifields. The presence of zero-mode cohomology at ghost number -1 signals, as usually, the presence of equations of motion. We see that the representations match the ones of the equations of motion of the spinor and scalar components. The multiplet is an on-shell multiplet in the traditional sense, and $Q\Phi = 0$ implies the component equations of motion. This is in complete agreement with a traditional superspace formulation of the hypermultiplet, where the scalar multiplet consists of the ghost number 0 part of Φ.

4 Batalin-Vilkovisky actions

With the description of the fields from section 3, we are now ready to write down BV actions. We will begin with the linearised theory, and then give the full interacting theory in section 4.4. A necessary ingredient will be certain operators, which are first given in section 4.1.

The BV action will of course be a scalar. The consistency condition is the BV master equation $(S, S) = 0$. Some care has to be taken to define the antibracket $(\cdot,\cdot)$, especially since the "Lagrangian" carries an SU(2) index. With the field Ψ and its antifield Ψ^*_i, the antibracket between $A = \int [dZ]_i a_i$ and $B = \int [dZ]_i b_i$ is

$$(A, B)_{\text{vector}} = \int \left(a_i \frac{\overleftarrow{\partial}}{\partial \Psi} [dZ]_j \frac{\overrightarrow{\partial}}{\partial \Psi^*_j} b_i - a_i \frac{\overleftarrow{\partial}}{\partial \Psi^*_j} [dZ]_j \frac{\overrightarrow{\partial}}{\partial \Psi} b_i \right) . \tag{4.1}$$

For the self-conjugate matter field Φ,

$$(A, B)_{\text{matter}} = \int a_i \frac{\overleftarrow{\partial}}{\partial \Phi} e_j [dZ]_j \frac{\overrightarrow{\partial}}{\partial \Phi^{\dagger}} b_i \,. \tag{4.2}$$

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4.1 Some useful operators

It was already observed that, in order to write the equations of motion for the physical fields (in the cohomology of Q), a triplet operator $\hat{H}_i$ with dimension 2 and ghost number -1 is needed. The rôle of the operator is essentially to create a new (triplet) pure spinor superfield which in the minimal picture would have the auxiliary field H_i as its λ- and θ-independent component. In ref. [8], similar operators were formed (in the context of maximally supersymmetric SYM) corresponding to various physical fields).

The first observation is that there are other nilpotent operators than Q. Also the expressions $q_i = (\lambda\sigma_i D)$ are nilpotent modulo the pure spinor constraint. They can be extended to

$$Q_i = (\lambda\sigma_i D) + \left(d\bar{\lambda}\sigma_i\frac{\partial}{\partial\bar{\lambda}}\right) \tag{4.3}$$

in order to act non-trivially in the non-minimal sector. Then, $\{Q, Q_i\} = 0$, $\{Q_i, Q_j\} = 0$.

A commonly used type of operator in pure spinor field (and string) theory is the b-operator. It has the property

$$\{Q, b\} = -\Box\,, \tag{4.4}$$

and clearly has ghost number -1 and dimension 2. An explicit form of b is

$$\begin{aligned} b = {} & \frac{1}{2}(\lambda\bar{\lambda})^{-1}(\bar{\lambda}\gamma^a D)\partial_a \\ & -\frac{1}{4}(\lambda\bar{\lambda})^{-2}(\bar{\lambda}\gamma^a\sigma^i d\bar{\lambda})\left(N_i\partial_a - \frac{1}{8}(D\gamma_a\sigma_i D)\right) \\ & -\frac{1}{16}(\lambda\bar{\lambda})^{-2}(\bar{\lambda}\gamma^{abc}d\bar{\lambda})\left(N_{ab}\partial_c - \frac{1}{24}(D\gamma_{abc}D)\right) \\ & -\frac{1}{32}(\lambda\bar{\lambda})^{-3}((\bar{\lambda}d\bar{\lambda}^2)^a\gamma^b D)N_{ab} - \frac{1}{16}(\lambda\bar{\lambda})^{-3}((\bar{\lambda}d\bar{\lambda}^2)^i D)N_i \\ & -\frac{1}{64}(\lambda\bar{\lambda})^{-4}(\bar{\lambda}d\bar{\lambda}^3)^{abi}N_{ab}N_i - \frac{1}{64}(\lambda\bar{\lambda})^{-4}(\bar{\lambda}d\bar{\lambda}^3)^{ij}N_iN_j \end{aligned} \tag{4.5}$$

(see appendix A for notation for antisymmetric products of spinors).

