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Manifestly T-dual formulation of AdS space

Machiko Hatsuda,^{a,b} Kiyoshi Kamimura^a and Warren Siegel^{c,1}

^a*Physics Division, Faculty of Medicine, Juntendo University,
Chiba 270-1695, Japan*

^b*KEK Theory Center, High Energy Accelerator Research Organization,
Tsukuba, Ibaraki 305-0801, Japan*

^c*C.N. Yang Institute for Theoretical Physics, Stony Brook University,
Stony Brook, NY 11794-3840, U.S.A.*

E-mail: mhatsuda@juntendo.ac.jp, kamimura@ph.sci.toho-u.ac.jp,
siegel@insti.physics.sunysb.edu

ABSTRACT: We present a manifestly T-dual formulation of curved spaces such as an AdS space. For group manifolds related by the orthogonal vielbein fields the three form $H = dB$ in the doubled space is universal at least locally. We construct an affine nondegenerate doubled bosonic AdS algebra to define the AdS space with the Ramond-Ramond flux. The non-zero commutator of the left and right momenta leads to that the left momentum is in an AdS space while the right momentum is in a dS space. Dimensional reduction constraints and the physical AdS algebra are shown to preserve all the doubled coordinates.

KEYWORDS: String Duality, AdS-CFT Correspondence, Space-Time Symmetries

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¹<http://insti.physics.sunysb.edu/siegel/plan.html>

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1 Introduction and conclusions

T-duality is one of the most characteristic features of string theories. The T-duality symmetry exists in its low energy effective theory described by the massless modes. Such a stringy gravity theory is a theory of the gravitational field G_{mn} , the B_{mn} field and the dilaton field. The general coordinate transformation is generalized in a T-duality covariant way. It is shown to be generated by the zero mode of the affine nondegenerate doubled Lie algebra [1–3]. This manifestly T-dual formulation is the procedure to construct gravity theories and it is being developed in [4]–[10]. The procedure contains roughly two steps: doubling the d-dimensional coordinates to manifest the $O(d,d)$ T-duality symmetry and imposing constraints to reduce a half of the doubled coordinates preserving the T-duality symmetry. For a flat space the procedure is straightforward, however for curved spaces it becomes nontrivial.

T-duality along a non-abelian isometry had been proposed [11–15] and non-abelian T-duality in AdS spaces has been investigated in for example [16–19], in which there remain many interesting problems to solve. Recently the equivalence between the integrable deformation of the AdS superstring and the non-abelian T-duality was proposed in [20] and has been developed in [21–23]. It was shown that the nonlocal charges of a string are equal to the Noether charges of a string in the T-dualized space for a flat space and the pp-wave space [24] which gives a relation between the integrability and the abelian T-duality. The superstring in the $AdS_5 \times S^5$ space has integrability [25], and the nonlocal charges generate the Yangian algebra as shown in [26, 27] based on the Hamiltonian formulation of the AdS string [28, 29]. In order to clarify the features of the non-abelian T-duality and its integrability the manifestly T-dual formulation of AdS space will be useful as the doubled space analysis. The superspace approach to the AdS space with manifestly T-duality is presented in [30] based on the super-AdS algebra in [8].

The double field theory on group manifolds has been studied in [31–34]. Our formulation is different from theirs in the following points, though it has some overlap:

- Vielbein fields are used to express the scalar curvature of the stringy gravity theory rather than the generalized metric. This is necessary to couple spacetime fermions.
- First class constraints are used to reduce a half of the doubled coordinates rather than the gauge fixing condition for the section constraint. This preserves the T-duality covariant local symmetry.
- The Wess-Zumino term in the worldsheet action is also doubled in order to give both the covariant derivative currents and the symmetry generator currents for both the left and right modes. The usual Wess-Zumino-Witten procedure gives the covariant derivative current for the right mode and the symmetry generator current for the left mode. This is necessary to describe $N=2$ supersymmetric theories.

The manifestly T-dual formulation is a gauge theoretical formulation of the stringy gravity: the gauge theory is a theory of the gauge field A_m^I with a gauge group generated by G_I . The covariant derivative ∇_m is an essential operator which gives the gauge

transformation rule, and its commutator gives the field strength F_{mn}^I

$$\begin{aligned} p_m &= \frac{1}{i} \partial_m \rightarrow \nabla_m = p_m + A_m^I G_I \\ \delta_\lambda \nabla_m &= i[\Lambda, \nabla_m], \quad \Lambda = \lambda^I G_I \Rightarrow \delta_\lambda A_m = -\partial_m \lambda \\ i[\nabla_m, \nabla_n] &= F_{mn}^I G_I. \end{aligned} \quad (1.1)$$

The gravity theory is a gauge theory of the vielbein e_a^m [35]. The covariant derivative ∇_a gives the general coordinate transformation rule as

$$\begin{aligned} p_m &\rightarrow \nabla_a = e_a^m p_m \\ \delta_\lambda \nabla_a &= i[\Lambda, \nabla_a], \quad \Lambda = \lambda^m p_m \Rightarrow \delta_\lambda e_a^m = \mathcal{L}_\lambda e_a^m = \lambda^n \partial_n e_a^m - e_a^n \partial_n \lambda^m. \end{aligned} \quad (1.2)$$

In order to obtain the curvature tensor the Lorentz generator s_{mn} must be taken into account as the generator. The covariant derivative includes the Lorentz connection ω_a^{mn} and its commutator gives the Riemannian curvature tensor R_{ab}^{cd} as

$$\nabla_a = e_a^m p_m + \frac{1}{2} \omega_a^{mn} s_{mn}, \quad i[\nabla_a, \nabla_b] = T_{ab}^c \nabla_c + \frac{1}{2} R_{ab}^{cd} s_{cd}, \quad (1.3)$$

where the torsion constraint $T_{ab}^c = 0$ relates e_a^m and ω_a^{mn} . This is extended to the stringy gravity by doubling the momentum including the winding mode as $p_m \rightarrow (p_m(\sigma), \partial_\sigma x^m(\sigma))$ [1–3]. The stringy gravity is a gauge theory of the doubled vielbein field $e_{\underline{a}}^{\underline{m}} \in \text{O}(d, d)/\text{O}(d-1, 1)^2$ with the doubled indices $\underline{m} = (m, {}^m)$ and $\underline{a}$. $e_{\underline{a}}^{\underline{m}}$ is considered as a function of $x^{\underline{m}}$ here, although it will be considered as a function of the doubled coordinates later. The stringy covariant derivative $\triangleright_{\underline{a}}(\sigma)$ gives the gauge transformation rule $\delta_\lambda e_{\underline{a}}^{\underline{m}}$ as

$$\begin{aligned} p_m &\rightarrow \nabla_a \rightarrow \triangleright_{\underline{a}}(\sigma) = e_{\underline{a}}^{\underline{m}} p_m + e_{\underline{a}m} \partial_\sigma x^m \\ \delta_\lambda \triangleright_{\underline{a}}(\sigma) &= i[\Lambda, \triangleright_{\underline{a}}(\sigma)], \quad \Lambda = \int d\sigma (\lambda^{\underline{m}} p_m + \lambda_m \partial_\sigma x^m). \end{aligned} \quad (1.4)$$

If $e_{\underline{a}}^{\underline{m}}$ is written in terms of G_{mn} and B_{mn} with the doubled Minkowski metric $\hat{\eta}^{\underline{a}\underline{b}}$, then the T-duality covariantized general coordinate transformation is given by

$$e_{\underline{a}}^{\underline{m}} \hat{\eta}^{\underline{a}\underline{b}} e_{\underline{b}}^{\underline{n}} = \begin{pmatrix} G^{mn} & G^{ml} B_{ln} \\ -B_{ml} G^{ln} & G_{mn} - B_{ml} G^{lk} B_{kn} \end{pmatrix} \Rightarrow \begin{cases} \delta_\lambda G_{mn} = \mathcal{L}_\lambda G_{mn} \\ \delta_\lambda B_{mn} = \mathcal{L}_\lambda B_{mn} - \partial_{[m} \lambda_{n]} \end{cases} \quad (1.5)$$

The curvature tensors are obtained by taking into account the Lorentz generator. The left and right Lorentz generators $S_{\underline{mn}} = (S_{mn}, \sqrt{2} s_{m'n'})$ are introduced in the left and right basis, $P_{\underline{m}} = (P_m, P_{m'})$ with $P_m = (p_m + \partial_\sigma x^m)/\sqrt{2}$, $P_{m'} = (p_m - \partial_\sigma x^m)/\sqrt{2}$ in the unitary gauge. The consistency of the “affine” Lie algebra requires the nondegeneracy of the group metric. The resultant algebra is the affine nondegenerate doubled Poincaré algebra generated by $\overset{\square}{\triangleright}_{\underline{M}}(\sigma) = (S_{\underline{mn}}, P_{\underline{m}}, \Sigma^{\underline{mn}})$ where $\Sigma^{\underline{mn}}$ is necessary for the nondegeneracy. The stringy covariant derivative $\triangleright_{\underline{A}}$ includes the doubled vielbein $E_{\underline{A}}^{\underline{M}}$, and its commutator becomes

$$\triangleright_{\underline{A}} = E_{\underline{A}}^{\underline{M}} \overset{\square}{\triangleright}_{\underline{M}}, \quad i[\triangleright_{\underline{A}}, \triangleright_{\underline{B}}] = T_{\underline{AB}}^{\underline{C}} \triangleright_{\underline{C}} \quad (1.6)$$

where the curvature tensors are included in the torsions as $T_{PP}^{\underline{S}}$.

In this paper we extend this manifestly T-dual formulation in the asymptotically flat space [1–3], [5]–[10] to curved spaces such as an AdS space. It is interesting and important to discuss supersymmetric case. However we restrict bosonic algebra in this paper leaving the supersymmetric case for future works. Our main result is the affine nondegenerate doubled bosonic AdS algebra (5.18)–(5.23) which defines the AdS space with manifest T-duality and generates the T-duality covariant general coordinate transformation.

The results are based on the following observations which we found in this paper:

- **Local universality of the three form $H = dB$ in the doubled space**

For curved spaces described by Lie algebras the three form $H = dB$ in the doubled space is universal at least locally. The doubled space three form of a group manifold is given by $H = \frac{1}{3!} J^{\underline{I}} \wedge J^{\underline{J}} \wedge J^{\underline{K}} f_{\underline{I}\underline{J}\underline{K}}$ with the left-invariant current $J^{\underline{I}}$, the structure constant $f_{\underline{I}\underline{J}\underline{K}} = f_{\underline{I}\underline{J}}^{\underline{L}} \eta_{\underline{L}\underline{K}}$ and the nondegenerate group metric $\eta_{\underline{I}\underline{J}}$. Doubled indices run over the left and right indices $\underline{I} = (I, I')$. The doubled space three form H is the sum of fluxes chained by T-duality, $H_{\underline{I}\underline{J}\underline{K}} \rightarrow f_{\underline{I}\underline{J}}^{\underline{K}} \rightarrow Q_{\underline{I}}^{\underline{J}\underline{K}} \rightarrow R^{\underline{I}\underline{J}\underline{K}}$ in the base of $(p_m, \partial_\sigma x^m)$ as $\underline{I} = (I, I')$. The flat space operators are denoted with “ $\square$ ”. The stringy covariant derivatives in curved and flat spaces are by the vielbein $E_{\underline{I}}^{\underline{M}}$. We show that the doubled space three form H in curved and flat spaces are the same

$$\triangleright_{\underline{I}} = E_{\underline{I}}^{\underline{M}} \triangleright_{\underline{M}}^{\square}, \quad E_{\underline{I}}^{\underline{M}} E_{\underline{J}}^{\underline{N}} \eta_{\underline{M}\underline{N}} = \eta_{\underline{I}\underline{J}} \rightarrow H = \overset{\square}{H}. \quad (1.7)$$

The gauge transformation of B field is also recognized as a T-duality rotation. The dilaton factor may play a role for a different value of the doubled space three form.

The doubled flat space is described by the coset group G/H where G is the nondegenerate doubled Poincaré group and H is the doubled Lorentz group $\times$ (dimensional reduction constraint) [9]. The doubled space three form $\overset{\square}{H} = d\overset{\square}{B}$ in the flat space belongs to a trivial class of the Chevalley-Eilenberg (CE) cohomology [36, 37] of the coset G/H . We have shown that $\overset{\square}{B}_{\underline{I}\underline{J}}$ in the doubled flat space is constant, so the Wess-Zumino term $\overset{\square}{B} = \overset{\square}{B}_{\underline{I}\underline{J}} \overset{\square}{J}^{\underline{I}} \wedge \overset{\square}{J}^{\underline{J}}$ is bilinears of the left-invariant currents [9, 10]. For the nondegenerate doubled AdS coset group, the three form $\overset{\circ}{H}$ is closed, $d\overset{\circ}{H} = 0$, but it belongs to a nontrivial class of the CE cohomology. The supersymmetry will change the situation as the sup-AdS group in the non-doubled space [38].

- **Spontaneous symmetry breaking by the Ramond-Ramond flux**

When the Ramond-Ramond (RR) flux has a non-zero vacuum expectation value, $\langle 0 | F_{\text{RR}}^{\alpha\beta'} | 0 \rangle \neq 0$, the Lorentz symmetry is broken; the full Lorentz symmetry is broken into its subgroup and the left and right Lorentz symmetries in the doubled space are broken into a linear combination of them. It is natural to expect non-zero commutator of the left and right momenta p_a and $p_{a'}$ as well as the non-zero anticommutator of the left and right supercovariant derivatives. We found that the nondegenerate doubled

bosonic AdS algebra includes

$$\begin{aligned} [p_a, p_b] &= i \left(\frac{1}{r_{\text{AdS}}^2} s_{ab} + \sigma_{ab} \right), \quad [p_{a'}, p_{b'}] = i \left(\frac{1}{r_{\text{AdS}}^2} s_{a'b'} + \sigma_{a'b'} \right) \\ [p_a, p_{b'}] &= i \left(\frac{1}{r_{\text{AdS}}^2} s_{ab'} + \sigma_{ab'} \right) \end{aligned} \quad (1.8)$$

where s_{ab} 's and σ^{ab} 's are nondegenerate partners. r_{AdS} is the AdS radius. The momentum p_a is a d-dimensional vector and the doubled momentum must be a $\text{SO}(d,d)$ vector. Therefore the third equation in (1.8) leads to that the left and right momenta are embedded in $\text{SO}(d,d+1)$. The left moving momentum is in an AdS space while the right moving momentum is forced to be in a dS space. This phenomena is similar to the point discussed in [16]. Now the doubled Lorentz group is $\text{SO}(d,d)$ instead of $\text{SO}(d-1,1) \times \text{SO}(1,d-1)$. Similarly the d-dimensional sphere is described by the coset $\text{SO}(d+1,d)/\text{SO}(d,d)$ in the doubled space.

The RR flux of the $\text{AdS}_5 \times S^5$ in the type IIB superstring theory breaks the $\text{SO}(9,1)$ Lorentz symmetry into $\text{SO}(4,1) \times \text{SO}(5)$. Naive doubling of the Lorentz subgroup does not give the correct number of degrees of freedom of G_{mn} and B_{mn} . The number of dimensions of the naive coset, $\text{O}(10,10)/[\text{SO}(4,1) \times \text{SO}(1,4) \times \text{SO}(5)^2]$ is not 10^2 . We solve this puzzle; now the doubled Lorentz group is $\text{O}(5,5)^2$ so the coset becomes $\text{O}(10,10)/\text{SO}(5,5)^2$ whose number of dimensions coincides with the number of degrees of freedom of G_{mn} and B_{mn} .

- **Nondegenerate non-abelian group**

A general method to construct a nondegenerate group is the followings: copy the subgroup H_0 of a coset group G/H_0 to H_1 and take the direct product: $G \rightarrow G \times H_1$ [39, 40]. Make subgroups by the semidirect product of H and $\check{H}$ from H_0 and H_1 , $H_0 \times H_1 \rightarrow H \ltimes \check{H}$, where H is generated by the vector type currents and $\check{H}$ is generated by the axial vector type currents. Then the nondegenerate group metric for H and $\check{H}$ is introduced as

$$\begin{aligned} G &\rightarrow G \times H_1 \\ H_0 &\rightarrow H_0 \times H_1 \rightarrow H \ltimes \check{H}, \quad \text{with } \text{tr}(h\check{h}) = \eta_{h\check{h}}, \quad h, \check{h} \in \text{Lie algebras of } H, \check{H}. \end{aligned} \quad (1.9)$$

H and $\check{H}$ correspond to the Lorentz group and its nondegenerate partner.

