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On the $gl(1|1)$ Wess-Zumino-Witten model

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ABSTRACT: We continue the study of the $gl(1|1)$ Wess-Zumino-Witten model. The Knizhnik-Zamolodchikov equations for the one, two, three and four point functions are analyzed, for vertex operators corresponding to typical and projective representations. We demonstrate their interplay with the logarithmic global conformal Ward identities. We compute the four point function for one projective and three typical representations. Three coupled first order Knizhnik-Zamolodchikov equations are integrated consecutively in terms of generalized hypergeometric functions, and we assemble the solutions into a local correlator. Moreover, we prove crossing symmetry of the four point function of four typical representations at generic momenta. Throughout, the map between the $gl(1|1)$ Wess-Zumino-Witten model and symplectic fermions is exploited and extended.

KEYWORDS: Conformal Field Models in String Theory, Conformal Field Theory, AdS-CFT Correspondence

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1 Introduction

We study a logarithmic conformal field theory with a supergroup symmetry, a supergroup Wess-Zumino-Witten model. The simple example we focus on is the $gl(1|1)$ Wess-Zumino-Witten model. It has been studied in [1, 2] where the four point functions of typical representations were established and a check on locality was performed. The supergeometry and the spectrum were clarified in [3], while the relation to symplectic fermions [4, 5] has been made more manifest in [6]. Many other remarkable properties of the model have been obtained (for instance in [7, 8]).

In this paper, we revisit the solution of the model, and take an algebraic perspective on the calculation of the correlation functions. We exploit the symmetries of the theory maximally, combine techniques from the logarithmic conformal field theory literature for solving the global conformal Ward identities with those available for solving Wess-Zumino-Witten models, and analyze the interplay between them.

The left regular representation of the $GL(1|1)$ group acting on itself gives rise to typical representations as well as four-dimensional projective representations of the $gl(1|1)$ alge-

bra.¹ That suggests that these are the current algebra primaries arising in the spectrum of the Wess-Zumino-Witten model [3]. Thus, we take an approach in which we manifestly work with both typical and projective representations.

The state of the art in the calculation of correlation functions of the model uses representations of the vertex operators in terms of free fields. In [3] a ghost system is used. In [6] a map is introduced between the $gl(1|1)$ Wess-Zumino-Witten degrees of freedom and two free bosons coupled to symplectic fermions [5] via an orbifold. The latter free field representation has the advantage of containing the fermionic zero modes that are an important property of the model on the supergroup.² Thus, we will revisit the symplectic fermion approach to the free field representation of the correlators, and we will observe a close analogy between the differential constraints exploited by [4, 5] to solve for the correlation functions of symplectic fermions, and the Knizhnik-Zamolodchikov equations [10]. This correspondence may be expected from the map between the models detailed in [6] and from the relation of the two models through gauging the bosonic directions in the supergroup [11].

An underlying goal of our analysis is to render the space-time supergroup symmetry of the model more manifest. We may hope to export the knowledge we gain into more complicated Wess-Zumino-Witten supergroup models, relevant to string theory on $AdS_3 \times S^3$, such as the $PSU(1,1|2)$ model [12–18]. Indeed, it should be noted that part of the motivation of the formalism that gives rise to supergroup Wess-Zumino-Witten models in string theory is to render the space-time symmetry manifest [13], and in this paper we carry this motivation over into the analysis of the world sheet model.

In our manifestly covariant framework, we will find occasion to derive new technical results. These include results on the number of invariants in various tensor product representations, and their explicit identification, the check of crossing symmetry of the four point functions of the typical representations at all values of the external momenta, and four point correlators for a projective top state and three typical insertions, in terms of generalized hypergeometric functions. The latter are reminiscent of results on other field theories with supergroup symmetries [19]. The results we derive also contain within them new results on symplectic fermion correlators.

Section 2 of our paper discusses symmetries and resulting Ward identities in a more general context. In section 3 we introduce the $gl(1|1)$ Wess-Zumino-Witten model on which we concentrate. The logarithmic conformal Ward identities are solved in section 4, and in section 5 we start studying the consequences of the Knizhnik-Zamolodchikov equations in more detail. Finally, in section 6 we integrate the four point functions of four typical, or three typical and one projective representation. In section 7, we combine the left- and right-moving correlators into local four point functions. We offer concluding remarks in section 8. Representation theoretic results on the $gl(1|1)$ algebra are gathered in appendix A, while appendix B reviews useful properties of the symplectic fermion conformal field theory.

¹The typical representations are also projective, strictly speaking. We reserve the word projective for the four-dimensional representations, for ease of expression.

²See e.g. [9] for a detailed discussion of fermionic zero modes in the bc ghost system, in contrast to the symplectic fermion model.

2 Symmetries and identities

The supergroup $gl(1|1)$ Wess-Zumino-Witten model is a logarithmic conformal field theory [1, 2]. Therefore, it satisfies conformal Ward identities that take into account the logarithmic nature of the spectrum [20]. Moreover, the energy-momentum tensor of the theory can be constructed using a generalization of the Sugawara construction [1]. As a consequence, the correlators also satisfy generalized Knizhnik-Zamolodchikov equations [10]. In this section, we review these constraints on correlators arising from the symmetries of the theory in a general context. We add remarks on the algebraic interplay between the logarithmic aspects of the model, and the global supergroup symmetry.

2.1 Logarithmic conformal field theory

When we bring the chiral scaling operator L_0 in our $gl(1|1)$ model into standard Jordan form, it contains Jordan cells of rank two. Thus we have a logarithmic conformal field theory [1]. We will have conformal group transformation laws for the logarithmic conformal primary fields ϕ_h that read [20]

$$\phi_h(z) = (\partial f)^h (1 + \log(\partial f) \hat{\delta}_h) \phi_h(f(z)). \quad (2.1)$$

The operator $\hat{\delta}_h$ codes the relation between primaries in a given Jordan cell. For example, when we have two logarithmic primary operators Ψ_h and Φ_h in a non-trivial Jordan cell of generalized conformal dimension h , then they can be chosen such that the operator $\hat{\delta}_h$ acts as

$$\hat{\delta}_h \Psi_h = \Phi_h, \quad \hat{\delta}_h \Phi_h = 0. \quad (2.2)$$

The logarithmic action of the Virasoro generators that follows from the transformation law (2.1) is

$$L_n \langle \phi_1(z_1) \dots \phi_n(z_n) \rangle = \sum_i (z_i^{n+1} \partial_i + z_i^n (n+1)(h_i + \hat{\delta}_{h_i})) \langle \phi_1(z_1) \dots \phi_n(z_n) \rangle. \quad (2.3)$$

On the sphere, we have three globally defined chiral Ward identities, namely, for $n = -1, 0, 1$ the action (2.3) of the generators L_n equals zero.

Before we move on, let us remark that results on correlation functions in logarithmic conformal field theory are often valid for correlators that involve only fields that are logarithmic themselves, or that cannot generate a logarithmic field through operator multiplication. Our correlators will fall outside the domain of validity of these general results, since we will have correlators of pre-logarithmic fields [20]. These pre-logarithmic fields are not logarithmic themselves, but do generate logarithmic fields through their operator product.³

We have briefly reviewed the consequences of chiral global conformal symmetry in logarithmic models, and turn to the implications of the affine current algebra symmetry.

³Fields in typical $\widehat{gl(1|1)}$ primary current algebra representations will occasionally multiply into fields in projective representations and while the first class is of Jordan rank one, the second is of Jordan rank two.

2.2 The Knizhnik-Zamolodchikov equation

We will examine a theory with a generalized Sugawara energy-momentum tensor [1]. The holomorphic energy-momentum tensor component $T(z)$ is quadratic in the holomorphic currents J^a that capture the affine symmetry of the Wess-Zumino-Witten model,

$$T = \frac{1}{2} Q_{ab} J^a J^b, \quad (2.4)$$

where Q_{ab} is a constant set of coefficients to be determined in due course. We define current algebra primaries ϕ through the equations

$$J^a(z)\phi(w) \approx \frac{t^a}{z-w}\phi(w) + \text{regular} \quad (2.5)$$

where t^a are representation matrices associated to the representation of the global symmetry group under which the current algebra primary ϕ transforms. Given these algebraic structures, we can derive the Knizhnik-Zamolodchikov equation for the model as a consequence of the identity between the first mode of the energy momentum tensor L_{-1} and the bi-linear combination of the first and zeroth mode of the current algebra [10] (when acting on current algebra primaries). The Knizhnik-Zamolodchikov equation on the current algebra primary correlators on the sphere reads

$$\left(\partial_i - \sum_{j \neq i} \frac{Q_{ab} t_i^a \otimes t_j^b}{z_i - z_j} \right) \langle \phi_1(z_1) \dots \phi_n(z_n) \rangle = 0, \quad (2.6)$$

where ∂_i denotes the derivative with respect to the holomorphic coordinate z_i of the i -th insertion, and the summation is over all other insertions, at the points z_j .

2.3 Common consequences

Often, global conformal Ward identities and the Knizhnik-Zamolodchikov equation are exploited conjointly. The conformal symmetry can be used to reduce the dependence of the four point functions to a dependence on one variable, while the Knizhnik-Zamolodchikov equations can then be exploited to integrate that dependence. In this subsection, we pause to observe the interplay between logarithmic conformal Ward identities and the Knizhnik-Zamolodchikov equation. Firstly, we note that the Knizhnik-Zamolodchikov equation implies the global conformal Ward identity for translations. This is seen by summing over the choice of insertion i in the equation (2.6). More interestingly, we can multiply the Knizhnik-Zamolodchikov equation (2.6) by z_i and sum over the insertion points i , to obtain the equation

$$\left(\sum_i z_i \partial_i - \sum_{i < j} Q_{ab} t_i^a \otimes t_j^b \right) \langle \phi_1(z_1) \dots \phi_n(z_n) \rangle = 0. \quad (2.7)$$

The scaling Ward identity on the other hand reads

$$\left(\sum_i z_i \partial_i + \frac{1}{2} Q_{ab} t_i^a t_i^b \right) \langle \phi_1(z_1) \dots \phi_n(z_n) \rangle = 0. \quad (2.8)$$

We have used that L_0 acts as the zero mode of the energy-momentum tensor on the current algebra primaries. The action of this operator captures both the conformal dimension h_i and the logarithmic term $\hat{\delta}_{h_i}$. Combining these two equations gives rise to the identity

$$\left(\frac{1}{2} Q_{ab} \sum_i t_i^a \otimes \sum_j t_j^b \right) \langle \phi_1(z_1) \dots \phi_n(z_n) \rangle = 0, \quad (2.9)$$

which is also implied by the global symmetry of the model. Indeed, the quadratic Casimir⁴ $C_2^\otimes$ in the tensor product representation of a correlator equals zero on that correlator

$$C_2^\otimes \langle \phi_1(z_1) \dots \phi_n(z_n) \rangle = 0. \quad (2.10)$$

For the supergroup conformal field theory this simple equation has interesting consequences.

3 The $gl(1|1)$ Wess-Zumino-Witten model

In this section, we focus on the $gl(1|1)$ Wess-Zumino-Witten model. It is a conformal non-linear sigma-model on a supergroup with Lie algebra $gl(1|1)$. To specify the structure of the model, we review the algebra, the current algebra and the energy-momentum tensor. We discuss their representations and a free field realization.

3.1 The $gl(1|1)$ algebra

The $gl(1|1)$ Lie algebra has four generators, of which two are bosonic, and two fermionic.⁵ We denote them E, N and $\Psi^\pm$ respectively. The non-trivial commutation relations are

$$\begin{aligned} \{\Psi^+, \Psi^-\} &= E, \\ [N, \Psi^\pm] &= \pm \Psi^\pm. \end{aligned} \quad (3.1)$$

The supertrace in the product of the defining representation gives a non-degenerate bilinear form κ that satisfies

$$\begin{aligned} \kappa(N, E) &= \kappa(E, N) = 1, \\ \kappa(\Psi^+, \Psi^-) &= -\kappa(\Psi^-, \Psi^+) = 1. \end{aligned} \quad (3.2)$$

A quadratic Casimir is $C_2(\alpha) = EN + NE + \Psi^-\Psi^+ - \Psi^+\Psi^- + \alpha E^2$ where α is any constant since the generator E is central.

3.2 The current algebra

The Wess-Zumino-Witten model has a holomorphic current algebra $\widehat{gl(1|1)}$ generated by the conserved current components $J^a(z)$. For each generator in the algebra, we obtain a

⁴Namely, the particular quadratic Casimir fixed by the energy-momentum tensor.

