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Expanding Einstein-Yang-Mills by Yang-Mills in CHY frame

Fei Teng^a and Bo Feng^{b,c}

^a*Department of Physics and Astronomy, University of Utah,
115 South 1400 East, Salt Lake City, UT 84112, U.S.A.*

^b*Zhejiang Institute of Modern Physics, Department of Physics, Zhejiang University,
No. 38 Zheda Road, Hangzhou 310027, P.R. China*

^c*Center of Mathematical Science, Zhejiang University,
No. 38 Zheda Road, Hangzhou 310027, P.R. China*

E-mail: Fei.Teng@utah.edu, fengbo@zju.edu.cn

ABSTRACT: Using the Cachazo-He-Yuan (CHY) formalism, we prove a recursive expansion of tree level single trace Einstein-Yang-Mills (EYM) amplitudes with an arbitrary number of gluons and gravitons, which is valid for general spacetime dimensions and any helicity configurations. The recursion is written in terms of fewer-graviton EYM amplitudes and pure Yang-Mills (YM) amplitudes, which can be further carried out until we reach an expansion in terms of pure YM amplitudes in Kleiss-Kuijf (KK) basis. Our expansion then generates naturally a spanning tree structure rooted on gluons whose vertices are gravitons. We further propose a set of graph theoretical rules based on spanning trees that evaluate directly the pure YM expansion coefficients.

KEYWORDS: Scattering Amplitudes, Gauge Symmetry

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1 Introduction

Recently, non-trivial relations between Einstein-Yang-Mills (EYM) and pure Yang-Mills (YM) amplitudes¹ have been studied in several papers [1–8]. These works have more or less the same goal: to find the relations that expand the EYM amplitude with n gluons and m gravitons in terms of a linear sum of $(n + m)$ -point pure YM amplitudes, although it is studied from different points of view. The starting point of our paper is the Cachazo-He-Yuan (CHY) formulation [9–14], which is the same as most of the above mentioned discussions. The key idea in this pursuit is to use various identities to reformulate the CHY integrand for EYM into an appropriate form. In particular, the cross-ratio identity and other off-shell identities of integrands [15, 16] play a very crucial rule.² With the help of these techniques, the explicit expansions of EYM amplitudes with up to three gravitons have been provided in [2]. Based on these explicit results, an recursive expansion for EYM amplitudes with an arbitrary number of gravitons has been conjectured in [8], using the gauge invariance principle advocated in [18–21].

One of the main subjects in this paper is to investigate directly in the CHY frame the aforementioned recursive structure proposed in [8]. Especially, we want to see how this

¹Unless otherwise noted, we only consider tree level single trace color ordered EYM amplitudes.

²See appendix of [17] for summary of these identities.

structure appears at the level of CHY integrands. Intuitively, we believe that the CHY formalism sets one of the best frameworks to understand such a recursive structure. The reason is that the information related to graviton polarizations is packed into a Pfaffian, which can be recursively evaluated by the Laplace expansion: for a $2n \times 2n$ antisymmetric matrix $M = (m_{ij})$, its Pfaffian can be calculated by

$$\text{Pf}(M) = \sum_{j=1, j \neq i}^{2n} (-1)^{i+j+1+\theta(i-j)} m_{ij} \text{Pf}(M_{ij}^{ij}), \quad (1.1)$$

where $\theta(i-j)$ is a step function and M_{ij}^{ij} denotes the matrix obtained from M by deleting the i -th and j -th row and column. As we will see later in section 3 and 4, such a Laplace expansion of the m -graviton EYM integrand always contains an $(m-1)$ -graviton integrand, plus other algebraic minors. Then if we can prove that those additional algebraic minors reduce to a linear combination of pure YM and EYM integrands with $m-2$ gravitons at most, we essentially derive a recursive structure! The proof requires an elegant arrangement of the Laplace expansion for Pfaffian, and a proper use of various on-shell and off-shell identities, to be spelled out later. This technique can possibly be generalized to a variety of theories constructed in, for example, [13, 14, 22].

Knowing the recursive relation, we can perform the expansion of generic EYM amplitudes level by level in a well-controlled way. Eventually, this process leads to an expansion in terms of pure YM Kleiss-Kuijf (KK) basis [23]. Although the final result involves a huge number of terms, it is actually very well-organized and enjoys a very elegant spanning tree structure. We find that our recursive relation derives an algorithm that evaluates directly the spanning trees, yielding the expansion coefficients we want. Spanning tree structures have appeared in the literature before for EYM and gravity in four dimensional spacetime with the MHV helicity configuration [5, 24, 25]. In contrary, the algorithm we propose here works in arbitrary spacetime dimensions for EYM amplitudes with general helicity configurations. Since EYM amplitudes can be realized as a double copy of YM and YM-scalar, even at multitrace levels [14, 26, 27], a very promising application of this technique will be the direct evaluation of Bern-Carrasco-Johansson (BCJ) numerators [28, 29] for YM and YM-scalar.

The structure of this paper is the following. In section 2, we give the essential background knowledge of the CHY formalism. The emphasis is put on the definition of the tree level YM and single trace EYM integrand. Then in section 3 we give several explicit examples, up to four gravitons, on how to expand the EYM integrands, such that one can clearly observe the pattern and summarize it into a recursive relation for generic cases, which is then proved by induction in section 4. We propose in section 5 a set of graphic rules based on spanning trees to evaluate directly the expansion coefficients of EYM amplitudes in terms of pure YM ones in the KK basis [23]. These rules of course come from the recursive relation we proved in section 4, which is demonstrated in appendix A. Finally, we give the conclusion and discussion in section 6.

2 CHY integrand for Yang-Mills, Einstein-Yang-Mills and gravity

The central object we are going to study is the tree-level single trace color ordered EYM amplitude with n gluons and m gravitons, denoted by

$$A_{n,m}^{\text{EYM}}(12\dots n | h_m h_{m-1} \dots h_1). \quad (2.1)$$

For convenience, we define the gluon and graviton set as

$$\mathbf{G} = \{12\dots n\} \quad \mathbf{H} = \{h_m h_{m-1} \dots h_1\}.$$

The CHY integrand for this amplitude is [13]

$$\mathcal{I}_{\text{EYM}}(12\dots n | h_m h_{m-1} \dots h_1) = \text{PT}(12\dots n) \text{Pf}(\Psi_{\mathbf{H}}) \text{Pf}'(\Psi), \quad (2.2)$$

where the Parke-Taylor factor:

$$\text{PT}(12\dots n) = \frac{1}{\sigma_{12}\sigma_{23}\dots\sigma_{n1}} \quad \sigma_{ij} \equiv \sigma_i - \sigma_j \quad (2.3)$$

encodes the color ordering of the gluons. The $2n \times 2n$ matrix Ψ is defined as:

$$\Psi = \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix}. \quad (2.4)$$

The three $n \times n$ matrices A , B and C have the form:

$$A_{ab} = \begin{cases} \frac{k_a \cdot k_b}{\sigma_{ab}} & a \neq b \\ 0 & a = b \end{cases} \quad B_{ab} = \begin{cases} \frac{\epsilon_a \cdot \epsilon_b}{\sigma_{ab}} & a \neq b \\ 0 & a = b \end{cases} \quad C_{ab} = \begin{cases} \frac{\epsilon_a \cdot k_b}{\sigma_{ab}} & a \neq b \\ -\sum_{c \neq a} \frac{\epsilon_a \cdot k_c}{\sigma_{ac}} & a = b \end{cases}, \quad (2.5)$$

where the indices a , b and c takes value within $\mathbf{G} \cup \mathbf{H}$. The reduced Pfaffian of Ψ is defined as:

$$\text{Pf}'(\Psi) = \frac{(-1)^{i+j}}{\sigma_{ij}} \text{Pf}(\Psi_{ij}^{ij}) \quad 1 \leq i < j \leq n+m. \quad (2.6)$$

The matrix Ψ_{ij}^{ij} is reduced from Ψ by deleting the i -th and j -th row and column. Finally, the $2m \times 2m$ matrix $\Psi_{\mathbf{H}}$ is defined as:

$$\Psi_{\mathbf{H}} = \begin{pmatrix} A_{\mathbf{H}} & -(C_{\mathbf{H}})^T \\ C_{\mathbf{H}} & B_{\mathbf{H}} \end{pmatrix}, \quad (2.7)$$

where $A_{\mathbf{H}}$, $B_{\mathbf{H}}$ and $C_{\mathbf{H}}$ are respectively the $m \times m$ submatrices of A , B and C whose rows and columns take value only within $\mathbf{H}$. In order to obtain the amplitude $A_{n,m}^{\text{EYM}}$, we integrate over the measure Ω_{CHY} that localizes all the σ 's to the solutions of the scattering equation for $n+m$ massless particles:

$$A_{n,m}^{\text{EYM}}(12\dots n | h_m h_{m-1} \dots h_1) = (-1)^{\frac{m(m+1)}{2}} \int \Omega_{\text{CHY}} \mathcal{I}_{\text{EYM}}(12\dots n | h_m h_{m-1} \dots h_1). \quad (2.8)$$

The expression of Ω_{CHY} is not needed in this work, and we refer the interested readers to the original works of CHY [9–14]. Similarly, we have the following integrands for tree-level YM and Einstein gravity:

$$\mathcal{I}_{\text{YM}}(12\dots n) = \text{PT}(12\dots n) \text{Pf}'(\Psi) \quad \mathcal{I}_{\text{GR}}(12\dots n) = \text{Pf}'(\Psi) \times \text{Pf}'(\Psi), \quad (2.9)$$

from which the amplitudes are obtained through a similar integration as in eq. (2.8). Then eq. (2.2) and (2.9) suggest that there should exist an expansion of the tree-level EYM integrands in terms of the YM ones, namely:

$$\text{PT}(12\dots n) \text{Pf}(\Psi_{\text{H}}) = \sum_{\alpha \in S_{n+m-2}} C_{\alpha} \text{PT}(1, \alpha\{2\dots n-1, h_1\dots h_m\}, n), \quad (2.10)$$

where the equal sign holds when the σ_i 's are solutions to the scattering equations.

The general expansion³ at the amplitude level has been proposed in a recent paper [8] based on the gauge invariance. Let us first recall some examples:

- With only one graviton, the expansion is given by:

$$A_{n,1}^{\text{EYM}}(1, 2, \dots, n | p) = \sum_{\sqcup} (\epsilon_p \cdot Y_p) A_{n+1}^{\text{YM}}(1, \{2, \dots, n-1\} \sqcup \{p\}, n), \quad (2.11)$$

where $\sum_{\sqcup}$ is over the shuffle product $\boldsymbol{\rho} \sqcup \boldsymbol{\omega}$ of two ordered sets $\boldsymbol{\rho}$ and $\boldsymbol{\omega}$ (i.e., all the permutations of $\boldsymbol{\rho} \cup \boldsymbol{\omega}$ that preserving the ordering of $\boldsymbol{\rho}$ and $\boldsymbol{\omega}$ respectively), and Y_p denotes the sum of the momenta of all the gluons ahead of the leg p in the color-ordered YM amplitude. Thus Y_p depends on the orderings contained in the shuffle product. We keep this dependence implicit throughout this work in order not to make our notation too cluttered.

- With two gravitons, the expansion is given by:

$$\begin{aligned} A_{n,2}^{\text{EYM}}(1, 2, \dots, n | p, q) &= \sum_{\sqcup} (\epsilon_p \cdot Y_p) A_{n+1,1}^{\text{EYM}}(1, \{2, \dots, n-1\} \sqcup \{p\}, n | q) \\ &\quad + \sum_{\sqcup} (\epsilon_p \cdot F_q \cdot Y_q) A_{n+2}^{\text{YM}}(1, \{2, \dots, n-1\} \sqcup \{q, p\}, n). \end{aligned} \quad (2.12)$$

where we have used the field strength tensor:

$$(F_q)^{\mu\nu} \equiv (k_q)^{\mu} (\epsilon_q)^{\nu} - (\epsilon_q)^{\mu} (k_q)^{\nu}. \quad (2.13)$$

³The results with one, two and three gravitons were first worked out in [2], but the forms cited here were given in [8].

- With three gravitons, the result is:

$$\begin{aligned}
 A_{n,3}^{\text{EYM}}(1, \dots, n | p, q, r) = & \sum_{\sqcup} (\epsilon_p \cdot Y_p) A_{n+1,2}^{\text{EYM}}(1, \{2, \dots, n-1\} \sqcup \{p\}, n | q, r) \\
 & + \sum_{\sqcup} (\epsilon_p \cdot F_q \cdot Y_q) A_{n+2,1}^{\text{EYM}}(1, \{2, \dots, n-1\} \sqcup \{q, p\}, n | r) \\
 & + \sum_{\sqcup} (\epsilon_p \cdot F_r \cdot Y_r) A_{n+2,1}^{\text{EYM}}(1, \{2, \dots, n-1\} \sqcup \{r, p\}, n | q) \\
 & + \sum_{\sqcup} (\epsilon_p \cdot F_q \cdot F_r \cdot Y_r) A_{n+3}^{\text{YM}}(1, \{2, \dots, n-1\} \sqcup \{r, q, p\}, n) \\
 & + \sum_{\sqcup} (\epsilon_p \cdot F_r \cdot F_q \cdot Y_q) A_{n+3}^{\text{YM}}(1, \{2, \dots, n-1\} \sqcup \{q, r, p\}, n).
 \end{aligned} \tag{2.14}$$

All the above relations are written in terms of amplitudes. However, using eq. (2.8), we can rewrite them in terms of CHY integrands, making the pattern even more transparent. With one graviton, we get:

$$\text{PT}(12 \dots n) \text{Pf}(\Psi_p) = - \sum_{\sqcup} (\epsilon_p \cdot Y_p) \text{PT}(1, \{2, \dots, n-1\} \sqcup \{p\}, n), \tag{2.15}$$

which can be interpreted as turning the graviton into a gluon. With two gravitons, we get:

$$\begin{aligned}
 \text{PT}(12 \dots n) \text{Pf}(\Psi_{pq}) = & \sum_{\sqcup} (\epsilon_p \cdot Y_p) \text{PT}(1, \{2, \dots, n-1\} \sqcup \{p\}, n) \text{Pf}(\Psi_q) \\
 & - \sum_{\sqcup} (\epsilon_p \cdot F_q \cdot Y_q) \text{PT}(1, \{2, \dots, n-1\} \sqcup \{q, p\}, n).
 \end{aligned} \tag{2.16}$$

With three gravitons, we get:

$$\begin{aligned}
 \text{PT}(12 \dots n) \text{Pf}(\Psi_{pqr}) = & - \sum_{\sqcup} (\epsilon_p \cdot Y_p) \text{PT}(1, \{2, \dots, n-1\} \sqcup \{p\}, n) \text{Pf}(\Psi_{qr}) \\
 & - \sum_{\sqcup} (\epsilon_p \cdot F_q \cdot Y_q) \text{PT}(1, \{2, \dots, n-1\} \sqcup \{q, p\}, n) \text{Pf}(\Psi_r) \\
 & - \sum_{\sqcup} (\epsilon_p \cdot F_r \cdot Y_r) \text{PT}(1, \{2, \dots, n-1\} \sqcup \{r, p\}, n) \text{Pf}(\Psi_q) \\
 & + \sum_{\sqcup} (\epsilon_p \cdot F_q \cdot F_r \cdot Y_r) \text{PT}(1, \{2, \dots, n-1\} \sqcup \{r, q, p\}, n) \\
 & + \sum_{\sqcup} (\epsilon_p \cdot F_r \cdot F_q \cdot Y_q) \text{PT}(1, \{2, \dots, n-1\} \sqcup \{q, r, p\}, n).
 \end{aligned} \tag{2.17}$$

Now it is clear that the general patterns observed in [8] indicates a pattern for CHY integrand expansion. In section 3 and 4, we will derive explicitly how this pattern appears. We note that the sign difference between the amplitude/integrand expansion originates from the phase factor in eq. (2.8).

3 Some examples

In this section, we will use various on-shell and off-shell identities to show the pattern of CHY integrand expansion with some examples, from which one can easily find the

generalization. To make our discussion clear, we will carefully distinguish the identities we derive. The *off-shell identities* are those that hold at the pure algebraic level. The *on-shell identities* are true only when certain on-shell conditions have been imposed. These on-shell conditions can be further divided into two types: the *first type* is the momentum conservation and the transverse condition of polarization vectors, and the *second type* is that the σ 's in the integrands are solutions to the scattering equations. To manifest the distinction, we will use $=$ for off-shell identities, $\doteq$ for on-shell identities of the first type and $\cong$ for on-shell identities of the second type.