- **Dimensional reduction constraints for nondegenerate partners**

These dimensions of nondegenerate partners are unphysical and reduced by imposing dimensional reduction constraints. For an element a of a group A the covariant derivative and the symmetry generator are calculated from $a^{-1}da$ and $(da)a^{-1}$ respectively. We denote a group A which is generated by the covariant derivative and $\tilde{A}$ which is generated by the symmetry generator. $A \times \tilde{A}$ acts on a by $a \rightarrow \tilde{c}ab^{-1}$, $b \in A$ and $\tilde{c} \in \tilde{A}$. The nondegenerate coset group is obtained from (1.9) as $G/H_0 \rightarrow G \times H_1/H \ltimes \check{H}$. However H and $\check{H}$ can not be imposed as first class constraints because of the Schwinger

term for the nondegeneracy. Instead H and $\tilde{H}$ can be imposed as first class constraints, since the covariant derivative and the symmetry generator commute. So the obtained coset is

$$\frac{G}{H_0} \xrightarrow{\text{nondegenerate}} \frac{G \times H_1}{H \times \tilde{H}} \quad (1.10)$$

The d -dimensional AdS space is described in the doubled space with nondegeneracy as;

$$\frac{SO(d-1, 2)}{SO(d-1, 1)} \xrightarrow{\text{double}} \frac{SO(d, d+1)}{SO(d, d)_0} \xrightarrow{\text{nondegenerate}} \frac{SO(d, d+1) \times SO(d, d)_1}{SO(d, d) \times \tilde{SO}(d, d)} \quad (1.11)$$

Similarly the d -dimensional sphere is described in the doubled space with nondegeneracy as;

$$\frac{SO(d+1)}{SO(d)} \xrightarrow{\text{double}} \frac{SO(d+1, d)}{SO(d, d)_0} \xrightarrow{\text{nondegenerate}} \frac{SO(d+1, d) \times SO(d, d)_1}{SO(d, d) \times \tilde{SO}(d, d)} \quad (1.12)$$

For a special case of $AdS_5 \times S^5$ we find that the group structure of the bosonic part is

$$\frac{SO(5, 6) \times SO(5, 5)_1}{SO(5, 5) \times \tilde{SO}(5, 5)} \times \frac{SO(6, 5) \times SO(5, 5)_1}{SO(5, 5) \times \tilde{SO}(5, 5)}. \quad (1.13)$$

• Dimensional reduction constraints for doubled momenta

A half of the doubled momenta is reduced by the dimensional reduction constraint, $\phi_a = \tilde{P}_a - \tilde{P}_{a'} \delta_a^{a'} = 0$. The symmetry generators of the affine algebras are $\tilde{P}$ for momentum and $\tilde{S}$ for Lorentz generator. The physical AdS algebra is generated by the physical momentum $\tilde{P}_{\text{total};a}$ and the physical Lorentz generator $\tilde{S}_{\text{total};ab}$ without gauge fixing for a half coordinates;

$$\begin{aligned} \tilde{P}_{\text{total};a} &= \tilde{P}_a + \tilde{P}_{a'} \delta_a^{a'} + \dots, \quad \tilde{S}_{\text{total};ab} = \tilde{S}_{ab} - \tilde{S}_{a'b'} \delta_a^{a'} \delta_b^{b'} + \dots \\ \left[\int \tilde{P}_{\text{total};a}, \int \tilde{P}_{\text{total};b} \right] &= \frac{i}{r_{\text{AdS}}^2} \int \tilde{S}_{\text{total};ab} \end{aligned} \quad (1.14)$$

where $\dots$ includes first class constraints and the left-right mixing term. The equation (1.14) is the expected physical AdS algebra.

The organization of the paper is the following. In section 2 we explain the procedure of the manifestly T-dual formulation. Notations are listed there. The general method to construct a nondegenerate Lie algebra and to double the Lie group is presented. Then affine extension of the obtained Lie algebra is performed. The equation on the B field is obtained. The computation of the zero mode of the affine Lie algebra is demonstrated. In section 3 the manifestly T-dual formulation of the flat space is reviewed. The B field is constant where the dilatation operator plays a role. The relation between the dimensional reduction constraints and the section condition is explained. In section 4 the manifestly T-dual formulation of curved spaces is presented. After examining the relation between

the flat covariant derivative and the curved space covariant derivatives of group manifolds, the B field and the three form $H = dB$ are obtained. In section 5 the manifestly T-dual formulation of an AdS space is presented. It is explained that the RR flux naturally gives the left and right mixing Lorentz generators. The nondegenerate doubled AdS algebra is obtained, then affine extension is performed. The dimensional reduction constraints and the physical AdS algebra are obtained with manifest T-duality.

2 Manifestly T-dual formulation

At first we explain the procedure of the manifestly T-dual formulation. List of notations is also in subsection 2.1. In subsection 2.2.1 the general method to construct a nondegenerate Lie algebra is presented. In subsection 2.2.2 it is shown that doubled coordinates are convenient to describe the closed string mechanics and doubling the whole group gives simpler treatment of the system. In subsection 2.3 we extend the obtained nondegenerate doubled Lie algebra to affine Lie algebras generated by the string covariant derivative $\triangleright_I$ and the string symmetry generator $\tilde{\triangleright}_I$. The B field appears in the string covariant derivatives $\triangleright_I$ as the relative coefficient of the particle covariant derivative ∇_I and the σ component of the left-invariant current J_1^I . The affine Lie algebra gives the equation on the B field. The space with manifest T-duality is defined by the affine Lie algebra generated by the string covariant derivative. The gauge symmetry of the space is generated by the affine Lie derivative. The computation of the zero mode of the affine Lie algebra is demonstrated.

2.1 Procedure and notations

In this subsection we present the manifestly T-dual formulation and notations proposed in [8]–[10] based on [1–3]–[6, 7]. The procedure is the following:

1. Extend a Lie algebra to an affine doubled algebra.

Begin with a Lie algebra and extend it in such a way that the nondegenerate group metric can be defined in order to construct an affine Lie algebra consistently. Double the whole algebra in order to make T-duality symmetry manifest. Perform affine extension of the Lie algebra which include the nondegenerate group metric as the coefficient of the Schwinger term.

2. Construct the covariant derivative and the symmetry generator for a string action with manifestly T-duality.

There are two kinds of affine Lie algebras generated by the covariant derivative $\triangleright_I$ and the symmetry generator $\tilde{\triangleright}_I$. The covariant derivative defines the space which has the T-duality covariant diffeomorphism. The symmetry generator makes dimensional reduction constraints and the physical symmetry algebra.

3. Make the curved space covariant derivative for a gravity theory with manifestly T-duality.

The covariant derivative in a curved space is obtained by multiplying the vielbein field E_A^I on the asymptotic space covariant derivative $\triangleright_I$ as $\triangleright_A = E_A^I \triangleright_I$. The

commutator of the curved space covariant derivatives gives the torsion. Curvature tensors are included in torsions in this formalism.

4. Reduce unphysical dimensions.

A half of the doubled coordinates is reduced by dimensional reduction constraint. The auxiliary dimensions introduced for the nondegeneracy are also reduced by the dimensional reduction constraints. Since dimensional reduction constraints are written in terms of the symmetry generators, the local structure determined by the covariant derivative is still preserved so the T-duality is manifest.

Notations of covariant derivatives and symmetry generators are summarized as below.

$$\begin{array}{ccccc}
 \text{Covariant derivatives :} & & & & \\
 \text{space} & \xrightarrow{\text{Lie algebra structure const. (torsion)}} & \text{particle} & \rightarrow & \text{string} \\
 & & G_I, f_{IJK} & & \nabla_I \quad \triangleright_I \\
 \text{Poincaré} & & G_M, \overset{\square}{f}_{MNL} & & \overset{\square}{\nabla}_M \quad \overset{\square}{\triangleright}_M \\
 \downarrow & & & & \\
 \text{Curved} & & (T_{ABC}) & & \nabla_A = E_A^M \overset{\square}{\nabla}_M \quad \triangleright_A = E_A^M \overset{\square}{\triangleright}_M \\
 \text{AdS} & & G_A, \overset{\circ}{f}_{ABC} & & \overset{\circ}{\nabla}_A \quad \overset{\circ}{\triangleright}_A \\
 \downarrow & & & & \\
 \text{Curved} & & (T_{MNL}) & & \nabla_M = E_M^A \overset{\circ}{\nabla}_A \quad \triangleright_M = E_M^A \overset{\circ}{\triangleright}_A
 \end{array} \tag{2.1}$$

In curved backgrounds covariant derivatives couple to gravitational fields, E_A^I , and the commutator of the covariant derivatives gives torsions, T_{IJK} . The factorization of the vielbein, $\triangleright_A = E_A^I \triangleright_I$, is a general feature of a string theory explained in section 2.2.2.

$$\begin{array}{ccccc}
 \text{Symmetry generators :} & & & & \\
 \text{space} & \xrightarrow{\text{Lie algebra structure constant}} & \text{particle} & \rightarrow & \text{string} \\
 & & G_I, f_{IJK} & & \tilde{\nabla}_I \quad \tilde{\triangleright}_I \\
 \text{Poincaré} & & G_M, \overset{\square}{f}_{MNL} & & \overset{\square}{\tilde{\nabla}}_M \quad \overset{\square}{\tilde{\triangleright}}_M \\
 \downarrow & & & & \\
 \text{Curved} & & - & & - \\
 \text{AdS} & & G_A, \overset{\circ}{f}_{ABC} & & \overset{\circ}{\tilde{\nabla}}_A \quad \overset{\circ}{\tilde{\triangleright}}_A \\
 \downarrow & & & & \\
 \text{Curved} & & - & & -
 \end{array} \tag{2.2}$$

In curved backgrounds symmetry generators do not have to generate any global symmetry algebra in general.

2.2 Nondegenerate doubled Lie algebra

For affine extension of a Lie algebra the consistency requires the existence of the nondegenerate group metric η_{IJ} and the totally antisymmetric structure constant with lowered

indices $f_{IJK} = f_{IJ}{}^L \eta_{LK} = f_{[IJK]}/3!$. In subsection 2.2.1 we present a general method to construct a nondegenerate non-abelian group. In subsection 2.2.2 after reviewing the string sigma model we double the whole group in order to construct both the covariant derivatives and the symmetry generators for both the left and right modes.

2.2.1 Nondegenerate Lie algebra

We consider the space governed by the affine Lie algebra. The consistency of the affine Lie algebra requires the existence of the nondegenerate group metric in the space. This nondegenerate group metric is different from the Killing metric of the Lorentz group. The nondegenerate group metric is used to define the σ -diffeomorphism generator in the string worldsheet $\mathcal{H}_\sigma$, so η_{PP} must be nonzero. For the Poincaré group the canonical dimensions of the momentum and the Lorentz generator are 1 and 0 respectively. A nondegenerate partner of the Lorentz generator has the canonical dimension 2, so that the sum of canonical dimensions of a nondegenerate pair is 2. For the manifest covariance including the curvature tensors the Lorentz generator must be involved.

At first we present the general method to make a non-abelian group to be nondegenerate for a symmetric space given by a coset group G/H .

1. For a coset group G/H_0 a subgroup H_0 corresponds to the Lorentz group generated by h_0 and G/H_0 is generated by k . They satisfy the following algebra,

$$[h_0, h_0] = h_0, \quad [h_0, k] = k, \quad [k, k] = h_0. \quad (2.3)$$

2. Introduce another copy of the subgroup H_1 [39, 40] in order to make $G \times H_1$ to be nondegenerate. H_1 is generated by h_1 ,

$$[h_1, h_1] = h_1. \quad (2.4)$$

3. Make nondegenerate pair h and $\check{h}$ by linear combinations of h_0 and h_1 as

$$\begin{cases} h_0 + h_1 = h \\ h_0 - h_1 = \check{h} \\ k \rightarrow k/\sqrt{2} \end{cases} \Rightarrow \begin{cases} [h, h] = h, \quad [h, \check{h}] = \check{h}, \quad [\check{h}, \check{h}] = h \\ [h, k] = k, \quad [k, k] = h + \check{h}, \quad [\check{h}, k] = k \end{cases}. \quad (2.5)$$

h and $\check{h}$ are generators of H and $\check{H}$ which are subgroups of $G \times H_1$.

4. Non-zero components of the nondegenerate group metric are

$$\text{tr}(kk) = \eta_{kk}, \quad \text{tr}(h\check{h}) = \eta_{h\check{h}}. \quad (2.6)$$

The structure constant lowered by the nondegenerate group metric becomes totally antisymmetric

$$f_{h\check{h}\check{h}} = f_{\check{h}\check{h}h} = f_{hkk} = f_{\check{h}kk} = 1. \quad (2.7)$$

2.2.2 Doubled Lie algebra

The gravitational field is described by a closed string which has the left and right moving modes. We begin by the sigma model Lagrangian for a closed string

$$\mathcal{L} = -\frac{1}{2} \left(\sqrt{-h} h^{ij} \partial_i x^m \partial_j x^n G_{mn} + \epsilon^{ij} \partial_i x^m \partial_j x^n B_{mn} \right). \quad (2.8)$$

In the conformal gauge, the Lagrangian is rewritten in the doubled basis $\partial_{\pm} x^m = \frac{1}{\sqrt{2}}(\partial_{\tau} \pm \partial_{\sigma})x^m$ with the two vielbein fields e_a^m and e'_{ma} as [1–3]

$$\begin{aligned} \mathcal{L}_{\text{conformal gauge}} &= \frac{1}{2} j^{\underline{a}} \eta_{\underline{ab}} j^{\underline{b}}, \quad G_{mn} + B_{mn} = e_m^a e'_{na} \\ j^{\underline{a}} &= \begin{cases} j^a = \partial_+ x^m e_m^a \\ j_a = \partial_- x^m e'_{ma} \end{cases}, \quad \eta_{\underline{ab}} = \begin{pmatrix} 0 & \delta_a^b \\ \delta_b^a & 0 \end{pmatrix}. \end{aligned} \quad (2.9)$$

The left and right currents are written in term of the canonical momentum $p_m \equiv \frac{\partial \mathcal{L}}{\partial \partial_{\tau} x^m} = G_{mn} \partial_{\tau} x^n + B_{nm} \partial_{\sigma} x^n$

$$j^{\underline{a}} = \begin{cases} j^a = \frac{1}{\sqrt{2}} (\eta^{ab} e_b^m (p_m + B_{mn} \partial_{\sigma} x^n) + \partial_{\sigma} x^n e_n^a) \\ j_a = \frac{1}{\sqrt{2}} (e'_{la} G^{lm} (p_m + B_{mn} \partial_{\sigma} x^n) - \partial_{\sigma} x^n e'_{na}) \end{cases} \quad (2.10)$$

with $G^{mn} = e_a^m \eta^{ab} e_b^n$ and $e_a^m e_m^b = \delta_a^b$. The basis of the doubled space are essentially the left and right moving modes.

On the other hand the Hamiltonian with the two dimensional diffeomorphism invariance is given by

$$\begin{aligned} \mathcal{H} &= \frac{1}{\sqrt{-h} h^{00}} \mathcal{H}_{\tau} - \frac{h^{01}}{h^{00}} \mathcal{H}_{\sigma}, \quad \begin{cases} \mathcal{H}_{\sigma} = \frac{1}{2} \triangleright_{\underline{a}} \eta^{\underline{ab}} \triangleright_{\underline{b}} = \frac{1}{2} \triangleright_{\underline{m}}^{\square} \eta^{\underline{mn}} \triangleright_{\underline{n}}^{\square} \\ \mathcal{H}_{\tau} = \frac{1}{2} \triangleright_{\underline{a}} \hat{\eta}^{\underline{ab}} \triangleright_{\underline{b}} = \frac{1}{2} \triangleright_{\underline{m}}^{\square} \mathcal{M}^{\underline{mn}} \triangleright_{\underline{n}}^{\square} \end{cases} \\ \triangleright_{\underline{a}} &= e_{\underline{a}}^{\underline{m}} \triangleright_{\underline{m}}^{\square}, \quad \triangleright_{\underline{m}}^{\square} = \begin{pmatrix} p_m \\ \partial_{\sigma} x^m \end{pmatrix}, \quad e_{\underline{a}}^{\underline{m}} \eta^{\underline{ab}} e_{\underline{b}}^{\underline{n}} = \eta^{\underline{mn}}, \quad e_{\underline{a}}^{\underline{m}} \eta_{\underline{mn}} e_{\underline{b}}^{\underline{n}} = \eta_{\underline{ab}} \\ \hat{\eta}^{\underline{ab}} &= \begin{pmatrix} \eta^{\underline{ab}} \\ \eta_{\underline{ab}} \end{pmatrix}, \quad \mathcal{M}^{\underline{mn}} = e_{\underline{a}}^{\underline{m}} \hat{\eta}^{\underline{ab}} e_{\underline{b}}^{\underline{n}} = \begin{pmatrix} G^{mn} & G^{ml} B_{ln} \\ -B_{ml} G^{ln} & G_{mn} - B_{ml} G^{lk} B_{kn} \end{pmatrix} \end{aligned} \quad (2.11)$$

The conformal gauge is given by $\frac{1}{\sqrt{-h} h^{00}} = 1$ and $\frac{h^{01}}{h^{00}} = 0$. The covariant derivatives in arbitrary curved backgrounds are written as the vielbein multiplied on the flat space covariant derivative as $\triangleright_{\underline{a}} = e_{\underline{a}}^{\underline{m}} \triangleright_{\underline{m}}^{\square}$. The doubled vielbein field $e_{\underline{a}}^{\underline{m}}$ satisfies the orthogonal condition (2.11), so it is an element of the coset

$$\frac{\text{O(d, d)}}{\text{SO(d-1, 1)} \times \text{SO(1, d-1)}}. \quad (2.12)$$

The number of physical degrees of freedom for G_{mn} and B_{mn} is d^2 which is the number of the dimensions of the coset in (2.12). While $G_{mn} + B_{mn}$ is transformed fractionally linearly, the

vielbein $e_{\underline{a}}^{\underline{m}}$ is transformed linearly under the $O(d,d)$ T-duality symmetry transformation, $e_{\underline{a}}^{\underline{m}} \rightarrow h_{\underline{a}}^{\underline{b}} e_{\underline{b}}^{\underline{n}} \Lambda_{\underline{m}}^{\underline{n}}$ with $\Lambda^T \eta \Lambda = \eta$ and $h^T \hat{\eta} h = \hat{\eta}$.

