Quantum deformation of planar amplitudes

M. Movshev^a and A. Schwarz^b

^a*Stony Brook University,
Stony Brook, NY 11794-3651, U.S.A.*

^b*Department of Mathematics, University of California,
Davis, CA 95616, U.S.A.*

E-mail: mmovshev@math.sunysb.edu, schwarz@math.ucdavis.edu

ABSTRACT: In maximally supersymmetric four-dimensional gauge theories planar on-shell diagrams are closely related to the positive Grassmannian and the cell decomposition of it into the union of so called positroid cells. (This was proven by N. Arkani-Hamed, J. Bourjaily, F. Cachazo, A. Goncharov, A. Postnikov, and J. Trnka.) We establish that volume forms on positroids used to express scattering amplitudes can be q -deformed to Hochschild homology classes of corresponding quantum algebras. The planar amplitudes are represented as sums of contributions of some set of positroid cells; we quantize these contributions. In classical limit our considerations allow us to obtain explicit formulas for contributions of positroid cells to scattering amplitudes.

KEYWORDS: Extended Supersymmetry, Non-Commutative Geometry, Scattering Amplitudes, Supersymmetric Gauge Theory

ARXIV EPRINT: [1711.10053](https://arxiv.org/abs/1711.10053)

Contents

1	Cluster algebras	3
2	Quantum spaces	6
3	Quantum Grassmannian	7
4	Hochschild homology and cohomology	8
5	Hochschild homology of quantum spaces	11
6	Poisson and Hochschild homology	12
7	Contribution of positroid cell to scattering amplitudes	12
A	Hochschild homology and cohomology-details	13

It was realized recently that on-shell diagrams are objects of fundamental importance in the analysis of scattering amplitudes. In four-dimensional theories planar on-shell diagrams are closely related to the positive Grassmannian and the cell decomposition of it into the union of so called positroid cells [1]. (The most complete results were obtained for $\mathcal{N} = 4$ - supersymmetric Yang-Mills theory, but the relation to positive Grassmannian exists also in other cases.) From the other side it was discovered [19] that there exists a bijective correspondence between positroid cells in positive Grassmannian and prime ideals in quantum Grassmannian that are invariant with respect to the torus action. (This Grassmannian was constructed in the framework of quantum group theory.) We come to the idea that the theory of [1] can be quantized (more precisely, q -deformed); one can hope that the quantization can lead to physically interesting results. This idea is supported by the fact that the Grassmannian can be considered as cluster variety [10] with compatible Poisson structure and, as was shown in [9, 12] in this situation there exists a unique quantization procedure that should lead to the same quantum Grassmannian.

We prove some results in this direction. Namely, we establish that volume forms on positroids used in [1] can be q -deformed to Hochschild homology classes of corresponding quantum algebras. (It is well known that Hochschild homology classes can be considered as non-commutative deformation of differential forms. Notice, however, that so called twisted Hochschild homology also can be regarded as deformation of differential forms; our results can be applied to twisted Hochschild homology.)

This fact follows from much more general results that can be applied to quantum cluster algebras in the sense of [6] and to quantum spaces in the sense of [11].

We use the deformed volume form to q -deform the contribution of a positroid cell to scattering amplitude. Our construction can be used to simplify calculations also in the case of classical Grassmannian (see section 7).

The scattering amplitude can be described as a sum of contributions of some positroid cells. The nicest description of the amplitude can be given in terms of amplituhedron [2].

We hope that one can “quantize” results of [2, 3] applying our results to partial flag varieties. (It was shown in [10] that partial flag varieties can be considered as cluster varieties, see also [13].) The partial flag varieties arise naturally in the framework of [1] either as flags of subspaces of dimensions $2, m, n - 2$ or, in twistor formalism, as flags of subspaces of dimension $4, m$.

The physical interpretation of our results is not clear. It is possible that they are related to non-commutative gauge theories in the spirit of [8, 18, 26]. Another conjecture: quantizing planar on-shell diagrams with $q = e^{i\hbar}$ equal to the N -th root of unity (in other words $\hbar = \frac{2\pi}{N}$) we can obtain physical quantities associated with gauge theories with the gauge group $U(N)$. Here q is the parameter entering the definition of the quantum Grassmannian. (Recall that the planar diagrams correspond to the limit $N \rightarrow \infty$.) Due to existence of Seiberg-Witten map [26] these two conjectures can be true simultaneously; if this is the case the $U(N)$ gauge theory should interact with constant B -field. (Interaction with B -field can explain the violation of Lorentz invariance that should be present in $U(N)$ gauge theory.)

The second conjecture is supported by the remark that a similar statement is true for Chern-Simons theory. It is well known that Chern-Simons theory with gauge group $U(N)$ is equivalent to topological string with string coupling constant $g_s = \frac{2\pi}{k+N}$ [22, 31]. (Here k stands for the coefficient in front of Chern-Simons action functional—the level.) From the other side the topological string for any genus can be obtained from genus zero topological string by means of quantization and the string coupling constant plays the role of Planck constant [32]. This agrees with our conjecture. (Genus zero string theory corresponds to $N = \infty$ and planar diagrams.)

One more reason to expect that after the quantization of Grassmannian we should get $U(N)$ gauge theory is based on the following remark. The scattering amplitudes of s particles in $N = \infty$ case are functions on some manifold. (For example, in twistor formalism one can consider them as functions on the configuration space of s points on the manifold $\mathbb{CP}^3$.) After quantization of this manifold with q obeying $q^N = 1$ we should get functions depending of the points of the same manifold and additionally of some discrete variables. (This follows from the fact that for $q^N = 1$ the quantized algebra has a large center isomorphic to the original commutative algebra.) This agrees with the picture of $U(N)$ gauge theory where these discrete variables can be identified with N^2 “colors” for every particle. It is a non-trivial check of our conjecture that the number of discrete variables is precisely N^{2s} where s is the number of particles; under some assumptions we were able to confirm this statement.

Our considerations are based on consideration of quantum cluster algebras in the sense of [6] and of quantum spaces in the sense of [9, 12]. However, we do not need the full strength of the theory of cluster algebras and our exposition does not depend on this theory. We

introduce much simpler notions of poor man cluster algebra and poor man quantum space that are sufficient for our goals.

We will work with unital associative algebras over $\mathbb{C}$. Prime ideal of an algebra (or of a ring) $\mathcal{A}$ is a two-sided ideal $\mathcal{I} \neq \mathcal{A}$ such that for two ideals X, Y obeying $XY \subset \mathcal{I}$ at least one of these ideals is contained in $\mathcal{I}$. Completely prime ideal is a two-sided ideal $\mathcal{I} \neq \mathcal{A}$ such that for two elements x, y of the algebra $xy \in \mathcal{I}$ implies that at least one of these elements belongs to $\mathcal{I}$. For commutative algebras these two notions coincide, prime ideals of the algebras of polynomials correspond to subvarieties. In general every completely prime ideal is prime, but prime ideal is not necessarily completely prime. However, in algebras we are interested in these two notions coincide.

Every commutative ring $\mathcal{R}$ without zero divisors (entire ring) can be embedded in the so called field of fractions $\mathcal{F}$ (formal expressions of the form ab^{-1} with some identifications and natural operations). It is easy to check that any homomorphism of R into a field has a unique extension to F . For non-commutative rings without zero divisors we can take this property as a definition. We say that a skew field $\mathcal{F}$ is a skew field of fractions of the ring $\mathcal{A}$ if $\mathcal{A}$ is a subring of $\mathcal{F}$ and every homomorphism of $\mathcal{A}$ into a skew field can be extended in unique way to a homomorphism of $\mathcal{F}$. (Skew field is a non-commutative ring, where every non-zero element is invertible.) It is not always possible to construct the skew field of fractions, however, in many interesting situations it does exist. For example, it is sufficient to assume that $\mathcal{A}$ has no zero divisors and has at most polynomial growth, i.e. its Gelfand-Kirillov dimension (see [4] for definition) is finite ([5] Lemma A1). In particular, one can consider the fraction skew field of non-commutative torus T defined as the algebra with generators $x_i, x_i^{-1}, 1 \leq i \leq n$ satisfying $x_j x_i = q_{ij} x_i x_j, x_i x_i^{-1} = x_i^{-1} x_i = 1$).

1 Cluster algebras

Let us consider an associative unital algebra $\mathcal{A}$ with generators $\{A_i\}$. We denote the set of generators $\{A_i\}$ by $\mathcal{S}$; we use the same notation for the corresponding set of indices. Let us fix a subset $F \subset \mathcal{S}$. We assume that the generators in F quasi-commute, i.e.

$$A_i A_l = q_{il} A_l A_i.$$

The set F is called the set of frozen variables. We say that a finite set $K \subset \mathcal{S}$ is a cluster if the generators $A_i \in K$ quasi-commute, $K \supset F$ and all other quasi-commuting systems of generators containing F have cardinality $\leq |K|$. We denote $|K|$ (the number of elements in a cluster) by k . Algebra generated by $A_i \in K$ will be denoted by $\mathbf{K}$.

