Induced gravity II: grand unification

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ABSTRACT: As an illustration of a renormalizable, asymptotically-free model of induced gravity, we consider an SO(10) gauge theory interacting with a real scalar multiplet in the adjoint representation. We show that dimensional transmutation can occur, spontaneously breaking SO(10) to SU(5)⊗U(1), while inducing the Planck mass and a positive cosmological constant, all proportional to the same scale v. All mass ratios are functions of the values of coupling constants at that scale. Below this scale (at which the Big Bang may occur), the model takes the usual form of Einstein-Hilbert gravity in de Sitter space plus calculable corrections. We show that there exist regions of parameter space in which the breaking results in a local minimum of the effective action giving a positive dilaton (mass)^2 from two-loop corrections associated with the conformal anomaly. Furthermore, unlike the singlet case we considered previously, some minima lie within the basin of attraction of the ultraviolet fixed point. Moreover, the asymptotic behavior of the coupling constants also lie within the range of convergence of the Euclidean path integral, so there is hope that there will be candidates for sensible vacua. Although open questions remain concerning unitarity of all such renormalizable models of gravity, it is not obvious that, in curved backgrounds such as those considered here, unitarity is violated. In any case, any violation that may remain will be suppressed by inverse powers of the reduced Planck mass.

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Contents

1 Introduction

The Standard Model (SM) cannot be perturbatively ultra-violet complete simply because of the presence of a U(1) gauge coupling, inevitably leading to a Landau pole. However the SM, when made supersymmetric, or by inclusion of other suitably chosen light states, does suggest the possibility of a gauge unification scale M_X of around 10^{16} GeV, corresponding to new physics based on a gauge group containing as a subgroup SU(3)⊗SU(2)⊗U(1). Models based on this idea typically involve proton decay mediated by particles with unification scale masses; predicting rates close to if not violating experimental limits. The relationship between M_X and the Planck scale $M_P \sim 10^{19}$ GeV (or the reduced Planck mass or string scale $M_P/\sqrt{8\pi}$,) has long been a source of inquiry in the context of efforts to construct an ultimate theory.

One point of view is that the ratio M_X/M_P being $O(10^{-3})$ is a good thing in rendering perturbation theory valid at M_X; another is that the existence of the two nearby scales is un-aesthetic, and the low energy theory should be modified so as to move M_X up to M_P. In either case, the question of the nature of the ultimate theory remains. One approach to this question is string theory. We follow an older path, that of renormalizable quantum field theory (QFT), including gravity [1]. This theory of gravity, sometimes called "R^2" gravity or "higher-derivative" gravity, has another attractive feature inasmuch as it is asymptotically free (AF) [2, 3]. Under certain circumstances, these properties may be extended to include matter in its usual form of scalar, vector, and fermion fields, corresponding to spins $(0, 1, 1/2)$.

Since this paper is a sequel to others [4, 5] along these same lines, we limit describing the motivation for this work to a few other introductory remarks. In addition to renormalizability and AF for all couplings, we, as do the authors of ref. [6], restrict our attention to such extensions that are classically scale invariant. This bequeathes certain naturalness properties to the theory that are essential to avoid issues of fine-tuning, even in the presence of the breaking of scale invariance by the conformal anomaly [7, 8]. It is also aesthetically attractive in that there are no elementary masses to be accounted for, and all mass scales must ultimately be due to dimensional transmutation (DT), whether perturbatively [9], or nonperturbatively, as in Yang-Mills theory [10] or massless QCD.[1] We shall focus exclusively on the perturbative scenario.

In previous work [4, 5], we considered the simplest possible extension of renormalizable gravity, viz., to the inclusion of a single, real scalar field. We showed that such a model can simultaneously generate by DT a scalar vacuum expectation value (VEV) and nonzero scalar curvature R. Moreover the theory has a region of parameter space containing an ultra-violet stable fixed point (UVFP) for coupling constant ratios and is AF in all its coupling constants. Unfortunately, however, the region of parameter space corresponding to DT and a "right-sign" Einstein term ($\xi > 0$) was disjoint from the basin of attraction of the UVFP: starting from the DT region, the couplings did not flow to the UVFP.

In this paper, we extend the results of ref. [4] to the case in which the matter sector includes non-Abelian gauge interactions and non-singlet scalars and fermions for which all the couplings are AF. We show that, not only does the same DT phenomenon occur, but the disappointing outcome mentioned above does not hold; this time there *is* a region of parameter space such that both DT occurs at a local minimum from which the couplings flow to the UVFP. Moreover, both M_P and M_X can be understood in terms of the scalar VEV.

In flat space, if Yukawa couplings are AF, then they usually fall faster than the quartic scalar couplings. There is no guarantee, however, that they are negligible at the DT scale. Our goal in the present effort is not to obtain a completely realistic model but to determine whether we can find any model of this type that realizes all our many constraints,[2] so, for present purposes, we shall ignore possible Yukawa couplings.

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[1]For some recent speculations about strong coupling in this context, see ref. [11]. As we have remarked previously [5], even classically, a $1/q^4$-propagator corresponds to a linearly growing potential, which would therefore be confining.

[2]Models of GUTs within renormalizable gravity were considered long ago [12, 13], but that work did not consider induced gravity or any of the constraints that we impose other than AF in all couplings. Induced gravity in models of GUTs have been previously considered, e.g., in ref. [14], but not in the context of renormalizable gravity with dimensional transmutation.

To summarize our goals: we seek a model that

(1) is AF for values of the couplings that ensure convergence of the EPI at sufficiently high scales,

(2) undergoes DT at some scale, with a locally stable minimum,

(3) is such that a portion of the range of couplings satisfying the preceding constraint lies within the basin of attraction of the UVFP, so that the couplings run from DT solutions to the UVFP. This is where our previous attempts failed.

We shall, in fact, be successful in all these goals.

2 Classically scale invariant gravity

The basic framework for this paper is classically scale invariant quantum gravity, defined by the Lagrangian

$$S_{\text{ho}} = \int d^4x \sqrt{g} \left[\frac{C^2}{2a} + \frac{R^2}{3b} + cG \right], \tag{2.1}$$

where C is the Weyl tensor and G is the Gauss-Bonnet term.[3] Just about the simplest imaginable scale invariant theory involving gravity and matter fields consists of the above, coupled to a single scalar field with a $\lambda\phi^4$ interaction and non-minimal gravitational coupling $\xi R\phi^2$. In recent papers [4, 5], we argued that even this matter-free theory can undergo dimensional transmutation (DT) à la Coleman-Weinberg [9], leading to effective action extrema[4] with nonzero values for $\langle R \rangle$ and $\langle \Phi \rangle$ (or $\langle T_2 \rangle$). However, the extrema are unstable and consequently unacceptable. It is important to emphasize that, as with the original treatment [9] of scalar electrodynamics, we restrict ourselves to DT that can be demonstrated perturbatively; in other words, for values of the relevant dimensionless couplings such that neglect of non-leading quantum corrections can be justified. In this paper we shall take the matter action to be that of a gauge field of a simple group, with a real scalar field in the adjoint representation:[5]

$$S_m = \int d^4x \sqrt{g} \left[\frac{1}{4}\text{Tr}[F_{\mu\nu}^2] + \frac{1}{2}\text{Tr}[(D_\mu\Phi)^2] - \frac{\xi\text{Tr}[\Phi^2]}{2}R + V_J(\Phi) \right], \tag{2.2}$$

where $\Phi = \sqrt{2}T^a\phi^a$ with ϕ_a real, $D_\mu\Phi \equiv \partial_\mu\Phi + ig[A_\mu, \Phi]$, $A_\mu \equiv \sqrt{2}T^aA_\mu^a$, and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$. By definition, the generators T^a are Hermitian and conventionally taken to be in the defining or fundamental representation of the group, normalized so that $\text{Tr}[T^aT^b] = \delta^{ab}/2$. Thus, with our conventions, $\text{Tr}[\Phi^2] = \sum(\phi_a)^2$. We take the potential to

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[3]We work in Euclidean spacetime throughout with the curvature conventions given in ref. [5].

[4]We use the term "extrema" to refer to stationary points generally, not just maxima and minima.

[5]The generalization to a semi-simple gauge group is straightforward, but U(1) factors are not permitted, since an abelian gauge coupling cannot be asymptotically free.

be

$$V_J(\Phi) \equiv \frac{h_1}{24}T_2^2 + \frac{h_2}{96}T_4, \qquad \text{or} \tag{2.3a}$$

$$V_J(\Phi) \equiv \frac{h_3}{24}T_2^2 + \frac{h_2}{96}\widetilde{T}_4, \qquad \text{where } \widetilde{T}_4 \equiv \left[T_4 - \frac{1}{d_T}T_2^2\right], \tag{2.3b}$$

where $T_n \equiv \text{Tr}[\Phi^n]$ and d_T is the dimension of the fundamental representation T^a. For SO(N) (and SU(N)), $d_T = N$ for their fundamental representations. The relation between the couplings in the two expressions is $h_3 \equiv h_1 + h_2/(4d_T)$. It can be easily shown that $T_4 \geq T_2^2/d_T$, so that $\widetilde{T}_4 \geq 0$.

Classically, for the potential to be bounded below, one must have $h_2 > 0$ and $h_3 > 0$. In the QFT, it is unclear at what scale this is required of the renormalized couplings $\{h_2(\mu), h_3(\mu)\}$ or, equivalently, that this classical requirement is necessary for the effective action to be bounded below. In fact, because of AF, the classical form of the renormalized action is an increasingly good approximation the larger the scale μ so these constraints are reliable for μ sufficiently large.[6] As the scale μ decreases, one must determine from the renormalization-group-improved effective action how far down in the infrared (IR) direction these inequalities will continue to remain necessary, assuming that it remains within the realm of a perturbative calculation.

To eq. (2.2), we shall add a certain number of massless fermions in representations yet to be specified. For simplicity, we shall ignore possible Yukawa interactions. Without gravitational interactions, it was remarked long ago [15] that, so long as they are themselves asymptotically free, Yukawa couplings vanish more rapidly in the UV than gauge couplings and scalar self-couplings, so their presence does not affect the asymptotic behavior of the other couplings. This conclusion survives the inclusion of the gravitational couplings in the cases we shall consider, although the sign of their contribution does in fact act so as to make the Yukawa couplings vanish *less* rapidly.[7] They could in principle affect the equations for DT in important ways, but to keep things simple, we shall assume they can be neglected down to the DT scale.

3 Beta-functions for an SO(N) model and asymptotic freedom

One attractive property of renormalizable gravity defined by eq. (2.1) is that it is asymptotically free (AF), and this property can be extended to include a matter sector with an asymptotically free gauge theory, or even a non-gauge theory, such as the ones considered previously [4, 5]. This can be seen as follows: at one-loop order, the gauge coupling g and the gravitational couplings a and c do not mix with other couplings. In the general case, their β-functions are[8]

$$\beta_{g^2} = -b_g(g^2)^2, \qquad\qquad \beta_a = -b_2 a^2, \qquad \beta_c = -b_1, \tag{3.1}$$

$$b_g = 2\left(\frac{11}{3}C_G - \frac{2}{3}T_F - \frac{1}{6}T_S\right), \qquad b_2 = \frac{133}{10} + N_a, \qquad b_1 = \frac{196}{45} + N_c, \tag{3.2}$$

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[6]Precisely the same constraint results from demanding convergence of the path integral. See section 5.

[7]With the original form of the beta-functions given, e.g., in ref. [16], they vanish *more* rapidly. As we described in ref. [4], we have adopted the alternative beta-functions given in ref. [6].

[8]We suppress throughout a factor $1/(16\pi^2)$ from all one-loop β-functions.

where $N_a = [N_0 + 3N_F + 12N_V]/60$ and $N_c = \left[N_0 + \frac{11}{2}N_F + 62N_V\right]/360$. Here, N_0 represents the number of real scalars; N_V, the number of massless vector bosons; N_F, the number of Majorana or Weyl fermions. Our Lie algebra conventions are summarized in appendix A. Since $b_2 > 0$, the coupling a is always AF; we may estimate its rate of decline by noting the form of N_a above. Typically, $N_{a,c}$ are dominated by vector bosons and fermions, since scalars are down from vectors by a factor of 12. In the $SO(N)$ model that we consider below, with a single, real, adjoint scalar, $N_a = 13N(N-1)/120 + N_F/20$. As we shall explain shortly, it turns out that there are AF solutions for the scalar couplings only for $N \geq 9$, so $N_a \geq 39/5 + N_F/20$ and $b_2 \geq 211/10 + N_F/20$. (Obviously, this lower bound grows quadratically with increasing N.)

The evolution of the coupling b is more complicated:

$$\beta_b \equiv -a^2 b_3(x, \xi'), \quad b_3(x, \xi') \equiv \left[\frac{10}{3} - 5x + \left(\frac{5}{12} + \frac{3\xi'^2 N_0}{2}\right)x^2\right], \qquad (3.3)$$

where $x \equiv b/a$, and we have introduced $\xi' \equiv \xi + 1/6$. (Whereas $\xi = 0$ for minimal coupling, $\xi' = 0$ for conformal coupling.) Thus, b mixes with the couplings a and ξ', and $\beta_{\xi'}$ depends on the matter self-couplings. Therefore, unlike a, the evolution of b is sensitive to other features of the model.

For reasons explained in ref. [4], we adopt the beta-functions of Salvio and Strumia [6], which differ for matter couplings from earlier results [16]. For the $SO(N)$ case with a single adjoint scalar field, the remaining beta-functions are[9]

$$\beta_{h_1} = \frac{1}{3}\left(\frac{N(N-1)}{2} + 8\right)h_1^2 + \frac{2N-1}{12}h_1 h_2 + \frac{1}{32}h_2^2 - 6(N-2)h_1 g^2 \qquad (3.4a)$$
$$+ 27g^4 + 3\Delta\beta_1 + h_1\Delta\beta_2,$$

$$\beta_{h_2} = 4h_1 h_2 + \frac{2N-1}{24}h_2^2 - 6(N-2)h_2 g^2 + 36(N-8)g^4 + h_2\Delta\beta_2, \qquad (3.4b)$$

$$\Delta\beta_1 = a^2\left(\xi' - \frac{1}{6}\right)^2\left(5 + 9x^2\xi'^2\right), \quad \Delta\beta_2 = a\left(5 - 18x\xi'^2\right), \qquad (3.4c)$$

$$\beta_{\xi'} = \xi'\left(\left(\frac{N(N-1)+4}{6}\right)h_1 + \frac{2N-1}{24}h_2 - 3(N-2)g^2\right) + \Delta\beta_{\xi'}, \qquad (3.4d)$$

$$\Delta\beta_{\xi'} = a\left(\xi' - \frac{1}{6}\right)\left(\frac{10}{3x} - \frac{3}{2}\xi'(2\xi'+1)x\right) = \left(\xi' - \frac{1}{6}\right)\left(\frac{10a^2}{3b} - \frac{3}{2}\xi'(2\xi'+1)b\right). \qquad (3.4e)$$

It is interesting that the gravitational contribution to $\beta_{\xi'}$, viz. $\Delta\beta_{\xi'}$, vanishes for *minimal* coupling, whereas the matter contributions vanish for *conformal* coupling, about which we shall have more to say shortly. We want to examine the possibility of obtaining a theory in which all of the couplings are AF. We must demand $b_g > 0$, so that the gauge coupling is AF. In a certain sense, the evolution of the two couplings a and g^2 control the behavior of the other couplings. To see this, it is useful to rescale the other couplings by one of these two and to express their beta-functions in terms of these ratios; since neither coupling vanishes at any finite scale, we may choose to rescale by either one. In theories

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[9]The flat space beta-functions for $\{\beta_{h_1}, \beta_{h_2}\}$ can be found in ref. [15].

without AF gauge couplings, one must choose a, as we did in our previous papers. In gauge models, it is more convenient [16] to rescale by $\alpha \equiv g^2$ instead, replacing the conventional running parameter $dt = d\ln\mu$ by $du = \alpha(t)dt$. This enables us to easily investigate the impact of gravitational corrections on the flat-space beta-functions. Thus we introduce rescaled couplings:

$$z_1 \equiv \frac{h_1}{\alpha}, \quad z_2 \equiv \frac{h_2}{\alpha}, \quad z_3 \equiv \frac{h_3}{\alpha}, \quad \bar{a} \equiv \frac{a}{\alpha}, \quad \bar{b} \equiv \frac{b}{\alpha}. \tag{3.5}$$