3.1 One graviton

The relevant part of the EYM integrand $\mathcal{I}_{\text{EYM}}(12\dots n | h_1)$ is

$$\text{PT}(12\dots n) \text{Pf}(\Psi_{h_1}), \quad (3.1)$$

where $\text{Pf}(\Psi_{h_1})$ can be easily calculated as:

$$\text{Pf}(\Psi_{h_1}) = \text{Pf} \begin{pmatrix} 0 & -C_{h_1 h_1} \\ C_{h_1 h_1} & 0 \end{pmatrix} = -C_{h_1 h_1} = \sum_{i=1}^n \frac{\epsilon_{h_1} \cdot k_i}{\sigma_{h_1 i}}. \quad (3.2)$$

To move on, we need the following very useful identity:

$$\frac{\text{PT}(12\dots n)}{\sigma_{ih_1}} = \text{PT}(12\dots i, h_1, i+1\dots n) + \frac{\text{PT}(12\dots n)}{\sigma_{i+1, h_1}}. \quad (3.3)$$

The proof is very simple:

$$\begin{aligned} \frac{\text{PT}(12\dots n)}{\sigma_{ih_1}} &= \text{PT}(12\dots i, h_1, i+1\dots n) \left(\frac{\sigma_{h_1, i+1}}{\sigma_{i, i+1}} \right) \\ &= \text{PT}(12\dots i, h_1, i+1\dots n) + \frac{\sigma_{h_1 i}}{\sigma_{i, i+1}} \text{PT}(12\dots i, h_1, i+1\dots n) \\ &= \text{PT}(12\dots i, h_1, i+1\dots n) + \frac{\text{PT}(12\dots n)}{\sigma_{i+1, h_1}}. \end{aligned} \quad (3.4)$$

where we have used $\sigma_{h_1, i+1} = \sigma_{h_1 i} + \sigma_{i, i+1}$ to go from the first line to second. We want to remark that many papers use instead $C_{ii} = \sum_{j \neq i} \epsilon_i \cdot k_j \left(\frac{\sigma_{jt}}{\sigma_{ji}\sigma_{it}} \right)$. Although this equivalent form has the right $\text{SL}(2, \mathbb{C})$ weight for the i -th node, we find that our form of C_{ii} is more convenient for our purpose, for example, the recursive manipulation in eq. (3.4). Furthermore, we emphasize that eq. (3.3) holds at the algebraic level for arbitrary set of σ 's, not necessarily those of the solutions to the scattering equations. In this sense, it is an off-shell identity.

Now with the help of eq. (3.3), we find that

$$\begin{aligned} C_{h_1 h_1} \text{PT}(12\dots n) &= \text{PT}(12\dots n) \left(\frac{\epsilon_{h_1} \cdot k_1}{\sigma_{1h_1}} + \dots + \frac{\epsilon_{h_1} \cdot k_n}{\sigma_{nh_1}} \right) \\ &= \sum_{\sqcup} (\epsilon_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2\dots n-1\} \sqcup \{h_1\}, n) + \epsilon_{h_1} \cdot \left(\sum_{i=1}^n k_i \right) \frac{\text{PT}(12\dots n)}{\sigma_{n, h_1}} \\ &\doteq \sum_{\sqcup} (\epsilon_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2\dots n-1\} \sqcup \{h_1\}, n), \end{aligned} \quad (3.5)$$

where we have used the on-shell condition $\epsilon_{h_1} \cdot (\sum_{i=1}^n k_i) = -\epsilon_{h_1} \cdot k_{h_1} = 0$ at the second line. Therefore, eq. (3.5) holds under the on-shell condition of the first type, as indicated by the $\doteq$ sign in the last line. With the help of eq. (3.5), we arrive at

$$\text{Pf}(\Psi_{h_1}) \text{PT}(12 \dots n) \doteq - \sum_{\sqcup} (\epsilon_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1\}, n), \quad (3.6)$$

or, in terms of the amplitudes,

$$A_{n,1}^{\text{EYM}}(12 \dots n | h_1) = \sum_{\sqcup} (\epsilon_{h_1} \cdot Y_{h_1}) A_{n+1}^{\text{YM}}(1, \{2 \dots n-1\} \sqcup \{h_1\}, n). \quad (3.7)$$

This is nothing but (2.15). The sign change is accounted by the phase factor of eq. (2.8).

For future use, we define the following level-zero-current CHY integrand T^μ

$$T^\mu[1, \{2 \dots n-1\} \sqcup \{h_1 \dots h_i\}, n | \emptyset] \equiv \sum_{\sqcup} Y_{h_1}^\mu \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1 \dots h_i\}, n), \quad (3.8)$$

Like the usual CHY integrands, T^μ has weight two for all legs, so it is well defined even without imposing the scattering equations. However, T^μ has some special properties. First, its parameters are divided to two parts. The first part contains two ordered sets $\{2 \dots n-1\}$ and $\{h_1 \dots h_i\}$. The second part contains an unordered set. For eq. (3.8) it is the empty set $\emptyset$, and this is the reason why eq. (3.8) is called the level-zero-current. Secondly, eq. (3.8) contains the Lorentz indices through $Y_{h_1}^\mu$. It is worth to notice that h_1 is the first element of the ordered list $\{h_1, \dots, h_i\}$. With this new notation, eq. (3.6) can be written as:

$$\text{Pf}(\Psi_{h_1}) \text{PT}(12 \dots n) \doteq -\epsilon_{h_1} \cdot T[1, \{2 \dots n-1\} \sqcup \{h_1\}, n | \emptyset]. \quad (3.9)$$

3.2 Two gravitons

The relevant part of the EYM integrand $\mathcal{I}_{\text{EYM}}(12 \dots n | h_2 h_1)$ is given by

$$\text{PT}(12 \dots n) \text{Pf}(\Psi_{h_2 h_1}). \quad (3.10)$$

To continue, we expand $\text{Pf}(\Psi_{h_2 h_1})$ as

$$\begin{aligned} \text{Pf}(\Psi_{h_2 h_1}) &= \text{Pf} \begin{pmatrix} 0 & A_{h_2 h_1} & -C_{h_2 h_2} & -C_{h_1 h_2} \\ A_{h_1 h_2} & 0 & -C_{h_2 h_1} & -C_{h_1 h_1} \\ C_{h_2 h_2} & C_{h_2 h_1} & 0 & B_{h_2 h_1} \\ C_{h_1 h_2} & C_{h_1 h_1} & B_{h_1 h_2} & 0 \end{pmatrix} \\ &= C_{h_2 h_2} \text{Pf}(\Psi_{h_1}^{n+2}) - C_{h_2 h_1} \text{Pf} \begin{pmatrix} 0 & -C_{h_1 h_2} \\ C_{h_1 h_2} & 0 \end{pmatrix} + B_{h_2 h_1} \text{Pf} \begin{pmatrix} 0 & A_{h_2 h_1} \\ A_{h_1 h_2} & 0 \end{pmatrix}, \end{aligned} \quad (3.11)$$

where the superscript $n+2$ in $\Psi_{h_1}^{n+2}$ is to remind us that we are now working with the $(n+2)$ -particle kinematics, although the form of $\Psi_{h_1}^{n+2}$ resembles the Ψ_{h_1} in section 3.1.

Similar to eq. (3.5), when $C_{h_2 h_2}$ hits $\text{PT}(12 \dots n)$, we get:

$$\begin{aligned}
 C_{h_2 h_2} \text{PT}(12 \dots n) &= \text{PT}(12 \dots n) \left(\frac{\epsilon_{h_2} \cdot k_1}{\sigma_{1 h_2}} + \dots + \frac{\epsilon_{h_2} \cdot k_n}{\sigma_{n h_2}} + \frac{\epsilon_{h_2} \cdot k_{h_1}}{\sigma_{h_1 h_2}} \right) \\
 &= \sum_{\sqcup} (\epsilon_{h_2} \cdot Y_{h_2}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_2\}, n) \\
 &\quad + \frac{\epsilon_{h_2} \cdot \sum_{i=1}^n k_i}{\sigma_{n h_2}} \text{PT}(12 \dots n) + \text{PT}(12 \dots n) \frac{\epsilon_{h_2} \cdot k_{h_1}}{\sigma_{h_1 h_2}} \\
 &\doteq \sum_{\sqcup} (\epsilon_{h_2} \cdot Y_{h_2}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_2\}, n) - \frac{\sigma_{n h_1}}{\sigma_{n h_2}} C_{h_2 h_1} \text{PT}(12 \dots n).
 \end{aligned} \tag{3.12}$$

Now we give some remarks to this equation. First, when using eq. (3.3) to insert h_2 into gluons, only the gluon momenta contribute, so we get $\epsilon_{h_2} \cdot Y_{h_2}$. Secondly, going from the second equality to the third, we have used the momentum conservation and $\epsilon_{h_2} \cdot k_{h_2} = 0$, as indicated by the $\doteq$ symbol. Thirdly, the above manipulation is, in fact, the action of *changing one graviton to gluon*, which results in the first term of eq. (3.12). However, in the presence of other gravitons, we need some corrections [such as the second term in eq. (3.12)]. As we will see shortly, the recursive structure is hidden in these corrections.

The relation (3.12) can be easily generalized to the cases with more gravitons as

$$\begin{aligned}
 C_{h_i h_i} \text{PT}(12 \dots n) &\doteq \sum_{\sqcup} (\epsilon_{h_i} \cdot Y_{h_i}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_i\}, n) \\
 &\quad - \sum_{\substack{j=1 \\ j \neq i}}^m \frac{\sigma_{n h_j} C_{h_i h_j}}{\sigma_{n h_i}} \text{PT}(12 \dots n),
 \end{aligned} \tag{3.13}$$

for any $h_i \in \mathbf{H} = \{h_m h_{m-1} \dots h_1\}$. We will use (3.13) in later computations again and again. A further observation is that, when we make the replacement $\epsilon_{h_i} \rightarrow k_{h_i}$ and impose the scattering equations, we will make $C_{h_i h_i} \rightarrow 0$ at the left hand side, such that eq. (3.13) leads to:

$$\sum_{\sqcup} (k_{h_i} \cdot Y_{h_i}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_i\}, n) \cong \sum_{j=1}^m \frac{\sigma_{n h_j} A_{h_i h_j}}{\sigma_{n h_i}} \text{PT}(12 \dots n). \tag{3.14}$$

We have used the $\cong$ symbol to emphasize that this equation holds only under the on-shell condition of the second type, namely, localizing σ 's to be the solutions to the scattering equations.

Now plugging eq. (3.12) back into eq. (3.10), we can rewrite eq. (3.10) into:

$$\begin{aligned}
 \text{PT}(12 \dots n) \text{Pf}(\Psi_{h_2 h_1}) &\doteq \sum_{\sqcup} (\epsilon_{h_2} \cdot Y_{h_2}) \text{Pf}(\Psi_{h_1}^{n+2}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_2\}, n) \\
 &\quad - C_{h_2 h_1} \left[\frac{\sigma_{n h_1}}{\sigma_{n h_2}} \text{Pf}(\Psi_{h_1}^{n+2}) + \text{Pf} \begin{pmatrix} 0 & -C_{h_1 h_2} \\ C_{h_1 h_2} & 0 \end{pmatrix} \right] \text{PT}(12 \dots n) \\
 &\quad + B_{h_2 h_1} \text{Pf} \begin{pmatrix} 0 & A_{h_2 h_1} \\ A_{h_1 h_2} & 0 \end{pmatrix} \text{PT}(12 \dots n).
 \end{aligned} \tag{3.15}$$

The first line of eq. (3.15) is now in its final form, which gives:

$$-\sum_{\sqcup} (\epsilon_{h_2} \cdot Y_{h_2}) A_{n+1,1}^{\text{EYM}}(1, \{2 \dots n-1\} \sqcup \{h_2\}, n | h_1)$$

after the CHY integration. Now we consider the second line of (3.15), which contains two terms. The first term contains the factor

$$\text{Pf}(\Psi_{h_1}^{n+2}) \text{PT}(12 \dots n).$$

It would just be the one-graviton case solved in section 3.1, had h_2 joined in the Parke-Taylor factor. This observation guides us to turn $\text{PT}(12 \dots n)$ into $\text{PT}(12 \dots nh_2)$ by inserting the factor $\sigma_{n1}/(\sigma_{nh_2}\sigma_{h_21})$, such that the previous result can be used directly:

$$\begin{aligned} \frac{\sigma_{nh_1}}{\sigma_{nh_2}} \text{Pf}(\Psi_{h_1}^{n+2}) \text{PT}(12 \dots n) &= \frac{\sigma_{nh_1}\sigma_{h_21}}{\sigma_{n1}} \text{Pf}(\Psi_{h_1}^{n+2}) \text{PT}(12 \dots nh_2) \\ &\doteq -\frac{\sigma_{nh_1}\sigma_{h_21}}{\sigma_{n1}} \sum_{\sqcup} (\epsilon_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n\} \sqcup \{h_1\}, h_2) \\ &\doteq -\frac{\sigma_{nh_1}}{\sigma_{nh_2}} \sum_{\sqcup} (\epsilon_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1\}, n) + C_{h_1h_2} \text{PT}(12 \dots n). \end{aligned} \quad (3.16)$$

To go from the second line to third line, we have separated the orderings in the shuffle $\{2 \dots n\} \sqcup \{h_1\}$ into two groups, in which: (1) n being the last element; (2) h_1 being the last element. The first group gives the first term in the third line of (3.16). For the second group, we get

$$-\frac{\sigma_{nh_1}\sigma_{h_21}}{\sigma_{n1}} \left(\epsilon_{h_1} \cdot \sum_{i=1}^n k_i \right) \text{PT}(12 \dots nh_1h_2) = \frac{\epsilon_{h_1} \cdot k_{h_2}}{\sigma_{h_1h_2}} \text{PT}(12 \dots n) = C_{h_1h_2} \text{PT}(12 \dots n).$$

Putting (3.16) back into the second row of eq. (3.15), we get:

$$\begin{aligned} &-C_{h_2h_1} \left[\frac{\sigma_{nh_1}}{\sigma_{nh_2}} \text{Pf}(\Psi_{h_1}^{n+2}) - C_{h_1h_2} \right] \text{PT}(12 \dots n) \\ &\doteq \frac{\sigma_{nh_1}(\epsilon_{h_2} \cdot k_{h_1})}{\sigma_{nh_2}\sigma_{h_2h_1}} \sum_{\sqcup} (\epsilon_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1\}, n) \\ &\doteq -(\epsilon_{h_2} \cdot k_{h_1}) \sum_{\sqcup} (\epsilon_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1h_2\}, n), \end{aligned} \quad (3.17)$$

which is its final form. To derive the last line of (3.17), we have to use an insertion relation, which can be derived as follows:

$$\begin{aligned} \frac{\sigma_{h_1n}}{\sigma_{h_1h_2}\sigma_{h_2n}} &= \left(\int_{\sigma_{h_1}}^{\sigma_{i+1}} + \int_{\sigma_{i+1}}^{\sigma_{i+2}} + \dots + \int_{\sigma_{n-1}}^{\sigma_n} \right) \frac{d\sigma}{(\sigma - \sigma_{h_2})^2} \\ &= \frac{\sigma_{h_1,i+1}}{\sigma_{h_1h_2}\sigma_{h_2,i+1}} + \frac{\sigma_{i+1,i+2}}{\sigma_{i+1,h_2}\sigma_{h_2,i+2}} + \dots + \frac{\sigma_{n-1,n}}{\sigma_{n-1,h_2}\sigma_{h_2n}}, \end{aligned} \quad (3.18)$$

We want to emphasize that eq. (3.18) is an off-shell identity. This manipulation can be generalized to

$$\begin{aligned} &\frac{\sigma_{h_in}}{\sigma_{h_ih_j}\sigma_{h_jn}} \sum_{\sqcup} Y_{h_1}^\mu \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1 \dots h_i\}, n) \\ &= \sum_{\sqcup} Y_{h_1}^\mu \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1 \dots h_ih_j\}, n), \end{aligned} \quad (3.19)$$

where the set $\{h_1 \dots h_i\}$ does not contain h_j . This identity will be very useful later.