We can consider an extension $\tilde{\mathcal{A}}$ of the algebra $\mathcal{A}$ where all generators A_i are invertible elements. Then for every cluster $K = \{A_{i_1}, \dots, A_{i_k}\}$ we can consider a subalgebra of $\tilde{\mathcal{A}}$ generated by $A_{i_1}, \dots, A_{i_k}, A_{i_1}^{-1}, \dots, A_{i_k}^{-1}$ (a non-commutative torus denoted by T).

Let us suppose that the subset L of cluster K_1 contains $k - 1$ elements, hence one can obtain K_1 from L adding a generator A_n . We assume that $L \supset F$. We will describe how one can obtain a new cluster K_2 from L adding a generator A_m (in other words $K_1 = L \cup \{A_n\}, K_2 = L \cup \{A_m\}$). We will consider monomials with respect to L , i.e. expressions of the form $r \prod A_i^{\alpha_i}$ where $r \in \mathbb{C}$. (We assume that the product as well as the

products below run over L . We should fix the order of factors in some way, the change of the order can be absorbed into the change of the constant factor r .) We say that the transition from K_1 to K_2 is a mutation (more precisely an A-mutation) in the direction A_n if $A_m A_n$ is a sum of two monomials with respect to elements of L

$$A_m A_n = r \prod A_i^{\alpha_i} + s \prod A_i^{\beta_i} = rV + sU \quad (1.1)$$

and the following conditions are satisfied

- a) α^i, β^i are non-negative integers obeying $\alpha^i \beta^i = 0$ (i.e. each factor A_i enters only in one of the monomials),
- b) For $j \in K_1, j \neq n$ (i.e. $j \in L$)

$$\begin{aligned} \prod q_{ji}^{\alpha_i} &= \prod q_{ji}^{\beta_i}, \text{ but} \\ \prod q_{ni}^{\alpha_i} &\neq \prod q_{ni}^{\beta_i}. \end{aligned} \quad (1.2)$$

It is easy to check that these conditions guarantee that the element A_m defined by the formula (1.1) in terms of the elements of the cluster K_1 quasicommutates with $A_i, i \in L$. If A_m is a generator of $\mathcal{A}$ we get a new cluster K_2 .

We will be writing $B = VU^{-1}$ for an element $\prod A_i^{\alpha_i} (\prod A_i^{\beta_i})^{-1} \in T_1$ in the noncommutative torus. We can express conditions (1.2) saying that B commutes with all $A_i \in L$, but does not commute with A_n (i.e. B does not belong to the center of the non-commutative torus T_1).

If the clusters K_1 and K_2 are related by mutation then they are embedded in the same non-commutative torus: $T_1 = T_2$. We will embed this torus into its skew field of fractions denoted by $\mathcal{F}$. There exists a unique homomorphism ϕ_B of the torus into $\mathcal{F}$ obeying $\phi_B(A_i) = A_i$ for $A_i \in L$, $\phi_B(A_n) = A_n(r + sB)^{-1}$, $\phi_B(A_i^{-1}) = A_i^{-1}$ for $A_i \in L$, $\phi_B(A_n^{-1}) = (r + sB)A_n^{-1}$. (The homomorphism is unique, because it is defined on generators of T_1 , it exists because the above formulas preserve the commutation relations between generators.) It can be extended in unique way to an automorphism of $\mathcal{F}$ denoted by the same symbol.

It is easy to check that

$$\phi_B(UA_n^{-1}) = (rV + sU)A_n^{-1} = A_m. \quad (1.3)$$

We say that $\mathcal{A}$ is a poor man cluster algebra if

- a) Every quasi-commuting family of generators containing F belongs to a cluster;
- b) One can get any cluster from any other cluster by means of a sequence of mutations.

Let us assume that the defining relations of the algebra $\mathcal{A}$ depend on parameter q (e.g. q_{ij} are powers of q) in such a way that for $q = 1$ we obtain a commutative algebra. It is convenient to assume that as we vary q linear spaces $\mathcal{A}_a := \mathcal{A}|_{q=a}$ form a continuous family. Technically it means that $\mathcal{A}$ is flat over $\mathbb{C}[q, q^{-1}]$. Under certain conditions $\mathcal{A}_1$ can

be identified with the algebra of polynomial homogeneous functions on a projective variety P . Then we write the quasi-commutation relations in the form $A_j A_l = q^{\lambda_{jl}} A_l A_j$ where $\lambda_{jl} = -\lambda_{lj} \in \mathbb{Z}$ and the mutation should have the form

$$A_m A_n = r q^a \prod A_i^{\alpha_i} + s q^b \prod A_i^{\beta_i}.$$

The conditions imposed on the monomials in this formula will be specified later (formula (1.4)). It is convenient to work in non-commutative torus generated by $A_{i_1}, \dots, A_{i_k}, A_{i_1}^{-1}, \dots, A_{i_k}^{-1}$. We will be writing $M(\rho), \rho = (\rho^1, \dots, \rho^k) \in \mathbb{Z}^k$ for the elements

$$M(\rho) = q^{\frac{1}{2} \sum_{l < i} \lambda_{il} \rho^i \rho^l} A_{i_1}^{\rho^1} \dots A_{i_k}^{\rho^k}.$$

These elements coincide with A_{i_j} when $\rho = e_j$ is an element of the standard basis for $\mathbb{Z}^k$ and satisfy the following relations

$$M(\rho) M(\sigma) = q^{\frac{1}{2} \Lambda(\rho, \sigma)} M(\rho + \sigma),$$

(Here $\Lambda(\rho, \sigma) = \sum_{j,l=1}^k \lambda_{jl} \rho^j \sigma^l$.)

In these notations

$$A_m = \text{const} M(-e_n + \alpha) + \text{const} M(-e_n + \beta)$$

where $\alpha = \sum_{i=1}^k \alpha^i e_i, \beta = \sum_{i=1}^k \beta^i e_i, \alpha^n = \beta^n = 0, \alpha^i \geq 0, \beta^i \geq 0$ and for every i either α^i or β^i vanishes.

We introduce the notation $b = \alpha - \beta$ and impose the conditions

$$\Lambda(b, e_i) = 0 \text{ for } i \neq n, \Lambda(b, e_n) = -d(b). \quad (1.4)$$

Here $d(b)$ denotes the minimal positive value for $\Lambda(b, \rho)$ where ρ runs over $\mathbb{Z}^k$.

In the construction of the automorphism $\phi_B : \mathcal{F} \rightarrow \mathcal{F}$ we use the same formulas with $B = M(b)$.

Notice that in the limit $q \rightarrow 1$ we obtain Poisson structure on the variety P and on classical limits of clusters, these Poisson structures agree. (The embedding of the algebra generated by cluster into the algebra $\mathcal{A}$ induces a Poisson map.)

A quantum cluster algebra in the sense of [6] can be described as poor man cluster algebra where a mutation exists in the direction of any non-frozen element of the cluster. In the construction of such an algebra Berenstein and Zelevinsky start with integer-valued skew-symmetric form Λ on $\mathbb{Z}^k$. Then they assume that there exists a solution $b_j \in \mathbb{Z}^k$ of the equation

$$\Lambda(b_j, e_l) = -\delta_{lj} d(b_l)$$

for every non-frozen variable. (As earlier $d(b)$ denotes the minimal positive value for $\Lambda(b, \rho)$ where ρ runs over $\mathbb{Z}^k$.) This allows them to define a mutation in every non-frozen direction e_j using the formula

$$A_m = \text{const} M(-e_j + \alpha_j) + \text{const} M(-e_j + \beta_j)$$

where α_j stands for the positive part of b_j and β_j stands for its negative part.

Let us consider prime ideals in $\mathcal{A}$ that are generated by subsets of the set $\mathcal{S}$ (set of generators of $\mathcal{A}$). For every ideal we consider the subset of $\mathcal{S}$ belonging to this ideal; such subsets will be called ideal subsets (of course, an ideal subset generates the corresponding ideal, but the ideal can be generated also by smaller subset). One can hope that taking a quotient of a cluster algebra with respect to the ideal of this kind we obtain a cluster algebra.

2 Quantum spaces

We have introduced the notion of non-commutative torus T as an algebra generated by $x_i, x_i^{-1}, 1 \leq i \leq n$ which satisfy $x_j x_i = q_{ij} x_i x_j, x_i x_i^{-1} = x_i^{-1} x_i = 1$. We say that an algebra T is a based non-commutative torus if the system of generators is fixed. An isomorphism of two non-commutative tori is called base change if it transforms generators of one torus into monomials of generators of another torus: $\tilde{x}_i = c_i x_1^{a_{i1}} \dots x_n^{a_{in}}$ where a_{ij} is an invertible matrix over $\mathbb{Z}$, i.e. both the matrix and its inverse have integer entries. (One can prove that all isomorphisms of non-commutative tori have this form.) Every isomorphism of non-commutative tori induces an isomorphism of their skew-fields of fractions, hence we can talk about base change in these skew fields. (Keep in mind that skew fields have also other isomorphisms.)

