On holographic Rényi entropy in some modified theories of gravity

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ABSTRACT: We perform a detailed analysis of holographic entanglement Rényi entropy in some modified theories of gravity with four dimensional conformal field theory duals. First, we construct perturbative black hole solutions in a recently proposed model of Einsteinian cubic gravity in five dimensions, and compute the Rényi entropy as well as the scaling dimension of the twist operators in the dual field theory. Consistency of these results are verified from the AdS/CFT correspondence, via a corresponding computation of the Weyl anomaly on the gravity side. Similar analyses are then carried out for three other examples of modified gravity in five dimensions that include a chemical potential, namely Born-Infeld gravity, charged quasi-topological gravity and a class of Weyl corrected gravity theories with a gauge field, with the last example being treated perturbatively. Some interesting bounds in the dual conformal field theory parameters in quasi-topological gravity are pointed out. We also provide arguments on the validity of our perturbative analysis, whenever applicable.

KEYWORDS: AdS-CFT Correspondence, Black Holes in String Theory

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1 Introduction

Entanglement in quantum mechanical systems is a topic that has intrigued and engaged physicists ever since the discovery of quantum mechanics. Two quantum subsystems that make up a composite system are said to be entangled when measurements performed on the subsystems are correlated, i.e the states of the individual subsystems are not independent. Broadly speaking, a measure of entanglement for a subsystem can be obtained by tracing out the complementary degrees of freedom from the total density matrix. Applications of the idea of entanglement to quantum many body systems and quantum field theories [1, 2] are relatively recent, but these have witnessed a lot of activity in the recent past, in the context of condensed matter physics, quantum information theory, etc. [3–5].

In the context of the AdS/CFT correspondence [6, 7], the calculation of the entanglement entropy of a strongly coupled quantum field theory holographically from a dual gravitational description arose out of the seminal work by Ryu and Takayanagi [8]. The

Ryu-Takayangi prescription and its generalisations [9–14] (see also [15, 16] for related discussions) have been at the forefront of research over the last decade and promises to yield a deep understanding of entanglement in strongly coupled quantum systems. Following up on this line of research, in recent times, there has been a surge of interest in investigating higher curvature theories of gravity, like Gauss-Bonnet gravity, Lovelock gravity [17], and quasi-topological gravity [18–20]. We will call such theories generically as modified theories of gravity (in the sense of being modified from the standard Einstein gravity). It is well known that such theories often suffer from the problem of negative energy states in their spectrum, but nonetheless might provide important clues towards the quantization of gravity. Viewed from a string theory perspective, these theories might be viewed as higher order corrections to the standard Einstein-Hilbert action, the latter being a low energy effective theory, and the corrections becoming important at large energy scales. Indeed, CFT duals to these modified theories of gravity have been extensively studied by holographic methods by now (see, for example, [21–25]).

Understanding entanglement in modified theories of gravity is important and interesting in its own right. Apart from providing insights into the nature of entanglement in novel examples of strongly coupled quantum field theories, these often provide us with extra tuneable physical parameters that substantially enrich the phase structure of the system, and is expected to be of use to model realistic situations. Further, these theories often provide useful insights in terms of novel physical bounds on the parameters of the dual CFT.

With these motivations, the purpose of this paper is to understand the physics of entanglement in some examples of modified theories of gravity. Recall the popular notion of entanglement entropy (EE) for a field theory which is a composite system of two subsystems, A and its complement B . Consider the reduced density matrix of A , given by ρ_A , obtained by tracing over the degrees of freedom corresponding to B out of the density matrix of the full system. The entanglement entropy of the subsystem A is given by,

$$S_{\text{EE}} = -\text{tr}(\rho_A \log \rho_A), \quad (1.1)$$

and is by definition, the von Neumann entropy of subsystem A . The entanglement Rényi entropy (ERE) [26, 27] on the other hand is known to be the generalisation of the above, and is believed to be crucial for a complete understanding of quantum states in a physical system. This is defined as

$$S_n = \frac{1}{1-n} \log \text{tr}(\rho_A^n). \quad (1.2)$$

As explained in the next section, this can be cast into the form

$$S_n = \frac{n}{n-1} \frac{1}{T_0} \int_{T_0/n}^{T_0} S dT, \quad (1.3)$$

where, S denotes the thermal entropy of the CFT living on $R \times H^{d-1}$. Here, n , called the order of the ERE is a positive real number. By construction, $S_{\text{EE}} = \lim_{n \rightarrow 1} S_n$, and hence we can obtain the EE from the ERE. The focus of the current work will be on the

ERE. We will in particular, study a spherical entangling surface, first considered in the work of [28] (for related recent results, see, e.g [29]).

After briefly introducing the necessary notations and conventions in section 2, in the first part of this paper, we study in section 3 the ERE associated to the recently discovered Einsteinian cubic gravity (ECG) [30], focusing on a five dimensional example. ECG is in general characterised by a Lagrangian density which we schematically write as

$$\mathcal{L} = R - 2\Lambda + \mathcal{L}_{\text{GB}} + \mathcal{L}_{\text{cubic}} \quad (1.4)$$

where $\mathcal{L}_{\text{GB}}$ is the standard Gauss-Bonnet term and $\mathcal{L}_{\text{cubic}}$ is a new cubic interaction term found in [30] that differs from ones considered in the literature before, for the following reasons. First, while other higher derivative theories of gravity (for example quasi-topological gravity) that exist in the literature have the characteristic that the coupling constants associated with different terms in the Lagrangian are dimension dependent, in ECG on the other hand, the coupling constants are dimension independent. Secondly, unlike say Gauss-Bonnet gravity, the extra cubic terms in the Lagrangian are not topological in four spacetime dimensions, i.e Einsteinian cubic gravity provides an interesting generalisation of pure Einstein gravity and an useful laboratory to apply the holographic principle in three dimensional dual CFTs. This novel theory is therefore important and interesting, and has attracted attention of late. In particular, one might compute useful physical quantities in ECG in the context of the gauge gravity duality. In the first part of this paper, we initiate such a study.

We construct the first known example of a black hole solution in the theory in five dimensions perturbatively, and are naturally led to the study of measures of entanglement of the resulting CFT, in particular the ERE. We also compute the scaling dimensions of the twist operators (to be elaborated upon in the next section) and verify the central charges of the dual CFT via a computation of the Weyl anomaly. We believe that ours is historically the first attempt at such a holographic analysis for these types of theories. We should mention here that after our paper appeared on the arXiv, the work of [31] found a generic class of black hole solutions in these theories and studied their thermodynamics. Several papers have appeared on the topic since then, and it is now known that in four dimensions, Einsteinian cubic gravity belongs to a general class of models called “Generalized quasi-topological gravity” theories [32] that has the same linearised spectrum as that of pure Einstein gravity. For recent developments in holographic studies of ECG, we refer the reader to [33] and references therein.

