

A representation basis for the quantum integrable spin chain associated with the $\mathfrak{su}(3)$ algebra

Kun Hao,^a Junpeng Cao,^{b,c} Guang-Liang Li,^d Wen-Li Yang,^{a,e,1} Kangjie Shi^a
and Yupeng Wang^{b,c,1}

^a*Institute of Modern Physics, Northwest University,
Xian 710069, China*

^b*Beijing National Laboratory for Condensed Matter Physics,
Institute of Physics, Chinese Academy of Sciences,
Beijing 100190, China*

^c*Collaborative Innovation Center of Quantum Matter,
Beijing, China*

^d*Department of Applied Physics, Xian Jiaotong University,
Xian 710049, China*

^e*Beijing Center for Mathematics and Information Interdisciplinary Sciences,
Beijing 100048, China*

E-mail: haoke72@163.com, junpengcao@iphy.ac.cn,
leegl@mail.xjtu.edu.cn, wlyang@nwu.edu.cn, kjshi@nwu.edu.cn,
yupeng@iphy.ac.cn

ABSTRACT: An orthogonal basis of the Hilbert space for the quantum spin chain associated with the $\mathfrak{su}(3)$ algebra is introduced. Such kind of basis could be treated as a nested generalization of separation of variables (SoV) basis for high-rank quantum integrable models. It is found that all the monodromy-matrix elements acting on a basis vector take simple forms. With the help of the basis, we construct eigenstates of the $\mathfrak{su}(3)$ inhomogeneous spin torus (the trigonometric $\mathfrak{su}(3)$ spin chain with antiperiodic boundary condition) from its spectrum obtained via the off-diagonal Bethe Ansatz (ODBA). Based on small sites (i.e. $N = 2$) check, it is conjectured that the homogeneous limit of the eigenstates exists, which gives rise to the corresponding eigenstates of the homogenous model.

KEYWORDS: Bethe Ansatz, Lattice Integrable Models

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¹Corresponding author.

Contents

1	Introduction	1
2	$su(n)$ spin torus and its spectrum	2
3	Nested SoV basis	6
4	Eigenstates of the transfer matrix	10
5	Conclusions	12
A	Exchange relations	13
B	$su(n)$ case	14
C	Proof of the operator decomposition	14
D	Proof of (4.7)	15

1 Introduction

Quantum integrable system has played an important role in understanding the physical contents of the planar $\mathcal{N} = 4$ super-symmetric Yang-Mills (SYM) theory and the planar AdS/CFT [1, 2] (see also references therein). Moreover, it has also provided valuable insight into important universality class in condensed matter physics [3] and cold atom systems [4]. In the past several decades, the integrable quantum spin chains with $U(1)$ -symmetry (with periodic boundary or with diagonal open boundaries [5]) and with some constrained open boundaries [6–21] have been extensively studied by various Bethe ansatz methods for a finite lattice and by the vertex operator method [22] in an infinite or a half-infinite lattice [23–28].

Very recently, an important progress has been achieved in solving the eigenvalue problem of integrable models without $U(1)$ -symmetry [29] (i.e., the off-diagonal Bethe Ansatz (ODBA), for comprehensive introduction we refer the reader to [30]). Several long-standing models [29, 31–36] have since been solved. It should be noted that besides ODBA [37, 38] some other methods such as the q-Onsager algebra method [39–42], the separation of variables (SoV) method [43–49] and the modified algebraic Bethe ansatz method [50–53] were also used to obtain the eigenstates of the XXZ spin chains with generic boundary conditions. Remarkably, ODBA allows us to obtain eigenvalues of the $U(1)$ -broken models associated with higher-rank algebras such as the $su(n)$ spin chain with generic integrable boundary

fields [35], the Izergin-Korepin model¹ with generic boundary conditions [36], the Hubbard model [34] and the supersymmetric $t - J$ model [54] with unparallel boundary fields, and the open chain related to AdS/CFT [55]. However, the corresponding eigenstates for these models are still missing.

According to Liouville’s theorem, a key feature of integrable models is that their variables are completely separable. This concept was generalized to quantum integrable models by Sklyanin [56–58] and provided a promising approach to construct eigenstates of quantum integrable models without $U(1)$ -symmetry. Nevertheless, Sklyanin’s SoV procedure has only succeeded for some rank-one quantum integrable models and a proper SoV scheme for the high-rank quantum integrable models is still absent. The main task of the present paper is to propose a nested SoV basis for the $\mathfrak{su}(n)$ spin chain model. As an example of application, we construct exact eigenstates of the $\mathfrak{su}(3)$ spin torus (i.e., the trigonometric $\mathfrak{su}(3)$ spin chain with anti-periodic boundary condition), an archetype high-rank quantum integrable model without highest weight reference state, based on its spectrum recently obtained in [59] via ODBA.

The paper is organized as follows. Section 2 serves as an introduction to our notations for the inhomogeneous $\mathfrak{su}(n)$ spin torus and its spectrum. In section 3, we introduce a nested SoV basis of the Hilbert space of the $\mathfrak{su}(3)$ spin chain. It is found that the actions of the monodromy matrix elements on a basis vector have no compensating exchange terms on the level of the local operators (i.e., polarization free) and therefore become drastically simple. In section 4, with the help of the basis, as an example, we construct eigenstates of the transfer matrix for the $\mathfrak{su}(3)$ spin torus from its spectrum obtained via ODBA [59]. Concluding remarks are given in section 5. Some detailed technical proofs are given in appendices A–D.

2 $\mathfrak{su}(n)$ spin torus and its spectrum

Let \mathbf{V} denote an n -dimensional linear space with an orthonormal basis $\{|i\rangle|i = 1, \dots, n\}$. We introduce the Hamiltonian H as follows:

$$H = \sum_{j=1}^N h_{j,j+1}, \tag{2.1}$$

where N is the number of sites and $h_{j,j+1}$ is the local Hamiltonian given by

$$h_{j,j+1} = \frac{\partial}{\partial u} \{P_{j,j+1} R_{j,j+1}(u)\} \Big|_{u=0}. \tag{2.2}$$

Here $P_{j,j+1}$ is the permutation operator on the tensor space and the R -matrix $R(u) \in \text{End}(\mathbf{V} \otimes \mathbf{V})$ is the trigonometric R -matrix associated with the quantum group [60]

¹It is a model beyond A type.

$U_q(\widehat{\mathfrak{su}(n)})$, which was first proposed by Perk and Shultz [61] and further studied in [62–66]²

$$\begin{aligned}
 R(u) = & \sinh(u + \eta) \sum_{k=1}^n E^{k,k} \otimes E^{k,k} + \sinh u \sum_{k \neq l}^n E^{k,k} \otimes E^{l,l} \\
 & + \sinh \eta \left(\sum_{k < l}^n e^{\frac{n-2(l-k)}{n}u} + \sum_{k > l}^n e^{-\frac{n-2(k-l)}{n}u} \right) E^{k,l} \otimes E^{l,k}, \tag{2.3}
 \end{aligned}$$

where the n^2 fundamental matrices $\{E^{k,l} | k, l = 1, \dots, n\}$ are all $n \times n$ matrices with matrix entries $(E^{k,l})_{\beta}^{\alpha} = \delta_{\alpha}^k \delta_{\beta}^l$ and η is the crossing parameter. The R -matrix satisfies the quantum Yang-Baxter equation (QYBE)

$$R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2), \tag{2.4}$$

and possesses the properties:

Initial condition: $R_{12}(0) = \sinh \eta P_{1,2},$ (2.5)

Unitarity: $R_{12}(u)R_{21}(-u) = \rho_1(u) \times \text{id}, \quad \rho_1(u) = -\sinh(u+\eta) \sinh(u-\eta),$ (2.6)

Crossing-unitarity: $R_{12}^{t_1}(u)R_{21}^{t_1}(-u-n\eta) = \rho_2(u) \times \text{id}, \quad \rho_2(u) = -\sinh u \sinh(u+n\eta),$ (2.7)

Fusion conditions: $R_{12}(-\eta) = -2 \sinh \eta P_{1,2}^{(-)}.$ (2.8)