Let us consider a family of non-commutative based tori $T_{\mathbf{i}}$ and their skew fields of fractions $\mathcal{F}_{\mathbf{i}}$, labeled by index $\mathbf{i}$. We define an A-automorphism of a skew field corresponding to a torus with generators $x_i, x_i^{-1}, 1 \leq i \leq k$ by the formula $\phi_B(x_i) = x_i$ for $i \neq n$, $\phi_B(x_n) = x_n(r + sB)^{-1}$, $\phi_B(x_i^{-1}) = x_i^{-1}$ for $i \neq n$, $\phi_B(x_n^{-1}) = (r + sB)x_n^{-1}$ where B is a monomial that does not contain x_n, x_n^{-1} . We say that an isomorphism $\mathcal{F}_{\mathbf{i}} \rightarrow \mathcal{F}_{\mathbf{i}'}$ is an A-mutation if it can be represented as a composition of an A-automorphism and base change. The family is called (a poor man) quantum A-space if every two elements of it can be connected by a sequence of A-mutations.

Every poor man cluster algebra (hence every quantum cluster algebra in the sense of [6]) generates a quantum A-space: tori $T_{\mathbf{i}}$ correspond to clusters. The representation of mutation of clusters as a composition of an A-automorphism and base change follows immediately from (1.3).

We say that automorphism ρ of $F_{\mathbf{i}}$ is an X-automorphism if it leaves one of the generators (say x_n) intact and multiplies other generators by a polynomial of x_n or by an inverse of such a polynomial: $\rho(x_n) = x_n, \rho(x_j) = x_j P_j(x_n)$ or $\rho(x_j) = x_j Q_j(x_n)^{-1}$ for $j \neq n$. A composition of an X-automorphism with base change is called X-mutation (in the direction n). By definition, a family of based tori $T_{\mathbf{i}}$ and their skew fields of fractions $F_{\mathbf{i}}$ is a poor man quantum X-space if every two fields of fractions in this family are connected by a sequence of mutations.

The definitions above are downgraded versions of the definitions in [9, 12]: we singled out only the properties of quantum spaces that are necessary for our proofs. Fock and Goncharov gave a construction of quantum spaces starting with some algebraic data. They [12] start with “feeds” $\mathbf{i} = (I, \epsilon_{ij}, d_i)$ where I is a finite set, $d_i \in \mathbb{Q}_{>0}$, and $\epsilon_{ij}, i, j \in I$, is an integral valued matrix such that $d_i \epsilon_{ij}$ is skew-symmetric. Matrix $\hat{\epsilon}_{ik} := d_i \epsilon_{ik}$ defines a skew-symmetric pairing $\Lambda : \mathbb{Z}^I \times \mathbb{Z}^I \rightarrow \frac{1}{N} \mathbb{Z}$, where N is the least common multiple of

the $\{d_i\}$. We introduce a notation $q_k := q^{1/d_k} \in \mathbb{Z}[q^{1/N}, q^{-1/N}]$. An X-automorphism ν_k of a fraction field F of a torus with generators $x_1, \dots, x_n$ can be defined by the formula:

$$x_i \rightarrow \begin{cases} x_i(1 + q_k x_k)(1 + q_k^3 x_k) \cdots \left(1 + q_k^{2|\epsilon_{ik}|-1} x_k\right) \\ x_i \left((1 + q_k^{-1} x_k)(1 + q_k^{-3} x_k) \cdots \left(1 + q_k^{1-2|\epsilon_{ik}|} x_k\right) \right)^{-1} \end{cases} \quad (2.1)$$

(formula (60) from [12]). The first line of this formula corresponds to $\epsilon_{ik} > 0$, the second line corresponds to $\epsilon_{ik} < 0$. The X-mutation is a composition of X-automorphism with base change.

The corresponding quantum X-space can be obtained from the initial feed by means of consecutive application of X-mutations. The same initial data can be used also to construct a quantum A-space.

3 Quantum Grassmannian

The quantum Grassmannian $\mathbb{C}_q[\mathbb{G}_{m,n}]$ can be defined as an algebra with generators Δ_I where I is an ordered subset of the set $[n] = \{1, 2, \dots, n\}$ consisting of m elements. (Hence the set of generators $\mathcal{S}$ can be identified with $\{I \subset [n] | |I| = m\}$). These generators (quantum minors) satisfy quantum Plücker relations

$$\sum_{i \in I-J} (-q)^{\text{inv}(i,I) - \text{inv}(i,J)} \Delta_{I-\{i\}} \Delta_{J \sqcup \{i\}} = 0$$

for any $(m+1)$ -subset I and $(m-1)$ -subset J . Here $\text{inv}(i, X)$ is the number of $x \in X \subset [1, \dots, n]$ such that $i > x$.

In the limit $q \rightarrow 1$ quantum Plücker relations give conventional Plücker relations. Hence the underlying projective variety P coincides with the classical Grassmannian in this case.

Two minors $\Delta_{I_1}, \Delta_{I_2}$ are quasi-commuting iff $I_1 = R \sqcup A_1, I_2 = R \sqcup A_2$ where A_1, A_2 are cyclically separated (i.e. after some cyclic shift they belong to non-overlapping intervals). This condition (called weak separation) can be represented as a requirement that either $I_1 = R \sqcup B \sqcup C, I_2 = R \sqcup A_2, B < A_2 < C$ or $I_1 = R \sqcup A_1, I_2 = R \sqcup D \sqcup E, D < A_1 < E$. In these notations $\Delta_{I_1} \Delta_{I_2} = q^{\epsilon_{12}} \Delta_{I_2} \Delta_{I_1}$ where $\epsilon_{12} = |C| - |B|$ in the first case and $\epsilon_{12} = |D| - |E|$ in the second case [25].

A maximal collection of quasi-commuting minors (a cluster) consists of $m(n-k) + 1$ elements. (We assume that the set $\mathcal{F}$ of frozen variables is empty.) Every collection of quasi-commuting minors can be extended to a maximal collection [23].

In the case when $m = 2, n = 4$ we have two clusters $(1, 2), (2, 3), (3, 4), (4, 1), (1, 3)$ and $(1, 2), (2, 3), (3, 4), (4, 1), (2, 4)$. The mutation can be written in the form

$$\Delta_{13} \Delta_{24} = q \Delta_{12} \Delta_{34} + q^{-1} \Delta_{14} \Delta_{23} = 0$$

Let us consider the general case. We will use the notation Rab for a union of some set $R \subset [n]$ and two-element set $\{a, b\}$. Let us assume now that a cluster contains minors with $I = Rab, Rbc, Rcd, Rda, Rac$ where a, b, c, d are cyclically ordered elements of $[n] \setminus R$.

Then we can get a new cluster replacing the minor Δ_{Rac} with the minor Δ_{Rbd} . The new cluster is related with the old one by mutation

$$\Delta_{Rac}\Delta_{Rbd} = q\Delta_{Rab}\Delta_{Rcd} + q^{-1}\Delta_{Rad}\Delta_{Rbc}$$

It is easy to check that the conditions (1.4) are satisfied.

Every two clusters are connected by a sequence of mutations of this kind [23], hence $\mathbb{C}_q[\mathbb{G}_{m,n}]$ can be considered as poor man cluster algebra.

For generic q the prime ideals generated by a subset of the set of generators of quantum Grassmannian were classified in [19]. (This classification was stated there as a conjecture, the proof was given in [20].) Recall that for every ideal of this kind we define the ideal set as the subset of $\mathcal{S}$ consisting of generators belonging to this ideal. It was shown in [19] that for quantum Grassmannian the ideal sets are complements to positroid subsets of $\mathcal{S} = \binom{[n]}{m}$ defined by Postnikov [24]. (Positroid subsets correspond to the cells of positive Grassmannians - positroid cells.) This means that the algebra obtained from $\mathbb{C}_q[\mathbb{G}_{m,n}]$ by means of factorization with respect to prime ideal generated by a subset of family of minors quantizes the algebra of functions on a positroid cell. (All these ideals are completely prime, hence the quotient algebra is an entire ring.)

