Cosmology in Poincaré gauge gravity with a pseudoscalar torsion

Jianbo Lu and Guoying Chee

Department of Physics, Liaoning Normal University,
Dalian 116029, P.R. China

E-mail: lvjianbo819@163.com, qgy8475@sina.com

Abstract: A cosmology of Poincar$é$ gauge theory is developed, where several properties of universe corresponding to the cosmological equations with the pseudoscalar torsion function are investigated. The cosmological constant is found to be the intrinsic torsion and curvature of the vacuum universe and is derived from the theory naturally rather than added artificially, i.e. the dark energy originates from geometry and includes the cosmological constant but differs from it. The cosmological constant puzzle, the coincidence and fine tuning problem are relieved naturally at the same time. By solving the cosmological equations, the analytic cosmological solution is obtained and can be compared with the ΛCDM model. In addition, the expressions of density parameters of the matter and the geometric dark energy are derived, and it is shown that the evolution of state equations for the geometric dark energy agrees with the current observational data. At last, the full equations of linear cosmological perturbations and the solutions are obtained.

Contents

1 Introduction

The discovery of the accelerated expansion of the universe motivates a large variety of theoretical works to explain it. In order to account for the acceleration the Einstein equation has to be modified and then two approaches are developed. One is to introduce "dark energy" in the right-hand side in the framework of general relativity (see [1–10] for recent reviews). The another is to modify the left-hand side of the equation, called modified gravitational theories, e.g., $f(R)$ gravity (see [11–20] for recent reviews). A large amount of literature in every approach has been accumulated in the past years. However, none of them offers a convincing explanation of the observed results, and most of them were introduced to explain the acceleration phenomenologically, rather than emerging naturally out of fundamental physics principles. The most popular model in the fist approach is the Λ cold dark matter (ΛCDM) model which is plagued by the cosmological constant problem and the coincidence and fine tuning problem. Meanwhile, there is not the enough evidence on the validity of this model. It is shown that the thermal and mechanical stability requirement provides an evidence against the dark energy hypothesis [21]. Adding dark energy to the content of the Universe may not be the answer to the cosmic acceleration problem. In the second approach the Einstein-Hilbert Lagrangian is usually generalized to a function f of the Ricci scalar R. However, at present there are no fully realized and empirically viable $f(R)$ theories that explain the observed level of cosmic acceleration. Furthermore, the $f(R)$ theories suffer from a long-standing controversy about which frame (Einstein or Jordan) is the physical one [22–26]. It should be noted that although we have strong observational evidence for accelerated cosmic expansion but no compelling evidence that the cause of this

acceleration is really a new energy component. At the same time we do not have enough independent data yet to clarify the nature of dark energy. This provides further motivation for a deeper investigation of the nature of dark energy or the origin of the accelerated cosmic expansion. In the framework of $f(R)$ gravity, the field equation can be written as the Einstein equation with an effective energy-momentum tensor that contains all the modifications and the energy-momentum tensor of matter fields. The contributions of the modifications of gravity can be identified with some kind of geometric dark energy. This is specially advantageous since one can define an equation of state associated with such dark energy and compare it with the ΛCDM model [27–29]. However, the function $f(R)$ is not known a priori, none introduces a new fundamental principle that can be used as a guiding line, it is usually constructed by trial and error. In fact, as a geometric theory a modified gravity should be formulated in a gauge theoretical framework. A famous example is the Poincaré gauge theory of Gravity [30–37]. Some works have been done to develop a model of geometric dark energy in the Poincaré gauge theory framework [38–56]. In [38–43] the effect of torsion is to introduce an extra-term into matter density and pressure which gives rise to an accelerated behavior of the universe. However, the torsion contributes only a constant density, it is not possible to solve the coincidence and fine tuning problem. The torsion model in [44–48] contributes an oscillating aspect to the expansion rate of the universe and displays features similar to those of only the presently observed accelerating universe. In [49–56] the Lagrangian involves too many terms and indefinite parameters, which make the field equations complicated and difficult to solve and the role of each term obscure. In order to simplify the field equations, some restrictions on indefinite parameters have to be imposed. Under these restrictions, especially, all the higher derivatives of the scale factor are excluded from the cosmological equations.

In fact, starting from a well behaved Lagrangian $\frac{1}{2}R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu}$ in quadratic gravity [57–63] and string theory [64] and adding a quadratic term of torsion $\gamma T^{\mu}{}_{\nu\rho}T_{\mu}{}^{\nu\rho}$ a good toy model can be obtained [65], where we derive to give the gravitational field equation and the cosmological evolutional equations. When the macroscopic spacetime average of the spin vanishes, the cosmological equations are found to split into two families. Each of them is related with only one torsion function, the scalar or the pseudoscalar torsion function. It has been argued [44–48] that only these two scalar torsion modes are physically acceptable and no-ghosts. This model has a free-ghost dynamics. It has a well posed initial value problem without any ghost or tachyonic propagation. Also, the field equations are allowed to contain higher derivatives in [65], which is different from [49–56], where some restrictions on indefinite parameters are imposed in order to exclude higher order derivatives. In addition, for the first family we solve the cosmological equations corresponding to the scalar torsion function by using the dynamical system approach in ref. [65]. In this paper, we study the second-family cosmological equations corresponding to the pseudoscalar torsion function in detail, with using a totally different way. Some meaningful consequences can be inferred, such as the geometrical interpretation of cosmological constant is investigated, the cosmological constant problem and the coincidence and fine tuning problem are solved naturally, the state equation of the geometrical dark energy is derived and its evolution is consistent with the current observations, the analytic solution $a = a(t)$ in cosmology is obtained and the perturbation analysis is given, etc.

In section 2, starting from the Poincaré gauge principle and the simple Lagrangian $\frac{1}{2}R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma T^{\mu}{}_{\nu\rho}T_{\mu}{}^{\nu\rho}$, which is the sum of the Starobinsky Lagrangian [66] and Yang-Mills type terms of the local rotation and translation field strength, we introduce the main equations in this Poincaré gauge cosmology, including the gravitational field equations and the cosmological equations, etc. In order to evade any unnecessary discussion regarding frames (i.e. Einstein vs. Jordan) the theory is treated using the original variables instead of transforming to a scalar-tensor theory in contrast to $f(R)$ theories. Furthermore, a set of cosmological equations corresponding to the pseudoscalar torsion function are discussed, where it is found that although we do not introduce a cosmology constant in the action it automatically emerges in the derivation of the cosmological equations and then is endowed with intrinsic character. The dark energy is identified with the geometry of the spacetime and is a function of the density and the pressure of the matter. It includes the cosmological constant but can not be identified with it. It is nothing but the intrinsic torsion or curvature of the vacuum universe. In section 3, the analytic expressions of the state equation and the density parameters of the matter and the geometric dark energy are derived and used to determine the values of α, β and γ. Then a theoretical value of the cosmological constant is computed and compared with the observed datum. The cosmological constant problem and the coincidence and fine tuning problem are solved naturally. In section 4 an analytic integral of the cosmological equation is obtained and used to evaluate the age of the universe which can be compared with observed data. In section 5 the full equations of linear cosmological perturbations and the solutions are obtained. In addition, the behavior of perturbations for the sub-horizon modes relevant to large-scale structures is discussed. It is shown that our model can be distinguished from others by considering the evolution of matter perturbations and gravitational potentials. Section 6 is devoted to conclusions.

2 Cosmological equations

The discussion in this paper are entirely classical. We consider a Poincaré gauge theory of gravity [30–56], in which there are two sets of local gauge potentials, the orthonormal frame field (tetrad) $e_I{}^{\mu}$ and the metric-compatible connection $\Gamma^{IJ}{}_{\mu}$ associated with the translation and the Lorentz subgroups of the Poincaré gauge group, respectively. We use the Greek alphabet (μ, ν, ρ, $\ldots = 0, 1, 2, 3$) to denote (holonomic) indices related to spacetime, and the Latin alphabet ($I, J, K, \ldots = 0, 1, 2, 3$) to denote algebraic (anholonomic) indices, which are raised and lowered with the Minkowski metric $\eta_{IJ} = \text{diag}\,(-1, +1, +1, +1)$. The field strengths associated with the tetrad and connection are the torsion $T^{\lambda}{}_{\mu\nu}$ and the curvature $R_{\mu\nu}{}^{\lambda\tau}$. We use the geometrized system of units in which $8\pi G = 1$, $c = 1$, and start from the action

$$S = \int d^4x \sqrt{-g}\left[\left(\frac{1}{2}R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma T^{\mu}{}_{\nu\rho}T_{\mu}{}^{\nu\rho}\right) + \mathcal{L}_m\right], \qquad (2.1)$$

where $\mathcal{L}_m$ denotes the Lagrangian of the source matter including baryonic matter, cold dark matter and radiation, α and β are two parameters with the dimension of $[L]^2$, γ is a dimensionless parameter. The vales of α, β and γ can be determined by experiment and

observational data. The terms $\frac{1}{2}R$ and $\gamma T^\mu{}_{\nu\rho}T_\mu{}^{\nu\rho}$ represent weak gravity, while αR^2 and $\beta R_{\mu\nu}R^{\mu\nu}$ represent strong gravity with the dimensionless strong gravity constant α and β according to Hehl et al. [30–37].

The variational principle yields the field equations for the tetrad $e_I = e_I{}^\mu \partial_\mu$ and the connection $\Gamma^{IJ} = \Gamma^{IJ}{}_\mu dx^\mu$ [30–37]:

$$DH_I - T_{(g)I} = T_I \ (first),$$
$$DH_{IJ} - s_{(g)IJ} = s_{IJ} \ (second),$$

with the covariant derivatives (D) of the translation excitation H_I and the Lorentz excitation H_{IJ}, the gauge currents of energy-momentum $T_{(g)I}$ and spin $s_{(g)IJ}$ and the canonical matter currents of energy-momentum T_I and spin (angular momentum) s_{IJ}. The reduced explicit form of these field equations are [65]:

$$R_{\nu\mu} - \frac{1}{2}g_{\nu\mu}R = T_{\nu\mu} + T_{(g)\nu\mu}, \tag{2.2}$$

and

$$T^\nu{}_{\tau\nu}\delta^\mu_\lambda - T^\nu{}_{\lambda\nu}\delta^\mu_\tau + T^\mu{}_{\nu\tau}\delta^\nu_\lambda = e^I{}_\lambda e^J{}_\tau \left(s_{IJ}{}^\mu + s_{(g)IJ}{}^\mu\right), \tag{2.3}$$

where $T_{\nu\mu} := e_{I\mu}\partial\left(\sqrt{-g}\mathcal{L}_m\right)/\partial e_I{}^\nu$ and $s_{IJ}{}^\mu := \partial\left(\sqrt{-g}\mathcal{L}_m\right)/\partial\Gamma^{IJ}{}_\mu$ are energy-momentum tensor and spin tensor of the source matter, respectively, while

$$T_{(g)\nu\mu} = -\alpha\left(4R_{\nu\mu} - g_{\nu\mu}R\right)R - \beta\left(2R^\rho{}_\nu R_{\rho\mu} + 2R^{\rho\sigma}R_{\nu\rho\mu\sigma} - g_{\nu\mu}R_{\rho\sigma}R^{\rho\sigma}\right)$$
$$+ \gamma\left(4\partial_\tau e^{I\lambda}\left(e_{I\nu}T_{\mu\lambda}{}^\tau - e_{I\lambda}T_{\mu\nu}{}^\tau\right) + 4\partial_\tau T_{\mu\nu}{}^\tau + g_{\nu\mu}T^\lambda{}_{\rho\sigma}T_\lambda{}^{\rho\sigma} - 4T^\lambda{}_{\nu\tau}T_{\lambda\mu}{}^\tau\right), \tag{2.4}$$

and

$$s_{(g)IJ}{}^\mu = -4\alpha\left(e_{[I}{}^\nu e_{J]}{}^\tau\Gamma^\mu{}_{\nu\tau}R + e_{[J}{}^\mu e_{I]}{}^\nu\left(\Gamma^\lambda{}_{\lambda\nu}R - \partial_\nu R\right) + e_{[I}{}^\nu e_{J]}{}^\mu Re^K{}_\tau\partial_\nu e_K{}^\tau\right)$$
$$- 4\beta e_J{}^\lambda\left(e_I{}^{[\mu}\partial_\nu R_\lambda{}^{\nu]} + e_I{}^{[\nu}R_\lambda{}^{\mu]}e^K{}_\tau\partial_\nu e_K{}^\tau + e_I{}^\tau\Gamma^{[\nu}{}_{\nu\tau}R_\lambda{}^{\mu]} + e_I{}^{[\nu}R_\tau{}^{\mu]}\Gamma^\tau{}_{\nu\lambda}\right)$$
$$- 4\gamma e_{I\nu}e_J{}^\tau T^{\nu\mu}{}_\tau, \tag{2.5}$$

are the energy-momentum and the spin of this kind of "geometric dark energy" corresponding to the terms $\alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma T^\mu{}_{\nu\rho}T_\mu{}^{\nu\rho}$ in (2.1). Note that the energy-momentum tensor $T_{\nu\mu}$ of type $(0, 2)$ should not be confused with the torsion tensor $T^\lambda{}_{\mu\nu}$ of type $(1, 2)$. If $\alpha = \beta = \gamma = 0$, these equations become the field equations of Einstein-Cartan-Sciama-Kibble theory. Furthermore, for $T^\lambda{}_{\mu\nu} = 0$ we come back to General Relativity. The eqs. (2.2)–(2.5) can be rewritten in terms of the covariant derivative. Here given that they are convenient in the concrete calculation for deriving the following cosmological equations, we exhibit these expressions (2.2)–(2.5) directly.