Now we come to calculate the last row of eq. (3.15). Actually, the result can be immediately read out from our previous calculations. After we replace ϵ_{h_1} by k_{h_1} into eq. (3.16), we get zero in the left hand side, since $\text{Pf}(\Psi_{h_1}^{n+2}) \rightarrow 0$ under this replacement, if σ 's are the solutions to the scattering equations. Thus we obtain:

$$A_{h_1 h_2} \text{PT}(12 \dots n) \cong \frac{\sigma_{nh_1}}{\sigma_{nh_2}} \sum_{\sqcup} (k_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1\}, n), \quad (3.20)$$

With this equation, the last row of eq. (3.15) becomes

$$\begin{aligned} & -B_{h_2 h_1} A_{h_1 h_2} \text{PT}(12 \dots n) \\ & \cong (\epsilon_{h_2} \cdot \epsilon_{h_1}) \frac{\sigma_{h_1 n}}{\sigma_{h_1 h_2} \sigma_{h_2 n}} \sum_{\sqcup} (k_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1\}, n) \\ & \cong (\epsilon_{h_2} \cdot \epsilon_{h_1}) \sum_{\sqcup} (k_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1 h_2\}, n). \end{aligned} \quad (3.21)$$

In fact, the relation between the second line and the third line of eq. (3.15) can be more easily seen by noticing that $C_{h_2 h_1}|_{k_{h_1} \rightarrow \epsilon_{h_1}} = B_{h_2 h_1}$, and consequently,

$$\left[\frac{\sigma_{nh_1}}{\sigma_{nh_2}} \text{Pf}(\Psi_{h_1}^{n+2}) + \text{Pf} \begin{pmatrix} 0 & -C_{h_1 h_2} \\ C_{h_1 h_2} & 0 \end{pmatrix} \right] \Big|_{\epsilon_{h_1} \rightarrow k_{h_1}} \cong \text{Pf} \begin{pmatrix} 0 & A_{h_2 h_1} \\ A_{h_1 h_2} & 0 \end{pmatrix}. \quad (3.22)$$

Putting together eq. (3.17) and (3.21), we can write down the final result for eq. (3.15):

$$\begin{aligned} \text{PT}(12 \dots n) \text{Pf}(\Psi_{h_2 h_1}) & \cong \sum_{\sqcup} (\epsilon_{h_2} \cdot Y_{h_2}) \text{Pf}(\Psi_{h_1}^{n+2}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_2\}, n) \\ & - \sum_{\sqcup} (\epsilon_{h_2} \cdot F_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1 h_2\}, n), \end{aligned} \quad (3.23)$$

where we have used the definition (2.13) to combine the second and third line of eq. (3.15). Namely, we can expand the two-graviton EYM integrand in terms of one-graviton integrands and pure gluon ones. Eq. (3.23) leads to the following amplitude relation:

$$\begin{aligned} A_{n,2}^{\text{EYM}}(12 \dots n | h_2 h_1) & = \sum_{\sqcup} (\epsilon_{h_2} \cdot Y_{h_2}) A_{n+1,1}^{\text{EYM}}(1, \{2 \dots n-1\} \sqcup \{h_2\}, n | h_1) \\ & + \sum_{\sqcup} (\epsilon_{h_2} \cdot F_{h_1} \cdot Y_{h_1}) A_{n+2}^{\text{YM}}(1, \{2 \dots n-1\} \sqcup \{h_1 h_2\}, n) \end{aligned} \quad (3.24)$$

The sign difference between eq. (3.24) and eq. (3.23) is correctly adjusted according to the phase factor in the CHY integration (2.8) for EYM amplitudes.

Similar to what we have done in eq. (3.8), we define the following level-one-current CHY integrand

$$\begin{aligned} & T^\mu [1, \{2 \dots n-1\} \sqcup \{h_2 \dots h_i\}, n | \{h_1\}] \\ & = \sum_{\sqcup} Y_{h_2}^\mu \text{Pf}(\Psi_{h_1}^{n+i}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_2 \dots h_i\}, n) \\ & - (F_{h_1})^{\mu\nu} T_\nu [1, \{2 \dots n-1\} \sqcup \{h_1 h_2 \dots h_i\}, n | \emptyset]. \end{aligned} \quad (3.25)$$

It is easy to see that after using eq. (3.8) at the second line, we can re-express eq. (3.23) as:

$$\text{PT}(12\dots n) \text{Pf}(\Psi_{h_2 h_1}) \cong \epsilon_{h_2} \cdot T[1, \{2\dots n-1\} \sqcup \{h_2\}, n | \{h_1\}] , \quad (3.26)$$

which holds for an arbitrary number of gluons.

3.3 Three gravitons

As the first nontrivial example, we come to expand the three-graviton EYM integrand:

$$\text{PT}(12\dots n) \text{Pf}(\Psi_{h_3 h_2 h_1}) .$$

We first expand $\text{Pf}(\Psi_{h_3 h_2 h_1})$ into the following form:⁴

$$\begin{aligned} \text{Pf}(\Psi_{h_3 h_2 h_1}) &= \text{Pf} \begin{pmatrix} 0 & A_{h_3 h_2} & A_{h_3 h_1} & -C_{h_3 h_3} & -C_{h_2 h_3} & -C_{h_1 h_3} \\ A_{h_2 h_3} & 0 & A_{h_2 h_1} & -C_{h_2 h_2} & -C_{h_2 h_1} & -C_{h_1 h_2} \\ A_{h_1 h_3} & A_{h_1 h_2} & 0 & -C_{h_3 h_1} & -C_{h_2 h_1} & -C_{h_1 h_1} \\ C_{h_3 h_3} & C_{h_3 h_2} & C_{h_3 h_1} & 0 & B_{h_3 h_2} & B_{h_3 h_1} \\ C_{h_2 h_3} & C_{h_2 h_2} & C_{h_2 h_1} & B_{h_2 h_3} & 0 & B_{h_2 h_1} \\ C_{h_1 h_3} & C_{h_1 h_2} & C_{h_1 h_1} & B_{h_1 h_3} & B_{h_1 h_2} & 0 \end{pmatrix} \\ &= -C_{h_3 h_3} \text{Pf}(\Psi_{h_2 h_1}^{n+3}) + C_{h_3 h_2} \text{Pf}[\psi(32|1)] + C_{h_3 h_1} \text{Pf}[\psi(31|2)] \\ &\quad - B_{h_3 h_2} \text{Pf}[\bar{\psi}(32|1)] - B_{h_3 h_1} \text{Pf}[\bar{\psi}(31|2)] . \end{aligned} \quad (3.27)$$

In this equation, the matrix $\psi(32|1)$ is given by:

$$\psi(32|1) \equiv \begin{pmatrix} 0 & A_{h_3 h_1} & -C_{h_2 h_3} & -C_{h_1 h_3} \\ A_{h_1 h_3} & 0 & -C_{h_2 h_1} & -C_{h_1 h_1} \\ C_{h_2 h_3} & C_{h_2 h_1} & 0 & B_{h_2 h_1} \\ C_{h_1 h_3} & C_{h_1 h_1} & B_{h_1 h_2} & 0 \end{pmatrix} . \quad (3.28)$$

Namely, $\psi(32|1)$ is obtained from $\Psi_{h_3 h_2 h_1}$ by deleting the row and column intersected at $\pm C_{h_3 h_2}$. Then $\psi(31|2)$ is obtained from $\psi(32|1)$ by the exchange $h_2 \leftrightarrow h_1$. On the other hand, the matrix $\bar{\psi}(32|1)$ is obtained from $\psi(32|1)$ by the replacement $\epsilon_{h_2} \rightarrow k_{h_2}$, and similarly for $\bar{\psi}(31|2)$. Now using eq. (3.13) for the first term of (3.27) to turn the graviton h_3 into a gluon, we arrive at:

$$\begin{aligned} \text{PT}(12\dots n) \text{Pf}(\Psi_{h_3 h_2 h_1}) &\doteq - \sum_{\sqcup} (\epsilon_{h_3} \cdot Y_{h_3}) \text{Pf}(\Psi_{h_2 h_1}^{n+3}) \text{PT}(1, \{2\dots n-1\} \sqcup \{h_3\}, n) \\ &\quad + C_{h_3 h_2} \left[\frac{\sigma_{nh_2}}{\sigma_{nh_3}} \text{Pf}(\Psi_{h_2 h_1}^{n+3}) + \text{Pf}[\psi(32|1)] \right] \text{PT}(12\dots n) \\ &\quad + C_{h_3 h_1} \left[\frac{\sigma_{nh_1}}{\sigma_{nh_3}} \text{Pf}(\Psi_{h_2 h_1}^{n+3}) + \text{Pf}[\psi(31|2)] \right] \text{PT}(12\dots n) \\ &\quad - B_{h_3 h_2} \text{Pf}[\bar{\psi}(32|1)] \text{PT}(12\dots n) \\ &\quad - B_{h_3 h_1} \text{Pf}[\bar{\psi}(31|2)] \text{PT}(12\dots n) . \end{aligned} \quad (3.29)$$

⁴This expansion comes from the standard Laplace formula eq. (1.1), with some additional rearrangements in the algebraic minors so that those associated with C and B elements share the same sign.

At this moment, after comparing eq. (3.29) with (3.15), one can see a pattern starting to emerge. Another thing we want to point out is the relative sign difference between the first line of (3.29) and (3.15).

In eq. (3.29), the first row is already in our desired form. We then focus on the second row, while the third row can be obtained by simply exchanging h_1 and h_2 . We would like to reuse eq. (3.23) to expand $\text{Pf}(\Psi_{h_2 h_1}^{n+3}) \text{PT}(12 \dots n)$. To do so, we again use the trick of inserting h_3 into the Parke-Taylor factor. Then using eq. (3.23), we obtain

$$\begin{aligned} \text{Pf}(\Psi_{h_2 h_1}^{n+3}) \text{PT}(12 \dots n h_3) &\cong \sum_{\sqcup} (\epsilon_{h_2} \cdot Y_{h_2}) \text{Pf}(\Psi_{h_1}^{n+3}) \text{PT}(1, \{2 \dots n\} \sqcup \{h_2\}, h_3) \\ &\quad - \sum_{\sqcup} (\epsilon_{h_2} \cdot F_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n\} \sqcup \{h_1 h_2\}, h_3). \end{aligned} \quad (3.30)$$

In this equation, we treat h_3 as a gluon, playing the same role as n in eq. (3.23). In the first line of eq. (3.30), the shuffle product can be divided into two groups: one with n near h_3 and the other with h_2 near h_3 :

$$\{2 \dots n\} \sqcup \{h_2\} = \{2 \dots n-1\} \sqcup \{h_2\}n + \{2 \dots n h_2\}.$$

After this separation, the first line of eq. (3.30) becomes:

$$\begin{aligned} &\sum_{\sqcup} (\epsilon_{h_2} \cdot Y_{h_2}) \text{Pf}(\Psi_{h_1}^{n+3}) \text{PT}(1, \{2 \dots n\} \sqcup \{h_2\}, h_3) \\ &\quad \doteq \sum_{\sqcup} (\epsilon_{h_2} \cdot Y_{h_2}) \text{Pf}(\Psi_{h_1}^{n+3}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_2\}, n, h_3) \\ &\quad \quad - [\epsilon_{h_2} \cdot (k_{h_1} + k_{h_3})] \text{Pf}(\Psi_{h_1}^{n+3}) \text{PT}(12 \dots n h_2 h_3) \\ &\quad \doteq \sum_{\sqcup} (\epsilon_{h_2} \cdot Y_{h_2}) \text{Pf}(\Psi_{h_1}^{n+3}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_2\}, n, h_3) \\ &\quad \quad + [\epsilon_{h_2} \cdot (k_{h_1} + k_{h_3})] \sum_{\sqcup} (\epsilon_{h_1} \cdot X_{h_1}) \text{PT}(1, \{2 \dots n h_2\} \sqcup \{h_1\}, h_3), \end{aligned} \quad (3.31)$$

where the momentum conservation $\sum_{i=1}^n k_i = -k_{h_1} - k_{h_2} - k_{h_3}$ has been used to derive the expression after the first equal sign. The final expression is then obtained by acting $\text{Pf}(\Psi_{h_1}^{n+3})$ onto $\text{PT}(12 \dots n h_2 h_3)$ using eq. (3.6). In the last line of eq. (3.31), we have introduced a new symbol X_{h_1} . This is because when we apply (3.6) here, h_1 will view h_2 as a gluon, so we use the X_{h_1} instead of Y_{h_1} to emphasize this difference. From now on, we will use X_i to stand for the sum of momenta of all the legs at the left hand side of leg i in the color ordering. With this explanation, we can expand the last line of (3.31) as:

$$\begin{aligned} &\sum_{\sqcup} (\epsilon_{h_1} \cdot X_{h_1}) \text{PT}(1, \{2 \dots n h_2\} \sqcup \{h_1\}, h_3) \\ &\quad \doteq \sum_{\sqcup} (\epsilon_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1\}, n, h_2, h_3) \\ &\quad \quad - \epsilon_{h_1} \cdot (k_{h_2} + k_{h_3}) \text{PT}(12 \dots n h_1 h_2 h_3) \\ &\quad \quad - (\epsilon_{h_1} \cdot k_{h_3}) \text{PT}(12 \dots n h_2 h_1 h_3), \end{aligned} \quad (3.32)$$

so that the final result of eq. (3.31) is:

$$\begin{aligned}
 & \sum_{\sqcup} (\epsilon_{h_2} \cdot Y_{h_2}) \text{Pf}(\Psi_{h_1}^{n+3}) \text{PT}(1, \{2 \dots n\} \sqcup \{h_2\}, h_3) \\
 & \doteq \sum_{\sqcup} (\epsilon_{h_2} \cdot Y_{h_2}) \text{Pf}(\Psi_{h_1}^{n+3}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_2\}, n, h_3) \\
 & \quad + [\epsilon_{h_2} \cdot (k_{h_1} + k_{h_3})] \sum_{\sqcup} (\epsilon_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1\}, n, h_2, h_3) \\
 & \quad - [\epsilon_{h_2} \cdot (k_{h_1} + k_{h_3})] [\epsilon_{h_1} \cdot (k_{h_2} + k_{h_3})] \text{PT}(12 \dots n h_1 h_2 h_3) \\
 & \quad - [\epsilon_{h_2} \cdot (k_{h_1} + k_{h_3})] (\epsilon_{h_1} \cdot k_{h_3}) \text{PT}(12 \dots n h_2 h_1 h_3). \tag{3.33}
 \end{aligned}$$

Now we move to the second row of eq. (3.30). Again we separate the shuffle product into three groups

$$\{2 \dots n\} \sqcup \{h_1 h_2\} = \{2 \dots n-1\} \sqcup \{h_1 h_2\} n + \{2 \dots n-1\} \sqcup \{h_1\} n h_2 + \{2 \dots n h_1 h_2\}.$$

This separation leads to:

$$\begin{aligned}
 & - \sum_{\sqcup} (\epsilon_{h_2} \cdot F_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n\} \sqcup \{h_1 h_2\}, h_3) \\
 & = - \sum_{\sqcup} (\epsilon_{h_2} \cdot F_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1 h_2\}, n, h_3) \\
 & \quad - \sum_{\sqcup} (\epsilon_{h_2} \cdot F_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1\}, n, h_2, h_3) \\
 & \quad + [\epsilon_{h_2} \cdot F_{h_1} \cdot (k_{h_2} + k_{h_3})] \text{PT}(12 \dots n h_1 h_2 h_3) \tag{3.34}
 \end{aligned}$$

Summing up eq. (3.33) and (3.34), we find that after a few algebras, eq. (3.30) becomes:

$$\begin{aligned}
 & \text{Pf}(\Psi_{h_2 h_1}^{n+3}) \text{PT}(12 \dots n h_3) \\
 & \cong \sum_{\sqcup} (\epsilon_{h_2} \cdot Y_{h_2}) \text{Pf}(\Psi_{h_1}^{n+3}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_2\}, n, h_3) \\
 & \quad - \sum_{\sqcup} (\epsilon_{h_2} \cdot F_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1 h_2\}, n, h_3) \\
 & \quad + (\epsilon_{h_2} \cdot k_{h_3}) \sum_{\sqcup} (\epsilon_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1\}, n, h_2, h_3) \tag{3.35a}
 \end{aligned}$$

$$- (\epsilon_{h_2} \cdot k_{h_3}) \epsilon_{h_1} \cdot (k_{h_2} + k_{h_3}) \text{PT}(12 \dots n h_1 h_2 h_3) \tag{3.35b}$$

$$- (\epsilon_{h_2} \cdot k_{h_3}) (\epsilon_{h_1} \cdot k_{h_3}) \text{PT}(12 \dots n h_2 h_1 h_3) \tag{3.35c}$$

$$- (\epsilon_{h_2} \cdot k_{h_1}) (\epsilon_{h_1} \cdot k_{h_3}) \text{PT}(12 \dots n h_2 h_1 h_3) \tag{3.35d}$$

$$+ (\epsilon_{h_2} \cdot \epsilon_{h_1}) \sum_{\sqcup} (k_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1\}, n, h_2, h_3) \tag{3.35e}$$

$$- (\epsilon_{h_2} \cdot \epsilon_{h_1}) k_{h_1} \cdot (k_{h_2} + k_{h_3}) \text{PT}(12 \dots n h_1 h_2 h_3). \tag{3.35f}$$

The first two lines of this equation are in our desired form, while the rest of them look very messy at the first glance. Very remarkably, they can actually be organized into $\text{Pf}[\psi(32|1)]$.