Here $R_{21}(u) = P_{1,2}R_{12}(u)P_{1,2}; P_{1,2}^{(-)}$ is the q -deformed anti-symmetric project operator [59] in the tensor product space $\mathbf{V} \otimes \mathbf{V}$; and t_i denotes the transposition in the i -th space. Here and below we adopt the standard notation: for any matrix $A \in \text{End}(\mathbf{V})$, A_j is an embedding operator in the tensor space $\mathbf{V} \otimes \mathbf{V} \otimes \dots$, which acts as A on the j -th space and as an identity on the other factor spaces; $R_{ij}(u)$ is an embedding operator of R -matrix in the tensor space, which acts as an identity on the factor spaces except for the i -th and j -th ones. For the $\mathfrak{su}(3)$ case the R -matrix reads

$$R(u) = \left(\begin{array}{c|cc|c} \bar{a}(u) & & & \\ \hline & \bar{b}(u) & \bar{c}(u) & \\ & & & \bar{d}(u) \\ \hline & \bar{d}(u) & \bar{b}(u) & \\ & & \bar{a}(u) & \\ & & & \bar{c}(u) \\ \hline & \bar{c}(u) & & \bar{b}(u) \\ & & \bar{d}(u) & \bar{b}(u) \\ & & & \bar{a}(u) \end{array} \right), \tag{2.9}$$

where the matrix elements are

$$\begin{aligned}
 \bar{a}(u) &= \sinh(u + \eta), & \bar{b}(u) &= \sinh u, \\
 \bar{c}(u) &= e^{\frac{u}{3}} \sinh \eta, & \bar{d}(u) &= e^{-\frac{u}{3}} \sinh \eta.
 \end{aligned} \tag{2.10}$$

²The R -matrix given by (2.3) corresponds to the so-called principal gradation, which is related to the R -matrix in homogeneous gradation by some gauge transformation [67].

Let us introduce the $n \times n$ twist matrix g

$$g = \begin{pmatrix} & & 1 \\ 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}, \quad \text{and } g^n = 1. \quad (2.11)$$

For the $\text{su}(3)$ case, it reads

$$g = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \text{and } g^3 = 1. \quad (2.12)$$

It is found that the R -matrix (2.3) is invariant with g ,

$$g_0 g_{0'} R_{00'}(u) g_0^{-1} g_{0'}^{-1} = R_{00'}(u). \quad (2.13)$$

This property enables us to construct the integrable $\text{su}(n)$ spin torus model [59].

Similar to the $\text{su}(2)$ spin torus (or the XXZ spin chain with anti-periodic boundary condition) [68], the $\text{su}(n)$ spin torus is described by the Hamiltonian H given by (2.1) with anti-periodic boundary conditions

$$E_{N+1}^{k,l} = g_1 E_1^{k,l} g_1^{-1}, \quad k, l = 1, \dots, n. \quad (2.14)$$

Let us introduce the ‘‘row-to-row’’ monodromy matrix $T(u)$, an $n \times n$ matrix with operator-valued elements acting on $\mathbf{V}^{\otimes N}$,

$$T_0(u) = R_{0N}(u - \theta_N) R_{0N-1}(u - \theta_{N-1}) \cdots R_{01}(u - \theta_1). \quad (2.15)$$

Here $\{\theta_j | j = 1, \dots, N\}$ are generic free complex parameters usually called as inhomogeneity parameters. The transfer matrix $t(u)$ of the associated spin chain with antiperiodic boundary condition (2.14) can be constructed similarly as [68]

$$t(u) = \text{tr}_0 \{ g_0 T_0(u) \}. \quad (2.16)$$

The QYBE and the relation (2.13) lead to the fact that the transfer matrices $t(u)$ given by (2.16) with different spectral parameters are mutually commuting: $[t(u), t(v)] = 0$. The Hamiltonian (2.1) with the anti-periodic boundary condition (2.14) can be obtained from the transfer matrix as

$$H = \sinh \eta \left. \frac{\partial \ln t(u)}{\partial u} \right|_{u=0, \{\theta_j\}=0}. \quad (2.17)$$

The eigenvalues $\Lambda(u)$ of the transfer matrix $t(u)$ in case of $\text{su}(3)$ are given in terms of an inhomogeneous $T - Q$ relation [59]

$$\begin{aligned} \Lambda(u) = e^{\frac{u}{3}} \left\{ e^{\phi_1} e^u a(u) \frac{Q^{(1)}(u - \eta)}{Q^{(2)}(u)} + e^{-\phi_1} \omega e^{-u - \frac{2\eta}{3}} d(u) \frac{Q^{(2)}(u + \eta) Q^{(3)}(u - \eta)}{Q^{(1)}(u) Q^{(4)}(u)} \right. \\ + \omega^2 e^{-u - \frac{4\eta}{3}} d(u) \frac{Q^{(4)}(u + \eta)}{Q^{(3)}(u)} + a(u) d(u) \frac{Q^{(3)}(u - \eta) f_1(u)}{Q^{(1)}(u) Q^{(2)}(u)} \\ \left. + a(u) d(u) \frac{Q^{(2)}(u + \eta) f_2(u)}{Q^{(3)}(u) Q^{(4)}(u)} \right\}, \quad (2.18) \end{aligned}$$

where

$$\begin{aligned}
 a(u) &= \prod_{l=1}^N \sinh(u - \theta_l + \eta), & d(u) &= \prod_{l=1}^N \sinh(u - \theta_l) = a(u - \eta), & (2.19) \\
 Q^{(i)}(u) &= \prod_{l=1}^N \sinh(u - \lambda_l^{(i)}), & & i = 1, 2, 3, 4,
 \end{aligned}$$

$\omega = e^{\frac{2i\pi}{3}}$ and the functions $f_1(u)$ and $f_2(u)$ are given by

$$f_1(u) = f_1^{(+)} e^u + f_1^{(-)} e^{-u}, \quad f_2(u) = f_2^{(-)} e^{-u}.$$

The $4N + 4$ parameters $\{\lambda_l^{(i)} | l = 1, \dots, N; i = 1, 2, 3, 4\}$, $f_1^{(\pm)}$, $f_2^{(-)}$ and e^{ϕ_1} satisfy the associated BAEs:

$$\omega e^{-\phi_1} e^{-\lambda_j^{(1)} - \frac{2\eta}{3}} \frac{Q^{(2)}(\lambda_j^{(1)} + \eta)}{Q^{(4)}(\lambda_j^{(1)})} + a(\lambda_j^{(1)}) \frac{f_1(\lambda_j^{(1)})}{Q^{(2)}(\lambda_j^{(1)})} = 0, \quad j = 1, \dots, N, \quad (2.20)$$

$$e^{\phi_1} e^{\lambda_j^{(2)}} Q^{(1)}(\lambda_j^{(2)} - \eta) + d(\lambda_j^{(2)}) \frac{Q^{(3)}(\lambda_j^{(2)} - \eta) f_1(\lambda_j^{(2)})}{Q^{(1)}(\lambda_j^{(2)})} = 0, \quad j = 1, \dots, N, \quad (2.21)$$

$$\omega^2 e^{-\lambda_j^{(3)} - \frac{4\eta}{3}} Q^{(4)}(\lambda_j^{(3)} + \eta) + a(\lambda_j^{(3)}) \frac{Q^{(2)}(\lambda_j^{(3)} + \eta) f_2(\lambda_j^{(3)})}{Q^{(4)}(\lambda_j^{(3)})} = 0, \quad j = 1, \dots, N, \quad (2.22)$$

$$\omega e^{-\phi_1} e^{-\lambda_j^{(4)} - \frac{2\eta}{3}} \frac{Q^{(3)}(\lambda_j^{(4)} - \eta)}{Q^{(1)}(\lambda_j^{(4)})} + a(\lambda_j^{(4)}) \frac{f_2(\lambda_j^{(4)})}{Q^{(3)}(\lambda_j^{(4)})} = 0, \quad j = 1, \dots, N, \quad (2.23)$$

$$e^{\phi_1} e^{-\Theta - \chi^{(1)} + \chi^{(2)}} + e^{-2\Theta + \chi^{(1)} + \chi^{(2)} - \chi^{(3)}} f_1^{(+)} = 0, \quad (2.24)$$