For the spatially flat Friedmann-Robertson-Walker (FRW) metric

$$g_{\mu\nu} = \text{diag}\left(-1, a\left(t\right)^2, a\left(t\right)^2, a\left(t\right)^2\right), \tag{2.6}$$

the non-vanishing torsion components with holonomic indices are given by two functions, the scalar torsion h and the pseudoscalar torsion f [44–48, 67–69]:

$$T_{ij0} = a^2 h\delta_{ij}, T_{ijk} = 2a^3 f\epsilon_{ijk}, \quad i, j, k, \ldots = 1, 2, 3. \tag{2.7}$$

The equations (2.2) and (2.3) yields the cosmological equations [65][1]

$$H^2 = \frac{1}{3} \left(\rho + \rho_g \right), \tag{2.8}$$

$$2\,\dot{H} + 3H^2 = -\left(p + p_g \right), \tag{2.9}$$

$$(\beta + 6\alpha) \left(\ddot{H} + \ddot{h} \right) + 6 \left(\beta + 4\alpha \right) (H + h) \, \dot{H} + (5\beta + 18\alpha) (H + h) \, \dot{h} - 4 \left(\beta + 3\alpha \right) f \, \dot{f}$$
$$+ 3 \left(\beta + 4\alpha \right) hH^2 + (5\beta + 18\alpha) h^2 H + 2 \left(\beta + 3\alpha \right) h^3 - 2 \left(\beta + 3\alpha \right) hf^2 + \frac{1}{4} h + \frac{1}{2} s_{01}{}^1 = 0, \tag{2.10}$$

and

$$f \{ 2 \left(\beta + 6\alpha \right) \left(\dot{H} + \dot{h} \right) + 6 \left(\beta + 4\alpha \right) H^2 + 2 \left(5\beta + 18\alpha \right) Hh$$
$$+ \left(\beta + 3\alpha \right) \left(4h^2 - 4f^2 \right) - 4\gamma + \frac{1}{2} \} - \frac{1}{2} s_{12}{}^3 = 0, \tag{2.11}$$

where $H = \dot{a}(t)/a(t)$ is the Hubble parameter, $\dot{H} = dH/dt$, while

$$\rho_g = -6Hh - 3h^2 + 3f^2$$
$$+ 12 \left(3\alpha + \beta \right) \left(\dot{H} + \dot{h} - Hh - h^2 + f^2 \right) \left(\dot{H} + \dot{h} + 2H^2 + 3Hh + h^2 - f^2 \right)$$
$$- 2\gamma \left(3h^2 + 4f^2 \right), \tag{2.12}$$

and

$$p_g = 4Hh + h^2 - f^2 + 2\dot{h}$$
$$+ 4 \left(3\alpha + \beta \right) \left(\dot{H} + \dot{h} - Hh - h^2 + f^2 \right) \left(\dot{H} + \dot{h} + 2H^2 + 3Hh + h^2 - f^2 \right)$$
$$- 2\gamma \left(2\dot{h} + 8Hh + h^2 + 4f^2 \right), \qquad i = 1, 2, 3, \tag{2.13}$$

are the density and the pressure of the geometric dark energy. Eqs. (2.12) and (2.13) indicate that the geometric dark energy is just the gravitational field itself described by h, H and f. The source matter is a fluid characterized by the energy density $\rho = T_{00}$, the pressure $p = T_{ij}$ $(i = j)$ and the spin $s_{IJ}{}^\mu$. Eqs. (2.8) and (2.9) lead to

$$\frac{\ddot{a}}{a} = -\frac{1}{6} \left(\rho + \rho_g + 3p + 3p_g \right). \tag{2.14}$$

It is easy to see that when $\alpha = \beta = \gamma = 0$ and $h = f = 0$, (2.8), (2.9) and (2.14) reduce to the Friedmann cosmology. Eqs. (2.8) and (2.9) correspond to the Friedmann equation and

---

[1]It can be seen that the symbol of p (denotes pressure) in eq. (2.9) is opposite to the one in eq. (15) of ref. [65]. The reason is that there is a error on symbol of p in [65]. The wrong symbol on p in eq. (15) of ref. [65] results that p (or $\dot{p}$ or $w = p/\rho$) in all formulas of ref. [65] should be corrected to $-p$ (or $-\dot{p}$ or $-w$). However, the main results and conclusions of ref. [65] remain unaffected, since the main discussions are performed in vacuum universe in ref. [65]. This wrong sign of p has been corrected in this paper. Also, the corrected equations can be found in the errata regarding ref. [65].

the Raychaudhuri equation respectively, while (2.14) is the acceleration equation, which represent the Einstein frame of the theory.

Since the spin orientation of particles for ordinary matter is random, the macroscopic spacetime average of the spin vanishes, we suppose $s_{IJ}{}^{\lambda} = 0$, henceforth. Then, the equation (2.11) has the solutions

$$f = 0, \tag{2.15}$$

and

$$f^2 = \frac{(\beta + 6\alpha)}{2(\beta + 3\alpha)}\left(\dot{H} + \dot{h}\right) + \frac{3(\beta + 4\alpha)}{2(\beta + 3\alpha)}H^2 + \frac{(5\beta + 18\alpha)}{2(\beta + 3\alpha)}Hh + h^2$$
$$- \frac{\gamma}{(\beta + 3\alpha)} + \frac{1}{8(\beta + 3\alpha)}. \tag{2.16}$$

The solution (2.15) has been investigated in [65]. We concentrate on the equation (2.16) now. Differentiating (2.16) gives

$$f\,\dot{f} = \frac{\beta + 6\alpha}{4(\beta + 3\alpha)}\left(\ddot{H} + \ddot{h}\right) + \frac{3(\beta + 4\alpha)}{2(\beta + 3\alpha)}H\,\dot{H} + \frac{5\beta + 18\alpha}{4(\beta + 3\alpha)}\,\dot{H}\,h + \frac{5\beta + 18\alpha}{4(\beta + 3\alpha)}H\,\dot{h} + h\,\dot{h}\,.$$

Substituting it and (2.16) into (2.10) gives (when $s_{01}{}^1 = 0$)

$$2h\gamma = 0,$$

and then

$$h = 0. \tag{2.17}$$

In this case (2.8), (2.9), (2.12), (2.13) and (2.16) lead to

$$12(3\alpha + \beta)\left(\dot{H} + f^2\right)\left(\dot{H} + 2H^2 - f^2\right) - 3H^2 + 3f^2 - 24\gamma f^2 + \rho = 0,$$

$$4(3\alpha + \beta)\left(\dot{H} + f^2\right)\left(\dot{H} + 2H^2 - f^2\right) + 2\dot{H} + 3H^2 - f^2 - 8\gamma f^2 + p = 0,$$

$$(\beta + 6\alpha)\dot{H} + 3(\beta + 4\alpha)H^2 - 2(\beta + 3\alpha)f^2 - 2\gamma + \frac{1}{4} = 0,$$

which further give

$$f^2 = \frac{\Lambda}{24\gamma} + \frac{\rho}{24\gamma} + \frac{3(4\alpha + \beta) - 16\gamma(3\alpha + \beta)}{48\beta\gamma}(\rho - 3p) + \frac{(3\alpha + \beta)(4\alpha + \beta)}{24\gamma\beta}(\rho - 3p)^2, \tag{2.18}$$

$$H^2 = \frac{(1 - 8\gamma)}{24\gamma}\Lambda + \frac{\rho}{24\gamma} - \frac{(8\gamma - 1)(4\alpha + \beta)}{16\gamma\beta}(\rho - 3p) + \frac{(3\alpha + \beta)(4\alpha + \beta)}{24\gamma\beta}(\rho - 3p)^2, \tag{2.19}$$

$$\dot{H} = \frac{16\gamma - 1}{24\gamma}\Lambda - \frac{\rho}{24\gamma}$$
$$- \frac{3(4\alpha + \beta) - 8\gamma(18\alpha + 5\beta)}{48\beta\gamma}(\rho - 3p) - \frac{(3\alpha + \beta)(4\alpha + \beta)}{24\gamma\beta}(\rho - 3p)^2, \tag{2.20}$$

where

$$\Lambda = \frac{3(1 - 8\gamma)}{4\beta}, \tag{2.21}$$

is the geometric cosmological constant coming from the terms $\beta R_{\mu\nu}R^{\mu\nu} + \gamma T^{\mu}{}_{\nu\rho}T_{\mu}{}^{\nu\rho}$ in (2.1). It is easy to see that the higher order derivative $\ddot{H}$ and $\dot{f}$ in (2.10) disappear. We also note that although we do not introduce a cosmological constant Λ in the action (2.1), it automatically emerges in these equations. In (2.18) the pseudoscalar torsion f is a function of ρ and p rather than a constant, in contrast to [27–29]. It should be noted that although (2.8) and (2.9) have the same form as the Friedmann equations, the solutions (2.19) and (2.20) are different. The reason is that in (2.8) and (2.9) ρ_g and p_g are functions of H, $\dot{H}$, h, $\dot{h}$, and f as indicated by (2.12) an (2.13). In other words, this is a different model from the ΛCDM model essentially.

Eqs. (2.12) and (2.13) become now

$$\rho_g = \frac{(1-8\gamma)}{8\gamma}\Lambda + \frac{(1-8\gamma)\rho}{8\gamma} + \frac{3(1-8\gamma)(4\alpha+\beta)}{16\beta\gamma}(\rho-3p) + \frac{(3\alpha+\beta)(4\alpha+\beta)}{8\beta\gamma}(\rho-3p)^2, \tag{2.22}$$

$$p_g = -\frac{(1+8\gamma)}{24\gamma}\Lambda - \frac{(8\gamma+1)\rho}{24\gamma} - \frac{3(4\alpha+\beta)-8\beta\gamma}{48\beta\gamma}(\rho-3p) - \frac{(3\alpha+\beta)(4\alpha+\beta)}{24\beta\gamma}(\rho-3p)^2, \tag{2.23}$$

which mean that *the geometrical dark energy includes the cosmological constant Λ but can not be identified with it.* The cosmological constant Λ is really a constant determined by β and γ as indicated by (2.20) while the geometrical dark energy ρ_g is a function of the density ρ and the pressure p of the matter. *The cosmological constant problem and the coincidence and fine tuning problem are relieved naturally, as shown in the next section.*

Substituting (2.22) and (2.23) into (2.14) yields

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3} + \frac{3\alpha+\beta}{3\beta}(\rho-3p). \tag{2.24}$$

Furthermore, (2.18), (2.19), (2.20) and (2.24) mean that the vacuum universe has the torsion

$$f_{\text{vac}}^2 = \frac{\Lambda}{24\gamma}, \tag{2.25}$$

the curvature

$$R_{\text{vac}} = 6\,\dot{H} + 12H^2 - 3f^2 = \frac{\Lambda}{8\gamma}. \tag{2.26}$$

and the acceleration

$$\left(\frac{\ddot{a}}{a}\right)_{\text{vac}} = \frac{\Lambda}{3}. \tag{2.27}$$

This means that the cosmological constant is nothing but the intrinsic torsion or curvature of the vacuum universe.