To show this, we first use the identity:

$$\begin{aligned}
 \text{Pf}(\Psi_{h_1}^{n+3}) \text{PT}(12 \dots nh_2h_3) &\doteq - \sum_{\sqcup} (\epsilon_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1\}, n, h_2, h_3) \\
 &\quad + \epsilon_{h_1} \cdot (k_{h_2} + k_{h_3}) \text{PT}(12 \dots nh_1h_2h_3) \\
 &\quad + (\epsilon_{h_1} \cdot k_{h_3}) \text{PT}(12 \dots nh_2h_1h_3)
 \end{aligned} \tag{3.36}$$

to group eq. (3.35a), (3.35b) and (3.35c) into:

$$\begin{aligned}
 (3.35a) + (3.35b) + (3.35c) &\doteq - (\epsilon_{h_2} \cdot k_{h_3}) \text{Pf}(\Psi_{h_1}^{n+3}) \text{PT}(12 \dots nh_2h_3) \\
 &= - \left(\frac{\sigma_{n1}}{\sigma_{nh_2}\sigma_{h_31}} \right) C_{h_2h_3} \text{Pf}(\Psi_{h_1}^{n+3}) \text{PT}(12 \dots n).
 \end{aligned} \tag{3.37}$$

Next, by extracting h_2 and h_3 out of the Parke-Taylor factor, we can write eq. (3.35d) as:

$$\begin{aligned}
 (3.35d) &= - \left(\frac{\sigma_{n1}}{\sigma_{nh_2}\sigma_{h_31}} \right) C_{h_2h_1} C_{h_1h_3} \text{PT}(12 \dots n) \\
 &= \left(\frac{\sigma_{n1}}{\sigma_{nh_2}\sigma_{h_31}} \right) C_{h_2h_1} \text{Pf} \begin{pmatrix} 0 & -C_{h_1h_3} \\ C_{h_1h_3} & 0 \end{pmatrix} \text{PT}(12 \dots n).
 \end{aligned} \tag{3.38}$$

As the last step, we replace the ϵ_{h_1} by k_{h_1} in eq. (3.36), which leads to the on-shell identity:

$$\begin{aligned}
 \sum_{\sqcup} (k_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1\}, n, h_2, h_3) \\
 \cong k_{h_1} \cdot (k_{h_2} + k_{h_3}) \text{PT}(12 \dots nh_1h_2h_3) \\
 + (k_{h_1} \cdot k_{h_3}) \text{PT}(12 \dots nh_2h_1h_3).
 \end{aligned} \tag{3.39}$$

With the help of this result, eq. (3.35e) and (3.35f) combine into:

$$\begin{aligned}
 (3.35e) + (3.35f) &\cong - \left(\frac{\sigma_{n1}}{\sigma_{nh_2}\sigma_{h_31}} \right) B_{h_2h_1} A_{h_3h_1} \text{PT}(12 \dots n) \\
 &= - \left(\frac{\sigma_{n1}}{\sigma_{nh_2}\sigma_{h_31}} \right) B_{h_2h_1} \text{Pf} \begin{pmatrix} 0 & A_{h_3h_1} \\ A_{h_1h_3} & 0 \end{pmatrix} \text{PT}(12 \dots n).
 \end{aligned} \tag{3.40}$$

Now collecting eq. (3.37), (3.38) and (3.40), one can show with some algebras that the last six lines of eq. (3.35) can be combined to:

$$\begin{aligned}
 (3.35a) + (3.35b) + (3.35c) + (3.35d) + (3.35e) + (3.35f) \\
 \cong - \frac{\sigma_{n1}}{\sigma_{nh_2}\sigma_{h_31}} \left[C_{h_2h_3} \text{PT}(\Psi_{h_1}^{n+3}) - C_{h_2h_1} \text{Pf} \begin{pmatrix} 0 & -C_{h_1h_3} \\ C_{h_1h_3} & 0 \end{pmatrix} \right. \\
 \quad \left. + B_{h_2h_1} \text{Pf} \begin{pmatrix} 0 & A_{h_3h_1} \\ A_{h_1h_3} & 0 \end{pmatrix} \right] \text{PT}(12 \dots n) \\
 = - \left(\frac{\sigma_{n1}}{\sigma_{nh_2}\sigma_{h_31}} \right) \text{Pf}[\psi(32|1)] \text{PT}(12 \dots n).
 \end{aligned} \tag{3.41}$$

Namely, we have proved the following identity:

$$\begin{aligned}
 & \text{Pf}(\Psi_{h_2 h_1}^{n+3}) \text{PT}(12 \dots n h_3) + \frac{\sigma_{n1}}{\sigma_{nh_2} \sigma_{h_3 1}} \text{Pf}[\psi(32|1)] \text{PT}(12 \dots n) \\
 & \cong \sum_{\sqcup} (\epsilon_{h_2} \cdot Y_{h_2}) \text{Pf}(\Psi_{h_1}^{n+3}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_2\}, n, h_3) \\
 & \quad - \sum_{\sqcup} (\epsilon_{h_2} \cdot F_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1 h_2\}, n, h_3). \quad (3.42)
 \end{aligned}$$

With all these preparations, we can now return to tackle the second row of eq. (3.29). Using eq. (3.42), we find that:

$$\begin{aligned}
 & C_{h_3 h_2} \left[\frac{\sigma_{nh_2}}{\sigma_{nh_3}} \text{Pf}(\Psi_{h_2 h_1}^{n+3}) + \text{Pf}[\psi(32|1)] \right] \text{PT}(12 \dots n) \\
 & = C_{h_3 h_2} \left(\frac{\sigma_{nh_2} \sigma_{h_3 1}}{\sigma_{n1}} \right) \left[\text{Pf}(\Psi_{h_2 h_1}^{n+3}) \text{PT}(12 \dots n h_3) + \frac{\sigma_{n1}}{\sigma_{nh_2} \sigma_{h_3 1}} \text{Pf}[\psi(32|1)] \text{PT}(12 \dots n) \right] \\
 & \cong -(\epsilon_{h_3} \cdot k_{h_2}) \left(\frac{\sigma_{h_2 n}}{\sigma_{h_2 h_3} \sigma_{h_3 n}} \right) \left[\sum_{\sqcup} (\epsilon_{h_2} \cdot Y_{h_2}) \text{Pf}(\Psi_{h_1}^{n+3}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_2\}, n) \right. \\
 & \quad \left. - \sum_{\sqcup} (\epsilon_{h_2} \cdot F_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1 h_2\}, n) \right]. \quad (3.43)
 \end{aligned}$$

Finally, using the insertion identity (3.18), we get:

$$\begin{aligned}
 & C_{h_3 h_2} \left[\frac{\sigma_{nh_2}}{\sigma_{nh_3}} \text{Pf}(\Psi_{h_2 h_1}^{n+3}) + \text{Pf}[\psi(32|1)] \right] \text{PT}(12 \dots n) \\
 & \cong -(\epsilon_{h_3} \cdot k_{h_2}) \left[\sum_{\sqcup} (\epsilon_{h_2} \cdot Y_{h_2}) \text{Pf}(\Psi_{h_1}^{n+3}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_2 h_3\}, n) \right. \\
 & \quad \left. - \sum_{\sqcup} (\epsilon_{h_2} \cdot F_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1 h_2 h_3\}, n) \right]. \quad (3.44)
 \end{aligned}$$

Next, noting that the fourth line of (3.29) can be obtained from the second line by the exchange $\epsilon_{h_2} \leftrightarrow k_{h_2}$ [just like eq. (3.22)], we can write down the result directly from eq. (3.44) as:

$$\begin{aligned}
 & B_{h_3 h_1} \text{Pf}[\bar{\psi}(32|1)] \text{PT}(12 \dots n) \\
 & \cong -(\epsilon_{h_3} \cdot \epsilon_{h_2}) \left[\sum_{\sqcup} (k_{h_2} \cdot Y_{h_2}) \text{Pf}(\Psi_{h_1}^{n+3}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_2 h_3\}, n) \right. \\
 & \quad \left. - \sum_{\sqcup} (k_{h_2} \cdot F_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1 h_2 h_3\}, n) \right]. \quad (3.45)
 \end{aligned}$$

Thus combining (3.44) and (3.45) together, we arrive at:

$$\begin{aligned}
 C_{h_3 h_2} & \left[\frac{\sigma_{nh_2}}{\sigma_{nh_3}} \text{Pf}(\Psi_{h_2 h_1}^{n+3}) + \text{Pf}[\psi(32|1)] \right] \text{PT}(12 \dots n) - B_{h_3 h_1} \text{Pf}[\bar{\psi}(32|1)] \text{PT}(12 \dots n) \\
 & \cong - \sum_{\sqcup} (\epsilon_{h_3} \cdot F_{h_2} \cdot Y_{h_2}) \text{Pf}(\Psi_{h_1}^{n+3}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_2 h_3\}, n) \\
 & \quad + \sum_{\sqcup} (\epsilon_{h_3} \cdot F_{h_2} \cdot F_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1 h_2 h_3\}, n). \tag{3.46}
 \end{aligned}$$

Then the third and fifth line of eq. (3.29) can be obtained by replacing $h_2 \leftrightarrow h_1$ in the above equation. Finally, putting everything altogether, we reach the following result:

$$\begin{aligned}
 & \text{PT}(12 \dots n) \text{Pf}(\Psi_{h_3 h_2 h_1}) \\
 & \doteq - \sum_{\sqcup} (\epsilon_{h_3} \cdot Y_{h_3}) \text{Pf}(\Psi_{h_2 h_1}^{n+3}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_3\}, n) \\
 & \quad - \sum_{\sqcup} (\epsilon_{h_3} \cdot F_{h_2} \cdot Y_{h_2}) \text{Pf}(\Psi_{h_1}^{n+3}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_2 h_3\}, n) \\
 & \quad - \sum_{\sqcup} (\epsilon_{h_3} \cdot F_{h_1} \cdot Y_{h_2}) \text{Pf}(\Psi_{h_2}^{n+3}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1 h_3\}, n) \\
 & \quad + \sum_{\sqcup} (\epsilon_{h_3} \cdot F_{h_2} \cdot F_{h_1} \cdot Y_{h_1}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_1 h_2 h_3\}, n) \\
 & \quad + \sum_{\sqcup} (\epsilon_{h_3} \cdot F_{h_1} \cdot F_{h_2} \cdot Y_{h_2}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_2 h_1 h_3\}, n) \tag{3.47}
 \end{aligned}$$

It is worth noting that after considering the relative signs encoded in eq. (2.8), this result indeed gives eq. (2.14) after the CHY integration.

Eq. (3.47) demonstrates a very clear recursive structure, while we can make it more compact by defining the level- $(m-1)$ -current CHY integrand [like the one given in eq. (3.25)]:

$$\begin{aligned}
 & T^\mu [1, \{2 \dots n-1\} \sqcup \{h_m \dots h_i\}, n | \{h_1, \dots, h_{m-1}\}] \tag{3.48} \\
 & \equiv \sum_{\sqcup} Y_{h_m}^\mu \text{Pf}(\Psi_{h_1 \dots h_{m-1}}^{n+i}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_m \dots h_i\}, n) \\
 & \quad + (-1)^{m-1} \sum_{s=1}^{m-1} (F_{h_s})^{\mu\nu} T_\nu [1, \{2 \dots n-1\} \sqcup \{h_s h_m \dots h_i\}, n | \{h_{m-1} \dots h_s \dots h_{m-1}\}].
 \end{aligned}$$

This is a recursive definition, and one can move down by reducing the number of gravitons in the right list until it becomes the empty set. Then eq. (3.8) terminates the whole recursive construction. We can use T^μ to rewrite eq. (3.47) as:

$$\text{Pf}(\Psi_{h_3 h_2 h_1}) \text{PT}(12 \dots n) \cong (-1)^3 \epsilon_{h_3} \cdot T[1, \{2 \dots n-1\} \sqcup \{h_3\}, n | \{h_2 h_1\}]. \tag{3.49}$$

This three-graviton example makes the recursive pattern very clear. Our next job is of course to prove that it works for arbitrary number of gravitons. To get more confidence and insight, we do one more example with four gravitons. The techniques used in this calculation can be very easily generalized to prove the generic case, presented in section 4.

3.4 Four gravitons

As the last example, we expand the Pfaffian part of the four-graviton integrand $\text{Pf}(\Psi_{h_4 h_3 h_2 h_1}) \text{PT}(12 \dots n)$ as follows:

$$\begin{aligned} \text{Pf}(\Psi_{h_4 h_3 h_2 h_1}) &= C_{h_4 h_4} \text{Pf}(\Psi_{h_3 h_2 h_1}^{n+4}) \\ &\quad - C_{h_4 h_3} \text{Pf}[\psi(43|21)] - C_{h_4 h_2} \text{Pf}[\psi(42|31)] - C_{h_4 h_1} \text{Pf}[\psi(41|32)] \\ &\quad + B_{h_4 h_3} \text{Pf}[\bar{\psi}(43|21)] + B_{h_4 h_2} \text{Pf}[\bar{\psi}(42|31)] + B_{h_4 h_1} \text{Pf}[\bar{\psi}(41|32)] . \end{aligned} \quad (3.50)$$

where $\Psi_{h_3 h_2 h_1}^{n+4}$ has a very similar form to that of eq. (3.27). However, now the σ 's satisfy instead the $n+4$ point scattering equations, while in the diagonal C elements, the summation is over all the $n+4$ particles. The matrix $\psi(43|21)$ is obtained from $\Psi_{h_4 h_3 h_2 h_1}$ by deleting the column and row intersected at $\pm C_{h_4 h_3}$:

$$\psi(43|21) = \begin{pmatrix} 0 & A_{h_4 h_2} & A_{h_4 h_1} & -C_{h_3 h_4} & -C_{h_2 h_4} & -C_{h_1 h_4} \\ A_{h_2 h_4} & 0 & A_{h_2 h_1} & -C_{h_3 h_2} & -C_{h_2 h_2} & -C_{h_1 h_2} \\ A_{h_1 h_4} & A_{h_1 h_2} & 0 & -C_{h_3 h_1} & -C_{h_2 h_1} & -C_{h_1 h_1} \\ C_{h_3 h_4} & C_{h_3 h_2} & C_{h_3 h_1} & 0 & B_{h_3 h_2} & B_{h_3 h_1} \\ C_{h_2 h_4} & C_{h_2 h_2} & C_{h_2 h_1} & B_{h_2 h_3} & 0 & B_{h_2 h_1} \\ C_{h_1 h_4} & C_{h_1 h_2} & C_{h_1 h_1} & B_{h_1 h_3} & B_{h_1 h_2} & 0 \end{pmatrix}, \quad (3.51)$$

while $\psi(42|31)$ and $\psi(41|32)$ are obtained from this matrix by exchanging the index h_3 with h_2 and h_1 respectively. The matrix $\bar{\psi}(\dots s | \dots)$ is obtained from the corresponding $\psi(\dots s | \dots)$ by the replacement $\epsilon_{h_s} \rightarrow k_{h_s}$.

