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# Off-shell scalar supermultiplet in the unfolded dynamics approach

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**ABSTRACT:** We show how manifestly supersymmetric action for Wess-Zumino model can be constructed within the unfolded dynamics approach. The off-shell unfolded system for  $\mathcal{N} = 1$ ,  $D = 4$  scalar supermultiplet is found. The action is presented in the form of integral of a closed 4-form over any  $(4, 0)$  surface in superspace as well as a superspace integral of an integral form or a chiral integral form. The proposed method is argued to provide a most general tool for the analysis of manifestly supersymmetric functionals.

**KEYWORDS:** Superspaces, Differential and Algebraic Geometry

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## 1 Introduction

Unfolded dynamics approach, originally developed for the description of higher-spin field dynamics [1], implies rewriting field equations in the form of some generalized covariant constancy conditions. In principle, any theory can be reformulated in such a way (e.g., in [2] this has been done for gravity and Yang-Mills theory). Unfolded formulation of a dynamical system allows one to control its gauge symmetries. The coordinate-free language of differential forms is particularly convenient for theories of gravity. Moreover, so-called universal unfolded equations [3], to which class belong all relevant examples, are insensitive to a particular space-time where the fields live. The latter property allows one in particular to derive different versions of the superspace formulations directly from the unfolded formulation in the usual space-time.

A related remarkable feature of the unfolded dynamics approach is that it provides a tool for the search for Lagrangians and conserved currents in terms of certain  $Q$ -cohomology associated with the system of unfolded equations [2]. The aim of this paper is to illustrate this method by systematic derivation of the manifestly supersymmetric superspace actions for the simplest supersymmetric model, namely  $4d$  Wess-Zumino model [4, 5]. Our results provide an off-shell extension of the on-shell results of [6].

Naively, the proposed scheme may look obstructed by the fact that superforms do not support integration over superspace. This is avoided once, as we proceed in this paper, the action is either defined as an integral of a superform over an even submanifold arbitrarily embedded into the full superspace or as a Bernstein-Leites integral form [7]. It should be noted that the Bernstein-Leites integration was applied in the context of D-branes [8] in which case it was used that the integrands in the model in question had a specific Gaussian form. In this paper we extend the class of integral forms to those behaving as  $\delta$ -functions of the supervielbeins. The respective superspace integrals turn out to be well-defined, providing the manifestly supersymmetric formulation for supersymmetric theories. Being useful in practical computation this extension of the class of superspace Lagrangians may open new possibilities for the construction of supersymmetric actions in superspace. In particular, for the Wess-Zumino model considered in this paper integral forms having the form of  $\delta$ -functions of supervielbeins provide explicit solutions for the equations that determine invariant actions.

The results of this paper provide an example of the application of the unfolded machinery to supersymmetric models, in which the superspace constraints naturally arise via uplifting the unfolded system to the superspace. We construct the most general unfolded Lagrangians of the Wess-Zumino model in the form of a 4-superform, integral form and a chiral integral form. They contain all possible supersymmetric Lagrangians of the  $4d$  Wess-Zumino model, i.e., besides the standard Wess-Zumino [4] and Salam-Strathdee [9] Lagrangians (see also [5]), they also contain higher-derivative Lagrangians. We expect that the list of supersymmetric Lagrangians presented in this paper does not go beyond the general supersymmetric Lagrangian  $\mathcal{L} = \int d^2\theta d^2\bar{\theta} K + [\int d^2\theta W + \text{h.c.}]$  for the chiral superfield  $\Phi$  ( $\bar{D}_{\dot{\alpha}}\Phi = 0$ ) with arbitrary Lorentz-invariant superpotential  $W(\Phi)$  and a (real) Kahler potential  $K$  which can depend on  $\Phi$  and  $\bar{\Phi}$  along with their (super)derivatives [10]. Nevertheless we believe that the obtained results can be useful for the practical analysis of the higher-derivative component supersymmetric actions since the manifestly supersymmetric superform unfolded action in our approach directly reproduces the component action upon identification of Minkowski space with an integration surface in the full superspace. In the context of the Wess-Zumino model higher-derivative Lagrangians were studied e.g. in [11–14].

In application to supersymmetric models, our method has much in common with the group manifold approach [15, 16] which treats actions as invariant functionals on hypersurface embedded into the group manifold, as well as with the “ectoplasm” approach of [17–20] in which supersymmetric component actions result from the  $d$ -closure condition on the superLagrangian. The  $Q$ -cohomology used in the unfolded dynamics is related to de Rham cohomology by virtue of unfolded equations.

The nice feature of the  $Q$ -cohomology approach is that it reformulates the problem in a coordinate-independent algebraic way, making the procedure more systematic compared to the analysis of differential operators as in the ectoplasm approach. As such, the  $Q$ -cohomology approach is more general, being applicable to any (not necessarily supersymmetric) model and/or geometry.

One of the key properties of the unfolded approach is that it makes gauge and global symmetries manifest (the latter as residual symmetries of the gauge symmetries of the field equations for the vacuum background fields). In particular, local or global supersymmetry is one of such symmetries depending on the model in question. It is this property that makes the unfolded approach particularly useful for the study of higher-spin gauge theories where it was originally developed. One of the aims of this paper is to stress that it can be useful for the analysis of usual (lower-spin) field theories as well. The feature that this approach involves infinite towers of field variables sometimes considered as a complication is, in fact, a simplification because these towers of fields just form a basis of all off-shell or on-shell nontrivial higher derivatives in the model. If one is not interested in the analysis of certain higher derivatives it is possible to truncate the unfolded equations appropriately. The benefit is that unfolded formulation provides a well-defined basis of fields for the analysis of higher-order and/or higher-derivative terms. The only condition is the nilpotency of the operator  $Q$ , which guarantees both consistency of the system and its symmetries. This condition provides a tool for the search of manifestly supersymmetric formulation of the theory. In this context, it should be noted that because the dynamical equations of an unfolded system are classified in terms of the so-called  $\sigma_-$ -cohomology [21], this makes it possible to derive an off-shell unfolded system directly from the on-shell one by finding such a completion of the unfolded equations in which the corresponding sector of the  $\sigma_-$ -cohomology vanishes. In principle, this provides a general approach for the search of manifestly supersymmetric versions of the theories in question.

Though gauge symmetries do not play an essential role in the Wess-Zumino model, we expect that the unfolded formalism may be useful for the analysis of more complicated models like super Yang-Mills theory despite the significant progress already achieved by different methods [22–27]. In this work we are mainly interested in the analysis of specificities of the application of the unfolded technique to supersymmetric models in superspace which is a modest step toward the future study of the gauge supersymmetric theories.

The paper is organized as follows. In section 2 unfolded dynamics approach is overviewed and the cohomological method of computation of Lagrangians for a graded system is proposed. In section 3 relevant aspects of superform integration are considered. In section 4 we recall the formulation of Minkowski superspace in terms of flat superconnections. Section 5 contains an overview of the results of [6] for the on-shell massless scalar supermultiplet as well as its off-shell generalization. In section 6 we explore the operator  $Q$  of the system in question and compute the cohomology of its highest grade part. In section 7 we derive and solve equations which determine supersymmetric invariant functionals of the model and find the particular solutions associated with action of the Wess-Zumino model. In the end of section 7 it is explained how conventional field-theoretic Lagrangians result from the found unfolded Lagrangians and is argued that the latter describe all possible

supersymmetric Lagrangians in the model in question including the standard expressions of [5] along with their higher-derivative generalizations. Conventions and notations are collected in appendix A. The full system of equations from section 7 is stored in appendix B.

## 2 Unfolded formulation

### 2.1 Unfolded equations

Let  $M^d$  be  $d$ -dimensional space-time manifold with local coordinates  $x^{\underline{n}}$ ,  $\underline{n} = 0, \dots, d-1$ . Unfolding of equations implies their reformulation in the form of generalized zero curvature equations

$$R^\Omega(x) := dW^\Omega(x) + G^\Omega(W(x)) = 0, \quad (2.1)$$

where  $d = dx^{\underline{m}} \frac{\partial}{\partial x^{\underline{m}}}$  is de Rham differential,  $W^\Omega(x)$  are degree  $p_\Omega$  differential forms and

$$G^\Omega(W^\Upsilon) := \sum_{n=1}^{\infty} f^\Omega_{\Upsilon_1 \dots \Upsilon_n} W^{\Upsilon_1} \dots W^{\Upsilon_n} \quad (2.2)$$

are degree  $p_\Omega + 1$  differential forms built from exterior products of forms  $W^\Upsilon(x)$  (wedge symbol is omitted in this paper). Here  $\Omega$  and  $\Upsilon$  are indices carried by differential forms.

The identity  $d^2 \equiv 0$  implies the compatibility condition

$$G^\Upsilon(W) \frac{\delta G^\Omega(W)}{\delta W^\Upsilon} \equiv 0, \quad (2.3)$$

which has to be satisfied for all  $W^\Omega$ . It can be equivalently rewritten as

$$Q^2 = 0, \quad Q = G^\Upsilon(W) \frac{\delta}{\delta W^\Upsilon}. \quad (2.4)$$

Unfolded equations are called universal [2, 3] if compatibility condition (2.3) holds independently of the fact that any  $p$ -form with  $p > d$  is zero in  $d$ -dimensional space. In this case one can differentiate freely over  $W^\Omega(x)$ , and equation (2.1) is invariant under gauge transformation

$$\delta W^\Omega = d\varepsilon^\Omega - \varepsilon^\Upsilon \frac{\delta G^\Omega(W)}{\delta W^\Upsilon}, \quad (2.5)$$

where  $(p_\Omega - 1)$ -form gauge parameter  $\varepsilon^\Omega(x)$  is related to the  $p_\Omega > 0$  form  $W^\Omega(x)$  (0-forms do not give rise to gauge parameters).

For universal unfolded equations, condition (2.3) holds independently of the choice of a space-time manifold. Full information about local physical degrees of freedom of the unfolded system is contained in 0-forms at any given point of space-time. Since these data remain the same in any space, universal unfolded systems provide an equivalent description in a larger (super)space simply via addition of extra coordinates. Particular examples of this phenomenon have been presented in [6, 28–30].

The following terminology is used. The fields that can neither be expressed via derivatives of some other fields nor gauged away are called dynamical. The rest of the fields are referred to as auxiliary. (Let us note that the decomposition of fields into dynamical

and auxiliary is not necessarily unambiguous.) Differential conditions imposed by unfolded equations on dynamical fields are called dynamical equations. Other equations are either consequences of dynamical equations or constraints which express auxiliary fields via derivatives of the dynamical ones.

An example of unfolded equation can be constructed as follows. Let  $g$  be a Lie algebra with a basis  $\{T_a\}$ . Consider a  $g$ -valued 1-form  $\Omega_0 = \Omega_0^a T_a$ . For  $G = \Omega_0 \Omega_0$ , equation (2.1) reads as

$$d\Omega_0 + \Omega_0 \Omega_0 = 0. \tag{2.6}$$

The compatibility condition (2.3) gives usual Jacobi identity for the algebra  $g$ . Eq. (2.6) means that the connection  $\Omega_0$  is flat which is the standard way to describe  $g$ -invariant vacuum. Eq. (2.5) gives usual gauge transformations of the connection  $\Omega_0$

$$\delta\Omega_0 = d\varepsilon_0(x) + \Omega_0 \varepsilon_0(x) - \varepsilon_0(x) \Omega_0, \tag{2.7}$$

where  $\varepsilon_0(x)$  is a 0-form valued in  $g$ . Given flat connection  $\Omega_0$  is invariant under the transformations with parameters obeying

$$d\varepsilon_0(x) + \Omega_0 \varepsilon_0(x) - \varepsilon_0(x) \Omega_0 = 0. \tag{2.8}$$

This equation is formally consistent by virtue of (2.3). Solutions of equations (2.8) describe the leftover global symmetry  $g$  of any solution of (2.6).

Let us linearize unfolded equations (2.1) around fixed connection  $\Omega_0$  satisfying (2.6),

$$W = \Omega_0 + C,$$

where  $C$  are differential forms treated as small perturbations and hence contributing linearly to the equations. Let  $\{C_p^i\}$  be a subset of forms of a fixed degree  $p$ , enumerated by index  $i$ . In the linear approximation, the part of  $G$  which is bilinear in  $\Omega_0$  and  $C_p^i$  contributes, i.e.  $G = \Omega_0^a (T_a)^i_j C_p^j$ . In this case, eq. (2.3) implies that the matrices  $(T_a)^i_j$  form a representation of the algebra  $g$  in the space  $V$  where  $p$ -forms  $C_p^i$  are valued. Corresponding equation (2.1) is the covariant constancy condition

$$D_{\Omega_0} C_p^i = 0, \tag{2.9}$$

where  $D_{\Omega_0} \equiv d + \Omega_0$  is the covariant derivative in the  $g$ -module  $V$ .  $C_p^i$  transform properly under  $g$  gauge transformations. Indeed, eq. (2.5) gives for (2.9)

$$\delta C_p^i = d\varepsilon_p^i - \varepsilon_0 C_p^i + \Omega_0 \varepsilon_p^i, \tag{2.10}$$

where  $\varepsilon_p^i$  are gauge parameters related to  $C_p^i$  (for  $p > 0$ ) and  $\varepsilon_0$  are global  $g$ -symmetry parameters obeying (2.8).

## 2.2 $\sigma_-$ -cohomology

Classification of dynamical fields, gauge symmetries and dynamical equations of the unfolded systems can be performed in terms of so-called  $\sigma_-$ -cohomology [2, 3, 6, 21]. Let a linear unfolded system be of the form

$$\left(d + \sum_i \sigma_i\right) C(x) = 0, \tag{2.11}$$

where  $C(x)$  are some differential form fields and operators  $\sigma_i$  act algebraically (i.e. do not differentiate  $x^n$ ).

In the  $\sigma_-$ -cohomology technics, the decomposition of the fields into dynamical and auxiliary is controlled by the  $\mathbb{Z}$ -grading  $\mathcal{G}$  with respect to which auxiliary fields have higher grade than dynamical ones. The grading operator  $\mathcal{G}$  has to be diagonalizable on the space of fields and to be bounded from below.  $d$  has grade zero. Usually,  $\mathcal{G}$  counts a number of tensor indices of the fields.

$\sigma_-$ -cohomology technics applies if  $\sigma_i$  contain operators of negative grades. Then  $\sigma_-$  is the operator of the lowest grade and eq. (2.11) takes the form

$$(d + \sigma_- + \Sigma)C(x) = 0 \tag{2.12}$$

with  $\Sigma$  denoting all operators that act algebraically and have  $\mathcal{G}$ -grade higher than  $\sigma_-$ . Since  $\sigma_-$  has the lowest  $\mathcal{G}$ -grade, from compatibility condition (2.3)

$$(d + \sigma_- + \Sigma)^2 = 0 \tag{2.13}$$

it follows that

$$(\sigma_-)^2 = 0. \tag{2.14}$$

Using that the gauge transformation (2.5) for equation (2.12) is

$$\delta C(x) = (d + \sigma_- + \Sigma)\varepsilon(x), \tag{2.15}$$

it can be shown [2, 3, 21] that, for  $p$ -forms  $C_p$  from the space  $V$ , the cohomology  $H^{p-1}(\sigma_-, V)$ ,  $H^p(\sigma_-, V)$  and  $H^{p+1}(\sigma_-, V)$  are, respectively, the spaces of differential gauge symmetries, dynamical fields and dynamical equations.