3 The state equation of the geometrical dark energy

Eqs. (2.22) and (2.23) gives the state equation of the dark energy:

$$w_g = \frac{p_g}{\rho_g} \tag{3.1}$$

$$= \frac{-2\left(1+8\gamma\right)\beta\Lambda - 2\left(8\gamma+1\right)\beta\rho - \left(3\left(4\alpha+\beta\right)-8\beta\gamma\right)\left(\rho-3p\right) - 2\left(3\alpha+\beta\right)\left(4\alpha+\beta\right)\left(\rho-3p\right)^2}{6\left(1-8\gamma\right)\beta\Lambda + 6\left(1-8\gamma\right)\beta\rho + 9\left(1-8\gamma\right)\left(4\alpha+\beta\right)\left(\rho-3p\right) + 6\left(3\alpha+\beta\right)\left(4\alpha+\beta\right)\left(\rho-3p\right)^2}.$$

The source matter includes ordinary baryon matter, dark matter and radiation:

$$\rho = \rho_{\mathrm{m}} + \rho_{\mathrm{r}}, p_{\mathrm{m}} = 0, p = \frac{1}{3}\rho_{\mathrm{r}}. \tag{3.2}$$

The equation (2.8) can be written as

$$\Omega = \Omega_{\mathrm{r}} + \Omega_{\mathrm{m}} + \Omega_g = 1,$$

where

$$\Omega_{\mathrm{m}} := \frac{\rho_{\mathrm{m}}}{3H^2}, \Omega_{\mathrm{r}} := \frac{\rho_{\mathrm{r}}}{3H^2}, \Omega_g := \frac{\rho_g}{3H^2}, \tag{3.3}$$

are the dimensionless density parameters of the matter, the radiation and the geometrical dark energy, respectively.

Suppose

$$\alpha = -\frac{\beta}{2}. \tag{3.4}$$

Eqs. (2.22), (2.23) and (2.24) become

$$\rho_g = \frac{1-8\gamma}{8\gamma}\Lambda + \frac{\left(1-8\gamma\right)\rho_{\mathrm{r}}}{8\gamma} - \frac{1-8\gamma}{16\gamma}\rho_{\mathrm{m}} + \frac{\beta}{16\gamma}\rho_{\mathrm{m}}^2, \tag{3.5}$$

$$p_g = -\frac{1+8\gamma}{24\gamma}\Lambda - \frac{\left(1+8\gamma\right)\rho_{\mathrm{r}}}{24\gamma} + \frac{1-8\gamma}{48\gamma}\rho_{\mathrm{m}} - \frac{\beta}{48\gamma}\rho_{\mathrm{m}}^2, \tag{3.6}$$

$$w_g = \frac{p_g}{\rho_g} = \frac{-2\left(1+8\gamma\right)\Lambda - 2\left(1+8\gamma\right)\rho_{\mathrm{r}} + \left(1-8\gamma\right)\rho_{\mathrm{m}} - \beta\rho_{\mathrm{m}}^2}{6\left(1-8\gamma\right)\Lambda + 6\left(1-8\gamma\right)\rho_{\mathrm{r}} - 3\left(1-8\gamma\right)\rho_{\mathrm{m}} + 3\beta\rho_{\mathrm{m}}^2}, \tag{3.7}$$

and

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \frac{1}{6}\left(\rho - 3p\right) = \frac{\Lambda}{3} - \frac{1}{6}\rho_{\mathrm{m}}. \tag{3.8}$$

Eqs. (3.3) and (2.19) give

$$\Omega_{\mathrm{m}} = \frac{16\gamma\rho_{\mathrm{m}}}{2\left(1-8\gamma\right)\Lambda + 2\rho_{\mathrm{r}} + \left(24\gamma-1\right)\rho_{\mathrm{m}} + \beta\rho_{\mathrm{m}}^2}, \tag{3.9}$$

$$\Omega_g = \frac{2\left(1-8\gamma\right)\Lambda + 2\left(1-8\gamma\right)\rho_{\mathrm{r}} - \left(1-8\gamma\right)\rho_{\mathrm{m}} + \beta\rho_{\mathrm{m}}^2}{2\left(1-8\gamma\right)\Lambda + 2\rho_{\mathrm{r}} + \left(24\gamma-1\right)\rho_{\mathrm{m}} + \beta\rho_{\mathrm{m}}^2}. \tag{3.10}$$

From the observed data

$$\rho_{\mathrm{crit}} = 1.88h^2 \times 10^{-29} gcm^{-3} = 7.2402 \times 10^{-58} cm^{-2},$$

$$\Omega_{\mathrm{m}} = 0.3, \Omega_{\mathrm{r}} = 1.8035 \times 10^{-4}\Omega_{\mathrm{m}} \tag{3.11}$$

$$w_g = -1, \tag{3.12}$$

using (3.4), (3.7) and (3.9) we can determine the parameters

$$\alpha = -4.1969 \times 10^{56} cm^2,$$
$$\beta = 8.3937 \times 10^{56} cm^2,$$
$$\gamma = 0.0576. \tag{3.13}$$

Then (2.21), (2.25), (2.26) and (2.27) give

$$\Lambda = 4.8179 \times 10^{-58} cm^{-2}, \tag{3.14}$$
$$f_{\text{vac}}^2 = 3.4852 \times 10^{-58} cm^{-2}, \tag{3.15}$$
$$R_{\text{vac}} = 1.0456 \times 10^{-57} cm^{-2}, \tag{3.16}$$

and

$$\left(\frac{\ddot{a}}{a}\right)_{\text{vac}} = 1.606 \times 10^{-58} cm^{-2} = 1.4425 \times 10^{-37} s^{-2}. \tag{3.17}$$

The value given by (3.14) can be compared with the observed datum

$$\Lambda^{(\text{obs})} = 8\pi G \rho_\Lambda^{(\text{obs})} = 8\pi G \left(10^{-12} GeV\right)^4 \sim 8\pi G \times 2 \times 10^{-10} erg/cm^3 = 4.1574 \times 10^{-58} cm^{-2}.$$

Since

$$\rho_{\text{r}} = \frac{\overline{\rho}_{\text{r}}}{a^4}, \overline{\rho}_{\text{r}} = \rho_{\text{r},a=1}, \rho_{\text{m}} = \frac{\overline{\rho}_{\text{m}}}{a^3}, \overline{\rho}_{\text{m}} = \rho_{\text{m},a=1},$$

using (3.13) and

$$\overline{\rho}_{\text{m}} = 0.3\rho_{\text{crit}}, \overline{\rho}_{\text{r}} = 1.8035 \times 10^{-4}\overline{\rho}_{\text{m}}$$

the state equation of the dark energy (3.7) can be written as

$$w_g(a) = \frac{p_g}{\rho_g} = \frac{-0.78766 - 6.4044 \times 10^{-5} a^{-4} + 6.5538 \times 10^{-2} a^{-3} - 2.216 \times 10^{-2} a^{-6}}{0.87221 + 7.0919 \times 10^{-5} a^{-4} - 0.19661 a^{-3} + 6.6481 \times 10^{-2} a^{-6}}, \tag{3.18}$$

or

$$w_g(z) = \frac{-0.78766 - 6.4044 \times 10^{-5} (1+z)^4 + 6.5538 \times 10^{-2} (1+z)^3 - 2.216 \times 10^{-2} (1+z)^6}{0.87221 + 7.0919 \times 10^{-5} (1+z)^4 - 0.19661 (1+z)^3 + 6.6481 \times 10^{-2} (1+z)^6}. \tag{3.19}$$

Figure 1 plots the evolution history of $w_g(a)$ given by (3.18).

In observation and experiments it is conventional to phrase constraints or projected constraints on $w(z)$ in terms of a linear evolutional model [70]:

$$w(a) = w_0 + w_a(1-a) = w_{\text{p}} + w_a(a_{\text{p}} - a),$$

where w_0 is the value of w at $z = 0$ ($a = 1$), and w_{p} is the value of w at a "pivot" redshift z_{p}. For typical data combinations, $z_{\text{p}} \approx 0.5$. To this end we give the linear approximation of (3.18). When $a = 1$,

$$w_{g0} = -1,$$
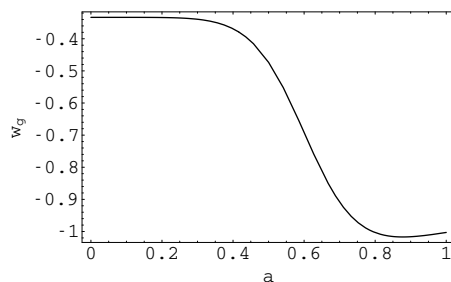
and

$$\frac{dw_g}{da}|_{a=1} = 0.19936, \tag{3.20}$$

– 9 –

Figure 1. The evolution of $w_g(a)$.

so we have

$$w_g(a) = w_{g0} + \frac{dw_g}{da}|_{a=1}(a-1) = -1 - 0.19936(1-a). \tag{3.21}$$

When $z_p = 0.5$, $a = \frac{2}{3}$,

$$w_{gp} = -0.84781, \tag{3.22}$$

and

$$\frac{dw_g}{da}|_{a=\frac{2}{3}} = -0.7653, \tag{3.23}$$

then we have

$$w_{gp}(a) = w_{gp} + \frac{dw_g}{da}|_p\left(a - \frac{2}{3}\right) = -0.8478 + 0.7653\left(\frac{2}{3} - a\right). \tag{3.24}$$

Using (3.8) and $\rho_m = \overline{\rho}_m/a^3$, one finds that when

$$a = a_{\text{trans}} = \left(\frac{2\beta\overline{\rho}_m}{3(1-8\gamma)}\right)^{\frac{1}{3}} = 0.60859, \tag{3.25}$$

$$z_{\text{trans}} = 0.64314, \tag{3.26}$$

the expansion of the universe transforms from deceleration to acceleration. Using (3.10) one can compute that when

$$a = 0.75817, \quad z = 0.31897, \quad \Omega_g = 0.5, \tag{3.27}$$

the universe transforms from the matter dominating phase into the dark energy dominating phase. In a flat ΛCDM universe with $(\Omega_m, \Omega_\Lambda) = (0.3, 0.7)$ acceleration begins at $z = 0.67$, while dark energy doesn't dominate the energy density of the universe until $z = 0.33$ [71].