$$\begin{aligned} & \omega e^{-\phi_1} e^{-\frac{2\eta}{3} + \Theta - \chi^{(1)} + \chi^{(2)} + \chi^{(3)} - \chi^{(4)}} + \omega^2 e^{-\frac{4\eta}{3} + \Theta - \chi^{(3)} + \chi^{(4)} - N\eta} \\ & + e^{2\Theta - N\eta} \{ e^{-\chi^{(1)} - \chi^{(2)} + \chi^{(3)} + N\eta} f_1^{(-)} + e^{+\chi^{(2)} - \chi^{(3)} - \chi^{(4)} - N\eta} f_2^{(-)} \} = 0, \end{aligned} \quad (2.25)$$

$$\begin{aligned} & \omega e^{-\Theta - \chi^{(3)} + \chi^{(4)}} + \omega^2 e^{\phi_1} e^{-\frac{2\eta}{3} - \Theta - \chi^{(1)} + \chi^{(2)} + \chi^{(3)} - \chi^{(4)} + N\eta} \\ & + e^{-2\Theta + N\eta} \{ \omega^2 e^{-\frac{2\eta}{3} + \chi^{(1)} + \chi^{(2)} - \chi^{(4)}} f_1^{(+)} + e^{\phi_1} e^{\frac{2\eta}{3} - \chi^{(1)} + \chi^{(3)} + \chi^{(4)} + N\eta} f_2^{(-)} \} = 0, \end{aligned} \quad (2.26)$$

$$e^{-\phi_1} e^{-\frac{4\eta}{3} + \Theta - \chi^{(1)} + \chi^{(2)} - N\eta} + \omega^2 e^{-\frac{2\eta}{3} + 2\Theta - \chi^{(1)} - \chi^{(2)} + \chi^{(4)} - N\eta} f_1^{(-)} = 0, \quad (2.27)$$

where

$$\Theta = \sum_{l=1}^N \theta_l, \quad \chi^{(i)} = \sum_{l=1}^N \lambda_l^{(i)}, \quad i = 1, 2, 3, 4. \quad (2.28)$$

In homogeneous limit: $\{\theta_j \rightarrow 0\}$, the resulting $T - Q$ relation (2.18) and the associated BAEs (2.20)–(2.27) give rise to the eigenvalue and BAEs of the corresponding homogeneous spin chain (i.e., the $\text{su}(3)$ spin torus).

3 Nested SoV basis

In this section, we propose a convenient basis of the Hilbert space parameterized by the N generic inhomogeneity parameters $\{\theta_j | j = 1, \dots, N\}$. It is found that actions of all the monodromy matrix elements on a basis vector take drastically simple forms like those in the so-called F-basis³ [69–73]. All these ingredients allow us to construct exact eigenstates of the $\text{su}(n)$ spin torus model.

For convenience, let us introduce the notations

$$A(u) = T_1^1(u), \quad B_i(u) = T_i^1(u), \quad C^i(u) = T_1^i(u), \quad \text{for } i = 2, \dots, n, \quad (3.1)$$

$$D_j^i(u) = T_j^i(u), \quad \text{for } i, j = 2, \dots, n. \quad (3.2)$$

The exchange relations among the above operators are listed in appendix A. Let us introduce further the left quasi-vacuum state $\langle 0|$ and the right quasi-vacuum state $|0\rangle$

$$\langle 0| = \langle 1, \dots, 1|, \quad |0\rangle = |1, 1, \dots, 1\rangle. \quad (3.3)$$

The operators (3.1)–(3.2) acting on the states give rise to

$$\langle 0| A(u) = a(u) \langle 0|, \quad \langle 0| D_i^l(u) = d(u) \delta_i^l \langle 0|, \quad i, l = 2, \dots, n, \quad (3.4)$$

$$\langle 0| B_i(u) = 0, \quad \langle 0| C^i(u) \neq 0, \quad i = 2, \dots, n, \quad (3.5)$$

$$A(u) |0\rangle = a(u) |0\rangle, \quad D_i^l(u) |0\rangle = d(u) \delta_i^l |0\rangle, \quad i, l = 2, \dots, n, \quad (3.6)$$

$$C^i(u) |0\rangle = 0, \quad B_i(u) |0\rangle \neq 0, \quad i = 2, \dots, n, \quad (3.7)$$

where the functions $a(u)$ and $d(u)$ are given by (2.19).

In the following part of this section, taking the $\text{su}(3)$ spin chain as an example, we construct a nested SoV basis of the Hilbert space. The generalization to the $\text{su}(n)$ case is given in appendix B. For two non-negative integers m_2 and m such that $m_2 \leq m \leq N$, let us introduce m positive integers $P = \{p_1, \dots, p_m\}$ such that

$$1 \leq p_1 < p_2 < \dots < p_{m_2} \leq N, \quad 1 \leq p_{m_2+1} < \dots < p_m \leq N, \quad \text{and } p_j \neq p_l. \quad (3.8)$$

For each P satisfies the above condition, let us introduce left and right states parameterized by the N inhomogeneity parameters $\{\theta_j\}$ as follows:

$$\langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} | = \langle 0| C^2(\theta_{p_1}) \dots C^2(\theta_{p_{m_2}}) C^3(\theta_{p_{m_2+1}}) \dots C^3(\theta_{p_m}), \quad (3.9)$$

$$| \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} \rangle = B_3(\theta_{p_m}) \dots B_3(\theta_{p_{m_2+1}}) B_2(\theta_{p_{m_2}}) \dots B_2(\theta_{p_1}) |0\rangle, \quad (3.10)$$

where m_2 (resp. $m - m_2$) is the number of the operators $C^2(u)$ or $B_2(u)$ (resp. $C^3(u)$ or $B_3(u)$).

³It is interesting to study the relation between this basis and the F-basis [71, 73].

It is easy to check that the states (3.9) and the states (3.10) are eigenstates of the operator $D_3^3(u)$, namely,

$$\begin{aligned} \langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} | D_3^3(u) = d(u) \prod_{l=m_2+1}^m \frac{\sinh(u - \theta_{p_l} + \eta)}{\sinh(u - \theta_{p_l})} \\ \times \langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} |, \end{aligned} \quad (3.11)$$

$$\begin{aligned} D_3^3(u) | \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} \rangle = d(u) \prod_{l=m_2+1}^m \frac{\sinh(u - \theta_{p_l} + \eta)}{\sinh(u - \theta_{p_l})} \\ \times | \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} \rangle. \end{aligned} \quad (3.12)$$

Noting the fact that $d(\theta_l) = 0$, $l = 1, \dots, N$ and using the exchange relations (A.1)–(A.10), we can derive some useful relations

$$\langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} | D_j^i(\theta_{p_l}) = 0, \quad l = m+1, \dots, N, \text{ and } i, j = 2, 3, \quad (3.13)$$

$$\langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} | B_i(\theta_{p_l}) = 0, \quad l = m+1, \dots, N, \text{ and } i = 2, 3, \quad (3.14)$$

$$D_j^i(\theta_{p_l}) | \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} \rangle = 0, \quad l = m+1, \dots, N, \text{ and } i = 2, 3, \quad (3.15)$$

$$C^i(\theta_{p_l}) | \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} \rangle = 0, \quad l = m+1, \dots, N, \text{ and } i = 2, 3. \quad (3.16)$$