Continuing our analysis of the ERE in modified gravity theories, in the next part of this work, we consider a few examples of such theories whose dual CFTs have a conserved global charge. In this case, we need to understand how the entanglement between the two subsystems A and B depends on the charge distribution among themselves. For a grand canonical ensemble, the charged ERE is defined [34, 35] by simply generalizing eq. (1.2),

$$S_n(\mu_c) = \frac{1}{1-n} \log \text{tr} \left[\rho_A \frac{e^{\mu_c Q_A}}{n_A(\mu_c)} \right]^n, \quad (1.5)$$

where, μ_c is the chemical potential conjugate to the charge Q_A and $n_A(\mu_c) = \text{tr}(\rho_A e^{\mu_c Q_A})$ is introduced to normalize the new density matrix with unit trace. One can derive the formula for charged ERE which resembles eq. (1.3) with S_n being a function of μ_c ,

$$S_n(\mu_c) = \frac{n}{n-1} \frac{1}{T_0} \int_{T_0/n}^{T_0} S(T, \mu_c) dT. \quad (1.6)$$

In section 4 of this paper, we consider ERE of charged quasi-topological gravity, whose thermodynamics was studied in [38]. Our aim here is to see whether the ERE can reveal any novel features on the parameter space. Indeed, we are able to provide interesting bounds on the central charges as well as other parameters of the dual CFT via our analysis. This arises upon studying the constraints that must be satisfied by the ERE (elaborated in the next section). In section 5, for the sake of completeness, we consider briefly ERE in two other extended gravity theories. We first consider Born-Infeld gravity [36, 37], and elaborate upon the behavior of the charged ERE both as a function of the chemical potential for fixed values of the Born-Infeld parameter, as well as a function of the Born-Infeld parameter with fixed chemical potential. We then briefly study Einstein's gravity with a Weyl-corrected gauge field [39, 40].

We should mention here that our studies of the ERE in Einsteinian cubic gravity and Weyl-corrected gravity contain perturbative analysis, and in order to justify the numerical values of the coupling constant that we have chosen, we have to check that the second order corrections to our linear order results in this paper are indeed small. Two appendices of this paper address this issue. For ECG, the relevant second order black hole solutions for ECG are summarised in appendix A, and these have been used in the main text to justify our linear order results. For Weyl-corrected gravity, the second order expressions are detailed in appendix B, but the analysis become tedious, and the second order corrections could not be performed satisfactorily, due to numerical issues, as we explain towards the end of the paper.

A word about the notation used: we consider four different theories and consider similar physical quantities in each of them. However, using different symbols for different theories will unnecessarily clutter the notation and we avoid doing it. It has to be thus remembered that a particular symbol used in a section remains relevant for that section only.

2 Holographic entanglement Rényi entropy

In this section, we will briefly review the relevant details of the computation of various quantities associated with holographic ERE. Since the material is quite well established by now, we will be brief here, and point the reader to the references herein for further details.

The calculation of the Rényi entropy in CFTs depends on the replica trick [1, 2] which involves computing a path integral over n -fold cover consisting of replicas of the original CFT. The different copies are separated by twist operators [41] inserted at branch points. Recently, holographic methods have been used to compute the ERE for a variety of CFTs with different bulk duals. Extending the idea of [42], the authors of [28] derived a formula for holographic ERE with spherical entangling surfaces and calculated the ERE explicitly

for Einstein gravity as well as some higher derivative theories like Gauss-Bonnet gravity and quasi-topological gravity. In a related work, [43] computed the ERE for four dimensional $\mathcal{N} = 4$ super-Yang-Mills holographically, using the same method. This proposal of ERE was then generalized by [34] to include a chemical potential in the field theory. They constructed the charged entanglement Rényi entropy in a grand canonical ensemble where one has to compute the Euclidean path integral by inserting a Wilson line encircling the entangling surface. For some recent interesting works on holographic Rényi entropy we refer the reader to [44–48].

Let us briefly recapitulate the computation of the ERE, closely following [42] and [28]. We consider a spherical entangling surface S^{d-1} in a flat d -dimensional CFT. From [42], it can be shown that the CFT can be mapped to a hyperbolic cylinder $R \times H^{d-1}$ by a conformal transformation. So, effectively the correlators in the vacuum CFT are mapped to a thermal ensemble on $R \times H^{d-1}$, with the temperature being fixed by the curvature scale $\mathcal{R}$ of the cylinder (which is also the radius of the original sphere S^{d-1}), given by,

$$T_0 = \frac{1}{2\pi\mathcal{R}}. \quad (2.1)$$

Now, using the AdS/CFT correspondence, we can infer that the CFT on the hyperbolic cylinder $R \times H^{d-1}$ has a bulk dual which is a black hole with a hyperbolic horizon. The above argument implies a simple relation between the thermal density matrix and the reduced density matrix ρ_A as,

$$\rho_A = U^{-1} \frac{e^{-H/T_0}}{Z(T_0)} U, \quad (2.2)$$

where U is a unitary transformation and $Z(T_0) = \text{tr}(e^{-H/T_0})$ and hence we have,

$$\text{tr}(\rho_A^n) = \frac{Z(T_0/n)}{Z(T_0)^n}. \quad (2.3)$$

Now making use of the expression of free energy F in thermodynamics, $F(T) = -T \log Z(T)$, the ERE of the CFT can be calculated as,

$$S_n = \frac{n}{1-n} \frac{1}{T_0} \left[F(T_0) - F\left(\frac{T_0}{n}\right) \right]. \quad (2.4)$$

Since $S = -\frac{\partial F}{\partial T}$, the above relation can be immediately recast as eq. (1.3), where using the gauge/gravity duality, we can identify the thermal entropy arising in that equation with the Wald entropy S_{Wald} [49] of the hyperbolic black hole and hence we can compute the Rényi entropy using eq. (1.3).¹ An entirely similar analysis can be used to arrive at the formula of eq. (1.6) for the cases with a global conserved charge.

The Rényi entropy satisfies the following inequalities [28, 51],

$$\begin{aligned} \frac{\partial S_n}{\partial n} &\leq 0, & \frac{\partial}{\partial n} \left(\frac{n-1}{n} S_n \right) &\geq 0, \\ \frac{\partial}{\partial n} \left((n-1) S_n \right) &\geq 0, & \frac{\partial^2}{\partial n^2} \left((n-1) S_n \right) &\leq 0, \end{aligned} \quad (2.5)$$

¹Recently, [50] considered corrections to Wald entropy of topological black holes and found logarithmic corrections to Rényi entropy.

where it can be shown from the gravity side that satisfying the inequalities in the first line of eq. (2.5) automatically leads to the ones in the second line of that equation [35]. As was argued in [28], these inequalities will hold for any CFT, as long as we have a stable thermal ensemble. Nevertheless, in all the examples considered in this paper, we will explicitly check the above inequalities. This is particularly important for the perturbative solutions that we consider.

Now, as we have mentioned, the replica trick for computing the Rényi entropy can be understood as the insertion of a surface operator, known as the twist operator (σ_n), at the entangling surface [28, 34, 52]. The scaling dimension of this operator is determined from the leading singularity in the correlator $\langle T_{\mu\nu} \sigma_n \rangle$. The leading singular behavior is fixed by the symmetry, tracelessness and the conservation properties of $T_{\mu\nu}$.