The situation with several operators of negative grade is more complicated. As shown in [6], in this case usual  $\sigma_-$ -analysis should be extended to the spectral sequence analysis of all such operators. The full field-theoretical pattern of the system is determined by the cohomology  $H(\sigma'_{-1} | \dots | \sigma''_- | \sigma'_- | \sigma_-)$  where the operators  $\sigma'_{-i}$  are arranged in the order of increase of their  $\mathcal{G}$ -grade and  $H(\sigma'_- | \sigma_-)$  means the cohomology of  $\sigma'_-$  restricted to  $H(\sigma_-)$ .

### 2.3 Unfolded actions and charges

Invariants of a general unfolded system such as actions and conserved charges are encoded by cohomology of the operator  $Q$  (2.4) [2].

Suppose that system (2.1) is off-shell, i.e. it does not contain any dynamical equations, describing only a set of constraints. In the language of  $\sigma_-$ -cohomology, this means that unfolded equations for  $(p-1)$ -forms  $W^\Omega$  have  $H^p(\sigma_-) = 0$ . Following [2], the action  $S$  of this system is defined as an integral over a manifold  $M^d$

$$S = \int_{M^d} \mathcal{L} \tag{2.16}$$

of some  $d$ -form  $\mathcal{L}(W)$  which is a  $Q$ -closed function of the fields  $W^\Omega$

$$Q\mathcal{L} = 0 : \quad G^{\Upsilon}(W) \frac{\partial}{\partial W^\Upsilon} \mathcal{L}(W) = 0. \tag{2.17}$$

Taking into account that  $\delta\mathcal{L} = (\partial\mathcal{L}/\partial W^\Omega)\delta W^\Omega$  and using (2.5), one easily obtains

$$\delta\mathcal{L} = d\left(\varepsilon^\Omega \frac{\partial\mathcal{L}}{\partial W^\Omega}\right). \tag{2.18}$$

Assuming that  $M^d$  has no boundary (or that fields decrease fast enough at infinity), the action remains invariant under gauge transformations (2.5).

If the Lagrangian  $\mathcal{L}$  is  $Q$ -exact, i.e.  $\mathcal{L} = G^\Omega \frac{\partial\mathcal{F}}{\partial W^\Omega}$ , by virtue of (2.1)

$$\mathcal{L} = -dW^\Omega \frac{\partial\mathcal{F}}{\partial W^\Omega} = -d\mathcal{F} \tag{2.19}$$

and hence  $Q$ -exact Lagrangians lead to trivial local actions. Thereby nontrivial invariant actions of the off-shell system (2.1) are in one-to-one correspondence with its  $Q$ -cohomology.

If system (2.1) is on-shell (i.e., contains some dynamical equations) and a  $p$ -form  $\mathcal{L}$  is a representative of the nonzero  $Q$ -cohomology class, the same formula (2.16) describes a conserved charge as an integral over a  $p$ -cycle  $\Sigma$

$$q = \int_{\Sigma} \mathcal{L}. \tag{2.20}$$

Let  $M^d$  be embedded into some ambient space,  $M^d \subset \widetilde{M}^{\tilde{d}}$ ,  $\tilde{d} > d$ . Extending (2.1) to  $\widetilde{M}^{\tilde{d}}$ , by virtue of (2.17), which is equivalent to  $d$ -closure of  $\mathcal{L}$ , action (2.16) is independent of the local form of this embedding.

The case where the algebra of functions of fields from which a Lagrangian is built admits a grading  $G$  bounded from below, is of particular interest. Let  $Q$  and  $\mathcal{L}$  admit decompositions into finite sums of  $G$ -homogeneous parts

$$Q = \sum_{i=0}^n Q_i, \quad \mathcal{L} = \sum_{i=0}^k \mathcal{L}_i, \tag{2.21}$$

where  $G(Q_i) = G(\mathcal{L}_i) = i$ . It can be shown that the space of nontrivial  $Q$ -closed Lagrangians is isomorphic to some subspace of  $H(Q_n)$ , where  $Q_n$  is the part of  $Q$  of maximal  $G$ -grade.

Indeed, from  $Q^2 = 0$  at different  $G$ -grades it follows that

$$(Q_n)^2 = 0, \tag{2.22}$$

$$\{Q_n, Q_{n-1}\} = 0, \tag{2.23}$$

...

$$(Q_0)^2 = 0. \tag{2.24}$$

Equation  $Q\mathcal{L} = 0$  gives

$$Q_n\mathcal{L}_k = 0, \tag{2.25}$$

$$Q_{n-1}\mathcal{L}_k + Q_n\mathcal{L}_{k-1} = 0, \tag{2.26}$$

...

$$Q_0\mathcal{L}_0 = 0. \tag{2.27}$$



Let the highest grade components be denoted as  $\mathbb{Q} := Q_n$  and  $\mathbb{L} := \mathcal{L}_k$ . Since non-trivial Lagrangians are represented by  $Q$ -cohomology, if  $\mathbb{L} = \mathbb{Q}f$  it can be removed by the redefinition

$$\mathcal{L}' = \mathcal{L} - Qf \tag{2.28}$$

so that  $G(\mathbb{L}') < G(\mathbb{L})$ , where  $\mathbb{L}'$  is the highest grade part of  $\mathcal{L}'$ . If  $\mathbb{L}' = \mathbb{Q}g$ , the subtraction  $\mathcal{L}'' = \mathcal{L}' - Qg$  reduces the maximal grade further. The process stops in a finite number of steps because  $G$ -grading is bounded below. Eventually, either the Lagrangian vanishes or its highest grade part  $\mathbb{L}$  belongs to  $H(\mathbb{Q})$ . Thus, any nontrivial Lagrangian is represented by some  $\mathbb{L} \in H(\mathbb{Q})$ .

This does not mean however that any  $\mathbb{L} \in H(\mathbb{Q})$  is associated with some nontrivial Lagrangian  $\mathcal{L} \in H(Q)$ . Two related phenomena may happen.

One is that  $\mathbb{L}$  cannot be supplemented with the terms of the lowest degrees to form a  $Q$ -closed Lagrangian  $\mathcal{L}$ . Indeed,  $Q_{n-1}\mathbb{L}$  in eq. (2.26) is  $\mathbb{Q}$ -closed by virtue of (2.22), (2.23) and (2.25). If it is not  $\mathbb{Q}$ -exact however, eq. (2.26) admits no solutions. In other words, that  $Q_{n-1}\mathbb{L}$  is in nontrivial  $\mathbb{Q}$ -cohomology provides an obstruction for reconstruction of  $\mathcal{L}$  in terms of  $\mathbb{L}$ .

Another phenomenon is that if the same element of the  $\mathbb{Q}$ -cohomology, that provides an obstruction for the extension to full  $Q$ -cohomology, is interpreted as a highest grade part of some other Lagrangian with  $\mathbb{L}' = Q_{n-1}\mathbb{L}$ , then such a highest grade part can be removed by adding a  $Q$ -exact term  $-Q\mathbb{L}$ .

More generally, a similar phenomenon may occur at any step of the analysis of eqs. (2.25)–(2.27). In particular, the corresponding  $Q$ -trivial highest grade components have the form

$$\mathbb{L} = \sum_{i=i_0}^n Q_i \mathcal{F}_{k-i} = Q_{i_0} \mathcal{F}_{k-i_0} + Q_{i_0+1} \mathcal{F}_{k-i_0-1} + \dots + Q_{n-1} \mathcal{F}_{k-n+1} + \mathbb{Q} \mathcal{F}_{k-n}, \tag{2.29}$$

with  $\mathcal{F}_i$  obeying

$$Q_{i_0+1} \mathcal{F}_{k-i_0} + Q_{i_0+2} \mathcal{F}_{k-i_0-1} + \dots + \mathbb{Q} \mathcal{F}_{k-n+1} = 0, \tag{2.30}$$

...

$$Q_{n-1} \mathcal{F}_{k-i_0} + \mathbb{Q} \mathcal{F}_{k-i_0-1} = 0, \tag{2.31}$$

$$\mathbb{Q} \mathcal{F}_{k-i_0} = 0, \tag{2.32}$$

where  $i_0 \in [0, n]$  is some fixed integer. If  $\mathcal{F}_{k-i_0} = \mathbb{Q}g$ , it can be removed by the redefinition  $\mathcal{F}'_i = \mathcal{F}_i - Q_{n-k+i_0+i}g$  with  $\mathcal{F}'_i$  which, by virtue of (2.22)–(2.24), obey the same system (2.29)–(2.32) with  $i'_0 = i_0 + 1$ . As a result one is left either with a trivial highest grade term  $\mathbb{L} = \mathbb{Q}\tilde{\mathcal{F}}_{k-n}$  (if all  $\mathcal{F}_i$  in (2.30)–(2.32) can be removed) or with  $\tilde{\mathcal{F}}_{k-i_0}$  from (2.32) that belongs to  $H(\mathbb{Q})$ . In the latter case, though being non- $\mathbb{Q}$ -exact,  $\mathbb{L}$  can be removed by adding  $Q$ -exact terms  $-\sum_i Q\mathcal{F}_i$ .

Note that somewhat similar situation took place in [31], where the deformation of Minkowski higher-spin vertices to AdS space was studied. There nontrivial vertices belong to cohomology of the nilpotent operator  $Q = Q^{\text{fl}} + \lambda^2 Q^{\text{sub}}$ , where  $-\lambda^2$  is the cosmological constant. The grading  $G$  counts the number of derivatives in vertices,  $G(Q^{\text{fl}}) = 1$ ,

$G(Q^{\text{sub}}) = -1$ . The AdS deformation (if exists) of a nontrivial vertex  $F$  in Minkowski space (where  $\lambda = 0$  and  $Q = Q^{\text{fl}}$ ) may in principle turn out to be trivial in AdS.

As a result, the space of nontrivial  $Q$ -closed Lagrangians is isomorphic to subspace of  $H(\mathbb{Q})$ , which is formed by some highest grade terms  $\mathbb{L}$  that cannot be represented in the form (2.29) with  $\mathcal{F}_i$  obeying (2.30)–(2.32).

The construction of invariant functionals presented so far works nicely for usual manifolds but is less obvious in the case of superspace which is of most interest in this paper. Since, naively, the differential superforms are not integrable over supermanifolds (see e.g., [32]), we have to specify the notion of an unfolded action in superspace.

### 3 Integration in superspace

Most of differential geometry admits straightforward generalization to supermanifolds. However extension of integration of differential forms over a supermanifold is not quite straightforward. One way to see this is to observe that superform transformation law does not match Berezin integral. Indeed, consider a supermanifold  $M^{p|q}$  with local coordinates  $z^M = (x^m, \theta^\mu)$ . To be coordinate-independent, the integration measure has to transform according to Berezin formula

$$\int_{M^{p|q}} f(x^m, \theta^\mu) d^p x d^q \theta = \int_{M^{p|q}} f(y^m, \xi^\mu) \text{Ber } J d^p y d^q \xi, \tag{3.1}$$

where

$$J = \begin{pmatrix} \frac{\partial x}{\partial y} & \frac{\partial x}{\partial \xi} \\ \frac{\partial \theta}{\partial y} & \frac{\partial \theta}{\partial \xi} \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad \text{Ber } J = \frac{\det(A - BD^{-1}C)}{\det D}.$$

Superforms resulting from naive extension to odd coordinates obviously do not satisfy this condition and hence are not integrable.

However, even forms can still be integrated over even cycles in superspace. Indeed, consider an even  $n$ -dimensional surface  $\mathcal{S}^n$  on  $M^{p|q}$ :  $z^M(t^a) = (x^m(t^a), \theta^\mu(t^a))$ ,  $a = 1, \dots, n$ , parametrized by some even parameters  $t^a$  (surface coordinates). Let the integral of a  $n$ -superform  $\omega = \omega_{M_1 \dots M_n} dz^{M_1} \dots dz^{M_n}$  over  $\mathcal{S}^n$  be defined as

$$\int_{\mathcal{S}^n} \omega = \int \omega_{M_1 \dots M_n} \left( \frac{\partial z^{M_1}}{\partial t^{a_1}} dt^{a_1} \right) \dots \left( \frac{\partial z^{M_n}}{\partial t^{a_n}} dt^{a_n} \right) = \int \omega'_{1 \dots n} dt^1 \dots dt^n. \tag{3.2}$$

It is neither dependent on the choice of coordinates  $t^i$  nor of  $z^M$ . Obviously if  $\omega$  is exact in superspace, the integration over  $\mathcal{S}^n$  amounts to that over its boundary  $\partial \mathcal{S}^n$ . By Cartan formula  $\mathcal{L}_V = \{d, i_V\}$  for the Lie derivative of a vector field  $V^N(z)$ , the integral of a closed form  $\omega$  ( $d\omega = 0$ ) is independent of local variations of  $\mathcal{S}^n$ . This allows us to define unfolded superfield action as the integral of a closed  $p$ -superform  $\mathcal{L}$  over an even  $p$ -dimensional surface in superspace.

More generally, integration in superspace can be defined in terms of integral forms, as was originally proposed in [7] (see also [32]). To this end an integral of superfunction

$f(x, \theta)$  in superspace can be rewritten as

$$\int f(x, \theta) d^p x d^q \theta = \int F(x, \theta, \varsigma, s) d^p x d^q \theta d^p \varsigma d^q s, \quad (3.3)$$

where  $\varsigma^m$ ,  $m = 1, \dots, p$  and  $s^\mu$ ,  $\mu = 1, \dots, q$  are treated as additional anticommuting and commuting integration variables, respectively. To contribute,  $F(x, \theta, \varsigma, s)$  should have the form

$$F(x, \theta, \varsigma, s) = f(x, \theta) \delta^p(\varsigma) \delta^q(s). \quad (3.4)$$

To make the link with differential forms, one formally substitutes  $dx^m$  and  $d\theta^\mu$  for  $\varsigma^m$  and  $s^\mu$  in (3.4). Resulting objects are called integral forms. Note that  $\delta^p(dx)$  is just the usual volume form  $dx^1 \dots dx^p$  while  $\delta^q(d\theta)$  is the actual (even)  $\delta$ -function, which is essentially non-polynomial. On the other hand, integration of usual superforms polynomial in  $d\theta$  does not make sense, leading to divergent integral (3.3) with  $F(x, \theta, \varsigma, s)$  polynomial in  $s^\mu$ .

As mentioned in Introduction, our approach has much in common with the “ectoplasm” method [17–20] of construction of manifestly supersymmetric actions represented by integrals of superforms over space-time “hypersurface” in the full superspace  $M^{p|q}$  with coordinates  $z^M = (x^m, \theta^\mu)$ . Let a  $p$ -superform

$$J = J_{M_1 \dots M_p} dz^{M_1} \dots dz^{M_p} \quad (3.5)$$

be closed

$$dJ = 0 \quad \Rightarrow \quad \mathcal{D}_{[N} J_{M_1 \dots M_p]} - \frac{d}{2} T_{[NM]}^P J_{P|M_2 \dots M_p]} = 0, \quad (3.6)$$

with the covariant derivative  $\mathcal{D}_M$  and torsion tensor  $T_{MN}^P$ . Then the integral over space-time hypersurface

$$S = \int_{M^p} J_{m_1 \dots m_p} dx^{m_1} \dots dx^{m_p} \quad (3.7)$$

is independent of coordinates in  $M^{p|q}$  and, by virtue of (3.6), of a particular choice of the integration hypersurface, provided that the latter is even and  $J$  falls down fast enough at spatial infinity. It is invariant under the transformation

$$\delta J_{M_1 \dots M_p} = \partial_{[M_1} \lambda_{M_2 \dots M_p]}. \quad (3.8)$$

The more general case of a curved superspace can be considered analogously in terms of supervielbeins [18, 19]. Note that in some applications of the ectoplasm approach the Bernstein-Leites integration was also used [8]. However, in this case its applicability relied on the specific Gaussian form of the integrands, allowing to carry out the integration over the odd differentials. In this paper we will extend the application of the Bernstein-Leites integration to distributions of the background supervielbeins which provides a simple and efficient way for writing superinvariants.