The positroid subsets (as well as prime ideals) can be labeled by decorated permutations (permutations where every fixed point is decorated by an element of a two-point set). For a decorated permutation σ we denote the corresponding prime ideal by $\mathcal{I}(\sigma)$, the corresponding ideal set by $\{\mathcal{I}(\sigma)\}$ and the positroid $\mathcal{S} \setminus \{\mathcal{I}(\sigma)\}$ by $Pos(\sigma)$. The dimension of corresponding positroid cell is denoted by $l(\sigma)$. It is natural to assume that both the quotient algebra $\mathbb{C}_q[\mathbb{G}_{m,n}]/\mathcal{I}(\sigma)$ and the subalgebra of quantum Grassmannian generated by $Pos(\sigma)$ are cluster algebras with the size of cluster equal to $l(\sigma) + 1$. The first of these statements was not proved yet. The second one is proved under assumption that the set of frozen variables is defined as Grassmann necklace corresponding to the permutation σ [23]. (One can prove also that there exists a family of minors $Inn(\sigma) \subset Pos(\sigma)$ generating a cluster algebra with the same clusters and empty $\mathcal{F}$ [15].)

4 Hochschild homology and cohomology

In the section we give some basic definitions and formulate some results about Hochschild homology and cohomology of non-commutative tori. The proofs are relegated to appendix.

The standard complex used for definition of Hochschild homology $HH_*(A)$ (see e.g. [21]) is a direct sum of $C_n = A \otimes A^{\otimes n}$, $n \geq 0$. It is common practice to denote an element $a_0 \otimes a_1 \otimes \cdots \otimes a_n \in C_n$ by $a_0|a_1| \cdots |a_n$. The differential is defined by the formula

$$\partial a_0|a_1| \cdots |a_n = a_0a_1| \cdots |a_n + \sum_{i=1}^{n-1} (-1)^i a_0|a_1| \cdots |a_i a_{i+1}| \cdots |a_n + (-1)^n a_n a_0|a_1| \cdots |a_{n-1}$$

Similarly one can define Hochschild homology with coefficients in bimodule M (in this case the differential acts on the direct sum of $M \otimes A^{\otimes n}$, $n \geq 0$).

The Hochschild cohomology of the algebra A with coefficients in bimodule M can be defined by means of a complex represented as a direct sum of $C^n(A, M)$ where elements of

$C^n(A, M)$ (cochains) are n -linear functions on A with values in M . The differential d on the chain D is defined by the formula

$$(dD)(a_0, \dots, a_n) = a_0 D(a_1, \dots, a_n) + \sum_{i=0}^{n-1} (-1)^{i+1} D(a_0, \dots, a_i a_{i+1}, \dots, a_n) + (-1)^{n+1} D(a_0, \dots, a_n) a_n$$

In particular, one-dimensional Hochschild cocycles are derivations, i.e. linear maps $D : A \rightarrow M$ obeying $D(ab) = D(a)b + aD(b)$.

Notice that in the case when $M = A$ the multiplication of cochains induces graded commutative multiplication of Hochschild cohomology classes [14].

Proposition 1. *We assume that associative algebra A admits a skew field of fractions $\mathcal{F}$. There is an action of cohomology on homology*

$$H^i(A, A) \cap H_j(A, A) \rightarrow H_{j-i}(A, A), \quad H^i(\mathcal{F}, \mathcal{F}) \cap H_j(\mathcal{F}, \mathcal{F}) \rightarrow H_{j-i}(\mathcal{F}, \mathcal{F}). \quad (4.1)$$

See [28] for the analysis of this action that generalizes the pairing of polyvector fields and differential forms with values in differential forms. It can be described in the following way. Fix $D \in \text{Hom}(A^{\otimes i}, A)$ a Hochschild cochain. Then on the level of chains the pairing (A.7) up to a sign is given by the formula ([27], formula 2.32)

$$D \otimes a_0 |a_1| \cdots |a_j| \xrightarrow{\cap} a_0 D(a_1, \dots, a_i) |a_{i+1}| \cdots |a_j| \quad (4.2)$$

For smooth commutative algebras Hochschild homology classes can be identified with differential forms and Hochschild cohomology classes with polyvector fields.

Hochschild homology and cohomology of an algebra with quasi-commuting generators were calculated in [16]. One can use the considerations of [16] to construct some special elements of the Hochschild homology of non-commutative torus, i.e. of the algebra A generated by $x_i, x_i^{-1}, 1 \leq i \leq k$ which satisfy $x_j x_i = q_{ij} x_i x_j, x_i x_i^{-1} = x_i^{-1} x_i = 1$. (Later in this section A stands for this algebra.) We will give a direct construction of these elements.

Proposition 2. *We will be writing $c_{\sigma(1), \dots, \sigma(k)}, \sigma \in S_k$ for $(x_{\sigma(1)} \cdots x_{\sigma(n)})^{-1} |x_{\sigma(1)}| \cdots |x_{\sigma(k)}$. Here S_k stands for symmetric group on k letters. The element*

$$c = 1/k! \sum (-1)^{|\sigma|} c_{\sigma(1), \dots, \sigma(k)}$$

is a Hochschild cycle. (One can say that this cycle is obtained from $(x_{\sigma(1)} \cdots x_{\sigma(n)})^{-1} |x_{\sigma(1)}| \cdots |x_{\sigma(k)}$ by means of antisymmetrization.) Its homology class is denoted by $[c]$.

This statement can be derived from [16] (see appendix for the derivation) or proved by direct calculation.

Notice that in the case of commutative torus the class $[c]$ can be identified with the standard volume form ω . In other words $[c]$ should be considered as q -deformed volume form.

Remark 3. *There is a simple way to construct more cycles. Fix a subalgebra $B \subset A$ isomorphic to a quantum torus with generators $x'_1, \dots, x'_{n'}$. Then the image of the antisymmetrization of $(x'_{\sigma(1)} \cdots x'_{\sigma(n')})^{-1} |x'_{\sigma(1)}| \cdots |x'_{\sigma(n')}|$ in $HH_*(A)$ will be a nontrivial homological class. Applying this construction to subalgebras generated by a subset of generators*

of A we get a family of cycles c_R labelled by subsets $R \subset [k] = \{1, \dots, k\}$. It follows from the results of [16] that every class of the Hochschild homology $HH_*(A)$ contains precisely one cycle represented as a linear combination of cycles c_R with coefficient from the center of the algebra A .

Hochschild cohomology and homology of A are related:

Proposition 4. $HH^i(A, A) \cong HH_{k-i}(A, A)$

Proof. See Corollary 14. □

If q_{ij} are not roots of unity non-commutative torus A can be embedded in its field of fractions $\mathcal{F}$.

Proposition 5. *Inclusion $A \rightarrow \mathcal{F}$ induces isomorphisms $H_*(A, \mathcal{F}) \cong H_*(\mathcal{F}, \mathcal{F})$, $H^*(A, \mathcal{F}) \cong H^*(\mathcal{F}, \mathcal{F})$.*

See appendix for the proof.

Let us consider derivations D_i obeying $D_i x_i = x_i, D_i x_j = 0$ for $j \neq i$. (Recall that x_i are generators of the torus A .) Their cohomology classes will be denoted by $[D_i] \in H^1(A, A)$. We use cup-product in Hochschild cohomology to define an element

$$[\mu] = [D_1] \cup \dots \cup [D_n] \in H^n(A, A)$$

represented by cocycle $\mu = D_1 \cup \dots \cup D_n$. (More explicitly, $\mu(\xi_1, \dots, \xi_n) = D_1(\xi_1) \dots D_n(\xi_n)$.) The derivations D_i can be extended to the skew field of fractions $\mathcal{F}$ therefore we can use the same formula to define a cocycle μ and an element $[\mu] \in H^n(\mathcal{F}, \mathcal{F})$.

Proposition 6. $H_n(\mathcal{F}, \mathcal{F})$ is a one-dimensional vector space over the center of $\mathcal{F}$.

Proof. Follows from Corollary 14. □

Proposition 7. $[\mu] \neq 0 \in H^n(\mathcal{F}, \mathcal{F})$

This statement follows from

Proposition 8. $[c] \cap [\mu] = [1], [1] \neq [0] \in H_0(\mathcal{F}, \mathcal{F})$

The first formula can be obtained by direct application of the formula (A.8) for cap-product. The second follows from Lemma 20.

Proposition 9. *Let us consider two elements of the group $H^n(\mathcal{F}, \mathcal{F})$ represented by cocycles ν_1, ν_2 regarded as polylinear $\mathcal{F}$ -valued functionals on $\mathcal{F}$. Assume that the cocycles $A\nu_1, A\nu_2$ obtained from these functionals by means of anti-symmetrization coincide when the arguments are chosen as generators $x_1, \dots, x_n$ of A . Then $[\nu_1] = [\nu_2]$.*

In this proposition we use the notation $(A\nu)(\xi_1, \dots, \xi_n) = 1/n! \sum_{\sigma \in S_n} (-1)^{|\sigma|} \nu(\xi_{\sigma(1)}, \dots, \xi_{\sigma(n)})$. The proof is based on Proposition 6 and on the remark that the anti-symmetrization does not change cohomology class.