Let us consider a four dimensional CFT where a planar twist operator is positioned at $x^1 = x^2 = 0$ in flat Euclidean space. The twist operator extends along the other two directions x^3 and x^4 . Now, one inserts a stress tensor at (y^1, y^2, y^3, y^4) and thus the orthogonal distance between the twist operator and the stress tensor operator is given by, $l = \sqrt{(y^1)^2 + (y^2)^2}$. With this notation, the leading singularity in the correlator of the twist operator and the stress tensor can be calculated as [28, 34],

$$\begin{aligned} \langle T_{ij} \sigma_n \rangle &= -\frac{h_n}{2\pi} \frac{\delta_{ij}}{l^4}, & \langle T_{ia} \sigma_n \rangle &= 0, \\ \langle T_{ab} \sigma_n \rangle &= \frac{h_n}{2\pi} \frac{3\delta_{ab} - 4n_a n_b}{l^4}, \end{aligned} \quad (2.6)$$

where, $(a, b) = (1, 2)$ are the normal directions and $(i, j) = (3, 4)$ are the tangential directions to the twist operator. Here $n^a = \frac{y^a}{l}$ is the orthogonal unit vector from the twist operator σ_n to the point where the stress tensor $T_{\mu\nu}$ has been inserted. From eq. (2.6), one should notice that the leading singularity is now fixed up to a constant h_n , known as the scaling dimension of σ_n . Following [28, 34], we quote the final expression of the scaling dimension h_n (in the four dimensional boundary CFT) in terms of the thermal energy density,

$$h_n(\mu_c) = \frac{n}{3} \frac{\mathcal{R}^3}{T_0} \left[\mathcal{E}(T_0, 0) - \mathcal{E}\left(\frac{T_0}{n}, \mu_c\right) \right], \quad (2.7)$$

where, $\mathcal{E}(T, \mu_c)$ is the thermal energy density given by,

$$\mathcal{E}(T, \mu_c) = \frac{E(T, \mu_c)}{\mathcal{R}^3 V_\Sigma}. \quad (2.8)$$

$E(T, \mu_c)$ being the total thermal energy and V_Σ is the regulated volume of the hyperbolic plane H^3 . We should point out here that the first term in eq. (2.7) appears due to the anomalous contribution from the stress tensor under conformal transformations. One can also express eq. (2.7) in terms of the thermal entropy density $\mathcal{S}$ using the first law of thermodynamics,

$$d\mathcal{E} = T d\mathcal{S} + T_0 \mu_c d\rho, \quad (2.9)$$

where, $\mathcal{S}$ and ρ are respectively the thermal entropy density and the charge density of the dual CFT.

The conformal dimension $h_n(\mu_c)$ possesses an interesting universal property, when expanded around $n = 1$ and $\mu_c = 0$,

$$h_n(\mu_c) = \sum_{\lambda, \delta} \frac{1}{\lambda! \delta!} h_{\lambda\delta} (n-1)^\lambda \mu_c^\delta, \quad \text{where } h_{\lambda\delta} = (\partial_n)^\lambda (\partial_{\mu_c})^\delta h_n(\mu_c)|_{n=1, \mu_c=0}. \quad (2.10)$$

Note that $h_{00} = 0$, while the authors of [28, 34] showed that the quantity h_{10} is related to the central charge $\tilde{C}_T$ for any d -dimensional CFT

$$h_{10} = \frac{2}{d-1} \pi^{1-\frac{d}{2}} \Gamma\left(\frac{d}{2}\right) \tilde{C}_T. \quad (2.11)$$

As mentioned in [34, 35], since we are discussing charged Rényi entropy, one may think about the correlator of the twist operator σ_n and the current operator J_μ . Here also one can extract the leading singular behavior of this correlator using the conservation of the current J_μ as

$$\langle J_a \sigma_n(\mu_c) \rangle = -\frac{ik_n(\mu_c)}{2\pi} \frac{\epsilon_{ab} n^b}{l^3}, \quad \langle J_a \sigma_n \rangle = 0, \quad (2.12)$$

where ϵ_{ab} stands for the volume form and $k_n(\mu_c)$ represents the magnetic response characterizing the response of the current to the magnetic flux. Finally, one can derive the magnetic response $k_n(\mu_c)$ using a conformal mapping [34, 35] as,

$$k_n(\mu_c) = 2\pi n \mathcal{R}^3 \rho(n, \mu_c). \quad (2.13)$$

In a similar way as argued with the scaling dimension, here one may expand the magnetic response $k_n(\mu_c)$ around $n = 1$ and $\mu_c = 0$,

$$k_n(\mu_c) = \sum_{\lambda, \delta} \frac{1}{\lambda! \delta!} k_{\lambda\delta} (n-1)^\lambda \mu_c^\delta, \quad \text{where } k_{\lambda\delta} = (\partial_n)^\lambda (\partial_{\mu_c})^\delta k_n(\mu_c)|_{n=1, \mu_c=0}. \quad (2.14)$$

In this work, we will broadly study the concepts introduced above. Having set up the notations and conventions, we will now proceed to the main body of this paper.

3 ERE for Einsteinian cubic gravity

Our first example is the analysis of ERE in ECG. This is an interesting proposal recently put forward in [30]. As stated in the introduction, modified theories of gravity often suffer from ghost modes in the spectrum. The work of [30] on the other hand attempts to formulate a theory which at the linearized level has no ghost modes, and where the coupling constants are independent of spacetime dimensions. This theory has the following action

$$\begin{aligned} S = \frac{1}{2\ell_p^3} \int d^5x \sqrt{-g} & \left[R - 2\Lambda + \frac{\lambda}{2} L^2 \chi_4 + \beta_1 L^4 \left(R_\mu{}^\rho{}_\nu{}^\sigma R_\rho{}^\gamma{}_\sigma{}^\delta R_\gamma{}^\mu{}_\delta{}^\nu + \frac{1}{12} R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\gamma\delta} R_{\gamma\delta}{}^{\mu\nu} \right. \right. \\ & \left. \left. - R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma} + \frac{2}{3} R_\mu{}^\nu{}_\nu{}^\rho R_\rho{}^\mu \right) + \beta_2 L^4 \left(\frac{14}{3} R_{\mu\nu}{}^{\rho\sigma} R_{\rho\sigma}{}^{\gamma\delta} R_{\gamma\delta}{}^{\mu\nu} - 24 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho}{}_\gamma R^{\sigma\gamma} \right. \right. \\ & \left. \left. + 3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} R + 16 R_{\mu\nu\rho\sigma} R^{\mu\rho} R^{\nu\sigma} + \frac{64}{3} R_\mu{}^\nu{}_\nu{}^\rho R_\rho{}^\mu - 12 R_{\mu\nu} R^{\mu\nu} R + R^3 \right) \right] \quad (3.1) \end{aligned}$$

where, R is the Ricci scalar and L is the AdS length, related to the cosmological constant Λ by, $\Lambda = -\frac{6}{L^2}$. χ_4 is the standard four-derivative Gauss-Bonnet term,

$$\chi_4 = R_{\mu\nu\rho\lambda}R^{\mu\nu\rho\lambda} - 4R_{\mu\nu}R^{\mu\nu} + R^2, \quad (3.2)$$

while β_1 and β_2 are the coupling constants of two new sets of six derivative terms. As stated earlier, our goal is to study the ERE of ECG using holographic methods. As in any other case of extended theories of gravity, this provides a novel dual CFT, which is interesting to analyze.