As we shall see, because of the nature of the symmetry breaking of the $SO(N)$ group in this model, it is usually simpler to use the pair $\{z_2, z_3\}$ than $\{z_1, z_2\}$. Of course, $x \equiv b/a = \bar{b}/\bar{a}$, and need not be rescaled. Note that ξ' is not rescaled.[10] If ξ' and the ratios $\{\bar{a}, \bar{b}, z_2, z_3\}$ approach a finite UVFP, then the original couplings $\{\alpha, a, b, h_1, h_3\}$ will all be AF. The rescaled beta-functions, $\overline{\beta}_{\lambda_i}$ correspond to $d\lambda_i/du$. Noting that $\beta_{h_3} = \alpha^2(\overline{\beta}_{z_3} - b_g z_3)$, and $\beta_{h_2} = \alpha^2(\overline{\beta}_{z_2} - b_g z_2)$, we find

$$\overline{\beta}_{\bar{a}} = \bar{a}\left(b_g - \bar{a}b_2\right), \tag{3.6a}$$

$$\overline{\beta}_{\bar{b}} - b_g\bar{b} = -\bar{a}^2 b_3(x,\xi') = \left[-\frac{10}{3}\bar{a}^2 + 5\bar{a}\bar{b} - \left(\frac{5}{12} + \frac{3N(N-1)\xi'^2}{4}\right)\bar{b}^2\right], \tag{3.6b}$$

$$\overline{\beta}_x - b_2 x\bar{a} = -b_3(x,\xi')\bar{a} = \bar{a}\left[-\frac{10}{3} + 5x - \frac{x^2}{12}\left(5 + 9N(N-1)\xi'^2\right)\right], \tag{3.6c}$$

$$\overline{\beta}_{z_2} - b_g z_2 = 36(N-8) + \frac{2N^2 - N - 24}{24N}z_2^2 + 4z_3 z_2 - 6(N-2)z_2 + \overline{\Delta\beta}_2 z_2, \tag{3.6d}$$

$$\overline{\beta}_{z_3} - b_g z_3 = \frac{36(N-2)}{N} + \frac{N(N-1)+16}{6}z_3^2 + \frac{N^2-4}{48N^2}z_2^2 + \frac{N^2-4}{12N}z_3 z_2$$
$$- 6(N-2)z_3 + \overline{\Delta\beta}_2 z_3 + 3\overline{\Delta\beta}_1, \tag{3.6e}$$

$$\overline{\Delta\beta}_1 = \bar{a}^2\left(\xi' - \frac{1}{6}\right)^2\left(5 + 9x^2\xi'^2\right) = \left(\xi' - \frac{1}{6}\right)^2\left(5\bar{a}^2 + 9\bar{b}^2\xi'^2\right), \tag{3.6f}$$

$$\overline{\Delta\beta}_2 = \bar{a}\left(5 - 18x\xi'^2\right) = 5\bar{a} - 18\bar{b}\xi'^2, \tag{3.6g}$$

$$\overline{\beta}_{\xi'} = \xi'\left[\frac{N^2-4}{24N}z_2 + \frac{N(N-1)+4}{6}z_3 - 3(N-2)\right] + \overline{\Delta\beta}_{\xi'}, \tag{3.6h}$$

$$\overline{\Delta\beta}_{\xi'} = \bar{a}\left(\xi' - \frac{1}{6}\right)\left[\frac{10}{3x} - \frac{3}{2}\xi'(2\xi'+1)x\right] = \left(\xi' - \frac{1}{6}\right)\left[\frac{10\bar{a}^2}{3\bar{b}} - \frac{3}{2}\xi'(2\xi'+1)\bar{b}\right]. \tag{3.6i}$$

All dependence on α has disappeared. For historical reasons, we retained the ratio $x \equiv b/a = \bar{b}/\bar{a}$, but it turns out that, to search for candidates for UVFPs, it is usually better to work with $\bar{b}$. Although redundant, we have given both $\overline{\beta}_{\bar{b}}$ and $\overline{\beta}_x$ and expressed the gravitational corrections $\overline{\Delta\beta}_k, (k = 1, 2, \xi')$ in two alternative forms, each of which is useful in different contexts. We shall see shortly that $\bar{b} \to \bar{b}^{(\mathrm{uv})} \sim O(b_g)$, so that $\bar{a}^{(\mathrm{uv})}/\bar{b}^{(\mathrm{uv})} \sim \mathcal{O}(1/b_2) \ll 1$. Inversely, $x = \bar{b}/\bar{a} \to x^{(\mathrm{uv})} = \bar{b}^{(\mathrm{uv})}/\bar{a}^{(\mathrm{uv})} \sim \mathcal{O}(b_2) \gg 1$.

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[10]For asymptotic freedom, we only require $\xi' \to \xi'^{(\mathrm{uv})}$, some finite constant. In that event, we could trivially replace ξ' by $\xi'' \equiv \xi' - \xi'^{(\mathrm{uv})}$, which approached zero. Thus, so long as ξ' approaches *any* finite constant asymptotically, the theory can be said to be AF. We shall also show however that $\xi'^{(\mathrm{uv})}$ is naturally extremely small but nonzero, so that such theories are never asymptotically conformal.
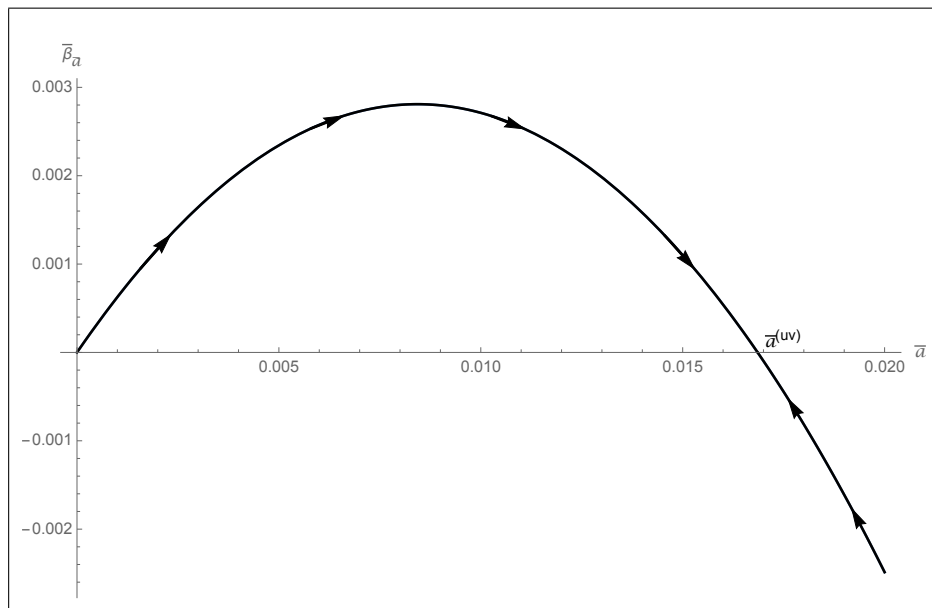
Figure 1. $\overline{\beta}_{\overline{a}}$ showing its UVFP at $\overline{a}^{(\mathrm{uv})}$.

Notice that the gravitational corrections $\{\overline{\Delta\beta}_1, \overline{\Delta\beta}_2\}$ do not depend upon N, so that the dependence of $\overline{\beta}_{z_3}$ and $\overline{\beta}_{z_2}$ on N is determined by the non-gravitational sector. We have shown that, without the gravitational couplings, $\mathrm{SO}(N)$ can in principle have asymptotically free scalar couplings only for[11] $N \geq 9$. (Similarly, $\mathrm{SU}(N)$ with an adjoint scalar is required to have $N \geq 7$.) These conclusions remain unaffected by including the gravitational interactions. As mentioned earlier, Yukawa couplings may usually be added without affecting the asymptotic behavior of the gauge or gravitational couplings $\{\alpha, a, b, \xi'\}$, so long as they themselves are AF.

The challenge now is to determine whether or not these beta-functions in eq. (3.6) have at least one finite, UV-stable, fixed point (FP) in all the parameters. In fact, substantial progress can be made in this simple model for arbitrary values of $N \geq 9$. In the remainder of this section, we discuss the general properties of a potential UVFP. In fact, we will show that the UVFP in $\{\overline{a}, \overline{b}, \xi'\}$ can, to a good approximation, be determined analytically, and further that, to determine the UVFP in $\{z_2, z_3\}$, we need only find the UVFP for their flat-space beta-functions with a gravitationally modified factor for b_g.[12]

The UV behavior of $\overline{a}$, eq. (3.6a), is easily discerned since, like α and a, it does not mix with other couplings at one-loop order. In figure 1, we plot this beta-function,[13] showing its UVFP at $\overline{a}^{(\mathrm{uv})} = b_g/b_2 > 0$. (Referring to eq. (3.2), we see that b_2 is always positive; we must require $b_g > 0$ for the gauge coupling to be AF.) If this were the only coupling in

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[11]In ref. [15], it was stated that $N = 8$ is also possible, but that resulted from the approximation $b_g = 0$, where b_g is the one-loop gauge beta-function coefficient (explicitly given in the next section.) In fact, in this class of models, asymptotic freedom mandates that $b_g \geq 1/6$.

[12]Readers interested only in seeing the results for $\mathrm{SO}(10)$ may safely skip forward to the next section.

[13]The actual numbers in figure 1 correspond to an example that will be used in subsequent figures and tables. An illustration of running $\overline{a}(u)$ from the DT-scale toward its UVFP is given in figure 4a.

the model, $\overline{a}^{(\mathrm{uv})}$ would be the dividing line between two phases. That is not the case here. Assuming that we find a UVFP, $\overline{a}^{(\mathrm{uv})}$ is simply one of its coordinates in the five-dimensional space of ratios $\{\overline{a}, \overline{b}, \xi', z_2, z_3\}$.

Nevertheless, because its beta-function is independent of the other couplings, the running of $\overline{a}$ can be understood easily. As the coupling $\overline{a}(u)$ runs from near the UVFP toward lower energy scales, $\overline{a}(u)$ increases if it starts from $\overline{a} > \overline{a}^{(\mathrm{uv})}$. On the other hand, if it starts at a value $\overline{a} < \overline{a}^{(\mathrm{uv})}$, then it decreases as the scale decreases. In the first case, $a(u) > \overline{a}^{(\mathrm{uv})}\alpha(u)$, so gravitational interactions are becoming relatively *stronger* than gauge interactions; in the second case, $a(u) < \overline{a}^{(\mathrm{uv})}\alpha(u)$, so gravitational interactions are becoming relatively *weaker* than gauge interactions. In both cases, $a(u)$ and $\alpha(u)$ are increasing, but there will be no breakdown of perturbation theory unless either gravitational interactions or gauge interactions actually become strong. The alternative, the one explored in this paper, is that DT occurs before strong interactions set in.

A priori, $b_g > 0$ could take values over a large range. For reasons to be explained in greater detail in section 9, it seems that $b_g \sim \mathcal{O}(1)$. The reason is the requirement that the scalar couplings be AF, to be discussed further in section 9. As a result, $\overline{a}^{(\mathrm{uv})} \lesssim \mathcal{O}(10^{-2})$. E.g., in the SO(10)-case discussed beginning in section 4, we find $\overline{a}^{(\mathrm{uv})}$ in the narrow range $0.015 \lesssim \overline{a}^{(\mathrm{uv})} \lesssim 0.019$.

Another implication is that $\overline{\beta}_{\xi'} \to \mathcal{O}(b_g/b_2^2) \lll 1$, so that $\xi'^{(\mathrm{uv})}$ will be nearly conformal but never exactly zero.[14] This can be seen as follows: as remarked in footnote 10, $\overline{\beta}_{\xi'}$, eqs. (3.6h), (3.6i), vanishes for neither conformal nor minimal coupling. For conformal coupling $\xi' = 0$ ($\xi = -1/6$), the contribution in eq. (3.6h) that is independent of gravitational corrections vanishes. This is the familiar property that, in a QFT in a fixed, background gravitational field, a free massless scalar field having $\xi' = 0$ is classically conformally invariant. This has been conjectured to remain true if scale-invariant interactions with other particles are added, but, with the inclusion of scale-invariant gravitational interactions, $\overline{a}, \overline{b} \neq 0$, that is in fact *not* correct, since $\overline{\Delta\beta}_{\xi'} \neq 0$, eq. (3.6i).

In contrast, the gravitational contribution $\overline{\Delta\beta}_{\xi'}$ to $\overline{\beta}_{\xi'}$ vanishes for minimal coupling ($\xi' = 1/6$), ($\xi = 0$). In Einstein-Hilbert gravity, it is well-known that gravitons in curved spacetime are minimally coupled to scalars. This is another way in which gravitons differ from vector bosons, which are conformally coupled (in a renormalizable theory.) The beta-functions respect the symmetry properties operative at very short distances where IR irrelevant operators may be neglected. This property is quite general for perturbation theory in curved spacetime backgrounds; because of the equivalence principle, the local coupling of gravitons to scalars is as if spacetime were flat.

These observations can be made more quantitative by developing a systematic expansion in $\overline{a}/\overline{b}$ near their UVFP. Given that $x = \overline{b}/\overline{a} \gg 1$ near the UVFP, as a zeroth approximation, we may neglect the terms in $\overline{a}$ in eq. (3.6b), giving

$$\overline{\beta}_{\overline{b}} \approx \overline{b}\left[b_g - \left(\frac{5}{12} + \frac{3N(N-1)\xi'^2}{4}\right)\overline{b}\right], \tag{3.7}$$

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[14]Of course, conformal or Weyl gravity is assigned different counterterms in an attempt to enforce conformal symmetry. It is unclear whether this is truly consistent.

As we shall confirm below, near the UVFP the ξ'^2 term is completely negligible, so eq. (3.7) is perfectly analogous to eq. (3.6a), with the replacements $\{\overline{a} \to \overline{b}, b_2 \to \frac{5}{12}\}$. Thus, in first approximation, $\overline{b}$ has a UVFP at $\overline{b}^{(\mathrm{uv})} \approx 12b_g/5$, and $\overline{\beta}_{\overline{b}} \approx b_g(\overline{b}^{(\mathrm{uv})} - \overline{b})$, which implies that, near their UVFPs, $\overline{b} \to \overline{b}^{(\mathrm{uv})}$ at the same rate as $\overline{a} \to \overline{a}^{(\mathrm{uv})}$.. As remarked earlier, typically, $12b_g/5 \sim \mathcal{O}(1)$.

In next approximation, suppose we neglect only the $\overline{a}^2$ term on the right-hand side of eq. (3.6b), then, neglecting the tiny term in ξ'^2, eq. (3.7) would be replaced by

$$\overline{\beta}_{\overline{b}} \approx \overline{b}\left[b_g + \frac{5\overline{a}}{\overline{b}} - \frac{5\overline{b}}{12} \right] \approx \overline{b}\left[\widetilde{b}_g - \frac{5\overline{b}}{12} \right], \quad \text{where} \quad \widetilde{b}_g \equiv b_g(1 + 5/b_2). \tag{3.8}$$

In the second step, we replaced $\overline{a}/\overline{b}$ by its asymptotic value and inserted our zeroth approximation for $\overline{b}^{(\mathrm{uv})}$. Thus, $\overline{b}^{(\mathrm{uv})} \approx 12\widetilde{b}_g/5$. Although $1/b_2$ is relatively small, because it enters multiplied by 5, this correction can be important for obtaining an accurate estimate. For example, if $b_2 = 50 \gg 1$, $5/b_2 = 1/10$, $\widetilde{b}_g/b_g = 1.1$, a 10% increase over the zeroth approximation! To first order in $\overline{a}/\overline{b}$, we then get[15]

$$\frac{\overline{a}^{(\mathrm{uv})}}{\overline{b}^{(\mathrm{uv})}} = \frac{1}{x^{(\mathrm{uv})}} \approx \frac{5b_g}{12b_2\widetilde{b}_g} = \frac{5}{12(b_2 + 5)} \ll 1, \tag{3.9}$$

independent of b_g! E.g., if $b_2 = 50$, $x^{(\mathrm{uv})} \approx 132$, or $\overline{a}^{(\mathrm{uv})}/\overline{b}^{(\mathrm{uv})} \approx 0.76 \times 10^{-2} \ll 1$.