Similarity of the ectoplasm and unfolded approaches is obvious. Indeed, in the former  $d$ -closure of Lagrangian guarantees its manifest SUSY, while in the latter the same is achieved via  $Q$ -closure condition. As shown in subsection 2.1,  $Q$  is an algebraic counterpart

of de Rham differential. Although for particular supersymmetric models both methods lead to similar results, the unfolded approach is more general being applicable to any (not necessarily supersymmetric) theory and (generalized) space-time like e.g., the Sp(8)-invariant space-time considered in [28, 30].

#### 4 Supersymmetric vacuum

Following [6], to obtain unfolded description of the flat superspace we start with the  $\mathcal{N} = 1$  SUSY algebra

$$[M_{ab}, M_{cd}] = -(\eta_{ac}M_{bd} + \eta_{bd}M_{ac} - \eta_{ad}M_{bc} - \eta_{bc}M_{ad}), \quad (4.1)$$

$$[P_a, M_{bc}] = \eta_{ab}P_c - \eta_{ac}P_b, \quad (4.2)$$

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = -2i(\sigma^a)_{\alpha\dot{\beta}}P_a, \quad (4.3)$$

$$[M_{ab}, Q_\alpha] = \frac{1}{2}(\sigma_{ab})_\alpha{}^\beta Q_\beta, \quad (4.4)$$

$$[M_{ab}, \bar{Q}^{\dot{\alpha}}] = \frac{1}{2}(\bar{\sigma}_{ab})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{\dot{\beta}}. \quad (4.5)$$

(All other (anti)commutators are zero.)

Gauge fields of supergravity are 1-forms of vierbein  $e^a = e^a_m dx^m$ , spin-connection  $\omega^{a,b} = \omega^{a,b}_m dx^m$  and gravitino  $\phi^\alpha = \phi^\alpha_m dx^m$  (see e.g. [33]). They are components of a 1-form connection  $\Omega_0$  valued in the SUSY algebra

$$\Omega_0 := e^a P_a + \frac{1}{2}\omega^{a,b}M_{ab} + \phi^\alpha Q_\alpha + \bar{\phi}_{\dot{\alpha}}\bar{Q}^{\dot{\alpha}}. \quad (4.6)$$

Supersymmetric flat background is represented by a connection  $\Omega_0$  obeying (2.6), which leads to the following component equations

$$D^L e^a + 2i\phi^\alpha \bar{\phi}^{\dot{\alpha}}(\sigma^a)_{\alpha\dot{\alpha}} := de^a + \omega^{a,b}e_b + 2i\phi^\alpha \bar{\phi}^{\dot{\alpha}}(\sigma^a)_{\alpha\dot{\alpha}} = 0, \quad (4.7)$$

$$D^L \omega^{a,b} := d\omega^{a,b} + \omega^{a,c}\omega_c{}^b = 0, \quad (4.8)$$

$$D^L \phi^\alpha := d\phi^\alpha + \frac{1}{4}\omega^{a,b}\phi^\beta(\sigma_{ab})_\beta{}^\alpha = 0, \quad (4.9)$$

$$D^L \bar{\phi}_{\dot{\alpha}} := d\bar{\phi}_{\dot{\alpha}} + \frac{1}{4}\omega^{a,b}\bar{\phi}_{\dot{\beta}}(\bar{\sigma}_{ab})^{\dot{\beta}}{}_{\dot{\alpha}} = 0, \quad (4.10)$$

where  $D^L \equiv d + \omega$  is the Lorentz covariant derivative.

As explained in subsection 2.1, to promote these unfolded equations (which are obviously universal) to superspace it suffices to add fermionic coordinates  $x^m \rightarrow z^M = (x^m, \theta^\mu, \bar{\theta}^{\dot{\mu}})$  extending properly indices of the differential forms

$$\begin{aligned} e^a_m(x)dx^m &\rightarrow E^a_M(z)dz^M, & \omega^{a,b}_m(x)dx^m &\rightarrow \Omega^{a,b}_M(z)dz^M, \\ \phi^\alpha_m(x)dx^m &\rightarrow E^\alpha_M(z)dz^M, & \bar{\phi}_{\dot{\alpha}}^m(x)dx^m &\rightarrow \bar{E}^{\dot{\alpha}}_M(z)dz^M. \end{aligned}$$

Flat superspace is described by the zero curvature equations

$$DE^a + 2iE^\alpha \bar{E}^{\dot{\alpha}} (\sigma^a)_{\alpha\dot{\alpha}} := dE^a + \Omega^{a,b} E_b + 2iE^\alpha \bar{E}^{\dot{\alpha}} (\sigma^a)_{\alpha\dot{\alpha}} = 0, \quad (4.11)$$

$$D\Omega^{a,b} := d\Omega^{a,b} + \Omega^{a,c} \Omega_c{}^b = 0, \quad (4.12)$$

$$DE^\alpha := dE^\alpha + \frac{1}{4} \Omega^{a,b} E^\beta (\sigma_{ab})_{\beta}{}^\alpha = 0, \quad (4.13)$$

$$D\bar{E}_{\dot{\alpha}} := d\bar{E}_{\dot{\alpha}} + \frac{1}{4} \Omega^{a,b} \bar{E}_{\dot{\beta}} (\bar{\sigma}_{ab})^{\dot{\beta}}{}_{\dot{\alpha}} = 0, \quad (4.14)$$

where  $D$  is the Lorentz covariant derivative in superspace.

Global SUSY transformations are described by those gauge transformations (2.8) of system (4.11)–(4.14), that leave invariant background connection

$$\delta E^a = d\varepsilon^a - \varepsilon^{a,b} E_b + \varepsilon_b \Omega^{a,b} - 2i(\varepsilon^\alpha \bar{E}^{\dot{\alpha}} (\sigma^a)_{\alpha\dot{\alpha}} + \bar{\varepsilon}^{\dot{\alpha}} E^\alpha (\sigma^a)_{\alpha\dot{\alpha}}) = 0, \quad (4.15)$$

$$\delta \Omega^{a,b} = d\varepsilon^{a,b} + \Omega^{a,c} \varepsilon_c{}^b - \varepsilon^{a,c} \Omega_c{}^b = 0, \quad (4.16)$$

$$\delta E^\alpha = d\varepsilon^\alpha + \frac{1}{4} \varepsilon^\beta \Omega^{a,b} (\sigma_{ab})_{\beta}{}^\alpha - \frac{1}{4} \varepsilon^{a,b} E^\beta (\sigma_{ab})_{\beta}{}^\alpha = 0, \quad (4.17)$$

$$\delta \bar{E}_{\dot{\alpha}} = d\bar{\varepsilon}_{\dot{\alpha}} + \frac{1}{4} \Omega^{a,b} \bar{\varepsilon}_{\dot{\beta}} (\bar{\sigma}_{ab})^{\dot{\beta}}{}_{\dot{\alpha}} - \frac{1}{4} \varepsilon^{a,b} \bar{E}_{\dot{\beta}} (\bar{\sigma}_{ab})^{\dot{\beta}}{}_{\dot{\alpha}} = 0. \quad (4.18)$$

Cartesian coordinate system is associated with the following solution of (4.11)–(4.14)

$$\begin{aligned} E^a &= dx^m \delta_m{}^a + d\theta^\mu (i\bar{\theta}^{\dot{\mu}} (\sigma^a)_{\mu\dot{\mu}}) + d\bar{\theta}_{\dot{\mu}} (i\theta^\mu (\sigma^a)_{\mu\dot{\mu}} \epsilon^{\dot{\mu}\mu}), \\ \Omega^{a,b} &= 0, \quad E^\alpha = d\theta^\mu \delta_\mu{}^\alpha, \quad \bar{E}_{\dot{\alpha}} = d\bar{\theta}_{\dot{\mu}} \delta^{\dot{\mu}}{}_{\dot{\alpha}}. \end{aligned} \quad (4.19)$$

In these coordinates, the explicit solution to system (4.15)–(4.18) is

$$\varepsilon^a = \xi^a + \xi^a{}_b x^b + i\xi^a{}_b \theta^\mu \bar{\theta}^{\dot{\mu}} (\sigma^b)_{\mu\dot{\mu}} + 2i(\xi^\alpha \bar{\theta}_{\dot{\mu}} (\sigma^a)_{\alpha\dot{\alpha}} \epsilon^{\dot{\mu}\alpha} + \theta_{\dot{\mu}} \bar{\xi}^{\dot{\alpha}} (\sigma^a)_{\alpha\dot{\alpha}} \epsilon^{\mu\alpha}), \quad (4.20)$$

$$\varepsilon^{a,b} = \xi^{a,b}, \quad (4.21)$$

$$\varepsilon^\alpha = \frac{1}{4} \xi^{a,b} \theta_{\dot{\mu}} \epsilon^{\mu\beta} (\sigma_{ab})_{\beta}{}^\alpha + \xi^\alpha, \quad (4.22)$$

$$\bar{\varepsilon}_{\dot{\alpha}} = \frac{1}{4} \xi^{a,b} \bar{\theta}_{\dot{\mu}} \delta^{\dot{\mu}}{}_{\dot{\beta}} (\bar{\sigma}_{ab})^{\dot{\beta}}{}_{\dot{\alpha}} + \bar{\xi}_{\dot{\alpha}} \quad (4.23)$$

with  $\xi^a$ ,  $\xi^{a,b} = -\xi^{b,a}$ ,  $\xi^\alpha$ ,  $\bar{\xi}_{\dot{\alpha}}$  being free constants, which are parameters of global symmetries.

Note that Lorentz covariant derivative in superspace can be rewritten in the form

$$D = E^a D_a + E^\alpha D_\alpha + \bar{E}_{\dot{\alpha}} D^{\dot{\alpha}}. \quad (4.24)$$

In Cartesian coordinates

$$D_a = \partial_a, \quad D_\alpha = \partial_\alpha - i(\sigma^b)_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \partial_b, \quad \bar{D}_{\dot{\alpha}} = \bar{\partial}_{\dot{\alpha}} - i\theta^\alpha (\sigma^b)_{\alpha\dot{\alpha}} \partial_b. \quad (4.25)$$

## 5 Unfolded free massless scalar supermultiplet

In this section we present an off-shell extension of the unfolded equations of motion for  $\mathcal{N} = 1$ ,  $D = 4$  free massless scalar supermultiplet, obtained in [6].

First we consider the problem in Minkowski space. It is described by a solution of system (4.7)–(4.10) with  $\phi^\alpha = \bar{\phi}^{\dot{\alpha}} = 0$ , i.e.

$$de^a + \omega^{a,b}e_b = 0, \quad d\omega^{a,b} + \omega^{a,c}\omega_c{}^b = 0. \quad (5.1)$$

Massless scalar field in Minkowski space is described by the unfolded equations [1, 21]

$$D^L C^{a(k)} + e_b C^{a(k)b} = 0, \quad (5.2)$$

where the 0-forms  $C^{a(k)}$  are symmetric traceless tensors of rank  $k$ .

Similarly, unfolded equations

$$D^L \chi_\alpha^{a(k)} + e_b \chi_\alpha^{a(k)b} = 0 \quad (5.3)$$

for the complex 0-forms  $\chi_\alpha^{a(k)}$  which are symmetric traceless rank- $k$  spinor-tensors, obeying  $\sigma$ -transversality condition

$$(\bar{\sigma}_b)^{\dot{\alpha}\alpha} \chi_\alpha^{a(k-1)b} = 0, \quad (5.4)$$

describe massless spin-1/2 field in Minkowski space [1, 34].

To unify systems (5.2) and (5.3) into supermultiplet the terms with connections  $\phi^\alpha$  and  $\bar{\phi}^{\dot{\alpha}}$  which mix bosons and fermions have to be introduced. This gives [6]

$$D^L C^{a(k)} + e_b C^{a(k)b} - \sqrt{2}\phi^\alpha \chi_\alpha^{a(k)} = 0, \quad (5.5)$$

$$D^L \chi_\alpha^{a(k)} + e_b \chi_\alpha^{a(k)b} - \sqrt{2}i\bar{\phi}^{\dot{\alpha}}(\sigma_b)_{\alpha\dot{\alpha}} C^{a(k)b} = 0. \quad (5.6)$$

Compatibility of the system is provided by flatness condition (2.6) and the identity  $(\sigma_b)_{\beta\dot{\alpha}} \chi_\alpha^{a(k)b} = (\sigma_b)_{\alpha\dot{\alpha}} \chi_\beta^{a(k)b}$  which follows from  $\sigma$ -transversality and the fact that spinor indices take two values.

Extension to superspace is trivially achieved via addition of fermionic coordinates  $x^m \rightarrow z^M = (x^m, \theta^\mu, \bar{\theta}^{\dot{\mu}})$

$$DC^{a(k)}(z) + E_b C^{a(k)b}(z) - \sqrt{2}E^\alpha \chi_\alpha^{a(k)}(z) = 0, \quad (5.7)$$

$$D\chi_\alpha^{a(k)}(z) + E_b \chi_\alpha^{a(k)b}(z) - \sqrt{2}i\bar{E}^{\dot{\alpha}}(\sigma_b)_{\alpha\dot{\alpha}} C^{a(k)b}(z) = 0. \quad (5.8)$$

The resulting system is universal. As explained in subsection 2.1, this implies that all its symmetries are preserved. Hence, system (5.7), (5.8) is supersymmetric.

To check that these equations indeed describe free massless scalar supermultiplet, one has to single out independent dynamical superfields and dynamical equations with the help of  $\sigma$ -cohomology technics. As shown in [6], this gives the following result. The only dynamical superfield is  $C(z)$ . All other fields are auxiliary, being expressed via its derivatives. For instance,  $\chi_\alpha(z) = \frac{1}{\sqrt{2}}D_\alpha C(z)$ . Independent superfield equations are

$$\bar{D}_{\dot{\alpha}} C(z) = 0, \quad (5.9)$$

$$D^\alpha D_\alpha C(z) = 0, \quad (5.10)$$

which are standard equations of motion of a massless scalar supermultiplet [5].

To construct the action, we should find an off-shell modification of system (5.7)–(5.8) which implies no dynamical equations. As a guiding example, first consider the off-shell formulation of system (5.2), (5.3). It results from relaxing the tracelessness condition for  $C^{a(k)}$  and for  $\chi_\alpha^{a(k)}$  as well as the  $\sigma$ -transversality condition for the latter. Then eqs. (5.2), (5.3) just represent a set of constraints which express higher rank tensors in terms of derivatives of the dynamical fields.

However, supersymmetric extension (5.5)–(5.6) of the resulting off-shell system via introducing connections for full SUSY algebra ceases to obey (2.3), i.e. becomes inconsistent. Indeed, the compatibility condition for eq. (5.6) requires

$$\phi_\alpha \bar{\phi}_{\dot{\alpha}} (\bar{\sigma}_b)^{\dot{\alpha}\beta} (\chi^{a(k)b})_\beta = 0. \quad (5.11)$$

In the on-shell case, it holds by virtue of  $\sigma$ -transversality, which is relaxed in the off-shell case.

