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On conserved charges and thermodynamics of the AdS_4 dyonic black hole

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Abstract: We consider four-dimensional gravity in the presence of a dilatonic scalar field and an Abelian gauge field. This theory corresponds to the bosonic sector of a Kaluza-Klein reduction of eleven-dimensional supergravity which induces a specific self-interacting potential for the scalar field. We compute the conserved charges and carry out the thermodynamics of an anti-de Sitter (AdS) dyonic black hole solution that was proposed recently. The charges coming from symmetries of the action are computed using the Regge-Teitelboim Hamiltonian approach. They correspond to the mass, which acquires contributions from the scalar field, and the electric charge. We introduce integrability conditions because the scalar field leads to non-integrable terms in the variation of the mass. These conditions are generically solved by introducing boundary conditions that relate the leading and subleading terms of the scalar field fall-off. The Hamiltonian Euclidean action, computed in the grand canonical ensemble, is obtained by demanding the action to have an extremum. Its value is given by a radial boundary term plus an additional polar angle boundary term due to the presence of a magnetic monopole. Remarkably, the magnetic charge can be identified from the variation of the additional polar angle boundary term, confirming that the first law of black hole thermodynamics is a consequence of having a well-defined and finite Hamiltonian action principle, even if the charge does not come from a symmetry of the action. The temperature and electrostatic potential are determined by demanding regularity of the black hole solution, whereas the value of the magnetic potential is determined by the variation of the additional polar angle boundary term. Consequently, the first law of black hole thermodynamics is identically satisfied by construction.

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Contents

1 Introduction

It has been recently conjectured that additional terms that are interpreted as scalar charges for a new class of dyonic black holes introduced in [1, 2], are needed in the first law of black hole thermodynamics. This black hole is asymptotically AdS, provided the system is endowed with a self-interacting potential. The theory is described by the action of the bosonic sector coming from a consistent truncation of a S^7 reduction of eleven-dimensional supergravity [3]. In the thermodynamic analysis of [1], the authors claimed that the first law is not satisfied unless one adds a term which is interpreted as a scalar charge [1, 2]. However, this argument conflicts with the fact that there is no gauge symmetry related to the presence of a single scalar field in this Lagrangian, i.e., there is no Noether charge (nor a topological charge as is the case for the magnetic charge). Moreover, it has been clearly identified that the scalar field generically contributes to the mass with non-integrable terms. This has been shown with general asymptotic conditions through Hamiltonian [4–6] and other methods [7, 8], and even with an explicit black hole example in the presence of gauge fields in three dimensions [9]. In this context [10] presented a new class of dyonic AdS black hole solutions of four-dimensional $\mathcal{N} = 8$ SO(8) gauged supergravity where the AdS_4 dyonic dilatonic black hole of [1] is included. It was found, in fact, that the missing term, supposedly related to a scalar charge, contributes to the variation of the mass. It was also pointed out in [10] that the non-integrability of this term leads to an ill-defined mass with independent electric and magnetic charges. However, as it will be shown in this manuscript, it is possible to impose general integrability conditions. One can impose some physical condition through suitable boundary conditions, which determines the arbitrary functions coming from the integrability conditions. One possible condition is to demand the preservation of the AdS symmetry of the scalar field fall-off [4–6, 11]. Then the scalar field contribution is cancelled by a term coming from the gravitational contribution to the mass.

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Another motivation for studying the black hole solution presented in [1] has to do with the computation of its Gibbs free energy. To do so, one has to have a well-defined action and also to present other thermodynamic features, e.g. to match all the charges of the solution with their respective chemical potentials. The Gibbs free energy presented in [1] fails to do that since it cannot recover the magnetic contribution to the Euclidean action, not even if one includes the additional scalar charge. The value of the action in [1] was obtained through the holographic renormalization method described in [12], by adding counterterms, which only include radial surface terms, to get a finite action principle. As we will show below, it is necessary to add an additional polar angle boundary term to obtain the magnetic contribution to the Euclidean action. To prove the latter, we formulate a well-defined and finite Hamiltonian action principle for the system and we prove that this additional boundary term comes from a total derivative in the polar angle which appears due to existence of a magnetic monopole.

The aim of this paper is to compute the conserved charges of the AdS_4 dyonic black hole and to formulate a well-defined and finite Hamiltonian action principle which enables one to obtain the value of the Hamiltonian Euclidean action. By imposing the grand canonical ensemble, the Gibbs free energy is chosen as our thermodynamic potential which, unlike the free energy computed in [1], exhibits all the conserved charges of the solution, i.e., the mass (with its respective scalar field contribution), the electric charge and the magnetic charge. The plan of this manuscript is the following, in the section 2 we present the Lagrangian and the AdS_4 dyonic dilatonic black hole solution of [1]. Section 3 focuses on the Hamiltonian analysis and the corresponding conserved charges. The mass and the electric charge are computed using the Regge-Teitelboim Hamiltonian approach. There are two contributions in the variation of the mass, the gravitational part and the scalar field part (already identified in [10]). Integrability conditions have to be imposed because the presence of the scalar field leads to a non-integrable term. Suitable boundary conditions are chosen in order to preserve the AdS symmetry of the scalar field fall-off. This implies a precise relation among the coefficients of the leading and subleading terms of the scalar field, as was noted in [6]. In section 4 we perform the thermodynamic analysis of the solution and introduce the Hamiltonian Euclidean action. For simplicity the calculations are done in a suitable Euclidean minisuperspace. To obtain the Gibbs free energy we compute the value of the Euclidean Hamiltonian action endowed with a suitable radial boundary term and an additional polar angle boundary term. These terms have to be added in order to have a well-defined and finite Hamiltonian action principle. It is possible to identify the variation of the Hamiltonian conserved charges of the system from the variation of the boundary term at infinity, which are the mass and the electric charge. On the other hand, the variation of the magnetic charge comes from the additional polar boundary term. This boundary term has to be considered due to the presence of a magnetic monopole. The chemical potentials associated to the Noether charges are the Lagrange multipliers of the system at infinity. Unlike the magnetic potential, they are obtained through regularity conditions at the horizon. Remarkably, the magnetic potential is already determined by the variation of the boundary term, together with the magnetic charge. It is worth noting that the first law of black hole thermodynamics is satisfied independently of the integrability conditions on the mass, since the relation only involves the variation of the conserved charges. Once the Gibbs free energy is obtained the value of the mass, the electric charge, the magnetic charge and the entropy are verified using the known thermodynamic relations. Finally, section 5 is devoted to some concluding remarks.