We can use these results to estimate $\xi'^{(\mathrm{uv})}$. From eqs. (3.6h), (3.6i), for conformal coupling ($\xi' = 0$), we have

$$\overline{\beta}_{\xi'}\Big|_{\xi'=0} = \overline{\Delta\beta}_{\xi'}\Big|_{\xi'=0} = -\frac{1}{6}\left[\frac{10\overline{a}}{3x} \right] = -\frac{5\overline{a}^2}{9\overline{b}} \approx -\frac{25b_g}{108b_2(b_2 + 5)} \sim \mathcal{O}\left(b_g/b_2^2\right), \tag{3.10}$$

an extremely small number. For example, for $b_g = 1, b_2 = 50$, this gives -0.8×10^{-4}. Since this is so small, it seems likely that $\xi'^{(\mathrm{uv})}$ is nearby. In linear approximation,

$$\overline{\beta}_{\xi'} \approx \overline{\Delta\beta}_{\xi'}\big|_{\xi'=0} + \xi'\left[\overline{\beta}'_{\xi'}\right]_{\xi'=0}, \tag{3.11a}$$

$$\left[\overline{\beta}'_{\xi'}\right]_{\xi'=0} \approx \left[\frac{N^2-4}{24N}z_2 + \frac{N(N-1)+4}{6}z_3 - 3(N-2) \right] + \frac{3\widetilde{b}_g}{5}, \tag{3.11b}$$

$$\xi'^{(\mathrm{uv})} \approx -\frac{\overline{\Delta\beta}_{\xi'}}{\left[\overline{\beta}'_{\xi'}\right]}\Bigg|_{\xi'=0} \approx -\frac{25b_g}{108b_2(b_2 + 5)}\left[\left|\overline{\beta}'_{\xi'}\right|\right]^{-1}_{\xi'=0}. \tag{3.11c}$$

(Here, "betabar-prime" in $\overline{\beta}'_{\xi'}$ denotes the partial derivative of $\overline{\beta}_{\xi'}$ with respect to ξ'.) These formulae require further explanation. From eq. (3.10), we know that $\overline{\Delta\beta}_{\xi'}|_{\xi'=0}$ is very small and negative. Therefore, the linear approximation eq. (3.11a) will yield a UVFP if and only if $[\overline{\beta}'_{\xi'}]_{\xi'=0} < 0$, which has been assumed in eq. (3.11c). Once one obtains values for the UVFPs $\{z_2{}^{(\mathrm{uv})}, z_3{}^{(\mathrm{uv})}\}$, one must return to check this assumption, but it will be presumed

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[15]This process could be iterated to further improve these estimates by including the $\overline{a}^2$ term in $\overline{\beta}_{\overline{b}}$, eq. (3.6b), and expanding to higher orders in $\overline{a}/\overline{b}$.

to be true for the rest of this section. Then there is a UVFP at small negative ξ' given by eq. (3.11c). The first contribution to the slope in eq. (3.11b) comes from the first term in eq. (3.6h), which arises from matter contributions in the absence of quantum gravity, i.e., QFT in curved spacetime. The second term in eq. (3.11b) comes from the slope of $\overline{\Delta\beta}_{\xi'}$, eq. (3.6i). Even though the second term in square-brackets in that formula vanishes at $\xi' = 0$, it is the dominant contribution to the slope $\overline{\Delta\beta}'_{\xi'}|_{\xi'=0} = \overline{b}/4$, the last term in eq. (3.11b), $\overline{b}^{(\mathrm{uv})}/4 = 3\widetilde{b}_g/5$. This is always positive and often not negligible. E.g., with $b_2 = 50$, we have $\overline{b}^{(\mathrm{uv})}/4 \approx 0.66\, b_g \sim \mathcal{O}(1)$.

If the linear approximation breaks down, it is conceivable there could still be a UVFP of $\overline{\beta}_{\xi'}$, but, for our SO(10) model, section 4, we numerically determined *all* the FPs, which are listed in tables 1 and 2, and there was no other UVFP. The linear approximation works extraordinarily well in this case; in section 9, we provide a detailed comparison.

In general, to know the actual magnitude of $\xi'^{(\mathrm{uv})}$, eq. (3.11c), we must know that there are UVFPs for $\{z_2, z_3\}$ and be able to at least estimate their values for input. To that end, we take up $\overline{\beta}_{z_2}, \overline{\beta}_{z_3}$, eqs. (3.6d), (3.6e). Since the gravitational corrections $\overline{\Delta\beta}_1, \overline{\Delta\beta}_2$, eqs. (3.6f), (3.6g), do not depend explicitly on $\{z_2, z_3\}$, they may be estimated using the approximations in eqs. (3.9), (3.11c). Asymptotically, in each $\overline{\Delta\beta}_k$, we may replace $\{\overline{a}, x, \xi'\}$ by $\{\overline{a}^{(\mathrm{uv})}, x^{(\mathrm{uv})}, \xi'^{(\mathrm{uv})}\}$. First consider $\overline{\Delta\beta}_1$, eq. (3.6f), which consists of two terms, the second of which is suppressed by $(x\xi')^2$ with respect to the first. From eqs. (3.9), (3.11c), we see that

$$x\xi' \approx -\frac{5b_g}{9b_2}\left[\left|\overline{\beta}'_{\xi'}\right|\right]^{-1}_{\xi'=0} \ll 1. \tag{3.12}$$

That being the case, certainly $(x\xi')^2$ is completely negligible with respect to the first term, so $\overline{\Delta\beta}_1 \approx 5(\overline{a}^{(\mathrm{uv})}/6)^2 = 5(b_g/6b_2)^2 \lll 1$. E.g., for $b_g = 1, b_2 = 50$, $\overline{\Delta\beta}_1 \approx 0.6{\times}10^{-4}$. Similarly, the second term in $\overline{\Delta\beta}_2$, eq. (3.6g), is suppressed by $18(x\xi')\xi'/5$, also a negligible correction to the first term. Hence, $\overline{\Delta\beta}_2 \approx 5\overline{a}^{(\mathrm{uv})} = 5b_g/b_2$. For future reference, we note that both $\overline{\Delta\beta}_1$ and $\overline{\Delta\beta}_2$ are positive.

Before proceeding further with eqs. (3.6d), (3.6e), we need to understand how roots of $\{\overline{\beta}_{z_2}, \overline{\beta}_{z_3}\}$ come about. We are only interested in models for which the UVFPs satisfy certain convergence criteria, section 5, and stability constraints, section 6. In the present context, the constraint of interest is that $\{z_2^{(\mathrm{uv})}, z_3^{(\mathrm{uv})}\}$ must both be positive. In that case, every term in $\overline{\beta}_{z_k}$ is positive except for the linear term $-6(N-2)z_k$, $(k{=}2,3)$. This sole negative term must cancel the sum of all the other terms.[16] It cannot be that each term becomes small, because, setting both z_2 and z_3 to zero, both $\overline{\beta}_{z_2}$ and $\overline{\beta}_{z_3}$ are large and positive.

Returning to our estimated gravitational corrections, we see from eqs. (3.6d), (3.6e) that $\overline{\Delta\beta}_1$ contributes only to $\overline{\beta}_{z_3}$. This is a very small positive constant to be added to the much larger one already present; with negligible error, we may drop $\overline{\Delta\beta}_1$. Turning to $\overline{\Delta\beta}_2$, we see that it enters both beta-functions in the coefficient of the terms linear in z_k in the combination $(b_g{+}\overline{\Delta\beta}_2) \approx \widetilde{b}_g$, the same $\widetilde{b}_g$ that entered into the corrections to $\overline{\beta}_{\overline{b}}$, eq. (3.8). Therefore, to a very good approximation sufficiently near the UVFP, we may

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[16]We use these observations in section 9 to set lower and upper bounds on the $z_k^{(\mathrm{uv})}$.

replace eqs. (3.6d), (3.6e) with

$$\overline{\beta}_{z_2} = 36(N-8) + \frac{2N^2-N-24}{24N}z_2^2 + 4z_3z_2 + \left(\widetilde{b}_g - 6(N-2)\right)z_2, \tag{3.13a}$$

$$\overline{\beta}_{z_3} = \frac{36(N-2)}{N} + \frac{N(N-1)+16}{6}z_3^2 + \frac{N^2-4}{48N^2}z_2^2 + \frac{N^2-4}{12N}z_3z_2$$
$$+ \left(\widetilde{b}_g - 6(N-2)\right)z_3. \tag{3.13b}$$

These are identical to the flat-space beta-functions except for the replacement $b_g \to \widetilde{b}_g$! Since $\widetilde{b}_g > b_g > 0$, the effect of dynamical gravity is to increase the difficulty finding a UVFP of these two equations.[17] At least in the cases that we have examined, these equations are remarkably sensitive to the value of $\widetilde{b}_g$, and we give an example in section 9.

If there are real solutions for the roots of $\{\overline{\beta}_{z_2}, \overline{\beta}_{z_3}\}$, it remains to determine whether any of them is a UVFP by calculating the "stability matrix" $\left[\partial\overline{\beta}_{z_j}/\partial z_k\right]$ at each FP and showing it has only negative eigenvalues. If such a UVFP candidate is identified, then one must return to eq. (3.11b), insert the values of z_k at the FP, and check that $\overline{\Delta\beta}_{\xi'}|_{\xi'=0} < 0$, as we have assumed.

We shall return to considering these equations for arbitrary N elsewhere [22], but, in order to develop some intuition from experience with such models, we here restrict ourselves to SO(10), which, for $N \geq 9$, is the smallest SO(N) having complex spinor (i.e., chiral) representations. This is one reason why SO(10) has been of great interest as a possible GUT.

4 An SO(10) model

Although our primary interest is in the existence of a UVFP in all the couplings, this model is simple enough to determine numerically *all* the FPs of the exact one-loop beta-functions. Let us begin with $\overline{\beta}_{\overline{a}}$, eq. (3.6a). As we discussed in the preceding section, $\overline{a}$ has a UVFP at $\overline{a}^{(\mathrm{uv})} = b_g/b_2$, whose dependence on N is implicit through b_g and b_2. For $N=10$, their values are $b_g = 4(21 - T_F)/3, b_2 = (461 + N_F)/20$. Note that $T_F < 21$ in order to preserve AF for the gauge coupling. Setting $N=10$, the remaining beta-functions in eq. (3.6) are

$$\overline{\beta}_x = \overline{a}\left[-\frac{10}{3} + \frac{(6N_F + 3366)}{120}x - \left(\frac{5}{12} + \frac{135}{2}\xi'^2\right)x^2\right]. \tag{4.1a}$$

$$\overline{\beta}_{z_2} = 72 + \frac{83}{120}z_2^2 + 4z_2z_3 + (b_g - 48)z_2 + \overline{a}\left(5 - 18x\xi'^2\right)z_2, \tag{4.1b}$$

$$\overline{\beta}_{z_3} = \frac{144}{5} + \frac{53}{3}z_3^2 + \frac{1}{50}z_2^2 + \frac{4}{5}z_2z_3$$
$$+ (b_g - 48)z_3 + \overline{a}\left(5 - 18x\xi'^2\right)z_3 + \frac{\overline{a}^2}{12}(6\xi' - 1)^2\left(5 + 9x^2\xi'^2\right), \tag{4.1c}$$

$$\overline{\beta}_{\xi'} = \left(\frac{2}{5}z_2 + \frac{47}{3}z_3 - 24\right)\xi' + \frac{\overline{a}}{6}\left(6\xi' - 1\right)\left(\frac{10}{3x} - \frac{3}{2}x\xi'(2\xi' + 1)\right). \tag{4.1d}$$

Requiring that the gauge coupling be AF, ($b_g > 0$), it would seem that there are a large number of possibilities with $0 \leq T_F < 21$. In fact, for reasons not particularly transparent, it turns out that there is a UVFP only for b_g as small as permitted. Restricting the fermions

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[17]Using the beta-functions of ref. [6], we find the opposite sign of the effect reported in refs. [12, 13].

	$\bar{a}$	x	ξ'	z_2	z_3	Nature
1.	**0.016856**	**106.8451**	$\mathbf{-1.4399\times10^{-5}}$	**1.7235**	**1.0706**	**UV stable**
2.	0.016856	106.8450	1.0030×10^{-4}	1.80221	1.5129	saddle point
3.	0.016856	0.07497	0.10641	1.80221	1.5130	saddle point
4.	0.016856	0.07488	-0.02161	1.72354	1.0706	saddle point
5.*	0	n. a.	0	1.7180	1.0592	saddle line
6.*	0	n. a.	0	1.80134	1.5293	saddle line

Table 1. Fixed points for an SO(10) model for finite $\bar{a}$.

to be in the vector, spinor, or adjoint representations, $\{\mathbf{10}, \mathbf{16}, \mathbf{45}\}$, $T_F = 4n_1 + \frac{1}{2}n_2 + n_3$, and $N_F = 45n_1 + 10n_2 + 16n_3$, where n_i is the number of representations (flavors) of each type. Since b_g vanishes for $T_F = 21$, the first allowable case has $T_F = \frac{41}{2}$ ($b_g = \frac{2}{3}$). Even with T_F fixed at $\frac{41}{2}$, there are still 66 possible choices for the three integers (n_1, n_2, n_3), each with a different value for N_F, spanning $235 \leq N_F \leq 410$. This corresponds to the ranges $\frac{174}{5} \leq b_2 \leq \frac{871}{20}$, $0.015 \lesssim \bar{a}^{(\mathrm{uv})} \lesssim 0.019$. There is a UVFP for all values of N_F in this range, and it is easy to see that the FPs are rather insensitive to N_F. In section 9, we show that, for $T_f = 20$, there is no UVFP.

To illustrate, consider $(n_1, n_2, n_3) = (0, 1, 20)$, for which $N_F = 330$. Then, $\bar{a}^{(\mathrm{uv})} = \frac{40}{2373} \approx$ 0.016856. Inserting this value of $\bar{a}$ into eq. (4.1), we find there are still four FPs in the other coupling constants. In table 1, we show the values we found for these four.[18] To determine their "nature", we must calculate the stability matrix by taking the partial derivatives of the beta-functions with respect to each of the variables, evaluating them at the FP, and determining the eigenvalues. As claimed, one of the FPs is UV stable. (For a model in which 3 (and only 3) spinor representations do not acquire GUT-scale masses, the alternative $(n_1, n_2, n_3) = (0, 3, 19)$, for example, might be preferable, with very similar results.)

As expected from our discussion in the preceding section, at the UVFP, the value of $x^{(\mathrm{uv})}$ is large, while $\xi'^{(\mathrm{uv})}$ is extremely small. We will defer to section 9 a more detailed quantitative accounting, but these approximations work extraordinarily well.[19]

One may also explore whether there are FPs in the extreme IR limit. As mentioned earlier, the behavior of these equations in the IR limit is purely formal since, if weak coupling DT does not take place, then the gauge or gravitational interactions (or both) become strong, and perturbation theory breaks down. Nevertheless, understanding the IR behavior of the running couplings may help us more easily understand the range of couplings lying within the catchment basin of the UVFP. Since the determination of the IRFPs of these equations is not relevant to our main line of development, we have relegated that analysis to appendix B.

Having established the existence of a class of simple models with a UVFP, are there further restrictions on the allowed range of values of the coupling constants at the UVFP? In fact, as we shall discuss in the next section, there are.

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[18]For lines 5.* & 6.*, see appendix B.

[19]For readers who wish to jump ahead, see the discussion surrounding eq. (9.2).

5 Constraints on the coupling constants

We have adopted the point of view of Euclidean quantum gravity [23, 24], in which the theory is quantized starting from the Feynman path integral with Euclidean signature, the Euclidean path integral (EPI) for short. Strictly speaking, one must require this of the bare couplings defined in the presence of a cutoff, and then show that one may obtain a sensible renormalized theory as the cutoff is removed. As illustrated by the enterprise of lattice field theory, this may be taken as a starting point for a nonperturbative definition of a theory, but even so, it can be problematic to remove the cutoff. For example, it is generally believed that $\lambda\phi^4$ theory in four dimensions has no nontrivial continuum limit, the reason being that the renormalized interaction strength λ at any finite scale tends to zero as the cutoff is removed. One case in which we can expect to find a continuum limit is in models in which all the couplings are AF. These are especially amenable to a perturbative treatment at high energies because we are assured that the quantum corrections are small. This is precisely the situation that has been established for the class of theories under consideration here.