4 An exact analytic solution of cosmological equation

The equation (2.19) can be solved in two cases as follows. In the radiation dominated era

$$\rho_r = \frac{\overline{\rho}_r}{a^4}, \quad \overline{\rho}_r = \rho_{r,a=1} = \text{const}, \quad p_r = \frac{1}{3}\rho_r,$$

Eq. (2.19) reads

$$H^2 = \frac{(1-8\gamma)^2}{32\gamma\beta} + \frac{\overline{\rho}_r}{24\gamma a^4}, \tag{4.1}$$

and can be rewritten

$$\frac{da}{dt} = a\sqrt{\frac{(1-8\gamma)^2}{32\gamma\beta} + \frac{\overline{\rho}_{\mathrm{r}}}{24\gamma a^4}}. \qquad (4.2)$$

Its integration gives

$$\ln\left(a^2 + \sqrt{a^4 + \frac{4\beta\overline{\rho}_{\mathrm{r}}}{3(1-8\gamma)^2}}\right) - \ln\sqrt{\frac{4\beta\overline{\rho}_{\mathrm{r}}}{3(1-8\gamma)^2}} = \frac{(1-8\gamma)}{2\sqrt{2\gamma\beta}}t, \qquad (4.3)$$

or

$$a = \left(\frac{\beta\overline{\rho}_{\mathrm{r}}}{3(1-8\gamma)^2}\right)^{\frac{1}{4}}\sqrt{\left(e^{\frac{(1-8\gamma)}{2\sqrt{2\gamma\beta}}t} - e^{-\frac{(1-8\gamma)}{2\sqrt{2\gamma\beta}}t}\right)}$$

$$= \left(\frac{\beta\overline{\rho}_{\mathrm{r}}}{3(1-8\gamma)^2}\right)^{\frac{1}{4}}\sqrt{2\sinh\frac{(1-8\gamma)}{2\sqrt{2\gamma\beta}}t}. \qquad (4.4)$$

In the matter dominated era

$$\rho_{\mathrm{m}} = \frac{\overline{\rho}_{\mathrm{m}}}{a^3}, \overline{\rho}_{\mathrm{m}} = \rho_{\mathrm{m},a=1} = \mathrm{const}, \ p = 0,$$

eq. (2.19) reads

$$H^2 = \frac{(1-8\gamma)^2}{32\gamma\beta} + \frac{12\alpha + 5\beta - 24\gamma(4\alpha+\beta)}{48\gamma\beta}\frac{\overline{\rho}_{\mathrm{m}}}{a^3} + \frac{(3\alpha+\beta)(4\alpha+\beta)}{24\gamma\beta}\frac{\overline{\rho}_{\mathrm{m}}^2}{a^6}, \qquad (4.5)$$

and then

$$\frac{da}{dt} = a\sqrt{\frac{(1-8\gamma)^2}{32\gamma\beta} + \frac{12\alpha + 5\beta - 24\gamma(4\alpha+\beta)}{48\gamma\beta}\frac{\overline{\rho}_{\mathrm{m}}}{a^3} + \frac{(3\alpha+\beta)(4\alpha+\beta)}{24\gamma\beta}\frac{\overline{\rho}_{\mathrm{m}}^2}{a^6}}. \qquad (4.6)$$

Its integration gives

$$\ln\left(\sqrt{a^6 + 2\frac{12\alpha + 5\beta - 24\gamma(4\alpha+\beta)}{3(1-8\gamma)^2}\overline{\rho}_{\mathrm{m}}a^3 + \frac{4(3\alpha+\beta)(4\alpha+\beta)}{3(1-8\gamma)^2}\overline{\rho}_{\mathrm{m}}^2} + a^3 \right.$$

$$\left. + \frac{12\alpha + 5\beta - 24\gamma(4\alpha+\beta)}{3(1-8\gamma)^2}\overline{\rho}_{\mathrm{m}}\right)$$

$$- \ln\left(\sqrt{\frac{4(3\alpha+\beta)(4\alpha+\beta)}{3(1-8\gamma)^2}\overline{\rho}_{\mathrm{m}}^2} + \frac{12\alpha + 5\beta - 24\gamma(4\alpha+\beta)}{3(1-8\gamma)^2}\overline{\rho}_{\mathrm{m}}\right)$$

$$= 3\frac{(1-8\gamma)}{\sqrt{32\gamma\beta}}t, \qquad (4.7)$$

and then

$$
\begin{aligned}
a = \Bigg\{ &\frac{1}{2} \left(\sqrt{\frac{4\,(3\alpha+\beta)\,(4\alpha+\beta)}{3\,(1-8\gamma)^2}\overline{\rho}_{\mathrm{m}}^2} + \frac{12\alpha+5\beta-24\gamma\,(4\alpha+\beta)}{3\,(1-8\gamma)^2}\overline{\rho}_{\mathrm{m}} \right) e^{\frac{3(1-8\gamma)}{4\sqrt{2\gamma\beta}}t} \\
&- \frac{12\alpha+5\beta-24\gamma\,(4\alpha+\beta)}{3\,(1-8\gamma)^2}\overline{\rho}_{\mathrm{m}} \\
&+ \frac{36\alpha+13\beta-48\gamma\,(1+4\gamma)\,(4\alpha+\beta)}{18\,(1-8\gamma)^4}\beta\overline{\rho}_{\mathrm{m}} \\
&\times \left(\sqrt{\frac{4\,(3\alpha+\beta)\,(4\alpha+\beta)}{3\,(1-8\gamma)^2}} + \frac{12\alpha+5\beta-24\gamma\,(4\alpha+\beta)}{3\,(1-8\gamma)^2} \right)^{-1} e^{-\frac{3(1-8\gamma)}{4\sqrt{2\gamma\beta}}t} \Bigg\}^{\frac{1}{3}}. \quad (4.8)
\end{aligned}
$$

In the case (3.13), equations (4.4) and (4.8) become

$$
a = 0.67617\sqrt{2\sinh 2.7417\times 10^{-29}t}, \tag{4.9}
$$

and

$$
a = \left(0.17801e^{4.1125\times 10^{-29}t} - 9.8074\times 10^{-2}e^{-4.1125\times 10^{-29}t} - 7.9934\times 10^{-2} \right)^{\frac{1}{3}}, \tag{4.10}
$$

where the time t is in cm. This is a new exact analytic cosmological solution which resembles but differs from the ΛCDM solution [72].

Now we can evaluate the age of the universe using (3.13), (4.3) and (4.7). In the radiation dominating era $z \gtrsim 3000$ [72], we have the equation

$$
\ln\left(a^2 + \sqrt{a^4 + \frac{4\beta\overline{\rho}}{3\,(1-8\gamma)^2}} \right) - \ln\sqrt{\frac{4\beta\overline{\rho}}{3\,(1-8\gamma)^2}} = \frac{(1-8\gamma)}{2\sqrt{2\gamma\beta}}t.
$$

Choosing $z = 3000$, then $a = 1/3001$, this equation gives

$$
t = 3.298\times 10^{23}cm = 3.4895\times 10^5 Years. \tag{4.11}
$$

In the matter dominating era $z \lesssim 3000$, we have the equation

$$
\begin{aligned}
&\ln\left(\sqrt{1 + 2\frac{(24\gamma-1)}{3\,(1-8\gamma)^2}\beta\overline{\rho}a_2^3 + \frac{2}{3\,(1-8\gamma)^2}(\beta\overline{\rho})^2 + a_2^3} + \frac{(24\gamma-1)}{3\,(1-8\gamma)^2}\beta\overline{\rho} \right) \\
&- \ln\left(\sqrt{1 + 2\frac{(24\gamma-1)}{3\,(1-8\gamma)^2}\beta\overline{\rho}a_1^3 + \frac{2}{3\,(1-8\gamma)^2}(\beta\overline{\rho})^2 + a_1^3} + \frac{(24\gamma-1)}{3\,(1-8\gamma)^2}\beta\overline{\rho} \right) \\
&= 3\frac{(1-8\gamma)}{\sqrt{32\gamma\beta}}(t_2 - t_1),
\end{aligned}
$$

Choosing

$$
\begin{aligned}
z_1 &= 3000, \quad a_1 = \frac{1}{3001}, \\
z_2 &= 0, \qquad a_2 = 1,
\end{aligned}
$$

we have

$$
t_2 - t_1 = 1.6383\times 10^{28}cm = 1.7334\times 10^{10} Years = 17.334 Gy. \tag{4.12}
$$

5 Perturbation theory

In order to discriminate lots of dark energy models, it is interested to seek the additional information other than the background expansion history of the universe [73–76]. Now we discuss the dynamics of linear perturbations and the structure growth of universe.

5.1 Cosmological perturbations of gravitational potentials and the torsion

The perturbed equations can be derived by straightforward and tedious calculations, following the approach of [77–82]. The computer software Maple has been applied to work out the lengthy calculations. We focus on the scalar perturbations, since they are sufficient to reveal the basic features of the theory, allowing for a discussion of the growth of matter overdensities. The perturbed vierbein reads

$$e^0_\mu = \delta^0_\mu (1 + \phi), \quad e^a_\mu = a\delta^a_\mu (1 - \psi),$$
$$e^\mu_0 = \delta^\mu_0 (1 - \phi), \quad e^\mu_a = \frac{1}{a}\delta^\mu_a (1 + \psi). \tag{5.1}$$

in which we have introduced the scalar modes ϕ and ψ as functions of t. This induces a metric perturbation of the known form, namely

$$ds^2 = a^2(\eta)[-(1 + 2\phi)d\eta^2 + (1 - 2\psi)\gamma_{ij}dx^i dx^j], \tag{5.2}$$

in the longitudinal gauge and the conformal time η.

In order to preserve the global homogeneity and isotropy of the spacetime the perturbations are assumed to be small. It has been argued [44–48] that only two scalar torsion modes h and f are physically acceptable and no-ghosts. On the basis of the above theoretical tests (e.g., "no-ghosts" or "no-tachyons"), we use (2.7) to give the linear perturbation of the nonvanishing torsion components

$$\delta T_{ij0} = \delta_{ij}a^2\delta h, \quad \delta T_{ijk} = 2\epsilon_{ijk}a^3\delta f, \quad i,j,k,\ldots = 1,2,3.$$

In the case (2.16), $h = 0$, we have

$$\delta T_{ij0} = 0, \quad \delta T_{ijk} = 2\epsilon_{ijk}a^3\xi, \quad i,j,k,\ldots = 1,2,3. \tag{5.3}$$

where $\xi = \delta f$.

The unperturbed field equation (2.2) can be written as

$$G^\mu{}_\nu = T^\mu{}_\nu + T_{(g)}{}^\mu{}_\nu,$$

where $G^\mu{}_\nu$ is the Einstein tensor, $T^\mu{}_\nu$ is the energy-momentum of the ordinary matter and the radiation, $T_{(g)}{}^\mu{}_\nu$ is the energy-momentum of the "geometric dark energy" given by (2.4). The equations of motion for small perturbations linearized on the background metric are

$$\delta G^\mu{}_\nu = \delta T^\mu{}_\nu + \delta T_{(g)}{}^\mu{}_\nu. \tag{5.4}$$

For scalar type metric perturbations with a line element given in (5.2) (in conformal time), the perturbed field equations can be obtained following the approach of [77–82].

The cosmic fluid includes radiation, baryonic matter and dark matter, $\rho_{\mathrm{m}} = \rho_b + \rho_d$, we have

$$\rho = \rho_{\mathrm{m}} + \rho_{\mathrm{r}} = \rho_b + \rho_d + \rho_r, \quad p = p_r = \frac{1}{3}\rho_r. \tag{5.5}$$

Since

$$\rho_b \propto \frac{1}{a^3}, \quad \rho_r \propto \frac{1}{a^4}, \tag{5.6}$$

we suppose

$$\rho_d \propto \frac{1}{a^n}. \tag{5.7}$$

Then we have

$$\rho_r = r\rho_b, \quad \rho_d = v\rho_b, \tag{5.8}$$

where

$$r \propto a^{-1}, \quad v \propto a^{3-n}. \tag{5.9}$$

The equation

$$\delta G^0{}_0 = -\delta\rho - \delta\rho_g \tag{5.10}$$

takes the form

$$2a^{-2}\left(3\mathcal{H}\left(\mathcal{H}\phi + \psi'\right) - \nabla^2\psi\right) \tag{5.11}$$
$$+ \left(\frac{3\left(1 - 8\gamma\right)}{16\beta\gamma} + \frac{12\alpha + 5\beta - 16\gamma\left(3\alpha + \beta\right)}{8\beta\gamma}\left(1 + v\right)\rho_b + \frac{\left(3\alpha + \beta\right)\left(4\alpha + \beta\right)}{4\gamma\beta}\left(1 + v\right)^2\rho_b^2\right)\psi$$
$$= -\frac{1 + r + v}{3\gamma}\rho_b\delta - \frac{12\alpha + 5\beta - 72\alpha\gamma - 20\beta\gamma}{8\beta\gamma}\left(1 + v\right)\rho_b\delta - \frac{\left(3\alpha + \beta\right)\left(4\alpha + \beta\right)}{2\beta\gamma}\left(1 + v\right)^2\rho_b^2\delta,$$

where the growth of the baryonic matter density perturbation $\delta := \delta\rho_b/\rho_b$, $\mathcal{H} := a'/a = aH$, prime denotes derivative with respect to the conformal time η.