Inconsistency of the system means, in addition, that its gauge transformations (2.5) do not obey SUSY algebra (4.1)–(4.5) any more, i.e. the system lost SUSY. To restore both off-shell consistency and SUSY, a set of auxiliary fields  $F^{a(k)}$  should be introduced. Supersymmetric off-shell system of equations acquires the form

$$D^L C^{a(k)} + e_b C^{a(k)b} - \sqrt{2} \phi^\alpha \chi_\alpha^{a(k)} = 0, \quad (5.12)$$

$$D^L \chi_\alpha^{a(k)} + e_b \chi_\alpha^{a(k)b} - \sqrt{2} i \bar{\phi}^{\dot{\alpha}} (\sigma_b)_{\alpha\dot{\alpha}} C^{a(k)b} - \sqrt{2} \phi_\alpha F^{a(k)} = 0, \quad (5.13)$$

$$D^L F^{a(k)} + e_b F^{a(k)b} - \sqrt{2} i \bar{\phi}_{\dot{\alpha}} (\bar{\sigma}_b)^{\dot{\alpha}\alpha} \chi_\alpha^{a(k)b} = 0. \quad (5.14)$$

This system is consistent, obeying (2.3). By virtue of eq. (5.14), auxiliary fields  $F^{a(k)}$  are higher derivatives of the ground auxiliary field  $F$  familiar for Wess-Zumino model. Not surprisingly, the fields  $F^{a(k)}$  provide closure of SUSY algebra of the off-shell system.

As explained in subsection 2.1, consistency of system (5.12)–(5.14) implies its global SUSY invariance. Corresponding SUSY transformations in Minkowski space are

$$\delta C^{a(k)} = \sqrt{2} \varepsilon^\alpha \chi_\alpha^{a(k)}, \quad (5.15)$$

$$\delta \chi_\alpha^{a(k)} = \sqrt{2} i \bar{\varepsilon}^{\dot{\alpha}} (\sigma_b)_{\alpha\dot{\alpha}} C^{a(k)b} + \sqrt{2} \varepsilon_\alpha F^{a(k)}, \quad (5.16)$$

$$\delta F^{a(k)} = \sqrt{2} i \bar{\varepsilon}_{\dot{\alpha}} (\bar{\sigma}_b)^{\dot{\alpha}\alpha} \chi_\alpha^{a(k)b}. \quad (5.17)$$

In Cartesian coordinates with  $e^a_{\underline{m}} = \delta^a_{\underline{m}}$ ,  $\phi_\alpha = \bar{\phi}_{\dot{\alpha}} = 0$ ,  $D^L = d$  system (5.12)–(5.14) implies  $C_a = -\partial_a C$ ,  $(\chi_\alpha)_a = -\partial_a \chi_\alpha$ . As a result,

$$\delta C = \sqrt{2} \xi^\alpha \chi_\alpha, \quad (5.18)$$

$$\delta \chi_\alpha = -\sqrt{2} i \bar{\xi}^{\dot{\alpha}} (\sigma^a)_{\alpha\dot{\alpha}} \partial_a C + \sqrt{2} \xi_\alpha F, \quad (5.19)$$

$$\delta F = -\sqrt{2} i \bar{\xi}_{\dot{\alpha}} (\bar{\sigma}^a)^{\dot{\alpha}\alpha} \partial_a \chi_\alpha, \quad (5.20)$$

where  $\xi^\alpha$  and  $\bar{\xi}^{\dot{\alpha}}$  are global SUSY parameters. These are standard supertransformations of the chiral supermultiplet [5].

Extension of system (5.12)–(5.14) to superspace is again achieved via extension of all functions to superspace

$$DC^{a(k)} + E_b C^{a(k)b} - \sqrt{2} E^\alpha \chi_\alpha^{a(k)} = 0, \quad (5.21)$$

$$D\chi_\alpha^{a(k)} + E_b \chi_\alpha^{a(k)b} - \sqrt{2} i \bar{E}^{\dot{\alpha}} (\sigma_b)_{\alpha\dot{\alpha}} C^{a(k)b} - \sqrt{2} E_\alpha F^{a(k)} = 0, \quad (5.22)$$

$$DF^{a(k)} + E_b F^{a(k)b} - \sqrt{2} i \bar{E}_{\dot{\alpha}} (\bar{\sigma}_b)^{\dot{\alpha}\alpha} \chi_\alpha^{a(k)b} = 0. \quad (5.23)$$

Resulting system imposes, however, differential equations with respect to odd coordinates, i.e. strictly speaking it is not fully off-shell in superspace. Indeed, using (4.24) it is easy to obtain from (5.21)–(5.23) that

$$\bar{D}_{\dot{\alpha}} C^{a(k)} = 0, \quad D_\alpha F^{a(k)} = 0. \quad (5.24)$$

These are chirality condition for the fields  $C^{a(k)}$  and antichirality condition for the fields  $F^{a(k)}$ . As is well known [5], these conditions do not impose differential equations in Minkowski space, where the system remains off-shell.

Since the fields of Wess-Zumino model are complex, system (5.21)–(5.23) should be supplemented by the complex conjugated equations

$$D\bar{C}^{a(k)} + E_b \bar{C}^{a(k)b} + \sqrt{2} \bar{E}^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}}^{a(k)} = 0, \quad (5.25)$$

$$D\bar{\chi}_{\dot{\alpha}}^{a(k)} + E_b \bar{\chi}_{\dot{\alpha}}^{a(k)b} + \sqrt{2} i E^\alpha (\sigma_b)_{\alpha\dot{\alpha}} \bar{C}^{a(k)b} - \sqrt{2} \bar{E}_{\dot{\alpha}} \bar{F}^{a(k)} = 0, \quad (5.26)$$

$$D\bar{F}^{a(k)} + E_b \bar{F}^{a(k)b} - \sqrt{2} i E_\alpha (\bar{\sigma}_b)^{\dot{\alpha}\alpha} \bar{\chi}_{\dot{\alpha}}^{a(k)b} = 0. \quad (5.27)$$

Their consequences

$$D_\alpha \bar{C}^{a(k)} = 0, \quad \bar{D}_{\dot{\alpha}} \bar{F}^{a(k)} = 0 \quad (5.28)$$

imply that  $\bar{C}^{a(k)}$  are antichiral and  $\bar{F}^{a(k)}$  are chiral.

## 6 Operator $Q$

### 6.1 General properties

According to the general scheme of [2] recalled in subsection 2.3, Lagrangians of the unfolded system are associated with its  $Q$ -cohomology. The full set of unfolded equations of the system in question includes eqs. (4.11)–(4.14) describing flat superspace background and eqs. (5.21)–(5.23), (5.25)–(5.27) describing scalar supermultiplet. The operator  $Q$  of this system is

$$Q = Q_\Omega + \hat{Q}, \quad (6.1)$$

where

$$Q_\Omega = \Omega^{a,b} E_b \frac{\partial}{\partial E^a} + \frac{1}{4} \Omega^{a,b} E^\beta (\sigma_{ab})_{\beta\alpha} \frac{\partial}{\partial E^\alpha} + \frac{1}{4} \Omega^{a,b} \bar{E}_{\dot{\beta}} (\bar{\sigma}_{ab})^{\dot{\beta}\dot{\alpha}} \frac{\partial}{\partial \bar{E}_{\dot{\alpha}}} + \Omega_{a,b} \hat{q}^{ba} + \Omega^{a,c} \Omega_c{}^b \frac{\partial}{\partial \Omega^{a,b}}, \quad (6.2)$$

$$\hat{Q} = 2i E^\alpha (\sigma^a)_{\alpha\dot{\alpha}} \bar{E}^{\dot{\alpha}} \frac{\partial}{\partial E^a} + E_a \hat{q}^a + \sqrt{2} E_\alpha \hat{q}^\alpha + \sqrt{2} \bar{E}_{\dot{\alpha}} \hat{q}^{\dot{\alpha}}, \quad (6.3)$$



with

$$\begin{aligned}
 \hat{q}^b{}_c &= C^{a(k-1)b} \frac{\partial}{\partial C^{a(k-1)c}} + \bar{C}^{a(k-1)b} \frac{\partial}{\partial \bar{C}^{a(k-1)c}} + F^{a(k-1)b} \frac{\partial}{\partial F^{a(k-1)c}} + \bar{F}^{a(k-1)b} \frac{\partial}{\partial \bar{F}^{a(k-1)c}} \\
 &\quad + \chi_\alpha^{a(k-1)b} \frac{\partial}{\partial \chi_\alpha^{a(k-1)c}} - \frac{1}{4} \chi_\beta^{a(k)} (\sigma^b{}_c)^\beta{}_\alpha \frac{\partial}{\partial \chi_\alpha^{a(k)}} + \bar{\chi}_{\dot{\alpha}}^{a(k-1)b} \frac{\partial}{\partial \bar{\chi}_{\dot{\alpha}}^{a(k-1)c}} - \frac{1}{4} \bar{\chi}_{\dot{\beta}}^{a(k)} (\bar{\sigma}^b{}_c)^{\dot{\beta}}{}_{\dot{\alpha}} \frac{\partial}{\partial \bar{\chi}_{\dot{\alpha}}^{a(k)}}, \\
 \hat{q}^b &= C^{a(k)b} \frac{\partial}{\partial C^{a(k)}} + \bar{C}^{a(k)b} \frac{\partial}{\partial \bar{C}^{a(k)}} + \chi_\alpha^{a(k)b} \frac{\partial}{\partial \chi_\alpha^{a(k)}} + \bar{\chi}_{\dot{\alpha}}^{a(k)b} \frac{\partial}{\partial \bar{\chi}_{\dot{\alpha}}^{a(k)}} + F^{a(k)b} \frac{\partial}{\partial F^{a(k)}} + \bar{F}^{a(k)b} \frac{\partial}{\partial \bar{F}^{a(k)}}, \\
 \hat{q}^\alpha &= (\chi^\alpha)^{a(k)} \frac{\partial}{\partial C^{a(k)}} - F^{a(k)} \frac{\partial}{\partial \chi_\alpha^{a(k)}} - i \epsilon^{\alpha\beta} (\sigma_b)_{\beta\dot{\beta}} \bar{C}^{a(k)b} \frac{\partial}{\partial \bar{\chi}_{\dot{\beta}}^{a(k)}} - i \bar{\chi}_{\dot{\beta}}^{a(k)b} (\bar{\sigma}_b)^{\dot{\beta}\alpha} \frac{\partial}{\partial \bar{F}^{a(k)}}, \\
 \hat{q}^{\dot{\alpha}} &= -(\bar{\chi}^{\dot{\alpha}})^{a(k)} \frac{\partial}{\partial \bar{C}^{a(k)}} + \bar{F}^{a(k)} \frac{\partial}{\partial \bar{\chi}_{\dot{\alpha}}^{a(k)}} - i \epsilon^{\dot{\alpha}\dot{\beta}} (\bar{\sigma}_b)_{\dot{\beta}\beta} C^{a(k)b} \frac{\partial}{\partial \chi_\beta^{a(k)}} - i (\bar{\sigma}_b)^{\dot{\alpha}\beta} \chi_\beta^{a(k)b} \frac{\partial}{\partial F^{a(k)}}.
 \end{aligned}$$

$\hat{Q}$  can be represented in the form

$$\hat{Q} = Q_1 + Q_2^+ + Q_2^- + Q_3, \quad (6.4)$$

where the operators

$$Q_1 = E_a \hat{q}^a \quad (6.5)$$

$$Q_2^+ = \sqrt{2} E_\alpha \hat{q}^\alpha, \quad Q_2^- = \sqrt{2} \bar{E}_{\dot{\alpha}} \hat{q}^{\dot{\alpha}}, \quad (6.6)$$

$$Q_3 = 2i E^\alpha (\sigma^a)_{\alpha\dot{\alpha}} \bar{E}^{\dot{\alpha}} \frac{\partial}{\partial E^a}, \quad (6.7)$$

obey

$$(Q_1)^2 = (Q_2^+)^2 = (Q_2^-)^2 = (Q_3)^2 = 0, \quad (6.8)$$

$$\{Q_1, Q_3\} = -\{Q_2^+, Q_2^-\} = 2i E^\alpha \bar{E}^{\dot{\alpha}} (\sigma_a)_{\alpha\dot{\alpha}} \hat{q}^a, \quad (6.9)$$

$$\{Q_2^+, Q_3\} = \{Q_2^-, Q_3\} = \{Q_1, Q_2^+\} = \{Q_1, Q_2^-\} = 0. \quad (6.10)$$

Eq. (6.9) implies

$$\{\hat{q}^\alpha, \hat{q}^{\dot{\alpha}}\} = -i (\bar{\sigma}_a)^{\dot{\alpha}\alpha} \hat{q}^a. \quad (6.11)$$

## 6.2 Highest grades

Now we are in a position to look for Lagrangians, representing cohomology of the operator  $Q$ . These are built from background 1-forms  $\Omega^{a,b}$ ,  $E_a$ ,  $E_\alpha$ ,  $\bar{E}_{\dot{\alpha}}$  and supermultiplet 0-form fields  $C^{a(k)}$ ,  $\bar{C}^{a(k)}$ ,  $\chi_\alpha^{a(k)}$ ,  $\bar{\chi}_{\dot{\alpha}}^{a(k)}$ ,  $F^{a(k)}$ ,  $\bar{F}^{a(k)}$ .

First of all, we observe that Lagrangian should be  $\Omega$ -independent since the terms resulting from the action of  $\Omega^{a,c} \Omega_c^b \frac{\partial}{\partial \Omega^{a,b}}$  in (6.2) cannot be canceled against other terms. Indeed, consider for instance a function  $\Omega^{a,b} A_{ab}$ , where  $A_{ab}$  is built from 1-forms  $E_a$ ,  $E_\alpha$ ,  $\bar{E}_{\dot{\alpha}}$  and 0-forms of supermultiplet fields. Then the part of  $Q \Omega^{a,b} A_{ab}$  bilinear in  $\Omega$  contains three terms of the form  $\Omega^{a,c} \Omega_c^b A_{ab}$  which do not cancel. Similarly one proceeds with terms of higher orders in  $\Omega$ . More precisely, though nonlinear  $\Omega$ -dependent terms can be present, all of them can be removed by adding  $Q$ -exact terms, thus representing the trivial class of  $Q$ -cohomology.

Clearly, for  $\Omega$ -independent Lagrangian, the condition

$$Q_\Omega \mathcal{L} = 0 \tag{6.12}$$

implies that  $\mathcal{L}$  is Lorentz invariant, i.e. all indices are contracted with Lorentz-invariant flat metric and  $\sigma$ -matrices.

As a result  $Q$ -cohomology amounts to cohomology of the  $\Omega$ -independent part  $\hat{Q}$  of  $Q$ . It is convenient to introduce the following grading  $G$  of the background 1-forms

$$G(E_\alpha) = G(\bar{E}_{\dot{\alpha}}) = 2, \quad G(E_a) = 1. \tag{6.13}$$

Hence

$$G(Q_1) = 1, \quad G(Q_2^+) = G(Q_2^-) = 2, \quad G(Q_3) = 3. \tag{6.14}$$

According to subsection 2.3, nontrivial Lagrangians can be represented by cohomology  $H(Q_3)$ .