The above relations and the exchange relations (A.1)–(A.10) allow us to derive the orthogonal relations between the left states and the right states

$$\begin{aligned} \langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} | \theta_{q_1}, \dots, \theta_{q_{m'_2}}; \theta_{q_{m'_2+1}}, \dots, \theta_{q_{m'}} \rangle = \\ \delta_{m, m'} \delta_{m_2, m'_2} \prod_{k=1}^m \delta_{p_k, q_k} G_m(\theta_{p_1}, \dots, \theta_{p_{m_2}} | \theta_{p_{m_2+1}}, \dots, \theta_{p_m}), \end{aligned} \quad (3.17)$$

where the factor $G_m(\theta_{p_1}, \dots, \theta_{p_{m_2}} | \theta_{p_{m_2+1}}, \dots, \theta_{p_m})$ is given by

$$\begin{aligned} G_m(\theta_{p_1}, \dots, \theta_{p_{m_2}} | \theta_{p_{m_2+1}}, \dots, \theta_{p_m}) = \\ \prod_{k=1}^{m_2} \sinh \eta d_{p_k}(\theta_{p_k}) a(\theta_{p_k}) \prod_{l=1, l \neq k}^{m_2} \frac{\sinh(\theta_{p_k} - \theta_{p_l} + \eta)}{\sinh(\theta_{p_k} - \theta_{p_l})} \\ \times \prod_{k=m_2+1}^m \sinh \eta d_{p_k}(\theta_{p_k}) a(\theta_{p_k}) \left\{ \prod_{l=m_2+1, l \neq k}^m \frac{\sinh(\theta_{p_k} - \theta_{p_l} + \eta)}{\sinh(\theta_{p_k} - \theta_{p_l})} \right. \\ \left. \times \prod_{l=1}^{m_2} \frac{\sinh(\theta_{p_k} - \theta_{p_l} - \eta)}{\sinh(\theta_{p_k} - \theta_{p_l})} \right\}. \end{aligned} \quad (3.18)$$

Here the functions $\{d_l(u)\}$ are given by

$$d_l(u) = \prod_{k=1, k \neq l}^N \sinh(u - \theta_k), \quad l = 1, \dots, N. \quad (3.19)$$

On the other hand, we know that the total number of the linear-independent left (right) states given in (3.9) ((3.10)) is

$$\sum_{m=0}^N \frac{N!}{(N-m)!m!} \sum_{m_2=0}^m \frac{m!}{(m-m_2)!m_2!} = \sum_{m=0}^N \frac{N!}{(N-m)!m!} 2^m = 3^N. \quad (3.20)$$

Thus these right (left) states form an orthogonal right (left) basis of the Hilbert space, namely,

$$\begin{aligned} \text{id} &= \sum_{m=0}^N \sum_{m_2=0}^m \sum_P \frac{1}{G_m(\theta_{p_1}, \dots, \theta_{p_{m_2}} | \theta_{p_{m_2+1}}, \dots, \theta_{p_m})} \\ &\times |\theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m}\rangle \langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m}|, \end{aligned} \quad (3.21)$$

where the notation \sum_P indicates the sum over all P satisfying the condition (3.8). Hence any right (left) state can be decomposed as a unique linear combination of these basis. Moreover, direct calculation shows that actions of the monodromy matrix elements on this basis become drastically simple (see below (3.22)–(3.26)). Here we list some of them relevant for us to construct eigenstates of the transfer matrix in the next section,

$$\begin{aligned} \langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} | D_3^3(u) &= \\ d(u) \prod_{l=m_2+1}^m \frac{\sinh(u - \theta_{p_l} + \eta)}{\sinh(u - \theta_{p_l})} & \\ \times \langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} |, & \end{aligned} \quad (3.22)$$

$$\begin{aligned} \langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} | D_3^2(u) &= \\ \sum_{l=m_2+1}^m \frac{\sinh \eta e^{\frac{u-\theta_{p_l}}{3}} d(u)}{\sinh(u - \theta_{p_l})} & \\ \times \prod_{k=m_2+1, k \neq l}^m \frac{\sinh(u - \theta_{p_k} + \eta)}{\sinh(u - \theta_{p_k})} \frac{\sinh(\theta_{p_l} - \theta_{p_k} - \eta)}{\sinh(\theta_{p_l} - \theta_{p_k})} & \\ \times \langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_{l-1}}, \theta_{p_{l+1}}, \dots, \theta_{p_m} |, & \end{aligned} \quad (3.23)$$

$$\begin{aligned} \langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} | D_2^3(u) &= \\ \sum_{l=1}^{m_2} \frac{\sinh \eta e^{-\frac{u-\theta_{p_l}}{3}} d(u)}{\sinh(u - \theta_{p_l})} & \\ \times \left\{ \prod_{k=1, k \neq l}^{m_2} \frac{\sinh(\theta_{p_l} - \theta_{p_k} + \eta)}{\sinh(\theta_{p_l} - \theta_{p_k})} \prod_{k=m_2+1}^m \frac{\sinh(u - \theta_{p_k} + \eta)}{\sinh(u - \theta_{p_k})} \right\} & \\ \times \langle \theta_{p_1}, \dots, \theta_{p_{l-1}}, \theta_{p_{l+1}}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} |, & \end{aligned} \quad (3.24)$$

$$\begin{aligned}
 \langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} | B_3(u) = & \\
 & \sum_{l=m_2+1}^m \frac{\sinh \eta e^{-\frac{u-\theta_{p_l}}{3}} d(u)}{\sinh(u-\theta_{p_l})} a(\theta_{p_l}) \\
 & \times \prod_{k=m_2+1, k \neq l}^m \frac{\sinh(u-\theta_{p_k}+\eta)}{\sinh(u-\theta_{p_k})} \frac{\sinh(\theta_{p_l}-\theta_{p_k}-\eta)}{\sinh(\theta_{p_l}-\theta_{p_k})} \\
 & \times \prod_{\alpha=1}^{m_2} \frac{\sinh(\theta_{p_l}-\theta_{p_\alpha}-\eta)}{\sinh(\theta_{p_l}-\theta_{p_\alpha})} \langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_{l-1}}, \theta_{p_{l+1}}, \dots, \theta_{p_m} | \\
 + \sum_{l=m_2+1}^m \frac{\sinh \eta e^{-\frac{u-\theta_{p_l}}{3}} d(u)}{\sinh(u-\theta_{p_l})} & \prod_{k=m_2+1, k \neq l}^m \frac{\sinh(u-\theta_{p_k}+\eta)}{\sinh(u-\theta_{p_k})} \frac{\sinh(\theta_{p_l}-\theta_{p_k}-\eta)}{\sinh(\theta_{p_l}-\theta_{p_k})} \\
 \times \sum_{\alpha=1}^{m_2} \frac{\sinh \eta e^{-\frac{\theta_{p_\alpha}-\theta_{p_l}}{3}}}{\sinh(\theta_{p_l}-\theta_{p_\alpha})} & a(\theta_{p_\alpha}) \prod_{k=1, k \neq \alpha}^{m_2} \frac{\sinh(\theta_{p_\alpha}-\theta_{p_k}-\eta)}{\sinh(\theta_{p_\alpha}-\theta_{p_k})} \\
 \times \langle \theta_{p_1}, \dots, \theta_{p_{\alpha-1}}, \theta_{p_l}, \theta_{p_{\alpha+1}}, \dots, & \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_{l-1}}, \theta_{p_{l+1}}, \dots, \theta_{p_m} |, \quad (3.25)
 \end{aligned}$$

$$\begin{aligned}
 \langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} | C^3(u) = & \\
 & \sum_{l=m+1}^N \frac{e^{-\frac{u-\theta_{p_l}}{3}} d(u)}{\sinh(u-\theta_{p_l}) d_{p_l}(\theta_{p_l})} \\
 & \times \prod_{k=m_2+1}^m \frac{\sinh(u-\theta_{p_k}+\eta)}{\sinh(u-\theta_{p_k})} \frac{\sinh(\theta_{p_l}-\theta_{p_k})}{\sinh(\theta_{p_l}-\theta_{p_k}+\eta)} \\
 & \times \langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m}, \theta_{p_l} | \\
 + \sum_{l=m+1}^N \sum_{\alpha=1}^{m_2} \frac{e^{-\frac{u-\theta_{p_\alpha}}{3}}}{\sinh(u-\theta_{p_\alpha})} & \prod_{k=m_2+1}^m \frac{\sinh(u-\theta_{p_k}+\eta)}{\sinh(u-\theta_{p_k})} \frac{\sinh(\theta_{p_l}-\theta_{p_k})}{\sinh(\theta_{p_l}-\theta_{p_k}+\eta)} \\
 \times \frac{\sinh \eta d(u) e^{\frac{\theta_{p_l}-\theta_{p_\alpha}}{3}}}{d_{p_l}(\theta_{p_l}) \sinh(\theta_{p_\alpha}-\theta_{p_l}-\eta)} & \prod_{k=1, k \neq \alpha}^{m_2} \frac{\sinh(\theta_{p_l}-\theta_{p_k})}{\sinh(\theta_{p_l}-\theta_{p_k}+\eta)} \frac{\sinh(\theta_{p_\alpha}-\theta_{p_k}+\eta)}{\sinh(\theta_{p_\alpha}-\theta_{p_k})} \\
 \times \langle \theta_{p_1}, \dots, \theta_{p_{\alpha-1}}, \theta_{p_l}, \theta_{p_{\alpha+1}}, \dots, & \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m}, \theta_{p_\alpha} |. \quad (3.26)
 \end{aligned}$$

The sketch proof of the above operator decompositions is given in appendix C. Similarly, one may derive operator decompositions on the right basis which also have simple forms as (3.22)–(3.26).