The operators Q_i and b will not be used further in the present paper, but will be of use when gauge fixing is considered. We turn to the construction of the operator $\hat{H}_i$. The precise criterion on $\hat{H}_i$ is that $\{Q, \hat{H}_i\} = 0$ modulo the shift transformations of eq. (3.1). This is satisfied by the operators

$$\begin{aligned} \hat{H}_i = {} & (\lambda\bar{\lambda})^{-2}(\bar{\lambda}\gamma^a\sigma_i d\bar{\lambda})\partial_a - \frac{1}{2}(\lambda\bar{\lambda})^{-3}(\bar{\lambda}d\bar{\lambda}^2)_i^{\alpha}D_\alpha \\ & + (\lambda\bar{\lambda})^{-4}\left[\frac{1}{4}(\bar{\lambda}d\bar{\lambda}^3)_{ij}N^j + \frac{1}{8}(\bar{\lambda}d\bar{\lambda}^3)_{abi}N^{ab}\right]. \end{aligned} \tag{4.6}$$

Note that the minimal representative for the auxiliary field cohomology is at $\Psi \sim \lambda\theta^3$, a component yielding a non-vanishing regularised integral $\int \Omega_i \wedge \Psi \sim H_i$. It would seem that $\hat{H}_i$ should contain three spinorial derivatives.[2] Instead it contains terms with $Dd\bar{\lambda}^2$ and

[2] In ref. [26] such an operator was constructed using minimal pure spinor variables. It had the drawback of not being well-defined outside cohomology.

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$d\bar{\lambda}^3$, which in the integral with regularisation according to eq. (2.12) can be converted into fermionic derivatives. The expression (4.6), being linear in derivatives, follows the pattern of similar operators constructed in ref. [8].

The linearised equations of motion for Ψ, already subject to $Q\Psi = 0$, can now be written as $\hat{H}_i\Psi = 0$.

4.2 SYM action

We are now ready to write down the BV action for the SYM multiplet in Ψ and its antifield Ψ^*_i. The linearised action is

$$S_{0,\text{vector}} = \int [dZ]_i \text{Tr}\left(\Psi^*_i Q\Psi + \frac{1}{2}\Psi\hat{H}_i\Psi\right) . \tag{4.7}$$

As mentioned earlier, the use of the triplet integration is consistent with the shift symmetry of the antifield, and necessary to implement it. It is somewhat easier to check the master equation by repeated variations on the field and antifield than directly in the form $(S, S) = 0$. The equations of motion following from the action are

$$\begin{aligned} Q\Psi &= 0\,, \\ Q\Psi^*_i + \hat{H}_i\Psi &= 0\,. \end{aligned} \tag{4.8}$$

These equations follow from variation of the action (in the case of Ψ^*_i one also needs to interpret the equation as modulo shifts). The two equations are also directly obtained as $(S, \Psi) = 0$ and $(S, \Psi^*) = 0$, respectively. If the second equation is seen as a condition on Ψ (effectively, the vanishing of the auxiliary fields), the first term is trivial in the cohomology. The consistency, i.e., the master equation, amounts to the nilpotency of the operator

$$\mathcal{Q} = \begin{pmatrix} Q & 0 \\ \hat{H}_i & Q \end{pmatrix} , \tag{4.9}$$

acting on the vector $(\Psi, \Psi^*_i)^t$, again modulo shift symmetry in the second entry.

4.3 Matter action

The matter field is self-conjugate, $Q\Phi = 0$ puts the component fields on shell, and it is straightforward to write down an action. Suppressing indices for the representation of Φ under the gauge group,

$$S_{0,\text{matter}} = \frac{1}{2}\int [dZ]_i \Phi^\dagger e_i Q\Phi\,. \tag{4.10}$$

Here, we use the quaternionic formalism, with e_i being the imaginary quaternionic units, as explained in section 2.1. Note that the shift transformation $\delta_\rho\Phi = \lambda^\dagger\rho$ leads to a change in the action

$$\delta_\rho S_{0,\text{matter}} = \frac{1}{2}\int [dZ]_i(\Phi^\dagger e_i\lambda^\dagger Q\rho + \rho^\dagger\lambda e_i Q\Phi) = 0\,, \tag{4.11}$$

where both terms vanish due to the property (2.11) of the integration measure.