2 AdS$_4$ dyonic black hole solution

We consider four-dimensional gravity with negative cosmological constant in the presence of an Abelian gauge field and a dilatonic scalar field with a self-interacting potential. The action reads

$$I[g_{\mu\nu}, A_\mu, \phi] = \int d^4x\sqrt{-g}\left(\frac{R}{2\kappa} - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}e^{-\sqrt{3}\phi}F^{\mu\nu}F_{\mu\nu} - V\left(\phi\right)\right). \tag{2.1}$$

Hereafter the gravitational constant is chosen as $\kappa = 1/2$.[1] The self-interacting potential of the scalar field is given by

$$V\left(\phi\right) = -6g^2\cosh\left(\frac{\phi}{\sqrt{3}}\right), \tag{2.2}$$

where the coupling constant g determines the AdS radius as $\ell^2 = g^{-2}$. The theory given by (2.1) corresponds to the bosonic sector of two possible dimensional reductions, which depend on the coupling constant g in the following way. In the case of vanishing g the action is obtained after a S^1 reduction of five-dimensional pure gravity. On the other hand, if $g \neq 0$ the action can be obtained after a S^7 reduction of eleven-dimensional supergravity [3].

The gravitational field equations for the action (2.1) are

$$G_{\mu\nu} = T^\phi_{\mu\nu} + T^A_{\mu\nu}, \tag{2.3}$$

where the contributions to the energy-momentum tensor of the dilatonic scalar field and the gauge field are given by

$$T^\phi_{\mu\nu} = \frac{1}{2}\partial_\mu\phi\partial_\nu\phi - \frac{1}{4}g_{\mu\nu}\partial^\lambda\phi\partial_\lambda\phi + \frac{1}{2}g_{\mu\nu}V(\phi), \tag{2.4}$$

$$T^A_{\mu\nu} = \frac{1}{2}e^{-\sqrt{3}\phi}\left(F_\mu{}^\lambda F_{\nu\lambda} - \frac{1}{4}g_{\mu\nu}F^{\lambda\rho}F_{\lambda\rho}\right), \tag{2.5}$$

respectively. The equation for the scalar field is

$$\Box\phi + \frac{\sqrt{3}}{4}e^{-\sqrt{3}\phi}F^{\mu\nu}F_{\mu\nu} - \frac{dV}{d\phi} = 0, \tag{2.6}$$

and the equation for the gauge field reads

$$\nabla_\mu\left(e^{-\sqrt{3}\phi}F^{\mu\nu}\right) = 0. \tag{2.7}$$

[1]The vacuum permeability constant located in front of the Maxwell-like action in (2.1) turns out to be normalized to one after the dimensional reduction.

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This system admits an AdS dyonic black hole which is static and spherically symmetric [1]. The line element of this configuration can be written as

$$ds^2 = -\left(H_1H_2\right)^{-1/2} f dt^2 + \frac{dr^2}{\left(H_1H_2\right)^{-1/2} f} + \left(H_1H_2\right)^{1/2} r^2 \left(d\theta^2 + \sin^2\left(\theta\right) d\varphi^2\right), \tag{2.8}$$

where the functions H_1, H_2 and f are given by

$$f\left(r\right) = f_0\left(r\right) + g^2r^2H_1\left(r\right)H_2\left(r\right), \qquad f_0\left(r\right) = 1 - \frac{2\mu}{r}, \tag{2.9}$$

$$H_1\left(r\right) = \gamma_1^{-1}\left(1 - 2\beta_1 f_0\left(r\right) + \beta_1\beta_2 f_0\left(r\right)^2\right), \quad H_2\left(r\right) = \gamma_2^{-1}\left(1 - 2\beta_2 f_0\left(r\right) + \beta_1\beta_2 f_0\left(r\right)^2\right), \tag{2.10}$$

with $\gamma_1 = 1 - 2\beta_1 + \beta_1\beta_2$, and $\gamma_2 = 1 - 2\beta_2 + \beta_1\beta_2$. The dilatonic scalar field is given by

$$\phi\left(r\right) = \frac{\sqrt{3}}{2}\log\left(\frac{H_2\left(r\right)}{H_1\left(r\right)}\right), \tag{2.11}$$

whereas the one-form gauge field has the following form

$$A = A_t(r)dt + A_\varphi(\theta)d\varphi. \tag{2.12}$$

The time component of (2.12) is

$$A_t\left(r\right) = \frac{\sqrt{2}\left(1 - H_1\left(r\right) - \beta_1\left(f_0 - H_1\left(r\right)\right)\right)}{\sqrt{\beta_1\gamma_2}H_1\left(r\right)}, \tag{2.13}$$

while the definition of the angular component of the gauge potential depends on the hemisphere, in order to avoid the Dirac string [13]. Hence,

$$A_\varphi\left(\theta\right) = \begin{cases} p(1+\cos\left(\theta\right)) & , \quad 0 \le \theta < \frac{\pi}{2} - \delta, \\ p\left(-1+\cos\left(\theta\right)\right) & , \quad \frac{\pi}{2} + \delta < \theta \le \pi, \end{cases} \tag{2.14}$$

where $p = 2\sqrt{2}\mu\gamma_2^{-1}\sqrt{\beta_2\gamma_1}$ and $\delta \to 0$ (Wu-Yang monopole [14, 15]). In this solution the coordinate ranges are $0 < r < \infty$, $-\infty < t < \infty$, $0 \le \theta < \pi$ and $0 \le \varphi < 2\pi$. All the integration constants (μ, β_1, β_2 , γ_1, γ_2) are restricted to be positive.

In the case of $\beta_1 = \beta_2$, the dilatonic scalar field is decoupled and the solution turns out to be an AdS dyonic Reissner-Nordström black hole where the electric and magnetic charges have the same value. If $\beta_1 = 0$ the solution is purely magnetic and in the case of $\beta_2 = 0$ the configuration becomes purely electric. If $\mu = 0$ the solution turns out to be AdS spacetime.

3 Hamiltonian generator and surface integrals

The Hamiltonian generator for the Lagrangian (2.1) reads

$$H\left[\xi, \xi^A\right] = \int d^3x \left(\xi^\perp \mathcal{H}_\perp + \xi^i \mathcal{H}_i - \xi^A \mathcal{G}\right) + Q\left[\xi, \xi^A\right], \tag{3.1}$$

where the boundary term $Q\left[\xi,\xi^A\right]$, which corresponds to the conserved charges in the Regge-Teitelboim approach, ensures that the Hamiltonian generator has well-defined functional derivatives [16]. The bulk term appearing in (3.1) is a linear combination of the constraints $\mathcal{H}_\perp$, $\mathcal{H}_i$ and $\mathcal{G}$, where the first two are the energy and momentum densities and the last one corresponds to the Gauss constraint associated to the Abelian gauge field. The asymptotic surface deformations of the spacetime are given by the vector $\xi=\left(\xi^\perp,\xi^i\right)$ and ξ^A is the gauge parameter of the Abelian symmetry. The constraints are explicitly given by

$$\begin{aligned}\mathcal{H}_\perp &= \frac{1}{\sqrt{\gamma}}\left(\pi^{ij}\pi_{ij}-\frac{1}{2}\left(\pi^i{}_i\right)^2\right)-\sqrt{\gamma}R\\ &\quad+\frac{\pi_\phi^2}{2\sqrt{\gamma}}+\sqrt{\gamma}\left(\frac{1}{2}\partial^i\phi\partial_i\phi+V(\phi)\right)+e^{\sqrt{3}\phi}\frac{\pi^i\pi_i}{2\sqrt{\gamma}}+\frac{1}{4}\sqrt{\gamma}e^{-\sqrt{3}\phi}F^{ij}F_{ij}, \qquad (3.2)\\ \mathcal{H}_i &= 2\nabla_j\pi^j{}_i+\pi_\phi\partial_i\phi+\pi^jF_{ij}, \qquad (3.3)\\ \mathcal{G} &= \partial_i\pi^i. \qquad (3.4)\end{aligned}$$