An $O(d,d)$ T-duality transformation which produces nonzero B field, $B_{mn} = 0 \rightarrow B_{mn} = \lambda_{mn}$, is given by

$$\begin{aligned} e_{\underline{a}}^{\underline{m}} &= \begin{pmatrix} e^{-1} & \\ & e^T \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & -\lambda \\ & 1 \end{pmatrix}, \quad \lambda^T = -\lambda \\ &\rightarrow e_{\underline{a}}^{\underline{n}} (\Lambda^{-1})_{\underline{n}}^{\underline{m}} = \begin{pmatrix} e^{-1} & \\ & e^T \end{pmatrix} \begin{pmatrix} 1 & \lambda \\ & 1 \end{pmatrix} \end{aligned} \quad (2.13)$$

where indices are omitted as e, λ for e_m^a, λ_{mn} for simpler notation. Another $O(d,d) \ni \Lambda$ transformation which interchanges the momenta and the winding modes the vielbein is transformed as:

$$\begin{aligned} \underline{\triangleright}_{\underline{m}} &\rightarrow \Lambda_{\underline{m}}^{\underline{n}} \underline{\triangleright}_{\underline{n}}, \quad e_{\underline{a}}^{\underline{m}} \rightarrow h_{\underline{a}}^{\underline{b}} e_{\underline{b}}^{\underline{n}} (\Lambda^{-1})_{\underline{n}}^{\underline{m}} \\ e_{\underline{a}}^{\underline{m}} &= \begin{pmatrix} e^{-1} & \\ & e^T \end{pmatrix} \begin{pmatrix} 1 & B \\ & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} & (\lambda^{-1})^T \\ \lambda & \end{pmatrix}, \quad h = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix} \\ &\rightarrow h_{\underline{a}}^{\underline{b}} e_{\underline{b}}^{\underline{n}} (\Lambda^{-1})_{\underline{n}}^{\underline{m}} = \begin{pmatrix} (\lambda e)^T & \\ & (\lambda e)^{-1} \end{pmatrix} \begin{pmatrix} 1 & \\ \lambda B \lambda^T & 1 \end{pmatrix} \end{aligned} \quad (2.14)$$

where indices are omitted as B, G for B_{mn}, G_{mn} . This simple transformation rule corresponds to the following transformation rules of G_{mn} and B_{mn} as

$$\begin{cases} G_{mn} \rightarrow (\lambda^{-1})^T (G - B G^{-1} B)^{-1} \lambda^{-1} \\ B_{mn} \rightarrow [(\lambda^{-1})^T G^{-1} B (G - B G^{-1} B)^{-1} \lambda^{-1}]_{[mn]/2} \end{cases} \quad (2.15)$$

with antisymmetrizing indices, $\mathcal{O}_{[mn]/2} = (\mathcal{O}_{mn} - \mathcal{O}_{nm})/2$. They are generalizations of the Buscher's transformation rule.

It is known that doubled coordinates manifest T-duality symmetry, and the physical degrees of freedom is a half of it. The section condition is usually considered as $\partial_m \partial^m = 0$ where $\partial_m = \frac{\partial}{\partial x^m}$ and $\partial^m = \frac{\partial}{\partial y_m}$, and it is imposed on the spacetime field weakly as $\partial_m \partial^m \Psi(x^m, y_m) = 0$ and strongly $\partial_m \Phi(x^m, y_m) \partial^m \Psi(x^m, y_m) = 0$. The y_m -independence satisfies the section condition and the theory reduces to the usual coordinate space theory. This condition is the σ -diffeomorphism invariance constraint $\mathcal{H}_\sigma = \partial_m \partial^m = 0$ for a string on the worldsheet. The σ -diffeomorphism invariance constraint is imposed on fields as a matrix element of the second quantized level, $\langle \Phi | \mathcal{H}_\sigma | \Psi \rangle = 0$. In other words fields in the target space governed by the string theory should be σ -diffeomorphism invariant.

The doubled momenta $\underline{\triangleright}_{\underline{m}} = P_{\underline{m}} = (P_m, P_{m'})$ are independent, so we have doubled coordinates. Then we impose dimensional reduction constraints to reduce a half. We do not impose gauge fixing conditions on spacetime fields $\frac{\partial}{\partial y_m} \Psi = 0$, and they are written as $P_m = (p_m + \partial_\sigma x^m)/\sqrt{2}$ and $P_{m'} = (p_{m'} - \partial_\sigma x^{m'})/\sqrt{2}$ in a flat space with $x^m = (x^m + y_m)/\sqrt{2}$ and $x^{m'} = (x^m - y_m)/\sqrt{2}$. The dimensional reduction constraints are first class, so the

local gauge symmetry and all doubled coordinates are preserved. Therefore the T-duality covariant general coordinate invariance of the stringy gravity is manifest.

The dimensional reduction constraints are made from the right-invariant one form, while the local geometry is made from the left-invariant one form so that the auxiliary coordinates are reduced by the dimensional reduction constraints without modifying the local geometry. In order to construct the left-invariant one form and the right-invariant one form for both left and right moving modes we double the whole group

$$G \rightarrow G \times G' . \quad (2.16)$$

A group element of the direct product of these groups $G \times G' \ni \underline{g} = g(Z^M)g(z^{M'})$ gives both the left and right moving modes of the left-invariant and the right-invariant currents; $\underline{g}^{-1}d\underline{g} = g^{-1}dg(Z) + g'^{-1}dg'(Z') = iJ(Z) + iJ(Z')$ and $d\underline{g}\underline{g}^{-1} = dg g^{-1}(Z) + dg' g'^{-1}(Z') = i\tilde{J}(Z) + i\tilde{J}(Z')$. For the RR background this factorization is nontrivial as seen later.

2.3 Affine Lie algebras

Let us go back to the procedure of the manifestly T-dual formulation in arbitrary group manifolds. We begin by a Lie algebra generated by G_I

$$[G_I, G_J] = if_{IJ}{}^K G_K , \quad \text{tr}(G_I G_J) = \eta_{IJ} , \quad \det \eta_{IJ} \neq 0 . \quad (2.17)$$

For the Lie algebra in (2.17) its group element g is parametrized by Z^M where the number of Lie algebra generators G_I is equal to the number of the parameters Z^M . We extend it to affine Lie algebras as string algebras. The coordinates Z^M 's are functions of the two-dimensional worldsheet coordinates. Generators of affine Lie algebras are constructed from the left and right-invariant currents and the particle covariant derivative and the particle symmetry generator.

- **Left-invariant one form and the particle covariant derivative**

The left-invariant one form J satisfies the Maurer-Cartan equation, and the covariant derivative ∇_I satisfies the following equation

$$\begin{aligned} g^{-1}dg &= iJ^I G_I , \quad J^I = dZ^M R_M^I \Rightarrow dJ^I = -\frac{1}{2}f_{JK}{}^I J^J \wedge J^K \\ \nabla_I &= (R^{-1})_I{}^M \frac{1}{\partial_M} \Rightarrow (R^{-1})_{[I}{}^M \nabla_{J]} R_M^K = if_{IJ}{}^K . \end{aligned} \quad (2.18)$$

- **Right-invariant current and the particle symmetry generator**

The right-invariant one form $\tilde{J}$ satisfies the Maurer-Cartan equation, and the symmetry generator $\tilde{\nabla}_I$ satisfies the following equation

$$\begin{aligned} dg g^{-1} &= i\tilde{J}^I G_I , \quad \tilde{J}^I = dZ^M L_M^I \Rightarrow d\tilde{J}^I = \frac{1}{2}f_{JK}{}^I \tilde{J}^J \wedge \tilde{J}^K \\ \tilde{\nabla}_I &= (L^{-1})_I{}^M \frac{1}{\partial_M} \Rightarrow (L^{-1})_{[I}{}^M \nabla_{J]} L_M^K = -if_{IJ}{}^K . \end{aligned} \quad (2.19)$$

• **Algebras by particle covariant derivative and symmetry generator**

The covariant derivative and the symmetry generator together with $J_1^I = \partial_\sigma Z^M R_M^I$ and $\tilde{J}_1^I = \partial_\sigma Z^M L_M^I$ satisfy the following affine Lie algebras:

$$\left\{ \begin{array}{l} [\nabla_I(1), \nabla_J(2)] = -if_{IJ}^K \nabla_K \delta(2-1) \\ [\nabla_I(1), J_1^J(2)] = -iJ_1^K f_{KI}^J \delta(2-1) - i\delta_I^J \partial_\sigma \delta(2-1) \cdot \\ [J_1^I(1), J_1^J(2)] = 0 \end{array} \right. \quad (2.20)$$

$$\left\{ \begin{array}{l} [\tilde{\nabla}_I(1), \tilde{\nabla}_J(2)] = if_{IJ}^K \tilde{\nabla}_K \delta(2-1) \\ [\tilde{\nabla}_I(1), \tilde{J}_1^J(2)] = i\tilde{J}_1^K f_{KI}^J \delta(2-1) + i\delta_I^J \partial_\sigma \delta(2-1) \cdot \\ [\tilde{J}_1^I(1), \tilde{J}_1^J(2)] = 0 \end{array} \right. \quad (2.21)$$

$$\left\{ \begin{array}{l} [\nabla_I(1), \tilde{\nabla}_J(2)] = 0 \\ [\tilde{\nabla}_I(1), J_1^J(2)] = -iM_I^J(2) \partial_\sigma \delta(2-1) \\ [\nabla_I(1), \tilde{J}_1^J(2)] = -i(M^{-1})_I^J(2) \partial_\sigma \delta(2-1) \end{array} \right. \quad (2.22)$$

with

$$\begin{aligned} M_I^J &= (L^{-1})_I^M R_M^J, \quad \tilde{J}^I M_I^K = J^K, \quad \tilde{\nabla}_I = M_I^K \nabla_K \\ \eta_{IJ} &= M_I^L M_J^K \eta_{LK}, \quad f_{IJK} = M_I^L M_J^P M_K^Q f_{LPQ}. \end{aligned} \quad (2.23)$$

σ_1 and σ_2 are abbreviated as 1 and 2, and $\delta(2-1) = \delta(\sigma_2 - \sigma_1)$ and $\partial_\sigma \delta(2-1) = \partial_{\sigma_2} \delta(\sigma_2 - \sigma_1)$. From the relation between the left-invariant one form and the right-invariant one form $g^{-1} \tilde{J}g = J \rightarrow L_M^I g^{-1} G_I g = R_M^I G_I$, the nondegenerate group metric $\eta_{IJ} = \text{tr}(G_I G_J) = \text{tr}(g^{-1} G_I g g^{-1} G_J g)$ leads to that the matrix M_I^J satisfies the orthogonal condition and invariance of the structure constant (2.23).

The string covariant derivative $\triangleright_I(\sigma)$ is constructed with the B field from the particle covariant derivative $\nabla_I(\sigma)$ and the σ component of the left-invariant current $J_1^I(\sigma)$. The string symmetry generator $\tilde{\triangleright}_I(\sigma)$ is constructed with the $\tilde{B}$ field from the particle symmetry generator $\tilde{\nabla}_I(\sigma)$ and the σ component of the right-invariant current $\tilde{J}_1^I(\sigma)$ as;

• **Covariant derivative** ¹

$$\triangleright_I = \nabla_I + \frac{1}{2} J_1^K (\eta_{KI} + B_{KI}) \quad (2.24)$$

• **Symmetry generator**

$$\tilde{\triangleright}_I = \tilde{\nabla}_I + \frac{1}{2} \tilde{J}_1^K (-\eta_{KI} + \tilde{B}_{KI}) \quad (2.25)$$

The affine extension of the Lie algebra (2.17) is performed using (2.20), (2.21) and (2.22).

¹The coefficient $\frac{1}{2}$ arises from the normalization of the Schwinger term in the affine Lie algebra. The same normalization of the Schwinger term is satisfied by $\frac{1}{\sqrt{2}} (\nabla_I + J_1^K (\eta_{KI} + B_{KI}))$.

• Affine Lie algebras

$$\begin{aligned}
[\triangleright_I(1), \triangleright_J(2)] &= -if_{IJ}{}^K \triangleright_K \delta(2-1) - i\eta_{IJ} \partial_\sigma \delta(2-1) \\
[\tilde{\triangleright}_I(1), \tilde{\triangleright}_J(2)] &= if_{IJ}{}^K \tilde{\triangleright}_K \delta(2-1) + i\eta_{IJ} \partial_\sigma \delta(2-1) \\
[\triangleright_I(1), \tilde{\triangleright}_J(2)] &= 0
\end{aligned} \tag{2.26}$$

The antisymmetric tensor B_{IJ} field in the covariant derivative must satisfy the following equation [10]

$$i\nabla_{[I} B_{JK]} - f_{[IJ]}{}^L B_{L|K]} = 2f_{IJK} . \tag{2.27}$$

Another antisymmetric tensor $\tilde{B}_{IJ}$ field in the symmetry generator is related to B_{IJ} from (2.23) as

$$\tilde{B}_{IK} = M_I{}^J B_{JL} M_K{}^L . \tag{2.28}$$

The two form B gives the Wess-Zumino term for a fundamental string

$$\begin{aligned}
B &= \frac{1}{2} dZ^M \wedge dZ^N B_{MN} = \frac{1}{2} J^I \wedge J^J B_{IJ} = \frac{1}{2} \tilde{J}^I \wedge \tilde{J}^J \tilde{B}_{IJ} \\
B_{MN} &= R_M{}^I R_N{}^J B_{IJ} = L_M{}^I L_N{}^J \tilde{B}_{IJ} .
\end{aligned} \tag{2.29}$$

The three form $H = dB$ is calculated with (2.27) as

$$\begin{aligned}
H = dB &= \frac{1}{3!} dZ^M \wedge dZ^N \wedge dZ^L H_{MNL} = \frac{1}{3!} J^I \wedge J^J \wedge J^K f_{IJK} = \frac{1}{3!} \tilde{J}^I \wedge \tilde{J}^J \wedge \tilde{J}^K f_{IJK} \\
H_{MNP} &= R_M{}^I R_N{}^J R_P{}^K f_{IJK} = L_M{}^I L_N{}^J L_P{}^K f_{IJK} .
\end{aligned} \tag{2.30}$$

It is also denoted that the condition on B_{IJ} in (2.27) is expressed as $dB = H$ where H is given in [10]. B is determined from it up to its gauge freedom $d\lambda$. The existence of the solution is guaranteed by $dH = 0$, which is proven using Maurer-Cartan equations.

The σ diffeomorphism generator is defined by bilinears of the covariant derivatives contracted with the nondegenerate group metric as

$$\mathcal{H}_\sigma = \frac{1}{2} \triangleright_I \eta^{IJ} \triangleright_J . \tag{2.31}$$

For a field Φ which is a function of the group manifold coordinates, the σ derivative of Φ is determined as

$$\partial_\sigma \Phi = i \int d\sigma' [\mathcal{H}_\sigma(\sigma'), \Phi] = \triangleright_I \eta^{IJ} (i\nabla_J \Phi) . \tag{2.32}$$

If a field Φ is a function of both phase space coordinates $(Z^M, \frac{1}{i}\partial_M)$, then the derivative $(i\nabla_J \Phi)$ is replaced by the commutator as $[i\triangleright_J, \Phi]$ in (2.32).

Let us consider a space defined by the affine Lie algebra generated by the covariant derivative in the first line of (2.26). Two vectors in the space, $\hat{\Lambda}_i = \Lambda_i^I(Z^M) \triangleright_I(\sigma)$ with $i = 1, 2$, satisfy the commutator as

$$\begin{aligned}
 & [\Lambda_1^I \triangleright_I(1), \Lambda_2^J \triangleright_J(2)] \\
 &= -i\Lambda_{12}^I \triangleright_I \delta(2-1) - i \left\{ \left(\frac{1}{2} + \mathcal{K} \right) \Psi_{(12)}(1) + \left(\frac{1}{2} - \mathcal{K} \right) \Psi_{(12)}(2) \right\} \partial_\sigma \delta(2-1) \\
 & \Lambda_{12}^I = \Lambda_{[1]}^K (i\nabla_K \Lambda_{|2]}^I) - \frac{1}{2} \Lambda_{[1]}^K (i\nabla^I \Lambda_{|2]K}) + \Lambda_1^J \Lambda_2^K f_{JK}^I - \mathcal{K} (i\nabla^I \Psi_{(12)}) \\
 & \Psi_{(12)} = \Lambda_1^I \Lambda_2^J \eta_{IJ}
 \end{aligned} \tag{2.33}$$

where σ derivative is calculated by (2.32). The factors “ i ”’s come from the definition of covariant derivative $\nabla_I = R_I^M \frac{1}{i} \partial_M$. There is an ambiguity with parameter $\mathcal{K}$ caused from the Schwinger term including $\partial_\sigma \delta(2-1)$. The regular part of the algebra is a generalization of the “C-bracket”

$$([\Lambda_1, \Lambda_2]_T)^I = -i\Lambda_{12}^I \tag{2.34}$$

where we put “T” which stands for T-duality. The expression of Λ_{12} depends on the value of $\mathcal{K}$ as

$$\Lambda_{12}^I = \begin{cases} \Lambda_{[1]}^K (i\nabla_K \Lambda_{|2]}^I) - \frac{1}{2} \Lambda_{[1]}^K (i\nabla^I \Lambda_{|2]K}) + \Lambda_1^J \Lambda_2^K f_{JK}^I \cdots \mathcal{K} = 0 \\ \Lambda_1^K (i\nabla_K \Lambda_2^I) + \Lambda_2^K (i\nabla_{[J} \Lambda_{|K]} \eta^{JI} + \Lambda_1^J \Lambda_2^K f_{JK}^I \cdots \mathcal{K} = -\frac{1}{2} \end{cases} . \tag{2.35}$$

For the case with $\mathcal{K} = 0$ it is antisymmetric under $1 \leftrightarrow 2$ interchanging, while the case with $\mathcal{K} = -1/2$ gives the usual gauge transformation rules. The Jacobi identity of the T-bracket is not satisfied in general because of lack of the contribution from the Schwinger term. The Jacobi identity of the affine algebra (2.33) is the Bianchi identity giving a condition on Λ_{12} .