The computation considerably simplifies in spinor notations. The dictionary between vector and spinor indices is provided by  $\sigma$ -matrices. For example, for a Lorentz vector  $A_a$

$$A_a = \frac{1}{2}(\bar{\sigma}_a)^{\dot{\alpha}\alpha} A_{\alpha\dot{\alpha}}, \quad A_{\alpha\dot{\alpha}} = (\sigma_a)_{\alpha\dot{\alpha}} A^a.$$

In spinor notations

$$Q_1 = \frac{1}{2} E^{\alpha\dot{\alpha}} \hat{q}_{\alpha\dot{\alpha}}, \tag{6.15}$$

$$Q_3 = 2i \bar{E}^{\dot{\alpha}} E^\alpha \frac{\partial}{\partial E^{\alpha\dot{\alpha}}}. \tag{6.16}$$

An efficient tool to compute cohomology is provided by the homotopy lemma (see e.g. [35]). Consider  $V = \sum_{p=-\infty}^{\infty} \bigoplus V^p$  where linear spaces  $V^p$  are finite dimensional. Let  $Q$  be a grade one nilpotent operator

$$Q(V^p) \subset V^{p+1}, \quad Q^2 = 0,$$

and  $\tilde{Q}$  be a grade  $-1$  nilpotent operator

$$\tilde{Q}(V^p) \subset V^{p-1}, \quad \tilde{Q}^2 = 0.$$

The homotopy lemma states, that if the homotopy operator

$$\mathcal{H} := \{Q, \tilde{Q}\} \tag{6.17}$$

is diagonalizable in  $V$ , then cohomology  $H(Q, V) \subset \text{Ker } \mathcal{H}$ . Indeed, let  $v \in V$  be a  $Q$ -closed eigenvector of  $\mathcal{H}$

$$\mathcal{H}v = \lambda v, \quad Qv = 0 \tag{6.18}$$

with  $\lambda \neq 0$ . Then  $v$  is  $Q$ -exact because

$$v = \lambda^{-1} \mathcal{H}v = Q\beta, \quad \beta = \lambda^{-1} \tilde{Q}v. \tag{6.19}$$

Hence, only  $v$ , that are eigenvectors of  $\mathcal{H}$  with zero eigenvalue, can belong to  $H(Q, V)$ .

To apply this technique let us introduce the operator

$$\tilde{Q}_3 = \frac{1}{2i} E^{\alpha\dot{\alpha}} \frac{\partial^2}{\partial \bar{E}^{\dot{\alpha}} \partial E^\alpha}, \quad (\tilde{Q}_3)^2 = 0, \quad G(\tilde{Q}_3) = -3. \quad (6.20)$$

The homotopy operator associated with (6.16) and (6.20) is

$$\mathcal{H} = E^\alpha \bar{E}^{\dot{\alpha}} \frac{\partial^2}{\partial \bar{E}^{\dot{\alpha}} \partial E^\alpha} + E^{\alpha\dot{\alpha}} \frac{\partial}{\partial E^{\alpha\dot{\alpha}}} + E^{\beta\dot{\alpha}} E^\alpha \frac{\partial^2}{\partial E^\beta \partial E^{\alpha\dot{\alpha}}} + E^{\alpha\dot{\beta}} \bar{E}^{\dot{\alpha}} \frac{\partial^2}{\partial \bar{E}^{\dot{\beta}} \partial E^{\alpha\dot{\alpha}}}. \quad (6.21)$$

As explained in section 3, in supersymmetric models one can look for different types of unfolded superLagrangians, depending on whether they are polynomials or distributions with respect to commuting odd differentials  $d\theta$  or, in terms of gauge fields, with respect to gravitino 1-forms  $E^\alpha$  and  $\bar{E}^{\dot{\alpha}}$ . In the both cases  $\mathcal{H}$  has zero  $G$ -grade, and is diagonalizable so that the homotopy lemma applies.

It is convenient to characterize expressions in question by the number of 1-forms  $E^{\alpha\dot{\alpha}}$  they contain. Being anticommutative,  $E^{\alpha\dot{\alpha}}$  can appear only in the following combinations:  $E^{\alpha\dot{\alpha}} X_{\alpha\dot{\alpha}}$ ,  $E^\alpha_{\dot{\alpha}} E^{\beta\dot{\alpha}} X_{\alpha\beta} + \text{h.c.}$ ,  $E^\alpha_{\dot{\alpha}} E^{\beta\dot{\alpha}} E_{\beta\dot{\beta}} X_{\alpha\dot{\beta}}$ ,  $E^\alpha_{\dot{\alpha}} E^{\beta\dot{\alpha}} E_{\beta\dot{\beta}} E_{\alpha\dot{\beta}} X$ , where all  $X$  are  $E^{\alpha\dot{\alpha}}$ -independent. Note that for the last term in (6.21) the following relations hold

$$E^{\alpha\dot{\beta}} \bar{E}^{\dot{\alpha}} \frac{\partial^2}{\partial \bar{E}^{\dot{\beta}} \partial E^{\alpha\dot{\alpha}}} (E^\gamma_{\dot{\gamma}} E^{\delta\dot{\gamma}} \bar{E}^{\dot{\delta}}) = E^\gamma_{\dot{\gamma}} E^{\delta\dot{\gamma}} \bar{E}^{\dot{\delta}}, \quad (6.22)$$

$$E^{\alpha\dot{\beta}} \bar{E}^{\dot{\alpha}} \frac{\partial^2}{\partial \bar{E}^{\dot{\beta}} \partial E^{\alpha\dot{\alpha}}} \delta^2(\bar{E}_{\dot{\delta}}) E^{\gamma\dot{\gamma}} = -2\delta^2(\bar{E}_{\dot{\delta}}) E^{\gamma\dot{\gamma}}, \quad (6.23)$$

using (A.6) and that  $E^\gamma_{\dot{\gamma}} E^{\delta\dot{\gamma}}$  is symmetric in  $\gamma$  and  $\delta$  due to anticommutativity of  $E^{\alpha\dot{\alpha}}$ . Derivative of  $\delta$ -function is defined as usual via  $\bar{E}_{\dot{\gamma}} \delta'_{\dot{\beta}}(\bar{E}_{\dot{\alpha}}) = \epsilon_{\dot{\gamma}\dot{\beta}} \delta^2(\bar{E}_{\dot{\alpha}})$ . Analogous relations hold for the third term in (6.21).

Now let us use the homotopy lemma to compute  $Q_3$ -cohomology in the class of real superforms polynomial in  $E^\alpha$  and  $\bar{E}^{\dot{\alpha}}$ . Depending on the number of  $E^{\alpha\dot{\alpha}}$ , there are five options:

- $\Lambda_0 = E^\alpha \dots E^\alpha \bar{E}^{\dot{\alpha}} \dots \bar{E}^{\dot{\alpha}} \lambda_{\alpha(m),\dot{\alpha}(n)} + \text{h.c.}$ , where  $\lambda_{\alpha(m),\dot{\alpha}(n)}$  are 0-forms symmetric over indices  $\alpha$  and  $\dot{\alpha}$ . Here we have

$$\mathcal{H}\Lambda_0 = mn\Lambda_0 + \text{h.c.} = 0, \quad (6.24)$$

which is true when  $m = 0$  or  $n = 0$ , i.e.

$$\Lambda_0 = E^\alpha \dots E^\alpha \lambda_{\alpha(m)} + \text{h.c.} \quad (6.25)$$

- $\Lambda_1 = E^{\beta\dot{\beta}} E^\alpha \dots E^\alpha \bar{E}^{\dot{\alpha}} \dots \bar{E}^{\dot{\alpha}} \lambda_{\beta,\alpha(m),\dot{\beta},\dot{\alpha}(n)} + \text{h.c.}$

$$\begin{aligned} \mathcal{H}\Lambda_1 &= mn\Lambda_1 + \Lambda_1 + mE^{\alpha\dot{\beta}} E^\beta E^\alpha \dots E^{\alpha_m} \bar{E}^{\dot{\alpha}} \dots \bar{E}^{\dot{\alpha}} \lambda_{\beta,\alpha(m),\dot{\beta},\dot{\alpha}(n)} \\ &\quad + nE^{\beta\dot{\alpha}} E^\alpha \dots E^\alpha \bar{E}^{\dot{\beta}} \bar{E}^{\dot{\alpha}} \dots \bar{E}^{\dot{\alpha}} \lambda_{\beta,\alpha(m),\dot{\beta},\dot{\alpha}(n)} + \text{h.c.} = 0. \end{aligned} \quad (6.26)$$

This equation has nontrivial solutions if  $\beta$  is antisymmetrized with some  $\alpha_1$  in  $\lambda_{\beta,\alpha(m),\dot{\beta},\dot{\alpha}(n)}$  (and similarly for dotted indices). This gives the following solutions

$$\Lambda_1^1 = E^{\alpha\dot{\alpha}} E_\alpha \bar{E}_{\dot{\alpha}} \lambda, \quad (6.27)$$

$$\Lambda_1^2 = E^{\beta\dot{\alpha}} E_\beta E^\alpha \dots E^\alpha \lambda_{\alpha(m),\dot{\alpha}} + \text{h.c.} \quad (6.28)$$

- $\Lambda_2 = E^\beta{}_{\dot{\beta}} E^{\gamma\dot{\beta}} E^\alpha \dots E^\alpha \bar{E}^{\dot{\alpha}} \dots \bar{E}^{\dot{\alpha}} \lambda_{\beta\gamma,\alpha(m),\dot{\alpha}(n)} + \text{h.c.}$

$$\mathcal{H}\Lambda_2 = mn\Lambda_2 + 2\Lambda_2 + 2mE^\alpha{}_{\dot{\beta}} E^{\gamma\dot{\beta}} E^\beta E^\alpha \dots E^\alpha \bar{E}^{\dot{\alpha}} \dots \bar{E}^{\dot{\alpha}} \lambda_{\beta\gamma,\alpha(m),\dot{\alpha}(n)} + n\Lambda_2 + \text{h.c.} = 0. \quad (6.29)$$

Again, this equation has nontrivial solutions only if  $\lambda_{\beta,\gamma,\alpha(m),\dot{\alpha}(n)}$  is antisymmetrized over  $(\beta, \alpha_1)$  and  $(\gamma, \alpha_2)$ , i.e.

$$\Lambda_2 = E^\beta{}_{\dot{\alpha}} E^{\gamma\dot{\alpha}} E_\beta E_\gamma E^\alpha \dots E^\alpha \lambda_{\alpha(m)} + \text{h.c.} \quad (6.30)$$

- $\Lambda_3 = E^\beta{}_{\dot{\gamma}} E^{\gamma\dot{\gamma}} E_\gamma{}^{\dot{\beta}} E^\alpha \dots E^\alpha \bar{E}^{\dot{\alpha}} \dots \bar{E}^{\dot{\alpha}} \lambda_{\beta,\alpha(m),\dot{\beta},\dot{\alpha}(n)} + \text{h.c.}$

$$\begin{aligned} \mathcal{H}\Lambda_3 &= mn\Lambda_3 + 3\Lambda_3 + mE^\alpha{}_{\dot{\gamma}} E^{\gamma\dot{\gamma}} E_\gamma{}^{\dot{\beta}} E^\beta E^\alpha \dots E^\alpha \bar{E}^{\dot{\alpha}} \dots \bar{E}^{\dot{\alpha}} \lambda_{\beta,\alpha(m),\dot{\beta},\dot{\alpha}(n)} + m\Lambda_3 \\ &\quad + nE^\beta{}_{\dot{\gamma}} E^{\gamma\dot{\gamma}} E_\gamma{}^{\dot{\alpha}} E^\alpha \dots E^\alpha \bar{E}^{\dot{\beta}} \bar{E}^{\dot{\alpha}} \dots \bar{E}^{\dot{\alpha}} \lambda_{\beta,\alpha(m),\dot{\beta},\dot{\alpha}(n)} + n\Lambda_3 + \text{h.c.} = 0. \end{aligned} \quad (6.31)$$

This equation admits no nontrivial solutions.

- $\Lambda_4 = E^\beta{}_{\dot{\beta}} E^{\gamma\dot{\beta}} E_{\gamma\dot{\gamma}} E_{\beta\dot{\gamma}} E^\alpha \dots E^\alpha \bar{E}^{\dot{\alpha}} \dots \bar{E}^{\dot{\alpha}} \lambda_{\alpha(m),\dot{\alpha}(n)} + \text{h.c.}$  From (6.22) we find

$$\mathcal{H}\Lambda_4 = mn\Lambda_4 + 4\Lambda_4 + 2m\Lambda_4 + 2n\Lambda_4. \quad (6.32)$$

Hence,  $\mathcal{H}\Lambda_4 = 0$  admits no nontrivial solutions.

Straightforward calculation shows that expressions (6.25), (6.27), (6.28) and (6.30) are  $Q_3$ -closed. Thus cohomology of  $Q_3$  in the class of superforms is contained in

$$H_1 = E^\alpha \dots E^\alpha \lambda_{\alpha(m)} + \text{h.c.} \quad (6.33)$$

$$H_2 = E^{\alpha\dot{\alpha}} E_\alpha \bar{E}_{\dot{\alpha}} \lambda, \quad (6.34)$$

$$H_3 = E^{\beta\dot{\alpha}} E_\beta E^\alpha \dots E^\alpha \lambda_{\alpha(m),\dot{\alpha}} + \text{h.c.} \quad (6.35)$$

$$H_4 = E^\beta{}_{\dot{\alpha}} E^{\gamma\dot{\alpha}} E_\beta E_\gamma E^\alpha \dots E^\alpha \lambda_{\alpha(m)} + \text{h.c.} \quad (6.36)$$

According to section 3 nontrivial Lagrangians can be associated with the 4-superforms from  $H(Q_3)$  which have the form

$$\mathbb{L}_8 = E^\alpha E^\alpha E^\alpha E^\alpha \ell_{\alpha(4)} + \text{h.c.}, \quad (6.37)$$

$$\mathbb{L}_7 = E^{\beta\dot{\alpha}} E_\beta E^\alpha E^\alpha \ell_{\alpha(2),\dot{\alpha}} + \text{h.c.}, \quad (6.38)$$

$$\mathbb{L}_6 = E^\alpha{}_{\dot{\alpha}} E^{\beta\dot{\alpha}} E_\alpha E_\beta \ell + \text{h.c.}, \quad (6.39)$$

where  $G(\mathbb{L}_i) = i$  and  $\ell_{\alpha(4)}$ ,  $\ell_{\alpha(2),\dot{\alpha}}$ ,  $\ell$  are built from the supermultiplet fields. However, it can be shown that (6.37) and (6.38) do not lead to  $Q$ -closed expressions. Skipping details, here the phenomenon mentioned in subsection 2.3 occurs, namely  $Q_1\mathbb{L}_8$  and  $Q_1\mathbb{L}_7$  belong to nontrivial cohomology of  $Q_2^+$ , that obstructs the reconstruction of the full Lagrangian.

So the only candidate for Lagrangians is (6.39). To single out trivial highest grade terms among (6.39) consider the following 3-forms from  $H(Q_3)$  (6.33)–(6.36)

$$\mathcal{F}_6 = E^\alpha E^\alpha E^\alpha f_{\alpha(3)} + \text{h.c.}, \quad (6.40)$$

$$\mathcal{F}_5^1 = E^{\alpha\dot{\alpha}} E_\alpha \bar{E}_{\dot{\alpha}} f, \quad (6.41)$$

$$\mathcal{F}_5^2 = E^{\alpha\dot{\alpha}} E_\alpha E^\beta f_{\beta\dot{\alpha}} + \text{h.c.} \quad (6.42)$$

where  $G(\mathcal{F}_i) = i$  and  $f$  are 0-forms built from the supermultiplet fields. As shown in subsection 2.3, trivial Lagrangians are described by solutions of the system (2.29)–(2.32) for (6.39) and some  $\mathcal{F}$  (6.40)–(6.42).

First, we observe that (6.40) cannot contribute since its grade  $G(\mathcal{F}_6) = 6$  is the same as  $\mathbb{L}_6$  (6.39), so that the system (2.29)–(2.32) admits no solutions since all  $Q_i$  have positive  $G$ -grades. For  $\mathcal{F}_5^1$  (6.41) and  $\mathcal{F}_5^2$  (6.42) with  $G(\mathcal{F}_5^1) = G(\mathcal{F}_5^2) = 5$  the system (2.29)–(2.32) takes the form

$$Q_3 \mathcal{F}_5^i = 0, \quad (6.43)$$

$$(Q_2^+ + Q_2^-) \mathcal{F}_5^i + Q_3 \mathcal{F}_4^i = 0, \quad (6.44)$$

$$Q_1 \mathcal{F}_5^i + (Q_2^+ + Q_2^-) \mathcal{F}_4^i + Q_3 \mathcal{F}_3^i = E^\alpha{}_{\dot{\alpha}} E^{\beta\dot{\alpha}} E_\alpha E_\beta \ell + \text{h.c.} \quad (6.45)$$

where  $i = 1, 2$ .