⁵See also appendix A for more details.

current J^a with Laurent coefficients J_n^a , and we find that they satisfy the affine $\widehat{gl(1|1)}$ current algebra.⁶

$$\begin{aligned} [N_r, E_s] &= r\delta_{r+s} \\ [N_r, \Psi_s^\pm] &= \pm \Psi_{r+s}^\pm \\ \{\Psi_r^+, \Psi_s^-\} &= E_{r+s} + r\delta_{r+s}. \end{aligned} \quad (3.3)$$

The corresponding current algebra operator product expansions are

$$\begin{aligned} N(z)E(w) &\approx \frac{1}{(z-w)^2} \\ N(z)\Psi^\pm(w) &\approx \pm \frac{\Psi^\pm(w)}{z-w} \\ \Psi^+(z)\Psi^-(w) &\approx \frac{1}{(z-w)^2} + \frac{E(w)}{z-w}. \end{aligned} \quad (3.4)$$

3.3 The energy-momentum tensor

A holomorphic energy-momentum tensor component that ensures that the currents are primaries of conformal dimension one is [1]

$$T(z) = \frac{1}{2} : (NE + EN + \Psi^-\Psi^+ - \Psi^+\Psi^- + EE) : . \quad (3.5)$$

This requirement also fixes the constant in front of the term proportional to E^2 . The central charge of the model is zero. The energy-momentum tensor is a bi-linear in the currents, and is of Sugawara type. We equate the above energy-momentum tensor to the formula (2.4), thus fixing the tensor Q_{ab} which features in the Knizhnik-Zamolodchikov equation (2.6).

3.4 The representation theory and the spectrum

The representation theory of the $gl(1|1)$ algebra is briefly reviewed in appendix A. We will mostly work with typical and projective cover representations. The typical representations $T_{e,n}$ are two-dimensional, and have a highest weight state $|\uparrow\rangle$ and a lowest weight state $|\downarrow\rangle$. They are characterized by a non-zero eigenvalue e for the generator E and eigenvalues $n \pm 1/2$ for the other Cartan generator N . The projective covers P_n are four-dimensional representations that arise at eigenvalue $e = 0$ for the generator E . There are four states, denoted t, l, r, b (for top, left, right and bottom) with eigenvalues $n, n+1, n-1, n$ respectively for the generator N . See appendix A for more details.

The mini-superspace analysis of [3] shows that the spectrum of quadratically integrable functions on the non-compact version of the supergroup decomposes with respect to the left regular action of the $GL(1|1)$ group on itself as

$$L^2(GL(1|1)) = \int_{e \neq 0} de \, dn \, T_{e,n}^L \otimes T_{-e,-n}^R \oplus \int dn \, P_n^L. \quad (3.6)$$

⁶The normalization of the non-zero level is arbitrary for this current algebra.

In the typical sector, the left and right actions factorize. In the projective ($e = 0$) sector, the left projective representations P_n^L combine into infinite dimensional representations J_q of the left-right group action. (See e.g. [3] for more explanation, or [16, 21] for illustrative diagrams.) In [3] it was further argued that the mini-superspace analysis, and properties of the correlators suggest that the full conformal field theory spectrum follows the same pattern, with the representations of the $gl(1|1)$ algebra replaced by primary current algebra representations. The state space H of the Wess-Zumino-Witten model was argued to be

$$H = \int_{e \neq 0 \bmod 1} de \, dn \hat{T}_{e,n}^L \otimes \hat{T}_{-e,-n}^R \oplus \sum_m \int dq \hat{J}_q^m, \quad (3.7)$$

where $\hat{T}_{e,n}$ are affine generalizations of the typical representations, and $\hat{J}_q^m$ are affine generalizations of the entangled left-right projective representations. These spaces are described in detail in [3]. (See also [22] for further analysis of the current algebra representation theory.)

From the mini-superspace analysis various crucial features are manifest [3]. Firstly, that the scaling operator L_0 , which on a given current algebra primary acts as the quadratic Casimir, is not diagonalizable. In particular, it sends top states in projective representations to bottom states. Secondly, the chiral spectrum consists of current algebra generalizations of typical and projective representations.

Thus, it is natural to study the model in terms of these current algebra typical and projective representations. The typical primaries have conformal weight equal to the quadratic Casimir determined by the energy-momentum tensor T . The weight of the $\hat{T}_{e,n}$ primary is

$$h_{e,n} = en + \frac{1}{2}e^2. \quad (3.8)$$

The conformal weight of projective primaries is zero. However, the action of the quadratic Casimir (and of the scaling operator L_0) maps the top projective current algebra primary onto the bottom projective current algebra primary in the same representation.

3.5 A free field representation

The $c = -2$ symplectic fermions are akin to a bc ghost system with two extra conjugate zero modes [9]. These extra zero modes are crucial to realize the Jordan block structure of the logarithmic conformal field theory. Thus, in choosing a free field representation of our theory, we opt for the bosons plus symplectic fermion construction put forward in [6]. We remind the reader of the main formulas.

3.5.1 The action and currents

The action for the free bosons Y, Z and the symplectic fermions $\chi^{1,2}$ on a Riemann surface Σ is

$$S = \frac{1}{4\pi} \int_{\Sigma} d^2z (-\partial Z \bar{\partial} Y - \partial Y \bar{\partial} Z + \epsilon_{ab} \partial \chi^a \bar{\partial} \chi^b), \quad (3.9)$$

where we have the convention that $\epsilon_{12} = 1$. The action leads to the operator product expansion on the plane

$$\begin{aligned}\chi^a(z, \bar{z})\chi^b(w, \bar{w}) &\approx -\epsilon^{ab} \log|z - w|^2 \\ Z(z, \bar{z})Y(w, \bar{w}) &\approx \log|z - w|^2,\end{aligned}\tag{3.10}$$

where $\epsilon^{12} = -1$. We thus also have the operator product expansions

$$\begin{aligned}\partial Y(z) : e^{eZ} : &\approx \frac{e}{z - w} : e^{eZ} : (w, \bar{w}) \\ \partial Z(z) : e^{nY} : &\approx \frac{n}{z - w} : e^{nY} : (w, \bar{w}).\end{aligned}\tag{3.11}$$

We represent the $\widehat{gl(1|1)}$ affine algebra through the currents

$$\begin{aligned}E(z) &= \partial Y^L \\ N(z) &= \partial Z^L \\ \Psi^+(z) &= -e^{Y^L} \partial \chi^2 \\ \Psi^-(z) &= e^{-Y^L} \partial \chi^1.\end{aligned}\tag{3.12}$$

We need twist fields μ_λ , introduced in [5] which twist the symplectic fermions χ^a by a phase determined by λ :

$$\begin{aligned}\chi^{1,L}(e^{2\pi i} z) \mu_\lambda(0) &= e^{-2\pi i \lambda} \chi^{1,L}(z) \mu_\lambda(0) \\ \chi^{1,R}(e^{-2\pi i} \bar{z}) \mu_\lambda(0) &= e^{-2\pi i \lambda} \chi^{1,R}(\bar{z}) \mu_\lambda(0) \\ \chi^{2,L}(e^{2\pi i} z) \mu_\lambda(0) &= e^{2\pi i \lambda} \chi^{2,L}(z) \mu_\lambda(0) \\ \chi^{2,R}(e^{-2\pi i} \bar{z}) \mu_\lambda(0) &= e^{2\pi i \lambda} \chi^{2,R}(\bar{z}) \mu_\lambda(0).\end{aligned}\tag{3.13}$$

We have introduced the chiral parts of the symplectic fermions

$$\chi^a(z, \bar{z}) = \chi^{a,L}(z) + \chi^{a,R}(\bar{z}).\tag{3.14}$$

The following operator product expansions were derived in [6, 8]:

$$\begin{aligned}\partial \chi^1(z) \mu_\lambda^L(0) &\approx \frac{1}{z^\lambda} \mu_{\lambda-1}^L(0) \\ \partial \chi^2 \mu_\lambda^L &\approx \frac{\lambda}{z^{1-\lambda}} \mu_{\lambda+1}^L \\ \bar{\partial} \chi^1 \mu_\lambda^R &\approx \frac{\lambda}{\bar{z}^{1-\lambda}} \mu_{\lambda+1}^R \\ \bar{\partial} \chi^2 \mu_\lambda^R &\approx -\frac{1}{\bar{z}^\lambda} \mu_{\lambda-1}^R,\end{aligned}\tag{3.15}$$

where we have also split the twist field μ into holomorphic and anti-holomorphic parts $\mu^{L,R}$.

3.5.2 Free field representations

We have reviewed the action, the currents and the twist fields. We now recall the construction of the current algebra primary vertex operators. Firstly, we exhibit the operators associated to the representations $T_{e,n}$ [6]. We concentrate on the left chirality only, and then find the two-dimensional representation⁷

$$T_{e,n}^L = \begin{pmatrix} e^{eZ^L + (n-\frac{1}{2})Y^L} \mu_{-e}^L & e^{eZ^L + (n+\frac{1}{2})Y^L} \mu_{-e+1}^L \end{pmatrix} = \begin{pmatrix} \downarrow & \uparrow \end{pmatrix}. \quad (3.16)$$

The operator product expansions of the currents (3.12) with the vertex operator (3.16) are of the form (2.5) with the typical representation matrices inserted. For the projective representation for the left action, we propose (see also [8])

$$P_n^L = \begin{pmatrix} e^{nY^L} : \chi^{1,L} \chi^{2,L} : -e^{(n-1)Y^L} \chi^{1,L} \\ e^{(n+1)Y^L} \chi^{2,L} & -e^{nY^L} \end{pmatrix} = \begin{pmatrix} \text{top} & \text{right} \\ \text{left} & \text{bottom} \end{pmatrix}. \quad (3.17)$$

To obtain the spectrum, we need to assemble these fields into mutually local left-right vertex operators. For the typical representation, it is sufficient to tensor

$$T_{e,n}^L \otimes T_{-e,-n}^R = \begin{pmatrix} e^{eZ + (n-\frac{1}{2})Y} \mu_{-e} & e^{eZ + (n+\frac{1}{2})Y^L + (n-\frac{1}{2})Y^R} \mu_{-e+1}^L \mu_{-e}^R \\ e^{eZ + (n-\frac{1}{2})Y^L + (n+\frac{1}{2})Y^R} \mu_{-e}^L \mu_{-e+1}^R & e^{eZ + (n+\frac{1}{2})Y} \mu_{-e+1} \end{pmatrix},$$

and these vertex operators are mutually local for all $e \neq 0$ and all n . The top component of the left projective representation (3.17) is the left part of the vertex operator $e^{nY} : \chi^1 \chi^2 :$ which is mutually local with the other vertex operators. It is also the top component of a right projective representation. In general, the mutually local primary projective vertex operators will decompose into the infinite dimensional indecomposable representations $\hat{J}_q^m$ that make up the spectrum at $e = 0$.

We use the free field representation of the primary current algebra vertex operators intermittently throughout the rest of the paper.

4 Logarithmic aspects of the $gl(1|1)$ WZW model

In this section, we discuss consequences of the logarithmic nature of the Wess-Zumino-Witten model on $gl(1|1)$. We combine the information provided by the representation theoretic content of the model reviewed in section 3, and the global logarithmic conformal Ward identities recalled in section 2. We treat correlators in increasing order of the number of insertions.

4.1 One point functions

The one point functions of conformal primaries are constants by translational invariance. In non-logarithmic conformal field theories, the global conformal Ward identities imply that all one point functions are zero, except for operators with conformal dimension zero.

⁷Truly, these are infinite dimensional affine representations $\hat{T}_{e,n}$. We hope the reader will allow us to leave a multitude of hats on the hat rack.

In unitary theories, this implies that only the identity operator has a non-zero one point function in a $SL(2, \mathbb{C})$ invariant vacuum. In the logarithmic conformal field theory at hand, the standard wisdom is modified. The identity operator sits inside a Jordan block of rank two. From the conformal scaling Ward identity for the logarithmic partners $\Phi_0 = 1$ and Ψ_0 in the zero conformal dimension Jordan block it follows that

$$\hat{\delta}_0 \langle \Psi_0 \rangle = \langle 1 \rangle = 0, \quad (4.1)$$

where 1 indicates the unit operator corresponding to the $SL(2, \mathbb{C})$ invariant vacuum. The fact that the unit operator has zero expectation value is also a consequence of the existence of two fermionic zero modes in the $gl(1|1)$ path integral. Thirdly, it follows because the unit operator lies at the bottom of a projective representation, and therefore is also the result of the quadratic Casimir acting on the corresponding top component. Applying the fact that the quadratic Casimir acting on a correlation function is zero, we again derive that the unit operator has vanishing correlation function. These derivations are all closely related.

The logarithmic partner Ψ_0 of the identity operator, on the other hand, can have a non-zero expectation value. The corresponding state contains two fermionic zero modes conjugate to the global fermionic symmetry generators. These fermionic modes render the resulting correlation functions non-zero. From the fact that the identity operator has zero N and E eigenvalues, and the structure of projective representations under the action of the fermionic generators of $gl(1|1)$, we conclude that it is the top state in the projective representation P_0 that has a non-zero one point function. Indeed, it is the logarithmic partner Ψ_0 of the identity operator 1. Note though that only the identity operator is a true group invariant inside P_0 . By invariance under the bosonic subalgebra of $gl(1|1)$, and zero mode counting, $\langle \Psi_0 \rangle$ is also the only non-zero one point function. We can alternatively summarize these statements in the equations

$$\begin{aligned} \langle T_{e,n} \rangle &= 0, \\ \langle (P_n)_i \rangle &= \delta_n \delta_{i,top}, \end{aligned} \quad (4.2)$$

which summarize that the operators in typical representations have zero expectation values, while the top component in the momentum zero projective representation only can have a non-zero one point function.

These statements are in accord with the free field representations of the current algebra primaries. In detail, in the symplectic fermion language, the operator $:\chi^1 \chi^2:$ will correspond to the operator $\omega =: \chi^1 \chi^2:$ of [5], and is the only operator with a non-zero vacuum expectation value.