3 Flat space

3.1 Dilatation operator and B field

We begin by the Poincaré algebra as a flat space, then introduce the nondegenerate partner of the Lorentz generator following to the previous section. The nondegenerate Poincaré algebra is generated by G_M . In this case there exists a dilatation operator $\hat{N}$ and the canonical dimensions of generator G_M is n_M as

$$[G_M, G_N] = i \overset{\square}{f}_{MN}{}^L G_L, \quad [\hat{N}, G_M] = i N_M{}^N G_N = i n_M G_M. \tag{3.1}$$

The generator of the nondegenerate Poincaré algebra, $G_M = (s_{mn}, p_m, \sigma^{mn})$, and the dilatation operator $\hat{N}$ satisfy the following algebra

$$\begin{aligned}
 [s_{mn}, s_{lk}] &= i\eta_{[k|[m} s_{n]l}], & [s_{mn}, p_l] &= ip_{[m} \eta_{n]l} \\
 [s_{mn}, \sigma_{lk}] &= i\eta_{[k|[m} \sigma_{n]l}], & [p_m, p_n] &= i\sigma_{mn} \\
 [\hat{N}, s_{mn}] &= 0, & [\hat{N}, p_m] &= ip_m, & [\hat{N}, \sigma^{mn}] &= 2i\sigma^{mn}.
 \end{aligned} \tag{3.2}$$

The nondegenerate group metric is

$$\eta_{IJ} = \begin{matrix} & s & p & \sigma \\ \begin{matrix} s \\ p \\ \sigma \end{matrix} & \begin{pmatrix} & & 1 \\ & 1 & \\ 1 & & \end{pmatrix} \end{matrix} \quad (3.3)$$

The sum of the canonical dimensions of the nondegenerate pair is 2; $(n_I + n_J)\eta_{IJ} = 2\eta_{IJ}$. The Jacobi identity among $\hat{N}$ and two G_M 's leads to an identity

$$\bar{f}_{MN}^{\square K} N_K^L + N_{[M}^K \bar{f}_{N]K}^{\square L} = (n_L - n_M - n_N) \bar{f}_{MN}^{\square L} = 0, \quad (3.4)$$

so the sum of the canonical dimensions of the lowered indices of the non-zero component of the structure constant is also 2; $(n_M + n_N + n_L) \bar{f}_{MNL}^{\square} = 2 \bar{f}_{MNL}^{\square}$. This identity gives a constant B field solution of the equation (2.27) for the nondegenerate Poincaré group as

$$\bar{B}_{NM} = -\frac{1}{2} N_{[N}^L \eta_{L|M]} = \frac{1}{2} (-n_N + n_M) \eta_{NM}. \quad (3.5)$$

As a result the stringy covariant derivative for the flat space $\bar{\mathbb{D}}_N$ is written in terms of the particle covariant derivative $\bar{\nabla}_N$ and the σ -component of the left-invariant current $\bar{J}_1^N$ in the flat space as²

$$\bar{\mathbb{D}}_M = \bar{\nabla}_M + \frac{1}{2} \bar{J}_1^L (\eta_{LM} + \bar{B}_{LM}) = \bar{\nabla}_M + \frac{n_M}{2} \bar{J}_{1;M} \quad (3.7)$$

with $\bar{J}_{1;M} \equiv \bar{J}_1^L \eta_{LM}$. It satisfies the affine nondegenerate Poincaré algebra

$$[\bar{\mathbb{D}}_M(1), \bar{\mathbb{D}}_N(2)] = -i \bar{f}_{MN}^{\square L} \bar{\mathbb{D}}_L \delta(2-1) - i \eta_{MN} \partial_\sigma \delta(2-1). \quad (3.8)$$

3.2 Affine Poincaré algebras

Next the nondegenerate Poincaré algebra is doubled as described in the previous section

$$\begin{aligned} G_M &\rightarrow G_{\underline{M}} = (G_M, G_{M'}) \\ \bar{f}_{MN}^{\square L} &\rightarrow \bar{f}_{\underline{MN}}^{\square L} = (\bar{f}_{MN}^{\square L}, \bar{f}_{M'N'}^{\square L'} = -\bar{f}_{MN}^{\square L}) \\ \eta_{MN} &\rightarrow \begin{cases} \eta_{\underline{MN}} = (\eta_{MN}, \eta_{M'N'} = -\eta_{MN}) \\ \hat{\eta}_{\underline{MN}} = (\eta_{MN}, \hat{\eta}_{M'N'} = \eta_{MN}) \end{cases}. \end{aligned} \quad (3.9)$$

Covariant derivatives and symmetry generators for the nondegenerate doubled Poincaré algebra are given as follows.

²The coefficient $\frac{1}{2}$ arises from the normalization of the Schwinger term in the affine Lie algebra (3.8). Another normalization gives the usual stringy covariant derivative

$$\bar{\mathbb{D}}_M = \frac{1}{\sqrt{2}} (\bar{\nabla}_M + n_M \bar{J}_M). \quad (3.6)$$

- **Flat covariant derivatives:** $\overset{\square}{\mathbb{D}}_{\underline{M}} = \overset{\square}{\nabla}_{\underline{M}} + \frac{1}{2} \overset{\square}{J}_1^L (\eta_{\underline{LM}} + \overset{\square}{B}_{\underline{LM}}) = (\overset{\square}{\mathbb{D}}_M, \overset{\square}{\mathbb{D}}_{M'})$

$$\begin{aligned} \text{Flat left } \overset{\square}{\mathbb{D}}_M &= (S_{mn}, P_m, \Sigma^{mn}); \quad \text{Flat right } \overset{\square}{\mathbb{D}}_{M'} = (S_{m'n'}, P_{m'}, \Sigma^{m'n'}) \\ \left\{ \begin{array}{l} S_{mn} = \nabla_S \\ P_m = \nabla_P + \frac{1}{2} J_{1;P} \\ \Sigma^{mn} = \nabla_\Sigma + J_{1;\Sigma} \end{array} \right. & \quad \left\{ \begin{array}{l} S_{m'n'} = \nabla_{S'} \\ P_{m'} = \nabla_{P'} - \frac{1}{2} J_{1;P'} \\ \Sigma^{m'n'} = \nabla_{\Sigma'} - J_{1;\Sigma'} \end{array} \right. \end{aligned} \quad (3.10)$$

with $J_{1;M'} = J_1^{N'} \eta_{NM}$.

- **Flat symmetry generators:** $\overset{\square}{\mathbb{D}}_{\underline{M}} = \overset{\square}{\nabla}_{\underline{M}} + \frac{1}{2} \overset{\square}{J}_1^L (-\eta_{\underline{LM}} + \overset{\square}{B}_{\underline{LM}}) = (\overset{\square}{\mathbb{D}}_M, \overset{\square}{\mathbb{D}}_{M'})$

$$\begin{aligned} \text{Flat left } \overset{\square}{\mathbb{D}}_M &= (\tilde{S}_{mn}, \tilde{P}_m, \tilde{\Sigma}^{mn}) \\ \left\{ \begin{array}{l} \tilde{S}_{mn} = \tilde{\nabla}_S - (\tilde{J}_{1;S} + c_S^P \tilde{J}_{1;P} + c_S^\Sigma \tilde{J}_{1;\Sigma}) \\ \tilde{P}_m = \tilde{\nabla}_P - \frac{1}{2} (\tilde{J}_{1;P} + c_P^\Sigma \tilde{J}_{1;\Sigma}) \\ \tilde{\Sigma}^{mn} = \tilde{\nabla}_\Sigma \end{array} \right. & \end{aligned} \quad (3.11)$$

$$\begin{aligned} \text{Flat right } \overset{\square}{\mathbb{D}}_{M'} &= (\tilde{S}_{m'n'}, \tilde{P}_{m'}, \tilde{\Sigma}^{m'n'}) \\ \left\{ \begin{array}{l} \tilde{S}_{m'n'} = \tilde{\nabla}_{S'} + (\tilde{J}_{1;S'} + c_{S'}^{P'} \tilde{J}_{1;P'} + c_{S'}^{\Sigma'} \tilde{J}_{1;\Sigma'}) \\ \tilde{P}_{m'} = \tilde{\nabla}_{P'} + \frac{1}{2} (\tilde{J}_{1;P'} + c_{P'}^{\Sigma'} \tilde{J}_{1;\Sigma'}) \\ \tilde{\Sigma}^{m'n'} = \tilde{\nabla}_{\Sigma'} \end{array} \right. \end{aligned}$$

with $\tilde{J}_{1;M'} = \tilde{J}_1^{N'} \eta_{NM}$. Coefficients c_M^N 's are given by M_I^J determined from (2.24) and (2.28). Their explicit forms, in a particular parametrization, have been given in [10].

The affine nondegenerate doubled Poincaré algebras generated by the covariant derivatives and the symmetry generators are given as:

- **Affine flat algebra by covariant derivatives:** $\overset{\square}{\mathbb{D}}_{\underline{M}} = (\overset{\square}{\mathbb{D}}_M, \overset{\square}{\mathbb{D}}_{M'})$

$$\left\{ \begin{array}{l} [\overset{\square}{\mathbb{D}}_M(1), \overset{\square}{\mathbb{D}}_N(2)] = -i f_{MN}^L \overset{\square}{\mathbb{D}}_L \delta(2-1) - i \eta_{MN} \partial_\sigma \delta(2-1) \\ [\overset{\square}{\mathbb{D}}_{M'}(1), \overset{\square}{\mathbb{D}}_{N'}(2)] = -i f_{M'N'}^{L'} \overset{\square}{\mathbb{D}}_{L'} \delta(2-1) - i \eta_{M'N'} \partial_\sigma \delta(2-1) \\ \quad = i f_{MN}^L \overset{\square}{\mathbb{D}}_{L'} \delta(2-1) + i \eta_{MN} \partial_\sigma \delta(2-1) \\ [\overset{\square}{\mathbb{D}}_M(1), \overset{\square}{\mathbb{D}}_{N'}(2)] = 0 \end{array} \right. \quad (3.12)$$

- **Affine flat algebra by symmetry generators:** $\overset{\square}{\mathbb{D}}_{\underline{M}} = (\overset{\square}{\mathbb{D}}_M, \overset{\square}{\mathbb{D}}_{M'})$

$$\left\{ \begin{array}{l} [\overset{\square}{\mathbb{D}}_M(1), \overset{\square}{\mathbb{D}}_N(2)] = i f_{MN}^L \overset{\square}{\mathbb{D}}_L \delta(2-1) + i \eta_{MN} \partial_\sigma \delta(2-1) \\ [\overset{\square}{\mathbb{D}}_{M'}(1), \overset{\square}{\mathbb{D}}_{N'}(2)] = i f_{M'N'}^{L'} \overset{\square}{\mathbb{D}}_{L'} \delta(2-1) + i \eta_{M'N'} \partial_\sigma \delta(2-1) \\ \quad = -i f_{MN}^L \overset{\square}{\mathbb{D}}_{L'} \delta(2-1) - i \eta_{MN} \partial_\sigma \delta(2-1) \\ [\overset{\square}{\mathbb{D}}_M(1), \overset{\square}{\mathbb{D}}_{N'}(2)] = 0 \end{array} \right. \quad (3.13)$$

• **Commutativity:**

$$[\overset{\square}{\triangleright}_{\underline{M}}(1), \overset{\square}{\triangleright}_{\underline{N}}(2)] = 0 \quad (3.14)$$

The flat space is defined by the affine nondegenerate doubled Poincaré algebra generated by the covariant derivative (3.10). The symmetry generators (3.11) become physical symmetry generators and dimensional reduction constraints.

3.3 Dimensional reduction constraints and the section condition

The symmetry generators obtained in (3.11) satisfying in (3.13) become the physical total momentum and the physical total Lorentz generators

$$\tilde{P}_{\text{total};m} = \tilde{P}_m + \tilde{P}_{n'}\delta_m^{n'}, \quad \tilde{S}_{\text{total};mn} = \tilde{S}_{mn} - \tilde{S}_{m'n'}\delta_m^{m'}\delta_n^{n'}, \quad (3.15)$$

and dimensional reduction constraints

$$\begin{aligned} \phi_m &= \tilde{P}_m - \tilde{P}_{n'}\delta_m^{n'} = 0 \\ [\phi_m(1), \phi_n(2)] &= -i(\tilde{\Sigma}_{mn} - \tilde{\Sigma}_{m'n'}\delta_m^{m'}\delta_n^{n'})\delta(2-1) \\ \Rightarrow \tilde{\Sigma}_{mn} &= \tilde{\Sigma}_{m'n'} = 0. \end{aligned} \quad (3.16)$$

The worldsheet τ/σ -diffeomorphism generators constructed with the metrics in (3.9) and the Virasoro algebra are given as

$$\begin{aligned} \mathcal{H}_\sigma &= \frac{1}{2} \overset{\square}{\triangleright}_{\underline{M}} \eta^{\underline{MN}} \overset{\square}{\triangleright}_{\underline{N}}, \quad \mathcal{H}_\tau = \frac{1}{2} \overset{\square}{\triangleright}_{\underline{M}} \hat{\eta}^{\underline{MN}} \overset{\square}{\triangleright}_{\underline{M}} \\ \left\{ \begin{aligned} [\mathcal{H}_\sigma(1), \mathcal{H}_\sigma(2)] &= i(\mathcal{H}_\sigma(1) + \mathcal{H}_\sigma(2))\partial_\sigma\delta(2-1) \\ [\mathcal{H}_\sigma(1), \mathcal{H}_\tau(2)] &= i(\mathcal{H}_\tau(1) + \mathcal{H}_\tau(2))\partial_\sigma\delta(2-1) \\ [\mathcal{H}_\tau(1), \mathcal{H}_\tau(2)] &= i(\mathcal{H}_\sigma(1) + \mathcal{H}_\sigma(2))\partial_\sigma\delta(2-1) \end{aligned} \right. \end{aligned} \quad (3.17)$$

These Virasoro constraints are imposed on the physical states for strings. This σ -diffeomorphism constraint written in the doubled coordinates is imposed on the fields in the doubled target space, as the section condition.

The relation between the section condition and the dimensional reduction constraint is the following: the σ -diffeomorphism constraint is satisfied on the constrained surface

$$\begin{aligned} \mathcal{H}_\sigma &= \frac{1}{2} \left(P_m^2 - P_{m'}^2 + \frac{1}{2} S_{mn} \Sigma^{mn} - \frac{1}{2} S_{m'n'} \Sigma^{m'n'} \right) \approx \frac{1}{2} (P_m^2 - P_{m'}^2) \\ &= \frac{1}{2} \overset{\square}{\triangleright}_{\underline{M}} \eta^{\underline{MN}} \overset{\square}{\triangleright}_{\underline{N}} \approx \tilde{P}_{\text{total},m} \phi^m = 0 \end{aligned} \quad (3.18)$$

where weak equalities $\approx$ in the first and the second lines are equal up to the constraints, local Lorentz constraints $S_{mn} = S_{m'n'} = 0$, and the dimensional reduction constraints, $\tilde{\Sigma}_{mn} = \tilde{\Sigma}_{m'n'} = 0$. In our formulation the first class constraint in (3.16) is imposed, which

reduces to $\frac{\partial}{\partial y_m} \Phi = 0$ only after the unitary gauge with $y_m = x^m - x^{m'}$. The section condition is automatically satisfied.

The zero-modes of the symmetry generators satisfy the Poincaré algebra as

$$\begin{aligned} \mathcal{P}_{\text{total};m} &= \int d\sigma \tilde{P}_{\text{total};m}(\sigma), \quad \mathcal{S}_{\text{total};mn} = \int d\sigma \tilde{S}_{\text{total};mn}(\sigma) \\ \left\{ \begin{aligned} [\mathcal{S}_{\text{total};mn}, \mathcal{S}_{\text{total};lk}] &= i\eta_{[k|[m} \mathcal{S}_{\text{total};n]l]} \\ [\mathcal{S}_{\text{total};mn}, \mathcal{P}_{\text{total};l}] &= i\mathcal{P}_{\text{total};[m} \eta_{n]l} \\ [\mathcal{P}_{\text{total};m}, \mathcal{P}_{\text{total};n}] &= i(\tilde{\Sigma}_{mn} - \tilde{\Sigma}_{m'n'} \delta_m^{m'} \delta_n^{n'}) \approx 0 \end{aligned} \right. \end{aligned} \quad (3.19)$$

where the dimensional reduction constraints (3.16) are used in the last equality.

4 Curved backgrounds in the asymptotically flat space

4.1 Curved space covariant derivative and torsion

The gravitational fields are coupled to closed string modes as given in (2.11)

$$\overset{\square}{\triangleright}_M \rightarrow \triangleright_{\underline{A}} = E_{\underline{A}}^{\underline{M}} \overset{\square}{\triangleright}_M, \quad E_{\underline{A}}^{\underline{M}} \eta_{\underline{MN}} E_{\underline{B}}^{\underline{N}} = \eta_{\underline{AB}}, \quad E_{\underline{A}}^{\underline{M}} \eta^{\underline{AB}} E_{\underline{B}}^{\underline{N}} = \eta^{\underline{MN}}. \quad (4.1)$$

The vielbein fields $E_{\underline{A}}^{\underline{M}}$ satisfies the orthogonal condition with respect to $\eta_{\underline{MN}}$. The generators includes Lorentz generators so the vielbein includes not only G_{mn} and B_{mn} but also the Lorentz connection ω_m^{nl} [6, 7].

While the σ -diffeomorphism generator in a curved space is unchanged from the one in a flat space because of the orthogonality (4.1), the τ -diffeomorphism generator $\mathcal{H}_\tau$ in a curved space is given by

$$\begin{aligned} \mathcal{H}_\sigma &= \frac{1}{2} \triangleright_{\underline{A}} \eta^{\underline{AB}} \triangleright_{\underline{B}} = \frac{1}{2} \overset{\square}{\triangleright}_{\underline{M}} \eta^{\underline{MN}} \overset{\square}{\triangleright}_{\underline{N}} \\ \mathcal{H}_\tau &= \frac{1}{2} \triangleright_{\underline{A}} \hat{\eta}^{\underline{AB}} \triangleright_{\underline{B}} = \frac{1}{2} \overset{\square}{\triangleright}_{\underline{M}} \mathcal{M}^{\underline{MN}} \overset{\square}{\triangleright}_{\underline{N}}, \quad \mathcal{M}^{\underline{MN}} = E_{\underline{A}}^{\underline{M}} \hat{\eta}^{\underline{AB}} E_{\underline{B}}^{\underline{N}} \end{aligned} \quad (4.2)$$

with the generalized metric $\mathcal{M}^{\underline{MN}}$ as a generalization of the third line of (2.11). Since the σ -diffeomorphism generator $\mathcal{H}_\sigma$ is independent on the background, it is possible to impose $\mathcal{H}_\sigma = 0$ as a first class constraint even in curved spaces.