4.4 Interactions

Interactions are introduced by "covariantisation" of the linearised action, so that the "field strength" $Q\Psi$ is replaced by $Q\Psi + \Psi^2$. At the same time, $Q\Phi \to (Q + \Psi\cdot)\Phi$ (the dot denoting action of the gauge algebra in the representation of Φ). This gives the complete action for SYM coupled to matter:

$$S = \int [dZ_i] \mathrm{Tr} \left(\Psi_i^*(Q\Psi + \Psi^2) + \frac{1}{2}\Psi \hat{H}_i \Psi \right) + \frac{1}{2} \int [dZ]_i \Phi^\dagger e_i (Q + \Psi\cdot)\Phi \,. \tag{4.12}$$

Note that although the component interactions, both between gauge fields and between scalars in the matter multiplets, include quartic terms, the present formalism only gives 3-point couplings. The quartic terms will arise when the superfield identities are solved, i.e., when non-physical components are eliminated. This is a typical feature of the pure spinor superfield formalism, and the present behaviour mirrors that of maximally supersymmetric SYM. Even more drastic reduction of the order of the interactions are seen in the actions for BLG and ABJM models [2–4], in the Born-Infeld deformation of $D = 10$ SYM [8, 11, 12], and in $D = 11$ supergravity [5, 6].

The equations of motion following from the action (4.12) are

$$\begin{aligned}
(S, \Psi) &= Q\Psi + \Psi^2 = 0 \,, \\
(S, \Psi_i^*) &= Q\Psi_i^* + [\Psi, \Psi_i^*] + \hat{H}_i \Psi - \frac{1}{2}\Phi^\dagger \circ e_i \Phi = 0 \,, \\
(S, \Phi) &= (Q + \Psi\cdot)\Phi = 0 \,,
\end{aligned} \tag{4.13}$$

where "$\circ$" is shorthand for formation of the adjoint of the gauge algebra, and $[\cdot,\cdot]$ denotes adjoint action. Note that gauge field interactions are introduced by deformation (covariantisation) of the cohomology, while the matter current back-reacts on the SYM fields through a deformation of the condition on the auxiliary fields (the current multiplet).

When checking that the master equation $(S, S) = 0$ is satisfied, one finds that it relies on $\{Q, \hat{H}_i\} = 0$, but also on the distributivity of $\hat{H}_i$, $\hat{H}_i(\Psi^2) = \hat{H}_i\Psi\Psi - \Psi\hat{H}_i\Psi$. This holds thanks to the linearity of $\hat{H}_i$ in derivatives.

Concerning other modifications, it should be straightforward to apply the method of ref. [8] in order to write possible higher-derivative interaction terms. Then there is no need to deform the gauge transformations, which should mean that the first equation in (4.13) can be left unchanged, i.e., additional terms do not contain the antifield. All new interaction then comes through modification of the on-shell condition $\hat{H}_i\Psi \sim \text{trivial} + J_i$.

5 Conclusions

We have presented a classical Batalin-Vilkovisky action for chiral $D = 6$ SYM theory. The gauge multiplet is not maximally supersymmetric, and consequently its equations of motion are not implied by the cohomology of the pure spinor superspace BRST operator. The hypermultiplet, on the other hand, is maximally supersymmetric, and supersymmetric action requires this kind of action. The construction may stand model for superspace formulations of other half-maximal models, like e.g. $D = 10$, $N = 1$ supergravity.