The dynamical variables of the system are the spatial components of the fields $\{\gamma_{ij},\, A_i,\, \phi\}$, where γ_{ij} is the spatial metric of the ADM decomposition. Here R stands for the scalar curvature of the three-dimensional spatial metric γ_{ij} and the self-interacting potential of the scalar field $V(\phi)$ is defined in eq. (2.2). The momentum conjugated to the three-dimensional metric γ_{ij} is

$$\pi^{ij}=-\sqrt{\gamma}\left(K^{ij}-\gamma^{ij}K\right), \qquad (3.5)$$

where the extrinsic curvature is given by

$$K_{ij}=\frac{1}{2N^\perp}\left(\nabla_iN_j+\nabla_jN_i-\dot{\gamma}_{ij}\right). \qquad (3.6)$$

The momentum for the dilatonic field ϕ reads

$$\pi_\phi=\frac{\sqrt{\gamma}}{N^\perp}\left(\dot{\phi}-N^i\partial_i\phi\right), \qquad (3.7)$$

and for the gauge field A_i,

$$\pi^i=-\frac{\sqrt{\gamma}e^{-\sqrt{3}\phi}}{N^\perp}\left(-\gamma^{ij}F_{0j}+N^j\gamma^{ik}F_{jk}\right). \qquad (3.8)$$

The variation of the surface term gets different contributions according to the field content of the theory, such that

$$\delta Q\left[\xi,\xi^A\right]=\delta Q^G\left[\xi,\xi^A\right]+\delta Q^\phi\left[\xi,\xi^A\right]+\delta Q^A\left[\xi,\xi^A\right], \qquad (3.9)$$

where δQ was obtained after demanding that $\delta H = 0$ on the constraint surface. The explicit expressions for the surface integrals are given by

$$\delta Q^G = \int dS_l G^{ijkl} \left(\xi^\perp \nabla_k \delta\gamma_{ij} - \partial_k \xi^\perp \delta\gamma_{ij}\right) + \int dS_l \left[2\xi_k \delta\pi^{kl} + \left(2\xi^k \pi^{jl} - \xi^l \pi^{kj}\right) \delta\gamma_{jk}\right], \tag{3.10}$$

$$\delta Q^\phi = -\int dS_i \left(\xi^\perp \sqrt{\gamma} \partial^i \phi \delta\phi + \xi^i \pi_\phi \delta\phi\right), \tag{3.11}$$

$$\delta Q^A = -\int dS_i \left[\xi^\perp \sqrt{\gamma} e^{-\sqrt{3}\phi} F^{ij} \delta A_j + \left(\xi^i \pi^j - \pi^j \xi^i\right) \delta A_j - \xi^A \delta\pi^i\right], \tag{3.12}$$

with

$$G^{ijkl} = \frac{1}{2}\sqrt{\gamma}\left(\gamma^{ik}\gamma^{jl} + \gamma^{il}\gamma^{jk} - 2\gamma^{ij}\gamma^{kl}\right). \tag{3.13}$$

Note that δQ in (3.9) stands for the surface term after taking the functional derivatives with respect to the canonical variables in the phase space of the Hamiltonian generator H. This surface term determines the conserved charges in the Regge-Teitelboim approach [16]. A priori, for a generic configuration δQ is a non-integrable quantity and one must then also provide the asymptotic behaviour of the fields representing the space of solutions at infinity. In some cases the latter is not enough for integrating δQ and some additional integrability conditions must be imposed on the phase space (for all practical purposes, on the integration constants of the solution). Indeed, this is the case for the mass of the AdS_4 dyonic black hole analyzed in this manuscript. This will be shown in detail in the next subsection.

3.1 Conserved charges of the AdS_4 dyonic black hole

In order to obtain the above surface integrals let us consider a static and spherically symmetric minisuperspace in which the AdS_4 dyonic black hole (2.8) is included. For simplicity we perform the following change of variable in the radial coordinate

$$\rho^2 = \sqrt{H_1(r) H_2(r)} r^2. \tag{3.14}$$

The line element then reads

$$ds^2 = -N^\perp(\rho)^2 dt^2 + \frac{d\rho^2}{F(\rho)} + \rho^2\left(d\theta^2 + \sin^2(\theta) d\varphi^2\right). \tag{3.15}$$

The gauge field ansatz is given by

$$A = A_t(\rho) dt + A_\varphi(\theta) d\varphi, \tag{3.16}$$

and the scalar field also depends on the radial coordinate $\phi = \phi(\rho)$. Taking this into consideration the only nonvanishing momentum in the minisuperspace is the radial component of the electromagnetic one, where $\pi^\rho = p^\rho(\rho, \theta)$. Therefore, the value of the Hamiltonian charges, computed on the sphere S^2 of infinite radius, is given by

$$\delta Q = \Bigg[-\xi^t \Bigg(\frac{8\pi\rho N^\perp \delta F}{\sqrt{F}} + 4\pi\sqrt{F} N^\perp \rho^2 \partial_\rho \phi \delta\phi + \pi \left[\left(\int \frac{N^\perp e^{-\sqrt{3}\phi}}{\sqrt{F}\rho^2} d\rho\right) \csc(\theta) \delta A_\varphi \partial_\theta A_\varphi\right]_{\theta=0}^{\theta=\pi} \Bigg) + 2\pi\xi^A \int_0^\pi \delta p^\rho d\theta \Bigg]_{\rho\to\infty}. \tag{3.17}$$

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Here, we have applied the definition of the deformation vectors $\xi^{\perp}$ and ξ^{i} in terms of the Killing vectors ξ^{t} and $\bar{\xi}^{i}$, which read

$$\xi^{\perp} = N^{\perp}\xi^{t}, \tag{3.18}$$
$$\xi^{i} = \bar{\xi}^{i} + N^{i}\xi^{t}. \tag{3.19}$$