The covariant derivative in a curved space $\triangleright_{\underline{A}}$ given in (4.1) satisfies the following algebra

$$\begin{aligned} [\triangleright_{\underline{A}}(1), \triangleright_{\underline{B}}(2)] &= -iT_{\underline{AB}}^{\underline{C}} \triangleright_{\underline{C}} \delta(2-1) - i\eta_{\underline{AB}} \partial_\sigma \delta(2-1) \\ T_{\underline{ABC}} &\equiv T_{\underline{AB}}^{\underline{D}} \eta_{\underline{DC}} = \frac{1}{2} (i\nabla_{[\underline{A}} E_{\underline{B}}^{\underline{M}}) E_{\underline{C}]}^{\underline{N}} + E_{\underline{A}}^{\underline{M}} E_{\underline{B}}^{\underline{N}} E_{\underline{C}}^{\underline{L}} f_{\underline{MNL}}^{\square} \\ i\nabla_{[\underline{A}} T_{\underline{BCD}]} &+ \frac{3}{4} T_{\underline{AB}}^{\underline{E}} T_{\underline{CD}]}^{\underline{E}} = 0. \end{aligned} \quad (4.3)$$

The orthogonality of the vielbein is used to give the same Schwinger term as the flat case, and the torsion $T_{\underline{AB}}^{\underline{D}}$ with lowered indices is totally antisymmetric. The Bianchi identity leads to the totally antisymmetric equation.

4.2 Group manifolds and the three form $H = dB$

We focus on the cases where the curved space is a group manifold so the torsion becomes constant $T_{AB}^C \rightarrow f_{IJ}^K$. The covariant derivative $\triangleright_I = E_I^M \bar{\triangleright}_M$ satisfies the affine algebra as

$$[\triangleright_I(1), \triangleright_J(2)] = -if_{IJ}^K \triangleright_K \delta(2-1) - i\eta_{IJ} \partial_\sigma \delta(2-1). \quad (4.4)$$

From the expression of the torsion in (4.3) the structure constant of the group manifold f_{IJK} is written in terms of vielbein field and the structure constant of the nondegenerate doubled Poincaré algebra $\bar{f}_{MNL}$ as

$$f_{IJK} = \frac{i}{2} (\nabla_I E_J^M) E_{K\bar{M}} + E_I^M E_J^N E_{K\bar{L}} \bar{f}_{MNL}. \quad (4.5)$$

The currents and the particle covariant derivatives are given by

$$\begin{cases} \bar{J}_1^{\bar{L}} = \partial_\sigma Z^M \bar{R}_{M\bar{L}} \\ J_1^I = \partial_\sigma Z^M R_{M\bar{I}} \end{cases}, \quad \begin{cases} \bar{\nabla}_{\bar{L}} = (\bar{R}^{-1})_{\bar{L}}^{\bar{M}} \partial_{\bar{M}} \\ \nabla_I = (R^{-1})_I^{\bar{M}} \partial_{\bar{M}} \end{cases} \\ \triangleright_I = E_I^{\bar{L}} \bar{\triangleright}_{\bar{L}} \Rightarrow E_I^{\bar{L}} = (R^{-1})_I^{\bar{M}} \bar{R}_{M\bar{L}}. \quad (4.6)$$

From equations in (4.6) the relation between structure constants in (4.5) becomes

$$J^I \wedge J^{\bar{J}} \wedge J^K f_{IJK} + dJ^I \wedge J^{\bar{J}} \eta_{IJ} = \bar{J}^{\bar{M}} \wedge \bar{J}^{\bar{N}} \wedge \bar{J}^{\bar{L}} \bar{f}_{MNL} + d\bar{J}^{\bar{M}} \wedge \bar{J}^{\bar{N}} \eta_{MN}.$$

Using Maurer-Cartan equations in (2.18) the three forms in the doubled space in (2.30) are shown to be equal as

$$\bar{H} = H, \quad \begin{cases} \bar{H} = \frac{1}{3!} \bar{J}^{\bar{M}} \wedge \bar{J}^{\bar{N}} \wedge \bar{J}^{\bar{L}} \bar{f}_{MNL} \\ H = \frac{1}{3!} J^I \wedge J^{\bar{J}} \wedge J^K f_{IJK} \end{cases}. \quad (4.7)$$

In the doubled space the three form $H = dB$ is universal at least locally. This is a consequence of the orthogonality in (4.1) in the doubled formalism.

For the three form $\bar{H} = H$ in (4.7) the two form B in the covariant derivatives in curved spaces $\triangleright_A$ are equal up to the gauge transformation

$$\begin{aligned} \bar{B} &= B + d\Lambda \\ &= \frac{1}{2} dZ^M \wedge dZ^N (\bar{R}_{M\bar{L}} \bar{R}_{N\bar{K}} \bar{B}_{\bar{L}\bar{K}}) = \frac{1}{2} dZ^M \wedge dZ^N (B_{MN} + \partial_{[\bar{M}} \Lambda_{\bar{N}]}) \\ &= \frac{1}{2} \bar{J}^{\bar{L}} \wedge \bar{J}^{\bar{K}} \bar{B}_{\bar{L}\bar{K}} = \frac{1}{2} J^I \wedge J^{\bar{J}} (B_{I\bar{J}} + (R^{-1})_I^{\bar{M}} (R^{-1})_{\bar{J}}^{\bar{N}} \partial_{[\bar{M}} \Lambda_{\bar{N}]}) \end{aligned} \quad (4.8)$$

$\bar{B}_{\bar{L}\bar{K}}$ is the constant solution given in (3.5) and $B_{MN} = R_{M\bar{L}} R_{N\bar{J}} B_{\bar{L}\bar{J}}$. The B field in the group manifold and $\bar{B}$ field in the flat sapce are introduced in covariant derivatives as

$$\begin{cases} \bar{\triangleright}_{\bar{M}} = \bar{\nabla}_{\bar{M}} + \frac{1}{2} \bar{J}_1^{\bar{N}} (\eta_{\bar{N}\bar{M}} + \bar{B}_{\bar{N}\bar{M}}) \\ \triangleright_I = \nabla_I + \frac{1}{2} J_1^{\bar{J}} (\eta_{I\bar{J}} + B_{I\bar{J}}) \end{cases} \quad (4.9)$$

which are related by the vielbein as in (4.6). The gauge symmetry of B_{MN} field in the covariant derivative $\triangleright_{\underline{I}}$ is realized by the rotation between the momentum and the winding mode as

$$\begin{aligned} \delta_{\Lambda} \triangleright_{\underline{I}} &= (R^{-1})_{\underline{I}}^{\underline{M}} \delta_{\Lambda} \left(\frac{1}{i} \partial_{\underline{M}} \right) + \frac{1}{2} (R^{-1})_{\underline{I}}^{\underline{N}} \partial_{\sigma} Z^{\underline{M}} \partial_{[\underline{M}} \Lambda_{\underline{N}]} \\ &\Leftrightarrow \begin{pmatrix} \frac{1}{i} \partial_{\underline{M}} \\ \partial_{\sigma} Z^{\underline{M}} \end{pmatrix} \rightarrow \begin{pmatrix} \delta_{\underline{M}}^{\underline{N}} & \partial_{[\underline{N}} \Lambda_{\underline{M}]} \\ 0 & \delta_{\underline{N}}^{\underline{M}} \end{pmatrix} \begin{pmatrix} \frac{1}{i} \partial_{\underline{N}} \\ \partial_{\sigma} Z^{\underline{N}} \end{pmatrix}. \end{aligned} \quad (4.10)$$

This transformation is a T-duality symmetry transformation of the doubled momenta. In other words the B field in the doubled space is also recognized as a gauge field of the T-duality symmetry transformation given in (4.10).

5 AdS space

5.1 Spontaneous symmetry breaking by the RR flux

In this section we examine a bosonic AdS space as a group manifold. The $\text{AdS}_5 \times S^5$ is a solution of the type IIB supergravity theory. For the N=2 superalgebra the RR D3-brane charge appears in $\Upsilon_{\alpha\beta'}$ [9]

$$\{D_{\alpha}, D_{\beta'}\} = \Upsilon_{\alpha\beta'} \quad (5.1)$$

where D_{α} and $D_{\beta'}$ are the left and right supersymmetry charges with $\alpha, \beta' = 1, \dots, 16$. On the other hand the $\text{AdS}_5 \times S^5$ superalgebra includes the Lorentz terms $\frac{1}{r_{\text{AdS}}} (S \cdot \gamma)_{\alpha\beta}$ in the anticommutator of the left and the right supercharges. The $\text{AdS}_5 \times S^5$ space is obtained in the large D3-brane charge limit. In the limit the right hand side of (5.1) becomes the product of the $1/r_{\text{AdS}}$ times the Lorentz generator $\Upsilon_{\alpha\beta'} \rightarrow \frac{1}{r_{\text{AdS}}} (S \cdot \gamma)_{\alpha\beta} \epsilon_{12}$ with $S \cdot \gamma = S_{ab} \gamma^{ab} + S_{\bar{a}\bar{b}} \gamma^{\bar{a}\bar{b}}$, $a, b = 0, 1, \dots, 4$ and $\bar{a}, \bar{b} = 5, \dots, 9$, and the vacuum expectation value of the RR flux becomes nonzero, $\langle 0 | F_{\text{RR}}^{\alpha\beta'} | 0 \rangle = \frac{1}{r_{\text{AdS}}} (\gamma_{01234} + \gamma_{56789})^{\alpha\beta'}$.

For a bosonic algebra a left-right mixing term will be introduced instead of the central extension of the superalgebra (5.1) as

$$[P_a, P_{b'}] = \Upsilon_{ab'} \quad (5.2)$$

The existence of $f_{PP'}{}^{\Upsilon}$ and introducing the nondegenerate pair as $\eta_{\Upsilon\mathbb{F}} = \mathbf{1}$ lead to the existence of $f_{PP'\mathbb{F}}$,

$$[\mathbb{F}^{ab'}, P_c] = \delta_c^a P^{b'}, \quad [\mathbb{F}^{ab'}, P_{c'}] = -\delta_{c'}^{b'} P^a. \quad (5.3)$$

This suggests that $\Upsilon_{ab'}$ and $\mathbb{F}^{ab'}$ [9] correspond to the left-right mixing Lorentz generators, $\Sigma^{ab'}$ and $S_{ab'}$ respectively. The algebra is determined by the Jacobi identity.

As a result the number of generators of the doubled 10-dimensional flat space and the one for the doubled $\text{AdS}_5 \times \text{S}^5$ coincide as follows.

	Flat	number	$\text{AdS}_5 \times \text{S}^5$	number
Lorentz	S_{mn}	45	$S_{ab}, S_{\bar{a}\bar{b}}$	10 + 10
	$S_{m'n'}$	45	$S_{a'b'}, S_{\bar{a}'\bar{b}'}$	10 + 10
			$S_{ab'}, S_{\bar{a}\bar{b}'}$	25 + 25
Momenta	P_m	10	$P_a, P_{\bar{a}}$	5 + 5
	$P_{m'}$	10	$P_{a'}, P_{\bar{a}'}$	5 + 5
Lorentz nondegenerate partner	Σ^{mn}	45	$\Sigma^{ab}, \Sigma^{\bar{a}\bar{b}}$	10 + 10
	$\Sigma^{m'n'}$	45	$\Sigma^{a'b'}, \Sigma^{\bar{a}'\bar{b}'}$	10 + 10
			$\Sigma^{ab'}, \Sigma^{\bar{a}\bar{b}'}$	25 + 25

(5.4)

The indices in this table are the followings: 10-d. flat indices are $m, m' = 0, \dots, 9$, AdS_5 indices are $a, a' = 0, 1, 2, 3, 4$ and S^5 indices are $\bar{a}, \bar{a}' = 5, 6, 7, 8, 9$. The subgroup H of the coset G/H is modified by the spontaneous symmetry breaking. The subgroup is given schematically as follows;

$$\begin{array}{c}
 \begin{array}{cc} \text{left} & \text{right} \\ \text{Poincaré} & \\ P_b & P_{b'} \end{array} \\
 \begin{array}{|c|c|} \hline P_a & S_{ab} \\ \hline P_{a'} & S_{a'b'} \\ \hline \end{array} \Rightarrow
 \begin{array}{cc} \text{left} & \text{right} \\ \text{AdS} & \text{S} \\ P_b & P_{\bar{b}} \quad P_{b'} & P_{\bar{b}'} \end{array} \\
 \begin{array}{|c|c|c|c|} \hline P_a & S_{ab} & & S_{ab'} \\ \hline P_{\bar{a}} & & S_{\bar{a}\bar{b}} & S_{\bar{a}\bar{b}'} \\ \hline P_{a'} & S_{a'b} & & S_{a'b'} \\ \hline P_{\bar{a}'} & & S_{\bar{a}'\bar{b}} & S_{\bar{a}'\bar{b}'} \\ \hline \end{array} =
 \begin{array}{cc} \text{left} & \text{right} \\ \text{AdS} & \text{S} \\ P_b & P_{b'} \quad P_{\bar{b}} & P_{\bar{b}'} \end{array} \\
 \begin{array}{|c|c|c|c|} \hline P_a & S_{ab} & S_{ab'} & \\ \hline P_{a'} & S_{a'b} & S_{a'b'} & \\ \hline P_{\bar{a}} & & & S_{\bar{a}\bar{b}} \quad S_{\bar{a}\bar{b}'} \\ \hline P_{\bar{a}'} & & & S_{\bar{a}'\bar{b}} \quad S_{\bar{a}'\bar{b}'} \\ \hline \end{array}
 \end{array} \quad (5.5)$$

The number of degrees of freedom of G_{mn} and B_{mn} is d^2 which now coincides with the number of the dimension of the coset $O(d, d)/O(\frac{d}{2}, \frac{d}{2})^2$. In this paper we focus on the doubled bosonic AdS part of the $\text{AdS}^5 \times \text{S}^5$ space which is the upper-left part of the third figure in (5.5) from now on. The doubled bosonic sphere part of the $\text{AdS}^5 \times \text{S}^5$ space is analyzed similarly which is the lower-right part of the third figure in (5.5).

5.2 Nondegenerate doubled AdS algebra

At first we make an AdS algebra doubled and nondegenerate in this section. In next subsection affine extension is performed. The criteria of the AdS algebra with manifest T-duality are followings:

- Dimensional reduction of the doubled space algebra gives to the AdS algebra in the usual single coordinate space.

- Doubled AdS algebra has a flat limit in the large AdS radius, $r_{\text{AdS}} \rightarrow \infty$.
- Doubled AdS algebra has the nondegenerate group metric and the totally antisymmetric structure constant.

We focus on the bosonic 5-dimensional AdS part in $\text{AdS}_5 \times S^5$. As seen in the previous section the existence of the RR flux leads to the left/right mixing Lorentz generators. The doubled d-dimensional AdS space is described by $\text{SO}(d, d+1)$ group. Next the nondegenerate pair of the Lorentz generators are introduced by direct product of another Lorentz group $\text{SO}(d, d)$. The obtained group $\text{SO}(d, d+1) \times \text{SO}(d, d)$ is the doubled AdS algebra with the nondegenerate group metric and the totally antisymmetric structure constant.

5.2.1 Doubled AdS algebra

We double the AdS group into the ones for left and right AdS groups, in addition to them we include the left/right mixing as seen in the previous section. So the doubled AdS group will be $\text{SO}(d, d+1)$.

The doubled d-dimensional AdS algebra is given by $\text{so}(d, d+1)$ generated by doubled momenta $\underline{p}_a = (p_a, p_{a'})$, doubled Lorentz $\underline{s}_{ab} = (s_{ab}, s_{a'b'}; s_{ab'})$ where a and a' runs 0 to $d-1$. The proposed doubled algebra is

$$\begin{aligned} [G_A, G_B] &= i f_{AB}^C G_C, \quad [G_{A'}, G_{B'}] = i f_{A'B'}^{C'} G_{C'}, \quad [G_A, G_{B'}] = i f_{AB'}^\Upsilon G_\Upsilon \\ [G_\Upsilon, G_A] &= i f_{\Upsilon A}^{B'} G_{B'}, \quad [G_\Upsilon, G_{A'}] = i f_{\Upsilon A'}^B G_B \\ [G_\Upsilon, G_\Upsilon] &= i f_{\Upsilon\Upsilon}^A G_A + i f_{\Upsilon\Upsilon}^{A'} G_{A'} \end{aligned} \quad (5.6)$$

where the left/right mixed index is denoted by Υ including its nondegenerate partner $\mathbf{F}$ [9]. The doubled AdS algebra is given by

$$\begin{aligned} \text{Left :} \quad [s_{ab}, s_{cd}] &= i \eta_{[d|[a} s_{b]|c]}, \quad [s_{ab}, p_c] = i p_{[a} \eta_{b]c}, \quad [p_a, p_b] = i \frac{1}{r_{\text{AdS}}^2} s_{ab} \\ \text{Right :} \quad [s_{a'b'}, s_{c'd'}] &= i \eta_{[d'|[a'} s_{b']c'}], \quad [s_{a'b'}, p_{c'}] = i p_{[a'} \eta_{b']c'}, \quad [p_{a'}, p_{b'}] = i \frac{1}{r_{\text{AdS}}^2} s_{a'b'} \\ \text{Mixed :} \quad [s_{ab'}, s_{cd'}] &= -i (\eta_{b'd'} s_{ac} + \eta_{ac} s_{b'd'}) \\ [s_{ab}, s_{cd'}] &= -i \eta_{c[a} s_{b]d'}, \quad [s_{a'b'}, s_{cd}] = -i \eta_{d'[a'} s_{c]b'} \\ [s_{ab'}, p_c] &= -i \eta_{ac} p_{b'}, \quad [s_{ab'}, p_{c'}] = i \eta_{b'c'} p_a, \quad [p_a, p_{b'}] = i \frac{1}{r_{\text{AdS}}^2} s_{ab'} \end{aligned} \quad (5.7)$$

The spacetime metric of the enlarged space is

$$\eta_{\underline{ab}} = (\eta_{\mathfrak{ab}}; \eta_{ab}; \eta_{a'b'}) = (-1; -1, 1, 1, 1, 1; 1, -1, -1, -1, -1). \quad (5.8)$$

The left moving mode is in an AdS space while the right moving mode is in a dS space. This phenomena is similar to the point discussed in [16]. The structure constants with lowered indices f_{ABC} are totally antisymmetric.