4.2 Two point functions

Typical representation primaries are ordinary (i.e. non-logarithmic) Virasoro primaries. To have a non-zero two point function amongst two typicals, we need the tensor product representation to land in the identity Jordan block, i.e. to combine into a projective cover P_0 . The top component of the identity block then leads to a non-zero two point function. Thus, typical representation primary vertex operators can have a non-zero two

point function when they have total momentum equal to zero, and indeed form a projective representation. Typical vertex operators are thus pre-logarithmic. The tensor product of a typical and a projective representation is typical, and therefore there can be no non-zero two point function between a typical and a projective representation. This also follows from bosonic group invariance. A projective representation (with regard to the left action) contains three ordinary primaries, and one logarithmic partner. The two point function of the logarithmic partners with themselves is logarithmic, while other combinations that saturate the fermionic zero modes will have two point functions of standard form. We provide further details in section 5.

4.3 Three point functions

In this section we concentrate on determining the consequences of the logarithmic Ward identities for the correlation functions of one projective and two typical current algebra primaries.⁸ See [20, 23, 24] for the original analysis. Let's suppose we concentrate on a projective representation primary Ψ_{h_1} that is mapped to Φ_{h_1} by the scaling operator L_0 .⁹ We thus want to study the global conformal Ward identity in the case in which we have a first insertion with a logarithmic partner

$$\hat{\delta}_{h_1} \Psi_{h_1}(z_1) = c_1 \Phi_{h_1}(z_1), \quad (4.3)$$

where we allowed for a general normalization constant c_1 . We first define a partner three point function of the ordinary Virasoro primaries

$$C_{3,0}^{\text{orig}}(z_{1,2,3}) = \langle \Phi_{h_1}(z_1) \Phi_{h_2}(z_2) \Phi_{h_3}(z_3) \rangle \quad (4.4)$$

as well as the three point function under study

$$C_{3,1}^{\text{orig}}(z_{1,2,3}) = \langle \Psi_{h_1}(z_1) \Phi_{h_2}(z_2) \Phi_{h_3}(z_3) \rangle. \quad (4.5)$$

The only crucial property of the fields Φ_{h_i} is that they are ordinary Virasoro primaries. The lower indices refer to the number of insertions, and the number of logarithmic insertions. As we do for the standard global conformal Ward identities, we eliminate the conformal dimension h_i dependence by taking out a pre-factor

$$C_{3,1}^{\text{orig}}(z_{1,2,3}) = \prod_{i < j} (z_i - z_j)^{\mu_{ij}} C_{3,1}(z_{kl}), \quad (4.6)$$

where we set

$$\mu_{12} = h_3 - h_1 - h_2, \quad \mu_{13} = h_2 - h_1 - h_3, \quad \mu_{23} = h_1 - h_2 - h_3. \quad (4.7)$$

⁸The typical representations can be replaced by any ordinary Virasoro primary for the reasoning that follows.

⁹The projective representation will necessarily have conformal dimension zero. We postpone the use of this knowledge to illustrate the generality of the reasoning.

The logarithmic global conformal Ward identities then read

$$\begin{aligned}\sum_i \partial_i C_{3,1} &= 0 \\ \sum_i z_i \partial_i C_{3,1} &= -c_1 C_{3,0} \\ \sum_i z_i^2 \partial_i C_{3,1} &= -2z_1 c_1 C_{3,0}.\end{aligned}\tag{4.8}$$

We split the solution to the logarithmic Ward identities into a homogeneous and a particular solution. The homogeneous solution, as is familiar from standard three point functions, is a constant. We thus obtain the sum

$$C_{3,1} = c_{3,1} + C_{3,1}^{\text{part}}(z_{kl}),\tag{4.9}$$

while for the particular solution we adopt the ansatz

$$C_{3,1}^{\text{part}} = c_{3,0} \sum_{i<j} c_{ij} \log z_{ij},\tag{4.10}$$

where we used the fact that the rescaled ordinary three point function $C_{3,0} = c_{3,0}$ is a constant. To find the particular solution then, we have to solve the equations

$$\sum_{i<j} c_{ij} = -c_1, \quad \sum_j c_{1j} = -2c_1, \quad c_{12} + c_{23} = 0 = c_{13} + c_{23},\tag{4.11}$$

which is easily done:

$$-c_{23} = c_{12} = c_{13} = -c_1.\tag{4.12}$$

We have thus derived the chiral three point logarithmic correlator

$$C_{3,1}^{\text{orig}} = \prod_{i<j} z_{ij}^{\mu_{ij}} \left(c_{3,1} + c_1 c_{3,0} \log \frac{z_{23}}{z_{12} z_{13}} \right),\tag{4.13}$$

in terms of the constants $c_{3,1}$, $c_{3,0}$, c_1 and the conformal dimensions of the operators. This is the form that the three point function of one top component in a projective representation and two (e.g. typical) Virasoro primaries will take.

4.4 Four point functions

We turn to determining the form of four point functions. We remind the reader that the standard global $\text{SL}(2, \mathbb{C})$ conformal Ward identities are solved by the ansatz:

$$\begin{aligned}\langle \Phi_{h_1}(z_1) \dots \Phi_{h_n}(z_n) \rangle &= \prod_{i<j} (z_i - z_j)^{\mu_{ij}} C_n(z_{mp}^{kl}) \\ \sum_{j \neq i} \mu_{ij} &= -2h_i,\end{aligned}\tag{4.14}$$

where we introduced the function C_n of all cross ratios

$$z_{mn}^{kl} = \frac{(z_m - z_p)(z_k - z_l)}{(z_m - z_l)(z_k - z_p)}.\tag{4.15}$$

In the case of four point functions, there is a single independent cross ratio. The solution for the powers μ_{ij} appearing in the prefactor are not unique. We can for instance pick the solution:

$$\begin{aligned}
 \mu_{12} &= \mu_{13} = 0, \\
 \mu_{14} &= -2h_1 \\
 \mu_{23} &= h_4 - h_1 - h_2 - h_3 \\
 \mu_{24} &= h_1 + h_3 - h_2 - h_4 \\
 \mu_{34} &= h_1 + h_2 - h_3 - h_4.
 \end{aligned} \tag{4.16}$$

We denote the cross ratio by x

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} \tag{4.17}$$

We can then write the solution to the global conformal Ward identity as

$$\langle \Phi_{h_1} \Phi_{h_2} \Phi_{h_3} \Phi_{h_4} \rangle = \prod_{i < j} z_{ij}^{\mu_{ij}} C_{4,0}(x). \tag{4.18}$$

The analysis applies when all four insertions are ordinary primaries of the Virasoro algebra.

We now solve an example of generalized global Ward identities, applicable to a four point function with one logarithmic insertion and three ordinary primaries. We denote the four point function by

$$C_{4,1}^{\text{orig}}(z_1, z_2, z_3, z_4) = \langle \Psi_{h_1}(z_1) \Phi_{h_2}(z_2) \Phi_{h_3}(z_3) \Phi_{h_4}(z_4) \rangle, \tag{4.19}$$

and it satisfies the equations

$$\begin{aligned}
 \sum_i \partial_i C_{4,1}^{\text{orig}} &= 0 \\
 \left(\sum_i z_i \partial_i + (h_i + \hat{\delta}_{h_i}) \right) C_{4,1}^{\text{orig}} &= 0 \\
 \left(\sum_i z_i^2 \partial_i + 2z_i(h_i + \hat{\delta}_{h_i}) \right) C_{4,1}^{\text{orig}} &= 0.
 \end{aligned} \tag{4.20}$$

We again denote and normalize the logarithmic partner as

$$\hat{\delta}_{h_1} \Psi_{h_1}(z_1) = c_1 \Phi_{h_1}(z_1). \tag{4.21}$$

The equations to solve become

$$\begin{aligned}
 \sum_i \partial_i C_{4,1}^{\text{orig}} &= 0 \\
 \left(\sum_i z_i \partial_i + h_i \right) C_{4,1}^{\text{orig}} &= -c_1 C_{4,0}^{\text{orig}} \\
 \left(\sum_i z_i^2 \partial_i + 2h_i z_i \right) C_{4,1}^{\text{orig}} &= -2c_1 z_1 C_{4,0}^{\text{orig}}.
 \end{aligned} \tag{4.22}$$

Since the correlator $C_{4,0}^{\text{orig}}$ contains only Virasoro primaries, we know that $C_{4,0}^{\text{orig}}$ is of the form of the standard four point functions

$$C_{4,0}^{\text{orig}} = \prod_{i<j} z_{ij}^{\mu_{ij}} C_{4,0}(x). \quad (4.23)$$

As a first simplification of the logarithmic four point function, we take out a factor

$$C_{4,1}^{\text{orig}}(z_i) = \prod_{i<j} z_{ij}^{\mu_{ij}} C_{4,1}(z_{ij}), \quad (4.24)$$

as well, with $C_{4,1}$ a function of the differences z_{ij} of the positions. In this way, we kill the homogeneous term on the left hand side, and can cancel the common factors on both sides of the equation to find

$$\begin{aligned} \sum_i z_i \partial_i C_{4,1}(z_{jk}) &= -c_1 C_{4,0}(x) \\ \sum_i z_i^2 \partial_i C_{4,1}(z_{jk}) &= -2c_1 z_1 C_{4,0}(x). \end{aligned} \quad (4.25)$$

To any solution for $C_{4,1}$, we can add a solution to the homogeneous equation, which is a function of the cross ratio. We thus write

$$C_{4,1}(z_{ij}) = C_{4,1}^{\text{hom}}(x) + C_{4,1}^{\text{part}}(z_{ij}). \quad (4.26)$$

We use as an ansatz for the particular solution

$$C_{4,1}^{\text{part}} = \sum_{i<j} c_{ij} \log z_{ij} C_{4,0}(x), \quad (4.27)$$

which leads to the equations

$$\begin{aligned} \sum_{i<j} c_{ij} &= -c_1, & \sum_j c_{1j} &= -2c_1, & c_{12} + c_{23} + c_{24} &= 0, \\ c_{13} + c_{23} + c_{34} &= 0, & c_{14} + c_{24} + c_{34} &= 0. \end{aligned} \quad (4.28)$$

The generic solution is

$$\begin{aligned} c_{14} &= -2c_1 - c_{12} - c_{13} \\ c_{23} &= -c_1 - c_{12} - c_{13} \\ c_{24} &= c_1 + c_{13} \\ c_{34} &= c_1 + c_{12}. \end{aligned} \quad (4.29)$$

A particular solution is

$$C_{4,1}^{\text{part}}(z_i) = c_1 \log \frac{z_{24} z_{34}}{z_{14}^2 z_{23}} C_{4,0}(x). \quad (4.30)$$

We note that this particular solution goes to zero for $z_4 \rightarrow \infty$. Moreover, the z_1 derivative will also go to zero when z_4 goes to infinity. Therefore, one practical conclusion we draw from our analysis is that there is a solution to the logarithmic conformal Ward identities that has the property that for the choice of points $(z_1, z_2, z_3, z_4) = (x, 0, 1, \infty)$ the non-homogeneous logarithmic term will not contribute to the Knizhnik-Zamolodchikov equation for the z_1 derivative of the four point function. We can, so to speak, throw away the ladder after we have climbed it.

5 Correlation functions

In the previous section, we analyzed constraints on the correlation functions that arise from combining the logarithmic rank two nature of the $gl(1|1)$ Wess-Zumino-Witten model with the global conformal Ward identities. In this section, we study the further constraints on the correlation functions implied by the Knizhnik-Zamolodchikov equations. We start out with two- and three point functions to illustrate the known power of the method, and to ponder the peculiarities of the logarithmic model at hand. In section 6, we treat four point functions.

5.1 Two point functions

For the two point functions, we distinguish two cases. We analyze the two point function of primary operators in typical representations, and the two point function of primary operators in projective representations.

5.1.1 Two typical

As remarked previously, two typical representations need to combine into a projective representation in order to find a non-zero two point function. There will be a single such projective representation, and since the non-top states have zero one point function, we find that typical representations have a single N, E invariant two point function that is non-zero. The Knizhnik-Zamolodchikov equation for this two point function reads

$$\left(\partial_1 - \frac{Q_{1\otimes 2}}{z_1 - z_2}\right) \langle T_{e_1, n_1}(z_1) T_{e_2, n_2}(z_2) \rangle = 0. \quad (5.1)$$

We denote by $Q_{i\otimes j}$ the operator $Q_{i\otimes j} = Q_{ab} t_i^a \otimes t_j^b$ acting on the tensor product representation of the i th and j th factor in the correlation function. For a non-zero two point function, we demand invariance under the bosonic generators in the tensor product representation, and therefore that $e_1 + e_2 = 0$. The fact that only the top state in the projective tensor product representation has a non-zero one point function combined with invariance under the N -generator implies that also $n_1 + n_2 = 0$. The only coefficients of the state that survive projection by the tensor product quadratic Casimir with respect to the left action are given by

$$\delta_{1\uparrow} \delta_{2\downarrow} + \delta_{1\downarrow} \delta_{2\uparrow}, \quad (5.2)$$

and therefore the two point function is proportional to this tensor structure, on the left hand side. Finally, let us calculate its value. We can use translation invariance to write the Knizhnik-Zamolodchikov equation as

$$\left(\partial_z - \frac{Q_{1\otimes 2}}{z}\right) \langle T_{e_1, n_1}(z) T_{e_2, n_2}(0) \rangle = 0. \quad (5.3)$$

Since there is a single top channel in the tensor product, the action of $Q_{1\otimes 2}$ necessarily leads to a result proportional to the single top channel up to states that give rise to zero two point function. The proportionality constant is such that

$$\left(\partial_z + \frac{h_{e_1, n_1} + h_{e_2, n_2}}{z}\right) \langle T_{e_1, n_1}(z) T_{e_2, n_2}(0) \rangle = 0. \quad (5.4)$$

We recuperate the derivation of the two point function from the global scaling conformal Ward identity, when it is non-zero. Thus, the two point function takes the value

$$\langle T_{e_1, n_1}(z) T_{e_2, n_2}(0) \rangle = c \delta_{e_1+e_2} \delta_{n_1+n_2} (\delta_{1\uparrow} \delta_{2\downarrow} + \delta_{1\downarrow} \delta_{2\uparrow}) z^{-h_{e_1, n_2} - h_{e_2, n_2}}, \quad (5.5)$$

in as far as the left-moving sector is concerned. We find the full correlator from an identical analysis on the right hand side, and combining them consistently with locality.