The preceding considerations do not guarantee the existence of a sensible QFT. For example, $\lambda\phi^4$ in four dimensions with $\lambda<0$ is AF. We must require a convergent EPI at sufficiently high scales where the effective action may be approximated by the form of the "classical" action with small couplings. Consider the action defined by eq. (2.1) plus eq. (2.2) with the potential given in eq. (2.3b). We gather the result together here:

$$S_{\rm cl}=\int d^4x\sqrt{g}\left[\frac{1}{4}{\rm Tr}[F_{\mu\nu}^2]+\frac{1}{2}{\rm Tr}[(D_\mu\Phi)^2]+\frac{h_3}{24}T_2^2+\frac{h_2}{96}\widetilde{T}_4-\frac{\xi T_2 R}{2}+\frac{C^2}{2a}+\frac{R^2}{3b}+cG\right]. \quad (5.1)$$

It is not clear what constraint, if any, is implied by the presence of the G-B term G. For now, we follow custom and ignore it, but we shall return to this question below. (It is certainly not ignorable in the determination of the effective action in de Sitter space.)

Euclidean signature of the metric ensures that ${\rm Tr}[F_{\mu\nu}^2]\geq 0$ and ${\rm Tr}[(D_\mu\Phi)^2]\geq 0$. For the integral over metrics at fixed other fields, the quadratic operators C^2, R^2 dominate for large fields. Therefore, both $a>0$ and $b>0$ since there are field configurations where one operator becomes large while the other does not. For the same reason, integration over the scalar fields implies both $h_2\geq 0$ and $h_3\geq 0$. More generally, we must require that the quadratic form

$$\frac{R^2}{3b}-\frac{\xi T_2 R}{2}+\frac{h_3}{24}T_2^2\geq 0 \quad (5.2)$$

for all field configurations. Since $b>0$, this form is positive as $T_2\rightarrow 0$, and it will have no real roots provided $h_3\geq 9b\xi^2/2>0$. This also implies that if either or both Φ and R condense, i.e., develop classical VEVs, then the associated cosmological constant will be positive.

Altogether, we conclude that, at sufficiently large scales, we must have

$$a>0,\quad b>0,\quad h_2>0,\quad h_3\geq\frac{9}{2}b\xi^2>0, \quad (5.3a)$$

$$\text{or } \overline{a}>0,\quad x>0,\quad z_2>0,\quad z_3\geq\frac{9}{2}x\overline{a}\xi^2=\frac{9}{2}\overline{b}\xi^2>0, \quad (5.3b)$$

where, in the second form, we have rewritten the constraints in terms of the rescaled couplings after dividing by $\alpha \equiv g^2$. From the one-loop beta-functions, we know that, if $a > 0$ at some high scale, then it will remain positive for all lower scales at which perturbation theory remains valid. (Obviously, the same is true for α.) Note that the sign of ξ is not constrained asymptotically, which is fortunate since eq. (3.11) implied that the asymptotic value $\xi'^{(\mathrm{uv})} \lesssim 0$, however tiny, so $\xi^{(\mathrm{uv})} \lesssim -1/6$. (On the other hand, we demand $\xi > 0$ at the DT scale in order to induce normal Einstein gravity.)

Returning to the G-B term, cG, one may write G in the form $\nabla_\mu B^\mu$, where B_μ is a one-form, not a vector. In a smooth, compact background, the integral is proportional to the Euler number, which can take either sign, depending on the topology of the manifold. Thus, it is hard to imagine finding a constraint on the sign of c. If one quantizes the theory using the background field method (BFM), then because $\sqrt{g}G$ has zero variation, it makes no contribution to the integral over the quantum fields. From this point of view, it is unnecessary to constrain the coupling c. Since the BFM is simply a change of field variables, it ought to be true in general. It is not entirely clear that this unambiguously defines the EPI nonperturbatively, but if we confine ourselves to perturbation theory, then perhaps this argument is sufficient to dispense with any constraints on the G-B coupling c.

Perturbatively, the value of c is determined up to a constant c_0 by the other couplings in the theory. In fact, we showed [19] that, in leading order, i.e., at tree level,

$$c = c_0 - b_1/(b_2 a) = c_0 - b_1/(b_2 \alpha \bar{a}) \tag{5.4}$$

where the constants b_1, b_2 are given in eq. (3.2). Remarkably, at one-loop order, it, like β_a and β_α, c is independent of all the other couplings, including b. Since a and α go to zero asymptotically, clearly $c \to -\infty$. (It appears as if c_0 would arise as a one-loop correction, but it is actually renormalization group invariant, i.e., it is scale independent.) Perhaps we should interpret AF to require $c_0 = 0$, but it is not entirely clear how c_0 affects observables. In any case, it seems that it is not necessary to impose a constraint on c, but this may not be the final word on this subject.

6 Spontaneous symmetry breaking

After this lengthy discussion concerning fixed points and UV behavior, we begin this section with an overview of the induced-gravity scenario that we have in mind. One solution of the classical field equations is the trivial solution $g_{\mu\nu} = \eta_{\mu\nu}$, $A_\mu = 0$, $\Phi = 0$. One might think that this is the "symmetric" phase in which none of the symmetries, including scale invariance, are broken, but, of course, scale invariance is explicitly broken in the QFT by the conformal anomaly, leading to a renormalizable theory rather than a conformal theory. Nevertheless, with all couplings AF, the theory does ultimately approximate a free field theory asymptotically, so this solution may be a possibility in the UV limit. For vector, scalar, and fermion fields, the elementary excitations are the familiar ones, but it isn't clear what particle-like excitations are to be associated with fluctuations in the metric, inasmuch as their propagators behave as $1/q^4$. Despite that, this theory in the trivial background

may be used to calculate beta-functions [6] and correlation functions at very short distances. There is nothing obviously problematic with this scalar-tensor theory so long as one realizes that it is limited in scope.[20]

The trivial solution is not however a solution for long-distances or low-energies, where, as we have described previously, there will be symmetry-breaking by DT, whether at weak or strong coupling. In order to realize something that looks more like our universe, it is crucial for consistency that scale invariance is anomalous and that the couplings run, so that we may entertain different approximate descriptions of the same underlying theory. (A strictly conformal theory with zero beta-functions is of little interest in this respect.) At a certain energy scale, set by DT, classical condensates form. If this occurs at weak coupling, as we assume in this paper, it is more nearly analogous to traditional GUT or electroweak symmetry breaking than to QCD: some of the massless particles simply acquire mass as a result of the formation of a scalar condensate, but also the curvature may become nonzero. Because the metric is associated with the geometry, the classical background may appear very different from Minkowski spacetime.

We can see how this works by reflecting on the form of the matter action, eq. (2.2). From the scalar condensate, the coefficient of the scalar curvature becomes nonzero. This can be identified with the (reduced) Planck mass $\widetilde{M}_P^2 \equiv \xi \left\langle \mathrm{Tr}[\Phi^2] \right\rangle$, where $\widetilde{M}_P \equiv M_P/\sqrt{8\pi}$. At the same time, the condensate gives a nonzero value for the potential, which acts like a positive cosmological constant, $\Lambda \equiv \left\langle V_J(\Phi) \right\rangle / \widetilde{M}_P^2 > 0$. Via the equations of motion, the curvature in first approximation has $\langle R \rangle = 4\Lambda$, as in Einstein-Hilbert theory. The simplest scenario would be a maximally symmetric background that approximates (half of) de Sitter spacetime. It is not so clear what happens in a cosmological situation. It may be that the Big Bang begins when this condensate first forms, but we leave such questions for future research.

This is the induced-gravity mechanism; it is obviously generic, independent of the particular symmetry group or scalar content. One might think that it could not occur in perturbation theory, and, indeed, it may not. Spelling out the conditions under which that may occur is the subject of the remainder of this paper.

The low-energy effective field theory, which includes a massless graviton in addition to massless matter, looks like ordinary general relativity plus matter. To leading order, of course, the graviton would appear to decouple, with interactions proportional to $1/\widetilde{M}_P$. However, in de Sitter background, there may be an exception to decoupling [32]. With a nonzero cosmological constant, there remains an essential [33] dimensionless coupling of the form $\Lambda/\widetilde{M}_P^2$.

The formation of a condensate $\langle \Phi \rangle \neq 0$ will also break the symmetry group SO(10), and we shall see that the direction of the breaking *can* be determined classically. Thus the mechanism that gives rise to the Planck mass and cosmological constant is also associated with the unification of gauge couplings. The particle physics will follow the familiar development, with some of the gauge bosons and scalars of SO(10) acquiring masses, and others remaining massless. If fermions are added together with Yukawa couplings, some of them

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[20]We shall return to the question of whether this theory is unitary in section 10.

will also get masses. In the remainder of this section, we consider the classical breaking of SO(10), which, it turns out, must be to SU(5)⊗U(1). In the next section, section 7, we shall determine the DT scale, the energy at which these condensates form, while in section 8, we shall investigate the stability requirements at the DT scale.

Let us begin by analysing the extrema of the classical action to determine how SO(10) might undergo spontaneous symmetry breaking (SSB). To solve the classical field equations in general is challenging when the background is curved and variable. To simplify the task, we shall assume that the background is approximately de Sitter space and that any fluctuations in the curvature may be neglected in first approximation. (This is generally the case in inflationary models of the very early universe.) By dimensional analysis, $\int d^4x \sqrt{g} = V_4/R^2$, where V_4 is an angular volume. In a de Sitter-like background, the Weyl term contributes nothing, but the Gauss-Bonnet (G-B) operator takes the value $G = R^2/6$. Therefore, for constant R and constant Φ, the value of the classical action takes the form

$$\frac{S_{\text{cl}}}{V_4} = \frac{1}{3b} + \frac{c}{6} + \frac{h_1}{24}\frac{T_2^2}{R^2} + \frac{h_2}{96}\frac{T_4}{R^2} - \frac{\xi}{2}\frac{T_2}{R}. \tag{6.1}$$

Since the action is dimensionless, it can depend only on the ratio $\Phi/\sqrt{R}$, where we suppose that the relevant range of the scalar curvature has $\langle R \rangle > 0$. Classically, extremizing this action with respect to Φ or R will never yield a scale but it may fix their ratio. The form of the action in eq. (2.2) has been treated many times, at least as far back as ref. [20]. One may employ the representation used therein, based on the standard form of the SO(N) generators, or one may make a unitary transformation to bring the generators to a form in which the Cartan subalgebra is represented by diagonal matrices. (See, e.g., ref. [21].) The latter are particularly simple. The generators take the form

$$R^a = \left(\begin{array}{c|c} \mathcal{R}_1 & \mathcal{R}_2 \\ \hline -\mathcal{R}_2^* & -\mathcal{R}_1^t \end{array}\right), \tag{6.2}$$

where the $\mathcal{R}_i$ are 5×5 matrices with the properties $\mathcal{R}_1$ is Hermitian and $\mathcal{R}_2$ is antisymmetric. (Here, $\mathcal{R}_1^t$ denotes the transpose.) We shall regard the elements of R_1 and the nonzero elements of R_2 as our 25+20=45 independent dynamical real variables. Defining $\varphi \equiv \Phi/\sqrt{R}$, the first variation of the action eq. (6.1) is[21]

$$\frac{\delta S_{\text{cl}}}{V_4} = \frac{h_1 t_2}{6}\text{Tr}[\varphi \delta\varphi] + \frac{h_2}{24}\text{Tr}[\varphi^3 \delta\varphi] - \xi\text{Tr}[\varphi\delta\varphi], \tag{6.3}$$

where $t_2 \equiv \text{Tr}[\varphi^2]$. The vanishing of this equation for arbitrary $\delta\varphi$ determines the extrema $\langle \varphi \rangle$.

Assuming that $\langle \varphi \rangle$ is constant and nonzero, one may apply an SO(10) transformation to bring $\langle \varphi \rangle$ into diagonal form. Calling the five real entries in $\langle \varphi_1 \rangle \equiv \text{Diag}(r_1, r_2, r_3, r_4, r_5)$, then $\langle t_2 \rangle = 2\sum_1^5 r_i^2$, and the vanishing of eq. (6.3) takes the form

$$\left(\frac{h_1 \langle t_2 \rangle}{3} - 2\xi\right)\text{Tr}[\langle\varphi_1\rangle \delta\varphi_1] + \frac{h_2}{12}\text{Tr}[\langle\varphi_1^3\rangle \delta\varphi_1] = 0. \tag{6.4}$$

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[21]$\delta\varphi$ is shorthand for a matrix of the form of eq. (6.2) with Hermitian $\delta\varphi_1$ and antisymmetric $\delta\varphi_2$.

Clearly, only the diagonal elements of $\delta\varphi_1$ enter this equation; since they are independent, this implies

$$r_j \left[\frac{h_1 \langle t_2 \rangle}{3} - 2\xi + \frac{h_2}{12} r_j^2 \right] = 0, \qquad (6.5)$$

for each element $r_j, j = \{1, \ldots, 5\}$. Consequently, $\langle \varphi_1 \rangle$ has diagonal entries either $r_j = 0$ or $r_j \equiv \pm r_0^{[k]}$, with $r_0^{[k]}$ satisfying

$$\frac{h_1 \langle t_2 \rangle}{3} - 2\xi + \frac{h_2}{12} r_0^{[k]2} = 0. \qquad (6.6)$$

Here, k denotes the number of zero elements along the diagonal $k = \{0, \ldots, 4\}$. All nonzero elements have the same magnitude, $r_0^{[k]}$, so there are five possible nontrivial extrema with $r_j = r_0^{[k]} \omega_k$ with[22] $\omega_0 \equiv \mathrm{Diag}(1,1,1,1,1)$, $\omega_1 = \mathrm{Diag}(1,1,1,1,0)$, $\ldots$, $\omega_4 = \mathrm{Diag}(1,0,0,0,0)$. Correspondingly, $\langle t_2 \rangle = 2(5-k) r_0^{[k]2}$, so

$$r_0^{[k]} = \sqrt{\frac{24\xi}{8(5-k)h_1 + h_2}}. \qquad (6.7)$$

As already remarked in section 5, we require $\xi > 0$ at the DT scale in order to generate a "right sign" Einstein term; moreover, as we shall see shortly, we must in any event have $\xi > 0$ at the DT scale for classical stability of the symmetry breaking. So we must also require that $h_2 + 8(5-k)h_1 > 0$ in order to have real solutions for $r_0^{[k]}$. We previously argued that, for the EPI to converge, eq. (5.3), we must have $h_2 > 0$ and $h_3 = h_1 + h_2/40 > 0$ *asymptotically*, but these constraints are not necessarily true at the DT scale. In fact, however, we shall see below that in this simple model, stability of the SO(10) breaking requires the number of zero elements $k=0$, with $(\xi, h_2, h_3) > 0$.

To explore local stability of these five extrema, we must determine the second variation of the action. Returning to eq. (6.3), the second variation is

$$\frac{\delta^2 S_{\mathrm{cl}}}{V_4} = \left\{ \frac{h_1}{3} (\mathrm{Tr}[\varphi \delta\varphi])^2 + \frac{h_2}{24} \left[2\mathrm{Tr}[\varphi^2 \delta\varphi^2] + \mathrm{Tr}[(\varphi \delta\varphi)^2] \right] \right\} + \left(\frac{h_1 t_2}{6} - \xi \right) \mathrm{Tr}[\delta\varphi^2]. \qquad (6.8)$$

To determine whether the candidate vacua are stable, we must evaluate eq. (6.8) for $\varphi \to \langle \varphi^{[k]} \rangle$ and arbitrary $\delta\varphi$. This is a rather complicated equation involving four distinct traces. We shall simply state the result here and refer the interested reader to appendix C for details. We find that the only local minimum among the five extrema has the number of zeros $k=0$, provided that $\{\xi, h_2, h_3\}$ are all positive.[23] Thus, we have classical stability at the DT scale only for breaking to SU(5)⊗U(1). It is interesting that this specific breaking pattern is singled out in this approach and preferred to other popular alternatives, such as SU(4)⊗SU(2)⊗SU(2).

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[22]In fact, any of the nonzero entries could be -1 instead, but this is not really distinct. WLOG, one may exchange the negative entry in $\mathcal{R}_1$ with the corresponding positive element in $-\mathcal{R}_1^t$.

[23]After including radiative corrections, these turn out to be necessary but not sufficient conditions, as we shall discuss in section 8.

Moreover, the maximal subgroup SU(5)⊗U(1) of SO(10) is precisely the group associated with "flipped" SU(5) models [34]. (Of course, in the absence of fermions, we do not distinguish this possibility from Georgi-Glashow SU(5)⊗U(1). For a recent analysis of "flipped" phenomenology, see, for example, ref. [35].)