It is difficult (if not impossible) to analytically solve the Einstein equations for this cubic theory taking into account all the three parameters λ , β_1 and β_2 . We will be interested in the cubic terms in the Lagrangian, and hence exclude the Gauss-Bonnet coupling from our analysis (the latter was studied in [28]). We will thus focus on one of the new coupling terms β_1 and β_2 . To simplify the computation, we will set $\beta_1 = \frac{3}{2}\beta$ and $\beta_2 = 0$ (the numerical factor of $\frac{3}{2}$ is just a choice which simplifies some constants that appear in our analysis in what follows. Any other choice will not alter the essential physics). Of course, one could alternatively set $\beta_1 = 0$ to begin with, as well. Now, even with a single coupling, we found it difficult to solve the system exactly. So, we proceed to solve the system up to linear order in the new coupling constant β .

In order to compute the holographic Rényi Entropy dual to this cubic gravity, we need to construct a hyperbolic black hole solution which can be done by choosing the following metric ansatz,

$$ds^2 = -\left(\frac{r^2}{L^2}f_0(r) - 1\right)(1 + \beta\mathcal{F}_1(r))\mathcal{N}(r)^2 dt^2 + \frac{dr^2}{\left(\frac{r^2}{L^2}f_0(r) - 1\right)(1 + \beta\mathcal{F}_2(r))} + r^2 d\Sigma_3^2 \quad (3.3)$$

where, $d\Sigma_3^2$ is the line element for the three dimensional hyperbolic plane H^3 having unit curvature. We will set

$$\mathcal{N}(r) = \frac{\tilde{L}}{\mathcal{R}}, \quad (3.4)$$

where $\tilde{L}$ is the AdS curvature scale which is to be determined and $\mathcal{R}$ is the curvature of the hyperbolic spatial slice.

Here $f_0(r)$ is the leading order solution which represents a hyperbolic Schwarzschild black hole in five dimensional AdS space, and has the form,

$$f_0(r) = 1 - \frac{\omega^4}{r^4}. \quad (3.5)$$

The position of the horizon of this unperturbed solution can be obtained by demanding, $f_0(r_h) = \frac{L^2}{r_h^2}$. This in turn, fixes the constant ω in the leading order solution as,

$$\omega = (r_h^4 - L^2 r_h^2)^{\frac{1}{4}} \quad (3.6)$$

Now varying the action with respect to $\mathcal{F}_1(r)$ and $\mathcal{F}_2(r)$ yield the following differential equations at order β^2 (the absence of $\mathcal{O}(\beta)$ terms signifies that the ansatz satisfies the

Einstein's equations at linear order)

$$\begin{aligned} r^9 (2rF_2(r) (L^2 - 2r^2) + F_2'(r) (L^2 r^2 - r^4 + \omega^4)) - 4\mathcal{A}(r) &= 0 \\ r^9 (2rF_2(r) (L^2 - 2r^2) + F_1'(r) (L^2 r^2 - r^4 + \omega^4)) - 4\mathcal{B}(r) &= 0, \end{aligned} \quad (3.7)$$

where we have denoted

$$\begin{aligned} \mathcal{A}(r) &= -72L^2 r^2 \omega^8 + r^{12} + 69r^4 \omega^8 - 74\omega^{12} \\ \mathcal{B}(r) &= -48L^2 r^2 \omega^8 + r^{12} + 45r^4 \omega^8 - 50\omega^{12} \end{aligned} \quad (3.8)$$

from which we obtain the following analytical solutions

$$\begin{aligned} \mathcal{F}_1(r) &= \frac{36L^2 r^2 \omega^8 + r^{12} - 57r^4 \omega^8 + 25\omega^{12}}{r^8 (L^2 r^2 - r^4 + \omega^4)} - \frac{C_1}{L^2 r^2 - r^4 + \omega^4} + C_2 \\ \mathcal{F}_2(r) &= \frac{48L^2 r^2 \omega^8 + r^{12} - 69r^4 \omega^8 + 37\omega^{12}}{r^8 (L^2 r^2 - r^4 + \omega^4)} - \frac{C_1}{L^2 r^2 - r^4 + \omega^4} \end{aligned} \quad (3.9)$$

where C_1 and C_2 are constants of integration, to be determined using appropriate boundary conditions. At this point it is important to note that we have inserted an extra factor of $\frac{1}{\mathcal{R}^2}$ in the metric component g_{tt} . With this inclusion, we make sure that the boundary CFT dual to this black hole geometry resides on $R \times H^3$, where the hyperbolic spatial slice H^3 has curvature $\mathcal{R}$ as mentioned before,

$$ds_\infty^2 = -dt^2 + \mathcal{R}^2 d\Sigma_3^2. \quad (3.10)$$

The constant C_2 can be set to zero since it simply produces a rescaling of the time coordinate, and we demand that the metric of eq. (3.3) be conformally equivalent to the one of eq. (3.10). Then C_1 can be fixed by demanding that the position of the horizon r_h remains the same as with the unperturbed solution, and by the fact that the potential singularities in $\mathcal{F}_1(r_h)$ and $\mathcal{F}_2(r_h)$ need to be avoided. It can be checked that this can be obtained by setting

$$C_1 = \frac{11L^6}{r_h^2} - 54L^4 + 75L^2 r_h^2 - 31r_h^4 \quad (3.11)$$

Since the asymptotic form of the bulk metric of eq. (3.3) should represent the background metric for the dual boundary CFT given by eq. (3.10), we now determine the AdS curvature scale $\tilde{L}$ as,

$$\tilde{L} = \frac{L}{\sqrt{1-\beta}} \equiv \frac{L}{\sqrt{f_\infty}}, \quad (3.12)$$

where we have used the fact that

$$(1 + \beta \mathcal{F}_1(r)) \left(f_0(r) - \frac{L^2}{r^2} \right) \Big|_{r \rightarrow \infty} = 1 - \beta \equiv f_\infty. \quad (3.13)$$

After determining the perturbed black hole solution up to $\mathcal{O}(\beta)$, we proceed to compute the Hawking temperature of the black hole. At this stage, it is convenient to make a change

of variables and use a new dimensionless variable, $x = \frac{r_h}{L}$, instead of using r_h . The Hawking temperature of the black hole then can be expressed as,

$$T = \frac{2x^2 - 1}{2\pi x \mathcal{R}} - \beta \frac{(x^2 - 1)^3}{\pi x^5 \mathcal{R}} \quad (3.14)$$

Note that there is a $\mathcal{O}(\beta)$ correction in the temperature and as we take the limit $\beta \rightarrow 0$, we recover the expression of the temperature with pure Einstein gravity [28], as expected.