The equation

$$\delta G^i{}_j = \left(\delta p_r + \delta p_g\right)\delta^i{}_j, \tag{5.12}$$

reads

$$-2a^{-2}\left\{\left[\left(2\mathcal{H}' + \mathcal{H}^2\right)\phi + \mathcal{H}\phi' + \psi'' + 2\mathcal{H}\psi' + \frac{1}{2}\nabla^2\left(\phi - \psi\right)\right]\delta^i{}_j - \frac{1}{2}\partial_i\partial_j\left(\phi - \psi\right)\right\}$$
$$+ 2\left(f^2\psi + f\xi\right)\delta^i{}_j$$
$$= \left\{\frac{1}{3}r\rho_b\delta - \frac{\left(8\gamma + 1\right)\left(1 + r + v\right)}{24\gamma}\rho_b\delta + \frac{8\beta\gamma - 3\left(4\alpha + \beta\right)}{48\beta\gamma}\left(1 + v\right)\rho_b\delta$$
$$- \frac{\left(3\alpha + \beta\right)\left(4\alpha + \beta\right)}{12\beta\gamma}\left(1 + v\right)^2\rho_b^2\delta\right\}\delta^i{}_j, \tag{5.13}$$

where $i = 1, 2, 3$.

The equation

$$G^0{}_i = R^0{}_i$$
$$= T^0{}_i - 4\alpha R^0{}_i R - \beta\left(2R^{\rho 0}R_{\rho i} + 2R^{\rho\sigma}R^0{}_{\rho i\sigma}\right)$$
$$+ \gamma\left(4e_I{}^0\partial_\nu\left(e^{I\lambda}T_{i\lambda}{}^\nu\right) - 4e^K{}_\tau T_i{}^{0\nu}\partial_\nu e_K{}^\tau - 4T^{\lambda 0}{}_\tau T_{\lambda i}{}^\tau\right), \tag{5.14}$$

yields

$$2a^{-2}\left[\mathcal{H}\phi+\psi'\right]_{,i}-a^{-1}\left(8\gamma-1-4\alpha\left(1+r+v\right)\rho_b\right)\psi'_{,i}-2a^{-1}\left(1+\frac{4}{3}r+v\right)\rho_b V_{,i}$$

$$-4\beta a^{-1}\{\frac{\left(8\gamma-1\right)\left(4\gamma+1\right)}{16\beta\gamma}-\frac{12\alpha+5\beta-8\gamma\left(3\alpha+2\beta\right)}{24\beta\gamma}\left(1+v\right)\rho_b$$

$$-\frac{\left(3\alpha+\beta\right)\left(4\alpha+\beta\right)}{12\gamma\beta}\left(1+v\right)^2\rho_b^2\}\mathcal{H}\phi_{,i}+8\beta a^{-1}\mathcal{H}f^2\psi_{,i}+8a^{-1}\mathcal{H}f\xi_{,i}$$

$$=0. \tag{5.15}$$

In commoving orthogonal coordinates, the three-velocity of baryonic matter vanishes, $V_b^i=0$ [83]. For the potential V of the three-velocity field of the dark matter, the perturbed conservation law

$$\delta\left(\nabla_\mu T^\mu{}_\nu+\nabla_\mu T_g{}^\mu{}_\nu\right)=0, \tag{5.16}$$

leads to the equation

$$\dot{V}_{,i}+\left(\frac{\frac{4}{3}\dot{r}+\dot{v}}{1+\frac{4}{3}r+v}+H\right)V_{,i}=0,83 \tag{5.17}$$

when $\nu=i$.

In the case (2.15) and (2.16), using (5.1) and (5.3) we obtain the perturbation of the equation (2.3)

$$2f\xi=\left(\frac{1+r+v}{24\gamma}+\frac{3\left(4\alpha+\beta\right)-16\gamma\left(3\alpha+\beta\right)}{48\beta\gamma}\left(1+v\right)+\frac{\left(3\alpha+\beta\right)\left(4\alpha+\beta\right)}{12\gamma\beta}\left(1+v\right)^2\rho_b\right)\rho_b\delta. \tag{5.18}$$

In the Fourier space k, from (5.11) and (5.13) we obtain the equations of ϕ and ψ,

$$2a^{-2}\left(3\mathcal{H}\left(\mathcal{H}\phi+\psi'\right)+k^2\psi\right) \tag{5.19}$$

$$+\left(\frac{3\left(1-8\gamma\right)}{16\beta\gamma}+\frac{12\alpha+5\beta-16\gamma\left(3\alpha+\beta\right)}{8\beta\gamma}\left(1+v\right)\rho_b+\left(3\alpha+\beta\right)\frac{4\alpha+\beta}{4\gamma\beta}\left(1+v\right)^2\rho_b^2\right)\psi$$

$$=-\frac{1+r+v}{3\gamma}\rho_b\delta-\frac{12\alpha+5\beta-72\alpha\gamma-20\beta\gamma}{8\beta\gamma}\left(1+v\right)\rho_b\delta-\frac{\left(3\alpha+\beta\right)\left(4\alpha+\beta\right)}{2\beta\gamma}\left(1+v\right)^2\rho_b^2\delta,$$

and

$$-2a^{-2}\left\{\left[\left(2\mathcal{H}'+\mathcal{H}^2\right)\phi+\mathcal{H}\phi'+\psi''+2\mathcal{H}\psi'-\frac{1}{2}k^2\left(\phi-\psi\right)\right]\delta^i{}_j-\frac{1}{2}\partial_i\partial_j\left(\phi-\psi\right)\right\}$$

$$+2\left(f^2\psi+f\xi\right)\delta^i{}_j$$

$$=\left\{\frac{1}{3}r\rho_b\delta-\frac{\left(8\gamma+1\right)\left(1+r+v\right)}{24\gamma}\rho_b\delta+\frac{8\beta\gamma-3\left(4\alpha+\beta\right)}{48\beta\gamma}\left(1+v\right)\rho_b\delta\right.$$

$$\left.-\frac{\left(3\alpha+\beta\right)\left(4\alpha+\beta\right)}{12\beta\gamma}\left(1+v\right)^2\rho_b^2\delta\right\}\delta^i{}_j. \tag{5.20}$$

When $i\neq j$, (5.20) leads to

$$\phi=\psi, \tag{5.21}$$

– 15 –

agreeing with GR but in contrast to $f(R)$ theory [84]. Then we have the equations of ψ:

$$2a^{-2}\left(3\mathcal{H}\left(\mathcal{H}\psi + \psi'\right) + k^2\psi\right) \tag{5.22}$$
$$+\left(\frac{3\left(1-8\gamma\right)}{16\beta\gamma} + \frac{12\alpha + 5\beta - 16\gamma\left(3\alpha + \beta\right)}{8\beta\gamma}\left(1+v\right)\rho_b + \left(3\alpha + \beta\right)\frac{4\alpha + \beta}{4\gamma\beta}\left(1+v\right)^2\rho_b^2\right)\psi$$
$$= -\frac{1+r+v}{3\gamma}\rho_b\delta - \frac{12\alpha + 5\beta - 72\alpha\gamma - 20\beta\gamma}{8\beta\gamma}\left(1+v\right)\rho_b\delta - \frac{\left(3\alpha + \beta\right)\left(4\alpha + \beta\right)}{2\beta\gamma}\left(1+v\right)^2\rho_b^2\delta,$$

$$-2a^{-2}\left[\left(2\mathcal{H}' + \mathcal{H}^2\right)\psi + \psi'' + 3\mathcal{H}\psi'\right] + 2\left(f^2\psi + f\xi\right)$$
$$= \frac{1}{3}r\rho_b\delta - \frac{\left(8\gamma + 1\right)\left(1 + r + v\right)}{24\gamma}\rho_b\delta + \frac{8\beta\gamma - 3\left(4\alpha + \beta\right)}{48\beta\gamma}\left(1+v\right)\rho_b\delta$$
$$-\frac{\left(3\alpha + \beta\right)\left(4\alpha + \beta\right)}{12\beta\gamma}\left(1+v\right)^2\rho_b^2\delta. \tag{5.23}$$

One of the methods to measure the cosmic growth rate is redshift-space distortion that appears in clustering pattern of galaxies in galaxy redshift surveys. In order to confront the models with galaxy clustering surveys, we are interested in the modes deep inside the Hubble radius. In this case we can employ the quasistatic approximation on sub-horizon scales, under which, $\partial/\partial\eta \sim \mathcal{H} \ll k$. Then the perturbation equations (5.22), (5.23) give

$$\left(2a^{-2}k^2 - \frac{3\left(1-8\gamma\right)}{16\beta\gamma} - \frac{12\alpha + 5\beta - 16\gamma\left(3\alpha + \beta\right)}{8\beta\gamma}\left(1+v\right)\rho_b - \left(3\alpha + \beta\right)\frac{4\alpha + \beta}{4\gamma\beta}\left(1+v\right)^2\rho_b^2\right)\psi$$
$$= \left(\frac{1+r+v}{3\gamma}\rho_b + \frac{12\alpha + 5\beta - 72\alpha\gamma - 20\beta\gamma}{8\beta\gamma}\left(1+v\right)\rho_b + \frac{\left(3\alpha + \beta\right)\left(4\alpha + \beta\right)}{2\beta\gamma}\left(1+v\right)^2\rho_b^2\right)\delta, \tag{5.24}$$

and

$$-2a^{-2}\left[\left(2\mathcal{H}' + \mathcal{H}^2\right)\psi + \psi'' + 3\mathcal{H}\psi'\right] + 2\left(f^2\psi + f\xi\right)$$
$$= \frac{1}{3}r\rho_b\delta - \frac{\left(8\gamma + 1\right)\left(1 + r + v\right)}{24\gamma}\rho_b\delta + \frac{8\beta\gamma - 3\left(4\alpha + \beta\right)}{48\beta\gamma}\left(1+v\right)\rho_b\delta$$
$$-\frac{\left(3\alpha + \beta\right)\left(4\alpha + \beta\right)}{12\beta\gamma}\left(1+v\right)^2\rho_b^2\delta. \tag{5.25}$$

The equation (5.24) gives the expression of gravitational potential ψ. In the case $\alpha = -\frac{\beta}{2}$, if

$$a^{-2}k^2 \gg \rho_b, |\alpha\rho_b| \gg 1, \tag{5.26}$$

it reduces to the Poisson equation

$$\frac{k^2}{a^2}\psi = -4\pi G_{\text{eff}}\rho_b\delta, \tag{5.27}$$

where

$$G_{\text{eff}} = \frac{1}{16\pi\gamma}\left(1+v\right)^2\alpha\rho_b \tag{5.28}$$

is the effective gravitational coupling constant. In the framework of GR, G_{eff} is equivalent to the gravitational constant $G = 1$.

5.2 Equation of the structure growth and its solution

Since different theoretical models can achieve the same expansion history of universe, several methods should be used to discriminate the different models. The study on the growth of matter density perturbations may become the useful tool due to that theories with the same expansion history can have a different cosmic growth history. The perturbation quantities can be easily related to the cosmic observations [85–88].