We can reproduce the resulting correlator using the free field representation and the symplectic fermion correlator [5]

$$\langle \mu_{-e}(z_1, \bar{z}_1) \mu_{1+e}(z_2, \bar{z}_2) \rangle = -|z_{12}|^{-2e(e+1)}, \quad (5.6)$$

as well as the correlator

$$\langle \mu_{-e}^L \mu_{-e+1}^R(z_1, \bar{z}_1) \mu_{1+e}^L \mu_e^R(z_2, \bar{z}_2) \rangle = -z_{12}^{-e(e+1)} \bar{z}_{12}^{-e(e-1)}. \quad (5.7)$$

5.1.2 Two projectives

For two projective representations, we again have

$$\left(\partial_z - \frac{Q_{1\otimes 2}}{z} \right) \langle P_{n_1}(z) P_{n_2}(0) \rangle = 0. \quad (5.8)$$

There are four top states that can have a non-zero one point function in the tensor product of two projective P_n representations. For each of the conditions $\sum n_i \pm 1 = 0$, there is one combination of states with a non-zero two point function. These are mapped to (a state equivalent to) zero by the $Q_{1\otimes 2}$ operator and therefore have a constant two point function. We have two channels when the momenta sum to zero, $\sum n_i = 0$. We refer to appendix A.3 for an analysis of these representation theoretic facts in all cases of relevance to this paper. Moreover, in appendix A.4, we catalogue the corresponding bottom component invariants.¹⁰ We denote these invariants $I_{a_1, a_2; a_3}^b$ where a_1 is the number of typical insertions, a_2 the number of projective covers, a_3 the negative of the sum of the n_i quantum numbers, and b a label running over all invariants in the channel. In the case at hand, the $Q_{1\otimes 2}$ operator maps the invariant $I_{0,2;0}^2$ into the structure $I_{0,2;0}^1$ times 2. Furthermore, we introduce the notation $\tilde{I}_{a_1, a_2; a_3}^b$ for the matrix of coefficients of the insertions in the correlator expressed in terms of the invariants $I_{a_1, a_2; a_3}^b$, after acting by the quadratic Casimir $C_2^\otimes$.

We can then parameterize the two-point function as

$$\langle P_1(z) P_2(0) \rangle = G_a^{0,2;0}(z) \tilde{I}_{0,2;0}^a, \quad (5.9)$$

We find the coupled system of Knizhnik-Zamolodchikov differential equations

$$\begin{aligned} \partial_z G_1^{0,2;0} &= 0 \\ \partial_z G_2^{0,2;0} &= 2 \frac{G_1^{0,2;0}}{z}. \end{aligned} \quad (5.10)$$

¹⁰These bottom invariants are used as a stand-in for the corresponding top components. They are handy in that they already incorporate the fact that all non-top components have zero correlation function. They capture the image of the $C_2^\otimes$ map.

These equations have the solutions

$$\begin{aligned} G_1^{0,2;0}(z) &= g_1^{0,2;0} \\ G_2^{0,2;0}(z) &= g_2^{0,2;0} + 2g_1^{0,2;0} \log z. \end{aligned} \quad (5.11)$$

A short calculation shows that the two top states $\tilde{I}_{0,2;0}^a$ indeed form a Jordan block of rank two with respect to the diagonal L_0 operator, generating the standard structure of the two point functions for a logarithmic conformal field theory.

In the free field formalism, at $n_1 + n_2 = 0$, we have the possibility of combining the two top states in the projective representations into a symplectic fermion two point function $\langle : \chi^1 \chi^2 : (z_1, \bar{z}_1) : \chi^1 \chi^2 : (z_2, \bar{z}_2) \rangle$, or the other states in such a way as to cancel the two fermionic zero modes and obtain $\chi^1 \chi^2$ combinations. In the first case, we obtain a two point function proportional to $-2(Z + \log |z_{12}|^2)$, where Z is an arbitrary constant, and in the second case, proportional to one [5].

To familiarize the reader further with the algebraic properties of the model, let us analyze this simplest of examples in even further detail. From the free field representations, we find the left correlators

$$\begin{aligned} \langle tt \rangle_L &= -2 \left(\frac{Z}{2} + \log z_{12} \right), \\ \langle lr \rangle_L &= 1, \quad \langle rl \rangle_L = -1, \\ \langle tb \rangle_L &= -1, \quad \langle bt \rangle_L = -1. \end{aligned} \quad (5.12)$$

We can check that for instance the Knizhnik-Zamolodchikov equation

$$\partial_1 \langle tt \rangle_L + \langle lr - rl \rangle_L / z_{12} = 0 \quad (5.13)$$

is satisfied. We also find that the second Casimir acting on the $\langle tt \rangle_L$ correlator gives rise to a zero correlator

$$C_2 \langle tt \rangle_L = -\langle bt + lr - rl + tb \rangle_L = 0, \quad (5.14)$$

as required. In analyzing the consequences of the Knizhnik-Zamolodchikov equation, we mostly concentrate on the holomorphic, left-moving correlators. It will be manifest when we switch back to the full non-chiral correlators. We often drop the subscript L from the chiral correlators for ease of notation.

5.2 Three point functions

For the three point functions, we concentrate on an example correlator involving one projective and two typical representations. The corresponding Knizhnik-Zamolodchikov equation is

$$\left(\partial_1 - \frac{Q_{1\otimes 2}}{z_1 - z_2} - \frac{Q_{1\otimes 3}}{z_1 - z_3} \right) \langle P_{n_1} T_{e_2, n_2} T_{e_3, n_3} \rangle = 0. \quad (5.15)$$

In the most interesting $\sum n_i = 0$ channel, we have two invariants (see appendix A.4), and we find the equation

$$\left(\partial_1 - \frac{Q_{12}}{z_1 - z_2} - \frac{Q_{13}}{z_1 - z_3} \right) \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} = 0 \quad (5.16)$$

for the position dependent correlators $G_{1,2}$ corresponding to the invariants $I_{2,1,0}^1$ and $I_{2,1,0}^2$. The matrices Q_{12} and Q_{13} are recorded in appendix A.5. We find the solution

$$G_2 = g_2(z_1 - z_2)^{e_2 n_1} (z_1 - z_3)^{e_3 n_1}, \quad (5.17)$$

consistent with the global conformal Ward identity for the non-logarithmic mode in the projective representation. Analyzing the z_2 dependent Knizhnik-Zamolodchikov equation or the global conformal Ward identity would determine the z_{23} dependence as well. We also find the correlator

$$G_1 = (z_1 - z_2)^{e_2 n_1} (z_1 - z_3)^{e_3 n_1} (g_1 - g_2 e_2 \log(z_1 - z_2)(z_1 - z_3)),$$

again in accord with the global conformal Ward identity (4.13).

5.3 A free field representation

Using the free field representation of the $\langle PTT \rangle$ correlator, we can determine a coherent set of constants in the correlation functions. This calculation was performed in the bc ghost system in [3]. The free bosonic Y, Z correlators that occur are

$$\langle \prod e^{e_i^L Z^L + e_i^R Z^R + n_i^L Y^L + n_i^R Y^R} \rangle = \prod_{i < j} (z_i - z_j)^{e_i^L n_j^L + e_j^L n_i^L} (\bar{z}_i - \bar{z}_j)^{e_i^R n_j^R + e_j^R n_i^R}. \quad (5.18)$$

The relevant symplectic fermion correlators are reviewed in appendix B. For the $\langle PTT \rangle$ correlators, there are $4 \times 2 \times 2$ possibilities for left chiral correlators. We concentrate again on the channel $\sum n_i = 0$. We obtain a first free field correlator

$$\langle t \uparrow \downarrow \rangle = z_{12}^{n_1 e_1} z_{13}^{n_1 e_3} z_{23}^{e_2(n_3 - \frac{1}{2}) + e_3(n_2 + \frac{1}{2})} (z_{23})^{e_2(1-e_2)} \left(\frac{1}{2} Z_{e_2} + \log \frac{z_{12} z_{13}}{z_{23}} \right), \quad (5.19)$$

where the constant Z_{e_2} equals [5]

$$Z_{e_2} = Z + 2\psi(1) - \psi(e_2) - \psi(1 - e_2), \quad (5.20)$$

and $\psi(z)$ is the digamma function. The $\langle t \downarrow \uparrow \rangle$ correlator is found by exchanging the second and third entry. Moreover, we find the left correlators

$$\langle b \uparrow \downarrow \rangle = 1, \quad \langle b \downarrow \uparrow \rangle = 1 \quad (5.21)$$

$$\langle l \downarrow \downarrow \rangle = -1, \quad \langle r \uparrow \uparrow \rangle = \frac{1}{e_2}, \quad (5.22)$$

where we left out the canonical z_{ij} dependence for ease of notation. A correlation function is zero if it is not top (after tensor product decomposition), which gives rise to the constraints

$$\begin{aligned} \langle l \downarrow \downarrow \rangle + \langle b \downarrow \uparrow \rangle &= 0, & \langle b \uparrow \downarrow \rangle - \langle b \downarrow \uparrow \rangle &= 0, \\ e_2 \langle r \uparrow \uparrow \rangle - \langle b \downarrow \uparrow \rangle &= 0, & e_2 (\langle t \uparrow \downarrow \rangle - \langle t \downarrow \uparrow \rangle) &= \langle b \downarrow \uparrow \rangle. \end{aligned} \quad (5.23)$$

To check the last equation, we use the shift identity of the digamma function ψ

$$\psi(z+1) = \psi(z) + \frac{1}{z}. \quad (5.24)$$

We can also check the Knizhnik-Zamolodchikov equations as applied to individual correlators more directly. An representative example equation is

$$\partial_1 \langle t \uparrow \downarrow \rangle + \frac{1}{z_{12}} \langle l \downarrow \downarrow \rangle - \frac{e_2 n_1}{z_{12}} \langle t \uparrow \downarrow \rangle - \frac{e_2 \langle r \uparrow \uparrow \rangle}{z_{13}} - \frac{e_3 n_1}{z_{13}} \langle t \uparrow \downarrow \rangle = 0,$$

which is indeed satisfied.

We conclude with the remark that the Knizhnik-Zamolodchikov equations that we encountered are essentially identical to the differential equations on symplectic fermion correlators recorded in appendix B of [5]. The generic proof of this fact will follow from gauging the bosonic directions in the $gl(1|1)$ Wess-Zumino-Witten model. This structural map is valid for the four point function of typical representations to be analyzed next, and will also hold for the four point function involving one projective representation. The latter case therefore adds to our knowledge of symplectic fermion correlators.

6 The four point functions

We consider the four point functions in which there are zero or one projective representations amongst the initial current algebra primaries, and four or three typicals. In the first case, we closely mimic the original derivation of the four point function [1]. In section 7 we will perform a more general check of crossing symmetry. The second case will have features reminiscent of the calculation of correlators in other models with supergroup symmetries [19].

6.1 Four typicals

We first solve for the four typical representations four point function. These typical representations are ordinary Virasoro primaries and as such their correlation functions satisfy the standard (i.e. non-logarithmic) global conformal Ward identities. Thus, their four point functions are of the form (4.18). We consider the Knizhnik-Zamolodchikov equation with respect to the first insertion at z_1 . We wish to write the equation as a differential equation in the cross ratio x . The derivative with respect to z_1 gives terms arising from the prefactor, and we use the chain rule to find

$$\partial_1 C_4(x) = \left(\frac{x}{z_{12}} - \frac{x}{z_{14}} \right) \partial_x C_4. \quad (6.1)$$

We can then put $z_1 = x, z_2 = 0, z_3 = 1, z_4 = \infty$ to obtain the differential equation

$$\left(\partial_x - \frac{Q_{1\otimes 2}}{x} - \frac{Q_{1\otimes 3}}{x-1} \right) C_4(x) = 0. \quad (6.2)$$

The representation theoretic structure of the correlators is again exhibited in appendix A. We must satisfy the E -invariance constraint $\sum_{i=1}^4 e_i = 0$, and find projective representations P_0 if we have $\sum n_i = \pm 1$ or $\sum n_i = 0$. These cases give rise to either a single or to two channels.