To compute the thermal entropy of the hyperbolic black hole, we proceed to evaluate the Wald entropy following [49],

$$S_{\text{Wald}} = -2\pi \int_{\text{horizon}} d^3x \sqrt{h} \frac{\partial \mathcal{L}}{\partial R_{\mu\nu\rho\lambda}} \epsilon_{\mu\nu} \epsilon_{\rho\lambda} \quad (3.15)$$

This proposal was put forward to compute the black hole entropy in higher derivative gravity theories. Here, $\mathcal{L}$ represents the Lagrangian of the particular higher derivative gravity, and $\epsilon_{\mu\nu}$ is the binormal Killing vector normalized as $\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2$. The final expression of the Wald entropy turns out to be

$$S_{\text{Wald}} = 2\pi \left(\frac{L}{\ell_p} \right)^3 V_{\Sigma} \left[x^3 + \frac{3}{2}\beta \left(25x^3 - 48x + \frac{18}{x} \right) \right]. \quad (3.16)$$

The first term in eq. (3.16) represents the usual thermal entropy in case of a hyperbolic AdS black hole in pure Einstein gravity, while the second term stands for the correction in entropy due to the presence of the cubic term in the action. Here $V_{\Sigma} = \int_{H^3} d\Sigma_3$ is the volume of the hyperbolic plane. As shown in [28], V_{Σ} is divergent and this behavior mimics the UV divergence of the Rényi Entropy in the dual CFT. Hence, one needs to regularize the entropy and this is done by integrating out the hyperbolic volume element up to a maximum radius $\frac{\mathcal{R}}{\delta}$, where δ is the short-distance cut-off of the dual CFT. Now after expanding V_{Σ} in powers of $\frac{\mathcal{R}}{\delta}$, it is straightforward to single out the universal contribution coming from the sub-leading terms,

$$V_{\Sigma, \text{univ}} = -2\pi \log \left(\frac{\mathcal{R}}{\delta} \right). \quad (3.17)$$

Now, writing eq. (1.3) in terms of the dimensionless variable x , the entanglement Rényi entropy can be expressed as,

$$\begin{aligned} S_n &= \frac{n}{n-1} \frac{1}{T_0} \int_{x_n}^{x_1} S(x) \frac{dT}{dx} dx \\ &= \frac{n}{n-1} \frac{1}{T_0} \left[S(x) T(x) \Big|_{x_n}^{x_1} - \int_{x_n}^{x_1} \frac{dS}{dx} T(x) dx \right], \end{aligned} \quad (3.18)$$

where in the second line we have done an integration by parts.

It follows that x_1 and x_n (the upper and lower limits, respectively, in eq. (3.18)) are the only two remaining quantities that we need for the computation of the ERE. These two quantities can be determined by solving the following equation,

$$T(x_n) = \frac{T_0}{n} \quad (3.19)$$

where, $T_0 = \frac{1}{2\pi\mathcal{R}}$. From eqs. (3.14) and (3.19), note that at the $\beta \rightarrow 0$ limit, x_1 and x_n are given by,

$$\begin{aligned} (x_1)_{\beta \rightarrow 0} &= 1 \\ (x_n)_{\beta \rightarrow 0} &= \frac{1}{4n} \left(1 + \sqrt{1 + 8n^2} \right) \equiv \tilde{x}_n. \end{aligned} \quad (3.20)$$

Now substituting eq. (3.14) in eq. (3.19) yields a sixth order algebraic equation for x_n which can not be solved exactly for arbitrary n . However, with $n = 1$, we find $x_1 = 1$ is still a solution of the equation. To determine x_n for arbitrary n , we solve x_n up to linear order in β ,

$$x_n = \tilde{x}_n - 2\beta \frac{(1 - \tilde{x}_n^2)^3}{\tilde{x}_n^3(1 + 2\tilde{x}_n^2)} \quad (3.21)$$

where, $\tilde{x}_n$ is defined in eq. (3.20). We construct the solution in such a way that x_n agrees with eq. (3.20) when β vanishes.

Finally, we compute the ERE,

$$\begin{aligned} S_n &= \frac{n}{n-1} T_0 \int_{x_1}^{x_n} S_{\text{Wald}} T'(x) dx \\ &= \frac{n}{n-1} \pi V_\Sigma \left(\frac{L}{\ell_p} \right)^3 \left[(2 - \tilde{x}_n^2 - \tilde{x}_n^4) + \frac{3\beta}{2\tilde{x}_n^2} (1 - \tilde{x}_n^2) (14 - 56\tilde{x}_n^2 + 27\tilde{x}_n^4) \right] \end{aligned} \quad (3.22)$$

We can also calculate the entanglement entropy from the above expression of ERE by taking the $n \rightarrow 1$ limit, which yields,

$$S_1 = 2\pi \left(\frac{L}{\ell_p} \right)^3 V_\Sigma \left(1 - \frac{15}{2}\beta \right). \quad (3.23)$$

Notice that, the above expression in the $\beta \rightarrow 0$ limit corresponds to the entanglement entropy for any CFT in four space-time dimensions dual to a five dimensional pure Einstein gravity. The second term in the above expression represents the correction in entanglement entropy due to the presence of the cubic term in the bulk gravity. This correction in entanglement entropy is related to the central charge a for a four dimensional CFT. As we will show in section 3.3, this matches with a corresponding calculation from the gravity side, via a computation of the Weyl anomaly.

3.1 Numerical analysis and results

In this subsection, we present the results for the ERE in ECG, plotted against two quantities, namely the order of the ERE n and the coupling β . Since S_n contains the volume factor V_Σ involving the short-distance cut-off of the boundary CFT, we plot the ratio of Rényi entropy S_n to the entanglement entropy S_1 , instead of plotting S_n . Let us first consider the behavior of the ERE as a function of the coupling β for different values of n as shown. This is shown in figure 1. Since we have treated the problem perturbatively, we consider small values of β up to $\beta = 0.05$ (these numerical values will be justified in a while, towards the end of this section). In figure 2, we have shown the variation of $\frac{S_n}{S_1}$ with

Figure 1. $\frac{S_n}{S_1}$ is plotted as a function of β , where the red, orange, green, blue and purple lines correspond to the values $n = 1, 2, 3, 10$ and 100 , respectively.

Figure 2. $\frac{S_n}{S_1}$ is plotted as a function of n , where the red, orange, green, blue and purple lines denote $\beta = 0, 0.01, 0.02, 0.03$ and 0.04 , respectively.

Figure 3. $\frac{n-1}{n} \frac{S_n}{S_1}$ is plotted as a function of n for different values of β , where we follow the same color coding for β as used in figure 2.

n for different values of β , and the red, orange, green, blue and purple lines denote $\beta = 0, 0.01, 0.02, 0.03$ and 0.04 , respectively. The slope of all the lines are negative, i.e., $\frac{\partial S_n}{\partial n} \leq 0$. Hence, we notice that the first inequality of eq. (2.5) is obeyed for any small value of the coupling constant β in Einsteinian cubic gravity. In figure 3, we show the variation of the quantity $\frac{n-1}{n} \frac{S_n}{S_1}$ with n for different values of the coupling β . Here again we follow the same color coding for β as in figure 2. Note that the slope of the lines are positive for all chosen values of β , i.e., $\frac{\partial}{\partial n} \left(\frac{n-1}{n} S_n \right) \geq 0$, which supports the second inequality obeyed by Rényi entropy in eq. (2.5).