Using (4.5), the equations (2.8) and (2.9) can be rewritten as

$$H^2 = \frac{1}{3}\left(\rho_b + \rho_{\text{other}}\right), \tag{5.29}$$

$$\dot{H} = -\frac{1}{2}\left(\rho_b + \rho_{\text{other}} + p_{\text{other}}\right), \tag{5.30}$$

where

$$\begin{aligned}
\rho_{\text{other}} &= \rho_r + \rho_d + \rho_g, \\
p_{\text{other}} &= p_r + p_g.
\end{aligned} \tag{5.31}$$

We introduce the perturbations of ρ_b, ρ_{other}, p_{other} and H [89–96]:

$$\rho_b \longrightarrow \left(1+\delta\right)\rho_b, \qquad \rho_{\text{other}} \longrightarrow \rho_{\text{other}} + \delta\rho_{\text{other}},$$

$$p_{\text{other}} \longrightarrow p_{\text{other}} + \delta p_{\text{other}}, \quad H \longrightarrow H + \delta H,$$

with

$$\delta H \equiv \frac{1}{3a}\nabla \cdot \mathbf{u}, \ \ \mathbf{u} = \nabla V. \tag{5.32}$$

Following the approach of [89–96] and [77–82], using (5.29), (5.30) and the perturbed conservation law

$$\delta\left(\nabla_\mu T^\mu{}_\nu + \nabla_\mu T_g{}^\mu{}_\nu\right) = 0, \tag{5.33}$$

we obtain the equation for the growth of the baryonic matter density perturbation δ:

$$\begin{aligned}
&\rho_b\,\ddot{\delta} + \frac{\dot{\rho}_b\,\rho_{\text{other}} - \rho_b\,\dot{\rho}_{\text{other}} + \dot{\rho}_b\,p_{\text{other}} - \rho_b\,\dot{p}_{\text{other}}}{\rho_b + \rho_{\text{other}} + p_{\text{other}}}\,\dot{\delta} \\
&-\left(\ddot{\rho}_{\text{other}} - \frac{3}{2}\rho_b\rho_b - 3\left(\rho_{\text{other}} + p_{\text{other}}\right)\rho_b - \frac{3}{2}\left(\rho_{\text{other}} + p_{\text{other}}\right)^2\right)\delta \\
&+\frac{2\,\dot{\rho}_b + 2\,\dot{\rho}_{\text{other}}}{\rho_b + \rho_{\text{other}} + p_{\text{other}}}\,\dot{p}_{\text{other}}\,\delta + \frac{2\,\dot{\rho}_b + 2\,\dot{\rho}_{\text{other}}}{\rho_b + \rho_{\text{other}} + p_{\text{other}}}\,\dot{\rho}_{\text{other}}\,\delta \\
&-\frac{\left(\dot{\rho}_b + \dot{\rho}_{\text{other}}\right)^2 + \left(\dot{\rho}_b + \dot{\rho}_{\text{other}}\right)\dot{p}_{\text{other}}}{\left(\rho_b + \rho_{\text{other}} + p_{\text{other}}\right)^2}\left(\rho_{\text{other}} + p_{\text{other}}\right)\delta \\
&+\ddot{\delta\rho}_{\text{other}} - \frac{2\,\dot{\rho}_b + 2\,\dot{\rho}_{\text{other}} + \dot{p}_{\text{other}}}{\rho_b + \rho_{\text{other}} + p_{\text{other}}}\,\dot{\delta\rho}_{\text{other}} - \frac{\dot{\rho}_b + \dot{\rho}_{\text{other}}}{\rho_b + \rho_{\text{other}} + p_{\text{other}}}\,\dot{\delta p}_{\text{other}} \\
&-\left(3\left(\rho_b + \rho_{\text{other}} + p_{\text{other}}\right) - \frac{\left(\dot{\rho}_b + \dot{\rho}_{\text{other}}\right)^2}{\left(\rho_b + \rho_{\text{other}} + p_{\text{other}}\right)^2} - \frac{\left(\dot{\rho}_b + \dot{\rho}_{\text{other}}\right)\dot{p}_{\text{other}}}{\left(\rho_b + \rho_{\text{other}} + p_{\text{other}}\right)^2}\right)\left(\delta\rho_{\text{other}} + \delta p_{\text{other}}\right) \\
&= 0.
\end{aligned} \tag{5.34}$$

Up to now, the complete set of equations that describes the general linear perturbations has been obtained. It provides enough information about the behaviors of the perturbation and can be compared with the results of the ΛCDM model.

Eqs. (5.5), (5.8) and (5.9) yield

$$\dot{\rho}_b = -3H\rho_b, \quad \dot{\rho}_r = -4H\rho_r, \quad \dot{\rho}_d = -nH\rho_d,$$
$$\dot{p}_r = -\frac{4}{3}H\rho_r,$$
$$\dot{r} = -rH, \qquad \dot{v} = (3-n)\,vH, \tag{5.35}$$

and then (5.31), (2.22) and (2.23) give

$$\rho_{\text{other}} = \frac{3\,(8\gamma - 1)^2}{32\beta\gamma} + \left(r + v - \frac{(8\gamma - 1)\,(1 + r + v)}{8\gamma} - \frac{3\,(8\gamma - 1)\,(4\alpha + \beta)}{16\beta\gamma}\,(1 + v) \right)\rho_b$$
$$+ \frac{(3\alpha + \beta)\,(4\alpha + \beta)}{8\beta\gamma}\,(1 + v)^2\,\rho_b^2, \tag{5.36}$$

$$p_{\text{other}} = \frac{64\gamma^2 - 1}{32\beta\gamma} + \left(\frac{1}{3}r - \frac{(8\gamma + 1)\,(1 + r + v)}{24\gamma} + \frac{8\beta\gamma - 3\,(4\alpha + \beta)}{48\beta\gamma}\,(1 + v) \right)\rho_b$$
$$- \frac{(3\alpha + \beta)\,(4\alpha + \beta)}{24\beta\gamma}\,(1 + v)^2\,\rho_b^2, \tag{5.37}$$

Using (5.36) and (5.37), by straightforward and tedious calculations, the equation (5.34) can be written as

$$(1 + r + v + A + D\rho_b)\,\rho_b\,\ddot{\delta} + M\,(r, v, \rho_b)\,H\,\dot{\delta} + N\,(r, v, \rho_b)\,H^2\delta + Q\,(r, v, \rho_b)\,\delta = 0, \tag{5.38}$$

where A, D, $M\,(r, v, \rho_b)$, $N\,(r, v, \rho_b)$, and $Q\,(r, v, \rho_b)$ are given in appendix A.

Supposing

$$n = 3, \tag{5.39}$$

in the case (3.4) and (5.26), i.e. when $\rho_d \propto a^{-3}$, $\beta = -2\alpha$, and $|\alpha\rho_b| \gg 1$, the equation (5.38) becomes

$$\ddot{\delta} - 22H\,\dot{\delta} + 3H^2\delta = 0. \tag{5.40}$$

Introduce the logarithmic time variable

$$N = \ln a. \tag{5.41}$$

Eq. (5.40) takes the form

$$\frac{d^2\delta}{dN^2} - 23\frac{d\delta}{dN} + 3\delta = 0,$$

and gives the solution

$$\delta = \delta_{0+}a^{\frac{1}{2}\left(23 + \sqrt{517}\right)} + \delta_{0-}a^{\frac{1}{2}\left(23 - \sqrt{517}\right)}$$
$$\approx \delta_{0+}a^{22.869} + \delta_{0-}a^{0.13118}, \tag{5.42}$$

which can be compared with the result in GR [97].

6 Conclusions

A cosmology of Poincaré gauge theory has been developed. We focus on the case including a pseudoscalar scalar torsion function f as suggested by Baekler, Hehl and Nester [98]. The gravitational field equation and the two-family cosmological equations has been obtained in ref. [65]. In this paper, we focus on studying the second family cosmological equations corresponding to the pseudoscalar torsion function. It is found that although we do not introduce a cosmological constant in the action it automatically emerges in the derivation of the cosmological equations and then is endowed with intrinsic character. It is nothing but the intrinsic torsion and curvature of the vacuum universe. The dark energy is identified with the geometry of the spacetime. Now we are returning to the original idea of Einstein and Wheeler: gravity is a geometry [99, 100]. The cosmological constant puzzle and the coincidence and fine tuning problem are solved naturally. The point is that the dark energy is the functions of the density and pressure of the cosmic fluid and includes the cosmological constant but can not be identified with it. The analytic expressions of the state equation and the density parameters of the matter and the geometric dark energy are derived and used to determine the values of α , β and γ. Then a theoretical value of the cosmological constant is computed and compared with the observed datum. An analytic integral of the cosmological equation is obtained and used to evaluate the age of the universe which can be compared with observed data. The full equations of linear cosmological perturbations and the solutions are obtained. In addition, the behavior of perturbations for the sub-horizon modes relevant to large-scale structures is discussed. This model can be distinguished from others by considering the evolution of matter perturbations and gravitational potentials.

Acknowledgments

A   The growth of structures in linear perturbation theory

In the following, we give the derivation of the equation (5.38):

Letting

$$A = -\frac{(8\gamma - 1)(1 + r + v)}{8\gamma} - \frac{3(8\gamma - 1)(4\alpha + \beta)}{16\beta\gamma}(1 + v),$$

$$B = -\frac{(8\gamma + 1)(1 + r + v)}{24\gamma} + \frac{8\beta\gamma - 3(4\alpha + \beta)}{48\beta\gamma}(1 + v),$$

$$D = \frac{(3\alpha + \beta)(4\alpha + \beta)}{4\beta\gamma}(1 + v)^2.$$

$$E = -\frac{(8\gamma - 1)((3 - n)v - r)}{8\gamma} - \frac{3(8\gamma - 1)(4\alpha + \beta)}{16\beta\gamma}(3 - n)v,$$

$$F = -\frac{(8\gamma + 1)((3 - n)v - r)}{24\gamma} + \frac{8\beta\gamma - 3(4\alpha + \beta)}{48\beta\gamma}(3 - n)v,$$

(A.1)

$$K = \frac{(3\alpha + \beta)(4\alpha + \beta)}{2\beta\gamma}(1 + v)(3 - n)v,$$

$$L = \frac{(8\gamma - 1)\left((3 - n)^2 v + r\right)}{8\gamma} + \frac{3(8\gamma - 1)(4\alpha + \beta)}{16\beta\gamma}(3 - n)^2 v, \tag{A.2}$$

Eqs. (5.23), (5.24) and (5.25) give

$$\rho_{\text{other}} = \frac{3(1 - 8\gamma)^2}{32\beta\gamma} + (r + v + A)\rho_b + \frac{D}{2}\rho_b^2,$$

$$p_{\text{other}} = \frac{64\gamma^2 - 1}{32\beta\gamma} + \left(\frac{1}{3}r + B\right)\rho_b - \frac{D}{6}\rho_b^2. \tag{A.3}$$

Then we compute

$$\dot{\rho}_{\text{other}} = -(vn + 4r)H\rho_b + EH\rho_b + \frac{K}{2}H\rho_b^2,$$

$$\dot{p}_{\text{other}} = -\frac{4}{3}rH\rho_b + FH\rho_b - \frac{K}{6}H\rho_b^2, \tag{A.4}$$

$$\ddot{\rho}_{\text{other}} = \left(n^2 v + 16r - L - 3E\right)H^2\rho_b$$
$$- \frac{1}{2}(n + 3)KH^2\rho_b^2 + \frac{K}{2}\frac{(3 - n)v}{(1 + v)}H^2\rho_b^2$$
$$+ (E - (vn + 4r))\dot{H}\rho_b + \frac{K}{2}\dot{H}\rho_b^2, \tag{A.5}$$

$$\delta\rho_{\text{other}} = (r + v + A)\rho_b\delta + D\rho_b^2\delta,$$

$$\delta p_{\text{other}} = \left(\frac{1}{3}r + B\right)\rho_b\delta - \frac{D}{3}\rho_b^2\delta, \tag{A.6}$$

$$\dot{\delta\rho}_{\text{other}} = -(vn + 4r)H\rho_b\delta + (r + v)\rho_b\dot{\delta} + A\rho_b\dot{\delta} + D\rho_b^2\dot{\delta} - (3A - E)H\rho_b\delta - (2D - K)H\rho_b^2\delta,$$

$$\dot{\delta p}_{\text{other}} = -\frac{4}{3}rH\rho_b\delta + \frac{1}{3}r\rho_b\dot{\delta} + B\rho_b\dot{\delta} - \frac{D}{3}\rho_b^2\dot{\delta} - (3B - F)H\rho_b\delta + \left(2D - \frac{K}{3}\right)H\rho_b^2\delta, \text{(A.7)}$$

and

$$\ddot{\delta\rho}_{\text{other}} = (r + v + (A + D\rho_b))\rho_b\ddot{\delta}$$
$$- 2\left[(nv + 4r) - 2(E - 3A + (K - 6D)\rho_b)\right]H\rho_b\dot{\delta}$$
$$+ \left[n^2 v + 16r + 9A - 6E - L + \left(36D - 12K + \frac{1 + 2v}{1 + v}K(3 - n)\right)\rho_b\right]H^2\rho_b\delta$$
$$+ (E - 3A - (vn + 4r) + (K - 6D)\rho_b)\dot{H}\rho_b\delta. \tag{A.8}$$