6.1.1 Four typicals with $\sum n_i + 1 = 0$

We concentrate first on a case with a single channel, namely $\sum n_i + 1 = 0$. We wish to evaluate the $Q_{i\otimes j}$ operator on the top state in this channel. Since there is a single channel, there is only a proportionality constant q_{1j} to compute, which for Q_{1j} is equal to

$$q_{1j} = h_{e_1+e_j, n_1+n_j+1/2} - h_{e_1, n_1} - h_{e_j, n_j}, \quad (6.3)$$

where the $h_{e,n}$ are conformal dimensions of primaries. In the tensor product $T_{e_1, n_1} \otimes T_{e_j, n_j}$, we project onto the relevant channel (with highest N eigenvalue — see appendix A.5 for details). We then find the equation

$$\left(\partial_x - \frac{q_{12}}{x} - \frac{q_{13}}{x-1} \right) C_{4,0;0}(x) = 0. \quad (6.4)$$

We solve the first order differential equation to find the chiral correlator

$$C_{4,0;0}(x) = c_{4,0;0} x^{q_{12}} (1-x)^{q_{13}}, \quad (6.5)$$

with $c_{4,0;0}$ an arbitrary integration constant.

6.1.2 Four typicals with $\sum n_i = 0$

In the case where the N -momenta sum to zero, we have two channels. The action of the operators Q_{1i} in our basis is recorded in appendix A.5. The Knizhnik-Zamolodchikov equation becomes a combination of two equations. We solve one equation for one of the correlators, substitute it in the other equation, and then solve the resulting hypergeometric equation [1]. We can do this for both channels, and then again impose consistency with the original equations relating the chiral correlators. We find:

$$\begin{aligned} G_1 &= (x-1)(1-x)^{\frac{1}{2}(2e_1(e_3+n_3)+2e_3n_1+e_1+e_3)} x^{e_2(n_1-\frac{1}{2})+e_1(e_2+n_2-\frac{1}{2})} \\ &\quad \times \left(c_1 \frac{e_1+e_2+1}{e_2} x^{e_1+e_2} {}_2F_1(e_1+1, e_1+e_2+e_3+1; e_1+e_2+1; x) \right. \\ &\quad \left. + c_2 \frac{e_3}{e_1+e_2} {}_2F_1(1-e_2, e_3+1; -e_1-e_2+1; x) \right) \\ G_2 &= (1-x)^{\frac{1}{2}(2e_1(e_3+n_3)+2e_3n_1+e_1+e_3)} x^{\frac{1}{2}e_2(2n_1-1)+e_1(e_2+n_2-\frac{1}{2})} \\ &\quad \times \left(c_1 x^{e_1+e_2+1} {}_2F_1(e_1+1, e_1+e_2+e_3+1; e_1+e_2+2; x) \right. \\ &\quad \left. + c_2 {}_2F_1(-e_2, e_3; -e_1-e_2; x) \right). \end{aligned} \quad (6.6)$$

We will later work in terms of the basis where G_1^1 is the coefficient of c_1 in G_1 et cetera. In section 7 we assemble these chiral building blocks into a left-right local correlation function satisfying crossing symmetry.

6.2 One projective and three typicals

In the second part of this section, we compute the chiral correlators for a four point function containing one projective and three typical representations.

6.2.1 One projective and three typicals with $\sum n_i + \frac{1}{2} = 0$

In simplifying the Knizhnik-Zamolodchikov equation, we can pick the solution to the global conformal Ward identity in the form (4.30) such that when we fix the points $z_i = (x, 0, 1, \infty)$, the potential supplementary logarithmic contribution vanishes. Alternatively, we can directly analyze the Knizhnik-Zamolodchikov equation without using the global conformal Ward identity, and only fix the points z_i in a later stage. In the following, we choose one or the other technique, depending on convenience.

In any case, with the constraint $\sum n_i + 1/2 = 0$, we find the three channels given in the appendix, equation (A.17), and the action of the operators $Q_{1\otimes 2}$ and $Q_{1\otimes 3}$ in the Knizhnik-Zamolodchikov equations are then as in equation (A.21). We note that L_0 is not diagonalizable on the invariants. It has non-trivial Jordan structure, with equal diagonal entries and mapping the first to the second invariant. Fixing the insertion points, we find the three Knizhnik-Zamolodchikov differential equations

$$\begin{aligned} -\frac{e_2 n_1 G_1(x) + G_2(x)}{x} - \frac{e_3 n_1 G_1(x) - e_3 G_3(x) - e_3/e_2 G_2(x)}{x-1} + G_1'(x) &= 0 \\ -\frac{e_2(n_1+1)G_3(x)}{x} - \frac{(e_2+e_3)/e_2 G_2(x) + e_3(n_1+1)G_3(x)}{x-1} + G_3'(x) &= 0 \\ -\frac{e_2 n_1 G_2(x)}{x} - \frac{e_3 n_1 G_2(x)}{x-1} + G_2'(x) &= 0. \end{aligned} \quad (6.7)$$

We can solve these differential equations consecutively, from simple to hard, and find the solution

$$\begin{aligned} G_2(x) &= (1-x)^{e_3 n_1} x^{e_2 n_1} c_3 \\ G_3(x) &= (1-x)^{e_3(n_1+1)} x^{e_2(n_1+1)} (c_2 - B(x, 1-e_2, -e_3) c_3 (e_2+e_3)/e_2) \\ G_1(x) &= (1-x)^{e_3 n_1} x^{e_2 n_1} \left(c_1 + c_3 \log x - c_3 \frac{e_3}{e_2} \log(1-x) \right. \\ &\quad + \frac{e_3}{1+e_2} x^{1+e_2} c_2 {}_2F_1(1+e_2, 1-e_3, 2+e_2; x) \\ &\quad + \frac{e_3(e_2+e_3)}{e_2(1-e_2)} c_3 x \Gamma(2-e_2) {}_p\tilde{F}_q(\{1, 1-e_2-e_3, 1\}, \{2-e_2, 2\}; x) \\ &\quad - c_3 \frac{e_3(e_2+e_3)}{e_2(1-e_2)} \frac{\Gamma(2-e_2)}{\Gamma(1-e_2)\Gamma(1+e_3)\Gamma(1-e_2-e_3)} \\ &\quad \left. \times G(\{\{0, e_2+e_3\}, \{1\}\}, \{\{0, 0, e_3\}, \{\}\}; 1-x) \right). \end{aligned} \quad (6.8)$$

We made use of standard notations for the arguments of the regularized generalized hypergeometric function ${}_p\tilde{F}_q$ and the Meijer G -function which we denoted G . We named the incomplete beta-function B . These solutions contain all information on the chiral correlators.

We found it useful to gain more insight into how their staircase structure arises from $gl(1|1)$ representation theory. There are $4 \times 2^3 = 32$ $\langle PTTT \rangle$ chiral correlation functions. Let's concentrate on the current primary states that satisfy the N -channel constraint $\sum n_i + \frac{1}{2} = 0$ and N -invariance. We have ten combinations satisfying these conditions:

$$b \downarrow \uparrow \uparrow, b \uparrow \downarrow \uparrow, b \uparrow \uparrow \downarrow, t \downarrow \uparrow \uparrow, t \uparrow \downarrow \uparrow, t \uparrow \uparrow \downarrow, l \uparrow \downarrow \downarrow, l \downarrow \uparrow \downarrow, l \downarrow \downarrow \uparrow, r \uparrow \uparrow \uparrow. \quad (6.9)$$

We find constraints from the demand that all combinations that map to zero under the quadratic Casimir C_2 have zero correlator. By the channel count, this should leave us with only three non-zero combinations. The seven constraints are

$$\begin{aligned} e_3 \langle b \downarrow \uparrow \uparrow \rangle + e_2 \langle b \uparrow \downarrow \uparrow \rangle &= 0 \\ e_2 \langle b \uparrow \uparrow \downarrow \rangle - e_4 \langle b \downarrow \uparrow \uparrow \rangle &= 0 \end{aligned} \quad (6.10)$$

$$e_2 \langle r \uparrow \uparrow \uparrow \rangle - \langle b \downarrow \uparrow \uparrow \rangle = 0 \quad (6.11)$$

$$\langle b \downarrow \uparrow \uparrow \rangle - \langle l \downarrow \uparrow \downarrow \rangle + \langle l \downarrow \downarrow \uparrow \rangle = 0 \quad (6.12)$$

$$e_2 \langle l \uparrow \downarrow \downarrow \rangle + e_3 \langle l \downarrow \uparrow \downarrow \rangle + e_4 \langle l \downarrow \downarrow \uparrow \rangle = 0 \quad (6.13)$$

$$-\langle t \uparrow \downarrow \uparrow \rangle + \langle t \downarrow \uparrow \uparrow \rangle + \langle t \uparrow \uparrow \downarrow \rangle + \frac{1}{e_2} \langle b \downarrow \uparrow \uparrow \rangle = 0 \quad (6.14)$$

$$e_2 \langle t \uparrow \downarrow \uparrow \rangle + e_3 \langle t \downarrow \uparrow \uparrow \rangle + \langle l \downarrow \downarrow \uparrow \rangle = 0. \quad (6.15)$$

To study the staircase structure of the Knizhnik-Zamolodchikov equations, we first consider the correlator $\langle b \uparrow \downarrow \uparrow \rangle$ and its permutations in the last three entries. These correlators must satisfy the constraints (6.10). We first perform a check on our formalism using the free field representation of the correlator. It consists of a free field YZ correlator, and a three point function of symplectic fermion twist fields. For the diagonal combination, the full non-chiral correlator is given by

$$\begin{aligned} C_{b\downarrow\uparrow\uparrow}^{ff} &= \langle e^{n_1 Y} e^{e_2 Z + (n_2 - \frac{1}{2})Y} e^{e_3 Z + (n_3 + \frac{1}{2})Y} e^{e_4 Z + (n_4 + \frac{1}{2})Y} \rangle \langle \mu_{-e_2} \mu_{-e_3+1} \mu_{-e_4+1} \rangle \\ &= C_{-e_2, -e_3+1, -e_4+1} |z_{23}^{e_3(1+e_2)} z_{34}^{e_3 e_4} z_{24}^{(1+e_2)e_4}|^2 |z_{12}^{n_1 e_2} z_{13}^{n_1 e_3} z_{14}^{n_1 e_4}|^2 \\ &\quad \times |z_{23}^{e_2(n_3 + \frac{1}{2}) + e_3(n_2 - \frac{1}{2})} z_{24}^{e_2(n_4 + \frac{1}{2}) + e_4(n_2 - \frac{1}{2})} z_{34}^{e_3(n_4 + \frac{1}{2}) + e_4(n_3 + \frac{1}{2})}|^2. \end{aligned} \quad (6.16)$$

The three point function C determined in [5] picks up factors under the interchange of the insertions 2, 3, 4. This becomes manifest when we rewrite

$$C_{-e_2, -e_3+1, -e_4+1} = \sqrt{\Gamma\left(\frac{e_2, e_3, e_4}{-e_2, -e_3, -e_4}\right)} \frac{e_2}{e_3 e_4}. \quad (6.17)$$

Permuted three point functions differ by factors of $\sqrt{e_3^2/e_2^2}$ or $\sqrt{e_4^2/e_2^2}$. With the appropriate sign choices (which we make), we then satisfy the constraints (6.10). The correlator involving the right state r is then fixed through the constraint (6.11). All these correlators sit on the first step of our staircase.