3.2 Scaling dimension of twist operators

Recall that as mentioned in section 2, the field theoretic method for calculating entanglement entropy involves performing a path integral over n replicas of the original CFT, and that twist operators open branch cuts between these copies. In this example, there is no chemical potential, and hence using eq. (2.9), eq. (2.7) reduces to the result [28],

$$h_n = \frac{n}{3T_0 V_\Sigma} \int_{x_n}^{x_1} dx T(x) S'(x). \quad (3.24)$$

Using the above equation, the dimension of the twist operator can be computed up to linear order in β as,

$$\begin{aligned} h_n = & -\frac{\pi}{n^3} \left(\frac{L}{\ell_p} \right)^3 \left[\frac{1}{32} \frac{(1 + \sqrt{1 + 8n^2})^3}{3 + \sqrt{1 + 8n^2}} (1 - n^2) \right] \\ & - \frac{\pi}{n^3} \left(\frac{L}{\ell_p} \right)^3 \frac{\beta}{64\sqrt{1 + 8n^2}} \left[\sqrt{1 + 8n^2} (88n^4 - 340n^2 + 81) + 432n^2 + 81 \right] + \mathcal{O}(\beta^2). \end{aligned} \quad (3.25)$$

As expected, the leading order term in the expression corresponds to a CFT with a bulk Einstein dual. Also note that, h_n at linear order vanishes if one takes the limit $n \rightarrow 1$. We also find that

$$\partial_n h_n|_{n=1} = \frac{2}{3} \pi \left(\frac{L}{\ell_p} \right)^3 \left(1 + \frac{9}{2} \beta + \mathcal{O}(\beta^2) \right) = \frac{2}{3\pi} c \quad (3.26)$$

where $c = \pi^2 \left(\frac{L}{\ell_p} \right)^3 \left(1 + \frac{9}{2} \beta + \mathcal{O}(\beta^2) \right)$ is the central charge of the four dimensional CFT. The above relation between the central charge c and the first order coefficient of the expansion of the scaling dimension $\partial_n h_n|_{n=1}$ is a general property and we show that this holds for ECG, as expected. We will compute the central charge c in the next section by studying holographic Weyl anomaly and show that it matches with the result above. This is of course expected, as the symmetry of the boundary CFT implies eq. (3.26).

3.3 Weyl anomaly and central charges

In this section, we compute both the central charges c and a characterizing a four dimensional CFT dual to ECG using the gauge/gravity duality. It is well known that, although the trace of the energy momentum tensor $\langle T_\mu^\mu \rangle$ vanishes in flat space, it is no longer zero if we place the CFT in a curved background. This is known as Weyl anomaly or conformal anomaly [53, 54], which relates the CFT parameters to the coupling constants of the dual gravity theory. The trace anomaly for a four dimensional CFT is given by,

$$\langle T_\mu^\mu \rangle = \frac{c}{16\pi^2} I_4 - \frac{a}{16\pi^2} E_4, \quad (3.27)$$

where,

$$I_4 = R_{ijkl} R^{ijkl} - 2R_{ij} R^{ij} + \frac{1}{3} R^2, \quad (3.28)$$

is the square of the Weyl tensor C_{ijkl} , while E_4 represents the Euler density in four dimensions,

$$E_4 = R_{ijkl} R^{ijkl} - 4R_{ij} R^{ij} + R^2. \quad (3.29)$$

Following the prescription of [54], one can obtain the central charges c and a via the gauge/gravity duality. Starting with the gravity action and employing the Fefferman-Graham expansion [55] one can write down the bulk metric as,

$$ds^2 = \frac{\tilde{L}^2}{4\rho^2} d\rho^2 + \frac{\bar{g}_{ij}}{\rho} dx^i dx^j, \quad (3.30)$$

where,

$$\bar{g}_{ij} = \bar{g}_{(0)ij} + \rho \bar{g}_{(1)ij} + \rho^2 \bar{g}_{(2)ij} + \dots, \quad (3.31)$$

and $\bar{g}_{(0)ij}$ represents the boundary metric at $\rho = 0$. Now substituting the form of the metric of eq. (3.30) into the gravity action (eq. (3.1)), $\bar{g}_{(2)ij}$ can be eliminated using the field equations. After writing the action in terms of $\bar{g}_{(0)ij}$ and $\bar{g}_{(1)ij}$ one can further express $\bar{g}_{(1)ij}$ in terms of $\bar{g}_{(0)ij}$ and write down the action in terms of $\bar{g}_{(0)ij}$ only. Finally, one can read the anomaly coefficients by extracting the terms producing a log divergence,

$$S_{\text{ln}} \sim \frac{1}{2} \ln \epsilon \int d^4x \sqrt{\bar{g}_{(0)}} \langle T_\mu^\mu \rangle, \quad (3.32)$$

where ϵ is a short distance cut-off. This procedure of finding the anomaly coefficients is straightforward but the calculation becomes complicated in higher derivative gravity theories. So, instead of following this approach, we follow [56, 57] in order to compute the central charges.² We choose the following bulk metric,

$$ds^2 = \frac{\tilde{L}^2}{4\rho^2} d\rho^2 + \frac{1}{\rho} \left[u(1 + A\rho) \left(-R^2 dt^2 + \frac{dR^2}{R^2} \right) + v(1 + B\rho) (d\theta^2 + \sin^2 \theta d\phi^2) \right], \quad (3.33)$$

having a boundary $AdS_2 \times S^2$. After plugging this metric into the Lagrangian, we extract the coefficient of $\frac{1}{\rho}$ which gives rise to a log divergence. We call this term L_{ln} , given by,

$$\begin{aligned} L_{\text{ln}} = & -\frac{1}{2L^2 \tilde{L}^5 \ell_p^3} \left(-A^2 L^2 uv \tilde{L}^4 + 2ABL^2 uv \tilde{L}^4 - 6ABuv \tilde{L}^6 - 3A\beta L^6 u \tilde{L}^2 \right. \\ & - AL^2 u \tilde{L}^6 - B^2 L^2 uv \tilde{L}^4 + 3\beta BL^6 v \tilde{L}^2 + BL^2 v \tilde{L}^6 + 6\beta L^6 \tilde{L}^4 - 3A^2 \beta L^6 uv \\ & \left. - 6A\beta BL^6 uv - 3\beta B^2 L^6 uv \right) \sin \theta \end{aligned} \quad (3.34)$$

The equations of motion for A and B are given by,

$$\frac{\partial}{\partial A} L_{\text{ln}} = 0, \quad \text{and} \quad \frac{\partial}{\partial B} L_{\text{ln}} = 0, \quad (3.35)$$

solving which we can determine A and B . Now the four dimensional quantities I_4 and E_4 can be computed using the boundary metric $\bar{g}_{(0)ij}$ as,

$$I_4 = \frac{4(u-v)^2}{uv}, \quad E_4 = -\frac{8}{uv} \quad (3.36)$$

²The authors of [58] developed an elegant method to simplify the computation of holographic Weyl anomaly and obtain the central charges for CFTs dual to higher derivative gravity. Using this approach, one does not need to solve any equations of motion. Instead, one can expand the action around a “referenced curvature” and then derive the central charges from the coefficient in the expansion.