Eq. (6.43) is satisfied since  $\mathcal{F}_5^1$  and  $\mathcal{F}_5^2$  are  $Q_3$ -closed. For  $\mathcal{F}_5^1$ , eq. (6.44) admits a solution

$$\mathcal{F}_4^1 = \frac{i}{\sqrt{2}} E^{\alpha\dot{\alpha}} E^\beta{}_{\dot{\alpha}} \hat{q}_\beta f + \text{h.c.} \quad (6.46)$$

However (6.45) admits no nonzero solutions for  $\mathcal{F}_5^1$  (6.41) and  $\mathcal{F}_4^1$  (6.46) because its l.h.s. contains among others the terms of the form

$$i E^{\alpha\dot{\alpha}} E^\beta{}_{\dot{\alpha}} \bar{E}^{\dot{\beta}} E_\alpha \hat{q}_{\dot{\beta}} \hat{q}_\beta f + i E^{\alpha\dot{\alpha}} E_\alpha{}^{\dot{\beta}} E^\beta \bar{E}_{\dot{\alpha}} \hat{q}_\beta \hat{q}_{\dot{\beta}} f$$

that are not present on the r.h.s. of (6.45) and cannot be compensated by any  $Q_3 \mathcal{F}_3^1$  due to (6.11) which in spinor notations reads as

$$\{\hat{q}^\alpha, \hat{q}^{\dot{\alpha}}\} = -i \hat{q}^{\alpha\dot{\alpha}}. \quad (6.47)$$

For  $\mathcal{F}_5^2$  (6.42), eq. (6.44) gives

$$\hat{q}_\alpha f_{\beta\dot{\alpha}} = 0, \quad \hat{q}_{\dot{\alpha}} \bar{f}_{\beta\alpha} = 0, \quad (6.48)$$

$$\mathcal{F}_4^2 = i\sqrt{2} E^\alpha{}_{\dot{\alpha}} E^{\beta\dot{\alpha}} E_\alpha \hat{q}^{\dot{\beta}} f_{\beta\dot{\beta}} + \text{h.c.} \quad (6.49)$$

Then (6.45) is solved by

$$\mathcal{F}_3^2 = -\frac{2}{3} E^\alpha{}_{\dot{\beta}} E^{\beta\dot{\beta}} E_\beta{}^{\dot{\alpha}} \hat{q}_{\dot{\gamma}} \hat{q}^{\dot{\gamma}} f_{\alpha\dot{\alpha}} + \text{h.c.}, \quad (6.50)$$

$$\ell = -\frac{1}{2} \hat{q}^{\alpha\dot{\alpha}} f_{\alpha\dot{\alpha}}. \quad (6.51)$$

The conclusion is that Lagrangians generated by  $\mathbb{L}$  (6.39) are trivial if  $\ell$  has the form (6.51) (analogously for  $\bar{\ell}$ ) with  $f_{\alpha\dot{\alpha}}$  obeying (6.48).

## 7 Lagrangians

### 7.1 Four-form Lagrangian

We look for a Lagrangian as a  $Q$ -closed 4-superform

$$\begin{aligned} \mathcal{L} = & E_a E_b \{ (\bar{\sigma}^{ab})^{\dot{\alpha}\dot{\beta}} \bar{E}_{\dot{\alpha}} \bar{E}_{\dot{\beta}} \ell_6 + (\sigma^{ab})^{\alpha\beta} E_{\alpha} E_{\beta} \bar{\ell}_6 \} \\ & + \epsilon^{abcd} E_a E_b E_c \{ \bar{E}_{\dot{\alpha}} (\bar{\sigma}_d)^{\dot{\alpha}\alpha} \ell_{5\alpha} + E^{\alpha} (\sigma_d)_{\alpha\dot{\alpha}} \bar{\ell}_5^{\dot{\alpha}} \} + E_a E_b E_c E_d \epsilon^{abcd} \ell_4, \end{aligned} \quad (7.1)$$

where 0-forms  $\ell_i$  are built Lorentz-covariantly from the supermultiplet fields  $C^{a(k)}$ ,  $\bar{C}^{a(k)}$ ,  $\chi^{\alpha(k)}$ ,  $\bar{\chi}_{\dot{\alpha}}^{a(k)}$ ,  $F^{a(k)}$ ,  $\bar{F}^{a(k)}$ . That this is the most general Lorentz-invariant 4-superform Ansatz follows from the results of section 6.2. (For instance, the term  $\epsilon^{abcd} E_a E_b E^{\alpha} (\sigma_c)_{\alpha\dot{\alpha}} \bar{E}^{\dot{\alpha}} \ell_d$ , that has the same  $G$ -grade as (6.39), is not added as not representing nonzero  $Q_3$ -cohomology.)

The equation  $\hat{Q}\mathcal{L} = 0$  can now be analyzed in different grade sectors, starting from the highest one. The full set of equations is presented in appendix B.

There are two complex conjugated equations in the highest grade  $G = 9$

$$2iE^{\gamma}(\sigma^c)_{\gamma\dot{\gamma}} \bar{E}^{\dot{\gamma}} \frac{\partial}{\partial E^c} E_a E_b (\bar{\sigma}^{ab})^{\dot{\alpha}\dot{\beta}} \bar{E}_{\dot{\alpha}} \bar{E}_{\dot{\beta}} \ell_6 = 0, \quad (7.2)$$

$$2iE^{\gamma}(\sigma^c)_{\gamma\dot{\gamma}} \bar{E}^{\dot{\gamma}} \frac{\partial}{\partial E^c} E_a E_b (\sigma^{ab})^{\alpha\beta} E_{\alpha} E_{\beta} \bar{\ell}_6 = 0. \quad (7.3)$$

These hold true for any  $\ell_6$  and  $\bar{\ell}_6$ , because the corresponding terms belong to  $H(Q_3)$ . Indeed, e.g. eq. (7.2) is proportional to  $\epsilon^{abcd} (\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}} (\bar{\sigma}_c)^{\dot{\gamma}\alpha} \bar{E}_{\dot{\alpha}} \bar{E}_{\dot{\beta}} \bar{E}_{\dot{\gamma}}$ . From relation (A.9) it follows that to be symmetric over three dotted spinor indices  $\epsilon^{abcd} (\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}} (\bar{\sigma}_c)^{\dot{\gamma}\alpha}$  must be antisymmetric with respect to the three undotted indices, which is zero because the latter take just two values.

In the grade  $G = 8$  we obtain four equations. From the first two

$$\bar{E}_{\dot{\gamma}} \hat{q}^{\dot{\gamma}} E_a E_b (\bar{\sigma}^{ab})^{\dot{\alpha}\dot{\beta}} \bar{E}_{\dot{\alpha}} \bar{E}_{\dot{\beta}} \ell_6 = 0, \quad (7.4)$$

$$E_{\gamma} \hat{q}^{\gamma} E_a E_b (\sigma^{ab})^{\alpha\beta} E_{\alpha} E_{\beta} \bar{\ell}_6 = 0, \quad (7.5)$$

we find that  $\ell_6$  and  $\bar{\ell}_6$  have to obey  $\hat{q}_{\dot{\alpha}} \ell_6 = 0$  and  $\hat{q}_{\alpha} \bar{\ell}_6 = 0$ , respectively. Using (A.9), one easily finds that the last two equations

$$2iE^{\gamma}(\sigma^e)_{\gamma\dot{\gamma}} \bar{E}^{\dot{\gamma}} \frac{\partial}{\partial E^e} \epsilon^{abcd} E_a E_b E_c \bar{E}_{\dot{\alpha}} (\bar{\sigma}_d)^{\dot{\alpha}\alpha} \ell_{5\alpha} + \sqrt{2} E_{\gamma} \hat{q}^{\gamma} E_a E_b (\bar{\sigma}^{ab})^{\dot{\alpha}\dot{\beta}} \bar{E}_{\dot{\alpha}} \bar{E}_{\dot{\beta}} \ell_6 = 0, \quad (7.6)$$

$$2iE^{\gamma}(\sigma^e)_{\gamma\dot{\gamma}} \bar{E}^{\dot{\gamma}} \frac{\partial}{\partial E^e} \epsilon^{abcd} E_a E_b E_c E^{\alpha} (\sigma_d)_{\alpha\dot{\alpha}} \bar{\ell}_5^{\dot{\alpha}} + \sqrt{2} \bar{E}_{\dot{\gamma}} \hat{q}^{\dot{\gamma}} E_a E_b (\sigma^{ab})^{\alpha\beta} E_{\alpha} E_{\beta} \bar{\ell}_6 = 0 \quad (7.7)$$

are solved by  $\ell_{5\alpha} = \frac{\sqrt{2}}{6} \hat{q}_{\alpha} \ell_6$  and  $\bar{\ell}_{5\dot{\alpha}} = \frac{\sqrt{2}}{6} \hat{q}_{\dot{\alpha}} \bar{\ell}_6$ .

Continuation of this analysis gives the following  $Q$ -closed superfield Lagrangian

$$\begin{aligned} \mathcal{L} = & E_a E_b (\bar{\sigma}^{ab})^{\dot{\alpha}\dot{\beta}} \bar{E}_{\dot{\alpha}} \bar{E}_{\dot{\beta}} W + E_a E_b (\sigma^{ab})^{\alpha\beta} E_{\alpha} E_{\beta} \bar{W} \\ & + \frac{\sqrt{2}}{6} \epsilon^{abcd} E_a E_b E_c \bar{E}_{\dot{\alpha}} (\bar{\sigma}_d)^{\dot{\alpha}\alpha} \hat{q}_{\alpha} W + \frac{\sqrt{2}}{6} \epsilon^{abcd} E_a E_b E_c E^{\alpha} (\sigma_d)_{\alpha\dot{\alpha}} \hat{q}^{\dot{\alpha}} \bar{W} \\ & + E_a E_b E_c E_d \epsilon^{abcd} \left( \frac{i\sqrt{2}}{16} \hat{q}_{\dot{\alpha}} \hat{q}^{\dot{\alpha}} \bar{W} - \frac{i\sqrt{2}}{16} \hat{q}^{\alpha} \hat{q}_{\alpha} W \right) \end{aligned} \quad (7.8)$$

where the 0-forms  $W$  and  $\bar{W}$  are arbitrary functions of  $C^{a(k)}$ ,  $\bar{F}^{a(k)}$  and  $\bar{C}^{a(k)}$ ,  $F^{a(k)}$ , respectively, so that  $\hat{q}^{\dot{\alpha}}W = 0$  and  $\hat{q}^{\alpha}\bar{W} = 0$ . By virtue of (5.24), (5.28) this means that  $W$  is chiral and  $\bar{W}$  is antichiral. Note that, as follows from (6.51),  $W$  of the form  $W = \hat{q}_a f^a$  lead to trivial Lagrangians.

By construction, Lagrangian (7.8) is manifestly supersymmetric and the corresponding action is independent of the local variation of the integration surface. A particular solution, that reproduces free Wess-Zumino action [5], results from (7.8) with  $W = i2\sqrt{2}C\bar{F}$ ,  $\bar{W} = -i2\sqrt{2}\bar{C}F$

$$\begin{aligned} \mathcal{L}^{\text{WZ}} = & i2\sqrt{2}E_a E_b (\bar{\sigma}^{ab})^{\dot{\alpha}\dot{\beta}} \bar{E}_{\dot{\alpha}} \bar{E}_{\dot{\beta}} C \bar{F} - i2\sqrt{2}E_a E_b (\sigma^{ab})^{\alpha\beta} E_{\alpha} E_{\beta} \bar{C} F \\ & + \frac{2i}{3} \epsilon^{abcd} E_a E_b E_c \bar{E}_{\dot{\alpha}} (\bar{\sigma}_d)^{\dot{\alpha}\alpha} [\chi_{\alpha} \bar{F} - iC (\bar{\sigma}_e)_{\alpha\dot{\beta}} \bar{\chi}^{\dot{\beta}}] \\ & + \frac{2i}{3} \epsilon^{abcd} E_a E_b E_c E^{\alpha} (\sigma_d)_{\alpha\dot{\alpha}} [\bar{\chi}^{\dot{\alpha}} F + i\bar{C} (\bar{\sigma}_e)^{\dot{\alpha}\beta} \chi_{\beta}^e] \\ & + E_a E_b E_c E_d \epsilon^{abcd} \left[ \frac{1}{2} C^e {}_e \bar{C} + \frac{1}{2} C \bar{C}^e {}_e + \frac{i}{2} \bar{\chi}_{\dot{\alpha}}^e (\bar{\sigma}_e)^{\dot{\alpha}\alpha} \chi_{\alpha} - \frac{i}{2} \bar{\chi}_{\dot{\alpha}} (\bar{\sigma}_e)^{\dot{\alpha}\alpha} \chi_{\alpha}^e + F \bar{F} \right]. \end{aligned} \quad (7.9)$$

An important comment is that, though the unfolded on-shell system from section 5 describes free dynamics of massless scalar supermultiplet, its off-shell modification represents just an infinite set of constraints. The form of these constraints is independent of whether the model is free or nonlinear. As a result, it can be used for description of a massive interacting theory. Nonlinear (starting from cubic) representatives of  $Q$ -cohomology determine Lagrangians with interactions. In particular, if  $W = W(C)$  depends only on  $C$  (respectively  $\bar{W} = \bar{W}(\bar{C})$ ), (7.8) describes the superpotential.

## 7.2 Lagrangian as integral form

As explained in section 3, a superspace Lagrangian can also be formulated as an integral form. This can be written as

$$\mathcal{L} = E_{a_1} \dots E_{a_m} \delta^2(E_{\alpha}) \delta^2(\bar{E}_{\dot{\alpha}}) \ell^{[a_1 \dots a_m]}, \quad (7.10)$$

where  $\ell^{[a_1 \dots a_m]}$  is some Lorentz-covariant 0-form built from the supermultiplet fields.

Applying the homotopy lemma, in spinor notations we have

$$\mathcal{H}(E_{\alpha_1 \dot{\alpha}_1} \dots E_{\alpha_m \dot{\alpha}_m} \delta^2(E_{\beta}) \delta^2(\bar{E}_{\dot{\beta}}) \ell^{\alpha_1 \dots \alpha_m, \dot{\alpha}_1 \dots \dot{\alpha}_m}) = 4\mathcal{L} + m\mathcal{L} - m\mathcal{L} - m\mathcal{L} = 0, \quad (7.11)$$

which implies  $m = 4$ . So the only nonzero cohomology of  $Q_3$  is

$$\mathcal{L} = E^{\alpha} {}_{\dot{\alpha}} E^{\beta \dot{\alpha}} E_{\beta \dot{\beta}} E_{\alpha}^{\dot{\beta}} \delta^2(E_{\gamma}) \delta^2(\bar{E}_{\dot{\gamma}}) \ell. \quad (7.12)$$

It is elementary to see that this Lagrangian is  $Q$ -closed.