5 Hochschild homology of quantum spaces

We have defined quantum A-spaces and X-spaces as families of based tori $T_{\mathbf{i}}$ and their skew fields of fractions $\mathcal{F}_{\mathbf{i}}$ connected by mutations.

Proposition 2 gives a special element in Hochschild homology of every torus $T_{\mathbf{i}}$ and therefore an element $[c_{\mathbf{i}}]$ in the Hochschild homology of $F_{\mathbf{i}}$. Proposition 7 gives a special element $[\mu_{\mathbf{i}}]$ in Hochschild cohomology.

Theorem 10. *A-mutations and X-mutations preserve the elements $[c_{\mathbf{i}}]$ and $[\mu_{\mathbf{i}}]$ up to a sign.*

(This theorem is a “quantization” of the statements about behavior of volume elements with respect to A- and X-mutation in [1], formula (15.18)) We start with a *proof of this statement for $\mu_{\mathbf{i}}$* .

First of all we calculate the effect of base change on the derivations D_i . From the relation $D_i(x_1^{a_1} \dots x_n^{a_n}) = a_i x_1^{a_1} \dots x_n^{a_n}$ we obtain that the base change transforms these derivations into linear combinations of them with coefficients given by the matrix specifying the base change. To calculate the transformation of $[\mu]$ we use the fact that cup-product is graded symmetric. We see that this transformation is given by the multiplication by the determinant of the matrix of base change. This matrix has integer entries and is invertible over integers, hence the determinant of it is equal to ± 1 .

The next step is the consideration of the behavior of μ with respect to A-automorphism $\phi = \phi_B$. Again we start with action of this automorphism on the derivations D_i . To calculate $\tilde{D}_j = \phi D_j \phi^{-1}$ we notice that $\phi^{-1}(x_k) = x_k(r + sB)$, $\phi^{-1}(x_i) = x_i$ for $i \neq k$. We obtain $\tilde{D}_j(x_i) = 0$ for $j \neq i$, $\tilde{D}_i(x_i) = x_i$, $\tilde{D}_i(x_k) = x_k(r + sB)^{-1} s D_i B$, $\tilde{D}_k(x_k) = x_k$. (In these formulas $i \neq k$.)

The most important consequence of these relations is the remark that $\tilde{D}_j(x_j) = D_j(x_j)$ for all j , hence $\tilde{\mu}(x_1, \dots, x_n) = \mu(x_1, \dots, x_n)$. (Here $\tilde{\mu}$ stands for $\tilde{D}_1 \cup \dots \cup \tilde{D}_n$.) Notice that all terms that we add by anti-symmetrization vanish because $\tilde{D}_j(x_i) = D_j(x_i) = 0$ for $j \neq i$. Therefore the relation $[\tilde{\mu}] = [\mu]$ follows from Proposition 9.

The same proof works for X-automorphism ρ . We define $\tilde{D}_j = \rho D_j \rho^{-1}$ and check that $\tilde{D}_j(x_j) = D_j(x_j)$ for all j and $\tilde{D}_j(x_i) = D_j(x_i) = 0$ for $j \neq i$. Again this allows us to apply Proposition 9.

The proof that c is invariant under mutations. To prove the invariance of $[c]$ with respect to A- and X-mutations we notice that $[c] \cap [\mu] = [1]$ (Proposition 8). It is a general fact that cap product is compatible with homomorphisms: $[\lambda]^\psi \cap [\tau]^\psi = ([\lambda] \cap [\tau])^\psi$. It is clear that $[1]^\psi = [1]$ for any ψ . As we know $[\mu]^\phi = [\mu]$ and $[c] \cap [\mu] = [1]$ then

$$[c]^{\phi^{-1}} \cap [\mu] = [c] \cap [\mu]^\phi = [c] \cap [\mu] = [1] \quad (5.1)$$

Cup-product with μ defines a nonzero $Z(\mathcal{F})$ -linear map from one-dimensional linear space $H_k(\mathcal{F}, \mathcal{F})$ over center $Z(\mathcal{F})$ (Corollary 14) to $H_0(\mathcal{F}, \mathcal{F})$. One-dimensionality and (5.1) imply that $[c]^\phi = [c]$.

Notice that working in terms of power series with respect to $1 - q$ one can represent the X-automorphism as a conjugation with quantum dilogarithm [12]. This allows us to simplify the proof in this setting (Hochschild (co)homology is invariant with respect to inner automorphisms).

6 Poisson and Hochschild homology

If a non-commutative algebra $\mathcal{A}_q$ depends on parameter q and becomes commutative for $q = 1$ then the algebra $\mathcal{A}_1$ has a structure of Poisson algebra. We are interested in the case when $\mathcal{A}_1$ is an algebra of functions on a Poisson manifold (or an algebra of homogeneous functions on a projective Poisson variety).

One can define Poisson homology HP_i and cohomology HP^i of Poisson algebra basically “taking the limit $q \rightarrow 1$ ” in the definition of Hochschild homology and cohomology (a limit of bimodule over $\mathcal{A}_q$ as $q \rightarrow 1$ is a Poisson module over $\mathcal{A}_1$ hence one can talk about Poisson homology and Poisson cohomology with coefficients in Poisson module).

The limit of Hochschild homology class c constructed in Sec 3 as $q \rightarrow 1$ is closely related to the differential form constructed in [9] and used in [1]. It follows that this form is a Poisson cycle. Of course, this can be checked directly.

It follows from the results of section 3 that in the case when $\mathcal{A}_q$ is a non-commutative torus and q is generic the Hochschild homology $HH_i(\mathcal{A}_q)$ is canonically isomorphic to Poisson homology of commutative torus $\mathcal{A}_1$. Similar results are valid for Hochschild and Poisson cohomology. (Non-canonical isomorphism between Hochschild and Poisson is known in more general situations.) The results of section 4 imply in the limit $q \rightarrow 1$ that for two commutative tori related by mutation there exists a q -dependent family of isomorphisms of their Poisson homologies.

It was shown in [9] there exists a canonical quantization of Poisson cluster algebras, it seems that using this result one can give an independent construction of the q -dependent family of isomorphisms we have mentioned.

7 Contribution of positroid cell to scattering amplitudes

It was proved in [1] that the contribution of positroid cell $\sigma \subset G(k, n)$ to the amplitude can be described in momentum -twistor formalism as an integral (8.27) containing the delta-function of $\hat{C}\mathcal{Z}$ where $\hat{C} \in \hat{\sigma}$, $\hat{\sigma}$ stands for the positroid cell in $G(k - 2, n)$ and $\mathcal{Z}$ denotes a column of n supertwistors. (A supertwistor is considered as a pair (z, η) where z is momentum-twistor variable and η is an odd variable; it was shown in [3] that η should be identified with dz .) We will give reformulation of this construction that allows us to quantize it. Moreover, this reformulation opens new ways for calculation of amplitudes.

To calculate the integral (8.27) one should solve the equation $\hat{C}\mathcal{Z} = 0$. It is sufficient to solve this equation for purely bosonic $\mathcal{Z}$ (the equation for the odd part has precisely the same form). If there exists a unique $\hat{C}$ obeying the equation for generic $\mathcal{Z}$ the solution specifies a map α of the top cell $G(4, n + 4)$ into the orthogonal positroid cell $\hat{\sigma}^\perp$ (this top cell can be identified with the top cell of the space of $4 \times n$ matrices). Then the amplitude is

equal to $\alpha^*\omega$ (the pullback of the standard volume form ω on the positroid) for appropriate choice of α . If the equation has finite number of solutions (the map α is multivalued), then we can lift α to a single-valued map of some variety M into $\hat{\sigma}^\perp$, the natural projection of M onto $G(4, n+4)$ is a branched covering. (The variety M can be defined as the set of pairs $\hat{C}, \mathcal{Z}$ where $\mathcal{Z}$ is purely bosonic obeying $\hat{C}\mathcal{Z} = 0$.) Then we can construct the scattering amplitude taking first the pullback of the volume form ω to M and then the push-forward to $G(4, n+4)$. (Denoting by α the projection of M onto $\hat{\sigma}^\perp$ and by π the projection onto $G(4, n+4)$ we represent the differential form corresponding to the scattering amplitude as $\pi_!\alpha^*\omega$.)

The construction of scattering amplitude presented in this form can be easily quantized by means of the theorems we have proven. Instead of the positroid cell $\hat{\sigma}^\perp$ we consider the corresponding noncommutative torus T , instead of top positroid cell in the Grassmannian $G(4, n+4)$ we consider the noncommutative torus U corresponding to this cell. If there exists a homomorphism $T \rightarrow U$ the q -deformed scattering amplitude can be defined as an image of q -deformed volume form (of Hochschild homology class $[c]$ constructed in Theorem 2). Moreover, it is easy to describe explicitly all homomorphisms of noncommutative tori; if the homomorphism we need does exist we get an explicit expression for a single-valued map α in the classical limit. If such a homomorphism does not exist we should consider intermediate noncommutative variety V that models the variety M and construct homomorphisms $T \rightarrow V, U \rightarrow V$. All homomorphisms can be easily calculated. Taking the limit $q \rightarrow 1$ we obtain an explicit expression for the contribution of the positroid cell into scattering amplitude in the situation of [1].