Numerical results for $\frac{S_n}{S_1}$ in ECG		
β	S_n/S_1 at $\mathcal{O}(\beta)$	S_n/S_1 at $\mathcal{O}(\beta^2)$
0	0.625272	0.625272
0.01	0.617127	0.617041
0.02	0.607545	0.607182
0.03	0.596108	0.59524
0.04	0.58222	0.580575

Table 1. Numerical results for $\frac{S_n}{S_1}$ with different values of the coupling β . The middle and the right part of the table show results for $\frac{S_n}{S_1}$ with $\mathcal{O}(\beta)$ and $\mathcal{O}(\beta^2)$ corrections, respectively.

Now, by noting the form of I_4 and E_4 in (eq. (3.36)), it is easy to show that the central charges can be obtained by taking the following limits,

$$\begin{aligned}
c &= \lim_{v \rightarrow \infty} 24\pi^2 \frac{L_{\text{ln}}}{\sqrt{g(0)}}|_{u=1}, \\
a &= \lim_{v \rightarrow 1} 4\pi^2 \frac{L_{\text{ln}}}{\sqrt{g(0)}}|_{u=1}.
\end{aligned} \tag{3.37}$$

Finally, using eq. (3.37) along with eq. (3.12), we determine the central charges up to linear order in β as,

$$c = \pi^2 \left(\frac{L}{\ell_p} \right)^3 \left(1 + \frac{9}{2}\beta + \mathcal{O}(\beta^2) \right), \tag{3.38}$$

$$a = \pi^2 \left(\frac{L}{\ell_p} \right)^3 \left(1 - \frac{15}{2}\beta + \mathcal{O}(\beta^2) \right). \tag{3.39}$$

Note that the expressions of the entanglement entropy in section 3 and the scaling dimension of twist operator in section 3.2 agree with the values for central charges obtained by the computation of holographic Weyl anomaly.

3.4 Validity of Rényi entropy inequalities

Since we work perturbatively up to linear order in the coupling constant β , we should consider relatively small values of β in all our computations. However, to justify that our results are reliable (by taking into account the terms up to linear order in β), we should find the change in Rényi entropy due to the inclusion of next-to-leading order corrections and check that it is indeed small. For this purpose, we need to solve the Einstein equations at $\mathcal{O}(\beta^2)$. Following the same strategy as described earlier, we obtained a hyperbolic black hole solution at $\mathcal{O}(\beta^2)$ (see appendix A).

The numerical results for the ERE at different values of the cubic coupling with $\mathcal{O}(\beta)$ and $\mathcal{O}(\beta^2)$ corrections are shown in table 1. To note the deviation in the values with increasing n , we have chosen a large value of n ($= 1000$) to produce the table. However, even with this large value of n , the second order correction in the entropy ratio is small and hence the first-order correction term is sufficiently reliable. It can be observed from the

table that $\mathcal{O}(\beta^2)$ correction appears only at the third decimal place even when we increase the coupling β to 0.04.

Here we should mention that the inequalities obeyed by Rényi entropy are satisfied in this regime of the coupling β which are shown in figures 2 and 3.

Having studied ERE and its various features for ECG, we will now move over to examples with gauge fields. Our next analysis involves a theory of quasi-topological gravity with a chemical potential.

4 Charged Rényi entropy with quasi-topological gravity

In this section, we will discuss the properties of the charged Rényi Entropy of a four dimensional CFT, dual to five dimensional charged quasi-topological gravity (QTG). The action of the five dimensional QTG coupled to a U(1) gauge field A is given by,

$$S = \frac{1}{2\ell_p^3} \int d^5x \sqrt{-g} \left[R + \frac{12}{L^2} + \frac{\lambda}{2} L^2 \chi_4 + \frac{7\mu}{4} L^4 \mathcal{Z}_5 - \frac{\ell_*^2}{4} F_{\mu\nu} F^{\mu\nu} \right], \quad (4.1)$$

where ℓ_* is related to the five dimensional gauge coupling g_5 as, $g_5^2 = \frac{2\ell_p^3}{\ell_*^2}$. λ and μ are coupling constants, and χ_4 is the standard four-derivative Gauss-Bonnet term,

$$\chi_4 = R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} - 4R_{\mu\nu} R^{\mu\nu} + R^2, \quad (4.2)$$

while $\mathcal{Z}_5$ is a six-derivative interaction term [19, 25] given by,

$$\begin{aligned} \mathcal{Z}_5 = & R_{\mu}^{\rho} R_{\nu}^{\lambda} R_{\rho}^{\alpha} R_{\lambda}^{\beta} R_{\alpha}^{\mu} R_{\beta}^{\nu} + \frac{1}{56} (21 R_{\mu\nu\rho\lambda} R^{\mu\nu\rho\lambda} R - 72 R_{\mu\nu\rho\lambda} R^{\mu\nu\rho} R^{\lambda\alpha} \\ & + 120 R_{\mu\nu\rho\lambda} R^{\mu\rho} R^{\nu\lambda} + 144 R_{\mu}^{\nu} R_{\nu}^{\rho} R_{\rho}^{\mu} - 132 R_{\mu}^{\nu} R_{\nu}^{\mu} R + 15 R^3). \end{aligned} \quad (4.3)$$

In order to compute the holographic charged Rényi Entropy dual to this bulk gravity theory, we need to construct charged hyperbolic black hole solutions in this [38]. This can be done by choosing the following metric and gauge field ansatz,

$$\begin{aligned} ds^2 = & - \left(\frac{r^2}{L^2} f(r) - 1 \right) \mathcal{N}(r)^2 dt^2 + \frac{dr^2}{\left(\frac{r^2}{L^2} f(r) - 1 \right)} + r^2 d\Sigma_3^2, \\ A = & \phi(r) dt. \end{aligned} \quad (4.4)$$

Substituting the ansatz of eq. (4.4) in the action of eq. (4.1), and varying the action with respect to $f(r)$, we have $\mathcal{N}'(r) = 0$, yielding

$$\mathcal{N}(r) = \frac{\tilde{L}}{\mathcal{R}}, \quad (4.5)$$

where $\tilde{L}$ is the AdS curvature scale to be determined. Also, note that we have chosen the $\frac{1}{\mathcal{R}^2}$ factor in the same spirit as we did earlier. Now varying the action with respect to $\phi(r)$ and solving that equation, we get

$$\phi(r) = \frac{q\tilde{L}}{2r^2\ell_*\mathcal{R}} - \frac{\mu_c}{2\pi\mathcal{R}} \quad (4.6)$$

Here, the chemical potential is given by

$$\mu_c = \frac{\pi \tilde{L} q}{\ell_* r_h^2}. \quad (4.7)$$

In the above expression, q is an integration constant, related to the charge of the black hole and r_h is the position of the horizon. Note that, we have chosen the chemical potential μ_c in such a way that the gauge field $\phi(r)$ vanishes at the horizon.