Next, we use the Knizhnik-Zamolodchikov equation to determine the $\langle l \downarrow \downarrow \uparrow \rangle$ correlator. The simplest choice of prefactor $\prod_{i<j} z_{ij}^{\mu_{ij}}$ to extract from the differential equation corresponds again to the choice of μ_{ij}

$$\begin{aligned} \mu_{12} &= 0, & \mu_{23} &= h_4 - h_1 - h_2 - h_3, & \mu_{24} &= h_1 + h_3 - h_2 - h_4, \\ \mu_{13} &= 0, & \mu_{14} &= -2h_1, & \mu_{34} &= h_1 + h_2 - h_3 - h_4, \end{aligned} \quad (6.18)$$

and we have that $h_1 = 0$, such that the factors agree with those of an ordinary three point function. Concretely, before fixing the insertion points, we obtain for example the

Knizhnik-Zamolodchikov equation

$$\begin{aligned} \partial_1 \langle l \downarrow \downarrow \uparrow \rangle &= \frac{1}{z_{12}} (e_2 \langle b \uparrow \downarrow \uparrow \rangle + e_2 (n_1 + 1) \langle l \downarrow \downarrow \uparrow \rangle) + \frac{1}{z_{14}} e_4 (n_1 + 1) \langle l \downarrow \downarrow \uparrow \rangle \\ &+ \frac{1}{z_{13}} (e_3 \langle b \downarrow \uparrow \uparrow \rangle + e_3 (n_1 + 1) \langle l \downarrow \downarrow \uparrow \rangle). \end{aligned} \quad (6.19)$$

Factoring out the $z_{ij}^{\mu_{ij}}$ dependence, and denoting the chiral $\langle b \downarrow \uparrow \uparrow \rangle$ correlation function (up to this factor) by the constant $c_{\downarrow \uparrow \uparrow}$, we obtain from these and analogous differential equations the results

$$\begin{aligned} \langle l \downarrow \downarrow \uparrow \rangle &= (1-x)^{e_3 n_1} x^{e_2 n_1} ((-1)^{1+e_2 n_1} c_{\downarrow \uparrow \uparrow} + (1-x)^{e_3} x^{e_2} d_1 \\ &\quad + (-1)^{e_2 n_1} c_{\downarrow \uparrow \uparrow} (x-1) \Gamma(-e_2) {}_2\tilde{F}_1(1, 1-e_2-e_3, 1-e_2; x) (e_2+e_3)) \\ \langle l \downarrow \uparrow \downarrow \rangle &= (1-x)^{e_3(n_1+1)} x^{e_2(n_1+1)} (d_2 + (-1)^{e_2 n_1} c_{\downarrow \uparrow \uparrow} B(x; -e_2, 1-e_3) e_4) \\ \langle l \uparrow \downarrow \downarrow \rangle &= x^{e_2(n_1+1)} (1-x)^{e_3(n_1+1)} (d_3 + (-1)^{1+e_2 n_1} c_{\downarrow \uparrow \uparrow} B(x; 1-e_2, -e_3) e_4). \end{aligned} \quad (6.20)$$

We introduced integration constants d_i . We note the special function identity between the incomplete beta function and the regularized hypergeometric function

$$B(z; a, b) = \Gamma(a) (1-z)^b z^a {}_2\tilde{F}_1(1, a+b; a+1; z) \quad (6.21)$$

which allows to rewrite the first correlator as

$$\begin{aligned} \langle l \downarrow \downarrow \uparrow \rangle &= (1-x)^{e_3 n_1} x^{e_2 n_1} ((-1)^{1+e_2 n_1} c_{\downarrow \uparrow \uparrow} + d_1 (1-x)^{e_3} x^{e_2} \\ &\quad + (1-x)^{e_3} x^{e_2} (-1)^{e_2 n_1} c_{\downarrow \uparrow \uparrow} B(x; -e_2, 1-e_3) e_4). \end{aligned}$$

These correlators satisfy the equality (6.12) provided $d_1 = d_2$. The constraint (6.13) then holds if we also impose $d_3 = d_1$. To verify the latter result, we use properties of the incomplete beta function under shifts of its arguments

$$\begin{aligned} B(x, 1-e_2, -e_3) &= \frac{e_2}{e_2+e_3} B(x; -e_2, -e_3) + \frac{1}{e_2+e_3} x^{-e_2} (1-x)^{-e_3} \\ B(x, -e_2, 1-e_3) &= \frac{e_3}{e_2+e_3} B(x; -e_2, -e_3) - \frac{1}{e_2+e_3} x^{-e_2} (1-x)^{-e_3}. \end{aligned}$$

Alternatively, we can link the left correlators to the invariants we computed previously, and write the correlators in terms of the functions G_i . This links the integration constants c_2, c_3 to the constants $d_i, c_{\downarrow \uparrow \uparrow}$ used above. The following two correlators are sufficient to perform the matching

$$\begin{aligned} \langle b \downarrow \uparrow \uparrow \rangle &= -\frac{e_2}{e_3} c_3 x^{e_2 n_1} (1-x)^{e_3 n_1} \\ \langle l \uparrow \downarrow \downarrow \rangle &= (1-x)^{e_3(1+n_1)} x^{e_2(1+n_1)} (-c_2 e_2 - B(x, 1-e_2, -e_3) c_3 e_4). \end{aligned}$$

We have understood the constraint equations and the properties of our correlators at the second step of our Knizhnik-Zamolodchikov staircase. We climb the last step in our study

of the top correlators. An example Knizhnik-Zamolodchikov equation for a top correlator is

$$\begin{aligned} \partial_1 \langle t \uparrow \uparrow \downarrow \rangle = & -\langle l \downarrow \uparrow \downarrow \rangle / z_{12} + n_1 e_2 \langle t \uparrow \uparrow \downarrow \rangle / z_{12} + \langle l \uparrow \downarrow \downarrow \rangle / z_{13} + n_1 e_3 \langle t \uparrow \uparrow \downarrow \rangle / z_{13} \\ & + e_4 \langle r \uparrow \uparrow \uparrow \rangle / z_{14} + n_1 e_4 \langle t \uparrow \uparrow \downarrow \rangle / z_{14} . \end{aligned} \quad (6.22)$$

The relation between the top correlators and the coefficients of the (top partners of the) invariants is

$$\begin{aligned} \langle t \uparrow \uparrow \downarrow \rangle &= \frac{e_2 + e_3}{e_3} G_1 + G_3 \\ \langle t \uparrow \downarrow \uparrow \rangle &= G_1 + G_3 \\ \langle t \downarrow \uparrow \uparrow \rangle &= -\frac{e_2}{e_3} G_1 + \frac{1}{e_3} G_2 . \end{aligned} \quad (6.23)$$

From these equations we can check the relation

$$\langle t \uparrow \uparrow \downarrow \rangle + \langle t \downarrow \uparrow \uparrow \rangle - \langle t \uparrow \downarrow \uparrow \rangle = \frac{e_3}{e_2} (1 - x)^{e_3 n_1} x^{e_2 n_1} = -\frac{1}{e_2} \langle b \downarrow \uparrow \uparrow \rangle \quad (6.24)$$

and the final constraint (6.15) is satisfied. All top correlators involve the generalized hypergeometric function incorporated in the special function G_1 . We checked that the combination (6.23) satisfies the Knizhnik-Zamolodchikov equation (6.22). We thus reconstructed step-by-step the chiral correlation functions summarized in the beginning of this subsection.

7 Local correlators

In this section, we glue together the chiral four point correlators of section 6 to obtain local four point correlation functions of the conformal field theory.

7.1 Four typicals at $\sum n_i + 1 = 0$

A typical diagonal four point function is given by the modulus square of the diagonal holomorphic result

$$C_{4,0;1}(x) = c|x|^{2q_{12}}|1 - x|^{2q_{13}} . \quad (7.1)$$

We see that $2 \rightarrow 3$ interchange is equivalent to $x \leftrightarrow 1 - x$ interchange in the function $C(x)$. That is a check on crossing symmetry.

7.2 Four typicals at $\sum n_i = 0$

We approach the correlator in two ways. We first determine a local four point function by analyzing single-valuedness of a proposed left-right correlator near $x = 0$ and $x = \infty$. In a second approach, we check crossing symmetry of the correlator under the interchange of insertions 2 and 3.

7.2.1 Locality

We can determine the four point function of four typical representations by demanding locality at $x = 0$ and $x = \infty$ (and consequently, at $x = 1$). To that end, we use the hypergeometric function transformation formulas¹¹

$$F(a, b, c; z) = \Gamma\left(\frac{c, b-a}{b, c-a}\right) (-z)^{-a} F(a, 1-c+a, 1-b+a; z^{-1}) \\ + \Gamma\left(\frac{c, a-b}{a, c-b}\right) (-z)^{-b} F(b, 1-c+b, 1-a+b; z^{-1}). \quad (7.2)$$

For convenience, we concentrate on the second channel only. We work in the set-up where $e_2 + e_3 \neq 0$. We can identify combinations of left- and right-moving blocks that are local at $x = 0$ (which consists of linear combinations of the modulus of G_2^1 and the modulus of G_2^2), with the modulus of the combinations local at $x = \infty$. We find that the combination that is both local at zero and at infinity, equal to

$$G(x, \bar{x}) = X_{11}|G_2^1|^2 + X_{22}|G_2^2|^2 \quad (7.3)$$

has a ratio of coefficients

$$X_{11}/X_{22} = -\Gamma\left(\frac{1+e_1, 1-e_3, -e_1-e_2, 1+e_2, 1+e_1+e_2+e_3, -e_1-e_2}{-e_1, e_3, 2+e_1+e_2, 2+e_1+e_2, -e_2, -e_1-e_2-e_3}\right). \quad (7.4)$$

7.2.2 Crossing symmetry

It will be interesting to further study explicitly whether the four point function satisfies crossing symmetry. The four point function is given by

$$G(x, \bar{x}) = \sum_{i,j=1,2} \tilde{I}_i \tilde{I}_j G_{i,j}(x, \bar{x}) \\ = \sum_{i,j=1,2} \tilde{I}_i \tilde{I}_j \sum_{n,m=1,2} X_{nm} G_i^m(x) G_j^m(\bar{x}), \quad (7.5)$$

where $\tilde{I}_i$ denote again the matrix of coefficients of the two (top partners of the) invariants in this channel, and $G_i^m(x)$ the holomorphic solutions to the Knizhnik-Zamolodchikov equations. We will immediately exploit that for single-valuedness near $x = 0$, we must have that X_{nm} is diagonal in $1, 2$. We thus have

$$G(x, \bar{x}) = \sum_{i,j=1,2} \tilde{I}_i \tilde{I}_j \sum_{n=1,2} X_{nn} G_i^n(x) G_j^n(\bar{x}). \quad (7.6)$$

Crossing symmetry requires that

$$G(1-x, 1-\bar{x}; 32) = G(x, \bar{x}; 23), \quad (7.7)$$

where we indicated explicitly that we should exchange all quantities associated to insertions 2 and 3 in the correlation function. It is important to note now that exchanging insertions

¹¹We often denote a ratio of products of Gamma-functions by $\frac{\Gamma(a_1)\dots\Gamma(a_n)}{\Gamma(b_1)\dots\Gamma(b_n)} = \Gamma(\frac{a_1, \dots, a_n}{b_1, \dots, b_n})$.

2 and 3 has the effect of exchanging the two invariants of the global symmetry group that we handily chose in appendix A.4.5. We thus must have that

$$G_{i,j}(x, \bar{x}; 23) = G_{3-i,3-j}(1-x, 1-\bar{x}; 32). \quad (7.8)$$

To compute the transformation rule for our holomorphic or anti-holomorphic constituents, we use the hypergeometric function transformation formulas

$$\begin{aligned} F(a, b; c; x) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F(a, b; a+b-c+1; 1-x) \\ &\quad + \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} (1-x)^{c-a-b} F(c-a, c-b; c-a-b+1; 1-x). \end{aligned} \quad (7.9)$$

We can introduce coefficients

$$G_i^n(1-x; 2 \leftrightarrow 3) = \sum_m c_{nm} G_{3-i}^m(x), \quad (7.10)$$

which are independent of the coefficient i , and obtain

$$\begin{aligned} c_{11} &= -\frac{\Gamma(2+e_1+e_3)\Gamma(-e_1-e_2-1)}{\Gamma(1+e_3)\Gamma(-e_2)} \\ c_{12} &= -\frac{\Gamma(2+e_1+e_3)\Gamma(1+e_1+e_2)}{\Gamma(1+e_1)\Gamma(1+e_1+e_2+e_3)} \frac{1}{e_3} \\ c_{21} &= \frac{\Gamma(-e_1-e_3)\Gamma(-e_1-e_2-1)}{\Gamma(-e_1)\Gamma(-e_1-e_2-e_3)} e_2 \\ c_{22} &= \frac{\Gamma(-e_1-e_3)\Gamma(1+e_1+e_2)}{\Gamma(1-e_3)\Gamma(e_2)}. \end{aligned} \quad (7.11)$$

We require

$$\begin{aligned} G(x, \bar{x}) &= \sum_{i,j} X_{mm} G_i^m(x) G_j^m(x) \tilde{I}_i \tilde{\bar{I}}_j \\ &= \sum_{i,j} X_{mm} G_i^m(1-x) G_j^m(1-x) (2 \leftrightarrow 3) \tilde{I}_{3-i} \tilde{\bar{I}}_{3-j} \\ &= \sum_{i,j} X_{11} (2 \leftrightarrow 3) c_{1m} G_{3-i}^m(x) c_{1n} G_{3-j}^n(x) \\ &\quad + X_{22} (2 \leftrightarrow 3) c_{2m} G_{3-i}^m(1-x) c_{2n} G_{3-j}^n(1-x) \tilde{I}_{3-i} \tilde{\bar{I}}_{3-j}. \end{aligned} \quad (7.12)$$

From these constraints, we find the equalities

$$\begin{aligned} X_{11} (2 \leftrightarrow 3) c_{11} c_{12} + X_{22} (2 \leftrightarrow 3) c_{21} c_{22} &= 0 \\ X_{11} (2 \leftrightarrow 3) c_{11}^2 + X_{22} (2 \leftrightarrow 3) c_{21}^2 &= X_{11} \\ X_{11} (2 \leftrightarrow 3) c_{12}^2 + X_{22} (2 \leftrightarrow 3) c_{22}^2 &= X_{22}. \end{aligned} \quad (7.13)$$

The first equation has the solution

$$X_{11} = -X_{22} \Gamma \left(\frac{1+e_1, 1-e_3, 1+e_2, 1+e_1+e_2+e_3, -e_1-e_2, -e_1-e_2}{-e_1, e_3, 2+e_1+e_2, 2+e_1+e_2, -e_2, -e_1-e_2-e_3} \right).$$

This equation is identical to the constraint (7.4) we found from locality. The functional dependence of the four point function is fixed by both requirements. Crossing symmetry puts a further constraint on how the overall normalization behaves under the exchange of the second and third insertion. We have to solve the equation

$$X_{11} = X_{11}(2 \leftrightarrow 3) \frac{\Gamma(-e_1 - e_2 - 1)^2 \Gamma(2 + e_1 + e_3)^2}{\Gamma(-e_2)^2 \Gamma(1 + e_3)^2} \times \\ \times \csc(\pi e_2) \csc(\pi e_3) \sin(\pi(e_1 + e_2)) \sin(\pi(e_1 + e_3)). \quad (7.14)$$

One solution is

$$X_{11} = \Gamma \left(\frac{-e_3, 1 + e_2}{2 + e_1 + e_2, -e_1 - e_3 - 1} \right). \quad (7.15)$$

We can multiply it by any function symmetric in the second and third momentum. Our four point function satisfies both locality and crossing symmetry, for any value of the external momenta.