5.2.2 Nondegenerate doubled AdS algebra

We will construct a nondegenerate AdS group $SO(d,d+1) \times SO(d,d)$ in such a way that the subalgebra H of the coset has its nondegenerate partner by following the procedure given in subsection 2.2.1:

1. The doubled momenta are the generators of the coset G/H_0 denoted by k , where $G=SO(d,d+1)$ is the doubled AdS group and $H_0=SO(d,d)$ is the doubled Lorentz subgroups and the left/right mixed Lorentz as in (5.7).
2. Another Lorentz group $H_1=SO(d,d)$ is introduced to construct the nondegenerate pair of the Lorentz group.
3. Make nondegenerate pair s_{ab} and σ^{ab} by linear combinations of h_0 and h_1 which are Lie algebras of H_0 and H_1 as

$$\begin{cases} h_0 + h_1 = s \\ h_0 - h_1 = \frac{1}{r_{\text{AdS}}^2} \sigma \\ k = \frac{1}{\sqrt{2}r_{\text{AdS}}} p \end{cases} \Rightarrow \begin{cases} [s, s] = s, [s, \sigma] = \sigma, [\sigma, \sigma] = \frac{1}{r_{\text{AdS}}^4} s \\ [s, p] = p, [p, p] = \frac{1}{r_{\text{AdS}}^2} s + \sigma, [\sigma, p] = \frac{1}{r_{\text{AdS}}^2} p \end{cases} \quad (5.9)$$

4. Non-zero components of the nondegenerate doubled AdS group metrics are

$$\eta_{pp} = -\eta_{p'p'} = \mathbf{1} = \eta_{s\sigma} = \eta_{s'\sigma'} = \eta_{\mathbf{F}\Upsilon} \quad (5.10)$$

with $(p_a, p_{a'}) = (p, p')$, $(s_{ab}, s_{a'b'}; s_{ab'}) = (s, s'; \mathbf{F})$ and $(\sigma^{ab}, \sigma^{a'b'}; \sigma^{ab'}) = (\sigma, \sigma'; \Upsilon)$. The signature of nondegenerate group metric is determined from the Jacobi identity. The structure constant including constant torsions with lowered indices are totally antisymmetric:

$$\begin{aligned} f_{ss\sigma} &= -f_{s's'\sigma'} = f_{\mathbf{F}\Upsilon s} = -f_{\mathbf{F}\Upsilon s'} = f_{\mathbf{F}\mathbf{F}\sigma} = -f_{\mathbf{F}\mathbf{F}\sigma'} = f_{pps} = f_{p'p's'} = -f_{pp'\mathbf{F}} = \mathbf{1} \\ f_{pp\sigma} &= f_{p'p'\sigma'} = -f_{pp'\Upsilon} = \frac{1}{r_{\text{AdS}}^2} \mathbf{1}, \quad f_{\sigma\sigma\sigma} = -f_{\sigma'\sigma'\sigma'} = f_{\Upsilon\Upsilon\sigma} = -f_{\Upsilon\Upsilon\sigma'} = \frac{1}{r_{\text{AdS}}^4} \mathbf{1}. \end{aligned} \quad (5.11)$$

5.3 Affine AdS algebras

5.3.1 Covariant derivative and symmetry generator in the AdS space

The covariant derivatives and the symmetry generators in the AdS space are given by (2.24) and (2.25) as follows.

• **AdS covariant derivatives**

The covariant derivative in the AdS space is a linear combination of the AdS particle covariant derivative $\overset{\circ}{\nabla}_{\underline{A}}$ and the σ component of the left-invariant current $\overset{\circ}{J}^{\underline{A}}$ with the B field.

$$\overset{\circ}{\mathbb{D}}_{\underline{A}} = \overset{\circ}{\nabla}_{\underline{A}} + \frac{1}{2} \overset{\circ}{J}^{\underline{B}} (\eta_{\underline{B}\underline{A}} + \overset{\circ}{B}_{\underline{B}\underline{A}}) \quad (5.12)$$

The $\overset{\circ}{B}_{\underline{B}\underline{A}}$ field on the AdS space is a solution of the equation given in (2.27) and the existence of the solution is guaranteed by $d\overset{\circ}{H} = 0$. The B field on the AdS space is not a constant

$$i\overset{\circ}{\nabla}_{[\underline{A}} \overset{\circ}{B}_{\underline{B}\underline{C}]} - \overset{\circ}{f}_{[\underline{A}\underline{B}]}{}^{\underline{D}} \overset{\circ}{B}_{\underline{D}[\underline{C}]} = 2\overset{\circ}{f}_{\underline{A}\underline{B}\underline{C}}. \quad (5.13)$$

The covariant derivatives of the nondegenerate doubled AdS algebra is the Lie algebra of the group $\text{SO}(d, d+1) \times \text{SO}(d, d)$

$$\overset{\circ}{\mathbb{D}}_{\underline{A}}(\sigma) = (S_{\underline{a}\underline{b}}, P_{\underline{a}}, \Sigma^{\underline{a}\underline{b}}), \begin{cases} S_{\underline{a}\underline{b}} = (S_{ab}, S_{ab'}, S_{a'b'}) \\ P_{\underline{a}} = (P_a, P_{a'}) \\ \Sigma^{\underline{a}\underline{b}} = (\Sigma^{ab}, \Sigma^{ab'}, \Sigma^{a'b'}) \end{cases}. \quad (5.14)$$

• **AdS symmetry generators**

The symmetry generator in the AdS space is a linear combination of the AdS particle symmetry generator $\overset{\circ}{\nabla}_{\underline{A}}$ and the σ component of the right-invariant current $\overset{\circ}{\tilde{J}}^{\underline{A}}$ with the $\tilde{B}$ field.

$$\begin{aligned} \overset{\circ}{\mathbb{D}}_{\underline{A}} &= \overset{\circ}{\nabla}_{\underline{A}} + \frac{1}{2} \overset{\circ}{\tilde{J}}^{\underline{B}} (-\eta_{\underline{B}\underline{A}} + \overset{\circ}{\tilde{B}}_{\underline{B}\underline{A}}) \\ \overset{\circ}{\tilde{B}}_{\underline{B}\underline{A}} &= \overset{\circ}{M}_{\underline{B}}{}^{\underline{C}} \overset{\circ}{M}_{\underline{A}}{}^{\underline{D}} \overset{\circ}{B}_{\underline{C}\underline{D}}, \quad \overset{\circ}{M}_{\underline{A}}{}^{\underline{D}} = (\overset{\circ}{L}^{-1})_{\underline{A}}{}^{\underline{M}} \overset{\circ}{R}_{\underline{M}}{}^{\underline{D}}. \end{aligned} \quad (5.15)$$

The symmetry generators of the nondegenerate doubled AdS algebra is the Lie algebra of the group $\text{SO}(d, d+1) \times \text{SO}(d, d)$

$$\overset{\circ}{\mathbb{D}}_{\underline{A}}(\sigma) = (S_{\underline{a}\underline{b}}, \tilde{P}_{\underline{a}}, \tilde{\Sigma}^{\underline{a}\underline{b}}), \begin{cases} \tilde{S}_{\underline{a}\underline{b}} = (\tilde{S}_{ab}, \tilde{S}_{ab'}, \tilde{S}_{a'b'}) \\ \tilde{P}_{\underline{a}} = (\tilde{P}_a, \tilde{P}_{a'}) \\ \tilde{\Sigma}^{\underline{a}\underline{b}} = (\tilde{\Sigma}^{ab}, \tilde{\Sigma}^{ab'}, \tilde{\Sigma}^{a'b'}) \end{cases}. \quad (5.16)$$

5.3.2 Affine AdS algebras

The nondegenerate doubled AdS algebra in (5.7) and (5.9) is extended to affine AdS algebras generated by the AdS covariant derivative in (5.12) and the AdS symmetry generator in (5.15). In contrast to the flat case the left and right moving modes of the AdS algebra are not really separated because of the left/right mixing caused by the RR flux. Since the commutativity of the covariant derivative and the symmetry generator holds for the

AdS space, their roles hold in the AdS space; while the covariant derivative determine the local structure of the space, the symmetry generators are used to separate out physical dimensions from unphysical dimensions. The affine AdS algebras by the covariant derivative (5.12) and the symmetry generator (5.15) in components are listed as below.

- **Affine AdS algebras by covariant derivative $\overset{\circ}{\triangleright}_{\underline{A}}$ and symmetry generator $\overset{\circ}{\triangleright}_{\underline{A}}$:**

$$\begin{aligned}
 [\overset{\circ}{\triangleright}_{\underline{A}}(1), \overset{\circ}{\triangleright}_{\underline{B}}(2)] &= -if_{\underline{AB}}^{\underline{C}} \overset{\circ}{\triangleright}_{\underline{C}} \delta(2-1) - i\eta_{\underline{AB}} \partial_{\sigma} \delta(2-1) \\
 [\overset{\circ}{\triangleright}_{\underline{A}}(1), \overset{\circ}{\triangleright}_{\underline{B}}(2)] &= if_{\underline{AB}}^{\underline{C}} \overset{\circ}{\triangleright}_{\underline{C}} \delta(2-1) + i\eta_{\underline{AB}} \partial_{\sigma} \delta(2-1) \\
 [\overset{\circ}{\triangleright}_{\underline{A}}(1), \overset{\circ}{\triangleright}_{\underline{B}}(2)] &= 0
 \end{aligned} \tag{5.17}$$

- **Affine AdS algebra by covariant derivatives: $\overset{\circ}{\triangleright}_{\underline{A}} = (\overset{\circ}{\triangleright}_A, \overset{\circ}{\triangleright}_{A'}, \overset{\circ}{\triangleright}_{\Upsilon})$**

$$\begin{aligned}
 \text{AdS Left : } \overset{\circ}{\triangleright}_A &= (S_{ab}, P_a, \Sigma^{ab}) \\
 \left\{ \begin{aligned}
 [S_{ab}(1), S_{cd}(2)] &= r_{\text{AdS}}^4 [\Sigma_{ab}(1), \Sigma_{cd}(2)] = -i\eta_{[d|[a} S_{b]|c]} \delta(2-1) \\
 [S_{ab}(1), P_c(2)] &= r_{\text{AdS}}^2 [\Sigma_{ab}(1), P_c(2)] = -iP_{[a} \eta_{b]c} \delta(2-1) \\
 [P_a(1), P_b(2)] &= -i \left(\frac{1}{r_{\text{AdS}}^2} S_{ab} + \Sigma_{ab} \right) \delta(2-1) - i\eta_{ab} \partial_{\sigma} \delta(2-1) \\
 [S_{ab}(1), \Sigma_{cd}(2)] &= -i\eta_{[d|[a} \Sigma_{b]|c]} \delta(2-1) - i\eta_{d[a} \eta_{b]c} \partial_{\sigma} \delta(2-1)
 \end{aligned} \right. \tag{5.18}
 \end{aligned}$$

$$\begin{aligned}
 \text{AdS Right : } \overset{\circ}{\triangleright}_{A'} &= (S_{a'b'}, P_{a'}, \Sigma^{a'b'}) \\
 \left\{ \begin{aligned}
 [S_{a'b'}(1), S_{c'd'}(2)] &= r_{\text{AdS}}^4 [\Sigma_{a'b'}(1), \Sigma_{c'd'}(2)] = -i\eta_{[d'|[a'} S_{b']c']} \delta(2-1) \\
 [S_{a'b'}(1), P_{c'}(2)] &= r_{\text{AdS}}^2 [\Sigma_{a'b'}(1), P_{c'}(2)] = -iP_{[a'} \eta_{b']c'} \delta(2-1) \\
 [P_{a'}(1), P_{b'}(2)] &= -i \left(\frac{1}{r_{\text{AdS}}^2} S_{a'b'} + \Sigma_{a'b'} \right) \delta(2-1) - i\eta_{a'b'} \partial_{\sigma} \delta(2-1) \\
 [S_{a'b'}(1), \Sigma_{c'd'}(2)] &= -i\eta_{[d'|[a'} \Sigma_{b']c']} \delta(2-1) - i\eta_{d'[a'} \eta_{b']c'} \partial_{\sigma} \delta(2-1)
 \end{aligned} \right. \tag{5.19}
 \end{aligned}$$

$$\begin{aligned}
 \text{AdS Mixed : } \overset{\circ}{\triangleright}_{\Upsilon} &= (S_{ab'}, \Sigma^{ab'}) \\
 \left\{ \begin{aligned}
 [S_{ab'}(1), S_{cd'}(2)] &= r_{\text{AdS}}^4 [\Sigma_{ab'}(1), \Sigma_{cd'}(2)] = i(\eta_{b'd'} S_{ac} + \eta_{ac} S_{b'd'}) \delta(2-1) \\
 [S_{ab}(1), S_{cd'}(2)] &= r_{\text{AdS}}^4 [\Sigma_{ab}(1), \Sigma_{cd'}(2)] = i\eta_{c[a} S_{b]d'} \delta(2-1) \\
 [S_{a'b'}(1), S_{cd'}(2)] &= r_{\text{AdS}}^4 [\Sigma_{a'b'}(1), \Sigma_{cd'}(2)] = i\eta_{d'[a'} S_{c]b'} \delta(2-1) \\
 [S_{ab'}(1), P_c(2)] &= r_{\text{AdS}}^2 [\Sigma_{ab'}(1), P_c(2)] = i\eta_{ac} P_{b'} \delta(2-1) \\
 [S_{ab'}(1), P_{c'}(2)] &= r_{\text{AdS}}^2 [\Sigma_{ab'}(1), P_{c'}(2)] = -i\eta_{b'c'} P_a \delta(2-1) \\
 [P_a(1), P_{b'}(2)] &= -i \left(\frac{1}{r_{\text{AdS}}^2} S_{ab'} + \Sigma_{ab'} \right) \delta(2-1) \\
 [S_{ab'}(1), \Sigma_{cd'}(2)] &= i(\eta_{b'd'} \Sigma_{ac} + \eta_{ac} \Sigma_{b'd'}) \delta(2-1) + i\eta_{b'd'} \eta_{ac} \partial_{\sigma} \delta(2-1) \\
 [S_{ab}(1), \Sigma_{cd'}(2)] &= [\Sigma_{ab}(1), S_{cd'}(2)] = i\eta_{c[a} \Sigma_{b]d'} \delta(2-1) \\
 [S_{a'b'}(1), \Sigma_{cd'}(2)] &= [\Sigma_{a'b'}(1), S_{cd'}(2)] = i\eta_{d'[a'} \Sigma_{c]b'} \delta(2-1)
 \end{aligned} \right. \tag{5.20}
 \end{aligned}$$