In order to compute and perform a proper analysis of the charges, we must give suitable asymptotic conditions that determine the behavior of the fields at infinity. These conditions are specified up to the orders that contribute to the charges, such that

$$F(\rho) = g^2\rho^2 + 1 + F_0 + \frac{F_1}{\rho} + \mathcal{O}\left(\frac{1}{\rho^2}\right), \tag{3.20}$$
$$N^{\perp}(\rho) = g\rho + \mathcal{O}\left(\frac{1}{\rho}\right), \tag{3.21}$$
$$\phi(\rho) = \frac{\phi_1}{\rho} + \frac{\phi_2}{\rho^2} + \mathcal{O}\left(\frac{1}{\rho^3}\right), \tag{3.22}$$
$$p^{\rho}(\rho,\theta) = p_0\sin(\theta) + \mathcal{O}\left(\frac{1}{\rho^1}\right), \tag{3.23}$$
$$\xi^{A} = \xi_0^{A} + \mathcal{O}\left(\frac{1}{\rho}\right). \tag{3.24}$$

The coefficients in the expansions given above are parameters that depend on the integration constants of the corresponding solution. The variation of the charge obtained after inserting the proposed asymptotic behavior in (3.17) is given by[2]

$$\delta Q = \xi^{t}\left[-8\pi\delta F_1 + 4\pi g^2\left(2\phi_2\delta\phi_1 + \phi_1\delta\phi_2\right)\right] + 4\pi\xi_0^{A}\delta p_0. \tag{3.25}$$

The mass is the conserved charge associated to time translations, which in this approach is obtained from $\delta M = \delta Q\left[\xi^{t}\right]$, while the electric charge is the charge associated to the Abelian gauge transformations, where $\delta Q_e = \delta Q\left[\xi^{A}\right]$. Then, the variations of the mass and the electric charge read

$$\delta M = -8\pi\delta F_1 + 4\pi g^2\left(2\phi_2\delta\phi_1 + \phi_1\delta\phi_2\right), \tag{3.26}$$
$$\delta Q_e = 4\pi\delta p_0. \tag{3.27}$$

The electric charge can be directly integrated for the AdS_4 dyonic black hole, which in terms of the integration constants of the solution is written as

$$Q_e = \frac{16\pi\sqrt{2}\mu\sqrt{\beta_1\gamma_2}}{\gamma_1}. \tag{3.28}$$

In contrast, the mass is generically non-integrable and its variation is explicitly given by

$$\delta M = \delta\left(\frac{16\pi\left(1+\beta_1\right)\left(1-\beta_2\right)\left(1-\beta_1\beta_2\right)\mu}{\gamma_1\gamma_2} + \frac{64\pi g^2\mu^3\left(1-\beta_1\beta_2\right)\left(\beta_1-\beta_2\right)^2\gamma}{\gamma_1^3\gamma_2^3}\right) + \Phi, \tag{3.29}$$

[2]It has to be noted that a divergent term appears in the variation of the charge but it vanishes once it is evaluated on the solution. This is because the divergent part of the gravitational contribution is cancelled by the divergent part of the scalar field contribution by virtue of the relation $\delta F_0 = \frac{g^2}{2}\phi_1\delta\phi_1$.

with

$$\gamma=\beta_1+\beta_2-8\beta_1\beta_2+6\beta_1^2\beta_2+6\beta_1\beta_2^2-8\beta_1^2\beta_2^2+\beta_1^3\beta_2^2+\beta_1^2\beta_2^2\,. \tag{3.30}$$

Note that the variation of the mass coincides with the one computed in [10], which has the non-integrable term Φ that comes from the scalar field part of the energy density. This term is given by

$$\Phi=4\pi g^2\left(2\phi_2\delta\phi_1+\phi_1\delta\phi_2\right), \tag{3.31}$$

where the leading and subleading terms of the scalar field fall-off are respectively

$$\phi_1 = \frac{2\sqrt{3}\left(\beta_2\left(1+{\beta_1}^2\right)-\beta_1\left(1+{\beta_2}^2\right)\right)\mu}{\gamma_1\gamma_2}, \tag{3.32}$$

$$\phi_2 = \frac{2\sqrt{3}\left(-\beta_2^2\left(1-\beta_1^4\right)-2\beta_1\beta_2^2(-4+3\beta_2)-2\beta_1^3\beta_2(-3+4\beta_2)-{\beta_1}^2\left(-1+8\beta_2-8\beta_2^3+\beta_2^4\right)\right)\mu^2}{\gamma_1^2\gamma_2^2}. \tag{3.33}$$

The presence of a non-integrable term Φ in the variation of the mass (3.26) forces us to impose relations among the fall-off coefficients of the scalar field. If the variations are treated as exterior derivatives, the condition $\delta^2 M=0$ is a sufficient condition to ensure the existence of M. Indeed, this condition is equivalent to requiring that the second derivatives of the functional M with respect to the integration constants commute. Then,

$$\begin{aligned}\delta^2 M &= \delta\Phi && (3.34)\\ &= 4\pi g^2\left(2\delta\phi_2\wedge\delta\phi_1+\delta\phi_1\wedge\delta\phi_2\right) && (3.35)\\ &= 4\pi g^2\delta\phi_2\wedge\delta\phi_1\equiv 0\,. && (3.36)\end{aligned}$$

This implies the functional relation $\phi_2=\phi_2(\phi_1)$. Hence, the mass generically takes the form

$$M=-8\pi F_1+4\pi g^2\int\left(2\phi_2+\phi_1\frac{d\phi_2}{d\phi_1}\right)d\phi_1. \tag{3.37}$$

At this point it is necessary to impose a boundary condition that fixes a precise relation between the leading and subleading terms of the scalar field behavior at infinity. One possible condition is to demand preservation of the AdS symmetry of the scalar field's asymptotic fall-off, which can be done since the AdS_4 dyonic dilatonic black hole of [1] is within the asymptotic conditions for AdS spacetimes analyzed in [5, 6, 11]. These references construct a set of boundary conditions for having well-defined and finite Hamiltonian generators for all the elements of the AdS algebra in the case of gravity minimally coupled to scalar fields. We are allowed to impose certain relations on the leading and subleading terms of the scalar field fall-off provided the scalar field does not break the AdS symmetry at infinity. These boundary conditions are $(\phi_1=0,\quad \phi_2\neq 0)$, $(\phi_1\neq 0,\quad \phi_2=0)$ and $\phi_2=c\phi_1^2$, where c is not allowed to vary. In terms of the integration constants the relation $\phi_2=c\phi_1^2$ becomes

$$\begin{aligned}&-2(\beta_1-\beta_2)\mu^2\Big[-\left(\sqrt{3}+6c\right)\beta_2-\left(\sqrt{3}-6c\right){\beta_1}^3{\beta_2}^2\\ &\qquad-\beta_1\left(\sqrt{3}-6c-8\sqrt{3}\beta_2+6\left(\sqrt{3}-2c\right){\beta_2}^2\right)\\ &\qquad-{\beta_1}^2\beta_2\left(6\left(\sqrt{3}+2c\right)-8\sqrt{3}\beta_2+\left(\sqrt{3}+6c\right){\beta_2}^2\right)\Big]=0.\end{aligned} \tag{3.38}$$