Some remarks are in order. In the rational limit,⁴ the resulting basis serves as the SoV basis for the associated rational spin chain model.⁵ We have checked that each basis vector given by (3.9) and (3.10) for the su(3) case (the generalizations to the su(n) case are given in appendix B, see (B.1) and (B.2) below) is an off-shell Bethe state obtained via the nested algebraic Bethe Ansatz [75, 76] by replacing the Bethe roots with some sets

⁴Redefine: $u \rightarrow \epsilon u$, $\theta_j \rightarrow \epsilon \theta_j$ and $\eta \rightarrow \epsilon \eta$, then take the limit $\epsilon \rightarrow 0$.

⁵The resulting SoV basis for the rational spin chain model is different from that in [74]. It is interesting to study the relation between them.

of the inhomogeneity parameters.⁶ This observation provides an efficient way to construct similar nested SoV basis for general high-rank quantum integrable models. From explicit expressions (3.22)–(3.26), one can see that in the basis (3.9) the operators have no compensating exchange terms on the level of the local operators (i.e. polarization free), which have similar simple forms as those in the F-basis [70–73] and allow us to compute correlation functions [77] for quantum spin chains associated with higher-rank algebras [78–81].

4 Eigenstates of the transfer matrix

In this section, we adopt the method developed in [37, 38] (see also [30]) to construct eigenstates of the $\mathfrak{su}(3)$ spin torus based on the inhomogeneous $T - Q$ relations given by (2.18) [59] and the basis introduced in the previous section. For the $\mathfrak{su}(3)$ case, the monodromy matrix is expressed in terms of the operators (3.1)–(3.2) as

$$T(u) = \begin{pmatrix} A(u) & B_2(u) & B_3(u) \\ C^2(u) & D_2^2(u) & D_3^2(u) \\ C^3(u) & D_2^3(u) & D_3^3(u) \end{pmatrix}. \quad (4.1)$$

The corresponding transfer matrix (2.16) reads

$$t(u) = B_2(u) + D_3^2(u) + C^3(u). \quad (4.2)$$

The commutativity of the transfer matrices $t(u)$ with different spectral parameters implies that they have common eigenstates. Let $|\Psi\rangle$ be a common eigenstate of $t(u)$, which does not depend upon u , with an eigenvalue $\Lambda(u)$, i.e.,

$$t(u)|\Psi\rangle = \Lambda(u)|\Psi\rangle,$$

where the eigenvalue $\Lambda(u)$ of the transfer matrix $t(u)$ is given by the inhomogeneous $T - Q$ relation (2.18). Due to the fact that the left states $\{|\theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m}\rangle, m_2 = 0, \dots, m; m = 0, \dots, N\}$ given by (3.9) form a basis of the dual Hilbert space, the eigenstate $|\Psi\rangle$ is completely determined (up to an overall scalar factor) by the following scalar products [29, 37, 38]

$$F_{m_2, m-m_2}(\theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m}) = \langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} | \Psi \rangle, \quad (4.3)$$

$$1 \leq p_1 < \dots < p_{m_2}, \quad 1 \leq p_{m_2+1} < \dots < p_m \leq N, \quad p_j \neq p_k, \quad 0 \leq m_2 \leq m \leq N.$$

⁶A general off-shell Bethe state is $|\lambda_1, \dots, \lambda_m; \lambda_1^{(1)}, \dots, \lambda_{m-m_2}^{(1)}\rangle = B_{i_1}(\lambda_1) \dots B_{i_m}(\lambda_m) F^{i_1, \dots, i_m} |0\rangle$, where $\{F^{i_1, \dots, i_m} |i_l = 2, 3\}$ are the vector components of a nested off-shell Bethe state $B^{(1)}(\lambda_1^{(1)}) \dots B^{(1)}(\lambda_{m-m_2}^{(1)}) |0\rangle^{(1)} = \sum_{i_1, \dots, i_{m-m_2}=2}^3 F^{i_1, \dots, i_{m-m_2}} |i_1, \dots, i_{m-m_2}\rangle^{(1)}$, and the operator $B^{(1)}(u)$ and $|0\rangle^{(1)}$ are the corresponding creation operator and the reference state associated with the nested $\mathfrak{su}(2)$ spin chain with m sites and the corresponding inhomogeneous parameters being $\{\lambda_1, \dots, \lambda_m\}$ [75, 76]. For general values of $\lambda_1, \dots, \lambda_m$ and $\lambda_1^{(1)}, \dots, \lambda_{m-m_2}^{(1)}$, the Bethe state $|\lambda_1, \dots, \lambda_m; \lambda_1^{(1)}, \dots, \lambda_{m-m_2}^{(1)}\rangle$ is a linear combination of the vectors (3.10) [80, 81]. However, if the parameters $\{\lambda_l | l = 1, \dots, m\}$ are particularly chosen as $\{\lambda_l = \theta_{p_l} | l = 1, \dots, m\}$ and then the nested parameters $\{\lambda_n^{(1)} | n = 1, \dots, m-m_2\}$ have to take the values in the chosen set of $\{\lambda_l | l = 1, \dots, m\}$ (e.g., $\{\lambda_n^{(1)} = \theta_{p_n} | n = m_2 + 1, \dots, m\}$), the corresponding linear combination becomes drastically simple such that only one term such as (3.10) does remain.

Following [37, 38], let us consider the quantities $\langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} | t(\theta_{p_{m+1}}) | \Psi \rangle$. Acting $t(\theta_{p_{m+1}})$ to the right gives rise to the relation

$$\Lambda(\theta_{p_{m+1}}) F_{m_2, m-m_2}(\theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m}) = \langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} | t(\theta_{p_{m+1}}) | \Psi \rangle. \quad (4.4)$$

With the help of the expression (4.2) of the transfer matrix and the relations (3.13)–(3.14), by acting $t(\theta_{p_{m+1}})$ to the left we readily obtain

$$F_{m_2, m-m_2}(\theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m}) = \left\{ \prod_{l=m_2+1}^m \Lambda(\theta_{p_l}) \right\} F_{m_2}(\theta_{p_1}, \dots, \theta_{p_{m_2}}), \quad (4.5)$$

where the scalar products $F_m(\theta_{p_1}, \dots, \theta_{p_m})$ are given by

$$F_m(\theta_{p_1}, \dots, \theta_{p_m}) = \langle 0 | C^2(\theta_{p_1}) \dots C^2(\theta_{p_m}) | \Psi \rangle, \quad m = 0, \dots, N. \quad (4.6)$$