- **Affine AdS algebra by symmetry generators:** $\overset{\circ}{\triangleright}_{\underline{A}} = (\overset{\circ}{\triangleright}_A, \overset{\circ}{\triangleright}_{A'}, \overset{\circ}{\triangleright}_\Upsilon)$

$$\begin{aligned}
 \text{AdS Left : } \overset{\circ}{\triangleright}_A &= (\tilde{S}_{ab}, \tilde{P}_a, \tilde{\Sigma}^{ab}) \\
 \left\{ \begin{aligned} [\tilde{S}_{ab}(1), \tilde{S}_{cd}(2)] &= r_{\text{AdS}}^4 [\tilde{\Sigma}_{ab}(1), \tilde{\Sigma}_{cd}(2)] = i\eta_{[d|[a}\tilde{S}_{b]|c]}\delta(2-1) \\ [\tilde{S}_{ab}(1), \tilde{P}_c(2)] &= r_{\text{AdS}}^2 [\tilde{\Sigma}_{ab}(1), \tilde{P}_c(2)] = i\tilde{P}_{[a}\eta_{b]c}\delta(2-1) \\ [\tilde{P}_a(1), \tilde{P}_b(2)] &= i\left(\frac{1}{r_{\text{AdS}}^2}\tilde{S}_{ab} + \tilde{\Sigma}_{ab}\right)\delta(2-1) + i\eta_{ab}\partial_\sigma\delta(2-1) \\ [\tilde{S}_{ab}(1), \tilde{\Sigma}_{cd}(2)] &= i\eta_{[d|[a}\tilde{\Sigma}_{b]|c]}\delta(2-1) + i\eta_{d[a}\eta_{b]c}\partial_\sigma\delta(2-1) \end{aligned} \right. \quad (5.21)
 \end{aligned}$$

$$\begin{aligned}
 \text{AdS Right : } \overset{\circ}{\triangleright}_{A'} &= (\tilde{S}_{a'b'}, \tilde{P}_{a'}, \tilde{\Sigma}^{a'b'}) \\
 \left\{ \begin{aligned} [\tilde{S}_{a'b'}(1), \tilde{S}_{c'd'}(2)] &= r_{\text{AdS}}^4 [\tilde{\Sigma}_{a'b'}(1), \tilde{\Sigma}_{c'd'}(2)] = i\eta_{[d'|[a'}\tilde{S}_{b']c']}\delta(2-1) \\ [\tilde{S}_{a'b'}(1), \tilde{P}_{c'}(2)] &= r_{\text{AdS}}^2 [\tilde{\Sigma}_{a'b'}(1), \tilde{P}_{c'}(2)] = i\tilde{P}_{[a'}\eta_{b']c'}\delta(2-1) \\ [\tilde{P}_{a'}(1), \tilde{P}_{b'}(2)] &= i\left(\frac{1}{r_{\text{AdS}}^2}\tilde{S}_{a'b'} + \tilde{\Sigma}_{a'b'}\right)\delta(2-1) + i\eta_{a'b'}\partial_\sigma\delta(2-1) \\ [\tilde{S}_{a'b'}(1), \tilde{\Sigma}_{c'd'}(2)] &= i\eta_{[d'|[a'}\tilde{\Sigma}_{b']c']}\delta(2-1) + i\eta_{d'[a'}\eta_{b']c'}\partial_\sigma\delta(2-1) \end{aligned} \right. \quad (5.22)
 \end{aligned}$$

$$\begin{aligned}
 \text{AdS Mixed : } \overset{\circ}{\triangleright}_\Upsilon &= (\tilde{S}_{ab'}, \tilde{\Sigma}^{ab'}) \\
 \left\{ \begin{aligned} [\tilde{S}_{ab'}(1), \tilde{S}_{cd'}(2)] &= r_{\text{AdS}}^4 [\tilde{\Sigma}_{ab'}(1), \tilde{\Sigma}_{cd'}(2)] = -i(\eta_{b'd'}\tilde{S}_{ac} + \eta_{ac}\tilde{S}_{b'd'})\delta(2-1) \\ [\tilde{S}_{ab}(1), \tilde{S}_{cd'}(2)] &= r_{\text{AdS}}^4 [\tilde{\Sigma}_{ab}(1), \tilde{\Sigma}_{cd'}(2)] = -i\eta_{c[a}\tilde{S}_{b]d'}\delta(2-1) \\ [\tilde{S}_{a'b'}(1), \tilde{S}_{cd'}(2)] &= r_{\text{AdS}}^4 [\tilde{\Sigma}_{a'b'}(1), \tilde{\Sigma}_{cd'}(2)] = -i\eta_{d'[a'}\tilde{S}_{c]b']}\delta(2-1) \\ [\tilde{S}_{ab'}(1), \tilde{P}_c(2)] &= r_{\text{AdS}}^2 [\tilde{\Sigma}_{ab'}(1), \tilde{P}_c(2)] = -i\eta_{ac}\tilde{P}_{b'}\delta(2-1) \\ [\tilde{S}_{ab'}(1), \tilde{P}_{c'}(2)] &= r_{\text{AdS}}^2 [\tilde{\Sigma}_{ab'}(1), \tilde{P}_{c'}(2)] = i\eta_{b'c'}\tilde{P}_a\delta(2-1) \\ [\tilde{P}_a(1), \tilde{P}_{b'}(2)] &= i\left(\frac{1}{r_{\text{AdS}}^2}\tilde{S}_{ab'} + \tilde{\Sigma}_{ab'}\right)\delta(2-1) \\ [\tilde{S}_{ab'}(1), \tilde{\Sigma}_{cd'}(2)] &= -i(\eta_{b'd'}\tilde{\Sigma}_{ac} + \eta_{ac}\tilde{\Sigma}_{b'd'})\delta(2-1) - i\eta_{b'd'}\eta_{ac}\partial_\sigma\delta(2-1) \\ [\tilde{S}_{ab}(1), \tilde{\Sigma}_{cd'}(2)] &= [\tilde{\Sigma}_{ab}(1), \tilde{S}_{cd'}(2)] = -i\eta_{c[a}\tilde{\Sigma}_{b]d'}\delta(2-1) \\ [\tilde{S}_{a'b'}(1), \tilde{\Sigma}_{cd'}(2)] &= [\tilde{\Sigma}_{a'b'}(1), \tilde{S}_{cd'}(2)] = -i\eta_{d'[a'}\tilde{\Sigma}_{c]b']}\delta(2-1) \end{aligned} \right. \quad (5.23)
 \end{aligned}$$

5.3.3 Curved backgrounds in the asymptotically AdS space

The AdS space is spanned by the AdS covariant derivative $\overset{\circ}{\triangleright}_{\underline{A}}$ in (5.12) which satisfies the affine Lie algebra given in the first line of (5.18). Let us consider gravity theory as a fluctuation in the asymptotically AdS space as

$$\triangleright_{\underline{M}} = E_{\underline{M}}^{\underline{A}} \overset{\circ}{\triangleright}_{\underline{A}}. \quad (5.24)$$

The commutator of the covariant derivative gives the torsion and the Bianchi identity gives the torsion equations

$$\begin{aligned}
 [\triangleright_{\underline{M}}(1), \triangleright_{\underline{N}}(2)] &= -iT_{\underline{MNL}}\triangleright^L\delta(2-1) - i\eta_{\underline{MN}}\partial_\sigma\delta(2-1) \\
 T_{\underline{MNL}} &= T_{\underline{MN}}{}^K\eta_{\underline{KL}} = \frac{1}{2}(i\nabla_{[\underline{M}}E_{\underline{N}}{}^A)E_{\underline{L}]A} + E_{\underline{M}}{}^AE_{\underline{N}}{}^BE_{\underline{L}}{}^C\overset{\circ}{f}_{\underline{ABC}} \quad (5.25) \\
 i\nabla_{[\underline{M}}T_{\underline{NLK]} &+ \frac{3}{4}T_{[\underline{MN}}{}^ET_{\underline{LK]}E} = 0 .
 \end{aligned}$$

The general gauge transformations are calculated from T-bracket given in (2.33) and (2.34) by taking the vielbein field as $\hat{\Lambda}_2 = E_{\underline{M}}{}^A\overset{\circ}{\triangleright}_A$, the gauge parameters as $\hat{\Lambda}_1 = \int \Lambda^A\overset{\circ}{\triangleright}_A$. The structure constant and the covariant derivative are specified as the AdS structure constant $\overset{\circ}{f}_{\underline{AB}}{}^C$ and the AdS covariant derivative $\overset{\circ}{\triangleright}_A$. The vielbein field has gauge symmetries generated by the above bracket as

$$\begin{aligned}
 \delta_\Lambda E_{\underline{M}}{}^A\overset{\circ}{\triangleright}_A &= i[\hat{\Lambda}_1, E_{\underline{M}}{}^A\overset{\circ}{\triangleright}_A] \\
 (\delta_\Lambda E_{\underline{M}}{}^A)E_{\underline{N}A} &= i\nabla_{[\underline{M}}(E_{\underline{N}}{}^A\Lambda_A) - T_{\underline{MNL}}E^L{}_A\Lambda_A . \quad (5.26)
 \end{aligned}$$

In the asymptotically flat limit the gauge symmetry transformation (5.26) is reduced to the one with the structure constant of the nondegenerate Poincaré algebra.

5.4 Auxiliary dimensions and physical dimensions

In order to manifest T-duality symmetry we have enlarged the space not only by introducing the doubled coordinates but also by introducing auxiliary dimensions of the nondegeneracy. In this section dimensional reduction constraints are obtained to reduce such unphysical dimensions. We also construct the physical symmetry algebra in terms of the symmetry generators written by doubled coordinates on the constrained surface.

5.4.1 Dimensional reduction constraints

As discussed in section 5.1 the non-zero vacuum expectation value of the RR flux in the AdS space, $\langle 0|F_{\text{RR}}^{\alpha\beta'}|0\rangle \neq 0$, breaks two Lorentz symmetries preserving only a combination of the left and right Lorentz transformations as

$$\begin{aligned}
 &\left[\frac{1}{2}\lambda^{ab}\tilde{S}_{ab} + \frac{1}{2}\lambda^{a'b'}\tilde{S}_{a'b'}, \langle 0|F_{\text{RR}}^{\alpha\beta'}|0\rangle \right] \\
 &= \frac{1}{2}\lambda^{ab}(\gamma_{ab})^\alpha{}_\beta \langle 0|F_{\text{RR}}^{\beta\beta'}|0\rangle + \frac{1}{2}\lambda^{a'b'}\langle 0|F_{\text{RR}}^{\alpha\alpha'}|0\rangle (\gamma_{a'b'})^{\beta'}{}_{\alpha'} . \quad (5.27)
 \end{aligned}$$

In general $\langle 0|F_{\text{RR}}^{\alpha\beta'}|0\rangle$ depends on the Lorentz coordinates, so it is transformed under the Lorentz transformations as above. In a simple gauge where the left and right spinors are the same chirality for the total Lorentz group, the vacuum expectation value of the five form RR flux is represented as $\langle 0|F_{\text{RR}}^{\alpha\beta'}|0\rangle = \frac{1}{r_{\text{AdS}}}\mu^{\alpha\beta'}$ with $\mu^{\alpha\beta'} = \epsilon_{IJ}(\gamma_{01234} + \gamma_{56789})^{\alpha\beta}$ with $N = 2$ spinor indices I, J . Only one combination of the two Lorentz symmetries with parameters $\lambda_{ab} + \lambda_{a'b'} = 0$ preserves the vacuum symmetry from $[\gamma_{ab}, \gamma_{01234} + \gamma_{56789}] = 0$. Therefore the

preserved Lorentz symmetry will be $\tilde{S}_{ab} - \tilde{S}_{a'b'}$. We introduce a parameter as a left-right mixing coefficient defined by the vacuum expectation value of the following tensor

$$\langle 0 | F_{\text{RR}}^{\alpha\alpha'} F_{\text{RR}}^{\beta\beta'} | 0 \rangle (\gamma_a)_{\alpha\beta} (\gamma^{b'})_{\alpha'\beta'} = \frac{\text{tr} \mathbf{1}}{r_{\text{AdS}}^2} \chi_a^{b'}. \quad (5.28)$$

It is possible to choose $\chi_{aa'}$ which satisfies

$$\chi_{aa'} \chi_{bb'} \eta^{ab} = -\eta_{a'b'} \quad , \quad \chi_{aa'} \chi_{bb'} \eta^{a'b'} = -\eta_{ab} \quad , \quad (5.29)$$

and it is inert under the Lorentz rotations, for a case $\chi_a^{b'} = \delta_a^{b'}$.

The criteria of the dimensional reduction constraints are the followings:

- Constraints are written in terms of symmetry generators. The symmetry generators commute with the covariant derivatives, so the dimensional reduction constraints can reduce unphysical degrees of freedom without changing the local geometry.
- The survived symmetry generated by the total momentum and the total Lorentz is the usual AdS algebra.

Before examining the dimensional reduction constraints we analyze the non-abelian doubled algebra. If the doubled group is a direct product, generated by G and G' , it has Z_2 structure

$$\begin{aligned} [G, G] &= G, \quad [G', G'] = -G', \quad [G, G'] = 0 \\ \Theta_0 &= (G - G'), \quad \Theta_1 = (G + G') \\ \Rightarrow [\Theta_\mu, \Theta_\nu] &= \delta_{\mu\nu} \Theta_0 + \epsilon_{\mu\nu} \Theta_{\mu+\nu} = \Theta_{\mu+\nu}, \quad \text{mod } 2, \quad \mu=(0,1). \end{aligned} \quad (5.30)$$

However we have introduced the left-right mixed term Υ as in (5.5) and (5.6), then the Z_2 structure is generalized. The antisymmetric and symmetric parts of Υ are denoted as $[\Upsilon]$ and (Υ) . The generalized Z_2 structure is given as;

$$\begin{aligned} [p, p] &= s, \quad [p', p'] = -s', \quad [p, p'] = [\Upsilon] + (\Upsilon) \\ \Theta_0 &= (s - s'), \quad \Theta_1 = [\Upsilon], \quad \Theta_2 = (s + s'), \quad \Theta_3 = (\Upsilon) \\ \Rightarrow [\Theta_0, \Theta_0] &= \Theta_0, \quad [\Theta_0, \Theta_i] = \Theta_i, \quad [\Theta_i, \Theta_j] = \delta_{ij} \Theta_0 + \epsilon_{ijk} \Theta_k, \quad i, j, k=(1,2,3). \end{aligned} \quad (5.31)$$

There are three sets of representations of the above algebra (5.31):

- **Lorentz symmetry generator algebra with $\tilde{S}$**

The linear combinations of the left and right Lorentz symmetry generators in (5.23) satisfy the above structure:

$$\begin{aligned} \Theta_0 &= \tilde{S}_{ab} - \tilde{S}_{a'b'} \chi_a^{a'} \chi_b^{b'}, \quad \Theta_1 = \tilde{S}_{[a|b'|} \chi_b]^{b'} \\ \Theta_2 &= \tilde{S}_{ab} + \tilde{S}_{a'b'} \chi_a^{a'} \chi_b^{b'}, \quad \Theta_3 = \tilde{S}_{(a|b'|} \chi_b)^{b'} \end{aligned} \quad (5.32)$$

$$\begin{cases} [\Theta_{0;ab}, \Theta_{\mu;cd}] = -i\eta_{[c|[a} \Theta_{\mu;b]}|d], \quad \mu=0,1,2, & [\Theta_{0;ab}, \Theta_{3;cd}] = -i\eta_{[c|[a} \Theta_{3;b]}|d] \\ [\Theta_{i;ab}, \Theta_{i;cd}] = -i\eta_{[c|[a} \Theta_{0;b]}|d], \quad i=1,2, & [\Theta_{3;ab}, \Theta_{3;cd}] = -i\eta_{[c|[a} \Theta_{0;b]}|d] \\ [\Theta_{i;ab}, \Theta_{3;cd}] = -i\eta_{[c|[a} \Theta_{3-i;b]}|d], \quad i=1,2, & [\Theta_{2;ab}, \Theta_{1;cd}] = -i\eta_{[c|[a} \Theta_{3;b]}|d] \end{cases}$$

where the worldvolume argument σ is abbreviated.

- Affine Lorentz symmetry generator algebra of subgroup H_0 with $\tilde{S} + \tilde{\Sigma}$

$$\begin{aligned}
 \check{\Theta}_0 &= (\tilde{S}_{ab} + r_{\text{AdS}}^2 \tilde{\Sigma}_{ab}) - (\tilde{S}_{a'b'} + r_{\text{AdS}}^2 \tilde{\Sigma}_{a'b'}) \chi_a^{a'} \chi_b^{b'} \\
 \check{\Theta}_1 &= \tilde{S}_{[a|b'|\chi b]}^{b'} + r_{\text{AdS}}^2 \tilde{\Sigma}_{[a|b'|\chi b]}^{b'} \\
 \check{\Theta}_2 &= (\tilde{S}_{ab} + r_{\text{AdS}}^2 \tilde{\Sigma}_{ab}) + (\tilde{S}_{a'b'} + r_{\text{AdS}}^2 \tilde{\Sigma}_{a'b'}) \chi_a^{a'} \chi_b^{b'} \\
 \check{\Theta}_3 &= \tilde{S}_{(a|b'|\chi b)}^{b'} + r_{\text{AdS}}^2 \tilde{\Sigma}_{(a|b'|\chi b)}^{b'} \\
 \left\{ \begin{aligned}
 [\check{\Theta}_{0;ab}(1), \check{\Theta}_{\mu;cd}(2)] &= -2i\eta_{[c|[a} \check{\Theta}_{\mu;b]|d]} \delta(2-1) - 2ir_{\text{AdS}}^2 \delta_{\mu,0} \eta_{c[a} \eta_{b]d} \partial_\sigma \delta(2-1) \\
 [\check{\Theta}_{0;ab}(1), \check{\Theta}_{3;cd}(2)] &= -2i\eta_{(c|[a} \check{\Theta}_{3;b]|d)} \delta(2-1) \\
 [\check{\Theta}_{i;ab}(1), \check{\Theta}_{i;cd}(2)] &= -2i\eta_{[c|[a} \check{\Theta}_{0;b]|d]} \delta(2-1) - 2ir_{\text{AdS}}^2 \eta_{c[a} \eta_{b]d} \partial_\sigma \delta(2-1) \\
 [\check{\Theta}_{3;ab}(1), \check{\Theta}_{3;cd}(2)] &= -2i\eta_{(c|[a} \check{\Theta}_{0;b]|d)} \delta(2-1) - 2ir_{\text{AdS}}^2 \eta_{c(a} \eta_{b)d} \partial_\sigma \delta(2-1) \\
 [\check{\Theta}_{i;ab}(1), \check{\Theta}_{3;cd}(2)] &= -2i\eta_{(c|[a} \check{\Theta}_{3-i;b]|d)} \delta(2-1) \\
 [\check{\Theta}_{2;ab}(1), \check{\Theta}_{1;cd}(2)] &= -2i\eta_{[c|[a} \check{\Theta}_{3;b]|d]} \delta(2-1)
 \end{aligned} \right.
 \end{aligned} \tag{5.33}$$

with $\mu=0,1,2$ and $i=1,2$.