From eq. (3.38) we observe three cases, two of them being nontrivial. When $\mu = 0$ the mass, the electric charge and the magnetic charge vanish giving rise to the vacuum solution which turns out to be AdS_4 spacetime. The other two cases imply that $\beta_1 = \beta_1(\beta_2)$ in such a way, that they force the terms in (3.29) that are proportional to g^2 to vanish. Hence, the mass becomes the AMD mass [17, 18] obtained in [1],

$$M = \frac{16\pi\,(1-\beta_1)\,(1-\beta_2)\,(1-\beta_1\beta_2)\,\mu}{\gamma_1\gamma_2}. \tag{3.39}$$

This fact is in agreement with [19], where it was pointed out that some holographic prescriptions are suitable for computing the mass for hairy spacetimes when the scalar field respects the AdS invariance at infinity. In this context, different kinds of boundary conditions were considered in [20–22].

4 Thermodynamics of the AdS_4 dyonic black hole

The thermodynamic analysis of the AdS_4 dyonic dilatonic black hole is performed in this section. We define the Euclidean Hamiltonian action of the theory including a surface term and an additional polar boundary term to have a finite action principle. The presence of the latter is due to the existence of a magnetic monopole in the solution. For simplicity, we take a minisuperspace in which the AdS_4 dyonic black hole is included. The variation of the Euclidean Hamiltonian action is computed in the grand canonical ensemble, where the chemical potentials are fixed. Remarkably, the magnetic charge emerges from the additional polar boundary term accompanied by its respective chemical potential. The value of the temperature and the electric potential, on the other hand, are fixed by imposing regularity conditions. When the variations of the additional surface and polar boundary terms are determined, as was mentioned above, integrability conditions are needed to be imposed to determine the value of the Euclidean Hamiltonian action leading to the Gibbs free energy.

4.1 Hamiltonian action and Euclidean minisuperspace

Let us consider spacetimes with a manifold of topology $\mathbb{R}^2 \times S^2$. The plane $\mathbb{R}^2$ is centered at the event horizon r_+ and is parametrized by the periodic Euclidean time τ and the radial coordinate r. These plane coordinates range as

$$0 \le \tau < \beta, \tag{4.1}$$

$$r_+ \le r < \infty, \tag{4.2}$$

with β the inverse of the Hawking temperature and the 2-sphere S^2 stands for the topology of the base manifold. The Hamiltonian Euclidean action for the system is given by

$$I^E = \int_0^\beta d\tau \int_\Sigma d^3x \left[\dot{\gamma}_{ij}\pi^{ij} + \dot{A}_i\pi^i + \dot{\phi}\pi_\phi - \left(N^\perp\mathcal{H}_\perp + N^i\mathcal{H}_i - A_\tau\mathcal{G}\right)\right] + B, \tag{4.3}$$

where $\Sigma = \mathbb{R} \times S^2$ is the spatial section of the manifold. Note that the additional term B in (4.3) needs to be added to the action in order to have a well-defined variational principle, and it is crucial for determining the value of the action for stationary configurations.

The Euclidean continuation of the AdS_4 dyonic black hole (2.8) is considered. The line element reads

$$ds^2 = N^{\perp}(r)^2 d\tau^2 + \frac{dr^2}{F(r)} + H(r)\left(d\theta^2 + \sin^2(\theta)\, d\varphi^2\right), \tag{4.4}$$

where the gauge field ansatz and the scalar field are given by

$$A = A_\tau(r)\, d\tau + A_\varphi(\theta)\, d\varphi, \tag{4.5}$$

$$\phi = \phi(r). \tag{4.6}$$

The radial component of the electromagnetic field momentum is $\pi^r = p^r(r,\theta)$ (all the other momenta of the fields vanish). Hence, it is possible to obtain the following reduced action

$$I^E = -2\pi\beta \int_{r_+}^{\infty} dr \int_0^{\pi} d\theta \left(N^{\perp}(r)\,\mathcal{H}_\perp - A_\tau(r)\,\mathcal{G}\right) + B, \tag{4.7}$$

from (4.3), where the reduced constraints take the form

$$\begin{aligned}\mathcal{H}_\perp =& -\frac{e^{-\sqrt{3}\phi}\sin(\theta)}{2\sqrt{F}H}\Big[-\csc^2(\theta)\left(\partial_\theta A_\varphi\right)^2 - 2e^{\sqrt{3}\phi}H\left(\partial_r F\partial_r H + 2F\partial_r^2 H - 2\right) \\ &+ e^{\sqrt{3}\phi}H^2\left(12g^2\cosh\left(\frac{\phi}{\sqrt{3}}\right) - F\left(\partial_r\phi\right)^2\right) + e^{\sqrt{3}\phi}F\left(\partial_r H\right)^2 + \csc^2(\theta)e^{2\sqrt{3}\phi}\left(p^r\right)^2\Big],\end{aligned} \tag{4.8}$$

$$\mathcal{G} = \partial_r p^r. \tag{4.9}$$

The variation of the reduced action (4.7) with respect to the Lagrange multipliers $N^\perp$ and A_τ indicates that the constraints have to vanish

$$\mathcal{H}_\perp = 0, \qquad \mathcal{G} = 0. \tag{4.10}$$

These equations define the constraint surface. On the other hand, the variation of (4.7) with respect to the independent functions of the dynamical fields in the minisuperspace leads to the field equations. The field equations related to $F(r)$ and $H(r)$ are given by

$$\begin{aligned}&\frac{e^{-\sqrt{3}\phi}\sin(\theta)}{4F^{3/2}H}\Big(N^\perp\left(-\csc^2(\theta)\left(\partial_\theta A_\varphi\right)^2 - F\left(\partial_r H\right)^2 e^{\sqrt{3}\phi} + \csc^2(\theta)e^{2\sqrt{3}\phi}\left(p^r\right)^2\right) \\ &\quad + H^2 N^\perp e^{\sqrt{3}\phi}\left(F\left(\partial_r\phi\right)^2 + 12g^2\cosh\left(\frac{\phi}{\sqrt{3}}\right)\right) + 4He^{\sqrt{3}\phi}\left(N^\perp - F\partial_r H\partial_r N^\perp\right)\Big) = 0,\end{aligned} \tag{4.11}$$