Remarks. In [1], crossing symmetry and locality of the correlator of four typical representations is analyzed in the limit $e_2 + e_3 \rightarrow 0$. This limit gives rise to a logarithm in the chiral block. In this limit, two typical representations tensor into a projective representation. The factorization of the four point function into three point functions in this regime was analyzed in [3].

7.3 The local projective and three typical correlator

In this subsection, we construct local four point correlation functions for a top state in a projective representation accompanied by three typical current algebra primaries. In particular, we will study the correlator $\langle t \uparrow \uparrow \downarrow \rangle$. We already determined that on the left it is of the form

$$\langle t \uparrow \uparrow \downarrow \rangle_L = \frac{e_2 + e_3}{e_3} G_1 + G_3 \quad (7.16)$$

and we take it to have the same dependence on the right-movers, with x replaced by $\bar{x}$, i.e. we choose the diagonal vertex operator. The constants appearing in the left chiral correlator become $\bar{x}$ dependent in the full correlator. The full local four-point correlation function then satisfies both the left and the right Knizhnik-Zamolodchikov equations, and we need to glue the solutions into a local four-point function.

In order to find local four-point functions, we use again the technique of analyzing the chiral solutions near $x = 0$ and near $x = 1$, to find combinations with phase monodromy. We can then take a linear combination of the modulus squared of these solutions to have a local solution near a given point. When we find a combination that is local at both $x = 0$ and $x = 1$, we have a globally local solution. A small complication is that logarithms lead to shift monodromies, which we need to cancel through recombination.

In our three step staircase solutions (6.8), we find the following behaviours near $x = 0$ and near $x = 1$

$$\begin{aligned} \text{near } x = 0 : \quad & x^{e_2 n_1}, & x^{e_2(n_1+1)}, & x^{e_2 n_1} \log x \\ \text{near } x = 1 : \quad & (1-x)^{e_3 n_1}, & (1-x)^{e_3(n_1+1)}, & (1-x)^{e_3 n_1} \log(1-x). \end{aligned} \quad (7.17)$$

We have that the function multiplying c_2 corresponds to the $x^{e_2(n_1+1)}$ behaviour, the function multiplying c_1 to the $x^{e_2 n_1}$ limiting behaviour, and we tune the coefficients to

$$c_3 = c_1 \frac{e_2 \csc(\pi e_3)}{(e_2 + e_3) (\pi \csc(\pi e_2) \csc(\pi(e_2 + e_3)) + \csc(\pi e_3) (\psi(e_2) - \psi(e_2 + e_3)))} \quad (7.18)$$

and $c_2 = 0$ to obtain the logarithmic behaviour near $x = 0$. Near $x = 1$, we similarly find the simple block multiplying c_1 with $(1 - x)^{e_3 n_1}$ limiting behaviour, and the linear combinations obeying

$$c_2 = c_1 \frac{\sin(\pi e_3) \Gamma(-e_3) \Gamma(e_2 + e_3 + 1)}{\pi \Gamma(e_2 + 1)}, \quad (7.19)$$

with $c_3 = 0$ of $(1 - x)^{e_3(n_1+1)}$ behaviour, and

$$\begin{aligned} c_3 &= c_1 \frac{e_2}{\pi e_2 (\cot(\pi e_2) + \cot(\pi e_3)) + e_2 \psi(e_3) + e_3 \psi(-e_2 + 1) - (e_2 + e_3) \psi(-e_2 - e_3 + 1)} \\ c_2 &= \frac{c_1}{\pi} \Gamma(1 - e_2) \Gamma(-e_3) \Gamma(e_2 + e_3 + 1) \\ &\quad \times \left(\pi \csc(\pi e_2) \csc(\pi e_3) e_2 + \csc(\pi(e_2 + e_3)) \psi(e_3) e_2 \right. \\ &\quad \left. + \csc(\pi(e_2 + e_3)) \psi(-e_2 + 1) e_3 - \csc \pi(e_2 + e_3) \psi(-e_2 - e_3 + 1) (e_2 + e_3) \right)^{-1}, \end{aligned} \quad (7.20)$$

gives rise to a logarithmic limit. We then define the blocks with the above limiting behaviours (and c_1 equal to one, or $c_2 = 1$ when $c_1 = 0$), near $x = 0$ and near $x = 1$. We denote them $B_{\log}^{0,1}, B_{n_1}^{0,1}, B_{n_1+1}^{0,1}$. To obtain four-point functions without monodromy, we define and equate the combinations

$$\begin{aligned} c_{\log}^0 (B_{\log}^0 \bar{B}_{n_1}^0 + B_{n_1}^0 \bar{B}_{\log}^0) + c_{n_1}^0 |B_{n_1}^0|^2 + c_{n_1+1}^0 |B_{n_1+1}^0|^2 = \\ = c_{\log}^1 (B_{\log}^1 \bar{B}_{n_1}^1 + B_{n_1}^1 \bar{B}_{\log}^1) + c_{n_1}^1 |B_{n_1}^1|^2 + c_{n_1+1}^1 |B_{n_1+1}^1|^2. \end{aligned} \quad (7.21)$$

We find the constraints

$$\begin{aligned} c_{\log}^1 &= \frac{c_{\log}^0 \csc(\pi e_3)}{(e_2 + e_3) (\pi \csc(\pi e_2) \csc(\pi(e_2 + e_3)) + \csc(\pi e_3) (\psi(e_2) - \csc(\pi e_3) \psi(e_2 + e_3)))} \\ &\quad \times \left(e_3 (\psi(-e_2 + 1) - \psi(-e_2 - e_3 + 1)) \right. \\ &\quad \left. + e_2 (-\psi(-e_2 - e_3 + 1) + \pi \cot(\pi e_2) + \pi \cot(\pi e_3) + \psi(e_3)) \right) \\ c_{n_1+1}^1 &= \frac{\pi^2 c_{n_1+1}^0 \csc^2(\pi e_3) \Gamma(e_2 + 1)^2}{\Gamma(-e_3)^2 \Gamma(e_2 + e_3 + 1)^2} \\ c_{n_1}^1 &= -c_{n_1+1}^1 - 2c_{\log}^1 + 2c_{\log}^0 + c_{n_1}^0 \end{aligned} \quad (7.22)$$

which depend on our normalizations, as well as the final relation

$$\begin{aligned} c_{\log}^0 &= -\frac{\pi^2 c_{n_1+1}^0 (e_2 + e_3) \csc \pi(e_2 + e_3) \Gamma(e_2 + 1)}{\Gamma(1 - e_2) \Gamma(-e_3)^2 \Gamma(e_2 + e_3 + 1)^2} \\ &\quad \times (\pi \csc(\pi e_2) \csc(\pi(e_2 + e_3)) + \csc(\pi e_3) (\psi(e_2) - \psi(e_2 + e_3))) . \end{aligned}$$

The block $|B_{n_1}^0|^2$ is local at both 0 and 1, and we therefore find no constraint on the coefficient $c_{n_1}^0$ from locality. This term will also be crossing symmetric by itself. Otherwise, the full functional dependence of the four-point correlator is fixed by the requirement of locality.

We remark that the four-point function continues the pattern of the three-point function for one projective and two typical representations in that it is a combination of polygamma functions, and more particularly, harmonic numbers coded in digamma functions with given behaviour under integer shifts of the argument.

8 Concluding remarks

In this paper, we continued the study of the $gl(1|1)$ Wess-Zumino-Witten model. We exploited that the quadratic Casimir acting on correlators is equal to zero, and that the only non-zero one point functions are logarithmic partners of group invariants. The spectrum of the model consists of typical and projective representations, and we explicitly exhibited correlation functions of projective representations in a covariant manner. We provided a more general check of the crossing symmetry of typical four point functions and computed a local projective and three typical four point function, by solving a three step Knizhnik-Zamolodchikov staircase.

Although the $gl(1|1)$ Wess-Zumino-Witten model is well-studied, there remain a slew of open questions. An open task is to compute all three point functions and four point functions, and to show factorization on the proposed spectrum in full generality. It would also be instructive to map our Knizhnik-Zamolodchikov analysis onto the differential equations governing symplectic fermion correlators in detail, by gauging global symmetries. In short, there is more groundlaying work to do on even the simplest of supergroup Wess-Zumino-Witten models.

This task is important since what we learn in the most basic of logarithmic conformal field theories with supergroup symmetry is bound to bare fruit in its diverse fields of application, including e.g. string theory on AdS_3 backgrounds, possibly with Ramond-Ramond flux.

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A The $gl(1|1)$ algebra and representation theory

We recall the part of the representation theory of the $gl(1|1)$ algebra that we need in our analysis of the Wess-Zumino-Witten model. The algebra is the algebra of endomorphisms of $\mathbb{C}^{1|1}$. These can be represented by two by two matrices. We can provide a basis E_{ij} with $i, j = 1, 2$, and non-zero entry equal to one at the position ij . We define

$$\begin{aligned} E &= E_{11} + E_{22}, & N &= \frac{1}{2}(E_{11} - E_{22}) \\ \Psi^+ &= E_{12}, & \Psi^- &= E_{21}. \end{aligned} \tag{A.1}$$

The $gl(1|1)$ algebra is $\mathbb{Z}$ graded, with two bosonic generators E, N of grade zero, and two fermionic generators $\Psi^\pm$ of grade plus and minus one respectively. We recall the non-zero commutation relations of the $gl(1|1)$ algebra

$$\{\Psi^+, \Psi^-\} = E, \quad [N, \Psi^\pm] = \pm \Psi^\pm. \quad (\text{A.2})$$

A.1 The representations

We study representations in which we diagonalize the actions of the central generator E and the bosonic generator N . (See e.g. [22] for a summary.) The two-dimensional typical representation $T_{n-1/2, e}$ has a state with weights $(N, E) = (n, e)$ and one with eigenvalues $(n-1, e)$, where the eigenvalue e is non-zero. We can represent the generators by the two by two matrices

$$N = \begin{pmatrix} n-1 & 0 \\ 0 & n \end{pmatrix} \quad \Psi^+ = \begin{pmatrix} 0 & 0 \\ e & 0 \end{pmatrix} \quad \Psi^- = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \quad (\text{A.3})$$

The highest weight state can be a boson or a fermion. We take the convention that for the constituent vertex operators in the typical representations, the highest weight (up) state is a fermion and the down state is a boson.¹² We will denote the two states through

$$|\uparrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad |\downarrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad (\text{A.4})$$

such that

$$\Psi^+|\downarrow\rangle = e|\uparrow\rangle \quad \Psi^-|\uparrow\rangle = |\downarrow\rangle. \quad (\text{A.5})$$

Other projective representations arise when E has eigenvalue zero in the representation. The projective cover is then a four-dimensional representation. The generators can be represented by the four by four matrices:

$$N = \begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n+1 & 0 & 0 \\ 0 & 0 & n-1 & 0 \\ 0 & 0 & 0 & n \end{pmatrix} \quad \Psi^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$E = 0 \quad \Psi^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \quad (\text{A.6})$$

One notation that we will use for the states is

$$|\text{top}\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad |\text{left}\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad |\text{right}\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad |\text{bottom}\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad (\text{A.7})$$

¹²The tensor products representations inherit their parity from the constituting factors.

such that we have

$$\begin{aligned}\Psi^+|\text{top}\rangle &= |\text{left}\rangle & \Psi^-|\text{top}\rangle &= |\text{right}\rangle \\ \Psi^+|\text{right}\rangle &= |\text{bottom}\rangle & \Psi^-|\text{left}\rangle &= -|\text{bottom}\rangle.\end{aligned}\tag{A.8}$$

These projective representations are assembled from four atypical one-dimensional representations of the algebra. In our convention, top states are bosons. There is one state in these representations which is a trivial representation of the group. It is the bottom state in the projective representation P_0 .