Substituting these into (5.34) yields (5.38):

$$(1 + r + v + A + D\rho_b)\rho_b\ddot{\delta} + M(r, v, \rho_b)H\dot{\delta} + N(r, v, \rho_b)H^2\delta + Q(r, v, \rho_b)\delta = 0, \tag{A.9}$$

where

$$M\left(r,v,\rho_b\right) = -\left(3+4r+2vn-3v+9A-3B-4E\right)\rho_b + \left(4K-22D\right)\rho_b^2$$

$$+\frac{\left(-\left(1+r+v+A\right)\rho_b+D\rho_b^2\right)\left(-vn-\frac{16}{3}r+E+F-3\right)}{\frac{\left(8\gamma-1\right)\left(16\gamma-1\right)}{16\beta\gamma}+\left(1+\frac{4}{3}r+v+A+B\right)\rho_b+\frac{1}{3}D\rho_b^2}\rho_b$$

$$+\frac{-\frac{1}{3}\left(1+r+v+A\right)\rho_b+\frac{1}{3}D\rho_b^2}{\frac{\left(8\gamma-1\right)\left(16\gamma-1\right)}{16\beta\gamma}+\left(1+\frac{4}{3}r+v+A+B\right)\rho_b+\frac{1}{3}D\rho_b^2}K\rho_b^2, \tag{A.10}$$

$$N\left(r,v,\rho_b\right) = -\frac{\frac{3\left(8\gamma-1\right)\left(16\gamma-1\right)}{16\beta\gamma}+3A\rho_b+\left(D-\frac{K}{2}\right)\rho_b^2}{\frac{\left(8\gamma-1\right)\left(16\gamma-1\right)}{16\beta\gamma}+\left(1+\frac{4}{3}r+v+A+B\right)\rho_b+\frac{1}{3}D\rho_b^2}\left(3+\frac{16}{3}r+vn-E-F\right)\rho_b$$

$$+\left(-9B+36D\rho_b-3E+\frac{1}{2}\left(n-19\right)K\right)\rho_b+\frac{2+3v}{2\left(1+v\right)}\left(3-n\right)K\rho_b^2$$

$$+\frac{\frac{\left(8\gamma-1\right)\left(16\gamma-1\right)}{16\beta\gamma}+A\rho_b+\frac{1}{3}\left(D-\frac{K}{2}\right)\rho_b^2}{\frac{\left(8\gamma-1\right)\left(16\gamma-1\right)}{16\beta\gamma}+\left(1+\frac{4}{3}r+v+A+B\right)\rho_b+\frac{1}{3}D\rho_b^2}K\rho_b^2, \tag{A.11}$$

and

$$Q\left(r,v,\rho_b\right) = +\frac{3}{2}\left(\frac{\left(8\gamma-1\right)\left(16\gamma-1\right)}{16\beta\gamma}\right)^2-\frac{3}{2}\frac{\left(8\gamma-1\right)\left(16\gamma-1\right)}{16\beta\gamma}\rho_b$$

$$-\left(\frac{\left(8\gamma-1\right)\left(16\gamma-1\right)}{16\beta\gamma}D+\frac{3}{2}\left(1+\frac{4}{3}r+v+A+B\right)^2+\frac{3}{2}\left(1+\frac{4}{3}r+v+A+B\right)\right)\rho_b^2$$

$$-\left(2\left(1+\frac{4}{3}r+v+A+B\right)+\frac{1}{2}\right)D\rho_b^3-\frac{1}{2}D^2\rho_b^4$$

$$-3A\,\dot{H}\,\rho_b\delta+\left(\frac{K}{2}-6D\right)\dot{H}\,\rho_b^2. \tag{A.12}$$

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](#)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

[1] L. Amendola and S. Tsujikawa, *Dark Energy*, Cambridge University Press, Cambridge U.K. (2010).

[2] PLANCK collaboration, P.A.R. Ade et al., *Planck 2015 results. XIII. Cosmological parameters*, [arXiv:1502.01589](#) [INSPIRE].

[3] M.J. Mortonson, D.H. Weinberg and M. White, *Dark Energy: A Short Review*, [arXiv:1401.0046](#) [INSPIRE].

[4] R.-G. Cai, Z.-K. Guo and T. Yang, *Null test of the cosmic curvature using $H(z)$ and supernovae data*, *Phys. Rev.* **D 93** (2016) 043517 [arXiv:1509.06283] [INSPIRE].

[5] R.-G. Cai and S.-J. Wang, *Dark matter superfluid and DBI dark energy*, *Phys. Rev.* **D 93** (2016) 023515 [arXiv:1511.00627] [INSPIRE].

[6] R.A. Battye and A. Moss, *Evidence for Massive Neutrinos from Cosmic Microwave Background and Lensing Observations*, *Phys. Rev. Lett.* **112** (2014) 051303 [arXiv:1308.5870] [InSPIRE].

[7] J.-F. Zhang, M.-M. Zhao, Y.-H. Li and X. Zhang, *Neutrinos in the holographic dark energy model: constraints from latest measurements of expansion history and growth of structure*, *JCAP* **04** (2015) 038 [arXiv:1502.04028] [InSPIRE].

[8] J. Lu, Y. Xu and Y. Wu, *Cosmic constraint on the unified model of dark sectors with or without a cosmic string fluid in the varying gravitational constant theory*, *Eur. Phys. J.* **C 75** (2015) 473 [arXiv:1503.02439] [InSPIRE].

[9] J. Lu, D. Geng, L. Xu, Y. Wu and M. Liu, *Reduced modified Chaplygin gas cosmology*, *JHEP* **02** (2015) 071 [arXiv:1312.0779] [InSPIRE].

[10] T.M. Davis, *Cosmological constraints on dark energy*, *Gen. Rel. Grav.* **46** (2014) 1731 [arXiv:1404.7266] [InSPIRE].

[11] S. Capozziello, T. Harko, T.S. Koivisto, F.S.N. Lobo and G.J. Olmo, *Hybrid metric-Palatini gravity*, *Universe* **1** (2015) 199 [arXiv:1508.04641] [InSPIRE].

[12] K. Takahashi and J. Yokoyama, *Equation of state of dark energy in f(R) gravity*, *Phys. Rev.* **D 91** (2015) 084060 [arXiv:1503.07412] [InSPIRE].

[13] A.W.H. Preston and T.R. Morris, *Cosmological back-reaction in modified gravity and its implications for dark energy*, *JCAP* **09** (2014) 017 [arXiv:1406.5398] [InSPIRE].

[14] T. Clifton, P.G. Ferreira, A. Padilla and C. Skordis, *Modified Gravity and Cosmology*, *Phys. Rept.* **513** (2012) 1 [arXiv:1106.2476] [InSPIRE].

[15] S. Capozziello and M. De Laurentis, *Extended Theories of Gravity*, *Phys. Rept.* **509** (2011) 167 [arXiv:1108.6266] [InSPIRE].

[16] K. Bamba, S. Capozziello, S. Nojiri and S.D. Odintsov, *Dark energy cosmology: the equivalent description via different theoretical models and cosmography tests*, *Astrophys. Space Sci.* **342** (2012) 155 [arXiv:1205.3421] [InSPIRE].

[17] L.G. Jaime, L. Patino and M. Salgado, *f(R) Cosmology revisited*, arXiv:1206.1642 [InSPIRE].

[18] S. Nojiri and S.D. Odintsov, *Unified cosmic history in modified gravity: from F(R) theory to Lorentz non-invariant models*, *Phys. Rept.* **505** (2011) 59 [arXiv:1011.0544] [InSPIRE].

[19] S. Capozziello and V. Faraoni, *Beyond Einstein Gravity*, Springer, Heidelberg Germany (2010).

[20] A. De Felice and S. Tsujikawa, *f(R) theories*, *Living Rev. Rel.* **13** (2010) 3 [arXiv:1002.4928] [InSPIRE].

[21] E.M. Barboza Jr., R.C. Nunes, É.M.C. Abreu and J.A. Neto, *Is this the end of dark energy?*, arXiv:1501.03491 [InSPIRE].

[22] T. Multamaki and I. Vilja, *Spherically symmetric solutions of modified field equations in f(R) theories of gravity*, *Phys. Rev.* **D 74** (2006) 064022 [astro-ph/0606373] [InSPIRE].

[23] V. Faraoni, *Negative energy and stability in scalar-tensor gravity*, *Phys. Rev.* **D 70** (2004) 081501 [gr-qc/0408073] [InSPIRE].

[24] E.E. Flanagan, *The Conformal frame freedom in theories of gravitation*, *Class. Quant. Grav.* **21** (2004) 3817 [gr-qc/0403063] [InSPIRE].

[25] Y. Fujii and K. Maeda, *The Scalar-Tensor Theory of Gravitation*, Cambridge University Press, Cambridge U.K. (2003).

[26] G. Magnano and L.M. Sokolowski, *On physical equivalence between nonlinear gravity theories and a general relativistic selfgravitating scalar field*, *Phys. Rev.* **D 50** (1994) 5039 [gr-qc/9312008] [INSPIRE].

[27] S. Capozziello, M. De Laurentis and O. Luongo, *Curvature dark energy reconstruction through different cosmographic distance definitions*, *Annalen Phys.* **526** (2014) 309 [arXiv:1406.6996] [INSPIRE].

[28] L.G. Jaime, L. Patiño and M. Salgado, *Note on the equation of state of geometric dark-energy in f(R) gravity*, *Phys. Rev.* **D 89** (2014) 084010 [arXiv:1312.5428] [INSPIRE].

[29] L.G. Jaime, L. Patino and M. Salgado, *f(R) Cosmology revisited*, arXiv:1206.1642 [INSPIRE].

[30] F.W. Hehl, J.D. McCrea, E.W. Mielke and Y. Ne'eman, *Metric affine gauge theory of gravity: Field equations, Noether identities, world spinors and breaking of dilation invariance*, *Phys. Rept.* **258** (1995) 1 [gr-qc/9402012] [INSPIRE].

[31] F.W. Hehl, *Gauge Theory of Gravity and Spacetime*, arXiv:1204.3672 [INSPIRE].

[32] D. Puetzfeld and Y.N. Obukhov, *Equations of motion in metric-affine gravity: A covariant unified framework*, *Phys. Rev.* **D 90** (2014) 084034 [arXiv:1408.5669] [INSPIRE].

[33] Y.N. Obukhov and D. Puetzfeld, *Conservation laws in gravity: A unified framework*, *Phys. Rev.* **D 90** (2014) 024004 [arXiv:1405.4003] [INSPIRE].

[34] S.A. Ali, C. Cafaro, S. Capozziello and C. Corda, *On the Poincaré Gauge Theory of Gravitation*, *Int. J. Theor. Phys.* **48** (2009) 3426 [arXiv:0907.0934] [INSPIRE].

[35] C.-H. Wang and Y.-H. Wu, *Inflation in $R + R^2$ Gravity with Torsion*, *Class. Quant. Grav.* **26** (2009) 045016 [arXiv:0807.0069] [INSPIRE].

[36] D. Puetzfeld, *Status of non-Riemannian cosmology*, *New Astron. Rev.* **49** (2005) 59 [gr-qc/0404119] [INSPIRE].

[37] Y.-F. Cai, S. Capozziello, M. De Laurentis and E.N. Saridakis, *f(T) Teleparallel Gravity and Cosmology*, arXiv:1511.07586 [INSPIRE].

[38] S. Capozziello, S. Carloni, G. Lambiase, C. Stornaiolo and A. Troisi, *Curvature and torsion quintessence*, gr-qc/0111106 [INSPIRE].

[39] S. Capozziello, V.F. Cardone, E. Piedipalumbo, M. Sereno and A. Troisi, *Matching torsion Lambda-term with observations*, *Int. J. Mod. Phys.* **D 12** (2003) 381 [astro-ph/0209610] [INSPIRE].

[40] S. Capozziello, R. Cianci, C. Stornaiolo and S. Vignolo, *f(R) gravity with torsion: The Metric-affine approach*, *Class. Quant. Grav.* **24** (2007) 6417 [arXiv:0708.3038] [INSPIRE].

[41] N.J. Popłwski, *Cosmology with torsion: An alternative to cosmic inflation*, *Phys. Lett.* **B 694** (2010) 181 [*Erratum ibid.* **B 701** (2011) 672] [arXiv:1007.0587] [INSPIRE].

[42] N.J. Poplawski, *Nonsingular, big-bounce cosmology from spinor-torsion coupling*, *Phys. Rev.* **D 85** (2012) 107502 [arXiv:1111.4595] [INSPIRE].