Acknowledgments

We are indebted to N. Arkani-Hamed, A. Berenstein, A. Goncharov, S. Launois, T. Lenagan and J. Trnka for useful discussions.

A Hochschild homology and cohomology-details

There is a remarkable complex associated with the algebra whose generators satisfy $y_i y_j = q_{ij} y_j y_i$ [16], see also [29]. In fact the authors construct the complex in greater generality.

Introduce noncommutative algebras X_* and A over $\mathbb{C}[q] := \mathbb{C}[q_{ij}]/(q_{ij}q_{ji} - 1)$, $1 \leq i, j \leq n$. X_* is a quotient of a free associative algebra on generators $y_i, z_j, e_l, y_i^{-1}, z_j^{-1} 1 \leq i, j, l \leq k$. The ideal is generated by the relations

$$\begin{aligned} y_j y_i &= q_{ij} y_i y_j, & z_j z_i &= q_{ij} z_i z_j, & z_j y_i &= q_{ij} y_i z_j \\ y_i^{-1} y_i &= y_i y_i^{-1} = z_i^{-1} z_i = z_i z_i^{-1} = 1 \end{aligned} \tag{A.1}$$

$$e_j e_i = -q_{ij} e_i e_j, \quad e_i^2 = 0, \quad e_j y_i = q_{ij} y_i e_j, \quad e_j z_i = q_{ij} z_i e_j \tag{A.2}$$

In positively graded algebra X_* y_i, z_j have degree zero, e_k has degree one. The differential ∂ satisfies graded Leibniz rule and

$$\partial y_i = \partial z_j = 0, \quad \partial e_l = y_l - z_l.$$

Let A be a noncommutative torus. X_* is A -bimodule: $x_i \times a \times x_j := y_i a z_j$ for any $a \in X_*$. Algebra A^{op} shares linear space with A but has the product defined by the formula $a \times b := ba$. Ant A -bimodule M is the same thing as a left module over the algebra $A^e := A \otimes A^{op}$. $(a \otimes b)m = amb$. It is also a right module over the same algebra $m(a \otimes b) := bma$. A is naturally an A -bimodule

Proposition 11. X_* is a resolution of A by free A -bimodules. The augmentation map $\epsilon : X_* \rightarrow A$ sends y_i, z_i to x_i and e_i to zero.

Proof. See [16]. □

We can use this complex for computation of Hochschild homology. To this end we take the tensor product of A -bimodules $X_*(A) := X_* \otimes_{A^e} A$ and compute the cohomology of the resulting complex. Denote by Λ_q the subalgebra in X_* generated by e_i . The space $X_*(A)$, which not an algebra any more, is spanned by $a \otimes b, a \in \Lambda_q, b \in A$. $X_*(A)$ in addition to relations that come from (A.1), (A.2) has relation $a \otimes b = b \otimes a$. It comes from the tensor product of bimodules.

Note that elements $e_i y_i^{-1} e_i z_i^{-1}$ belong to the (graded) center of X_*

Proposition 12. Let $P \in X_*$ be a nonzero polynomial in $e_i y_i^{-1}$. The image of $P \otimes 1$ defines a nonzero ∂ -cycle in $X_*(A)$. It also defines a nontrivial homology class in $HH_*(A)$.

Proof. See [16]. □

The cycles corresponding to products of $e_i y_i^{-1}$ are described in the Proposition 2. Let us prove this Proposition.

Proof of Proposition 2.

Proof. According to Proposition 1.4, Remark 1.5 [16] there exists a chain map $\theta_* : X_*(A) \rightarrow C_*$.

$$\theta((x_1 \cdots x_n)^{-1} e_1 \cdots e_n) = \sum_{\sigma \in S_n} sg_q(\sigma) (x_1 \cdots x_n)^{-1} |x_{\sigma(1)}| \cdots |x_{\sigma(n)}| \quad (\text{A.3})$$

Here $sg_q(\sigma) = \prod_{h>j, \sigma(h)<\sigma(j)} (-q_{i_{\sigma(h)} i_{\sigma(j)}}) = (-1)^{|\sigma|} \prod_{h>j, \sigma(h)<\sigma(j)} (q_{i_{\sigma(h)} i_{\sigma(j)}}) = (-1)^{|\sigma|} \tilde{sg}_q$. The factor $\tilde{sg}_q$ can be absorbed by the element with negative exponent. We continue equation (A.3):

$$= \sum_{\sigma \in S_n} (-1)^{|\sigma|} (x_{\sigma(1)} \cdots x_{\sigma(n)})^{-1} |x_{\sigma(1)}| \cdots |x_{\sigma(n)}|$$

We obtained the cycle described in Proposition 2. □

The complex

$$X^*(A) := \text{Hom}_{A^e}(X_*, A) \quad (\text{A.4})$$

computes Hochschild cohomology $HH^i(A, A)$.

The following two-term complex P will be useful in our computations

$$\mathbb{C}[x, x^{-1}] \xleftarrow{x^{-1}} \mathbb{C}[x, x^{-1}]$$

Its only nontrivial nonzero cohomology is in degree zero and is to $\mathbb{C}$. We interpret $P^{\otimes n}$ as a projective resolution of $\mathbb{C}$ over $B = \mathbb{C}[x_1, x_1^{-1}, \dots, x_k, \xi_k^{-1}]$. It is isomorphic to Koszul complex $K_*(B) := B \otimes \Lambda[\xi_1, \dots, \xi_n]$ with differential $\sum_{i=1}^n (x_i - 1) \frac{\partial}{\partial \xi_i}$. We have $K_i(B) = B \otimes \Lambda^i$. There is also a closely related complex $K^*(B) := B \otimes \Lambda[\xi_1, \dots, \xi_n]$ with graded components $K^i(B) = B \otimes \Lambda^i$. We set $r = b_{\xi_{i_1}} \cdots \xi_{i_s}$. Then $dr = \sum_{i=1}^k (x_i - 1) \xi_i r$. The complexes are isomorphic under the map $\psi : K^s(B) \rightarrow K_{n-s}(B)$

$$\psi(b_{\xi_{i_1}} \cdots \xi_{i_s}) = b \frac{\partial^s}{\partial \xi_{i_1} \cdots \partial \xi_{i_s}} \xi_1 \cdots \xi_n.$$

B is the group algebra of the group $\mathbb{Z}^k$. We can use K_* and K^* for computation of homology and cohomology of $\mathbb{Z}^n$:

$$\begin{aligned} H_i(\mathbb{Z}^k, M) &= H_i(K(M)) \\ H^i(\mathbb{Z}^k, M) &= H^i(\text{Hom}_B(K(B), M) = H^i K^*(M) = H_{k-i} K_*(M) \end{aligned} \quad (\text{A.5})$$

It is convenient to choose $\eta_i = e_i y_i^{-1}, y_i, z_i$ as a generating set for X_* . The algebra A is a projective representation of $\mathbb{Z}^n$: $x_j x_i a = q_{ij} x_i x_j a$. Because of that it wouldn't be a surprise to learn that in adjoint action $x_i \times a = x_i a x_i^{-1}$ cocycle factors q_{ij} drop out and we have an ordinary (as opposed to projective) representation of $\mathbb{Z}^n$. We denote it by A^{ad}

Proposition 13. *A map $a\eta_{i_1} \cdots \eta_{i_s} \rightarrow a\xi_{i_1} \cdots \xi_{i_s}$ defines an isomorphism of complexes $X_*(A)$ and $K_*(A^{ad})$. Fix $g \in \text{Hom}_{A^e}(X_*, A)$. We will be writing $g_{i_1, \dots, i_s}$ for the value of g on $\eta_{i_1} \cdots \eta_{i_s}$. The map $g \rightarrow \sum g_{i_1, \dots, i_s} \xi_{i_1} \cdots \xi_{i_s}$ defines an isomorphism of complexes $X^*(A)$ and $K^*(A^{ad})$. The last two statements are still valid if we replace A^{ad} by $\mathcal{F}^{ad}$.*

Proof. Direct inspection. $\square$

Denote by $\mathcal{F}(x_1, \dots, x_k)$ the noncommutative field of fractions of $A(x_1, \dots, x_k)$.