A.2 Tensor product decompositions

It will be important to us to identify the occurrence of this invariant state (or, more precisely, its partner top state) inside the tensor product of various typical and projective representations. To find the projective representations P_0 , it suffices to use the tensor product rules (see e.g. [22])

$$\begin{aligned}T_{n_1,e_1} \otimes T_{n_2,-e_1} &= P_{n_1+n_2} \\ T_{n_1,e_1} \otimes T_{n_2,e_2} &= T_{n_1+n_2+1/2,e_1+e_2} \oplus T_{n_1+n_2-1/2,e_1+e_2} \\ T_{n_1,e_1} \otimes P_{n_2} &= T_{n_1+n_2+1,e_1} \oplus 2T_{n_1+n_2,e_1} \oplus T_{n_1+n_2-1,e_1} \\ P_{n_1} \otimes P_{n_2} &= P_{n_1+n_2+1} \oplus 2P_{n_1+n_2} \oplus P_{n_1+n_2-1}.\end{aligned}\tag{A.9}$$

A.3 Counting channels

We obtain the tensor product decompositions — assuming that the total sum of the E -momenta e_i is equal to zero:

$$\begin{aligned}T_{n_1,e_1} \otimes T_{n_2,e_2} \otimes T_{n_3,e_3} &= P_{n_1+n_2+n_3+\frac{1}{2}} \oplus P_{n_1+n_2+n_3-\frac{1}{2}} \\ T_{n_1,e_1} \otimes T_{n_2,-e_1} \otimes P_{n_3} &= P_{n_1+n_2+n_3+1} \oplus 2P_{n_1+n_2+n_3} \oplus P_{n_1+n_2+n_3-1} \\ T_{n_1,e_1} \otimes T_{n_2,e_2} \otimes T_{n_3,e_3} \otimes T_{n_4,e_4} &= P_{\sum n_i+1} \oplus 2P_{\sum n_i} \oplus P_{\sum n_i-1} \\ P_{n_1} \otimes T_{n_2,e_2} \otimes T_{n_3,e_3} \otimes T_{n_4,e_4} &= P_{\sum n_i+3/2} \oplus 3P_{\sum n_i+1/2} \oplus 3P_{\sum n_i-1/2} \oplus P_{\sum n_i-3/2}.\end{aligned}$$

These give rise to the following counting of possible non-zero channels in correlation functions: In appendix A.4 we explicitly identify the invariant states in the projectives P_0 that appear in the tensor products of these representations.

A.4 Invariant combinations

In this subsection, we identify states in tensor products which are invariant under the action of the diagonal algebra. These invariant states live at the bottom of projective P_0 representations. We counted the occurrences of the projective P_0 representations in table 1. We will exhibit invariant linear combinations of states below in each instance. These invariant combinations feature as specific choices of basis in the bulk of the paper.

$\sum n_i$	0	± 1	$\pm \frac{1}{2}$	$\pm \frac{3}{2}$
TT	1	-	-	-
PP	2	1	-	-
TTT	-	-	1	-
PTT	2	1	-	-
TTTT	2	1	-	-
PTTT	-	-	3	1

Table 1. Counting projective P_0 representations in tensor products.

A.4.1 Two typicals

In the case of the tensor product of two typical representations satisfying $e_1 + e_2 = 0$ and $n_1 + n_2 = 0$, the invariant combination is (proportional to)

$$I_{2,0;0} = \downarrow\uparrow - \uparrow\downarrow. \quad (\text{A.10})$$

The lower indices indicate the number of typical representations in the tensor product, the number of projective covers P , and the negative of the sum of all quantum numbers n_i labeling the representations.

A.4.2 Two projectives

In the case of two projective P_n representations, we distinguish three cases (and four invariants). We write down the invariant combinations for the cases $n_1 + n_2 + 1 = 0$ and $n_1 + n_2 = 0$ (with the case $n_1 + n_2 - 1 = 0$ mirroring what we find for $n_1 + n_2 + 1 = 0$). For $n_1 + n_2 + 1 = 0$, we have

$$I_{0,2;1} = bl - lb \quad (\text{A.11})$$

and for $n_1 + n_2 = 0$

$$\begin{aligned} I_{0,2;0}^1 &= bb \\ I_{0,2;0}^2 &= bt + lr - rl + tb. \end{aligned} \quad (\text{A.12})$$

In this case, there are two invariants, and we have explicitly chosen a basis of invariants, where the upper label serves as an enumeration of invariants.

A.4.3 Three typicals

For the cases $\sum n_i \pm 1/2 = 0$, we have the invariants

$$\begin{aligned} I_{0,3;-1/2} &= e_1 \uparrow\downarrow\downarrow + e_2 \downarrow\uparrow\downarrow + e_3 \downarrow\downarrow\uparrow \\ I_{0,3;1/2} &= \downarrow\uparrow\uparrow - \uparrow\downarrow\uparrow + \uparrow\uparrow\downarrow. \end{aligned} \quad (\text{A.13})$$

A.4.4 One projective and two typicals

We put the projective representation in the first factor. In the two channel $\sum n_i = 0$ case, we have the invariants

$$\begin{aligned} I_{2,1,0}^1 &= b\uparrow\downarrow + l\downarrow\downarrow - e_2(r\uparrow\uparrow + t\downarrow\uparrow - t\uparrow\downarrow) \\ I_{2,1,0}^2 &= b\downarrow\uparrow - b\uparrow\downarrow. \end{aligned} \quad (\text{A.14})$$

A.4.5 Four typicals

Let us recall that we assume the total sum of the eigenvalues e_i is zero. We have the cases $\sum n_i \pm 1 = 0$ with one invariant and $\sum n_i = 0$ with two. If we concentrate on the invariant state with $\sum n_i + 1 = 0$, we find that it is proportional to

$$I_{4,0;1} = -\downarrow\uparrow\uparrow\uparrow + \uparrow\downarrow\uparrow\uparrow - \uparrow\uparrow\downarrow\uparrow + \uparrow\uparrow\uparrow\downarrow. \quad (\text{A.15})$$

For $\sum n_i = 0$, we find two invariant channels. The invariant linear combinations of states are

$$\begin{aligned} I_{4,0;0}^1 &= e_3(\downarrow\downarrow\uparrow\uparrow - \downarrow\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow\downarrow) - e_1(\downarrow\uparrow\downarrow\uparrow - \uparrow\downarrow\downarrow\uparrow + \uparrow\uparrow\downarrow\downarrow) \\ I_{4,0;0}^2 &= -e_2(\downarrow\downarrow\uparrow\uparrow - \downarrow\uparrow\downarrow\uparrow + \downarrow\uparrow\uparrow\downarrow) - e_1(\downarrow\downarrow\uparrow\uparrow - \uparrow\downarrow\downarrow\uparrow + \uparrow\downarrow\uparrow\downarrow). \end{aligned} \quad (\text{A.16})$$

A.4.6 One projective and three typicals

Let us consider the case in which we have three possible channels, and where $\sum n_i + 1/2 = 0$. We then solve for the three invariant channels in this four factor tensor product space and find:

$$\begin{aligned} I_{3,1;1/2}^1 &= -\frac{e_3}{e_2}(-b\uparrow\downarrow\uparrow + b\uparrow\uparrow\downarrow - l\downarrow\downarrow\uparrow + l\downarrow\uparrow\downarrow + e_2(r\uparrow\uparrow\uparrow + t\downarrow\uparrow\uparrow - t\uparrow\downarrow\uparrow + t\uparrow\uparrow\downarrow)) \\ I_{3,1;1/2}^2 &= -e_3(b\downarrow\uparrow\uparrow - b\uparrow\downarrow\uparrow + b\uparrow\uparrow\downarrow) \\ I_{3,1;1/2}^3 &= \frac{1}{e_2}(e_3(-b\uparrow\downarrow\uparrow + b\uparrow\uparrow\downarrow - l\downarrow\downarrow\uparrow + l\downarrow\uparrow\downarrow) - e_2(b\uparrow\downarrow\uparrow + l\downarrow\downarrow\uparrow - l\uparrow\downarrow\downarrow)). \end{aligned} \quad (\text{A.17})$$

These parameterizations of channels are used in the bulk of the paper.

A.5 The tensor products of generators

In this subsection, we compute the action of the tensor products of generators appearing in the Knizhnik-Zamolodchikov equation on the invariants as parameterized in appendix A.4.

A.5.1 Two typicals

The matrix coefficient Q_{12} for acting on $I_{2,0;0}$ with the operator Q_{12} is equal to $Q_{12} = -(h_{e_1, n_1} + h_{e_2, n_2})$, namely the negative of the sum of conformal dimensions of the first and second insertion.

A.5.2 Two projectives

For the $n_1 + n_2 + 1$ channel, we find a zero coefficient, while for the $n_1 + n_2 = 0$ channel, we find the matrix

$$Q_{12} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} \quad (\text{A.18})$$

acting on the basis of invariants $I_{0,2;0}^i$, namely the operator $Q_{1\otimes 2}$ maps the second invariant into 2 times the first.

A.5.3 One projective and two typicals

In the two channel case we find the maps

$$\begin{aligned} Q_{12} &= \begin{pmatrix} e_2 n_1 & -e_2 \\ 0 & e_2 n_1 \end{pmatrix} \\ Q_{13} &= \begin{pmatrix} -e_2 n_1 & -e_2 \\ 0 & -e_2 n_1 \end{pmatrix}. \end{aligned} \quad (\text{A.19})$$

A.5.4 Four typicals

In the one channel $\sum n_i + 1 = 0$ case, we find that $Q_{12} = h_{e_1+e_2, n_1+n_2+1/2} - h_{e_1, n_1} - h_{e_2, n_2}$. In the two channel $\sum n_i = 0$ case, we have

$$\begin{aligned} Q_{12} &= \begin{pmatrix} \frac{e_2(2n_1+1)+e_1(2e_2+2n_2+1)}{2} & e_3 \\ 0 & \frac{e_2(2n_1-1)+e_1(2e_2+2n_2-1)}{2} \end{pmatrix} \\ Q_{13} &= \begin{pmatrix} \frac{e_3(2n_1-1)+e_1(2e_3+2n_3-1)}{2} & 0 \\ e_2 & \frac{e_3(2n_1+1)+e_1(2e_3+2n_3+1)}{2} \end{pmatrix}. \end{aligned} \quad (\text{A.20})$$

A.5.5 One projective and three typicals

The transformation matrices associated to the operators $Q_{1\otimes 2}$ and $Q_{1\otimes 3}$ are

$$\begin{aligned} Q_{12} &= \begin{pmatrix} e_2 n_1 & 1 & 0 \\ 0 & e_2 n_1 & 0 \\ 0 & 0 & e_2 (n_1 + 1) \end{pmatrix} \\ Q_{13} &= \begin{pmatrix} e_3 n_1 & -\frac{e_3}{e_2} & -e_3 \\ 0 & e_3 n_1 & 0 \\ 0 & \frac{e_3}{e_2} + 1 & e_3 (n_1 + 1) \end{pmatrix}. \end{aligned} \quad (\text{A.21})$$

We plug these matrices into the Knizhnik-Zamolodchikov equations in the bulk of the paper.

B Symplectic fermions

In this appendix, we remind the reader of results in [5], and complement them with some extensions. First of all, we note that we have

$$L_0 \cdot : \chi^1 \chi^2 := 1 \quad (\text{B.1})$$

and we can thus set the rank 2 pair denoted ω, Ω in [5] equal to $\omega =: \chi^1 \chi^2$ and $\Omega = 1$ in our framework. We have the two point functions:¹³

$$\begin{aligned} \langle : \chi^1 \chi^2 : \rangle &= 1 \\ \langle : \chi^1 \chi^2 : : \chi^1 \chi^2 : \rangle &= -2(Z + \log |z_{12}|^2) \\ \langle \mu_\lambda \mu_{1-\lambda} \rangle &= -|z_{12}|^{2\lambda(1-\lambda)} \end{aligned} \quad (\text{B.2})$$

and the three point functions

$$\begin{aligned} \langle \mu_\lambda \mu_{1-\lambda} : \chi^1 \chi^2 : \rangle &= |z_{12}|^{2\lambda(1-\lambda)} \left(Z_\lambda + \log \left| \frac{z_{13} z_{23}}{z_{12}} \right|^2 \right) \\ Z_\lambda &= Z + 2\psi(1) - \psi(\lambda) - \psi(1-\lambda) \\ \langle \mu_{\lambda_1} \mu_{\lambda_2} \mu_{\lambda_3} \rangle &= C_{\lambda_1, \lambda_2, \lambda_3} |z_{12}^{\lambda_1 \lambda_2} z_{13}^{\lambda_1 \lambda_3} z_{23}^{\lambda_2 \lambda_3}|^2 \quad \text{for } \lambda_1 + \lambda_2 + \lambda_3 = 1 \\ &= C_{\lambda_1, \lambda_2, \lambda_3} |z_{12}^{(1-\lambda_1)(1-\lambda_2)} z_{13}^{(1-\lambda_1)(1-\lambda_3)} z_{23}^{(1-\lambda_2)(1-\lambda_3)}|^2 \quad \text{for } \sum \lambda_i = 2 \\ C_{\lambda_1, \lambda_2, \lambda_3} &= \sqrt{\frac{\Gamma(\lambda_1) \Gamma(\lambda_2) \Gamma(\lambda_3)}{\Gamma(1-\lambda_1) \Gamma(1-\lambda_2) \Gamma(1-\lambda_3)}} \end{aligned} \quad (\text{B.3})$$

which we propose to supplement with the chiral expressions

$$\begin{aligned} \langle \chi^1 \mu_{-\lambda+1}^L \mu_{\lambda+1}^L \rangle &= -\frac{1}{\lambda} z_{12}^\lambda z_{13}^{-\lambda} z_{23}^{-\lambda^2} \\ \langle \chi^2 \mu_{-\lambda}^L \mu_\lambda^L \rangle &= -z_{12}^{-\lambda} z_{13}^\lambda z_{23}^{-\lambda^2}. \end{aligned} \quad (\text{B.4})$$

We remark that the results on the four point function of the top primaries in the projective representations discussed in detail in the bulk of the paper can be straightforwardly translated into new results for symplectic fermion correlators.

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¹³We fix the constant O_λ in [5] to $O_\lambda = 1$.

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