We start from the first line of eq. (3.50). When combining $C_{h_4 h_4}$ with the Parke-Taylor factor, we can turn the graviton h_4 into a gluon with some extra pieces:

$$\begin{aligned} C_{h_4 h_4} \text{PT}(12 \dots n) &\doteq \sum_{\sqcup} (\epsilon_{h_4} \cdot Y_{h_4}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_4\}, n) \\ &\quad - \sum_{s=1}^3 \frac{\sigma_{nh_s} C_{h_4 h_s}}{\sigma_{nh_4}} \text{PT}(12 \dots n). \end{aligned} \quad (3.52)$$

Putting it back to (3.50) and reorganizing, we get:

$$\begin{aligned} &\text{Pf}(\Psi_{h_4 h_3 h_2 h_1}) \text{PT}(12 \dots n) \\ &\doteq \sum_{\sqcup} (\epsilon_{h_4} \cdot Y_{h_4}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_4\}, n) \text{Pf}(\Psi_{h_3 h_2 h_1}^{n+4}) \\ &\quad + \left\{ -C_{h_4 h_3} \left[\frac{\sigma_{nh_3}}{\sigma_{nh_4}} \text{Pf}(\Psi_{h_3 h_2 h_1}^{n+4}) + \text{Pf}[\psi(43|21)] \right] + B_{h_4 h_3} \text{Pf}[\bar{\psi}(43|21)] \right\} \text{PT}(12 \dots n) \\ &\quad + (h_3 \leftrightarrow h_2) + (h_3 \leftrightarrow h_1). \end{aligned} \quad (3.53)$$

We only focus on the case $s=3$, as explicitly shown in the second line from the bottom in this equation. The other two terms in the last line can be obtained from the trivial replacements $h_3 \leftrightarrow h_2$ and $h_3 \leftrightarrow h_1$. First, according to eq. (3.49), we have:

$$\begin{aligned} \frac{\sigma_{nh_3}}{\sigma_{nh_4}} \text{Pf}(\Psi_{h_3 h_2 h_1}^{n+4}) \text{PT}(12 \dots n) &= \left(\frac{\sigma_{nh_3} \sigma_{h_4 1}}{\sigma_{n1}} \right) \text{Pf}(\Psi_{h_3 h_2 h_1}^{n+4}) \text{PT}(12 \dots nh_4) \\ &\cong - \left(\frac{\sigma_{nh_3} \sigma_{h_4 1}}{\sigma_{n1}} \right) \epsilon_{h_3} \cdot T[1, \{2 \dots n\} \sqcup \{h_3\}, h_4 | \{h_2 h_1\}]. \end{aligned} \quad (3.54)$$

The expansion of $T^\mu [1, \{2 \dots n\} \sqcup \{h_3\}, h_4 | \{h_2 h_1\}]$ involves the shuffle products of the form:

$$\{\dots n\} \sqcup \{\star \star \star h_3\} = \{\dots n-1\} \sqcup \{\star \star \star h_3\} n + \{\dots n\} \sqcup \{\star \star \star\} h_3,$$

such that it can be separated in the following way, according to the definition (3.48):

$$\begin{aligned} T^\mu [1, \{2 \dots n\} \sqcup \{h_3\}, h_4 | \{h_2 h_1\}] &\doteq T^\mu [1, \{2 \dots n-1\} \sqcup \{h_3\}, n, h_4 | \{h_2 h_1\}] \\ &\quad - \left[\sum_{s=1}^4 (k_{h_s})^\mu \right] \text{Pf}(\Psi_{h_2 h_1}^{n+4}) \text{PT}(12 \dots n h_3 h_4) \\ &\quad + (F_{h_2})^{\mu\nu} T_\nu [1, \{2 \dots n\} \sqcup \{h_2\}, h_3, h_4 | \{h_1\}] \\ &\quad + (F_{h_1})^{\mu\nu} T_\nu [1, \{2 \dots n\} \sqcup \{h_1\}, h_3, h_4 | \{h_2\}] \quad (3.55) \end{aligned}$$

Now using the following identities proved in the three-graviton example:⁵

$$\begin{aligned} -\epsilon_{h_2} \cdot T [1, \{2 \dots n\} \sqcup \{h_2 h_4\}, h_3 | \{h_1\}] &\cong \frac{1}{\sigma_{h_4 h_2}} \left[\frac{\sigma_{h_3 h_2}}{\sigma_{h_3 h_4}} \text{Pf}(\Psi_{h_2 h_1}^{n+4}) + \text{Pf}[\psi(432|1)] \right] \text{PT}(12 \dots n h_3) \\ -k_{h_2} \cdot T [1, \{2 \dots n\} \sqcup \{h_2 h_4\}, h_3 | \{h_1\}] &\cong \frac{1}{\sigma_{h_4 h_2}} \text{Pf}[\bar{\psi}(432|1)] \text{PT}(12 \dots n h_3) \\ -\epsilon_{h_1} \cdot T [1, \{2 \dots n\} \sqcup \{h_1 h_4\}, h_3 | \{h_2\}] &\cong \frac{1}{\sigma_{h_4 h_1}} \left[\frac{\sigma_{h_3 h_1}}{\sigma_{h_3 h_4}} \text{Pf}(\Psi_{h_2 h_1}^{n+4}) + \text{Pf}[\psi(431|2)] \right] \text{PT}(12 \dots n h_3) \\ -k_{h_1} \cdot T [1, \{2 \dots n\} \sqcup \{h_1 h_4\}, h_3 | \{h_2\}] &\cong \frac{1}{\sigma_{h_4 h_1}} \text{Pf}[\bar{\psi}(431|2)] \text{PT}(12 \dots n h_3), \quad (3.56) \end{aligned}$$

and eq. (3.19), we can derive that:

$$\begin{aligned} \epsilon_{h_2} \cdot T [1, \{2 \dots n\} \sqcup \{h_2\}, h_3, h_4 | \{h_1\}] &\cong \text{Pf}(\Psi_{h_2 h_1}^{n+4}) \text{PT}(12 \dots n h_3 h_4) \quad (3.57) \\ &\quad + \frac{\sigma_{h_3 1}}{\sigma_{h_3 h_2} \sigma_{h_4 1}} \text{Pf}[\psi(432|1)] \text{PT}(12 \dots n h_3) \\ k_{h_2} \cdot T [1, \{2 \dots n\} \sqcup \{h_2\}, h_3, h_4 | \{h_1\}] &\cong \frac{\sigma_{h_3 1}}{\sigma_{h_3 h_2} \sigma_{h_4 1}} \text{Pf}[\bar{\psi}(432|1)] \text{PT}(12 \dots n h_3), \end{aligned}$$

After we replace the T 's in the last two lines of eq. (3.55) by the above equation, and dot the entire eq. (3.55) with ϵ_{h_3} , it becomes:

$$\begin{aligned} \epsilon_{h_3} \cdot T [1, \{2 \dots n\} \sqcup \{h_3\}, h_4 | \{h_2 h_1\}] &\quad (3.58) \\ &\cong \epsilon_{h_3} \cdot T [1, \{2 \dots n-1\} \sqcup \{h_3\}, n, h_4 | \{h_2 h_1\}] \\ &\quad + \frac{\sigma_{h_3 1}}{\sigma_{h_4 1}} \left[-C_{h_3 h_4} \text{Pf}(\Psi_{h_2 h_1}^{n+4}) + C_{h_3 h_2} \text{Pf}[\psi(432|1)] + C_{h_3 h_1} \text{Pf}[\psi(431|2)] \right. \\ &\quad \left. - B_{h_3 h_2} \text{Pf}[\bar{\psi}(432|1)] - B_{h_3 h_1} \text{Pf}[\bar{\psi}(431|2)] \right] \text{PT}(12 \dots n h_3) \\ &\cong \epsilon_{h_3} \cdot T [1, \{2 \dots n-1\} \sqcup \{h_3\}, n, h_4 | \{h_2 h_1\}] + \frac{\sigma_{h_3 1}}{\sigma_{h_4 1}} \text{Pf}[\psi(43|21)] \text{PT}(12 \dots n h_3). \end{aligned}$$

⁵The matrix $\psi(432|1) [\psi(431|2)]$ is obtained from $\psi(43|21)$ by deleting the row and column intersected at $\pm C_{h_3 h_2} (\pm C_{h_3 h_1})$.

If we put them back into eq. (3.54) and then eq. (3.53), we get:

$$C_{h_4 h_3} \left[\frac{\sigma_{nh_3}}{\sigma_{nh_4}} \text{Pf}(\Psi_{h_3 h_2 h_1}^{n+4}) + \text{Pf}[\psi(43|21)] \right] \text{PT}(12 \dots n) \\ \cong (\epsilon_{h_4} \cdot k_{h_3}) \epsilon_{h_3} \cdot T[1, \{2 \dots n-1\} \sqcup \{h_3 h_4\}, n | \{h_2 h_1\}] . \quad (3.59)$$

If we make the replacement $\epsilon_{h_3} \rightarrow k_{h_3}$ above, we then get:

$$\frac{1}{\sigma_{h_4 h_3}} \text{Pf}[\bar{\psi}(43|21)] \cong k_{h_3} \cdot T[1, \{2 \dots n-1\} \sqcup \{h_3 h_4\}, n | \{h_2 h_1\}] . \quad (3.60)$$

The $(h_3 \leftrightarrow h_2)$ and $(h_3 \leftrightarrow h_1)$ term of eq. (3.53) can then be obtained by making the corresponding exchange in eq. (3.59) and (3.60). The final result of eq. (3.53) is thus:

$$\text{Pf}(\Psi_{h_4 h_3 h_2 h_1}) \text{PT}(12 \dots n) \cong \sum_{\sqcup} (\epsilon_{h_4} \cdot Y_{h_4}) \text{Pf}(\Psi_{h_3 h_2 h_1}^{n+4}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_4\}, n) \\ - \epsilon_{h_4} \cdot F_{h_3} \cdot T[1, \{2 \dots n-1\} \sqcup \{h_3 h_4\}, n | \{h_2 h_1\}] \\ - \epsilon_{h_4} \cdot F_{h_2} \cdot T[1, \{2 \dots n-1\} \sqcup \{h_2 h_4\}, n | \{h_3 h_1\}] \\ - \epsilon_{h_4} \cdot F_{h_1} \cdot T[1, \{2 \dots n-1\} \sqcup \{h_1 h_4\}, n | \{h_3 h_2\}] . \quad (3.61)$$

Finally, using the definition (3.48), one can express the four-graviton EYM integrand as:

$$\text{Pf}(\Psi_{h_4 h_3 h_2 h_1}) \text{PT}(12 \dots n) \cong (-1)^4 \epsilon_{h_4} \cdot T[1, \{2 \dots n-1\} \sqcup \{h_4\}, n | \{h_3 h_2 h_1\}] . \quad (3.62)$$

The recursive nature of this construction is now clear, and we are going to prove the generic recursive relation for expanding EYM integrands with an arbitrary number of gravitons.

4 General recursive relation

The calculations in section 3, especially the cases of three and four gravitons, suggest us to construct the recursive definition (3.48), which we recall here as:

$$T^\mu[1, \{2 \dots n-1\} \sqcup \{h_m \dots h_i\}, n | \{h_{m-1} \dots h_1\}] \\ = \sum_{\sqcup} Y_{h_m}^\mu \text{Pf}(\Psi_{h_{m-1} \dots h_1}^{n+i}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_m \dots h_i\}, n) \\ + (-1)^{m-1} \sum_{s=1}^{m-1} (F_{h_s})^{\mu\nu} T_\nu[1, \{2 \dots n-1\} \sqcup \{h_s h_m \dots h_i\}, n | \{h_{m-1} \dots \cancel{h_s} \dots h_1\}] . \quad (4.1)$$

This recursion finally lands on $T[\dots|\emptyset]$ defined in eq. (3.8), for arbitrary $i \geq m$. This definition gives a well defined weight two integrand for $n+i$ particles. One crucial point we want to mention is that

$$T^\mu[1, \{2 \dots n-1\} \sqcup \{h_m \dots h_i\}, n | \{h_{m-1} \dots h_1\}]$$

has the manifest gauge invariance for the gravitons $\{h_1 h_2 \dots h_{m-1}\}$. In this section, we are going to prove by induction a very important result:

$$\text{Pf}(\Psi_H) \text{PT}(12 \dots n) \cong (-1)^m \epsilon_{h_m} \cdot T[1, \{2 \dots n-1\} \sqcup \{h_m\}, n | \{h_{m-1} \dots h_1\}] \quad (4.2)$$

for $\mathbf{H} = \{h_m h_{m-1} \dots h_1\}$. In this representation, h_m is singled out as the row along which we choose to expand the Pfaffian. Because of this choice, the gauge invariance for $\{h_1 \dots h_{m-1}\}$ is manifest, but not for h_m . Only after we fully expand it to pure YM amplitudes, can we check the gauge invariance of h_m with various KK and generalized BCJ relations [28].

The $m = 1, 2, 3$ and 4 cases have been explicitly checked in the previous section. Now suppose that eq. (4.2) holds for p gravitons with $p \leq m - 1$, as our induction assumption, then we move to the case of m gravitons. The $2m \times 2m$ matrix $\Psi_{\mathbf{H}}$ has the following form:

$$\Psi_{\mathbf{H}} = \begin{pmatrix} 0 & A_{h_m h_{m-1}} & \cdots & -C_{h_m h_m} & -C_{h_{m-1} h_m} & \cdots \\ A_{h_{m-1} h_m} & 0 & \cdots & -C_{h_m h_{m-1}} & -C_{h_{m-1} h_{m-1}} & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \\ C_{h_m h_m} & C_{h_m h_{m-1}} & \cdots & 0 & B_{h_m h_{m-1}} & \cdots \\ C_{h_{m-1} h_m} & C_{h_{m-1} h_{m-1}} & \cdots & B_{h_{m-1} h_m} & 0 & \cdots \\ \vdots & \vdots & & \vdots & \vdots & \end{pmatrix}. \quad (4.3)$$

To calculate the Pfaffian, we expand along the row starting with $C_{h_m h_m}$ [which is the $(m + 1)$ -th row]:

$$\begin{aligned} (-1)^m \text{Pf}(\Psi_{\mathbf{H}}) &= C_{h_m h_m} \text{Pf}(\Psi_{h_{m-1} \dots h_1}^{n+m}) - \sum_{s=1}^{m-1} C_{h_m h_s} \text{Pf}[\psi(m, s | m - 1 \dots \not{s} \dots 1)] \\ &\quad + \sum_{s=1}^{m-1} B_{h_m h_s} \text{Pf}[\bar{\psi}(m, s | m - 1 \dots \not{s} \dots 1)]. \end{aligned} \quad (4.4)$$

In this expansion, $\Psi_{h_{m-1} \dots h_1}^{n+m}$ is obtained from $\Psi_{\mathbf{H}}$ by deleting the rows and columns intersected at the $\pm C_{h_m h_m}$. We note that $\Psi_{h_{m-1} \dots h_1}^{n+m}$ is part of the EYM integrand with $m - 1$ gravitons, while h_m has been turned into a gluon. The matrix $\psi(m, m - 1 | m - 2 \dots 1)$ is obtained from $\Psi_{\mathbf{H}}$ by deleting the rows and columns intersected at the $\pm C_{h_m h_{m-1}}$, while the other $\psi(m, s | m - 1 \dots \not{s} \dots 1)$ can be generated by exchanging h_{m-1} and h_s . In the last row of eq. (4.4), we have:

$$\bar{\psi}(m, s | m - 1 \dots \not{s} \dots 1) = \psi(m, s | m - 1 \dots \not{s} \dots 1)|_{\epsilon_{h_s} \rightarrow k_{h_s}}.$$

After multiplying eq. (4.4) with the Parke-Taylor factor $\text{PT}(12 \dots n)$, the first term yields:

$$\begin{aligned} &C_{h_m h_m} \text{Pf}(\Psi_{h_{m-1} \dots h_1}^{n+m}) \text{PT}(12 \dots n) \\ &\doteq \sum_{\sqcup} (\epsilon_{h_m} \cdot Y_{h_m}) \text{Pf}(\Psi_{h_{m-1} \dots h_1}^{n+m}) \text{PT}(1, \{2 \dots n - 1\} \sqcup \{h_m\}, n) \\ &\quad - \sum_{s=1}^{m-1} \frac{\sigma_{n h_s} C_{h_m h_s}}{\sigma_{n h_m}} \text{Pf}(\Psi_{h_{m-1} \dots h_1}^{n+m}) \text{PT}(12 \dots n), \end{aligned} \quad (4.5)$$

with the help of eq. (3.13). Then from eq. (4.4), we get:

$$\begin{aligned}
 & (-1)^m \text{Pf}(\Psi_H) \text{PT}(12 \dots n) \\
 & \doteq \sum_{\sqcup} (\epsilon_{h_m} \cdot Y_{h_m}) \text{Pf}(\Psi_{h_{m-1} \dots h_1}^{n+m}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_m\}, n) \\
 & - \sum_{s=1}^{m-1} \frac{\epsilon_{h_m} \cdot k_{h_s}}{\sigma_{h_m h_s}} \left[\frac{\sigma_{n h_s}}{\sigma_{n h_m}} \text{Pf}(\Psi_{h_{m-1} \dots h_1}^{n+m}) + \text{Pf}[\psi(m, s | m-1 \dots \not{s} \dots 1)] \right] \text{PT}(12 \dots n) \\
 & + \sum_{s=1}^{m-1} \frac{\epsilon_{h_m} \cdot \epsilon_{h_s}}{\sigma_{h_m h_s}} \text{Pf}[\bar{\psi}(m, s | m-1 \dots \not{s} \dots 1)] \text{PT}(12 \dots n). \tag{4.6}
 \end{aligned}$$