It follows that in order to obtain all the scalar products (4.3) it is sufficient to compute the scalar products (4.6). After a tedious calculation, we have

$$\begin{aligned} F_m(\theta_{p_1}, \dots, \theta_{p_m}) &= \sum_{1 \leq p'_1 < \dots < p'_m \leq N} g_m(\theta_{p_1}, \dots, \theta_{p_m} | \theta_{p'_1}, \dots, \theta_{p'_m}) \\ &\times \prod_{\alpha=1}^m \prod_{k=m+1}^N \sinh(\theta_{p'_\alpha} - \theta_{p_k} + \eta) \frac{\prod_{l=1}^m \Lambda(\theta_{p'_l})}{f_m(\theta_{p'_1}, \dots, \theta_{p'_m})} \frac{\prod_{l=1}^N a(\theta_k)}{\prod_{k=m+1}^N \Lambda(\theta_{p_k})} \langle \bar{0} | \Psi \rangle, \end{aligned} \quad (4.7)$$

where the state $\langle \bar{0} | = \langle 3, \dots, 3 |$ and the functions $g_m(v_1, \dots, v_m | u_1, \dots, u_m)$ and $f_m(\theta_{p_1}, \dots, \theta_{p_m})$ are given by

$$g_m(v_1, \dots, v_m | u_1, \dots, u_m) = \frac{\prod_{\alpha=1}^m \prod_{k=1}^m \sinh(u_\alpha - v_k + \eta) \sinh(u_\alpha - v_k)}{\prod_{k < l}^m \sinh(u_l - u_k) \sinh(v_k - v_l)} \det \mathcal{M}, \quad (4.8)$$

$$f_m(\theta_{p_1}, \dots, \theta_{p_m}) = \prod_{l=1}^m \sinh \eta d_{p_l}(\theta_{p_l}) a(\theta_{p_l}) \prod_{k=1, k \neq l}^m \frac{\sinh(\theta_{p_l} - \theta_{p_k} + \eta)}{\sinh(\theta_{p_l} - \theta_{p_k})}, \quad (4.9)$$

and \mathcal{M} is an $m \times m$ matrix with matrix elements

$$\mathcal{M}_{\alpha, k} = \frac{\sinh \eta e^{-\frac{u_\alpha - v_k}{3}}}{\sinh(u_\alpha - v_k + \eta) \sinh(u_\alpha - v_k)}, \quad \alpha, k = 1, \dots, m. \quad (4.10)$$

The proof of (4.7) is given in appendix D.

The identity decomposition (3.21) allows us to retrieve the eigenstate $|\Psi\rangle$ of the transfer matrix corresponding to an eigenvalue $\Lambda(u)$ as

$$\begin{aligned} |\Psi\rangle &= \sum_{m=0}^N \sum_{m_2=0}^m \sum_P \frac{\langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} | \Psi \rangle}{G_m(\theta_{p_1}, \dots, \theta_{p_{m_2}} | \theta_{p_{m_2+1}}, \dots, \theta_{p_m})} \\ &\times |\theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m}\rangle \\ &= \sum_{m=0}^N \sum_{m_2=0}^m \sum_P \frac{F_{m_2}(\theta_{p_1}, \dots, \theta_{p_{m_2}}) \prod_{k=m_2+1}^m \Lambda(\theta_{p_k})}{G_m(\theta_{p_1}, \dots, \theta_{p_{m_2}} | \theta_{p_{m_2+1}}, \dots, \theta_{p_m})} \\ &\times |\theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m}\rangle, \end{aligned} \quad (4.11)$$

where the factors $F_{m_2}(\theta_{p_1}, \dots, \theta_{p_{m_2}})$ and $G_m(\theta_{p_1}, \dots, \theta_{p_{m_2}} | \theta_{p_{m_2+1}}, \dots, \theta_{p_m})$ are given respectively by (4.7) and (3.18). It should be emphasized that the factor $F_{m_2}(\theta_{p_1}, \dots, \theta_{p_{m_2}})$ does depend upon the corresponding eigenvalue $\Lambda(u)$ associated with the eigenstate $|\Psi\rangle$, while $G_m(\theta_{p_1}, \dots, \theta_{p_{m_2}} | \theta_{p_{m_2+1}}, \dots, \theta_{p_m})$ does not.

Some remarks are in order. In the homogeneous limit, the resulting eigenstate (4.11) (if it exists) becomes the eigenstate of the homogeneous quantum spin chain (i.e. the $\text{su}(3)$ spin torus) due to the fact the $T - Q$ relation (2.18) and the associated BAEs (2.20)–(2.27) have well-defined homogeneous limits. We have checked that such a limit of the state (4.11) does exist for some small N . For an example, here we present the limit of the $N = 2$ case

$$\begin{aligned}
 \lim_{\theta_1, \theta_2 \rightarrow 0} |\Psi\rangle \propto & |0\rangle + \frac{1}{\sinh^3 \eta} [\Lambda' B_3 + \Lambda B'_3 - 2 \coth \eta \Lambda B_3] |0\rangle + \frac{\Lambda^2}{\sinh^8 \eta} B_3 B_3 |0\rangle \\
 & + \frac{\Lambda^2}{a^2(0)} \left\{ \left[\left(\frac{8}{9} - 2 \coth \eta \frac{\Lambda'}{\Lambda} + \left(\frac{\Lambda'}{\Lambda} \right)^2 \right) B_2 + \left(\frac{\Lambda'}{\Lambda} - \coth \eta - \frac{1}{3} \right) B'_2 \right] \right. \\
 & \quad + \frac{\Lambda}{\sinh^4 \eta} \left[\left(\coth \eta \frac{\Lambda'}{\Lambda} - \frac{\Lambda'}{3\Lambda} - \frac{8}{9} \right) B_3 B_2 \right. \\
 & \quad \quad \left. \left. + \left(\frac{\Lambda'}{\Lambda} - \coth \eta - \frac{1}{3} \right) (B'_3 B_2 - B_3 B'_2) \right] \right. \\
 & \quad \left. + \frac{\Lambda^2}{\sinh^8 \eta} B_2 B_2 \right\} |0\rangle, \tag{4.12}
 \end{aligned}$$

where

$$\begin{aligned}
 B_i &= B_i(0), & B'_i &= \left. \frac{\partial}{\partial u} B_i(u) \right|_{u=0}, & i &= 2, 3, \\
 \Lambda &= \Lambda(0), & \Lambda' &= \left. \frac{\partial}{\partial u} \Lambda(u) \right|_{u=0}.
 \end{aligned}$$

It is conjectured that the eigenstate (4.11) for generic N has a well-defined homogeneous limit. However, the direct proof remains an important open problem.

5 Conclusions

In this paper, we introduced a convenient basis of the Hilbert space, which could be treated as the SoV basis for the trigonometric spin chain model associated with the $\text{su}(3)$ algebra. We have demonstrated that the monodromy matrix elements acting on a generic basis vector take simple forms such as (3.22)–(3.26) without compensating exchange terms on the level of the local operators (i.e. polarization free). With the help of this basis, the corresponding eigenstates of the transfer matrix can be constructed by (4.11) via its ODBA solution [59]. In the rational limit, the resulting basis serves as the SoV basis for the associated rational spin chain model. Moreover, as each basis vector is an off-shell Bethe state with the Bethe roots replaced by the inhomogeneous parameters, this procedure provides an efficient way to construct nested SoV basis for generic high-rank quantum integrable models such as the one-dimensional Hubbard model [82] with algebraic Bethe Ansatz.

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A Exchange relations

The QYBE implies the following exchange relations among the monodromy matrix elements

$$C^l(v) D_i^k(u) = \sum_{\alpha, \beta=2}^n \frac{R_{\alpha\beta}^{kl}(u-v)}{\sinh(u-v)} D_i^\alpha(u) C^\beta(v) - \frac{R_{i1}^{l1}(u-v)}{\sinh(u-v)} D_i^l(v) C^k(u), \quad (\text{A.1})$$

$$C^k(v) A(u) = \frac{\sinh(u-v-\eta)}{\sinh(u-v)} A(u) C^k(v) + \frac{R_{1k}^{k1}(v-u)}{\sinh(u-v)} A(v) C^k(u), \quad (\text{A.2})$$

$$\begin{aligned} [C^i(u), B_l(v)] &= \frac{1}{\sinh(u-v)} (R_{1l}^{l1}(u-v) A(v) D_l^i(u) - R_{1i}^{l1}(u-v) A(u) D_l^i(v)) \\ &= \frac{1}{\sinh(u-v)} (R_{l1}^{l1}(v-u) D_l^i(u) A(v) - R_{i1}^{l1}(v-u) D_l^i(v) A(u)), \end{aligned} \quad (\text{A.3})$$

$$A(u) B_i(v) = \frac{\sinh(u-v-\eta)}{\sinh(u-v)} B_i(v) A(u) + \frac{R_{i1}^{i1}(v-u)}{\sinh(u-v)} B_i(u) A(v), \quad (\text{A.4})$$