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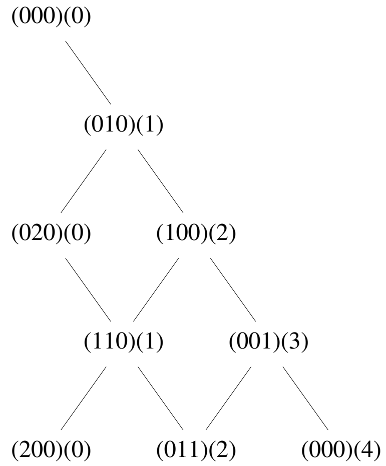

Figure 2. Irreducible representations in antisymmetric products of spinors.

The quantum theory has not been addressed. It seems likely that models of the present type could serve as an arena for the investigation of a complete and systematic gauge fixing procedure for theories formulated on pure spinor superspace. At the present level of understanding, the constraint "$b\Psi = 0$" reproduces Lorentz gauge and other appropriate conditions on antifields, but how it is to be incorporated in a systematic way in the BV formalism, using a gauge fixing fermion, remains to be investigated. Simplifications may occur when fields and antifields are separated. This will be the subject of future work.

A Some spinor relations

When constructing the operators of negative ghost number, completely antisymmetric products of spinors are needed. All terms in b and $\hat{H}_i$ contain $\bar{\lambda}_{[\alpha_1}d\bar{\lambda}_{\alpha_2}\dots d\bar{\lambda}_{\alpha_p]}$. The complete list of antisymmetrised spinors up to fourth order is given in figure 2.

The general antisymmetric bilinear Fierz identity, conveniently expressed with the help of a fermionic spinor s_α, is

$$s_\alpha s_\beta = \frac{1}{8}(\gamma_a\sigma_i)_{\alpha\beta}(s\gamma^a\sigma^i s) + \frac{1}{96}(\gamma_{abc})_{\alpha\beta}(s\gamma^{abc}s)\,. \tag{A.1}$$

expressing $\wedge^2(010)(1) = (100)(2) \oplus (020)(0)$. At third order, $\wedge^3(010)(1) = (110)(1) \oplus (001)(3)$, represented by

$$\begin{aligned}(s^3)^a_\alpha &= (\sigma_i s)_\alpha(s\gamma^a\sigma^i s)\,,\\ (s^3)^{i\alpha} &= (\gamma_a s)^\alpha(s\gamma^a\sigma^i s)\,.\end{aligned} \tag{A.2}$$

One also has the identity

$$(\gamma_{bc}s)_\alpha(s\gamma^{abc}s) = -4(\sigma_i s)_\alpha(s\gamma^a\sigma^i s)\,. \tag{A.3}$$

At fourth order, $\wedge^4(010)(1) = (200)(0) \oplus (011)(2) \oplus (000)(4)$. They can be constructed from the cubic or quadratic expressions as

$$\begin{aligned}(s^4)^{ab} &= (s\gamma^a(s^3)^b) = (s\gamma^a\sigma_i s)(s\gamma^b\sigma^i s)\,,\\ (s^4)^{abi} &= (s\gamma^a\sigma^i(s^3)^b) = -\epsilon^{ijk}(s\gamma^a\sigma_j s)(s\gamma^b\sigma_k s)\,,\\ (s^4)^{ij} &= (s\sigma^i(s^3)^j) = (s\gamma_a\sigma^i s)(s\gamma^a\sigma^j s)\,.\end{aligned} \tag{A.4}$$

A dependent expression for (011)(2) is

$$(s\gamma^{ab}(s^3)^i) = (s\gamma^{abc}s)(s\gamma_c\sigma^i s) = -2(s^4)^{abi}\,. \tag{A.5}$$

Since the dimension of the spinor module is 8, higher antisymmetric products follow. The construction of the measure relies on

$$\Omega_i = (\lambda\bar\lambda)^{-1}(\bar\lambda(d\lambda^5)_i) \tag{A.6}$$

with $(d\lambda^5)_i^\alpha = (\sigma^j d\lambda)^\alpha(d\lambda\gamma^a\sigma_j d\lambda)(d\lambda\gamma_a\sigma_i d\lambda)$ in (001)(3).

Acknowledgments

The author would like to thank Evgeny Ivanov for a discussion inspiring me to continue on an old project, and Bengt E.W. Nilsson for collaboration on that old project [26]. I would also like to thank Evgeny Ivanov and Ioseph Buchbinder for a discussion on gauge fixing.

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