$$\begin{aligned}&\frac{e^{-\sqrt{3}\phi}\sin(\theta)}{2\sqrt{F}H^2}\Big(N^\perp\left(-\csc^2(\theta)\left(\partial_\theta A_\varphi\right)^2 - F\left(\partial_r H\right)^2 e^{\sqrt{3}\phi} + \csc^2(\theta)e^{2\sqrt{3}\phi}\left(p^r\right)^2\right) \\ &\quad - H^2 e^{\sqrt{3}\phi}\left(N^\perp\left(12g^2\cosh\left(\frac{\phi}{\sqrt{3}}\right) - F\left(\partial_r\phi\right)^2\right) - 2\left(\partial_r F\partial_r N^\perp + 2F\partial_r^2 N^\perp\right)\right) \\ &\quad + He^{\sqrt{3}\phi}\left(N^\perp\left(\partial_r F\partial_r H + 2F\partial_r^2 H\right) + 2F\partial_r H\partial_r N^\perp\right)\Big) = 0,\end{aligned} \tag{4.12}$$

respectively. The field equations associated to $A_\varphi(r,\theta)$ and $p^r(r,\theta)$ are

$$\frac{N^\perp e^{-\sqrt{3}\phi}\csc(\theta)}{\sqrt{F(r,s)}H(r,s)}\left(\partial_\theta A_\varphi\cot(\theta) - \partial_\theta^2 A_\varphi\right) = 0, \qquad \partial_r A_\tau + \frac{\csc(\theta)N^\perp e^{\sqrt{3}\phi}p^r}{\sqrt{F}H} = 0, \tag{4.13}$$

and finally the scalar field equation reads

$$
\begin{aligned}
-\frac{e^{-\sqrt{3}\phi}\sin(\theta)}{2\sqrt{F}H}\Bigg(N^{\perp}\Bigg(\sqrt{3}\csc^{2}(\theta)\left(\partial_{\theta}A_{\varphi}\right)^{2}+2FHe^{\sqrt{3}\phi}\partial_{r}H\partial_{r}\phi \\
+H^{2}e^{\sqrt{3}\phi}\left(\partial_{r}F\partial_{r}\phi+2F\partial_{r}^{2}\phi+4\sqrt{3}g^{2}\sinh\left(\frac{\phi}{\sqrt{3}}\right)\right) \\
+\sqrt{3}\csc^{2}(\theta)e^{2\sqrt{3}\phi}\left(p^{r}\right)^{2}\Bigg)+2FH^{2}e^{\sqrt{3}\phi}\partial_{r}\phi\partial_{r}N^{\perp}\Bigg)=0.
\end{aligned} \tag{4.14}
$$

Then, the variation of the reduced action (4.7) on the constraint surface, evaluated on-shell (i.e. eqs. (4.11) to (4.14) have to be satisfied), becomes

$$
\begin{aligned}
\delta I^{E}\Big|_{\mathrm{on-shell}} = -2\pi\beta\int_{0}^{\pi}d\theta\Bigg[N^{\perp}\sin(\theta)\Bigg(\frac{\partial_{r}H\delta F+\partial_{r}F\delta H}{\sqrt{F}}-\frac{\sqrt{F}\partial_{r}H\delta H}{H} \\
+\frac{\sqrt{F}H\partial_{r}\phi\delta\phi}{2}+2\sqrt{F}\partial_{r}\delta H\Bigg)-\partial_{r}\left(2N^{\perp}\sin(\theta)\sqrt{F}\right)\delta H-A_{\tau}\delta p^{r}\Bigg]_{r_{+}}^{\infty} \\
-2\pi\beta\int_{r+}^{\infty}dr\left[\frac{N^{\perp}e^{-\sqrt{3}\phi}}{H\sqrt{F}\sin\theta}\partial_{\theta}A_{\varphi}\delta A_{\varphi}\right]_{0}^{\pi}+\delta B.
\end{aligned} \tag{4.15}
$$

If we demand that the action has an extremum, i.e., $\delta I^{E}\Big|_{on-shell}=0$, the variation of the additional term δB must necessarily be given by

$$
\begin{aligned}
\delta B = 2\pi\beta\int_{0}^{\pi}d\theta\Bigg[N^{\perp}\sin(\theta)\Bigg(\frac{\partial_{r}H\delta F+\partial_{r}F\delta H}{\sqrt{F}}-\frac{\sqrt{F}\partial_{r}H\delta H}{H} \\
+\frac{\sqrt{F}H\partial_{r}\phi\delta\phi}{2}+2\sqrt{F}\partial_{r}\delta H\Bigg)-\partial_{r}\left(2N^{\perp}\sin(\theta)\sqrt{F}\right)\delta H-A_{\tau}\delta p^{r}\Bigg]_{r_{+}}^{\infty} \\
+2\pi\beta\int_{r+}^{\infty}dr\left[\frac{N^{\perp}e^{-\sqrt{3}\phi}}{H\sqrt{F}\sin(\theta)}\partial_{\theta}A_{\varphi}\delta A_{\varphi}\right]_{0}^{\pi}.
\end{aligned} \tag{4.16}
$$

It is possible to recognize two kinds of terms in this expression. The surface term comes from a total derivative in the radial coordinate and a boundary term that comes from a total derivative in the polar angle. The latter is clearly not vanishing because of the presence of an angular component depending on the polar angle in the gauge field. The analysis of the variation of the term B and the evaluation on the AdS$_4$ dyonic black hole (4.16) will be performed in the following subsection.

4.2 Gibbs free energy and first law

From (4.16) we can identify different contributions, depending on whether the term comes from a total derivative in the radial coordinate, or whether the term comes from a total derivative in the polar angle, which will be identified as a polar boundary term. The surface term evaluated at infinity will be denoted by $\delta B(\infty)$ while $\delta B(r_+)$ will stand for the surface term at the horizon. The polar boundary term will be denoted by δB_θ. Hence, the variation of B, see (4.16), can be written as

$$\delta B = \delta B(\infty) + \delta B(r_+) + \delta B_\theta, \tag{4.17}$$

where the surface term at infinity is given by

$$\begin{aligned}\delta B(\infty) = 2\pi\beta \int_0^\pi d\theta \Bigg[N^\perp \sin(\theta) \Bigg(\frac{\partial_r H \delta F + \partial_r F \delta H}{\sqrt{F}} - \frac{\sqrt{F}\partial_r H \delta H}{H} \\ + \frac{\sqrt{F} H \partial_r \phi \delta\phi}{2} + 2\sqrt{F}\partial_r \delta H \Bigg) - \partial_r \left(2N^\perp \sin(\theta)\sqrt{F} \right) \delta H - A_\tau \delta p^r \Bigg]_\infty,\end{aligned} \tag{4.18}$$

the surface term at the horizon is

$$\begin{aligned}\delta B(r_+) = -2\pi\beta \int_0^\pi d\theta \Bigg[N^\perp \sin(\theta) \Bigg(\frac{\partial_r H \delta F + \partial_r F \delta H}{\sqrt{F}} - \frac{\sqrt{F}\partial_r H \delta H}{H} \\ + \frac{\sqrt{F} H \partial_r \phi \delta\phi}{2} + 2\sqrt{F}\partial_r \delta H \Bigg) - \partial_r \left(2N^\perp \sin(\theta)\sqrt{F} \right) \delta H - A_\tau \delta p^r \Bigg]_{r_+},\end{aligned} \tag{4.19}$$

and the polar boundary term reads

$$\delta B_\theta = 2\pi\beta \int_{r+}^\infty dr \left[\frac{N^\perp e^{-\sqrt{3}\phi}}{H\sqrt{F}\sin(\theta)} \partial_\theta A_\varphi \delta A_\varphi \right]_0^\pi . \tag{4.20}$$