To analyze whether or not such actions contain trivial parts one has to consider  $Q$ -images of the expressions containing derivatives of delta-functions. Indeed, for

$$\mathcal{F} = E^{\alpha} {}_{\dot{\alpha}} E^{\beta \dot{\alpha}} E_{\beta \dot{\beta}} E_{\alpha}^{\dot{\beta}} \delta'_{\delta} (E_{\gamma}) \delta^2(\bar{E}_{\dot{\gamma}}) f^{\delta} + \text{h.c.} \quad (7.13)$$

we obtain

$$Q\mathcal{F} = (Q_2^+ + Q_2^-)\mathcal{F} = \sqrt{2}E^\alpha{}_{\dot{\alpha}}E^{\beta\dot{\alpha}}E_{\beta\dot{\beta}}E_\alpha{}^{\dot{\beta}}\delta^2(E_\gamma)\delta^2(\bar{E}_{\dot{\gamma}})(\hat{q}^\delta f_\delta + \text{h.c.}). \quad (7.14)$$

As a result, (7.12) describes trivial Lagrangians if  $\ell = \hat{q}^\alpha f_\alpha + \text{h.c.}$  for some  $f_\alpha$ . In particular, Lagrangians with  $\ell = \hat{q}^{\alpha\dot{\alpha}} f_{\alpha\dot{\alpha}}$  are trivial because, as follows from (6.47), in this case

$$\ell = i\hat{q}^\alpha(\hat{q}^{\dot{\alpha}} f_{\alpha\dot{\alpha}}) + i\hat{q}^{\dot{\alpha}}(\hat{q}^\alpha f_{\alpha\dot{\alpha}}). \quad (7.15)$$

In tensor indices the Lagrangian reads as

$$\mathcal{L} = \epsilon^{abcd}E_aE_bE_cE_d\delta^2(E_\alpha)\delta^2(\bar{E}_{\dot{\alpha}})\ell. \quad (7.16)$$

The Lagrangian which reproduces the free Salam-Strathdee Lagrangian [5] is

$$\mathcal{L}^{\text{WB}} = \epsilon^{abcd}E_aE_bE_cE_d\delta^2(E_\alpha)\delta^2(\bar{E}_{\dot{\alpha}})(\bar{C}C), \quad (7.17)$$

as follows from the formula (5.24) which implies that  $C(z)$  is a chiral superfield.

To relate Lagrangians (7.16) and (7.8) one can choose the integration surface for (7.8) as

$$x^m = f^m(t^n), \quad \theta^\mu = \varphi^\mu(t^n), \quad \bar{\theta}^{\dot{\mu}} = \bar{\varphi}^{\dot{\mu}}(t^n), \quad (7.18)$$

where coordinates  $t^n$ ,  $n = 1, \dots, 4$  are even,  $f^m(t^n)$  are even and  $\varphi^\mu(t^n)$ ,  $\bar{\varphi}^{\dot{\mu}}(t^n)$  are odd. Extending  $t^n$  by odd variables  $\lambda^\mu$ ,  $\bar{\lambda}^{\dot{\mu}}$ ,  $\underline{\mu}, \underline{\dot{\mu}} = 1, 2$  so that  $\{t^n, \lambda^\mu, \bar{\lambda}^{\dot{\mu}}\}$  provide a full set of superspace coordinates, we extend (7.18) to

$$x^m = f^m + i\varphi^\mu(\sigma^m)_{\underline{\mu}\dot{\mu}}\bar{\lambda}^{\dot{\mu}} - i\lambda^\mu(\sigma^m)_{\underline{\mu}\dot{\mu}}\bar{\varphi}^{\dot{\mu}}, \quad \theta^\mu = \varphi^\mu + \lambda^\mu, \quad \bar{\theta}^{\dot{\mu}} = \bar{\varphi}^{\dot{\mu}} + \bar{\lambda}^{\dot{\mu}}. \quad (7.19)$$

Substitution of (7.19) into  $S = \int \mathcal{L}$  (7.16) and integration over  $\lambda^\mu, \bar{\lambda}^{\dot{\mu}}$  represents the action as an integral over the even surface (7.18) of the Lagrangian (7.8), where  $W = \hat{q}_{\dot{\alpha}}\hat{q}^{\dot{\alpha}}\ell$  and  $\bar{W} = \hat{q}^\alpha\hat{q}_\alpha\ell$  (plus  $Q$ -exact terms). In the process, 1-forms  $E_a, E_\alpha, \bar{E}_{\dot{\alpha}}$  from (7.16) get transformed into Cartesian 1-forms (4.19) in the resulting Lagrangian (7.8)

$$\begin{aligned} \tilde{E}^a &= df^a(t) + d\varphi^\mu(t)(i\bar{\varphi}^{\dot{\mu}}(t)(\sigma^a)_{\underline{\mu}\dot{\mu}}) + d\bar{\varphi}_{\dot{\mu}}(t)(i\varphi^\mu(t)(\sigma^a)_{\underline{\mu}\dot{\mu}}\epsilon^{\dot{\mu}\dot{\nu}}), \\ \tilde{E}^\alpha &= d\varphi^\alpha(t), \\ \tilde{\bar{E}}_{\dot{\alpha}} &= d\bar{\varphi}_{\dot{\alpha}}(t). \end{aligned} \quad (7.20)$$

For the free Salam-Strathdee Lagrangian (7.17), this gives the unfolded Wess-Zumino action with Lagrangian (7.9) hence showing their equivalence.

However, superpotentials are not represented in the integral form (7.16). This is expected since they represent chiral functions to be integrated over chiral subspaces [5]. In our approach such terms also are most conveniently represented in the form intermediate between the 4-form Lagrangians and integral-form Lagrangians, i.e. as integrals over chiral superspace.



### 7.3 Chiral superspace

To introduce superpotentials we introduce the following chiral integral forms,

$$\Lambda = \delta^2(E_\alpha) E_{a_1} \dots E_{a_m} \bar{E}_{\dot{\alpha}} \dots \bar{E}_{\dot{\alpha}} W^{[a_1 \dots a_m] \dot{\alpha}(n)}, \quad (7.21)$$

where  $W$  is the Lorentz-covariant 0-form built from chiral functions  $C^{a(k)}$  and  $\bar{F}^{a(k)}$  (so,  $Q_2^- \Lambda = 0$ ). Such forms are integrable over chiral superspace  $\mathbb{C}^{m+n|2}$  in a standard way described in section 3. Here  $\bar{E}_{\dot{\alpha}}$  without  $\delta$ -functions describe the pullback of the respective 1-forms to the chiral superspace.

Let us explore equation  $\mathcal{H}\Lambda_i = 0$  in spinor notations, where  $\Lambda_i$  from (7.21) contains  $i$  factors of  $E^{\alpha\dot{\alpha}}$ . For

$$\Lambda_0 = \delta^2(E_\alpha) \bar{E}^{\dot{\alpha}} \dots \bar{E}^{\dot{\alpha}} W_{\dot{\alpha}(m)}, \quad (7.22)$$

$$\Lambda_1 = \delta^2(E_\alpha) \bar{E}^{\dot{\alpha}} \dots \bar{E}^{\dot{\alpha}} E^{\beta\dot{\beta}} W_{\beta,\beta,\dot{\alpha}(m)}, \quad (7.23)$$

$$\Lambda_2 = \delta^2(E_\alpha) \bar{E}^{\dot{\alpha}} \dots \bar{E}^{\dot{\alpha}} (E^\beta_{\dot{\beta}} E^{\gamma\dot{\gamma}} W_{\beta\gamma,\dot{\alpha}(m)} + E^{\beta\dot{\beta}} E_{\beta\dot{\beta}}{}^{\dot{\gamma}} W_{\dot{\beta}\dot{\gamma},\dot{\alpha}(m)}), \quad (7.24)$$

$$\Lambda_3 = \delta^2(E_\alpha) E^{\beta\dot{\beta}} E_{\beta\dot{\beta}}{}^{\dot{\gamma}} E_{\gamma\dot{\gamma}}{}^{\dot{\beta}} \bar{E}^{\dot{\alpha}} \dots \bar{E}^{\dot{\alpha}} W_{\beta,\beta,\dot{\alpha}(m)}, \quad (7.25)$$

the equation  $\mathcal{H}\Lambda = 0$  has nontrivial solutions (with arbitrary 0-forms  $W$ ) only at  $m = 0$ . This is easy to understand by noting that, because  $\delta^2(E_\alpha) \bar{E}^{\dot{\alpha}} = Q_3(\frac{i}{4} \delta'_{\beta\dot{\beta}}(E_\alpha) E^{\beta\dot{\beta}})$  all such  $\Lambda$ , being  $Q_3$ -closed due to  $\delta$ -functions, are not  $Q_3$ -exact only if they are independent of  $\bar{E}^{\dot{\alpha}}$ .

However, for

$$\Lambda_4 = \delta^2(E_\alpha) E^{\beta\dot{\beta}} E_{\beta\dot{\beta}}{}^{\dot{\gamma}} E_{\gamma\dot{\gamma}}{}^{\dot{\beta}} E_{\beta\dot{\beta}}{}^{\dot{\gamma}} \bar{E}^{\dot{\alpha}} \dots \bar{E}^{\dot{\alpha}} W_{\dot{\alpha}(m)}, \quad (7.26)$$

$\mathcal{H}\Lambda_4 = -2m\Lambda_4 + 4\Lambda_4 - 4\Lambda_4 + 2m\Lambda_4 \equiv 0$ . Hence,  $\Lambda_4$  (7.26) can represent  $Q_3$ -cohomology at any  $m$ . Thus  $Q_3$ -cohomology in the class of chiral integral forms is represented by

$$\Lambda_0 = \delta^2(E_\alpha) W, \quad (7.27)$$

$$\Lambda_1 = \delta^2(E_\alpha) E^{\beta\dot{\beta}} W_{\beta\dot{\beta}}, \quad (7.28)$$

$$\Lambda_2 = \delta^2(E_\alpha) (E^\beta_{\dot{\beta}} E^{\gamma\dot{\gamma}} W_{\beta\gamma} + E^{\beta\dot{\beta}} E_{\beta\dot{\beta}}{}^{\dot{\gamma}} \bar{W}_{\dot{\beta}\dot{\gamma}}), \quad (7.29)$$

$$\Lambda_3 = \delta^2(E_\alpha) E^{\beta\dot{\beta}} E_{\beta\dot{\beta}}{}^{\dot{\gamma}} E_{\gamma\dot{\gamma}}{}^{\dot{\beta}} W_{\beta\dot{\beta}}, \quad (7.30)$$

$$\Lambda_4 = \delta^2(E_\alpha) E^{\beta\dot{\beta}} E_{\beta\dot{\beta}}{}^{\dot{\gamma}} E_{\gamma\dot{\gamma}}{}^{\dot{\beta}} E_{\beta\dot{\beta}}{}^{\dot{\gamma}} \bar{E}^{\dot{\alpha}} \dots \bar{E}^{\dot{\alpha}} W_{\dot{\alpha}(m)}. \quad (7.31)$$

To be integrated over  $\mathbb{C}^{4|2}$ , the only appropriate expression from (7.27)–(7.31) is

$$\mathcal{L}_0 = \delta^2(E_\alpha) E^{\beta\dot{\beta}} E_{\beta\dot{\beta}}{}^{\dot{\gamma}} E_{\gamma\dot{\gamma}}{}^{\dot{\beta}} E_{\beta\dot{\beta}}{}^{\dot{\gamma}} W. \quad (7.32)$$

Note that  $G(\delta^2(E_\alpha)) = -4$ , because the  $\delta$ -function has the degree of homogeneity  $-2$  and  $G(E_\alpha) = 2$ , as a result  $G(\mathcal{L}_0) = 0$ . To single out trivial Lagrangians, we write down eqs. (2.29)–(2.32) for

$$\mathcal{F}_{-1} = \delta^2(E_\alpha) E^{\beta\dot{\beta}} E_{\beta\dot{\beta}}{}^{\dot{\gamma}} E_{\gamma\dot{\gamma}}{}^{\dot{\beta}} f_{\beta\dot{\beta}}, \quad G(\mathcal{F}_{-1}) = -1, \quad (7.33)$$

which represents the only expression from those in (7.27)–(7.31), whose  $Q$ -image can contribute to (7.32). This gives

$$Q_3 \mathcal{F}_{-1} = 0, \tag{7.34}$$

$$(Q_2^+ + Q_2^-) \mathcal{F}_{-1} + Q_3 \mathcal{F}_{-2} = 0, \tag{7.35}$$

$$Q_1 \mathcal{F}_{-1} + (Q_2^+ + Q_2^-) \mathcal{F}_{-2} + Q_3 \mathcal{F}_{-3} = \delta^2(E_\alpha) E^\beta{}_{\dot{\beta}} E^{\gamma\dot{\beta}} E_{\gamma\dot{\gamma}} E_{\beta\dot{\gamma}} \dot{\gamma} W. \tag{7.36}$$

We immediately conclude that  $\mathcal{F}_{-2}$  with  $G = -2$  here can only have the form

$$\mathcal{F}_{-2} = \delta'_\delta(E_\alpha) E^\beta{}_{\dot{\beta}} E^{\gamma\dot{\beta}} E_{\gamma\dot{\gamma}} E_{\beta\dot{\gamma}} \dot{\gamma} f^\delta.$$

( $G(\delta'_\beta(E_\alpha)) = -6$  as follows from the definition  $E_\gamma \delta'_\beta(E_\alpha) = \epsilon_{\gamma\beta} \delta^2(E_\alpha)$  and  $G(\delta(E_\alpha)) = -4$ .) Eq. (7.34) is satisfied as  $\mathcal{F}_{-1} \in H(Q_3)$ . Term with  $\mathcal{F}_{-1}$  in (7.35) is zero because (7.33) contains  $\delta$ -function and  $f_{\beta\dot{\beta}}$  is built from  $C^{a(k)}$  and  $\bar{F}^{a(k)}$ , so  $\mathcal{F}_{-2} = 0$ . Finally, eq. (7.36) has a solution

$$\mathcal{F}_{-3} = 0, \quad W = \frac{1}{8} \hat{q}_{\alpha\dot{\alpha}} f^{\alpha\dot{\alpha}}.$$

We conclude that (7.32) describes nontrivial Lagrangians only if  $W \neq \hat{q}_{\alpha\dot{\alpha}} f^{\alpha\dot{\alpha}}$  for some  $f^{\alpha\dot{\alpha}}$ . Also, analogously to subsection 7.2, we have to consider expressions with derivatives of  $\delta$ -function whose  $Q$ -images can lead to trivial Lagrangians in (7.32). It is easy to see that the only appropriate elements from  $\text{Ker}(Q_3)$  are

$$\begin{aligned} K_1 &= \delta'_\beta(E_\alpha) \bar{E}^{\dot{\alpha}} \dots \bar{E}^{\dot{\alpha}} \bar{k}^{\beta}{}_{\dot{\alpha}(4)}, \\ K_2 &= \delta'_\beta(E_\alpha) E^{\gamma\dot{\alpha}} E_{\gamma\dot{\beta}} \bar{E}^{\dot{\alpha}} \bar{E}_{\dot{\beta}} \bar{k}^{\beta}. \end{aligned}$$

However the former has  $G(K_1) = 2$  while the latter has  $G(K_1) = 0$ . Since  $G(\mathcal{L}_0) = 0$  for (7.32), the system (2.29)–(2.32) admits no nonzero solutions in these cases.

Next, besides  $Q_3 \mathcal{L}_0 = 0$ ,  $\mathcal{L}_0$  (7.32) obeys

$$Q_2^- \mathcal{L}_0 = 0, \tag{7.37}$$

$$Q_2^+ \mathcal{L}_0 = 0, \tag{7.38}$$

$$Q_1 \mathcal{L}_0 = 0. \tag{7.39}$$

The first equation holds because  $W$  in (7.32) is built from chiral  $C^{a(k)}$  and  $\bar{F}^{a(k)}$ . The second one holds due to the  $\delta$ -function. The third one is true because  $\mathcal{L}_0$  contains the maximal number of  $E^{\alpha\dot{\alpha}}$ . So  $\mathcal{L}_0$  (7.32) is  $Q$ -closed and hence represents the general form of an unfolded chiral Lagrangian. (Note that (7.27)–(7.30) with nonzero  $W$  are not  $Q$ -closed, because they contain less than four  $E^{\alpha\dot{\alpha}}$  and hence are not annihilated by  $Q_1 = \frac{1}{2} E^{\alpha\dot{\alpha}} \hat{q}_{\alpha\dot{\alpha}}$ .)