As remarked in the preceding section, asymptotically, we also must have ($h_2{>}0$, $h_3{>}0$) for convergence of the EPI. In fact, the UVFP in table 1 fulfilled these conditions but has $\xi{<}0$, so that the sign of ξ must change while running from the DT scale (where we require $\xi{>}0$) to its UVFP. This turns out to be possible.

Even though we have determined the symmetry-breaking pattern, the actual value of the DT scale remains to be determined. We want to show that the RG evolution fixes the DT scale while allowing for all these stability conditions to be fulfilled. This is the topic to which we shall turn in the next section.

Before so doing, a final remark: for this particular symmetry breaking, the coupling constant h_3 is to be preferred to h_1, which is reinforced by noting that the value of the classical action on-shell after symmetry breaking is given by

$$\frac{S_{\text{cl}}^{(\text{os})}}{V_4} = \frac{1}{3b} + \frac{c}{6} - \frac{3\xi^2}{2h_3}. \tag{6.9}$$

This is because $\widetilde{T}_4{=}0$ for this breaking pattern, and $\langle\Phi\rangle$ is SU(5)⊗U(1) invariant.

7 Dimensional transmutation

In our paper on scale invariance [5], we derived the conditions for DT in models like this one. The effective action takes the generic form

$$\Gamma(\lambda_i, r, \rho/\mu) = S_{\text{cl}}(\lambda_i, r) + B(\lambda_i, r)\log(\rho/\mu) + \frac{C(\lambda_i, r)}{2}\log^2(\rho/\mu) + \dots, \tag{7.1}$$

where $\rho \equiv \sqrt{R}$. All coupling constants are denoted by the set $\{\lambda_i\}$. In writing the effective action in this form, we have assumed that Φ is spacetime independent and that the background metric is well-approximated by the de Sitter metric with constant scalar curvature R. (In general, we would have to return to the Lagrangian form analogous to eq. (5.1) rather than to this integrated action analogous to eqs. (6.1), (6.9).) The functions $B(\lambda_i, r)$, $C(\lambda_i, r)$ remain to be determined. In the loop-expansion, $B = B_1 + B_2 + \dots$, with the first nonzero contributions to B coming at one-loop. Similarly, $C = C_2 + C_3 + \dots$, with the first nonzero contributions to C starting at two-loops.

In this section, we shall evaluate $B_1^{(\text{os})}$ and, in the next section, $C_2^{(\text{os})}$; here "(os)" signifies "on-shell", that is to say evaluated with $r{=}r_0^{[0]}$ and $\mu{=}\langle\rho\rangle{=}v$. The classical action $S_{\text{cl}}(\lambda_i, r)$ plays a central role in these calculations, so we begin by reviewing some of its features in our SO(10) model. We shall need it off-shell in the next section, for which the general form was given in eq. (6.1), with the first and second variations in eqs. (6.3), (6.8).

For our purposes in this section, we may assume the breaking is in the SU(5)⊗U(1) direction, so that $r_i^2 \equiv r^2$ for all i. Then S_{cl} becomes

$$\frac{S_{\text{cl}}}{V_4} = \frac{1}{3b} + \frac{c}{6} + \left(\frac{25h_3}{6}\right)r^4 - 5\xi r^2, \tag{7.2}$$

where $r \equiv \sqrt{T_2/(10R)}$. Although we specified the direction of the breaking, we have not put the ratio r on-shell. The first and second derivatives of this expression are

$$\frac{S'_{\text{cl}}}{V_4} = 10r\left[\frac{5}{3}h_3 r^2 - \xi\right], \quad \frac{S''_{\text{cl}}}{V_4} = 10\left[5h_3 r^2 - \xi\right]. \tag{7.3}$$

As was previously noted toward the end of section 6, the first derivative vanishes for $r \to r_i^{[0]} = \sqrt{3\xi/(5h_3)}$, where the classical curvature becomes

$$\left.\frac{S''_{\text{cl}}}{V_4}\right|_{r_i} = 20\xi. \tag{7.4}$$

We see that, in order that the ratio of fields $\langle T_2(\Phi)\rangle/\langle R\rangle$ be classically stable, we must have $\xi > 0$.

The value of the scalar curvature $\langle R\rangle$ is undetermined classically, and the normalization scale of the couplings $h_3(\mu), \xi(\mu)$ is also unknown. We want to determine where the first derivative of the effective action eq. (7.1) with respect to ρ vanishes. Taking the couplings to be normalized at the scale of the breaking where $\rho \equiv v$, then the extrema are determined at one-loop order by the vanishing of B_1 on-shell, where it takes the generic form[24]

$$B_1^{(\text{os})}(\lambda_i(v), r_0) = \sum_i \beta_{\lambda_i(v)} \frac{\partial S_{\text{cl}}}{\partial \lambda_i}\bigg|_{r=r_0} = 0, \tag{7.5}$$

which is to be evaluated at its extremum (either before, as in eq. (6.9), or after taking its derivatives.) In eq. (7.5), the quantity $r_0^{[0]2}$, eq. (6.7), has been replaced by the rescaled ratio $r_0^2 \equiv 3\xi/(5z_3)$ to make manifest that $B_1^{(\text{os})}$ is a function of the ratios only!

Actually, we can pause here to ask whether the Yang-Mills $SO(N)$ without any other form of matter can undergo DT. The only couplings would then be $\{a, b, \alpha\}$, and $B_1^{(\text{os})}$ is a function of the two ratios $\{\overline{a}, \overline{b}\}$ only. This calculation is quite similar to the one carried out for pure gravity earlier in ref. [5]. The qualitative results are the same, viz., one can in fact satisfy the $B_1^{(\text{os})} = 0$ for a certain value of $w = \overline{a}/\overline{b}$, but it is always locally unstable ($C_2 < 0$ in the language of section 8.) This remains true if one adds an arbitrary number of fermions. We shall not stop to discuss this calculation.

Returning to eq. (7.5) for the present model, inserting the one-loop beta-functions for the couplings, and rewriting everything in terms of the rescaled variables $\{\overline{a}, \overline{b}, z_2, z_3\}$ defined in eq. (3.5), we find

$$B_1^{(\text{os})}(z_3, z_2, \xi', x, \overline{a}) = \frac{b_3(x, \xi')}{3x^2} - \frac{b_1}{6} - \frac{25r_0^4}{6}\left(\overline{\beta}_{z_3} - b_g z_3\right) - 5r_0^2 \overline{\beta}_{\xi'}, \tag{7.6}$$

where b_1, b_g may be found in eq. (3.1); b_3, in eq. (3.3). b_3 is essentially the beta-function for b and is closely related to $\overline{\beta}_x$, as can be seen in eq. (3.6c). Note that the G-B beta-function b_1 contributes in an important way.

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[24]Since the classical action and beta-functions are real, if the effective action had an imaginary part, this procedure would not find it. One would have to return to calculating the radiative corrections directly.
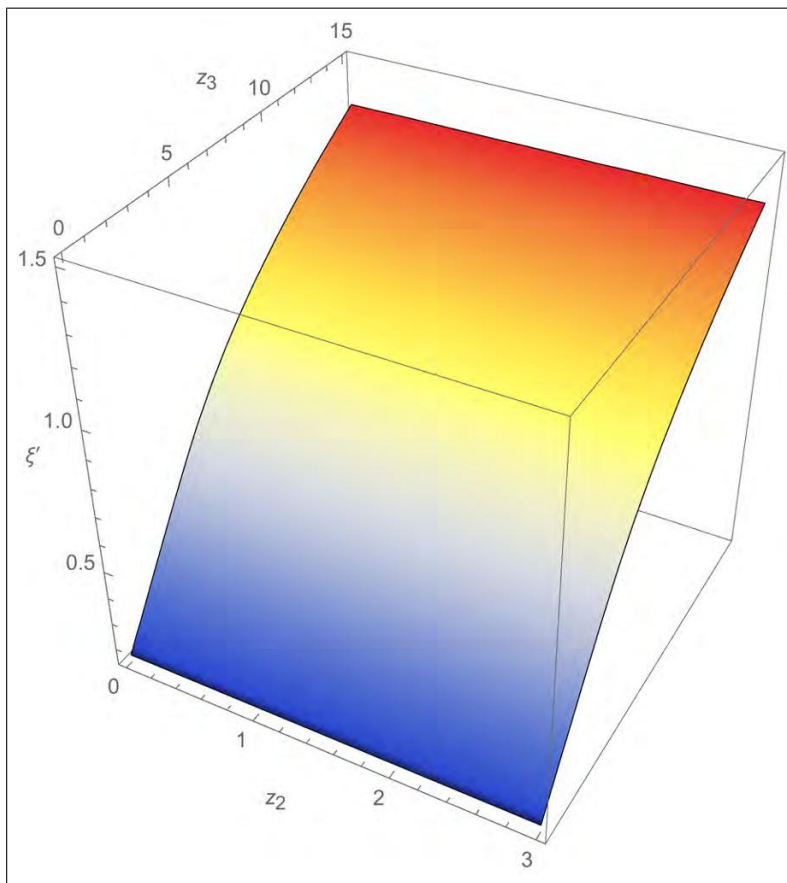
Figure 2. Portion of DT-surface for $x = 120, \overline{a} = 0.025$ as function of $\{z_2, z_3, \xi'\}$.

In our SO(10) model, with a single real adjoint scalar and $T_F=41/2$, the parameters take the values

$$b_g=\frac{2}{3}, \ b_1=\frac{(8806 + N_F)}{720}, \ b_2=\frac{(461 + N_F)}{20}, \ b_3=\frac{10}{3} - 5x + \left(\frac{5}{12} + \frac{135\xi'^2}{2} \right) x^2, \qquad (7.7)$$

and $\overline{\beta}_{z_3}$ and $\overline{\beta}_{\xi'}$ may be taken from eqs. (4.1c), (4.1d), respectively. Thus, $B_1^{(\text{os})}$ is independent of α and depends only on the ratios of couplings via the various $\overline{\beta}$'s. The absolute magnitudes of the couplings $\{\alpha, a, h_2, h_3\}$ are irrelevant so long as they are within the perturbative regime. The explicit form of eq. (7.6) is long and complicated; it is given in eq. (D.1) of appendix D.

There are also some constraints that we must apply from our discussion of SSB in section 6: in order for SSB of SO(10) to occur, we found that $h_3>0$, and, for local stability of that breaking pattern, $h_2>0$.

In sum, in addition to $B_1^{(\text{os})}=0$ at the DT scale, we require $\{\overline{a}, \xi, z_2, z_3\}$ positive, $(\xi'>1/6)$. We refer to the range of couplings satisfying all these requirements[25] as the *DT-surface* in the five-dimensional space $\{\overline{a}, x, \xi', z_3, z_2\}$. In figure 2, we display a small portion

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[25]We do not include the constraints of stability under radiative corrections (section 8) or lying in the catchment basis of the UVFP (section 9).

of the DT-surface in the case presented in table 1, viz., in which the fermion content corresponds to $T_F = 41/2, N_F = 330$. In the figure, we chose to portray a three-dimensional slice of the DT-surface having $x = 120, \bar{a} = 0.025$. (There is a continuum of other slices possible.)

There are two more crucial restrictions on the portions of the DT-surface that are acceptable candidates for symmetry breaking. The first is to determine the nature of the stationary point at $\rho = v$ and to find that subregion of the DT-surface for which this is a local *minimum* of the effective action,[26] a requirement equivalent to requiring the dilaton (mass)^2 to be positive. As with the running of the couplings, this nonzero mass is due to the conformal anomaly, but, unlike the DT scale, the leading contributions to it are two-loop order. In the next section, we shall determine this eigenvalue of the second variation of the action called ϖ_2. The second restriction is the nontrivial requirement that the couplings lie within the basin of attraction (or catchment) of the UVFP. (This is where our previous attempts [4] failed.) We take this up in section 9. Both of these additional restrictions are complicated, the first, because it occurs at two-loop order, and the second, because it involves the full nonlinearities of the beta-functions.

8 Local stability of the DT-surface

Our goal in this section is to determine the conditions under which portions of the DT-surface are locally stable. The effective action has the generic form [5] given in eq. (7.1). We shall replace B by B_1 and C by C_2, their leading non-zero contributions. Using the Renormalisation Group Equation for Γ as defined in eq. (7.1), we found that, off-shell, $B_1(\lambda_i, r)$ and $C_2((\lambda_i, r)$ satisfy:

$$B_1(\lambda_i, r) = \beta_{\lambda_i}^{(1)} \frac{\partial}{\partial \lambda_i} S_{\text{cl}}(\lambda_i, r) - \gamma_r^{(1)} r S'_{\text{cl}}(\lambda_i, r), \tag{8.1a}$$

$$B'_1(\lambda_i, r) = \beta_{\lambda_i}^{(1)} \frac{\partial}{\partial \lambda_i} S'_{\text{cl}}(\lambda_i, r) - \gamma_r^{(1)} \frac{\partial}{\partial r} \big(r S'_{\text{cl}}(\lambda_i, r) \big), \tag{8.1b}$$

$$C_2(\lambda_i, r) = \left[\beta_{\lambda_i}^{(1)} \frac{\partial}{\partial \lambda_i} - \gamma_r^{(1)} r \frac{\partial}{\partial r} \right] B_1(\lambda_i, r), \tag{8.1c}$$

where $\gamma_r^{(1)}$ is the one-loop anomalous dimension of the field, and we have suppressed other possible gauge-dependent terms that would contribute off-shell in gauges in which the RGE contains a gauge parameter. These equations are quite general and, in particular, do not require the classical action to be broken in the SU(5)⊗U(1) direction.

In our earlier paper [5], we showed that the second variation of the effective action, eq. (7.1), is given on-shell by

$$\delta^{(2)}\Gamma = \frac{1}{2} \left(\delta r \;\; \frac{\delta\rho}{\rho} \right) \begin{bmatrix} S''_m(\lambda_i, r_0) & B_1'(\lambda_i, r_0) \\ B_1'(\lambda_i, r_0) & C_2(\lambda_i, r_0) \end{bmatrix} \begin{pmatrix} \delta r \\ \frac{\delta\rho}{\rho} \end{pmatrix}. \tag{8.2}$$

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[26]It may be sufficient to be metastable, if the lifetime is longer than the age of the universe, but we would expect this to be only be a slight extension of the stable region.

This is a kind-of see-saw situation, since S''_m is $O(1)$; B'_1, $O(\hbar)$; to C_2, $O(\hbar^2)$. This stability matrix has eigenvalues equal to $S''_{cl}(\lambda_i)/2 + O(\hbar)$, and

$$\varpi_2(r_0, v) = \frac{1}{2}\left[C_2 - \frac{(B'_1)^2}{S''_{cl}}\right]_{r=r_0} + O(\hbar^3). \tag{8.3}$$

In the case of breaking to SU(5)⊗U(1), we have from eq. (7.4), $S''_{cl}/2=10\xi$, but we need to calculate the corresponding $\varpi_2(r_0, v)$ for our present theory. Although of two-loop order, C_2 is evidently computable from eq. (8.1) knowing only the one-loop results. Since the anomalous dimension $\gamma_r^{(1)}$ cancels out on-shell in $\varpi_2(r_0, v)$, we shall ignore it in the following and simply compute the terms we need to determine ϖ_2. In eq. (7.6), we only gave the form of B_1 on-shell, but here we need it off-shell in order to determine B'_1. In fact, the terms have essentially the same form as before with the replacement of r_0 by r. Then we can compute

$$B_1 = -\frac{\beta_b}{3b^2} - \frac{b_1}{6} + \frac{25r^4}{6}\left(\overline{\beta}_{z_3} - b_g z_3\right) - 5r^2\overline{\beta}_\xi + \dots, \tag{8.4a}$$

$$B'_1 = \frac{50r^3}{3}\left(\overline{\beta}_{z_3} - b_g z_3\right) - 10r\overline{\beta}_\xi + \dots \tag{8.4b}$$

$$C_2 = \beta_{\lambda_i}^{(1)}\frac{\partial}{\partial\lambda_i}B_1(\lambda_i, r) + \dots, \tag{8.4c}$$

where the ellipses represent gauge-dependent terms that are essentially irrelevant in that they cancel out in ϖ_2. The actual analytic expression is reproduced in eq. (D.2) of appendix D. The requirement that $\varpi_2(r_0, v)>0$ turns out to be a strong restriction on the portions of the DT-surface that are allowed.

As an illustration, in figure 3, we display a subsection of the DT-surface shown in figure 2, with the same parameters as given there. The impact of the restriction to $\varpi_2>0$ is shown by the cross-hatched region.