Once the different contributions to the variation of B are identified one can analyze their physical content. It is possible to find the variation of the charges coming from symmetries of the action together with their respective chemical potentials from the surface term at infinity $\delta B(\infty)$. The chemical potentials correspond to the Lagrange multipliers of the respective symmetry at infinity (as was shown in section 3). This is because at the end of the day the term (4.18) is obtained from the boundary term of the Hamiltonian, which ensures that the canonical generators have well-defined functional derivatives [16]. The variations of the mass and the electric charge of the AdS_4 dyonic dilatonic black hole will be identified from $\delta B(\infty)$. The entropy of the black hole, which corresponds to the Bekenstein-Hawking entropy, will be obtained from the surface term at the horizon $\delta B(r_+)$. Finally, the contribution of the topological charge of the system, leading to the variation of the magnetic charge multiplied by the magnetic potential, can be identified from the polar boundary term δB_θ .

Let us introduce the Euclidean continuation of the AdS_4 dyonic dilatonic black hole that satisfies the field equations (4.11)–(4.14) and the constraints (4.10). This is obtained after performing the identifications $t \to -i\tau$ and $\beta_1 \to -\beta_1$ in the Lorentzian solution. Then the black hole functions take the form

$$H_1(r) = \gamma_1^{-1}\left(1+2\beta_1 f_0(r) - \beta_1\beta_2 f_0(r)^2\right), \quad H_2(r) = \gamma_2^{-1}\left(1-2\beta_2 f_0(r) - \beta_1\beta_2 f_0(r)^2\right), \tag{4.21}$$

where $\gamma_1 = 1 + 2\beta_1 - \beta_1\beta_2$ and $\gamma_2 = 1 - 2\beta_2 - \beta_1\beta_2$. The functions $F(r)$ and $H(r)$ in the line element (4.4) are

$$F(r) = \frac{f(r)}{\sqrt{H_1(r) H_2(r)}}, \qquad H(r) = \sqrt{H_1(r) H_2(r)} r^2, \tag{4.22}$$

where the function $f(r)$ is the same as the one given in (2.9). The lapse function is $N^\perp(r) = \sqrt{F(r)}$. The scalar field is defined in (2.11) and the temporal component of the gauge field is given by

$$A_\tau(r) = -\frac{\sqrt{2}\left(1 - H_1(r) + \beta_1\left(f_0(r) - H_1(r)\right)\right)}{\sqrt{\beta_1\gamma_2} H_1(r)} + \Phi_e. \tag{4.23}$$

Note that the possibility of adding a constant Φ_e allows one to have a regular gauge field at the horizon. This constant is related to the electrostatic potential of the solution when the regularity conditions on the black hole horizon are established. The angular component of the gauge field takes the same definition as given in (2.14).

Inserting the Euclidean continuation of the AdS_4 dyonic dilatonic black hole in the surface term at infinity $\delta B(\infty)$, given in eq. (4.18), we get

$$\delta B(\infty) = -\beta\delta M - \beta\Phi_e\delta Q_e, \tag{4.24}$$

where the variations of the mass and the electric charge are given by

$$\delta M = \delta\left(\frac{16\pi(1+\beta_1)(1-\beta_2)(1+\beta_1\beta_2)\mu}{\gamma_1\gamma_2}\right) + \Theta, \tag{4.25}$$

$$\delta Q_e = 4\pi\delta\left(\frac{2\sqrt{2}\mu\sqrt{\beta_1\gamma_2}}{\gamma_1}\right). \tag{4.26}$$

The above variations coincide with the values computed in (3.29) and (3.28). In the variation of the mass we clearly obtain a contribution

$$\begin{aligned}\Theta &= \frac{64\pi g^2\mu^3(1+\beta_1\beta_2)(\beta_1+\beta_2)^2\gamma}{\gamma_1^3\gamma_2^3} + \Phi^E \\ &= -\frac{32\pi g^2\mu^3(\beta_1+\beta_2)}{\gamma_1^2\gamma_2^2}\left(\beta_2(1-2\beta_1-2\beta_2+\beta_1\beta_2)\delta\beta_1 - \beta_1(1+2\beta_1+2\beta_2+\beta_1\beta_2)\delta\beta_2\right),\end{aligned} \tag{4.27}$$

where Φ^E is the Euclidean continuation of Φ. Here Θ corresponds to the new scalar charge term in the context of [1].

The inverse of the temperature β and the electrostatic potential Φ_e are determined through the regularity conditions at the horizon. Indeed, we find

$$\beta = \frac{4\pi\sqrt{H_1(r_+) H_2(r_+)}}{f'(r_+)}, \qquad \Phi_e = -\sqrt{\frac{2}{\beta_1\gamma_2}}\left(1 + \beta_1 - \frac{1+\beta_1 f_0(r_+)}{H_1(r_+)}\right). \tag{4.28}$$

The value of the temperature is obtained by demanding absence of conical singularities around the event horizon, while the electrostatic potential comes from the trivial holonomy condition of the gauge field around a temporal cycle on the plane $r-\tau$ at the event horizon. Inserting the values of the chemical potentials (4.28) into the surface term at the horizon $\delta B\left(r_{+}\right)$, we get that this term exactly coincides with the Bekenstein-Hawking entropy

$$\delta B\left(r_{+}\right)=\delta\left(16\pi^{2}\sqrt{H_{1}\left(r_{+}\right)H_{2}\left(r_{+}\right)}r_{+}^{2}\right)=\delta S. \tag{4.29}$$