Corollary 14. *It follows from (A.5) and Proposition 13 that $HH_i(A, A) = HH^{k-i}(A, A)$, $HH_i(A, \mathcal{F}) = HH^{k-i}(A, \mathcal{F})$. In particular $HH_k(A, \mathcal{F}) = HH^0(A, \mathcal{F})$. Recall that $HH^0(A, \mathcal{F}) = HH^0(\mathcal{F}, \mathcal{F})$ is the center of $\mathcal{F}$.*

Fix a ring B and an injective automorphism ϕ of a B . Skew Laurent series ring $B((x, \phi))$ (see [7] for details) consists of formal series $b = \sum_{i=k}^{\infty} b_i x^i$ of an indeterminate x with an integer k (which could be negative) and coefficients $b_i \in B$. Addition in $B((x, \phi))$ is defined naturally and multiplication is defined by the rule $x^i b = \phi^i(b) x^i$ ($a \in B$). The ring $B((x, \text{id})) = B((x))$ is the ordinary Laurent series ring of B .

According to [7] Theorem 2.3.1 if B is a skew field then $B((x, \phi))$ is also a skew field. We can construct this way examples of skew fields inductively. Start with $\mathbb{C}((x_1))$ and define an automorphism ϕ that acts by $\phi(x_1) = q_{12}^{-1} x_1$. The skew field $\mathbb{C}((x_1))((x_2, \phi))$ will contain subalgebra $A(x_1, x_2)$ generated by $x_1^{\pm}, x_2^{\pm}$, which satisfy $x_2 x_1 = q_{12} x_1 x_2$. $\mathbb{C}((x_1))((x_2, \phi))$ will also contain the field of fractions $F(x_1, x_2)$. Suppose $\mathbb{C}((x_1))((x_2, \phi)) \cdots ((x_{n-1}, \phi))$ is given. Extend ϕ by the rule: $\phi(x_i) = q_{1n}^{-1} x_i$. The resulting skew field

$$SF(x_1, \dots, x_n) := \mathbb{C}((x_1))((x_2, \phi)) \cdots ((x_n, \phi)) \quad (\text{A.6})$$

contains $A(x_1, \dots, x_n)$ and its field of fractions $\mathcal{F}(x_1, \dots, x_n)$.

Proposition 15. $SF(x_1, \dots, x_n)$ is flat left (right) module over $A(x_1, \dots, x_n)$.

Proof. Let us carry the proof for the left structures. SF is a linear space over $\mathcal{F}$. It has a Hamel basis indexed by some set Y . Let Z be a set of finite subsets of Y ordered by inclusion. Then $SF = \lim_{\rightarrow Z} \mathcal{F}^z, z \in Z$. We already know that F is flat over A and by Lazard, Govorov theorem $\mathcal{F} = \lim_{\rightarrow Z} A^z, z \in U$. From this we conclude that SF is a direct limit of free A -modules indexed by the set $Z \times U$ and SF is flat A -module. $\square$

Proposition 16. The tensor product $\mathcal{F}^e \otimes_{A^e} X_*, (SF^e \otimes_{A^e} X_*)$ is a free $\mathcal{F}^e$ (SF^e) resolution of $\mathcal{F}, (SF)$.

Proof. We do the proof only for $\mathcal{F}$. The proof for SF is similar. Flatness in this case follows from Proposition 15.

$\mathcal{F}^e$ is A^e -flat. This is true because by Lazard, Govorov theorem $\mathcal{F} = \lim_{\rightarrow} \mathcal{F}_\alpha$ where $\mathcal{F}_\alpha$ free finitely generated A -modules indexed by directed system. $\mathcal{F}_\beta^{op}$ is a similar system of modules for $\mathcal{F}^{op}$ over A^{op} . $\mathcal{F}_{\alpha\beta} = \mathcal{F}_\alpha \otimes_{\mathbb{C}} \mathcal{F}_\beta^{op}$ is a bi-system over A^e for $\mathcal{F}^e$. $\mathcal{F}_{\alpha\beta}$ are obviously A^e -free. Thus $\lim_{\rightarrow} \mathcal{F}_{\alpha\beta} = \mathcal{F}^e$ is flat.

Lemma 17. Any domain of polynomial growth is an Ore domain. In particular A is an Ore domain

Proof. See [6] Proposition 11.1 $\square$

It follows from flatness of $\mathcal{F}^e$ and Lemma 17 that $\mathcal{F}^e \otimes_{A^e} X_*$ is acyclic away from zero degree. Its cohomology at zero is $\mathcal{F}^e \otimes_{A^e} A = \mathcal{F}$. Indeed, by Ore condition $\forall a, s \neq 0 \in A, \exists a', s' \neq 0 \in A$ $s' \otimes a \otimes 1 = 1 \otimes s' a \otimes 1 = 1 \otimes a' s \otimes 1 = 1 \otimes a' \otimes s$. After multiplication on s^{-1}, s'^{-1} it becomes $s'^{-1} \otimes a' \otimes 1 = 1 \otimes a \otimes s^{-1}$. We use this to identify $\mathcal{F}^e \otimes_{A^e} A$ with $\mathcal{F}$. $\square$

We do not expect that $\mathcal{F}^e \otimes_{A^e} X_*$ will be an algebra for arbitrary q . The reason is that the product ba $a \in \mathcal{F}(y_1, \dots, y_n) \subset \mathcal{F}(y_1, \dots, y_n, z_1, \dots, z_n)$ and $b \in \mathcal{F}(z_1, \dots, z_n) \subset \mathcal{F}(y_1, \dots, y_n, z_1, \dots, z_n)$ will in general lie in $\mathcal{F}(y_1, \dots, y_n, z_1, \dots, z_n)$, which is much bigger than $\mathcal{F}(y_1, \dots, y_n) \otimes \mathcal{F}(z_1, \dots, z_n)$.

Proof of Proposition 5.

Proof. By Proposition 16 $\mathcal{F} \otimes_{\mathcal{F}^e} \mathcal{F}^e \otimes_{A^e} X_*$ is the complex for calculation of $H_*(\mathcal{F}, \mathcal{F})$. It coincides with $\mathcal{F} \otimes_{A^e} X_*$ that computes $H_*(A, \mathcal{F})$. The argument for cohomology is similar. $\square$

In the following we will use freely these isomorphisms.

Proposition 18. (Proposition 1 of section 4) There is an action of cohomology on homology

$$H^i(A, A) \cap H_j(A, A) \rightarrow H_{j-i}(A, A), \quad H^i(\mathcal{F}, \mathcal{F}) \cap H_j(\mathcal{F}, \mathcal{F}) \rightarrow H_{j-i}(\mathcal{F}, \mathcal{F}). \quad (\text{A.7})$$

Proof. The proof follows from two facts. The first is the interpretation of $H^i(A, A)$ as $\text{Ext}_{A^e}^i(A, A)$ and $H_i(A, A)$ as $\text{Tor}_i^{A^e}(A, A)$ (see [30] Lemma 9.1.3). The second is that Ext^i is a group of morphisms in derived category $\text{Hom}_{D\text{Mod}(A^e)}(A, A[i])$. It must define morphism of derived functors $(A \overset{\mathbb{L}}{\otimes}_{A^e} ?$ in our case)(see [17] for details).

More explicitly the pairings have been described by [28] in the context of noncommutative calculus. Fix $D \in \text{Hom}(A^{\otimes i}, A)$ a Hochschild cochain. Then on the level of chains the pairing (A.7) up to a sign is given by the formula ([27] formula 2.32)

$$D \otimes a_0 |a_1| \cdots |a_j \overset{\cap}{\rightarrow} a_0 D(a_1, \dots, a_i) |a_{i+1}| \cdots |a_j \quad (\text{A.8})$$

□

Remark 19. On a A -bimodule M we define a new bimodule structure M_f by the formula $x_i \times a \times x_j := f x_i f^{-1} a x_j$, $f = x_1 \cdots x_n$. Then the complex $\{X^i(M_c), d\}$ is isomorphic to $\{X_{n-i}(M), d\}$. As a corollary we get Proposition 4.

Lemma 20. $[1] \neq [0] \in H_0(A, SF)$

Proof. If a_i are monomials $a_i = x_i^m := x_1^{m_1} \cdots x_n^{m_n}$ then $[x_i, a_i]$ will be a zero or non-constant monomial. From this we conclude that equation

$$\sum_{i=1}^n [x_i, a_i] = 1 \quad (\text{A.9})$$

has no solutions. Solution of this equation in SF is the same as a sequence $a_i(r) \in A$, $r = 1, \dots, \infty$, which converge in SF to a_i and such that $\sum_{i=1}^n [x_i, a_i(k)] - 1$ converge to zero, that is divisible by arbitrary high power of x_1 in formal Taylor series in x_1 . As -1 will never cancel with any term in $\sum_{i=1}^n [x_i, a_i(k)]$ such divisibility is impossible (see [7] for details on topology on SF).

□

Corollary 21. $[1] \neq [0] \in H_0(A, \mathcal{F})$ because $[1]$ is mapped to nonzero element in $H_0(A, SF)$

Proposition 22. For generic q_{ij} the center $Z(A)$ maps to $H_0(A, A) = A/[A, A]$ and the map is an isomorphism.