[43] N.J. Poplawski, *Spacetime torsion as a possible remedy to major problems in gravity and cosmology*, *Astron. Rev.* **8** (2013) 108 [arXiv:1106.4859] [INSPIRE].

[44] H.-j. Yo and J.M. Nester, *Hamiltonian analysis of Poincaré gauge theory scalar modes*, *Int. J. Mod. Phys.* **D 8** (1999) 459 [gr-qc/9902032] [INSPIRE].

[45] K.-F. Shie, J.M. Nester and H.-J. Yo, *Torsion Cosmology and the Accelerating Universe*, *Phys. Rev.* **D 78** (2008) 023522 [arXiv:0805.3834] [INSPIRE].

[46] H. Chen, F.-H. Ho, J.M. Nester, C.-H. Wang and H.-J. Yo, *Cosmological dynamics with propagating Lorentz connection modes of spin zero*, *JCAP* **10** (2009) 027 [arXiv:0908.3323] [INSPIRE].

[47] F.-H. Ho and J.M. Nester, *Poincaré Gauge Theory With Coupled Even And Odd Parity Dynamic Spin-0 Modes: Dynamic Equations For Isotropic Bianchi Cosmologies*, *Annalen Phys.* **524** (2012) 97 [arXiv:1106.0711] [INSPIRE].

[48] F.-H. Ho, H. Chen, J.M. Nester and H.-J. Yo, *General Poincaré Gauge Theory Cosmology*, *Chin. J. Phys.* **53** (2015) 110109 [arXiv:1512.01202] [INSPIRE].

[49] A.S. Garkun, V.I. Kudin and A.V. Minkevich, *To theory of asymptotically stable accelerating Universe in Riemann-Cartan spacetime*, *JCAP* **12** (2014) 027 [arXiv:1410.0460] [INSPIRE].

[50] A.V. Minkevich, *On theory of regular accelerating universe in Riemann-Cartan spacetime*, *Mod. Phys. Lett.* **A 28** (2013) 1350090 [arXiv:1309.6075] [INSPIRE].

[51] A.V. Minkevich, A.S. Garkun and V.I. Kudin, *On some physical aspects of isotropic cosmology in Riemann-Cartan spacetime*, *JCAP* **03** (2013) 040 [arXiv:1302.2578] [INSPIRE].

[52] A.S. Garkun, V.I. Kudin, A.V. Minkevich and Yu. G. Vasilevsky, *Numerical analysis of cosmological models for accelerating Universe in Poincaré gauge theory of gravity*, arXiv:1107.1566 [INSPIRE].

[53] A.V. Minkevich, *Gravitation and regular Universe without dark energy and dark matter*, arXiv:1102.0620 [INSPIRE].

[54] A.V. Minkevich, *de Sitter spacetime with torsion as physical spacetime in the vacuum*, *Mod. Phys. Lett.* **A 26** (2011) 259 [arXiv:1002.0538] [INSPIRE].

[55] A.V. Minkevich, *Accelerating Universe without dark energy and dark matter and spacetime torsion*, *Phys. Lett.* **B 678** (2009) 423 [arXiv:0902.2860] [INSPIRE].

[56] A.V. Minkevich, A.S. Garkun and V.I. Kudin, *Regular accelerating universe without dark energy*, *Class. Quant. Grav.* **24** (2007) 5835 [arXiv:0706.1157] [INSPIRE].

[57] H. Lü, A. Perkins, C.N. Pope and K.S. Stelle, *Black Holes in Higher-Derivative Gravity*, *Phys. Rev. Lett.* **114** (2015) 171601 [arXiv:1502.01028] [INSPIRE].

[58] H. Lü, A. Perkins, C.N. Pope and K.S. Stelle, *Spherically Symmetric Solutions in Higher-Derivative Gravity*, *Phys. Rev.* **D 92** (2015) 124019 [arXiv:1508.00010] [INSPIRE].

[59] K.S. Stelle, *Renormalization of higher-derivative quantum gravity*, *Phys. Rev.* **D 16** (1977) 953.

[60] A. Hindawi, B.A. Ovrut and D. Waldram, *Consistent spin-two coupling and quadratic gravitation*, *Phys. Rev.* **D 53** (1996) 5583.

[61] S. Deser and B. Tekin, *Gravitational energy in quadratic curvature gravities*, *Phys. Rev. Lett.* **89** (2002) 101101 [hep-th/0205318] [INSPIRE].

[62] J.D. Barrow and S. Hervik, *On the evolution of universes in quadratic theories of gravity*, *Phys. Rev.* **D 74** (2006) 124017 [gr-qc/0610013] [INSPIRE].

[63] J.D. Barrow and J.D. Middleton, *Stable isotropic cosmological singularities in quadratic gravity*, *Phys. Rev.* **D 75** (2007) 123515 [gr-qc/0702098] [INSPIRE].

[64] J. Polchinski, *String Theory*, Cambridge University Press, Cambridge U.K. (1998).

[65] G. Chee and Y. Guo, *Exact de Sitter solutions in quadratic gravitation with torsion*, *Class. Quant. Grav.* **29** (2012) 235022 [arXiv:1205.5419] [INSPIRE].

[66] A.A. Starobinsky, *A New Type of Isotropic Cosmological Models Without Singularity*, *Phys. Lett.* **B 91** (1980) 99 [INSPIRE].

[67] M. Tsamparlis, *Cosmological principle and torsion*, *Phys. Lett.* **A 75** (1979) 27.

[68] H. Gonner and F. Mueller-Hoissen, *Spatially Homogeneous and Isotropic Spaces in Theories of Gravitation With Torsion*, *Class. Quant. Grav.* **1** (1984) 651 [INSPIRE].

[69] P. Baekler, *Exact solutions in the Poincaré gauge field theory of gravitation*, Ph.D. Thesis, University of Cologne, Cologne Germany (1986).

[70] M.J. Mortonson, D.H. Weinberg and M. White, *Dark Energy: A Short Review*, arXiv:1401.0046 [INSPIRE].

[71] T.M. Davis, *Cosmological constraints on dark energy*, *Gen. Rel. Grav.* **46** (2014) 1731 [arXiv:1404.7266] [INSPIRE].

[72] J. Frieman, M. Turner and D. Huterer, *Dark Energy and the Accelerating Universe*, *Ann. Rev. Astron. Astrophys.* **46** (2008) 385 [arXiv:0803.0982] [INSPIRE].

[73] D. Comelli, M. Crisostomi and L. Pilo, *FRW Cosmological Perturbations in Massive Bigravity*, *Phys. Rev.* **D 90** (2014) 084003 [arXiv:1403.5679] [INSPIRE].

[74] N.A. Lima and A.R. Liddle, *Linear perturbations in viable $f(R)$ theories*, *Phys. Rev.* **D 88** (2013) 043521 [arXiv:1307.1613] [INSPIRE].

[75] F.G. Alvarenga, A. de la Cruz-Dombriz, M.J.S. Houndjo, M.E. Rodrigues and D. Sáez-Gómez, *Dynamics of scalar perturbations in $f(R,T)$ gravity*, *Phys. Rev.* **D 87** (2013) 103526 [arXiv:1302.1866] [INSPIRE].

[76] J. Matsumoto, *Cosmological perturbations in F(R) gravity*, *Phys. Rev.* **D 87** (2013) 104002 [arXiv:1303.6828] [INSPIRE].

[77] P. Peter, *Cosmological Perturbation Theory*, arXiv:1303.2509 [INSPIRE].

[78] J. Lesgourgues, *Cosmological Perturbations*, arXiv:1302.4640 [INSPIRE].

[79] F. Bernardeau, S. Colombi, E. Gaztanaga and R. Scoccimarro, *Large scale structure of the universe and cosmological perturbation theory*, *Phys. Rept.* **367** (2002) 1 [astro-ph/0112551] [INSPIRE].

[80] C.-P. Ma and E. Bertschinger, *Cosmological perturbation theory in the synchronous and conformal Newtonian gauges*, *Astrophys. J.* **455** (1995) 7 [astro-ph/9506072] [INSPIRE].

[81] V.F. Mukhanov, H.A. Feldman and R.H. Brandenberger, *Theory of cosmological perturbations*, *Phys. Rept.* **215** (1992) 203.

[82] J.M. Bardeen, *Gauge Invariant Cosmological Perturbations*, *Phys. Rev.* **D 22** (1980) 1882 [INSPIRE].

[83] K.A. Malik and D. Wands, *Cosmological perturbations*, *Phys. Rept.* **475** (2009) 1 [arXiv:0809.4944] [INSPIRE].

[84] D.H. Weinberg, M.J. Mortonson, D.J. Eisenstein, C. Hirata, A.G. Riess and E. Rozo, *Observational Probes of Cosmic Acceleration*, *Phys. Rept.* **530** (2013) 87 [arXiv:1201.2434] [inSPIRE].

[85] G.-B. Zhao, H. Li, E.V. Linder, K. Koyama, D.J. Bacon and X. Zhang, *Testing Einstein Gravity with Cosmic Growth and Expansion*, *Phys. Rev.* **D 85** (2012) 123546 [arXiv:1109.1846] [inSPIRE].

[86] R. Bean, D. Bernat, L. Pogosian, A. Silvestri and M. Trodden, *Dynamics of Linear Perturbations in f(R) Gravity*, *Phys. Rev.* **D 75** (2007) 064020 [astro-ph/0611321] [inSPIRE].

[87] T. Koivisto and H. Kurki-Suonio, *Cosmological perturbations in the palatini formulation of modified gravity*, *Class. Quant. Grav.* **23** (2006) 2355 [astro-ph/0509422] [inSPIRE].

[88] J.-c. Hwang and H.-r. Noh, *Gauge ready formulation of the cosmological kinetic theory in generalized gravity theories*, *Phys. Rev.* **D 65** (2002) 023512 [astro-ph/0102005] [inSPIRE].

[89] E. Lifshitz, *On the Gravitational stability of the expanding universe*, *J. Phys. (USSR)* **10** (1946) 116 [inSPIRE].

[90] W.B. Bonnor, *Jeans' formula for gravitational instability*, *Mon. Not. Roy. Astron. Soc.* **117** (1957) 104.

[91] E.M. Lifshitz and I.M. Khalatnikov, *Investigations in relativistic cosmology*, *Adv. Phys.* **12** (1963) 185 [inSPIRE].

[92] H. Nariai, *The Lagrangian Approach to the Gravitational Instability in an Expanding Universe*, *Prog. Theor. Phys.* **41** (1969) 686.

[93] H. Noh and J.-c. Hwang, *Second-order perturbations of the Friedmann world model*, *Phys. Rev.* **D 69** (2004) 104011 [inSPIRE].

[94] J.-c. Hwang and H. Noh, *Classical evolution and quantum generation in generalized gravity theories including string corrections and tachyon: Unified analyses*, *Phys. Rev.* **D 71** (2005) 063536 [gr-qc/0412126] [inSPIRE].

[95] J.-c. Hwang and H. Noh, *Second order perturbations of a zero-pressure cosmological medium: Proofs of the relativistic-Newtonian correspondence*, *Phys. Rev.* **D 72** (2005) 044011 [gr-qc/0412128] [inSPIRE].

[96] J.-c. Hwang and H. Noh, *Third order perturbations of a zero-pressure cosmological medium: Pure general relativistic nonlinear effects*, *Phys. Rev.* **D 72** (2005) 044012 [gr-qc/0412129] [inSPIRE].

[97] E.J. Copeland, M. Sami and S. Tsujikawa, *Dynamics of dark energy*, *Int. J. Mod. Phys.* **D 15** (2006) 1753 [hep-th/0603057] [inSPIRE].

[98] P. Baekler, F.W. Hehl and J.M. Nester, *Poincaré gauge theory of gravity: Friedman cosmology with even and odd parity modes. Analytic part*, *Phys. Rev.* **D 83** (2011) 024001 [arXiv:1009.5112] [inSPIRE].

[99] H.-Y. Chiu and W.F. Hoffmann, *Gravitation and Relativity*, W.A. Benjamin Inc., New York U.S.A. (1964).

[100] C.W. Misner, K.s. Thorne and J.A. Wheeler, *Gravitation*, W.H. Freeman and Co., San Francisco U.S.A. (1973).