$$D_i^j(u) B_l(v) = \sum_{\alpha, \beta=2}^n \frac{R_{i\alpha}^{j\beta}(u-v)}{\sinh(u-v)} B_\beta(v) D_\alpha^j(u) - \frac{R_{1j}^{j1}(u-v)}{\sinh(u-v)} B_i(u) D_l^j(v), \quad (\text{A.5})$$

$$B_i(u) B_j(v) = \sum_{\alpha, \beta=2}^n \frac{R_{ij}^{\alpha\beta}(u-v)}{\sinh(u-v+\eta)} B_\beta(v) B_\alpha(u), \quad (\text{A.6})$$

$$C^j(v) C^i(u) = \sum_{\alpha, \beta=2}^n \frac{R_{\alpha\beta}^{ij}(u-v)}{\sinh(u-v+\eta)} C^\alpha(u) C^\beta(v), \quad (\text{A.7})$$

$$[T_\beta^\alpha(u), T_\beta^\alpha(v)] = 0, \quad \alpha, \beta = 1, \dots, n, \quad (\text{A.8})$$

$$\begin{aligned} [T_\alpha^\alpha(u), T_\beta^\beta(v)] &= \frac{1}{\sinh(u-v)} \{R_{\alpha\beta}^{\beta\alpha}(u-v) T_\alpha^\beta(v) T_\beta^\alpha(u) - R_{\beta\alpha}^{\alpha\beta}(u-v) T_\alpha^\beta(u) T_\beta^\alpha(v)\}, \\ &\alpha \neq \beta, \quad \text{and} \quad \alpha, \beta = 1, \dots, n, \end{aligned} \quad (\text{A.9})$$

$$\begin{aligned} [T_\beta^\alpha(u), T_\alpha^\beta(v)] &= \frac{R_{\beta\alpha}^{\alpha\beta}(u-v)}{\sinh(u-v)} \{T_\beta^\beta(v) T_\alpha^\alpha(u) - T_\beta^\beta(u) T_\alpha^\alpha(v)\}, \\ &\alpha \neq \beta, \quad \text{and} \quad \alpha, \beta = 1, \dots, n. \end{aligned} \quad (\text{A.10})$$

B $\mathfrak{su}(n)$ case

For the $\mathfrak{su}(n)$ spin chain, let us introduce $n - 1$ non-negative integers m_2, m_3, \dots, m_n such that $\sum_{l=2}^n m_l \leq N$ and

$$\begin{aligned} & \langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \dots; \theta_{p_{m_2+\dots+m_{n-1}+1}}, \dots, \theta_{p_{m_2+\dots+m_n}} | = \\ & \quad \langle 0 | C^2(\theta_{p_1}) \cdots C^2(\theta_{p_{m_2}}) \cdots C^{m_2+\dots+m_{n-1}+1}(\theta_{p_{m_2+\dots+m_{n-1}+1}}) \cdots C^{m_2+\dots+m_n}(\theta_{p_{m_2+\dots+m_n}}), \end{aligned} \quad (\text{B.1})$$

$$\begin{aligned} & | \theta_{p_1}, \dots, \theta_{p_{m_2}}; \dots; \theta_{p_{m_2+\dots+m_{n-1}+1}}, \dots, \theta_{p_{m_2+\dots+m_n}} \rangle = \\ & \quad B_n(\theta_{p_{m_2+\dots+m_n}}) \cdots B_n(\theta_{p_{m_2+\dots+m_{n-1}+1}}) \cdots B_2(\theta_{p_{m_2}}) \cdots B_2(\theta_{p_1}) | 0 \rangle, \end{aligned} \quad (\text{B.2})$$

where $1 \leq p_1 < \dots < p_{m_2} \leq N, \dots, 1 \leq p_{m_2+\dots+m_{n-1}+1} < \dots < p_{m_2+\dots+m_n} \leq N$ and $p_j \neq p_k$. Note that the number of the operators $C^j(u)$ (or $B_j(u)$) in the above expression is m_j . Due to the fact that $d(\theta_j) = 0$, with the help of the exchange relations (A.1) and (A.5), we can show that these states are in fact eigenstates of the operator $D_n^n(u)$

$$\begin{aligned} & \langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \dots; \theta_{p_{m_2+\dots+m_{n-1}+1}}, \dots, \theta_{p_{m_2+\dots+m_n}} | D_n^n(u) = \\ & \quad d(u) \prod_{k=m_2+\dots+m_{n-1}+1}^{m_2+\dots+m_n} \frac{\sinh(u - \theta_{p_k} + \eta)}{\sinh(u - \theta_{p_k})} \\ & \quad \times \langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \dots; \theta_{p_{m_2+\dots+m_{n-1}+1}}, \dots, \theta_{p_{m_2+\dots+m_n}} |, \end{aligned} \quad (\text{B.3})$$

$$\begin{aligned} & D_n^n(u) | \theta_{p_1}, \dots, \theta_{p_{m_2}}; \dots; \theta_{p_{m_2+\dots+m_{n-1}+1}}, \dots, \theta_{p_{m_2+\dots+m_n}} \rangle = \\ & \quad d(u) \prod_{k=m_2+\dots+m_{n-1}+1}^{m_2+\dots+m_n} \frac{\sinh(u - \theta_{p_k} + \eta)}{\sinh(u - \theta_{p_k})} \\ & \quad \times | \theta_{p_1}, \dots, \theta_{p_{m_2}}; \dots; \theta_{p_{m_2+\dots+m_{n-1}+1}}, \dots, \theta_{p_{m_2+\dots+m_n}} \rangle. \end{aligned} \quad (\text{B.4})$$

For generic values of $\{\theta_j\}$, these right (left) states form an orthogonal right (left) basis of the Hilbert space, and any right (left) state can be decomposed as a unique linear combination of these basis.

Using the similar method in appendix C, we can obtain the explicit expressions for the operators $\{D_n^i(u), D_n^i(u) | i = 2, \dots, n\}$, $B_n(u)$ and $C^n(u)$ in the basis (B.1). Like (3.22)–(3.26), the operators take some simple forms without compensating exchange terms on the level of the local operators (i.e. polarization free) and hence have similar simple forms as those in the F-basis [70, 71, 73]. These resulting simple forms allow one to construct eigenstates of the transfer matrix of the $\mathfrak{su}(n)$ spin torus via its ODBA solution [59].

C Proof of the operator decomposition

Keeping the relations (3.11) and (3.13)–(3.14) in mind and using the exchange relations (A.1)–(A.10), one can easily check the actions (3.22)–(3.25) straightforwardly. In

order to prove (3.26), we apply the identity decomposition (3.21) to the l.h.s. of (3.26), giving rise to

$$\begin{aligned} \langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} | C^3(u) = \\ \sum_{P'} |\theta_{p'_1}, \dots, \theta_{p'_{m_2}}; \theta_{p'_{m_2+1}}, \dots, \theta_{p'_{m+1}} \rangle \\ \times \frac{\langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} | C^3(u) | \theta_{p'_1}, \dots, \theta_{p'_{m_2}}; \theta_{p'_{m_2+1}}, \dots, \theta_{p'_{m+1}} \rangle}{G_{m+1}(\theta_{p'_1}, \dots, \theta_{p'_{m_2}} | \theta_{p'_{m_2+1}}, \dots, \theta_{p'_{m+1}})}, \end{aligned} \tag{C.1}$$

where the sum is over $P' = \{p'_1, \dots, p'_{m+1}\}$ such that $1 \leq p'_1 < \dots < p'_{m_2} \leq N$, $1 \leq p'_{m_2+1} < \dots < p'_{m+1} \leq N$ and $p'_j \neq p'_k$. Since that the factor $G_{m+1}(\theta_{p'_1}, \dots, \theta_{p'_{m_2}} | \theta_{p'_{m_2+1}}, \dots, \theta_{p'_{m+1}})$ is already known (3.18), it is sufficient to compute the scalar products

$$\begin{aligned} \langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} | C^3(u) | \theta_{p'_1}, \dots, \theta_{p'_{m_2}}; \theta_{p'_{m_2+1}}, \dots, \theta_{p'_{m+1}} \rangle = \\ \langle \theta_{p_1}, \dots, \theta_{p_{m_2}}; \theta_{p_{m_2+1}}, \dots, \theta_{p_m} | C^3(u) B_3(\theta_{p'_{m+1}}) \dots B_3(\theta_{p'_{m_2+1}}) \\ \times B_2(\theta_{p'_{m_2}}) \dots B_2(\theta_{p'_1}) | 0 \rangle. \end{aligned} \tag{C.2}$$

The relations (3.12), (3.15)–(3.16) and the exchange relations (A.1)–(A.10) allow us to arrive at the operator decomposition (3.26) by direct calculation. Similarly, we can work out the explicit decomposition expressions for the operators $D_2^2(u)$, $B_2(u)$, $C^2(u)$ and $A(u)$.