Proof. The proof will simplify if we use an action of a commutative torus $\prod_{i=1}^k \mathbb{C}^\times$ on A : $z x_i = z_i x_i$ where $z = (z_1, \dots, z_k)$. Weight spaces of G^n action on A are spanned by monomials m in x_i . Weight grading of A descends to $A/[A, A]$. We have direct sum decomposition $A = \bigoplus_{m \in Z(A)} \mathbb{C}m + \bigoplus_{m \notin Z(A)} \mathbb{C}m = Z(A) + B$. If $m \notin Z(A)$, then there is $m' \notin Z(A)$ such that $mm' \neq m'm$. Still $mm' = F(q)m'm$, for some polynomial F in q_{ij} . Thus $(m(m')^{-1})m' - m'm(m')^{-1} = m - F^{-1}(q)m$. We conclude that for generic q_{ij} $B \subset [A, A]$. We conclude that the map $Z(A) \rightarrow A/[A, A]$ is onto.

By Corollary 21 $[1]$ maps to nonzero element in $\mathcal{F}/[\mathcal{F}, \mathcal{F}]$. The later group is a linear space over the field $Z(\mathcal{F})$. By construction $Z(A) \subset Z(\mathcal{F})$. Thus $Z(A) \rightarrow A/[A, A]$ has no kernel.

□

Let $RH^*(A, A)$ be subalgebra in $H^*(A, A)$ generated by D_i and $Z(A)$.

Proposition 23. *Inclusion $A \subset \mathcal{F}$ defines embeddings $H_*(A, A) \rightarrow H_*(\mathcal{F}, \mathcal{F})$ and $RH^*(A, A) \rightarrow H^*(A, \mathcal{F}) \cong H^*(\mathcal{F}, \mathcal{F})$.*

Proof. By Proposition 1.9 [16] homology classes from $H_*(A, A)$ are in one-to-one correspondence with linear combinations of c_R (see Remark 3) with coefficients in center $Z(A)$. Pairing (A.7) $H^i(A, A) \cap H_i(A, A) \rightarrow H_0(A, A) = Z(A)$ between such cycles with monomial coefficients from $Z(A)$ and cocycles $D_{i_1} \cup \dots \cup D_{i_s}$ also with monomial coefficients from $Z(A)$ is not degenerate. This can be seen by straightforward application of formula (A.8). The pairing (A.7) is compatible with embedding $A \subset \mathcal{F}$. As $A/[A, A] \rightarrow \mathcal{F}/[\mathcal{F}, \mathcal{F}]$ is an embedding we conclude that from nondegeneracy of the pairing the maps $H_i(A, A) \rightarrow H_i(\mathcal{F}, \mathcal{F})$ and $RH^i(A, A) \rightarrow H^i(\mathcal{F}, \mathcal{F})$ are an embedding. $\square$

It follows from Proposition 5 that any derivation of A can be extended in a unique way to derivation of $\mathcal{F}$. We think about D_{x_i} as elements of $H^1(\mathcal{F}, \mathcal{F})$.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo, A.B. Goncharov, A. Postnikov and J. Trnka, *Grassmannian Geometry of Scattering Amplitudes*, Cambridge University Press (2016) [[arXiv:1212.5605](https://arxiv.org/abs/1212.5605)] [[INSPIRE](#)].
- [2] N. Arkani-Hamed and J. Trnka, *The Amplituhedron*, *JHEP* **10** (2014) 030 [[arXiv:1312.2007](https://arxiv.org/abs/1312.2007)] [[INSPIRE](#)].
- [3] N. Arkani-Hamed, H. Thomas and J. Trnka, *Unwinding the Amplituhedron in Binary*, *JHEP* **01** (2018) 016 [[arXiv:1704.05069](https://arxiv.org/abs/1704.05069)] [[INSPIRE](#)].
- [4] M. Artin, *Noncommutative Rings*, class notes, Math 251, Berkeley, fall 1999 [<http://math.mit.edu/~etingof/artinnotes.pdf>].
- [5] A. Berenstein, *Group-Like Elements In Quantum Groups And Feigin's Conjecture*, [q-alg/9605016](https://arxiv.org/abs/q-alg/9605016).
- [6] A. Berenstein and A. Zelevinsky, *Quantum cluster algebras*, *Adv. Math.* **195** (2005) 405 [[math/0404446](https://arxiv.org/abs/math/0404446)].
- [7] P.M. Cohn, *Skew Fields Theory of General Division Rings*, Cambridge University Press (1995).
- [8] A. Connes, M.R. Douglas and A.S. Schwarz, *Noncommutative geometry and matrix theory: Compactification on tori*, *JHEP* **02** (1998) 003 [[hep-th/9711162](https://arxiv.org/abs/hep-th/9711162)] [[INSPIRE](#)].
- [9] V. Fock and A. Goncharov, *Cluster ensembles, quantization and the dilogarithm*, [math/0311149](https://arxiv.org/abs/math/0311149).
- [10] V.V. Fock and A.B. Goncharov, *Cluster X-varieties, amalgamation and Poisson-Lie groups*, in *Algebraic geometry and number theory*, Birkhäuser Boston (2006), pg. 27–68.

- [11] V.V. Fock and A.B. Goncharov, *Cluster ensembles, quantization and the dilogarithm II: The intertwiner*, [math/0702398](#).
- [12] V.V. Fock and A.B. Goncharov, *The quantum dilogarithm and representations of quantum cluster varieties*, *Invent. Math.* **175** (2008) 223 [[math/0702397](#)].
- [13] C. Geiss, B. Leclerc and J. Schröer, *Cluster structures on quantum coordinate rings*, *Sel. Math. New Ser.* **19** (2013) 337 [[arXiv:1104.0531](#)].
- [14] M. Gerstenhaber, *The cohomology structure of an associative ring*, *Ann. Math.* **78** (1963) 267.
- [15] V. Danilov, A. Karzanov and G. Koshevoy, *The purity of set-systems related to Grassmann necklaces*, [arXiv:1312.3121](#).
- [16] J.A. Guccione and J.J. Guccione, *Hochschild homology of some quantum algebras*, *J. Pure Appl. Algebra* **132** (1998) 129.
- [17] S.I. Gelfand and Y.I. Manin, *Methods of Homological Algebra*, 2nd edition, Springer Monographs in Mathematics, Springer (2003).
- [18] A. Konechny and A.S. Schwarz, *Introduction to M(atrix) theory and noncommutative geometry*, *Phys. Rept.* **360** (2002) 353 [[hep-th/0012145](#)] [[INSPIRE](#)].
- [19] S. Launois, T.H. Lenagan and L. Rigal, *Prime ideals in the quantum Grassmannian*, *Sel. Math. New Ser.* **13** (2008) 697. [[arXiv:0708.0744](#)].
- [20] S. Launois, T.H. Lenagan and B.M. Nolan, *Total positivity is a quantum phenomenon: the Grassmannian case*, in preparation.
- [21] J.L. Loday, *Cyclic Homology*, Springer-Verlag Berlin Heidelberg (1992).
- [22] M. Mariño, *Chern-Simons theory and topological strings*, *Rev. Mod. Phys.* **77** (2005) 675 [[hep-th/0406005](#)] [[INSPIRE](#)].
- [23] S. Oh, A. Postnikov and D.E. Speyer, *Weak separation and plabic graphs*, *Proc. Lond. Math. Soc.* **110** (2015) 721 [[arXiv:1109.4434](#)].
- [24] A. Postnikov, *Total positivity, Grassmannians and networks*, [math/0609764](#) [[INSPIRE](#)].
- [25] J. Scott, *Quasi-commuting families of quantum minors*, *J. Algebra* **290** (2005) 204 [[math/0008100](#)].
- [26] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032 [[hep-th/9908142](#)] [[INSPIRE](#)].
- [27] B. Tsygan, *Noncommutative calculus and operads*, [arXiv:1210.5249](#).
- [28] D. Tamarkin and B. Tsygan, *Noncommutative differential calculus, homotopy BV algebras and formality conjectures*, [math/0002116](#).
- [29] M. Wambst, *Hochschild and cyclic homology of the quantum multiparametric torus*, *J. Pure Appl. Algebra* **114** (1997) 321.
- [30] C.A. Weibel, *An Introduction to Homological Algebra*, Cambridge Studies in Advanced Mathematics, Book 38, [Cambridge University Press](#) (1994).
- [31] E. Witten, *Chern-Simons gauge theory as a string theory*, *Prog. Math.* **133** (1995) 637 [[hep-th/9207094](#)] [[INSPIRE](#)].
- [32] E. Witten, *Quantum background independence in string theory*, in *Conference on Highlights of Particle and Condensed Matter Physics (SalamFEST)*, Trieste, Italy, March 8–12, 1993, pp. 257–275 [[hep-th/9306122](#)] [[INSPIRE](#)].