The polar boundary term δB_{θ} has to be carefully computed using the definition of the angular component of the gauge field given in (2.14). Then,

$$\begin{aligned}\delta B_{\theta} &= 2\pi\beta\left(\int_{r+}^{\infty}dr\frac{e^{-\sqrt{3}\phi}}{H}\right)\left(\left[\frac{\partial_{\theta}A_{\varphi}\delta A_{\varphi}}{\sin\left(\theta\right)}\right]_{0}^{\pi/2-\delta}+\left[\frac{\partial_{\theta}A_{\varphi}\delta A_{\varphi}}{\sin\left(\theta\right)}\right]_{\pi/2+\delta}^{\pi}\right)_{\delta\to0}\\ &= -2\pi\beta\left(\int_{r+}^{\infty}dr\frac{e^{-\sqrt{3}\phi}}{H}\right)\left(\left[p\delta p\left(1+\cos\left(\theta\right)\right)\right]_{0}^{\pi/2-\delta}+\left[p\delta p\left(-1+\cos\left(\theta\right)\right)\right]_{\pi/2+\delta}^{\pi}\right)_{\delta\to0}\\ &= 4\pi\beta\left(\int_{r+}^{\infty}dr\frac{e^{-\sqrt{3}\phi}}{H}\right)p\delta p.\end{aligned} \tag{4.30}$$

This term can be conveniently written as

$$\delta B_{\theta}=-\beta\Phi_{m}\delta Q_{m}, \tag{4.31}$$

where we can identify the magnetic potential

$$\Phi_{m}=-\sqrt{\frac{2}{\beta_{2}\gamma_{1}}}\left(1-\beta_{2}-\frac{1-\beta_{2}f_{0}\left(r_{+}\right)}{H_{2}\left(r_{+}\right)}\right), \tag{4.32}$$

and also the value of variation of the magnetic charge

$$\delta Q_{m}=4\pi\delta\left(\frac{2\sqrt{2}\mu\sqrt{\beta_{2}\gamma_{1}}}{\gamma_{2}}\right). \tag{4.33}$$

As a consequence, the variation of the boundary term B is given by

$$\delta B=\delta S-\beta\delta M-\beta\Phi_{e}\delta Q_{e}-\beta\Phi_{m}\delta Q_{m}. \tag{4.34}$$

Note that once this term is integrated, the value of B corresponds to the Euclidean Hamiltonian action I^{E} evaluated on stationary configurations and on the constraint surface. In the grand canonical ensemble I^{E} is related to the Gibbs free energy by $I^{E}=-\beta G$. It is also worth to point out that since the first law of black hole thermodynamics,

$$\delta M=T\delta S-\Phi_{e}\delta Q_{e}-\Phi_{m}\delta Q_{m}, \tag{4.35}$$

is a consequence of the Euclidean action having an extremum, (4.35) is identically satisfied independently of the boundary conditions on the mass. This is because (4.35) is a relation that only involves the variation of the conserved charges. This can be shown explicitly by introducing the value for the charge variations (4.25), (4.26), (4.33) and the chemical potentials obtained by using the regularity conditions (4.28) into (4.35).

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Once the mass is integrated using arbitrary boundary conditions (see section 3), it is possible to find the value of the Gibbs free energy which is equivalent to the Euclidean Hamiltonian action evaluated on-shell,

$$I^E = S - \beta M - \beta\Phi_e Q_e - \beta\Phi_m Q_m. \tag{4.36}$$

Recalling that we have chosen the grand canonical ensemble and taking the Euclidean action as our thermodynamic potential, the values of the extensive quantities, the mass, the electric charge, the magnetic charge and the entropy are obtained through the following thermodynamic relations

$$M = -\left(\frac{\partial I^E}{\partial \beta}\right)_{\Phi_e,\Phi_m} + \frac{\Phi_e}{\beta}\left(\frac{\partial I^E}{\partial \Phi_e}\right)_{\beta,\Phi_m} + \frac{\Phi_m}{\beta}\left(\frac{\partial I^E}{\partial \Phi_m}\right)_{\beta,\Phi_e}, \tag{4.37}$$

$$Q_e = -\frac{1}{\beta}\left(\frac{\partial I^E}{\partial \Phi_e}\right)_{\beta,\Phi_m}, \tag{4.38}$$

$$Q_m = -\frac{1}{\beta}\left(\frac{\partial I^E}{\partial \Phi_m}\right)_{\beta,\Phi_e}, \tag{4.39}$$

$$S = I_E - \beta\left(\frac{\partial I^E}{\partial \beta}\right)_{\Phi_e,\Phi_m}. \tag{4.40}$$

The values of the charges and the entropy computed above coincide with (3.37), (3.28), (4.33) and (4.29), respectively.

5 Concluding remarks

We have carried out the thermodynamic analysis of a new class of AdS_4 dyonic dilatonic black holes recently proposed in [1], which are solutions of the bosonic sector of a Kaluza-Klein reduction of eleven-dimensional supergravity. The conserved Noether charges were computed using the Regge-Teitelboim Hamiltonian approach. These correspond to the mass, which acquires contributions from the scalar field and the electric charge. It was also shown that the mass acquires non-integrable contributions from the scalar field, in which case it was necessary to impose integrability conditions to have a definite mass. These conditions are generically solved by imposing boundary conditions that relate the leading and subleading terms of the scalar field fall-off. A possible physical condition to establish the arbitrary functions coming from the integrability condition is to preserve the AdS symmetry of the scalar field behavior at infinity as was established in [4–6, 11]. The Hamiltonian Euclidean action was computed by demanding that the action has an extremum, where its value was given by the corresponding radial boundary term plus an additional polar angle boundary term, because of the presence of a magnetic monopole. The computation was performed in the grand canonical ensemble. The conserved charges were identified from the thermodynamic analysis. The Noether charges, the mass and the electric charge, were obtained from the radial boundary term at infinity, unlike the magnetic charge. The latter one comes from the additional polar angle boundary term. Remarkably, the magnetic potential appeared already in the variation of the boundary term, unlike the

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chemical potentials associated to the Noether charges which are the Lagrange multipliers of the system at infinity. They are obtained by imposing regularity conditions at the horizon. Considering the above, it is possible to verify that the first law of black hole thermodynamics is identically satisfied. This is a consequence of having a well-defined and finite Hamiltonian action principle.

A different way to deal with the thermodynamics of dyonic black holes is to consider a manifestly duality invariant action that involves two U(1) symmetries, producing the appearance of electric and magnetic Gauss constraints [23]. The dyonic Reissner-Nordström black hole is a solution of the system proposed in [23], however in that case the magnetic and the electric fields appear as Coulomb potentials, hence the solution is devoid of stringy singularities. In this case, all the conserved charges that appear in the first law come from symmetries of the action.

It would be interesting to analyze the existence of phase transitions between the dyonic dilatonic black hole solution and the dyonic Reissner-Nordström black hole, i.e. studying the probability that below a critical temperature the dyonic Reissner-Nordström black hole spontaneously changes to a state that is dressed with a dilaton scalar field. This kind of results have been reproduced, for instance, in the case of four-dimensional topological black holes dressed with a scalar field in [24].

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