Of course, the dilaton (mass)^2 is proportional to[27] $\varpi_2(r_0, v)v^2$, so local stability is equivalent to requiring that the dilaton is not tachyonic. The gauge bosons of SU(10)/SU(5) ⊗U(1) obtain masses of $O(g_{DT}\langle\Phi\rangle)$, where g_{DT} is the gauge coupling at the DT scale (the gauge unification scale), and $\langle\Phi\rangle \sim v\sqrt{\xi/h_3}$.

The requirement $\varpi_2(r_0, v)>0$ completes the set of relations on the ratios of coupling constants[28] that must obtain on the DT-surface. However, we must also know which points in this subregion actually lie within the basin of attraction of the UVFP.

9 The catchment basin of the UVFP

In the preceding sections, we have specified all the requirements for the existence of a DT scale where symmetry-breaking occurs in a manner that is locally stable. To review, we

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[27]The exact relation depends on the normalisation of the ρ, Φ kinetic terms. It is most simply and reliably determined in the Einstein frame and will be spelled out in a future publication [37].

[28]Rather than repeat long phrases such as this one or "coupling constant ratios," we shall refer to them as "couplings" or "ratios" when it should be clear from the context what is intended.
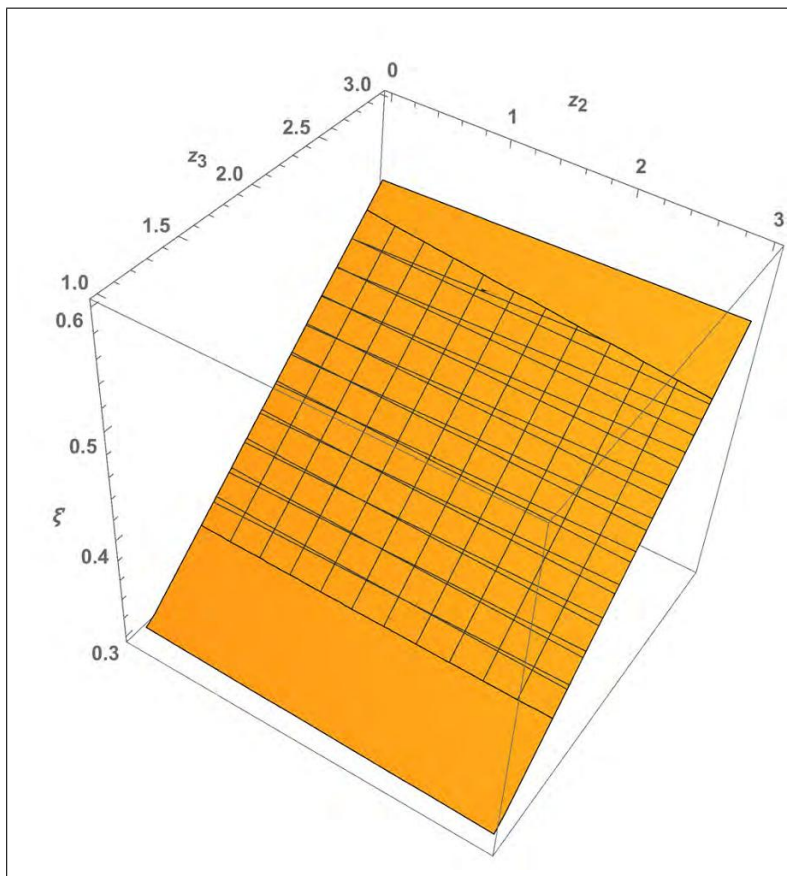
Figure 3. Section of DT-surface having $\varpi_2 > 0$. Cross-hatched portion has $\varpi_2 > 0$. (Same parameters as in figure 2.)

seek points on the DT-surface that, for classical stability, have $\{\bar{a}, \xi, z_2, z_3\}$, all positive; in addition, for stability under quantum fluctuations, $\varpi_2(r_0, v) > 0$. All of these conditions can be expressed in terms of these five ratios, but we tacitly assume that the original six couplings, $\{\alpha, a, b, \xi, h_2, h_3\}$, were small enough to justify the use of perturbation theory. Of course, if the five ratios are all less than one at their UVFP, then in the absence of data to the contrary, we may simply choose $\alpha(v)$ to be small at the scale v. Possibly relevant data comes from searches for proton decay[29] that place the scale of gauge coupling unification around 10^{16} GeV, where SU(5)⊗U(1) may have been broken, so the unification to SO(10) is at least that large. An estimate of the gauge coupling at that scale is $g^2/(4\pi) \approx 0.04$, or $g^2 \approx 0.5$.

Once one has a set of couplings ratios $\{\bar{a}, x, \xi', z_3, z_2\}$ fulfilling all the preceding conditions on the DT-surface, one must ascertain whether or not a given point flows to the UVFP so that the running couplings are AF. This is by no means trivial; often one or another of these ratios blows up rather than approaching the UVFP. With reference to table 1, we see, for example, that the saddle point on line 2 lies very near the UVFP.

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[29]For a review, see S. Raby, "Grand Unified Theories" in ref. [36].

A saddle repels couplings coming from one direction while attracting them from another. Thus, a linearized analysis is of little use over a large range of scales, and there is no alternative to starting at a point on the DT-surface that also is locally stable and running the couplings up to higher scales in order to determine whether the five coupling constant ratios approach their UVFP. This is exactly what is done in the SM from the electroweak scale to test for gauge coupling unification. Here we must test whether they flow to the UVFP or lead to a breakdown of perturbation theory.

We recall that the equations that must be solved for the running couplings take the form

$$16\pi^2 \frac{d\lambda_i}{du} = \overline{\beta}_{\lambda_i}, \tag{9.1}$$

where $du \equiv \alpha(t)dt$. Here, λ_i represents any of the five ratios of coupling constants $\{\overline{a}, x, \xi', z_3, z_2\}$. The corresponding $\overline{\beta}_{\lambda_i}$ are given in eqs. (3.6a), (4.1). We may infer certain general properties from the form of these beta-functions. The couplings $a(\mu)$ and $\alpha(\mu)$ do not mix with other couplings at one-loop, so being positive asymptotically, they remain so as the scale μ decreases. Consequently, their ratio $\overline{a}$ also remains positive at the DT scale. As discussed earlier in section 3, $\overline{\beta}_{\overline{a}}$, eq. (3.6a) has its UVFP at $\overline{a}^{(\mathrm{uv})} = b_g/b_2$. All $\overline{a}(\mu) > 0$ flow monotonically to this UVFP, so long as the initial values of a and α lie within the perturbative domain. As $\overline{a} \to \overline{a}^{(\mathrm{uv})}$, $\overline{\beta}_{\overline{a}} \approx b_g(\overline{a}^{(\mathrm{uv})} - \overline{a})$, so its final rate of approach is set by b_g.

All the other ratios $\{x, \xi', z_2, z_3\}$ mix with each other and with $\overline{a}$, and it is far more difficult to determine their running analytically. Despite the complexity of these beta-functions in five variables, it is not difficult to solve for the running couplings numerically. In figure 4, we present running couplings for one such case having the same parameters as in figure 2 and figure 3. It is worth keeping in mind several of the basic parameters of this example: $T_F = 41/2, N_a = 330, b_g = 2/3, b_2 = 791/20$, not so very different from the example used in section 3. From figure 3, we then selected a point at scale v from the cross-hatched region from which to run: $(\overline{a} = 0.025, x = 120, \xi' = 0.35231, z_2 = 1.66754, z_3 = 1.48330)$. These initial values and the associated UVFP from table 1 are given below each sub-figure. We display $\overline{a}(u)$ running over a very large range of scales, but, to keep the figures of manageable size and to display the behavior near the DT-surface, only a small portion of the running is shown for the other four ratios.

We shall comment on some of the properties of these figures and use them as points of departure to summarize some of the qualitative features in other cases. The nonlinearity of the beta-functions is evident in many ways. For example, in figure 4d, although the initial value of z_2 is not very far from its asymptotic value $z_2^{(\mathrm{uv})}$, it decreases rapidly at first before turning around and climbing back up. In other cases, where it starts at larger or smaller values, it may approach its UVFP far more directly. In figure 4e, although z_3 starts above its UVFP, asymptotically it approaches it from below. In other cases within this same slice $(\overline{a} = 0.025, x = 120)$, it approaches from above. In figure 4c, ξ' falls monotonically to its UVFP near zero, but one easily finds other cases where it rises initially before turning down. Finally, in figure 4b, we see that, although x starts at 120, not much larger than its UVFP value ≈ 107, it falls dramatically to ≈ 40 before turning upward again. That circuitous

(a) $\overline{a}$ from $0.025 \rightarrow 0.016856$.

(b) x from $120 \rightarrow 106.8451$.

(c) ξ' from $0.352307 \rightarrow -1.43995 \times 10^{-5}$.

(d) z_2 from $1.66754 \rightarrow 1.72354$.
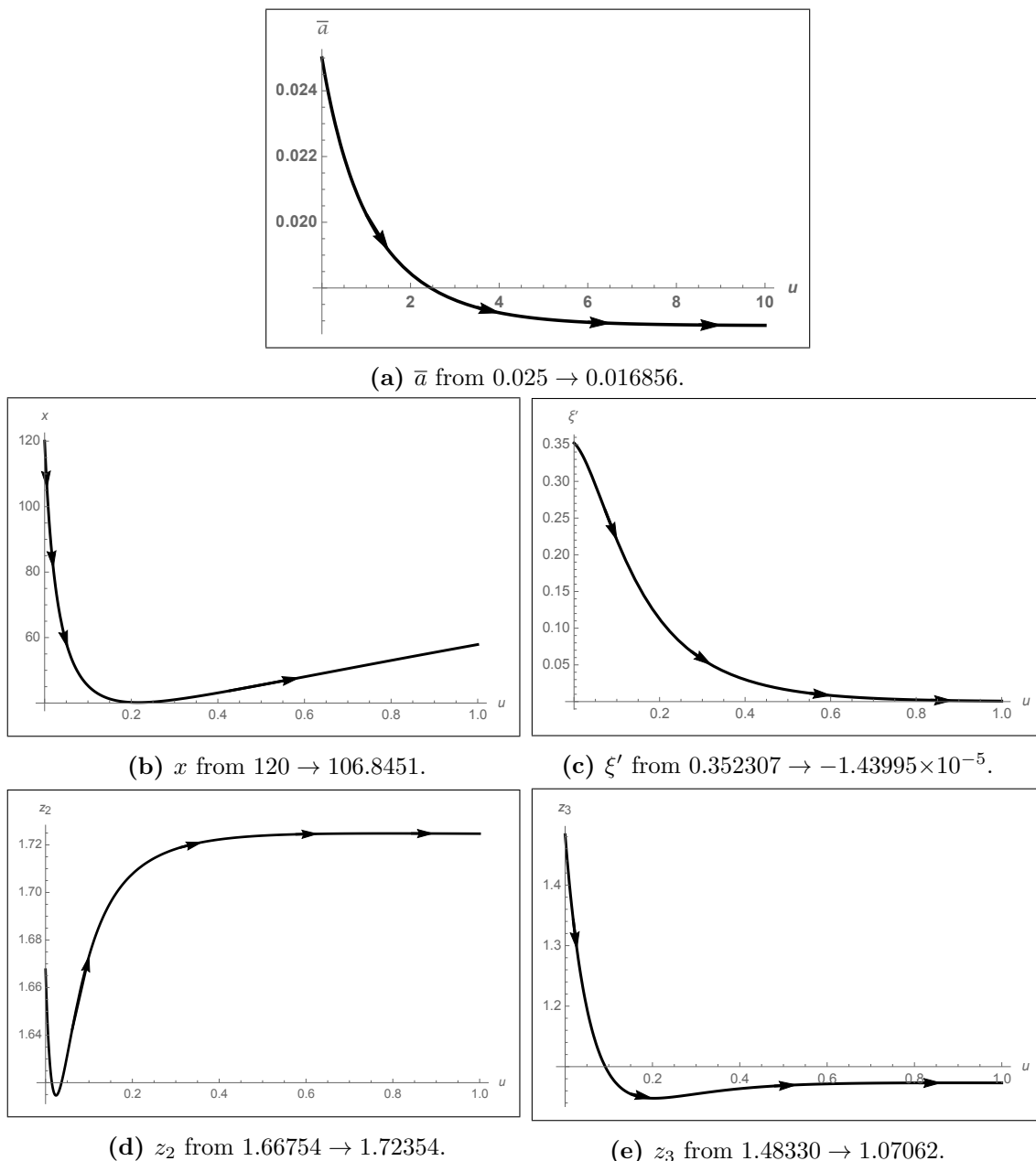

(e) z_3 from $1.48330 \rightarrow 1.07062$.

Figure 4. Running couplings up from a point on the DT-surface.

behavior is characteristic of this segment of the DT-surface, but it is not generic. When the initial value of $x \ll x^{(\mathrm{uv})}$, we may find it increasing monotonically to its asymptotic value. Its behavior also is sensitive to whether one is in this "stronger gravity" region, where initially $\overline{a} > \overline{a}^{(\mathrm{uv})}$, or in the "weaker gravity" region, where initially $\overline{a} < \overline{a}^{(\mathrm{uv})}$.

From the scale of figure 4b, it is not evident that x ever grows to $x^{(\mathrm{uv})}$. This is partly because we wanted to display the structure near the DT-surface but mostly because it runs much more slowly than the other couplings. The latter point is worth explaining. Note from eqs. (3.6c), (4.1a) that $\overline{\beta}_x$ has a factor of $\overline{a}$ in front. Because $\overline{a}$ is small, $\overline{\beta}_x$ is relatively

small, so that x runs slowly. This behavior is the result of the conventional definition of x; this was one of the motivations for our introduction of $\bar{b}$ in section 3. For the example presented in table 1, the value of $\bar{b}^{(\mathrm{uv})}=x^{(\mathrm{uv})}\bar{a}^{(\mathrm{uv})}=1.80101$. The corresponding figure that would replace figure 4b would show $\bar{b}$ running from $\bar{b}=3.0$ on the DT-surface to $\bar{b}=1.8$, finally converging near its UVFP at the same rate as $\bar{a} \to \bar{a}^{(\mathrm{uv})}$.

Near the UVFP in our example, $\bar{a}^{(\mathrm{uv})}/\bar{b}^{(\mathrm{uv})} \sim 10^{-2}$. In section 3, we showed that a small value is completely generic, this ratio depending only on b_2, eq. (3.9). We may also check our estimate of $\xi'^{(\mathrm{uv})}$. For this model, the first term in eq. (3.11b) takes the value ≈ -6.54, and $\tilde{b}_g=0.751$, so that eq. (3.11c) yields $\overline{\Delta\beta}_{\xi'}|_{\xi'=0} \approx -5.79<0$, negative, as required. Then from eq. (3.11c), we get $\xi'^{(\mathrm{uv})} \approx -1.51\times10^{-5}$, to be compared with the more precise value in table 1 of -1.44×10^{-5}, only about a 5% error.

Finally, we come to the UVFPs for $\{z_2, z_3\}$, approximated by the solutions of eq. (3.13), which, for SO(10), become

$$\frac{\overline{\beta}_{z_2}}{48} = \frac{3}{2} + \frac{83}{5760}z_2^2 + \frac{1}{12}z_2 z_3 + \frac{\tilde{b}_g z_2}{48} - z_2, \tag{9.2a}$$

$$\frac{\overline{\beta}_{z_3}}{48} = \frac{3}{5} + \frac{53}{144}z_3^2 + \frac{1}{2400}z_2^2 + \frac{1}{60}z_2 z_3 + \frac{\tilde{b}_g z_3}{48} - z_3. \tag{9.2b}$$

Here we divided eqs. (3.13a), (3.13b) by the factor of $6(N-2)$ so as to normalize the coefficient of the negative contribution to one. With this normalization, we see that, the leading constants are $O(1)$, and the coefficients of the quadratic terms are all less than one.

Let us first see how well these equations approximate the more precise solution in table 1. In that case, we have $T_F \to 41/2, N_F \to 330$, corresponding to $b_g \to 2/3, b_2 \to 791/20$. Therefore, $\tilde{b}_g \equiv b_g[1 + 5/b_2]=594/791 \approx 0.751$, significantly larger than b_g. Solving simultaneously eqs. (9.2a), (9.2b), we find that there are two FPs, of which one is a UVFP having the values ($z_2=1.7235, z_3=1.0706$), agreeing to five significant figures with the values in table 1 calculated from the exact beta-functions! There is little doubt that this approximation captures the bulk of the effects due to dynamical gravity.