D Proof of (4.7)

Let us introduce a subspace \mathcal{H}_m for a fixed non-negative integer m spanned by the states

$$\langle 0 | C^2(\theta_{p_1}) \dots C^2(\theta_{p_m}) C^3(\theta_{p_{m+1}}) \dots C^3(\theta_{p_N}), \tag{D.1}$$

where $1 \leq p_1 < \dots < p_m \leq N$, $1 \leq p_{m+1} < \dots < p_N \leq N$ and $p_j \neq p_k$. It is easy to check that the dimension of the subspace is $\frac{N!}{m!(N-m)!}$ and that the subspace can also be spanned by another set of states

$$\langle \bar{0} | D_3^2(\theta_{p_1}) \dots D_3^2(\theta_{p_m}), \quad \langle \bar{0} | = \langle 3, \dots, 3 |, \tag{D.2}$$

where $1 \leq p_1 < \dots < p_m \leq N$. Similar to the procedure for deriving (3.17), we have

$$\langle \bar{0} | D_3^2(\theta_{p_1}) \dots D_3^2(\theta_{p_m}) D_2^3(\theta_{q_1}) \dots D_2^3(\theta_{q_{m'}}) | \bar{0} \rangle = \delta_{m,m'} \prod_{k=1}^m \delta_{p_k, q_k} f_m(\theta_{p_1}, \dots, \theta_{p_m}), \tag{D.3}$$

where the state $|\bar{0}\rangle = |3, \dots, 3\rangle$ and the normalized factor $f_m(\theta_{p_1}, \dots, \theta_{p_m})$ is given by (4.9). The relations (3.13)–(3.14) and the operator decomposition (3.24) enable us to derive that

$$\begin{aligned} \langle 0 | C^2(\theta_{p_1}) \dots C^2(\theta_{p_m}) C^3(\theta_{p_{m+1}}) \dots C^3(\theta_{p_N}) D_2^3(u_1) \dots D_2^3(u_m) | \bar{0} \rangle \\ = \prod_{\alpha=1}^m \prod_{k=m+1}^N \sinh(u_\alpha - \theta_{p_k} + \eta) g_m(\theta_{p_1}, \dots, \theta_{p_m} | u_1, \dots, u_m) \langle 0 | C^3(\theta_1) \dots C^3(\theta_N) | \bar{0} \rangle \\ = \prod_{\alpha=1}^m \prod_{k=m+1}^N \sinh(u_\alpha - \theta_{p_k} + \eta) g_m(\theta_{p_1}, \dots, \theta_{p_m} | u_1, \dots, u_m) \prod_{k=1}^N a(\theta_k), \end{aligned} \tag{D.4}$$

where the function $g_m(v_1, \dots, v_m | u_1, \dots, u_m)$ is given by (4.8) and we have used the identity: $\langle 0 | C^3(\theta_1) \dots C^3(\theta_N) | \bar{0} \rangle = \prod_{k=1}^N a(\theta_k)$.

The corresponding matrix g given by (2.12) allows us to introduce an operator $U(g)$ acting on the Hilbert space as

$$U(g) = g_1 g_2 \dots g_N, \quad \{U(g)\}^3 = \text{id}. \quad (\text{D.5})$$

The invariant property (2.13) of the R -matrix and the definition (2.15) of the monodromy matrix $T_0(u)$ give rise to the relation

$$g_0 T_0(u) g_0^{-1} = U^{-1}(g) T_0(u) U(g), \quad (\text{D.6})$$

which implies that

$$U^{-1}(g) C^3(u) U(g) = D_3^2(u), \quad U^{-1}(g) t(u) U(g) = t(u). \quad (\text{D.7})$$

Direct calculation shows that

$$\langle 0 | U(g) = \langle \bar{0} |. \quad (\text{D.8})$$

The invariance (D.7) of the transfer matrix leads to that the state $U(g) |\Psi\rangle$ is also an eigenstate of the transfer matrix with the same eigenvalue, namely,

$$t(u) U(g) |\Psi\rangle = \Lambda(u) U(g) |\Psi\rangle. \quad (\text{D.9})$$

Hence we can simultaneously diagonalize the transfer matrix and the operator $U(g)$, i.e.,

$$U(g) |\Psi\rangle = \omega^{Z(|\Psi\rangle)} |\Psi\rangle, \quad Z(|\Psi\rangle) = 0, 1, 2. \quad (\text{D.10})$$

Moreover, with the help of the relations (2.5) and (2.6) we can show that

$$\prod_{j=1}^N t(\theta_j) = \left\{ \prod_{j=1}^N a(\theta_j) \right\} \times U(g), \quad (\text{D.11})$$

which gives rise to the identity

$$\frac{\prod_{j=1}^N \Lambda(\theta_j)}{\prod_{j=1}^N a(\theta_j)} = \omega^{Z(|\Psi\rangle)}. \quad (\text{D.12})$$

The above identity allows one to compute the eigenvalue of the operator $U(g)$ for an eigenstate $|\Psi\rangle$ from the associated Bethe ansatz solution given by (2.18)–(2.27).

The relations (D.7)–(D.9) allow us to derive that

$$\begin{aligned} \langle \bar{0} | D_3^2(\theta_{p_1}) \dots D_3^2(\theta_{p_m}) | \Psi \rangle &= \langle 0 | C^3(\theta_{p_1}) \dots C^3(\theta_{p_m}) U(g) | \Psi \rangle \\ &\stackrel{(4.5)}{=} \prod_{l=1}^m \Lambda(\theta_{p_l}) \langle 0 | U(g) | \Psi \rangle \\ &\stackrel{(D.10)}{=} \omega^{Z(|\Psi\rangle)} \prod_{l=1}^m \Lambda(\theta_{p_l}) \langle 0 | \Psi \rangle \\ &\stackrel{(D.12)}{=} \frac{\prod_{j=1}^N \Lambda(\theta_j)}{\prod_{j=1}^N a(\theta_j)} \times \prod_{l=1}^m \Lambda(\theta_{p_l}) \langle 0 | \Psi \rangle. \end{aligned} \quad (\text{D.13})$$

Now we are in position to prove (4.7). The relation (4.5) implies that

$$\begin{aligned}
 F_m(\theta_{p_1}, \dots, \theta_{p_m}) &= \langle 0 | C^2(\theta_{p_1}) \cdots C^2(\theta_{p_m}) | \Psi \rangle \\
 &= \frac{\langle 0 | C^2(\theta_{p_1}) \cdots C^2(\theta_{p_m}) C^3(\theta_{p_{m+1}}) \cdots C^3(\theta_{p_N}) | \Psi \rangle}{\prod_{k=m+1}^N \Lambda(\theta_{p_k})} \\
 &= \sum_{1 \leq p'_1 < \dots < p'_m \leq N} \frac{\langle \bar{0} | D_3^2(\theta_{p'_1}) \cdots D_3^2(\theta_{p'_m}) | \Psi \rangle}{f_m(\theta_{p'_1}, \dots, \theta_{p'_m}) \prod_{k=m+1}^N \Lambda(\theta_{p_k})} \\
 &\quad \times \langle 0 | C^2(\theta_{p_1}) \cdots C^2(\theta_{p_m}) C^3(\theta_{p_{m+1}}) \cdots C^3(\theta_{p_N}) D_2^3(\theta_{p'_1}) \cdots D_2^3(\theta_{p'_m}) | \bar{0} \rangle.
 \end{aligned}
 \tag{D.14}$$

Substituting the equations (D.4) and (D.13) into the above equation, we finally have the relation (4.7).

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