Our proof completes if we can prove the following two identities:

$$\begin{aligned}
 & \frac{1}{\sigma_{h_m h_s}} \left[\frac{\sigma_{n h_s}}{\sigma_{n h_m}} \text{Pf}(\Psi_{h_{m-1} \dots h_1}^{n+m}) + \text{Pf}[\psi(m, s | m-1 \dots \not{s} \dots 1)] \right] \text{PT}(12 \dots n) \\
 & \cong (-1)^m \epsilon_{h_s} \cdot T[1, \{2 \dots n-1\} \sqcup \{h_s h_m\}, n | \{h_{m-1} \dots \not{h}_s \dots h_1\}] , \tag{4.7}
 \end{aligned}$$

$$\begin{aligned}
 & \frac{1}{\sigma_{h_m h_s}} \text{Pf}[\bar{\psi}(m, s | m-1 \dots \not{s} \dots 1)] \text{PT}(12 \dots n) \\
 & \cong (-1)^m k_{h_s} \cdot T[1, \{2 \dots n-1\} \sqcup \{h_s h_m\}, n | \{h_{m-1} \dots \not{h}_s \dots h_1\}] . \tag{4.8}
 \end{aligned}$$

We still follow an inductive scheme and assume that they hold for p gravitons with $p \leq m-1$ gravitons. The $m=2, 3$, and 4 cases have been explicitly calculated in section 3. For details, one can see the derivation of eq. (3.17), (3.21), (3.44), (3.59) and (3.60). It is sufficient to only work with $s=m-1$, since other values of s can be obtained by a trivial exchange of indices. Like the calculation performed in section 3.4, we start with the expansion:

$$\begin{aligned}
 & \frac{\sigma_{n h_{m-1}}}{\sigma_{n h_m}} \text{Pf}(\Psi_{h_{m-1} \dots h_1}^{n+m}) \text{PT}(12 \dots n) \\
 & \cong \frac{\sigma_{n h_{m-1}} \sigma_{h_{m-1}}}{\sigma_{n1}} (-1)^{m-1} \epsilon_{h_{m-1}} \cdot T[1, \{2 \dots n\} \sqcup \{h_{m-1}\}, h_m | \{h_{m-2} \dots h_1\}] \\
 & \cong \frac{\sigma_{n h_{m-1}} \sigma_{h_{m-1}}}{\sigma_{n1}} (-1)^{m-1} \epsilon_{h_{m-1}} \cdot T[1, \{2 \dots n-1\} \sqcup \{h_{m-1}\}, n, h_m | \{h_{m-2} \dots h_1\}] \\
 & - (-1)^{m+1} C_{h_{m-1} h_m} \text{Pf}(\Psi_{h_{m-2} \dots h_1}^{n+m}) \text{PT}(12 \dots n) \\
 & + (-1)^m \left(\frac{\sigma_{n h_{m-1}} \sigma_{h_{m-1}}}{\sigma_{n1}} \right) \sum_{t=1}^{m-2} C_{h_{m-1} h_t} \left[\frac{\sigma_{h_{m-1} h_t}}{\sigma_{h_{m-1} h_m}} \text{Pf}(\Psi_{h_{m-2} \dots h_1}^{n+m}) \text{PT}(12 \dots n h_{m-1}) \right. \\
 & \quad \left. - \sigma_{h_m h_t} (-1)^{m-1} \epsilon_{h_t} \cdot T[1, \{2 \dots n\} \sqcup \{h_t h_m\}, h_{m-1} | \{h_{m-2} \dots \not{h}_t \dots h_1\}] \right] \\
 & - \left(\frac{\sigma_{n h_{m-1}} \sigma_{h_{m-1}}}{\sigma_{n1}} \right) \sum_{t=1}^{m-2} B_{h_{m-1} h_t} \sigma_{h_m h_t} k_{h_t} \\
 & \cdot T[1, \{2 \dots n\} \sqcup \{h_t h_m\}, h_{m-1} | \{h_{m-2} \dots \not{h}_t \dots h_1\}] \tag{4.9}
 \end{aligned}$$

Now our inductive assumption says:⁶

$$\begin{aligned}
 & \frac{1}{\sigma_{h_m h_t}} \left[\frac{\sigma_{h_{m-1} h_t}}{\sigma_{h_{m-1} h_m}} \text{Pf}(\Psi_{h_{m-2} \dots h_1}^{n+m}) + \text{Pf}[\psi(m, m-1, t|m-2 \dots \cancel{t} \dots 1)] \right] \\
 & \quad \cong (-1)^{m-1} \epsilon_{h_t} \cdot T[1, \{2 \dots n\} \sqcup \{h_t h_m\}, h_{m-1} | \{h_{m-2} \dots \cancel{h_t} \dots h_1\}] , \\
 & \frac{1}{\sigma_{h_m h_t}} \text{Pf}[\bar{\psi}(m, m-1, t|m-2 \dots \cancel{t} \dots 1)] \\
 & \quad \cong (-1)^{m-1} k_{h_t} \cdot T[1, \{2 \dots n\} \sqcup \{h_t h_m\}, h_{m-1} | \{h_{m-2} \dots \cancel{h_t} \dots h_1\}] . \quad (4.10)
 \end{aligned}$$

As a result, we get:

$$\begin{aligned}
 & \frac{\sigma_{n h_{m-1}}}{\sigma_{n h_m}} \text{Pf}(\Psi_{h_{m-1} \dots h_1}^{n+m}) \text{PT}(12 \dots n) \\
 & \quad \cong \frac{\sigma_{n h_{m-1}} \sigma_{h_m 1}}{\sigma_{n 1}} (-1)^{m-1} \epsilon_{h_{m-1}} \cdot T[1, \{2 \dots n-1\} \sqcup \{h_{m-1}\}, n, h_m | \{h_{m-2} \dots h_1\}] \\
 & \quad - (-1)^{m+1} C_{h_{m-1} h_m} \text{Pf}(\Psi_{h_{m-2} \dots h_1}^{n+m}) \text{PT}(12 \dots n) \\
 & \quad + (-1)^{m+1} \sum_{t=1}^{m-2} C_{h_{m-1} h_t} \text{Pf}[\psi(m, m-1, t|m-2 \dots \cancel{t} \dots 1)] \text{PT}(12 \dots n) \\
 & \quad - (-1)^{m+1} \sum_{t=1}^{m-2} B_{h_{m-1} h_t} \text{Pf}[\bar{\psi}(m, m-1, t|m-2 \dots \cancel{t} \dots 1)] \text{PT}(12 \dots n) \\
 & \quad \cong \sigma_{h_m h_{m-1}} (-1)^m \epsilon_{h_{m-1}} \cdot T[1, \{2 \dots n-1\} \sqcup \{h_{m-1} h_m\}, n | \{h_{m-2} \dots h_1\}] \\
 & \quad - \text{Pf}[\psi(m, m-1|m-2 \dots 1)] \text{PT}(12 \dots n) , \quad (4.11)
 \end{aligned}$$

which proves eq. (4.7). Then replacing $\epsilon_{h_{m-1}}$ by $k_{h_{m-1}}$, we also proves eq. (4.8):

$$\begin{aligned}
 & (-1)^m k_{h_{m-1}} \cdot T[1, \{2 \dots n-1\} \sqcup \{h_{m-1} h_m\}, n | \{h_{m-2} \dots h_1\}] \\
 & \quad \cong \frac{1}{\sigma_{h_m h_{m-1}}} \text{Pf}[\bar{\psi}(m, m-1|m-2 \dots 1)] \text{PT}(12 \dots n) .
 \end{aligned}$$

With the help of these two equations, we can immediately see that eq. (4.6) gives exactly our statement:

$$(-1)^m \text{Pf}(\Psi_H) \text{PT}(12 \dots n) \cong \epsilon_{h_m} \cdot T[1, \{2 \dots n-1\} \sqcup \{h_m\}, n | \{h_{m-1} \dots h_1\}] , \quad (4.12)$$

We have now completed the proof of our general recursive relation eq. (4.2).

5 Expansion in terms of pure YM amplitudes

If we keep using the recursive relation (4.2), we can finally expand generic EYM tree amplitudes in terms of pure YM amplitudes. In this representation, the final expression does not enjoy the explicit permutation invariance of the gravitons. The reason is that in eq. (4.2), we have defined a standard graviton order $\{h_m h_{m-1} \dots h_1\}$. At each level of recursion, we always convert the first graviton remained in this standard order into a gluon according to the definition of T , which breaks the explicit permutation invariance.⁷ In

⁶The matrix $\psi(m, m-1, t|m-2 \dots \cancel{t} \dots 1)$ is obtained from $\psi(m, m-1|m-2 \dots 1)$ by deleting the row and column intersected at $\pm C_{h_{m-1} h_t}$.

⁷The permutation invariant expression can be recovered after a straightforward, although tedious, symmetrization. We will not discuss it in this paper.

this section, we give these expansions a graph theory picture, derived from our recursive expansion (4.2). More details on this derivation is presented in appendix A

5.1 Some expansion examples

In this part, we will give the expansions of the EYM amplitudes with two and three gravitons. It serves two purposes: (1) establishing our convention; (2) providing explicit expressions to compare with known results as well as a better understanding of the graphic rules.

Let us start with the example of two gravitons, whose recursive construction is given in eq. (3.24). Since the last term of eq. (3.24) contains only YM amplitudes. We only need to further expand the first term using eq. (3.7):

$$\begin{aligned} & \sum_{\sqcup} (\epsilon_{h_2} \cdot Y_{h_2}) A_{n+1,1}^{\text{EYM}}(1, \{2 \dots n-1\} \sqcup \{h_2\}, n | h_1) \\ &= - \sum_{\sqcup} (\epsilon_{h_2} \cdot Y_{h_2}) (\epsilon_{h_1} \cdot X_{h_1}) A_{n+2}^{\text{YM}}(1, [\{2 \dots n-1\} \sqcup \{h_2\}] \sqcup \{h_1\}, n), \end{aligned} \quad (5.1)$$

where Y_{h_i} is defined as the sum of all original gluon momenta before the graviton h_i in each ordering of the shuffle product. The important difference between Y_{h_i} and X_{h_i} is that X_{h_i} is the sum of all momenta before h_i , including those gluons converted from gravitons. Now using the associativity of the shuffle product:

$$\begin{aligned} [\{2 \dots n-1\} \sqcup \{h_2\}] \sqcup \{h_1\} &= \{2 \dots n-1\} \sqcup [\{h_2\} \sqcup \{h_1\}] \\ &= \{2 \dots n-1\} \sqcup \{h_1 h_2\} + \{2 \dots n-1\} \sqcup \{h_2 h_1\}, \end{aligned} \quad (5.2)$$

we can write the second line of eq. (5.1) as

$$\begin{aligned} & - \sum_{\sqcup} (\epsilon_{h_2} \cdot Y_{h_2}) (\epsilon_{h_1} \cdot Y_{h_1}) A_{n+2}^{\text{YM}}(1, \{2 \dots n-1\} \sqcup \{h_1 h_2\}, n) \\ & - \sum_{\sqcup} (\epsilon_{h_2} \cdot Y_{h_2}) (\epsilon_{h_1} \cdot Y_{h_1} + \epsilon_{h_1} \cdot k_{h_2}) A_{n+2}^{\text{YM}}(1, \{2 \dots n-1\} \sqcup \{h_2 h_1\}, n). \end{aligned} \quad (5.3)$$

Therefore, if we further expand eq. (3.24) down to the pure YM level, we get:

$$\begin{aligned} (-1) A_{n,2}^{\text{EYM}}(12 \dots n | h_2 h_1) &= \sum_{\sqcup} (\epsilon_{h_2} \cdot Y_{h_2}) (\epsilon_{h_1} \cdot Y_{h_1}) A_{n+2}^{\text{YM}}(1, \{2 \dots n-1\} \sqcup \{h_1 h_2\}, n) \\ &+ \sum_{\sqcup} (\epsilon_{h_2} \cdot Y_{h_2}) (\epsilon_{h_1} \cdot Y_{h_1}) A_{n+2}^{\text{YM}}(1, \{2 \dots n-1\} \sqcup \{h_2 h_1\}, n) \\ &+ \sum_{\sqcup} (\epsilon_{h_2} \cdot F_{h_1} \cdot Y_{h_1}) A_{n+2}^{\text{YM}}(1, \{2 \dots n-1\} \sqcup \{h_1 h_2\}, n) \\ &+ \sum_{\sqcup} (\epsilon_{h_1} \cdot k_{h_2}) (\epsilon_{h_2} \cdot Y_{h_2}) A_{n+2}^{\text{YM}}(1, \{2 \dots n-1\} \sqcup \{h_2 h_1\}, n). \end{aligned} \quad (5.4)$$

We want to emphasize that in the above expansion, the coefficients contain only the kinematic variables ϵ_{h_i} , k_{h_i} and Y_{h_i} .

Similar manipulations can be performed for EYM amplitudes with three gravitons. By recursively using eq. (2.13) and (2.11) to eq. (2.14), one can write the final result as:⁸

$$A_{n,3}^{\text{EYM}}(12\dots n | h_3 h_2 h_1) = \sum_{\sigma \in S_3} \sum_{\sqcup} C_{[321]}(\sigma) A_{n+3}^{\text{YM}}(1, \{2\dots n-1\} \sqcup \{h_{\sigma(1)} h_{\sigma(2)} h_{\sigma(3)}\}, n). \quad (5.5)$$

For each graviton permutation σ , the coefficient $C_{[321]}(\sigma)$ has the following expression:

$$\begin{aligned} C_{[321]}(123) = & (\epsilon_{h_3} \cdot Y_{h_3}) (\epsilon_{h_2} \cdot Y_{h_2}) (\epsilon_{h_1} \cdot Y_{h_1}) + (\epsilon_{h_3} \cdot F_{h_2} \cdot Y_{h_2}) (\epsilon_{h_1} \cdot Y_{h_1}) \\ & + (\epsilon_{h_3} \cdot F_{h_1} \cdot Y_{h_1}) (\epsilon_{h_2} \cdot Y_{h_2}) + (\epsilon_{h_2} \cdot F_{h_1} \cdot Y_{h_1}) (\epsilon_{h_3} \cdot Y_{h_3}) \\ & + (\epsilon_{h_3} \cdot F_{h_1} \cdot Y_{h_1}) (\epsilon_{h_2} \cdot k_{h_1}) + (\epsilon_{h_3} \cdot F_{h_2} \cdot F_{h_1} \cdot Y_{h_1}) \end{aligned} \quad (5.6)$$

$$\begin{aligned} C_{[321]}(213) = & (\epsilon_{h_3} \cdot Y_{h_3}) (\epsilon_{h_1} \cdot Y_{h_1}) (\epsilon_{h_2} \cdot Y_{h_2}) + (\epsilon_{h_3} \cdot F_{h_1} \cdot Y_{h_1}) (\epsilon_{h_2} \cdot Y_{h_2}) \\ & + (\epsilon_{h_3} \cdot F_{h_2} \cdot Y_{h_2}) (\epsilon_{h_1} \cdot Y_{h_1}) + (\epsilon_{h_1} \cdot k_{h_2}) (\epsilon_{h_2} \cdot Y_{h_2}) (\epsilon_{h_3} \cdot Y_{h_3}) \\ & + (\epsilon_{h_3} \cdot F_{h_2} \cdot Y_{h_2}) (\epsilon_{h_1} \cdot k_{h_2}) + (\epsilon_{h_3} \cdot F_{h_1} \cdot F_{h_2} \cdot Y_{h_2}) \end{aligned} \quad (5.7)$$

$$\begin{aligned} C_{[321]}(321) = & (\epsilon_{h_1} \cdot Y_{h_1}) (\epsilon_{h_2} \cdot Y_{h_2}) (\epsilon_{h_3} \cdot Y_{h_3}) + (\epsilon_{h_1} \cdot k_{h_2}) (\epsilon_{h_2} \cdot Y_{h_2}) (\epsilon_{h_3} \cdot Y_{h_3}) \\ & + (\epsilon_{h_1} \cdot k_{h_3}) (\epsilon_{h_3} \cdot Y_{h_3}) (\epsilon_{h_2} \cdot Y_{h_2}) + (\epsilon_{h_2} \cdot k_{h_3}) (\epsilon_{h_3} \cdot Y_{h_3}) (\epsilon_{h_1} \cdot Y_{h_1}) \\ & + (\epsilon_{h_1} \cdot k_{h_3}) (\epsilon_{h_3} \cdot Y_{h_3}) (\epsilon_{h_2} \cdot k_{h_3}) + (\epsilon_{h_1} \cdot k_{h_2}) (\epsilon_{h_2} \cdot k_{h_3}) (\epsilon_{h_3} \cdot Y_{h_3}) \end{aligned} \quad (5.8)$$