More generally, the simultaneous solution of $(\overline{\beta}_{z_2}=0, \overline{\beta}_{z_3}=0)$, eqs. (9.2a), (9.2b), can be regarded as two constraints on the three parameters, $\{\tilde{b}_g, z_2, z_3\}$. Consequently, we can use this approximation to explore the range of solutions for all possible values of $\tilde{b}_g$. However, for reasons of stability and AF, we are only interested in solutions for which each parameter is positive. Solving numerically, we find UVFPs having positive values for the three parameters for the curve displayed in figure 5. In particular, there are real positive solutions for $\{z_2, z_3\}$ only for $0<\tilde{b}_g<1.406$, and they range over $1.679<z_2<1.788$, $0.9884<z_3<1.273$.

In a certain sense, the curve in figure 5 represents the entire range of conceivable UVFPs for the SO(10) model with a single real adjoint scalar field. In reality, this simple model is much more restrictive. In figure 5, we treated $\tilde{b}_g$ as a continuous parameter, but of course, it only takes discrete values for the allowed values of T_F, N_F. For $T_F = 41/2$, the range of N_F depends on the choice of fermion representations. As we discussed below eq. (4.1), restricting fermions to the $\{\mathbf{10}, \mathbf{16}, \mathbf{45}\}$, there are 66 possible choices for N_F,
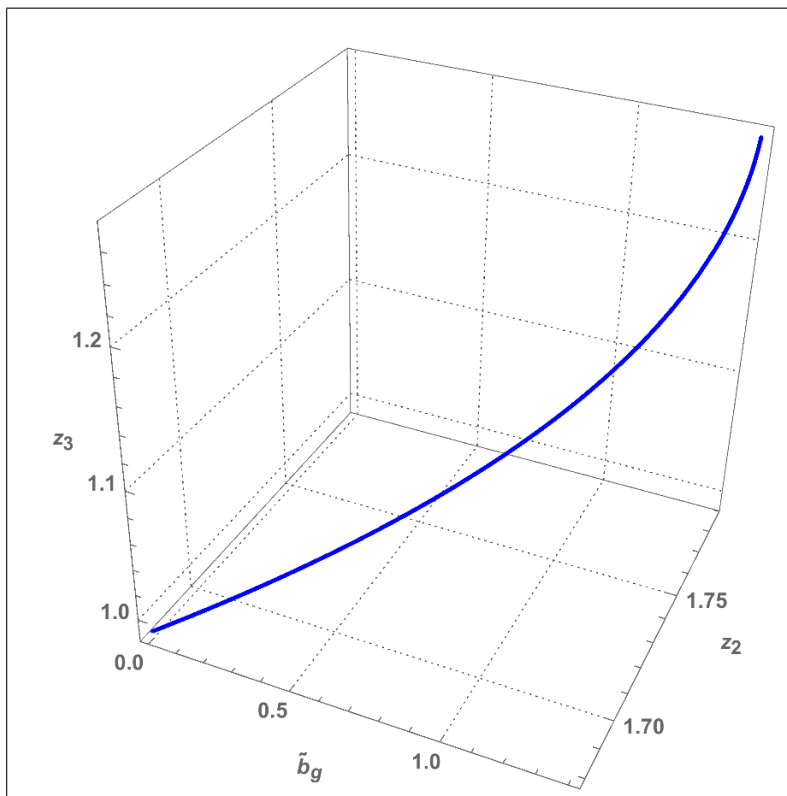
Figure 5. Potential range of UVFPs for SO(10) with adjoint scalar.

with $235 \leq N_F \leq 410$, corresponding to $\frac{343}{10} \leq b_2 \leq \frac{861}{20}$, which in turn implies that $0.743 < \widetilde{b}_g < 0.762$.

As a second example, consider the case when $T_F = 20$. In this case, $b_g = 4/3$ and, one quickly determines that there is indeed a UVFP in the absence of dynamical gravity. To account for gravitational corrections, we need to replace b_g with $\widetilde{b}_g$. With the restriction to the same fermion representations as before, there are again 66 cases with $225 \leq N_F \leq 400$, corresponding to $\frac{343}{10} \leq b_2 \leq \frac{861}{20}$, and $1.488 < \widetilde{b}_g < 1.528$. Even the minimum allowed value exceeds the upper limit of $\widetilde{b}_g = 1.406$. The effect of gravitational corrections has been to eliminate the UVFP![30] Smaller values of T_F (larger b_g) are obviously even worse. $T_F = 41/2$ gives the only possible value of b_g for which there is a UVFP for the scalar couplings!

These examples illustrate the power of these approximations, enabling the determination of whether a UVFP exists for a model and, if so, providing rather accurate values for $\{\overline{b}^{(\mathrm{uv})}, \xi^{(\mathrm{uv})}, z_2{}^{(\mathrm{uv})}, z_3{}^{(\mathrm{uv})}\}$, together with calculable estimates of their uncertainties.

As we have seen, the only place where nonlinearities become very important for estimating the UVFP is in eq. (9.2), which turns out to be extremely restrictive. We wish to conclude with a brief discussion of why that is. Because we must insist on finding solutions having positive (z_2, z_3), these beta-functions have the feature that every term is positive except the linear term, $-z_k$, which must offset the sum of all the other terms.

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[30]We have confirmed this conclusion with a more precise calculation using the exact beta-functions.

As a result, the range of solutions is quite limited. The z_k cannot be too small, because each formula, eqs. (9.2a), (9.2b), has a constant term of $\mathcal{O}(1)$. If we completely ignore all the positive terms except for the constants, we quickly arrive at lower bounds of $\mathcal{O}(1)$: $z_2^{(\mathrm{uv})} > 1.52$, $z_3^{(\mathrm{uv})} > 0.61$. At the same time, the solutions for $(z_2^{(\mathrm{uv})}, z_3^{(\mathrm{uv})})$ cannot be too large because the quadratic terms will overwhelm the sole negative term in each beta-function. Just as $z \gtrsim z^2 > 0$ allows one to conclude $z \lesssim 1$, one can make estimates of the upper limits coming from the quadratic terms here. Finally, $\widetilde{b}_g$ also contributes a positive, linear term that makes it even more difficult to have solutions. The upper limit on $\widetilde{b}_g$ may be far less than one might have guessed, but the $+\widetilde{b}_g z_k$ terms exacerbate a situation in which, even without them, it is already difficult to have AF scalar couplings.

10 Conclusions and outlook

We have succeeded in demonstrating within the context of a non-Abelian gauge theory coupled to renormalizable gravity that there exist regions of parameter space within which the three requirements listed at the conclusion of the introduction have been met: (1) having AF with values of the coupling constants that ensure convergence of the EPI, (2) manifesting DT perturbatively with a locally stable minimum, and (3) lying within the catchment basin of the UVFP. We regard these three requirements as necessary for a sensible theory of this type.

Providing a renormalizable and AF completion of Einstein gravity, this model provides a connection between the Planck mass M_P, the cosmological constant Λ, the unification scale, $M_U \equiv \sqrt{\langle T_2(\Phi) \rangle} = r_0 \sqrt{\langle R \rangle / \alpha} = r_0 v / \sqrt{\alpha}$, the masses $M_V \sim r_0 v$ of the vector bosons, and the masses of heavy scalars arising from SSB. These relations are technically natural; the ratios of masses are functions of the coupling constants at scale v. It remains to explore in more realistic models how great a range of values result.

To demonstrate local stability, we calculated the $O(\hbar^2)$ quantity $\varpi_2 \propto m_d^2$, where m_d is the mass of the dilaton [37]. We showed that there are regions of parameters space where $\varpi_2 > 0$ at the DT-scale, so extrema can be local minima. Since $m_d^2 > 0$ for some range of couplings, it may be that the usual conformal instability, characteristic of models starting from the Einstein-Hilbert action, is absent. This warrants further study.

Our discussion below figure 4 and elsewhere may make it sound as if, although the SO(10) model is technically natural, a good deal of cooking has gone into the stew to make everything work out. In fact, we regard the need to follow a recipe as a positive aspect of this approach. The dynamical requirements dictate much about the choices of compatible representations of scalars and fermions. Indeed, the need for scalar couplings to be AF favors representations with large values of T_F for fermions and even larger values of T_S for scalars, so long as AF of the gauge coupling is maintained. Values of $b_g \lesssim \mathcal{O}(1)$ seem to be strongly favored.

Given our limited goals for this paper, we have not included any mechanism describing further breaking of this symmetry down to the Standard Model. We hope it will be possible to do so, but it may be difficult to arrange that the splitting between the unification scale and the electroweak scale be naturally large. Perhaps there are supersymmetric extensions that would be technically natural, but we have not explored this possibility yet.

In principle, having overcome other limitations, we should now be in a position to begin to investigate whether such models respect unitarity. In the low energy theory below the DT scale, one might be concerned with the possibility of a negative norm state, generally believed to be a problem for "$R + R^2$" gravity. The identification of this issue relies on an expansion about flat space in order to write the (inverse) quadratic form of the graviton fluctuations as

$$\frac{1}{M^2}\left(\frac{1}{k^2} - \frac{1}{k^2 + M^2}\right) \tag{10.1}$$

However, an inevitable consequence of DT is the existence of a cosmological constant, so that flat space is not a solution to the equations of motion. Thus the question is far more complicated than it might naively appear, dealing as it does with spacetimes that are not asymptotically flat, such as de Sitter space. In fact, it has been known for more than 30 years [17, 18] that, to one-loop order, there are no unstable modes in de Sitter background provided the parameters of the model obey certain inequalities, which our present model satisfies.[31] To our knowledge, this has been most thoroughly investigated to by date by Ashtekar, Bonga, and Kesevan [38–41] who emphasize several distinct features of de Sitter space. No matter how small the cosmological constant, there are "no asymptotic Hilbert spaces in dynamical situations of semi-classical gravity" [39]. Further, they show on physical grounds that one must include non-normalizable growing modes among the gravitational waves on $\mathcal{I}^+$. With all Killing fields spacelike at and near $\mathcal{I}^+$, there is no way to define a conserved Hamiltonian and "... in the quantum theory, we cannot decompose fields into positive and negative frequency parts, even at $\mathcal{I}$ ..." [40]. It seems as if the infrared problems in such spacetimes are more serious than generally believed and not simple generalizations from QED. Theorems such as the Ostrogradsky instability [42], associated with Lagrangians containing higher than first-order time derivatives,[32] would seem not to apply.

Such spacetimes have no S-matrix and the attempts to generalize the ADM formalism to de Sitter space are inadequate. Another property of spacetimes that are not asymptotically flat is that the G-B operator cannot be discarded. Although its coupling constant is determined by the other couplings up to a constant [19], it certainly played an important role in our derivation of the conditions for DT. It seems as if a new approach to QFTs with a positive cosmological constant may be required, both to resolve these theoretical challenges and to understand the observed "Dark Energy".

Finally, the nature of measurement in theories with diffeomorphism invariance is complicated. It is conventional to say that there are no local, gauge-invariant observables. We take the view that, normally, this can be resolved once the measurement apparatus is included. Although the physical interpretation of a "particle" is frame-dependent, each piece of the experimental apparatus singles out a special reference frame.[33] Exactly how this is to be generalized to de Sitter space with strong curvature has not been precisely

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[31] There are however several zero modes to be dealt with. This is reviewed in ref. [37].

[32] For recent discussions, see, e.g., refs. [43–45].

[33] This is no more anthropic than the point of view of Gell-Mann and Hartle concerning decoherence, with which we agree. See ref. [46] and earlier papers cited therein.

formulated. Although correlation functions can be calculated in perturbation theory for any particular choice of coordinates, without a Hilbert space and well-defined norm, we are not quite sure how to define probability. Until that has been spelled out, unitarity will probably remain an open question.

Note also that in the high energy phase (where the Higgs VEV is zero and there is no cosmological constant) the graviton propagator has the form $1/k^4$ and it is an open question as to whether this theory is physically sensible. We will discuss all these issues at more length in a future publication, ref. [37].

The cosmological implications of these models, in particular, the details of inflation, also remain to be developed but should be very interesting. In a previous paper, [47], we showed that the Higgs inflation paradigm [48, 49] is in fact compatible with a simple SU(5) GUT structure, with the adjoint Higgs being responsible both for inflation and the breaking to SU(3)⊗SU(2)⊗U(1). Aside from the issue of reassessing this for the SO(10) case, more difficult is the large value of ξ associated with Higgs inflation. This large value caused controversy regarding unitarity, but in our framework is clearly incompatible with our use of perturbation theory at the DT scale, because of its effect on the various dimensionless coupling β-functions, when gravity is quantised.

Exactly what the nature of the medium is at scales much larger than v is not at all obvious. Is it a plasma of particles or something else? Would it be possible to associate a temperature in this region? Is it hot or cold?

Without fine-tuning, M_P, Λ, M_U, together with the dilaton mass m_d, are all associated with a single scale v, the scale of dimensional transmutation. This truly is a unification of gravity with particle physics. It appears as if the Big Bang may begin at the scale v, which may be too large to explain the order of magnitude of inhomogeneities in the CMB. However, this is only the beginning of an investigation into models of this type. It promises to be a very interesting development.

Acknowledgments

A   Lie algebra conventions

In this paper, we limit ourselves to considering simple groups, mostly SO(N). The generalization to semi-simple groups is straightforward, since their algebras are the direct sum of the algebras of simple groups. T_F, T_S are defined by the relation $\mathrm{Tr}[T^a T^b] \equiv T(\mathbf{R})\delta^{ab}$ for any representation $\mathbf{R}$. In general, the representation $\mathbf{R}$ will be reducible but expressible as the direct sum of irreducible representations.

For an irreducible representation, $\sum_a T^a T^a = C_2(\mathbf{R}) \mathbf{1}_{d(\mathbf{R})}$, where $d(\mathbf{R})$ is the dimension of $\mathbf{R}$, and $C_2(\mathbf{R})$ is the quadratic Casimir invariant. It follows that $d(\mathbf{R})C_2(\mathbf{R}) = d(G)T(\mathbf{R})$. C_G is equal to the quadratic Casimir $C_2(\mathbf{G})$ for the adjoint representation $\mathbf{G}$.

The precise relationship between $\{T(\mathbf{G}), C_G\}$ and the gauge coupling g depends upon the normalization convention for the generators. The convention in physics is $T_{\mathbf{N}} = 1/2$ for the defining representation $\mathbf{N}$. This choice gives for two classical series $C_{\mathrm{SO}(N)} = (N-2)/2$, $C_{\mathrm{SU}(N)} = N$.

For low-dimensional representations, this can be confusing. Even though the Lie algebras $\mathrm{SO}(3) \cong \mathrm{SU}(2)$, the fundamental for $\mathrm{SO}(3)$, [$\mathrm{SU}(2)$] is the vector $\mathbf{3}$ [spinor $\mathbf{2}$], respectively. For $\mathrm{SU}(2)$, $C_{\mathrm{SU}(2)} = C_2(\mathbf{3}) = 2$; with $T(\mathbf{2}) \equiv 1/2$, then $C_2(\mathbf{2}) = 3/4$. For $\mathrm{SO}(3)$, $C_{\mathrm{SO}(3)} = C_2(\mathbf{3}) = 1/2 \equiv T(\mathbf{3})$, so $T(\mathbf{2}) = 2C_2(\mathbf{2})/3 = 1/8$.

B Infrared fixed points

In this appendix, we explicate the analysis of the IRFPs of our SO(10) model.

Beginning again with $\bar{a}$, there are two possibilities for its IR behavior, viz., depending on whether initially $\bar{a} \to 0$ (weaker gravity region) or $\bar{a} \to \infty$ (stronger gravity region). In the first case, we may set $\bar{a} = 0$ in these equations. Then $\overline{\beta}_x = 0$ at any fixed x, and all dependence on x drops out of the remaining beta-functions. In this case, x is undetermined. In fact, all gravitational corrections drop out in the sense that all three $\Delta\beta_k = 0$. Both $\overline{\beta}_{z_2}$ and $\overline{\beta}_{z_3}$ take their flat space values, and these equations have two roots for (z_2, z_3), one a UVFP and the other a saddle for flat space. Inserting $\bar{a} = 0$ and either of these values of (z_2, z_3) into $\overline{\beta_{\xi'}}$, we see that $\overline{\beta_{\xi'}} = 0$ implies $\xi' = 0$, its conformal value. These two FPs have been included in the text in table 1, lines 5.* & 6.*, because they occur at finite $\bar{a}$. Since $\bar{a} = 0$ is an IRFP for $\overline{\beta}_{\bar{a}}$, both solutions are in fact at best saddle in nature in the larger space; we called them "saddle lines" since x is not determined at leading order.