In tensor notations, general chiral Lagrangian is

$$\mathcal{L} = \delta^2(E_\alpha) \epsilon^{abcd} E_a E_b E_c E_d W, \tag{7.40}$$

where Lorentz-invariant 0-form  $W$  is built from  $C^{a(k)}$  and  $\bar{F}^{a(k)}$ , and  $W \neq \hat{q}^a f_a$ . For the Lagrangian to be real, (7.40) should be supplemented by the complex conjugated expression to be integrated over antichiral superspace with the 0-form  $\bar{W}$  built from  $\bar{C}^{a(k)}$  and

$F^{a(k)}$ . Combination of (7.40) (plus conjugated expression) and (7.16) gives the unfolded superspace action of the interacting theory.

To reproduce superpotential of the Wess-Zumino model [5] we choose  $W = kC + \frac{m}{2}CC + \frac{g}{3}CCC$  with arbitrary constants  $k, m, g$ . Then superpotential takes the form

$$\mathcal{L} = \int \delta^2(E_\alpha)\epsilon^{abcd}E_aE_bE_cE_d\left(kC + \frac{m}{2}CC + \frac{g}{3}CCC\right) + \text{h.c.} \quad (7.41)$$

To write a full action containing both kinetic term and superpotential of the Wess-Zumino model in the chiral form we set  $W = -\frac{1}{16}C\bar{F} + kC + \frac{m}{2}CC + \frac{g}{3}CCC$ . This gives

$$S = \int \delta^2(E_\alpha)\epsilon^{abcd}E_aE_bE_cE_d\left(-\frac{1}{16}C\bar{F} + kC + \frac{m}{2}CC + \frac{g}{3}CCC\right) + \text{h.c.} \quad (7.42)$$

The kinetic term  $-\frac{1}{16}C\bar{F}$  in (7.42) results from integration of (7.17) over  $\bar{\theta}_\alpha$  taking into account that from (5.25)–(5.27) it follows that  $\bar{F} = 2\bar{D}\bar{D}\bar{C}$ .

As in section 7.2, one can map solution (7.40) to the 4-superform (7.8). The only difference is that instead of (7.19) even surfaces of the chiral superspace  $(x^m, \theta^\mu) \rightarrow (t^m, \lambda^\mu)$  are parametrized as

$$x^m = f^m(t) + i(\varphi^\mu(t) + \lambda^\mu)(\sigma^m)_{\mu\dot{\mu}}\bar{\varphi}^{\dot{\mu}}(t), \quad \theta^\mu = \varphi^\mu(t) + \lambda^\mu. \quad (7.43)$$

Integration over  $\lambda^\mu$ , gives Lagrangian (7.8) with the same function  $W$  as in (7.40). Analogously to section 7.2, the resulting Lagrangian is integrated over the surface (7.18) with 1-forms (7.20).

Expressions (7.8), (7.16) and (7.40) give the most general unfolded Lagrangians which can be written for the Wess-Zumino model as a 4-superform, integral form or a chiral integral form respectively. Besides the standard Wess-Zumino and Salam-Strathdee Lagrangians, they also describe the higher-derivative Lagrangians. Namely, by virtue of eqs. (5.21)–(5.23), the rank  $k$  tensors  $C^{a(k)}, \chi_\alpha^{a(k)}$  and  $F^{a(k)}$  describe the  $k$ -th derivatives of the dynamical fields  $C, \chi_\alpha$  and  $F$ . Plugging the higher-rank tensors into (7.8), (7.16) or (7.40) gives the higher-derivative unfolded actions. Unfolded Lagrangians contain all possible ordinary Lagrangians that can be written for the  $4d$  Wess-Zumino model. To obtain a conventional field-theoretic Lagrangian from the unfolded, one has to express all auxiliary fields  $C^{a(k)}, \bar{C}^{a(k)}, \chi_\alpha^{a(k)}, \bar{\chi}_{\dot{\alpha}}^{a(k)}, F^{a(k)}, \bar{F}^{a(k)}$  with  $k \geq 1$  in terms of the derivatives of the dynamical fields using unfolded equations (5.21)–(5.23), (5.25)–(5.27). Then, fixing an integration surface, the substitution of the resulting expressions for instance into (7.8) gives an ordinary space-time action.

In particular, doing this for (7.9) and choosing Minkowski space as an integration surface we see that the integral of (7.9) reproduces the component action of the free chiral supermultiplet [5]. Alternatively, one can use Lagrangians (7.17) or (7.42), arriving at the standard Salam-Strathdee superfield action. Generally, being manifestly supersymmetric the unfolded superform action leads directly to the component action.

## 8 Conclusion

In this paper, unfolded off-shell formulation of the free massless scalar supermultiplet is presented and the system of equations, that determines all superfield Lagrangians of the model, is derived and analyzed. The particular solutions leading to superfield actions of the Wess-Zumino model in the form of integrals of a 4-superform, integral form and chiral integral form are obtained. Explicit relations between these forms of superspace actions are established. It is shown in particular how usual superspace action for the Wess-Zumino model can be rewritten as an integral of a 4-superform.

In some sense, the construction of a chiral action is intermediate between the one of section 7.1 and that of section 7.2. In fact, this is a particular example of a very general phenomenon that the full action may have a form of integral over (super)manifolds of different dimensions. As long as even dimension is kept fixed, integration over supercoordinates will result in one or another space-time action. We expect that in more complicated theories like higher-spin theories and their further multiparticle extensions, invariant functionals resulting from integrals over space-times with different even dimensions may all contribute to the final result.

Being based on unfolded dynamics, the proposed method is most general, providing maximal flexibility in the construction of supersymmetric actions. Applied to on-shell unfolded system it provides a systematic tool for the analysis of on-shell counterterms in supersymmetric systems, the issue which was extensively studied during the recent years [22–27]. It would be interesting to explore its applications to more complicated models with extended SUSY and, in the first place, to the theories whose manifestly supersymmetric formulation is yet lacking, like  $\mathcal{N} = 1, D = 10$  or  $\mathcal{N} = 4, D = 4$  super Yang-Mills theories. It would also be interesting to clarify the relation of our approach to the harmonic superspace approach [36].

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## A Notations

We work with 4-dimensional Minkowski space with coordinates  $x^{\underline{m}}$ ,  $\underline{m} = 0 \dots 3$  and the superspace  $\mathbb{R}^{4|4}$  with coordinates  $z^{\underline{M}} = (x^{\underline{m}}, \theta^{\underline{\mu}}, \bar{\theta}^{\underline{\dot{\mu}}})$ ;  $\underline{m} = 0 \dots 3$ ;  $\underline{\mu}, \underline{\dot{\mu}} = 1, 2$ , which are denoted by the underlined letters from the middle of Latin and Greek alphabets. Supervielbeins  $E_a, E_\alpha, \bar{E}_{\dot{\alpha}}$  relate the base indices to the indices of the flat fiber space, denoted by the letters from the beginning of the respective alphabets:  $(x^a, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$ ;  $a = 0 \dots 3$ ;  $\alpha, \dot{\alpha} = 1, 2$ .

The fiber space Minkowski metric is  $\eta_{ab} = \text{diag}\{1, -1, -1, -1\}$ . We use condensed notations for symmetrized fiber indices writing  $a(k)$  instead of  $(a_1 \dots a_k)$ . Indices in brackets

$[a_1 \dots a_k]$  are antisymmetrized. Spinorial indices are raised and lowered by the matrices

$$\epsilon^{\alpha\beta} = \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}, \quad \epsilon_{\alpha\beta} = \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix}, \quad (\text{A.1})$$

$$\xi_\alpha = \epsilon_{\alpha\beta} \xi^\beta, \quad \xi^\alpha = \epsilon^{\alpha\beta} \xi_\beta, \quad \bar{\xi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \bar{\xi}_{\dot{\beta}}, \quad \bar{\xi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{\xi}^{\dot{\beta}}. \quad (\text{A.2})$$

$\sigma$ -matrices are

$$(\sigma^0)_{\alpha\dot{\beta}} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \quad (\sigma^1)_{\alpha\dot{\beta}} = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \quad (\sigma^2)_{\alpha\dot{\beta}} = \begin{vmatrix} 0 & -i \\ i & 0 \end{vmatrix}, \quad (\sigma^3)_{\alpha\dot{\beta}} = \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix}. \quad (\text{A.3})$$

Also we use anti-Hermitian matrices

$$\begin{aligned} (\sigma_{ab})_\alpha{}^\beta &= \frac{1}{2}((\sigma_a)_{\alpha\dot{\alpha}}(\bar{\sigma}_b)^{\dot{\alpha}\beta} - (\sigma_b)_{\alpha\dot{\alpha}}(\bar{\sigma}_a)^{\dot{\alpha}\beta}), \\ (\bar{\sigma}_{ab})^{\dot{\alpha}}{}_\beta &= \frac{1}{2}((\bar{\sigma}_a)^{\dot{\alpha}\alpha}(\sigma_b)_{\alpha\dot{\beta}} - (\bar{\sigma}_a)^{\dot{\alpha}\alpha}(\sigma_b)_{\alpha\dot{\beta}}). \end{aligned} \quad (\text{A.4})$$

Their tensorial indices are lowered/raised by Levi-Civita symbol

$$(\sigma^{ab})_\alpha{}^\beta = \frac{i}{2}\epsilon^{abcd}(\sigma_{cd})_\alpha{}^\beta, \quad (\bar{\sigma}^{ab})^{\dot{\alpha}}{}_\beta = -\frac{i}{2}\epsilon^{abcd}(\bar{\sigma}_{cd})^{\dot{\alpha}}{}_\beta. \quad (\text{A.5})$$

Following well-known relations are used

$$T_{\alpha\beta\gamma\dots} - T_{\beta\alpha\gamma\dots} = \epsilon_{\alpha\beta} T^\delta{}_{\delta\gamma\dots}, \quad (\text{A.6})$$

$$(\sigma^a)_{\alpha\dot{\alpha}}(\bar{\sigma}_a)^{\beta\dot{\beta}} = 2\delta_\alpha{}^\beta \delta_{\dot{\alpha}}{}^{\dot{\beta}}, \quad (\text{A.7})$$

$$(\sigma_a \bar{\sigma}_b)_\alpha{}^\beta = \eta_{ab} \delta_\alpha{}^\beta + (\sigma_{ab})_\alpha{}^\beta. \quad (\text{A.8})$$

Using them one obtains

$$(\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}} - (\sigma_b)_{\alpha\dot{\alpha}}(\sigma_a)_{\beta\dot{\beta}} = (\sigma_{ab})_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} + (\bar{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}} \epsilon_{\alpha\beta}, \quad (\text{A.9})$$

$$(\sigma_a)_{\alpha\dot{\alpha}}(\sigma_b)_{\beta\dot{\beta}} + (\sigma_b)_{\alpha\dot{\alpha}}(\sigma_a)_{\beta\dot{\beta}} = \eta_{ab} \epsilon_{\alpha\beta} \epsilon_{\dot{\alpha}\dot{\beta}} + (\sigma_{ca})_{\alpha\beta} (\bar{\sigma}_{cb})_{\dot{\alpha}\dot{\beta}}, \quad (\text{A.10})$$

$$(\sigma^a \bar{\sigma}^b \sigma^c)_{\alpha\dot{\alpha}} - (\sigma^c \bar{\sigma}^b \sigma^a)_{\alpha\dot{\alpha}} = 2i \epsilon^{abcd} (\sigma_d)_{\alpha\dot{\alpha}}. \quad (\text{A.11})$$

## B Equations for superform Lagrangian

Expanding  $\hat{Q}\mathcal{L} = 0$  for (7.1) and (6.4) into parts with different  $G$ -grade, one obtains a following chain of equations

$$G = 9 : \quad Q_3 [E_a E_b (\bar{\sigma}^{ab})^{\dot{\alpha}\dot{\beta}} \bar{E}_{\dot{\alpha}} \bar{E}_{\dot{\beta}} \ell_6] = 0,$$

$$Q_3 [E_a E_b (\sigma^{ab})^{\alpha\beta} E_\alpha E_\beta \bar{\ell}_6] = 0,$$

$$G = 8 : \quad Q_2^- [E_a E_b (\bar{\sigma}^{ab})^{\dot{\alpha}\dot{\beta}} \bar{E}_{\dot{\alpha}} \bar{E}_{\dot{\beta}} \ell_6] = 0,$$

$$Q_2^+ [E_a E_b (\sigma^{ab})^{\alpha\beta} E_\alpha E_\beta \bar{\ell}_6] = 0,$$

$$Q_3 [\epsilon^{abcd} E_a E_b E_c \bar{E}_{\dot{\alpha}} (\bar{\sigma}_d)^{\dot{\alpha}\alpha} \ell_{5\alpha}] + Q_2^+ [E_a E_b (\bar{\sigma}^{ab})^{\dot{\alpha}\dot{\beta}} \bar{E}_{\dot{\alpha}} \bar{E}_{\dot{\beta}} \ell_6] = 0,$$

$$Q_3 [\epsilon^{abcd} E_a E_b E_c E^\alpha (\sigma_d)_{\alpha\dot{\alpha}} \bar{\ell}_5^{\dot{\alpha}}] + Q_2^- [E_a E_b (\sigma^{ab})^{\alpha\beta} E_\alpha E_\beta \bar{\ell}_6] = 0.$$

$$\begin{aligned}
 G = 7 : \quad & Q_2^- [\epsilon^{abcd} E_a E_b E_c \bar{E}_{\dot{\alpha}} (\bar{\sigma}_d)^{\dot{\alpha}\alpha} \ell_{5\alpha}] + Q_1 [E_a E_b (\bar{\sigma}^{ab})^{\dot{\alpha}\beta} \bar{E}_{\dot{\alpha}} \bar{E}_{\dot{\beta}} \ell_6] = 0, \\
 & Q_2^+ [\epsilon^{abcd} E_a E_b E_c E^\alpha (\sigma_d)_{\alpha\dot{\alpha}} \bar{\ell}_5^{\dot{\alpha}}] + Q_1 [E_a E_b (\sigma^{ab})^{\alpha\beta} E_\alpha E_\beta \bar{\ell}_6] = 0, \\
 & Q_3 [E_a E_b E_c E_d \epsilon^{abcd} \ell_4] + Q_2^- [\epsilon^{abcd} E_a E_b E_c E^\alpha (\sigma_d)_{\alpha\dot{\alpha}} \bar{\ell}_5^{\dot{\alpha}}] \\
 & \quad + Q_2^+ [\epsilon^{abcd} E_a E_b E_c \bar{E}_{\dot{\alpha}} (\bar{\sigma}_d)^{\dot{\alpha}\alpha} \ell_{5\alpha}] = 0. \\
 G = 6 : \quad & Q_2^- [E_a E_b E_c E_d \epsilon^{abcd} \ell_4] + Q_1 [\epsilon^{abcd} E_a E_b E_c \bar{E}_{\dot{\alpha}} (\bar{\sigma}_d)^{\dot{\alpha}\alpha} \ell_{5\alpha}] = 0, \\
 & Q_2^+ [E_a E_b E_c E_d \epsilon^{abcd} \ell_4] + Q_1 [\epsilon^{abcd} E_a E_b E_c E^\alpha (\sigma_d)_{\alpha\dot{\alpha}} \bar{\ell}_5^{\dot{\alpha}}] = 0. \\
 G = 5 : \quad & Q_1 [E_a E_b E_c E_d \epsilon^{abcd} \ell_4] = 0.
 \end{aligned}$$

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