$$\begin{aligned} C_{[321]}(132) = & (\epsilon_{h_2} \cdot Y_{h_2}) (\epsilon_{h_3} \cdot Y_{h_3}) (\epsilon_{h_1} \cdot Y_{h_1}) + (\epsilon_{h_2} \cdot k_{h_3}) (\epsilon_{h_3} \cdot Y_{h_3}) (\epsilon_{h_1} \cdot Y_{h_1}) \\ & + (\epsilon_{h_2} \cdot F_{h_1} \cdot Y_{h_1}) (\epsilon_{h_3} \cdot Y_{h_3}) + (\epsilon_{h_3} \cdot F_{h_1} \cdot Y_{h_1}) (\epsilon_{h_2} \cdot Y_{h_2}) \\ & + (\epsilon_{h_2} \cdot k_{h_1}) (\epsilon_{h_3} \cdot F_{h_1} \cdot Y_{h_1}) + (\epsilon_{h_2} \cdot k_{h_3}) (\epsilon_{h_3} \cdot F_{h_1} \cdot Y_{h_1}) \end{aligned} \quad (5.9)$$

$$\begin{aligned} C_{[321]}(231) = & (\epsilon_{h_1} \cdot Y_{h_1}) (\epsilon_{h_3} \cdot Y_{h_3}) (\epsilon_{h_2} \cdot Y_{h_2}) + (\epsilon_{h_1} \cdot k_{h_3}) (\epsilon_{h_3} \cdot Y_{h_3}) (\epsilon_{h_2} \cdot Y_{h_2}) \\ & + (\epsilon_{h_1} \cdot k_{h_2}) (\epsilon_{h_2} \cdot Y_{h_2}) (\epsilon_{h_3} \cdot Y_{h_3}) + (\epsilon_{h_3} \cdot F_{h_2} \cdot Y_{h_2}) (\epsilon_{h_1} \cdot Y_{h_1}) \\ & + (\epsilon_{h_1} \cdot k_{h_2}) (\epsilon_{h_3} \cdot F_{h_2} \cdot Y_{h_2}) + (\epsilon_{h_1} \cdot k_{h_3}) (\epsilon_{h_3} \cdot F_{h_2} \cdot Y_{h_2}) \end{aligned} \quad (5.10)$$

$$\begin{aligned} C_{[321]}(312) = & (\epsilon_{h_2} \cdot Y_{h_2}) (\epsilon_{h_1} \cdot Y_{h_1}) (\epsilon_{h_3} \cdot Y_{h_3}) + (\epsilon_{h_2} \cdot F_{h_1} \cdot Y_{h_1}) (\epsilon_{h_3} \cdot Y_{h_3}) \\ & + (\epsilon_{h_2} \cdot k_{h_3}) (\epsilon_{h_3} \cdot Y_{h_3}) (\epsilon_{h_1} \cdot Y_{h_1}) + (\epsilon_{h_1} \cdot k_{h_3}) (\epsilon_{h_3} \cdot Y_{h_3}) (\epsilon_{h_2} \cdot Y_{h_2}) \\ & + (\epsilon_{h_2} \cdot k_{h_3}) (\epsilon_{h_3} \cdot Y_{h_3}) (\epsilon_{h_1} \cdot k_{h_3}) + (\epsilon_{h_2} \cdot F_{h_1} \cdot k_{h_3}) (\epsilon_{h_3} \cdot Y_{h_3}) . \end{aligned} \quad (5.11)$$

The subscript [321] stands for the standard reference order of converting the gravitons into gluons, whose role will be discussed later. Note that these coefficients under the standard order [321] do not enjoy the explicit permutation invariance, namely,

$$C_{[321]}(213) \neq C_{[321]}(123)|_{h_1 \leftrightarrow h_2} \quad C_{[321]}(321) \neq C_{[321]}(123)|_{h_1 \leftrightarrow h_3} ,$$

for example. This is because the expansion basis is KK, which has some redundancy, such that a certain gauge choice is allowed.

It is worth to compare our $C_{[321]}(\sigma)$ with the results given in a previous paper [8], where the kinematic variable Z has been used and the coefficients contain different number of terms. Here, although $C_{[321]}(\sigma)$ are not symmetric under relabeling, the total number of terms contained in each $C_{[321]}(\sigma)$ is the same, namely, $6 = 3!$ terms for this three-graviton example. In fact, with some further observations, we find the following features:

⁸From now on, we are going to use the boldface σ to stand for an ordered set, while $\sigma(i)$ denotes the i -th element in the set σ . Please do not confuse them with the solutions to the scattering equations.

- In terms of only the ϵ , F , and Y variables, each coefficient is given by a sum of six terms. Then each of these six terms is a product of a few scalar factors, enclosed in parentheses.
- Now we count the number of elements in each parenthesis. For examples, $(\epsilon_{h_3} \cdot F_{h_2} \cdot F_{h_1} \cdot Y_{h_1})$ has 4 elements, $(\epsilon_{h_2} \cdot F_{h_1} \cdot k_{h_3})$ has 3 and $(\epsilon_{h_3} \cdot Y_{h_3})$ has 2. Then we find that for each term that appears in any $C_{[321]}(\sigma)$, we have:

$$\sum_{\text{all factors}} (n_i - 1) = 3. \quad (5.12)$$

If we represent each graviton by a vertex, and supplement them with a common root, we can then associate each factor with a path starting at a graviton vertex labeled by ϵ and ending at either a graviton vertex (labeled by k) or the root (labeled by Y), while n_i is just the number of vertices along the path. One can immediately observe that each term of $C_{[321]}(\sigma)$ provides a tree structure. The sum $\sum_{\text{all factors}} (n_i - 1)$ gives nothing but the total number of gravitons. This observation motivates us to construct the following graph theory rules based on spanning trees.

5.2 Graphic rules for coefficients of pure YM expansion

In this part, we propose a set of graphic rules for a direct reading of the pure YM expansion coefficients, based on spanning trees. The recursive relation (4.2) indicates that for a generic m -graviton EYM amplitude, we can always expand it in terms of pure YM amplitudes in the KK basis, with the following form:⁹

$$A_{n,m}^{\text{EYM}}(12\dots n | h_1 h_2 \dots h_m) = \sum_{\sigma \in S_m} \sum_{\sqcup} C_{\rho}(\sigma) A_{n+m}^{\text{YM}}(1, \{2\dots n-1\} \sqcup \sigma, n), \quad (5.13)$$

where σ is a permutation of the m gravitons $\{h_1 \dots h_m\}$ and ρ is a reference order. The coefficients $C_{\rho}(\sigma)$ can be evaluated from the spanning trees of $m+1$ vertices. Among these vertices, m of them correspond to the gravitons $\{h_1, h_2 \dots h_m\}$, and a single vertex $\mathbf{g}$ as the root, representing the gluon set $\{2\dots n-1\}$. The coefficients $C_{\rho}(\sigma)$ can be read out by the following algorithm:

Step 1: constructing the trees contributing to the order σ . We construct the *increasing trees* with respect to the order σ , i.e., $\mathbf{g} \prec h_{\sigma(1)} \prec h_{\sigma(2)} \prec \dots \prec h_{\sigma(m)}$. Such trees can be obtained by arranging vertices $h_{\sigma(1)}, h_{\sigma(2)}, \dots, h_{\sigma(m)}$ from the left to right and then drawing all possible tree diagrams after avoiding the following situations: *if two vertices $A \prec B$ are connected by a path originated from the root, then B can not be the one closer to the root*. We emphasize that the construction of the trees only depends on σ , not the reference order ρ .

⁹As kept implicit here, $C_{\rho}(\sigma)$ also depends on the shuffle product, since it contains Y_{h_i} , the sum of gluon momenta before h_i in $\{2\dots n-1\} \sqcup \sigma$.

For example, when there are three gravitons, we have the following trees contributing to $\sigma = \{123\}$:

In these trees, all vertices in a path starting from the root and ending at a leaf respect the order σ . There are exactly $m!$ such trees. This can be easily seen by a recursive construction. For example, from each tree shown in eq. (5.14), we can construct exactly four increasing trees for $\{1234\}$ by either connecting h_4 to any h_i , with $i = 1, 2, 3$, or connecting h_4 directly to the root. More explicitly, the first tree gives:

Step2: reading out the expression for a tree. Having constructed the increasing trees, we assign an expression to each tree. At this moment, the choice of ordering ρ will play an important role. We will accomplish it by following steps:

- First, we draw ordered paths. We will start from $\rho(1)$, the first graviton in list ρ , and then draw a path from the root g to $\rho(1)$. If there are ℓ vertices along this path, we will set $\phi_1 = \rho(1)$, and the subsequent vertices ϕ_2, ϕ_3 , etc, until the root $\phi_\ell = g$. We will denote such a path by $\mathcal{P}[1] = \{\rho(1), \phi_2, \dots, \phi_{\ell-1}, g\}$. An illustration of such path $\mathcal{P}[1]$ is given as the following:

It is worth to emphasize that $\rho(1)$ may or may not be a leaf.

- As the first path is done, we construct the second path from the remaining vertices:

$$\rho_1 \equiv \rho \setminus \{\phi_1 \dots \phi_{\ell-1}, \mathbf{g}\}, \quad (5.17)$$

Now we construct $\mathcal{P}[2]$ by considering the path from the root $\mathbf{g}$ to the first element of ρ_1 , denoted as $\rho_1(1)$. Let us denote this tentative path as $\mathcal{T}_2$. Starting from the root, the path $\mathcal{T}_2$ will in general coincide with other previous constructed paths until the vertex V_2 , the last vertex (counted from the root) along $\mathcal{T}_2$ that belongs to previous constructed paths. The crossroad vertex V_2 can either be a graviton or just the root. Now the part of $\mathcal{T}_2$ from V_2 and beyond, until $\rho_1(1)$, is our $\mathcal{P}[2]$. It has the form $\mathcal{P}[2] = \{\tilde{\phi}_1, \tilde{\phi}_2, \dots, \tilde{\phi}_t, V_2\}$, where $\tilde{\phi}_i \in \rho_1$ but $V_2 \notin \rho_1$. This process is illustrated as:

- Repeat this procedure until we have exhausted all the graviton vertices.
- Now we assign each path a factor. The first element in the path is replaced by ϵ , the middle element by F . For the last element, if it is the root, then we replace it by Y_t if it is the root $\mathbf{g}$ (the t is the next-to-last element); if it is a graviton vertex a , we replace it by k_a . Finally, multiplying all these factors together, we get the term contributed by the tree under our consideration.
- We emphasize that given a tree, the evaluation only depends on the reference order ρ . Actually, a tree can contribute to more than one σ in general, which will be further clarified later in this section.

To demonstrate how these rules work, we again take $\rho = \{321\}$ and consider the six trees given in (5.14) as an example. These trees all contribute to $\sigma = \{123\}$. It is easy to see that we have

tree 1 :	tree 2 :	tree 3 :
$\mathcal{P}[1] = \{h_3, \mathbf{g}\};$	$\mathcal{P}[1] = \{h_3, \mathbf{g}\};$	$\mathcal{P}[1] = \{h_3, h_2, \mathbf{g}\};$
$\mathcal{P}[2] = \{h_2, \mathbf{g}\};$	$\mathcal{P}[2] = \{h_2, h_1, \mathbf{g}\};$	$\mathcal{P}[2] = \{h_1, \mathbf{g}\};$
tree 4 :	tree 5 :	tree 6 :
$\mathcal{P}[1] = \{h_3, h_1, \mathbf{g}\};$	$\mathcal{P}[1] = \{h_3, h_1, \mathbf{g}\};$	$\mathcal{P}[1] = \{h_3, h_2, h_1, \mathbf{g}\};$
$\mathcal{P}[2] = \{h_2, \mathbf{g}\};$	$\mathcal{P}[2] = \{h_2, h_1\};$	

(5.19)

Using our rules, we can write down six coefficients given in eq. (5.19)

$$\begin{aligned}
 \text{tree 1 : } & (\epsilon_{h_3} \cdot Y_{h_3}) \times (\epsilon_{h_2} \cdot Y_{h_2}) \times (\epsilon_{h_1} \cdot Y_{h_1}); \\
 \text{tree 2 : } & (\epsilon_{h_3} \cdot Y_{h_3}) \times (\epsilon_{h_2} \cdot F_{h_1} \cdot Y_{h_1}); \\
 \text{tree 3 : } & (\epsilon_{h_3} \cdot F_{h_2} \cdot Y_{h_2}) \times (\epsilon_{h_1} \cdot Y_{h_1}); \\
 \text{tree 4 : } & (\epsilon_{h_3} \cdot F_{h_1} \cdot Y_{h_1}) \times (\epsilon_{h_2} \cdot Y_{h_1}); \\
 \text{tree 5 : } & (\epsilon_{h_3} \cdot F_{h_1} \cdot Y_{h_1}) \times (\epsilon_{h_2} \cdot k_{h_1}); \\
 \text{tree 6 : } & (\epsilon_{h_3} \cdot F_{h_2} \cdot F_{h_1} \cdot Y_{h_1});
 \end{aligned} \tag{5.20}$$

One can check that they exactly reproduce the $C_{[321]}(123)$ given in eq. (5.6). With the reference order $\rho = \{321\}$, another example is

$$\begin{array}{c} h_1 \quad h_2 \quad h_3 \\ \bullet \quad \bullet \quad \bullet \\ \hline \bullet \\ g \end{array} = (\epsilon_{h_3} \cdot Y_{h_3}) (\epsilon_{h_2} \cdot k_{h_3}) (\epsilon_{h_1} \cdot k_{h_2}) . \tag{5.21}$$

This tree contributes to $\sigma = \{321\}$ only. As some less trivial examples, we consider several typical trees that appear in the evaluation of the coefficients for four-graviton EYM amplitudes. With the reference order $\rho = \{4321\}$, we have:

$$\begin{array}{c} h_1 \quad h_2 \quad h_3 \quad h_4 \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \hline \bullet \\ g \end{array} = (\epsilon_{h_4} \cdot F_{h_3} \cdot Y_{h_3}) (\epsilon_{h_2} \cdot k_{h_3}) (\epsilon_{h_1} \cdot k_{h_4}) \\
 \begin{array}{c} h_1 \quad h_2 \quad h_3 \quad h_4 \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \hline \bullet \\ g \end{array} = (\epsilon_{h_4} \cdot Y_{h_4}) (\epsilon_{h_3} \cdot k_{h_4}) (\epsilon_{h_2} \cdot k_{h_3}) (\epsilon_{h_1} \cdot k_{h_4}) \\
 \begin{array}{c} h_1 \quad h_2 \quad h_3 \quad h_4 \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \hline \bullet \\ g \end{array} = (\epsilon_{h_4} \cdot F_{h_3} \cdot Y_{h_3}) (\epsilon_{h_2} \cdot Y_{h_2}) (\epsilon_{h_1} \cdot k_{h_2}) \\
 \begin{array}{c} h_1 \quad h_2 \quad h_3 \quad h_4 \\ \bullet \quad \bullet \quad \bullet \quad \bullet \\ \hline \bullet \\ g \end{array} = (\epsilon_{h_4} \cdot Y_{h_4}) (\epsilon_{h_3} \cdot F_{h_2} \cdot Y_{h_2}) (\epsilon_{h_1} \cdot k_{h_3})
 \end{array} \tag{5.22}$$

In the above discussion, we have given the graphic rule to read out the corresponding coefficients of a given KK basis. In eq. (5.13), there are $m!$ coefficients $C_\rho(\sigma)$, each of which contains $m!$ terms given by $m!$ increasing trees. Thus naively we would expect that there

Figure 1. The rooted tree with three levels.

are in all $(m!)^2$ trees involved. However, there are only $(m+1)^{m-1}$ distinct spanning trees for $m+1$ vertices (m graviton vertices and a gluon root). Since $(m!)^2 > (m+1)^{m-1}$ for $m \geq 2$, some trees must contribute to multiple $C_{\rho}(\sigma)$ coefficients. The next problem we want to address is that given a spanning tree, how to find out all the $C_{\rho}(\sigma)$ coefficients it can contribute to.