- Affine Lorentz symmetry generator algebra of subgroup H_1 with $\tilde{S} - \tilde{\Sigma}$

$$\begin{aligned}
 \check{\check{\Theta}}_0 &= (\tilde{S}_{ab} - r_{\text{AdS}}^2 \tilde{\Sigma}_{ab}) - (\tilde{S}_{a'b'} - r_{\text{AdS}}^2 \tilde{\Sigma}_{a'b'}) \chi_a^{a'} \chi_b^{b'} \\
 \check{\check{\Theta}}_1 &= \tilde{S}_{[a|b'|\chi b]}^{b'} - r_{\text{AdS}}^2 \tilde{\Sigma}_{[a|b'|\chi b]}^{b'} \\
 \check{\check{\Theta}}_2 &= (\tilde{S}_{ab} - r_{\text{AdS}}^2 \tilde{\Sigma}_{ab}) + (\tilde{S}_{a'b'} - r_{\text{AdS}}^2 \tilde{\Sigma}_{a'b'}) \chi_a^{a'} \chi_b^{b'} \\
 \check{\check{\Theta}}_3 &= \tilde{S}_{(a|b'|\chi b)}^{b'} - r_{\text{AdS}}^2 \tilde{\Sigma}_{(a|b'|\chi b)}^{b'} \\
 \left\{ \begin{aligned}
 [\check{\check{\Theta}}_{0;ab}(1), \check{\check{\Theta}}_{\mu;cd}(2)] &= -2i\eta_{[c|[a} \check{\check{\Theta}}_{\mu;b]|d]} \delta(2-1) + 2ir_{\text{AdS}}^2 \delta_{\mu,0} \eta_{c[a} \eta_{b]d} \partial_\sigma \delta(2-1) \\
 [\check{\check{\Theta}}_{0;ab}(1), \check{\check{\Theta}}_{3;cd}(2)] &= -2i\eta_{(c|[a} \check{\check{\Theta}}_{3;b]|d)} \delta(2-1) \\
 [\check{\check{\Theta}}_{i;ab}(1), \check{\check{\Theta}}_{i;cd}(2)] &= -2i\eta_{[c|[a} \check{\check{\Theta}}_{0;b]|d]} \delta(2-1) + 2ir_{\text{AdS}}^2 \eta_{c[a} \eta_{b]d} \partial_\sigma \delta(2-1) \\
 [\check{\check{\Theta}}_{3;ab}(1), \check{\check{\Theta}}_{3;cd}(2)] &= -2i\eta_{(c|[a} \check{\check{\Theta}}_{0;b]|d)} \delta(2-1) + 2ir_{\text{AdS}}^2 \eta_{c(a} \eta_{b)d} \partial_\sigma \delta(2-1) \\
 [\check{\check{\Theta}}_{i;ab}(1), \check{\check{\Theta}}_{3;cd}(2)] &= -2i\eta_{(c|[a} \check{\check{\Theta}}_{3-i;b]|d)} \delta(2-1) \\
 [\check{\check{\Theta}}_{2;ab}(1), \check{\check{\Theta}}_{1;cd}(2)] &= -2i\eta_{[c|[a} \check{\check{\Theta}}_{3;b]|d]} \delta(2-1)
 \end{aligned} \right.
 \end{aligned} \tag{5.34}$$

The linear combinations of the doubled momenta

$$\phi_{\pm;a} = \tilde{P}_a \pm \tilde{P}_{a'} \chi_a^{a'} \tag{5.35}$$

satisfy the following algebras with $\check{\Theta}_\mu$ and $\check{\check{\Theta}}_\mu$

$$\begin{aligned}
 [\phi_{+;a}(1), \phi_{+;b}(2)] &= \frac{i}{r_{\text{AdS}}^2} (\check{\Theta}_{2;ab} + \check{\Theta}_{1;ab}) \delta(2-1) \\
 [\phi_{-;a}(1), \phi_{-;b}(2)] &= \frac{i}{r_{\text{AdS}}^2} (\check{\Theta}_{2;ab} - \check{\Theta}_{1;ab}) \delta(2-1) \\
 [\phi_{+;a}(1), \phi_{-;b}(2)] &= \frac{i}{r_{\text{AdS}}^2} (\check{\Theta}_{0;ab} - \check{\Theta}_{3;ab}) \delta(2-1) + 2i\eta_{ab} \partial_\sigma \delta(2-1) \\
 [\check{\Theta}_{0;ab}(1), \phi_{\pm;c}(2)] &= 2i\phi_{\pm;[a}\eta_{b]c} \delta(2-1) \\
 [\check{\Theta}_{i;ab}(1), \phi_{\pm;c}(2)] &= 2i\phi_{\mp;[a}\eta_{b]c} \delta(2-1), \quad i=1,2 \\
 [\check{\Theta}_{3;ab}(1), \phi_{\pm;c}(2)] &= 2i\phi_{\pm;[a}\eta_{b]c} \delta(2-1) \\
 [\check{\check{\Theta}}_{\mu;ab}(1), \phi_{\pm;c}(2)] &= [\check{\check{\Theta}}_{\mu;ab}(1), \check{\check{\Theta}}_{\nu;cd}(2)] = 0
 \end{aligned} \tag{5.36}$$

We choose a set of first class constraints to reduce unphysical dimensions as

$$\begin{aligned}
 \phi_{-;a} &= \tilde{P}_a - \tilde{P}_{a'} \chi_a^{a'} = 0 \\
 \psi_{ab} &= (\tilde{S}_{ab} + r_{\text{AdS}}^2 \tilde{\Sigma}_{ab}) + (\tilde{S}_{a'b'} + r_{\text{AdS}}^2 \tilde{\Sigma}_{a'b'}) \chi_a^{a'} \chi_b^{b'} - \tilde{S}_{[a|b'} \chi_{|b]}^{b'} - r_{\text{AdS}}^2 \tilde{\Sigma}_{[a|b'} \chi_{|b]}^{b'} \\
 &= \check{\Theta}_{2;ab} - \check{\Theta}_{1;ab} = 0 \\
 \varphi_{ab} &= (\tilde{S}_{ab} - r_{\text{AdS}}^2 \tilde{\Sigma}_{ab}) - (\tilde{S}_{a'b'} - r_{\text{AdS}}^2 \tilde{\Sigma}_{a'b'}) \chi_a^{a'} \chi_b^{b'} + \tilde{S}_{[a|b'} \chi_{|b]}^{b'} - r_{\text{AdS}}^2 \tilde{\Sigma}_{[a|b'} \chi_{|b]}^{b'} \\
 &= \check{\Theta}_{0;ab} + \check{\Theta}_{1;ab} = 0
 \end{aligned} \tag{5.37}$$

which satisfy the following algebra

$$\begin{aligned}
 [\phi_{-;a}(1), \phi_{-;b}(2)] &= i\psi_{ab} \delta(2-1) \\
 [\varphi_{ab}(1), \varphi_{cd}(2)] &= 4i\eta_{[d|[a} \varphi_{b][c]} \delta(2-1) \\
 \text{others} &= 0.
 \end{aligned} \tag{5.38}$$

The first class constraint $\phi_{-;a} = 0$ reduces the half of the degrees of freedom of doubled momenta. We also impose the local Lorentz constraints $S_{\underline{ab}} = 0$. The first class constraints $\psi_{ab} = \varphi_{ab} = 0$ can be imposed without conflicting with the local Lorentz constraints by the same reason.

5.4.2 Physical AdS algebra

The physical global AdS algebra is constructed as follows. We identify the total momentum and the total Lorentz generator as

$$\begin{aligned}
 \tilde{P}_{\text{total};a} &= \frac{1}{2} (\tilde{P}_a + \tilde{P}_{a'} \chi_a^{a'}) + \frac{1}{2} \phi_{-;a} = \frac{1}{2} (\phi_{+;a} + \phi_{-;a}) = \tilde{P}_a \\
 \tilde{S}_{\text{total};ab} &= \frac{1}{2} (\tilde{S}_{ab} - \tilde{S}_{a'b'} \chi_a^{a'} \chi_b^{b'} + \tilde{S}_{[a|b'} \chi_{|b]}^{b'}) + \frac{1}{4} (\psi_{ab} - \varphi_{ab}) \\
 &= \frac{1}{2} (\Theta_{0;ab} + \Theta_{1;ab}) + \frac{1}{4} (\psi_{ab} - \varphi_{ab}) = \frac{1}{2} (\tilde{S}_{ab} + r_{\text{AdS}}^2 \tilde{\Sigma}_{ab}).
 \end{aligned} \tag{5.39}$$

The total momentum and the total Lorentz symmetry generators in the flat space are the same as (5.39) with first class constraints, $\phi_{-;a} = \psi_{ab} = \varphi_{ab} = 0$ and $\tilde{S}_{ab'} = 0$. The

physical global AdS algebra is generated by the zero mode of the total momenta and the total Lorentz generator

$$\begin{aligned} \mathcal{P}_{\text{total};a} &= \int d\sigma \tilde{P}_{\text{total};a}(\sigma), \quad \mathcal{S}_{\text{total};ab} = \int d\sigma \tilde{S}_{\text{total};ab}(\sigma) \\ \left\{ \begin{aligned} [\mathcal{S}_{\text{total};ab}, \mathcal{S}_{\text{total};cd}] &= i\eta_{[d|[a}\mathcal{S}_{\text{total};b]|c]} \\ [\mathcal{S}_{\text{total};ab}, \mathcal{P}_{\text{total};c}] &= i\mathcal{P}_{\text{total};[a}\eta_{b]|c]} \\ [\mathcal{P}_{\text{total};a}, \mathcal{P}_{\text{total};b}] &= i\frac{2}{r_{\text{AdS}}^2}\mathcal{S}_{\text{total};ab} \end{aligned} \right. \quad (5.40) \end{aligned}$$

The doubled AdS momenta is not a simple sum of the left and the right momenta, because of the left moving AdS momentum and the right moving dS momentum. Although the physical global AdS spacetime generators coincide with the left moving symmetry generators, they are written in terms of the doubled coordinates so the T-duality symmetry is manifest.

The total dS algebra is obtained vice versa as follows: the constraint $\varphi_{ab} = 0$ in (5.37) is instead

$$\begin{aligned} \varphi_{-;ab} &= \check{\Theta}_{0;ab} - \check{\Theta}_{1;ab} = 0 \\ [\varphi_{-;ab}(1), \varphi_{-;cd}(2)] &= 4i\eta_{[d|[a}\varphi_{-;b]|c]}\delta(2-1) \quad (5.41) \end{aligned}$$

The total dS momentum and Lorentz generators are

$$\begin{aligned} \tilde{P}_{\text{dS};a'}\chi_a^{a'} &= \frac{1}{2}(\tilde{P}_a + \tilde{P}_{a'}\chi_a^{a'}) - \frac{1}{2}\phi_{-;a} = \frac{1}{2}(\phi_{+;a} - \phi_{-;a}) = \tilde{P}_{a'}\chi_a^{a'} \quad (5.42) \\ \tilde{S}_{\text{dS};a'b'}\chi_a^{a'}\chi_b^{b'} &= \frac{1}{2}(\Theta_{0;ab} + \Theta_{1;ab}) - \frac{1}{4}(\psi_{ab} - \varphi_{-;ab}) = -\frac{1}{2}(\tilde{S}_{a'b'} + r_{\text{AdS}}^2\tilde{\Sigma}_{a'b'})\chi_a^{a'}\chi_b^{b'}. \end{aligned}$$

The global dS algebra is generated by

$$\begin{aligned} \mathcal{P}_{\text{dS};a'} &= \int d\sigma \tilde{P}_{\text{dS};a'}(\sigma), \quad \mathcal{S}_{\text{dS};a'b'} = \int d\sigma \tilde{S}_{\text{dS};a'b'}(\sigma) \\ \left\{ \begin{aligned} [\mathcal{S}_{\text{dS};a'b'}, \mathcal{S}_{\text{dS};c'd'}] &= i\eta_{[d'|[a'}\mathcal{S}_{\text{dS};b']|c']} \\ [\mathcal{S}_{\text{dS};a'b'}, \mathcal{P}_{\text{dS};c'}] &= i\mathcal{P}_{\text{dS};[a'}\eta_{b']|c']} \\ [\mathcal{P}_{\text{dS};a'}, \mathcal{P}_{\text{dS};b'}] &= -i\frac{2}{r_{\text{AdS}}^2}\mathcal{S}_{\text{dS};a'b'} \end{aligned} \right. \quad (5.43) \end{aligned}$$

Unphysical coordinates for doubled dimensions, Lorentz and its nondegenerate partner can be gauged away by using local symmetries generated by the first class constraints. These first class constraints commute with the covariant derivatives, so our dimensional reduction procedure preserves the T-duality gauge symmetry manifestly.

5.4.3 Comparison with the non-doubled AdS algebra

We also mention the relation between the AdS algebra in this paper and our previous AdS algebra in [8, 28, 29]. In the previous paper the $\text{AdS}_5 \times \text{S}^5$ space is described by the $\text{PSU}(2,2|4)$ coordinates. A half of doubled coordinates are gauged away, and only

coordinates for the physical total momentum and the physical total Lorentz symmetry are used. Gauge fixing conditions and corresponding first class constraints are given:

$$\begin{aligned}
 &\text{Gauge fixing conditions} \quad x^{m'} - x^m = v^{mn} = v^{m'n'} = u^{mn'} = u^{m'n} + u^{mn} = 0 \\
 &\text{First class constraints} \quad \phi_{-;a} = \psi_{ab} = \varphi_{ab} = S_{ab} = S_{a'b'} = S_{ab'} = 0 \\
 &\text{Second class constraints} \quad v^{mn'} = \tilde{\Sigma}^{ab'} = 0
 \end{aligned} \tag{5.44}$$

After the gauge fixing (5.44) the covariant derivatives become as in [8]

$$\begin{cases} P_a = \frac{1}{2}(\overset{\circ}{\nabla}_P + J^P) \\ P_{a'} = \frac{1}{2}(\overset{\circ}{\nabla}_P - J^P) \end{cases} \Rightarrow \begin{cases} [P_a, P_b] = \overset{\circ}{\nabla}_S + J^S + \partial_\sigma \delta = \overset{\circ}{\nabla}_\Sigma + \partial_\sigma \delta \\ [P_a, P_{b'}] = \overset{\circ}{\nabla}_S = \overset{\circ}{\nabla}_{S;ab'} \\ [P_{a'}, P_{b'}] = \overset{\circ}{\nabla}_S - J^S - \partial_\sigma \delta = \overset{\circ}{\nabla}_{\Sigma'} - \partial_\sigma \delta \end{cases} \tag{5.45}$$

In the right hand sides of the first and third lines of the algebras the particle component of the Lorentz covariant derivatives, $\overset{\circ}{\nabla}_S$, are identified with $\overset{\circ}{\nabla}_\Sigma$ and $\overset{\circ}{\nabla}_{\Sigma'}$, rather than $\overset{\circ}{\nabla}_S$ and $\overset{\circ}{\nabla}_{S'}$. It is because S and S' satisfy the opposite sign structure constant in the doubled AdS algebra (5.7), so it cannot be equal consistently. This is the same reason that the naive sum of momenta $\tilde{P} + \tilde{P}'$ does not satisfy the AdS algebra globally in (5.40). Lorentz generators are coset constraints $\overset{\circ}{\nabla}_S = \overset{\circ}{\nabla}_{S'} = 0$, so they are included in $\overset{\circ}{\nabla}_{\Sigma'}$'s.

In the gauge (5.44) the covariant derivatives and the symmetry generators for the left and right moving modes in the flat case as become

$$\begin{aligned}
 &\text{Covariant derivatives : } S = -S' = \Xi_u \frac{1}{i} \partial_u \\
 &P = e^u \frac{1}{i} \partial_x + \frac{1}{2} e^{-u} \partial_\sigma x, \quad P' = e^u \frac{1}{i} \partial_x - \frac{1}{2} e^{-u} \partial_\sigma x \\
 &\Sigma = \Sigma' = e^{-u} \partial_\sigma u = \Xi_u^{-1} \partial_\sigma u \\
 &\text{Symmetry generators : } \tilde{S} = -\tilde{S}' = \Xi_{-u} \frac{1}{i} \partial_u + [x, \frac{1}{i} \partial_x] \\
 &\tilde{P} = P' e^{-u}, \quad \tilde{P}' = P e^{-u} \\
 &\tilde{\Sigma} = \tilde{\Sigma}' = 0
 \end{aligned} \tag{5.46}$$

Indices of generators and coordinates are abbreviated; the order of contraction of the indices are also omitted as, $e^u \frac{1}{i} \partial_x = (\partial_x)^n u_{nm}$. The left-invariant one form gives $e^{-iu \cdot s} d e^{iu \cdot s} = \Xi_u^{-1} i du \cdot s$ with $u \cdot s = \frac{1}{2} u^{mn} s_{mn}$. The left and right modes of the Lorentz and Σ generators are not independent respectively. The left and right modes of the momentum symmetry generators are not independent from the right and left modes of the momentum covariant derivatives. In this gauge it is easy to see that the commutator of the left and right AdS momenta gives the Lorentz generator which is nonzero [8, 28, 29]. The supergroup PSU(2,2|4) as the AdS₅ × S⁵ group is a gauge fixed version of the fully manifestly T-duality formulation. Both the left and right AdS₅ × S⁵ groups do not exist; only one kind of the momentum, Lorentz and no nondegenerate Lorentz partner exist. Although the covariant derivative of SO(5,5) × SO(5,5) exists as in (5.46), it is not manifestly doubled AdS covariant. Furthermore the gauge invariant superstring action in the AdS space with manifestly T-duality requires the formulation without the gauge fixing.

6 Summary

We have proposed a manifestly T-dual formulation of group manifolds. Especially the bosonic part of the AdS space with manifestly T-duality caused by the RR flux is examined in detail. It is a doubled AdS space which is defined by the affine nondegenerate doubled AdS algebra given by (5.18)–(5.23). Contrast to the flat case the left and right momenta do not commute by the existence of the AdS curvature term. This mixing leads to that the left and right momenta are in AdS and dS spaces respectively. We have found a set of first class constraints (5.37) to reduce unphysical dimensions of the doubled AdS space. This allows the manifest T-dual formulation without any gauge fixing such as dual coordinate independence, $\partial^m = 0$, on doubled space functions. Then the physical AdS algebra is realized preserving whole doubled space coordinates to manifest T-duality.

The B field is extended in the nondegenerate doubled space, and it gives the Wess-Zumino term for a string with the manifest T-duality. We have also shown that the doubled space three form $H = dB$ is at least locally universal for arbitrary group manifolds with the same dimension.

The obtained doubled AdS space will be useful to clarify the structure of integrability and the dual conformal symmetry which play important roles in AdS/CFT correspondence. The supersymmetric extension and analysis of the fermionic T-duality are important issues which are the future problems. The doubled group manifold and AdS spaces will have many applications.

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