As discussed above, the other possibility is $\bar{a} \to \infty$ as $t \to -\infty$. This simply means that $a(t)$ increases faster than $\alpha(t)$ in the IR and, in this case, our decision to rescale the couplings by α does not serve us well. To determine the correct behavior, we must re-express the beta-functions in terms of $\bar{\alpha} \equiv \alpha/a = 1/\bar{a}$ instead, and entertain the limit as $\bar{\alpha} \to 0$. At the same time, to seek IR fixed points, we must introduce the rescaled parameter $du' \equiv a(t)dt$ and re-express the beta-functions accordingly

$$\overline{\beta'}_{\overline{\alpha}} = \overline{\alpha}\left(b_2 - b_g\overline{\alpha}\right), \tag{B.1a}$$

$$\overline{\beta'} = \frac{1}{a}\beta = \overline{\alpha}\overline{\beta} \text{ for } \overline{\beta'}_x \text{ and } \overline{\beta'}_{\xi'}, \tag{B.1b}$$

$$\left(\overline{\beta'} - b_2 z'\right) = \overline{\alpha}^2\left(\overline{\beta} - b_g z\right) \text{ for } \overline{\beta'}_{z_2'} \text{ and } \overline{\beta'}_{z_3'}, \tag{B.1c}$$

where $z' = h/a = \overline{\alpha}z$, for any of the scalar couplings. We see that $\overline{\alpha}$ has a UVFP at b_2/b_g, which is of course precisely the equivalent result as for $\bar{a}$, and the behavior of the couplings near there is just as before. As anticipated, however, it also has an IRFP at $\overline{\alpha} = 0$.

Using eq. (B.1), and with $(n_1, n_2, n_3) = (0, 1, 20)$ as before, $\overline{\beta}'$ may be expressed in terms of rescaled scalar couplings:

$$\overline{\beta}'_x = \left[-\frac{10}{3} + \frac{891}{20}x - \left(\frac{5}{12} + \frac{135}{2}\xi'^2 \right) x^2 \right], \tag{B.2a}$$

$$\overline{\beta}'_{\xi'} = \left(\frac{2}{5}z'_2 + \frac{47}{3}z'_3 - 24\overline{\alpha} \right) \xi' + (\xi' - 1/6) \left(\frac{10}{3x} - \frac{3}{2}x\xi'(2\xi' + 1) \right), \tag{B.2b}$$

$$\overline{\beta}'_{z'_2} = 72\overline{\alpha}^2 + \frac{83}{120}z'^2_2 + 4z'_2 z'_3 + \left(\frac{791}{20} - \frac{142}{3}\overline{\alpha} \right) z'_2 + \left(5 - 18x\xi'^2 \right) z'_2, \tag{B.2c}$$

$$\overline{\beta}'_{z'_3} = \frac{144}{5}\overline{\alpha}^2 + \frac{53}{3}z'^2_3 + \frac{1}{50}z'^2_2 + \frac{4}{5}z'_2 z'_3 + \left(\frac{791}{20} - \frac{142}{3}\overline{\alpha} \right) z'_3$$
$$+ \left(5 - 18x\xi'^2 \right) z'_3 + 3(\xi' - 1/6)^2 \left(5 + 9x^2\xi'^2 \right). \tag{B.2d}$$

Note that $\overline{\beta}'_x$ is independent of $\overline{\alpha}$ and has two FPs in x for fixed ξ', the same two as it had previously for $\overline{a} \neq 0$:

$$x_{\pm} = \frac{2673 \pm \sqrt{7124929 - (1800\xi')^2}}{50(1 + 162\xi'^2)} \tag{B.3}$$

The larger one x_+, is a candidate UVFP in x; the smaller, x_-, a candidate IRFP. Of course, this only makes sense if there is a FP for ξ' in the range $0 \leq |\xi'| < \sqrt{7124929}/1800$, which we shall find is a correct assumption. Taking $\overline{\alpha} \to 0$ in the remaining equations, the other three $\overline{\beta}'$ are

$$\overline{\beta}'_{\xi'} = \left(\frac{2}{5}z'_2 + \frac{47}{3}z'_3 \right) \xi' + (\xi' - 1/6) \left(\frac{10}{3x} - \frac{3}{2}x\xi'(2\xi' + 1) \right), \tag{B.4a}$$

$$\overline{\beta}'_{z'_2} = z'_2 \left[\frac{83}{120}z'_2 + 4z'_3 + \frac{891}{20} - 18x\xi'^2 \right], \tag{B.4b}$$

$$\overline{\beta}'_{z'_3} = \frac{53}{3}z'^2_3 + \frac{1}{50}z'^2_2 + \frac{4}{5}z'_2 z'_3 + \left(\frac{891}{20} - 18x\xi'^2 \right) z'_3 + 3(\xi' - 1/6)^2 \left(5 + 9x^2\xi'^2 \right). \tag{B.4c}$$

These take the form of the theory without gauge interactions (even though both a and α blow up in this limit.)

We remarked earlier that, without the gauge coupling, the scalar couplings in eqs. (3.4), (3.6) have no UVFP, and that is the case here as well. On the other hand, they may well have other FPs. Setting each of these equal to zero, we can solve. We list the results in table 2. It is amusing that one of the FPs at $\overline{\alpha} = 0$ ($\overline{a} \to \infty$) is an IRFP, but of course, this perturbative calculation is not trustworthy in that limit.

C Stability of classical extrema

In this appendix, we include details concerning the classical stability of the extrema of the action, eq. (6.1). To review, the extrema take the form $\langle \varphi^{[k]} \rangle = r^{[k]} \text{Diag}(\omega_k, -\omega_k)$, $\{k = 0, \ldots, 4\}$, as described in the discussion surrounding eqs. (6.6), (6.7). Then $\langle t_2 \rangle = r^{[k]2}(10 - 2k)$, $\langle \varphi^{[k]3} \rangle = r^{[k]2} \langle \varphi^{[k]} \rangle$, and $\langle \varphi^{[k]4} \rangle = r^{[k]2} \langle \varphi^{[k]2} \rangle$. The second variation, or

	$\overline{\alpha}$	x	ξ'	z_2'	z_3'	Nature
1.	**0**	**0.0751125**	**0.166667**	**0**	**0**	**IRFP**
2.	0	19.3649	0.166667	0	0	saddle point
3.	0	0.0931182	1.17747	0	-1.93125	saddle point
4.	0	69.5124	-0.0575469	0	-1.33242	saddle point
5.	0	106.842	-4.13596×10^{-4}	0	-2.51221	saddle point
6.	0	106.844	1.95663×10^{-4}	0	-0.00937304	saddle point

Table 2. Infrared fixed points for an SO(10) model for $\overline{\alpha}{=}0$ $(\overline{a} \to \infty)$.

"stability matrix," is given in eq. (6.8); as noted earlier, it involves four distinct traces. Going on-shell by replacing $\varphi \to \left\langle \varphi^{[k]} \right\rangle$, they take the values

$$\mathrm{Tr}[\delta\varphi^2] = 2\sum_{mn}\left[|(\delta\varphi_1)_{mn}|^2 + |(\delta\varphi_2)_{mn}|^2\right], \tag{C.1a}$$

$$\mathrm{Tr}[\varphi\delta\varphi] = 2r^{[k]}\mathrm{Tr}[\omega_k\delta\varphi_1] = 2r^{[k]}{\sum_{mm}}'(\delta\varphi_1)_{mm}, \tag{C.1b}$$

$$\mathrm{Tr}[\varphi^2\delta\varphi^2] = r^{[k]2}\left[2\mathrm{Tr}[\omega_k\delta\varphi_1^2] + \mathrm{Tr}[\omega_k\{\delta\varphi_2, \delta\varphi_2^\dagger\}]\right]$$
$$= 2r^{[k]2}{\sum_{mn}}'\left[|(\delta\varphi_1)_{mn}|^2 + |(\delta\varphi_2)_{mn}|^2\right], \tag{C.1c}$$

$$\mathrm{Tr}[(\varphi\delta\varphi)^2] = 2r^{[k]2}{\sum_{mn}}'\left[|(\delta\varphi_1)_{mn}|^2 - |(\delta\varphi_2)_{mn}|^2\right], \tag{C.1d}$$

where the prime on the summation in the last three formulas denotes restricting m to the non-null components of ω_k (but summing over all n.) Pulling these pieces together, eq. (6.8) becomes

$$\frac{\delta^2 S_{\mathrm{cl}}^{(\mathrm{os})}}{V_4} = r^{[k]2}\left\{\frac{4h_1}{3}\left|{\sum_{mm}}'(\delta\varphi_1)_{mm}\right|^2 + \frac{h_2}{12}{\sum_{mn}}'\left[3\,|(\delta\varphi_1)_{mn}|^2 + |(\delta\varphi_2)_{mn}|^2\right]\right\}$$
$$+ \left(\frac{h_1 r^{[k]2}(10-2k)}{3} - 2\xi\right)\sum_{mn}\left[|(\delta\varphi_1)_{mn}|^2 + |(\delta\varphi_2)_{mn}|^2\right]. \tag{C.2}$$

Noting eq. (6.6), we can write the coefficient of the last term as $-h_2 r^{[k]2}/12$, in which form, it is simpler to combine with the other terms having coefficient h_2. However, to do so requires breaking up the sum into the restricted sum $\sum'$ plus the remaining terms $\sum''$. Then the second variation eq. (C.2) becomes

$$\frac{\delta^2 S_{\mathrm{cl}}^{(\mathrm{os})}}{V_4} = \frac{r^{[k]2}}{3}\left\{4h_1\left|{\sum_{mm}}'(\delta\varphi_1)_{mm}\right|^2 + h_2{\sum_{mn}}'\left[|(\delta\varphi_1)_{mn}|^2\right]\right\}$$
$$- \frac{h_2 r^{[k]2}}{12}{\sum_{mn}}''\left[|(\delta\varphi_1)_{mn}|^2 + |(\delta\varphi_2)_{mn}|^2\right]. \tag{C.3}$$

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We see from the second term in the first line that the off-diagonal contributions to $\delta\varphi_1$ are stable only if $h_2 > 0$. On the other hand, the second line (involving $\sum''$) restricts m to be in the null subspace of $\langle\varphi\rangle$. As a result, this sum contains fluctuations $\{(\delta\varphi_1)_{mn}, (\delta\varphi_2)_{mn}\}$ that occur nowhere else in eq. (C.3), and, since they enter with a minus sign, such fluctuations are stable only for $h_2 < 0$. Thus, for either sign of h_2, there is an instability.

Consequently, the only possibility of finding a nontrivial, stable minimum is for the case $k=0$, when $\langle\varphi\rangle$ has no zero eigenvalues and the second line is absent. In that case, the preceding equation simplifies to

$$\frac{\delta^2 S_{\text{cl}}^{(\text{os})}}{V_4} = \frac{r^{[0]2}}{6} \left\{ 8h_1 \left| \sum_m (\delta\varphi_1)_{mm} \right|^2 + h_2 \sum_{mn} \left[|(\delta\varphi_1)_{mn}|^2 \right] \right\}. \tag{C.4}$$

$\delta\varphi_2$ drops out, so those fluctuations do not get mass. These are the would-be Goldstone bosons that, in the gauge theory, get "eaten" to form the massive vectors. The remaining fluctuations are the SU(5)⊗U(1) invariant scalars that get masses. For stability, so that these particles are not tachyons, this expression must be nonnegative for all fluctuations $\delta\varphi_{mn}$. The off-diagonal elements contribute $\propto h_2 \sum_{n>m} |\delta\varphi_{1mn}|^2$, so we must have $h_2 > 0$.

Setting the off-diagonal elements zero, the diagonal elements of $\delta\varphi_1$ make up a homogeneous polynomial of degree two in five real variables. For $\delta\varphi_1$ diagonal , we can rewrite the curly brackets in eq. (C.4) as

$$(8h_1 + h_2/5) \text{Tr}[\delta\varphi_1]^2 + h_2 \text{Tr}\left[\left(\delta\varphi_1 - \frac{\text{Tr}[\delta\varphi_1]}{5} \right)^2 \right]. \tag{C.5}$$

Therefore, this is nonnegative provided $h_1 + h_2/40 = h_3 > 0$. In sum, this symmetry-breaking is stable provided both h_2, h_2 are positive.

D DT scale and stability

The formulas for the determination of the DT scale v and the nature of the extrema are simple in principle but quite complicated in practice, even in the oversimplified model of matter considered in this paper. For completeness, we present the formulas for the on-shell values of $B_1^{(\text{os})}$ and $\varpi_2^{(\text{os})}$ for SO(10) with an arbitrary number N_F of fermions and with contribution T_F to the gauge boson beta-function.

$$\begin{aligned} B_1^{(\text{os})} = {} & \frac{11N_f - 5026}{4320} + \frac{10}{9x^2} - \frac{5}{3x} + \xi'(2\xi'-1) + \frac{(6\xi'-1)(60-z_2)}{30z_3} \\ & + \frac{(6\xi'-1)^2(z_2^2+1440)}{1200z_3^2} + \frac{\bar{a}(6\xi'-1)^2\left(20-15x+9x^2\xi'(4\xi'-1)\right)}{72xz_3} \\ & + \frac{\bar{a}^2(6\xi'-1)^4(5+9x^2\xi'^2)}{288z_3^2}, \end{aligned} \tag{D.1}$$

$$\varpi_2^{(\text{os})} = \frac{\alpha}{2160000x^3z_3^3}\left[-625\,\bar{a}^4x^3(6\xi'-1)^6\left(5+9x^2\xi'^2\right)^2\right.$$

$$-1875\,\bar{a}^3x^2z_3(6\xi'-1)^4\left(-200+x\left(611+N_F+110x\xi'-720\,x\xi'^2\right.\right.$$

$$\left.\left.+3x^2\xi'^2\left(-2427+2x(1+33\xi'+8820\xi'^2)\right)\right)\right)$$

$$+150\,\bar{a}^2x(6\xi'-1)^2\left[-2x^2(6\xi'-1)^2\left(5+9x^2\xi'^2\right)\left(1440+z_2^2\right)\right.$$

$$+60\,x^2(6\xi'-1)\left(5+6x^2\xi'^2(1+3\xi')\right)(z_2-60)z_3$$

$$-25z_3^2\left(800+x\left((3x-8)N_F+1913x+x\xi'\left(7191x(4\xi'-1)\right.\right.\right.$$

$$\left.\left.\left.\left.+40(2907\xi'-22)+6x^2\left(1+4\xi'(8+3(769-3222\xi')\xi')\right)\right)-21268\right)\right)\right]$$

$$+100\,\bar{a}z_3\left(3x^2(6\xi'-1)^2\left(1440\left(40+9x\left(-5+2x\xi'(7\xi'-1)\right)\right)\right.\right. \tag{D.2}$$

$$\left.+z_2^2\left(3x(6x(\xi'-1)\xi'-5)+40\right)\right)-360\,x^2z_3(6\xi'-1)\left(10(z_2-60)\right.$$

$$\left.\left.+x\left(600-5z_2\right)+3x^2\xi'\left(60+24\xi'^2(z_2-60)-z_2-4\xi'(30+z_2)\right)\right)\right.$$

$$-50z_3^2\left(3x\left(37648+(8-6x)N_F+12x^3\xi'(6\xi'-1)(2\xi'(411\xi'-85)-3)\right.\right.$$

$$\left.\left.+45x^2\left(7+4\xi'(-2+949\xi')\right)-4x\left(7099+20\xi'(2859\xi'-11)\right)\right)-1600\right)\right)$$

$$+3x^3\left[5z_3(6\xi'-1)\left(230400((6\xi'-1)(T_F-3)-36)-8640z_2(42\xi'-23)\right.\right.$$

$$-2880(1+6\xi')z_2^2+(61+210\xi')z_2^3\right)+100z_3^2\left(1920\left(6(69+5T_F)\xi'-324\xi'^2\right.\right.$$

$$\left.-5(30+T_F)\right)+5760z_2+(6\xi'(35+72\xi')-95)z_2^2\right)-12(6\xi'-1)^2\left(1440+z_2^2\right)^2$$

$$\left.\left.+4000\left(3180+72\xi'(2\xi'(z_2-60)-z_2)-41z_2\right)z_3^3+5\,640\,000\xi'(4\xi'-1)z_3^4\right]\right].$$

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