The answer to this question is simple: *suppose we define the set of all the increasing trees respecting the order σ as $IT(\sigma)$, then given a spanning tree T , it will contribute to all those σ 's whose $IT(\sigma)$ contains T .* These σ 's can be read out in the following way. First let us define the level structure of a spanning tree, demonstrated by figure 1:

- From the root, there are three outgoing arrows, so we have three subsets in the first level.
- The subsets of a and b are simple, which are $\{a_1, a_2, a_3\}$ and $\{b_1, b_2, b_3\}$ respectively.
- For the middle subset, it is much more complicated. Along the arrow, first we have $\{c_1, c_2, c_3\}$. However, there are two outgoing branches starting from c_3 . Thus we have a second level with two branches: one is $\{d_1, d_2\}$ and the other is $\{e_1, e_2\}$. Again, starting from e_2 there are two branches: $\{f_1, f_2\}$ and $\{g_1, g_2\}$, which give us the third level.
- Putting everything together, we have the following level structure:

$$\left\{ \begin{array}{l} \{a_1, a_2, a_3\} \\ \{b_1, b_2, b_3\} \\ \left\{ c_1, c_2, c_3, \begin{array}{l} \{d_1, d_2\} \\ \left\{ e_1, e_2, \begin{array}{l} \{f_1, f_2\} \\ \{g_1, g_2\} \end{array} \right\} \end{array} \right\} \end{array} \right\} \quad (5.23)$$

After reading out the level structure, we can write down the orderings each level gives by shuffle actions. First, there are two branches at level 3, such that this level gives:

$$\text{level3} = \{f_1, f_2\} \sqcup \{g_1, g_2\},$$

Then at the second level, there are also two branches. The first is simply $\{d_1, d_2\}$, while the second one is $\{e_1, e_2\}$ concatenating **level3**. The orderings given by this level are:

$$\text{level2} = \{d_1, d_2\} \sqcup \{e_1, e_2, \text{level3}\}.$$

Finally, at the first level, we have:

$$\text{level1} = \{a_1, a_2, a_3\} \sqcup \{b_1, b_2, b_3\} \sqcup \{c_1, c_2, c_3, \text{level2}\}.$$

After the shuffle products are fully expanded, **level1** gives the set of orderings this tree contributes to.

Next, we check this rule with our three-graviton expressions. In eq. (5.14), the first tree only has one level, such that

contributes to

$$= (\epsilon_{h_1} \cdot Y_{h_1}) (\epsilon_{h_1} \cdot Y_{h_1}) (\epsilon_{h_1} \cdot Y_{h_1}) = \{h_1\} \sqcup \{h_2\} \sqcup \{h_3\} \\ = \{h_1 h_2 h_3\} + \{h_1 h_3 h_2\} + \{h_2 h_1 h_3\} \\ + \{h_2 h_3 h_1\} + \{h_3 h_1 h_2\} + \{h_3 h_2 h_1\}$$

(5.24)

Indeed, this term is contained in all six $C_\rho(\sigma)$, given in eq. (5.6) to (5.11). For tree 5 in eq. (5.14), there is only one branch at level one but two branches at level two, such that

contributes to

$$= (\epsilon_{h_3} \cdot F_{h_1} \cdot Y_{h_1}) (\epsilon_{h_2} \cdot k_{h_1}) \{h_1, \{h_2\} \sqcup \{h_3\}\} = \{h_1 h_2 h_3\} + \{h_1 h_3 h_2\}$$

(5.25)

This expression can only be found in $C_{[321]}(123)$ and $C_{[321]}(132)$ as expected. Last but not least, the tree in eq. (5.21) only has one level, contributing only to the order $\{321\}$. Again, the corresponding term can only be found in $C_{[321]}(321)$.

Using the above understanding, we also give two four-graviton examples:

contributes to

$$\{h_4, \{h_1\} \sqcup \{h_3 h_2\}\} \\ = \{h_4 h_1 h_3 h_2\} + \{h_4 h_3 h_1 h_2\} + \{h_4 h_3 h_2 h_1\}$$

contributes to

$$\{h_2 h_1\} \sqcup \{h_3 h_4\} \\ = \{h_2 h_1 h_3 h_4\} + \{h_2 h_3 h_1 h_4\} + \{h_2 h_3 h_4 h_1\} \\ + \{h_3 h_4 h_2 h_1\} + \{h_3 h_2 h_4 h_1\} + \{h_3 h_2 h_1 h_4\}$$

(5.26)

Figure 2. A tree whose first level contains N branches.

Before closing this section, we note that the above rules can be recursively defined for generic trees. First, the level splitting happens only when a branch in the tree splits. Then for a subtree with the structure of figure 2, it will contribute to the following set of orderings:

$$\{1, 2 \dots i, \text{Branch1} \sqcup \text{Branch2} \sqcup \dots \sqcup \text{BranchN}\}, \quad (5.27)$$

where, for example, **Branch1** is the set of orderings contributed by the first branch, etc, which can be recursively constructed using the above relation, until the highest level.

6 Conclusion and discussion

In this paper, we have established a recursive expansion (4.2) for tree level single trace EYM amplitudes with m gravitons and n gluons, facilitated by expanding the CHY integrand $\text{Pf}(\Psi_H)$ along one row. It works for arbitrary number of gravitons and gluons, as we have proved in section 4. This recursive construction agrees with the one proposed in a recent paper [8], which is observed simply by imposing gauge invariance. Our paper gives a firm proof of the observation. We note that our recursive expansion holds only on-shell, namely, the momentum conservation, transversal condition and scattering equations should all be satisfied.

Very remarkably, once we fully expand the m -graviton and n -gluon EYM amplitudes in terms of pure YM amplitudes with $n + m$ gluons in the KK basis, all the expansion coefficients can be written down through a set of elegant graph theory rules. To be specific, the coefficient of each KK basis amplitude contains exactly $m!$ terms, each of which corresponds to an increasing tree with respect to the order of the gravitons in a certain KK basis amplitude. Then the evaluation of each tree relies on a reference order. On one hand, the origin of this reference order is how we expand $\text{Pf}(\Psi_H)$ step by step. On the other hand, it serves as a “gauge choice” in the over-complete KK basis YM amplitudes.

From these results, one very promising direction is of course the construction BCJ numerators [28, 29]. It is pointed out in [8] that the YM integrand $\text{Pf}'(\Psi)$ can be readily expanded in terms of the EYM integrands with polynomial coefficients. Then our results essentially provides a working algorithm to write down the polynomial BCJ numerators for tree level YM, especially the spanning tree structure which has not been explicitly spelled out in this paper. This study may shed lights on new understandings of the numerator algebra, and also on possible new formulations of YM that manifest the numerator algebra.

Such a formulation already exists for NLSM [30], and it will be fascinating if it also exists for YM.

Our work shows that a controlled expansion of CHY integrands can produce very fruitful results. It will be interesting to see, for example, how the expansion of $\text{Pf}'(A)$, the CHY integrand for NLSM,¹⁰ interplays with [30]. One can also try the CHY integrand at the one-loop level for YM theory [34–36]. Another topic that worths more attention is the study of multitrace EYM integrands [2, 4, 14], and its relation with the known results constructed from double copy [26, 27].

Note added. During the final completion of this work, we came aware of [37], which partially overlaps with our results.

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A Understanding the graphic rules

This appendix is devoted to show how our graphic rules given in section 5 emerge from our recursive relation (4.2). Suppose we are about to expand the m -graviton integrand

$$(-1)^m \text{Pf}(\Psi_H) \text{PT}(12 \dots n),$$

we have to make a choice on along which row do we expand the Pfaffian. Throughout our discussion in this paper, we have consistently chosen h_m , namely,

$$(-1)^m \text{Pf}(\Psi_H) \text{PT}(12 \dots n) = \epsilon_{h_m} \cdot T[1, \{2 \dots n-1\} \sqcup \{h_m\}, n \mid \{h_{m-1} \dots h_1\}]. \quad (\text{A.1})$$

Of course the value of this integrand will not change should we choose any other graviton to perform the expansion, although the final pure YM expansion coefficients may have a different form. Now we carry out the expansion one step further, according to eq. (4.2):

$$\begin{aligned} & \epsilon_{h_m} \cdot T[1, \{2 \dots n-1\} \sqcup \{h_m\}, n \mid \{h_{m-1} \dots h_1\}] \\ &= \sum_{\sqcup} (\epsilon_{h_m} \cdot Y_{h_m}) \text{Pf}(\Psi_{h_{m-1} \dots h_1}^{n+m}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_m\}, n) \\ &+ (-1)^{m-1} \sum_{s=1}^{m-1} \epsilon_{h_m} \cdot F_{h_s} \cdot T[1, \{2 \dots n-1\} \sqcup \{h_s h_m\}, n \mid \{h_{m-1} \dots h_s \dots h_1\}]. \end{aligned} \quad (\text{A.2})$$

In the first term, the factor $(\epsilon_{h_m} \cdot Y_{h_m})$ can be viewed as connecting the graviton vertex h_m to the root, while in the second term, $(\epsilon_{h_m} \cdot F_{h_s})^\mu$ can be interpreted as connecting h_m to another graviton h_s , since in the ordering $\{h_s h_m\}$, h_s is at the left hand side of the h_m .

¹⁰Such an expansion of the NLSM integrand should give us a set of BCJ numerators in the KK basis. It is also interesting to compare this result with known forms of the NLSM BCJ numerators [31–33].

Figure 3. Schematic expansion of the EYM integrand.

For the second term, further expansion will again lead to two term, one with F_{h_s} connected to the root, giving $(\epsilon_{h_m} \cdot F_{h_s} \cdot Y_{h_s})$ and an $(m-2)$ -graviton EYM integrand:

$$\sum_{\sqcup} (\epsilon_{h_m} \cdot F_{h_s} \cdot Y_{h_s}) \text{Pf}(\Psi_{h_{m-1} \dots h_s \dots h_1}^{n+m}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{h_s h_m\}, n),$$

and the other with F_{h_s} connected to yet another graviton, giving

$$\epsilon_{h_m} \cdot F_{h_s} \cdot F_{h_t} \cdot T[1, \{2 \dots n-1\} \sqcup \{h_t h_s h_m\}, n \mid \{h_{m-1} \dots h_s \dots h_t \dots h_1\}].$$

We can do it recursively and this procedure gives, in fact, a construction of those *increasing trees* presented in **Step 1** in the section 5. Furthermore, the starting vertex h_m plays the role of $\rho(1)$ in the reference order ρ . For convenience, we show this expansion process schematically in figure 3.

Now we work out more details for the above construction by an example. We first keep following the red arrows in figure 3 to connect more gravitons until the vertex ϕ_ℓ , which is then connected to the gluon root. In this way, we get a chain evaluated as

$$\begin{array}{ccccccc} \bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \\ \mathbf{g} & \phi_\ell & \phi_{\ell-1} & & \phi_2 & \phi_1 & \end{array} = (\epsilon_{\phi_1} \cdot F_{\phi_2} \cdots F_{\phi_m} \cdot Y_{\phi_m}), \quad (\text{A.3})$$

where we have defined the first vertex $\phi_1 \equiv h_m$. In the scheme of figure 3, terminating a graviton chain at the gluon root essentially means that we switch to a blue arrow and arrive at the block

$$\sum_{\sqcup} (\epsilon_{\phi_1} \cdot F_{\phi_2} \cdots F_{\phi_m} \cdot Y_{\phi_m}) \text{Pf}(\Psi_{\theta_1 \dots \theta_t}^{n+m}) \text{PT}(1, \{2 \dots n-1\} \sqcup \{\phi_\ell \dots \phi_1\}, n),$$

where $\{\theta_1 \dots \theta_t\}$ is the complement of $\{\phi_1 \dots \phi_\ell\}$ in $H = \{h_1 \dots h_m\}$. Now we come across a Pfaffian again, and we have to choose another graviton along which we expand this Pfaffian. Thus here comes the origin of the reference order ρ : *it is a list of priority we give to each graviton, such that once we need to expand a Pfaffian, we always choose to expand along the one with the highest priority according to ρ* . In this case, we simply delete $\{\phi_1 \dots \phi_\ell\}$ from

ρ and pick up the first remaining graviton, say θ_1 . We then stay with the red arrows until we deplete all the gravitons, which gives us another chain. The final block we reach gives:

$$\sum_{\sqcup_1} \sum_{\sqcup_2} (\epsilon_{\phi_1} \cdot F_{\phi_2} \cdots F_{\phi_\ell} \cdot Y_{\phi_\ell}) (\epsilon_{\theta_1} \cdot F_{\theta_2} \cdots F_{\theta_t} \cdot X_{\theta_t}) \times \text{PT}(1, \{2 \dots n-1\} \sqcup_1 \{\phi_\ell \dots \phi_1\} \sqcup_2 \{\theta_t \dots \theta_1\}). \quad (\text{A.4})$$

Since we have treated $\{\phi_\ell \dots \phi_1\}$ as gluons when expanding the θ -chain, we need to include k_{ϕ_i} into X_{θ_t} if there are some ϕ_i before θ_t :

$$X_{\theta_t} = Y_{\theta_t} + \sum_{\phi_i \prec \theta_t} k_{\phi_i}. \quad (\text{A.5})$$

After fully expanding the shuffle $\{\phi_\ell \dots \phi_1\} \sqcup \{\theta_t \dots \theta_1\}$, one can notice that the factor $(\epsilon_{\theta_1} \cdot F_{\theta_2} \cdots F_{\theta_t} \cdot Y_{\theta_t})$ is contained in the coefficients of all the corresponding KK basis:

$$(\epsilon_{\phi_1} \cdot F_{\phi_2} \cdots F_{\phi_\ell} \cdot Y_{\phi_\ell}) (\epsilon_{\theta_1} \cdot F_{\theta_2} \cdots F_{\theta_t} \cdot Y_{\theta_t}) \subset \{\phi_\ell \dots \phi_1\} \sqcup \{\theta_t \dots \theta_1\}. \quad (\text{A.6})$$

In addition, the factor $(\epsilon_{\theta_1} \cdot F_{\theta_2} \cdots F_{\theta_t} \cdot k_{\phi_\ell})$ appears when ϕ_ℓ is ahead of θ_t :

$$(\epsilon_{\phi_1} \cdot F_{\phi_2} \cdots F_{\phi_\ell} \cdot Y_{\phi_\ell}) (\epsilon_{\theta_1} \cdot F_{\theta_2} \cdots F_{\theta_t} \cdot k_{\phi_\ell}) \subset \{\phi_\ell, \{\phi_{\ell-1} \dots \phi_1\} \sqcup \{\theta_t \dots \theta_1\}\}. \quad (\text{A.7})$$

Generally, we have:

$$(\epsilon_{\phi_1} \cdot F_{\phi_2} \cdots F_{\phi_\ell} \cdot Y_{\phi_\ell}) (\epsilon_{\theta_1} \cdot F_{\theta_2} \cdots F_{\theta_t} \cdot k_{\phi_i}) \subset \{\phi_\ell \dots \phi_i, \{\phi_{i-1} \dots \phi_1\} \sqcup \{\theta_t \dots \theta_1\}\}. \quad (\text{A.8})$$

Now one can understand how the structure of increasing tree emerges. We can graphically represent eq. (A.6) to (A.8) as follows:

$$= (\epsilon_{\phi_1} \cdot F_{\phi_2} \cdots F_{\phi_m} \cdot Y_{\phi_\ell}) (\epsilon_{\theta_1} \cdot F_{\theta_2} \cdots F_{\theta_t} \cdot Y_{\theta_t}), \quad (\text{A.9})$$

$$= (\epsilon_{\phi_1} \cdot F_{\phi_2} \cdots F_{\phi_m} \cdot Y_{\phi_\ell}) (\epsilon_{\theta_1} \cdot F_{\theta_2} \cdots F_{\theta_t} \cdot k_{\phi_i}). \quad (\text{A.10})$$

It is interesting to observe that the tree in (A.9) is an increasing tree respecting any $\sigma \in \{\phi_\ell \dots \phi_1\} \sqcup \{\theta_t \dots \theta_1\}$. Similarly, the tree in (A.10) is an increasing tree respecting any $\sigma \in \{\phi_\ell \dots \phi_i, \{\phi_{i-1} \dots \phi_1\} \sqcup \{\theta_t \dots \theta_1\}\}$. Our derivation also shows that the terms evaluated from eq. (A.9) and (A.10) do contribute to all such σ 's represented by the two trees. Finally, this example shows how we should close a chain that ends on a graviton vertex, as given in **Step 2** in section 5

The above two-chain calculation demonstrates how the graph theory rules presented in section 5, in their primitive forms, originates from our recursive relation. Of course, we have already refined these rules into a set of algorithms in section 5 such that they can be applied to the most generic cases.

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