The variation of the action with respect to $\mathcal{N}(r)$ yields a first order differential equation for $f(r)$, solving which we get a cubic equation for $f(r)$,

$$1 - f(r) + \lambda f(r)^2 + \mu f(r)^3 = \frac{L^2 m}{3r^4} - \frac{L^2 q^2}{12r^6}, \quad (4.8)$$

where m is an integration constant, related to the mass of the black hole. Using the fact that $f(r_h) = \frac{L^2}{r_h^2}$, one can also express the mass parameter in terms of the horizon radius r_h from eq. (4.8),

$$m = \frac{3r_h^4}{L^2} - 3r_h^2 + \frac{q^2}{4r_h^2} + 3\lambda L^2 + \frac{3\mu L^4}{r_h^2}. \quad (4.9)$$

Here we should mention that unlike the Einstein gravity, the AdS curvature scale $\tilde{L}$ is not equal to the length scale L related to the cosmological constant in QTG. These two are related by $\tilde{L} = \frac{L}{\sqrt{f_\infty}}$, where f_∞ is the asymptotic value of $f(r)$ and can be determined by solving eq. (4.8) at $r \rightarrow \infty$,

$$1 - f_\infty + \lambda f_\infty^2 + \mu f_\infty^3 = 0. \quad (4.10)$$

As discussed in [25, 59], the 3 + 1 dimensional boundary CFT dual to this cubic gravity, can be characterized by two central charges c and a , and an extra parameter t_4 which is required to describe certain scattering phenomena. Further, they can be expressed in terms of the coupling constants in the theory as,

$$\begin{aligned} t_4 &= \frac{2\mu f_\infty^2}{1 - 2\lambda f_\infty - 3\mu f_\infty^2}, \\ c &= \pi^2 \frac{\tilde{L}^3}{\ell_p^3} (1 - 2\lambda f_\infty - 3\mu f_\infty^2), \\ a &= \pi^2 \frac{\tilde{L}^3}{\ell_p^3} (1 - 6\lambda f_\infty + 9\mu f_\infty^2). \end{aligned} \quad (4.11)$$

It is to be noted that setting $\mu = 0$ yields $t_4 = 0$, while c and a reduce to the form of the central charges in Gauss-Bonnet gravity.

From the above equations one can write down the coupling constants in terms of t_4 and the central charges as,

$$\begin{aligned} \frac{\tilde{L}^3}{\ell_p^3} &= \frac{a}{2\pi^2} \left[3 \frac{c}{a} (1 + 3t_4) - 1 \right], \\ \lambda f_\infty &= \frac{1}{2} \frac{\frac{c}{a} (1 + 6t_4) - 1}{3 \frac{c}{a} (1 + 3t_4) - 1}, \\ \mu f_\infty^2 &= \frac{\frac{c}{a} t_4}{3 \frac{c}{a} (1 + 3t_4) - 1}. \end{aligned} \quad (4.12)$$

Using the above equations, one can further write down the form of f_∞ from eq. (4.10), in terms of t_4 and the central charges as,

$$f_\infty = 2 \frac{\frac{c}{a}(1 + 3t_4) - 1}{\frac{c}{a}(1 + 2t_4) - 1} \quad (4.13)$$

The two coupling constants and hence the central charges along with t_4 can be constrained to avoid negative energy excitations in the boundary CFT [25]:

$$\begin{aligned} \frac{c}{a}(1 - 84t_4) &\leq 2, \\ \frac{c}{a}(1 - 210t_4) &\geq \frac{1}{2}, \\ \frac{c}{a}(1 + 224t_4) &\geq \frac{2}{3}. \end{aligned} \quad (4.14)$$

As was done earlier with earlier examples, here we first define $r_h = \tilde{L}x$ and compute the Hawking temperature as,

$$T = \frac{\kappa}{2\pi} = \frac{1}{2\pi x \mathcal{R}} \left[1 + \frac{x^4 \ell_*^2 \mu_c^2 f_\infty^2 - 24\pi^2 L^2 (\mu f_\infty^3 - x^4 f_\infty + \lambda x^2 f_\infty^2 + x^6)}{12\pi^2 L^2 f_\infty (3\mu f_\infty^2 + 2\lambda x^2 f_\infty - x^4)} \right] \quad (4.15)$$

where κ is the surface gravity of the black hole. It is straightforward to analytically express the charge parameter q in terms of the chemical potential μ_c and hence in eq. (4.15) we have written down the final expression of T in terms of μ_c .

Also, the thermal entropy of the black hole can be computed using Wald's prescription [49],

$$S = 2\pi \left(\frac{\tilde{L}}{\ell_p} \right)^3 V_\Sigma x^3 \left(1 - \frac{6\lambda f_\infty}{x^2} + \frac{9\mu f_\infty^2}{x^4} \right) \quad (4.16)$$

In the process of computing the Rényi entropy S_n , we can express eq. (1.6) in terms of the dimensionless variable x such that it can be rewritten in a form similar to eq. (3.18) as,

$$\begin{aligned} S_n(\mu_c) &= \frac{n}{n-1} \frac{1}{T_0} \int_{x_n}^{x_1} S(\mu_c, x) \frac{dT(\mu_c, x)}{dx} dx \\ &= \frac{n}{n-1} \frac{1}{T_0} \left[S(\mu_c, x) T(\mu_c, x) \Big|_{x_n}^{x_1} - \int_{x_n}^{x_1} \frac{dS(\mu_c, x)}{dx} T(\mu_c, x) dx \right], \end{aligned} \quad (4.17)$$

where x_1 and x_n are the integration limits to be determined. Finally, we compute the charged Rényi entropy,

$$\begin{aligned} S_n(\mu_c) &= \left(\frac{\tilde{L}}{\ell_p} \right)^3 \pi V_\Sigma \frac{n}{n-1} \left[3(x_1^2 - x_n^2) \left(1 + \frac{\ell_*^2 \mu_c^2 f_\infty}{12\pi^2 L^2} + \frac{\mu f_\infty^2}{x_1^2 x_n^2} \right) - \frac{3}{f_\infty} (x_1^4 - x_n^4) \right. \\ &\quad \left. + 2(P_1 Q_1 - P_n Q_n) \right] \end{aligned} \quad (4.18)$$

where,

$$\begin{aligned} P_1 &= \frac{9\mu f_\infty^2 - 6\lambda x_1^2 f_\infty + x_1^4}{x_1^2 (3\mu f_\infty^2 + 2\lambda x_1^2 f_\infty - x_1^4)}, \quad Q_1 = x_1^4 \left(1 + \frac{\ell_*^2 \mu_c^2 f_\infty}{12\pi^2 L^2} \right) + \mu f_\infty^2 - \frac{2x_1^6}{f_\infty} \\ P_n &= \frac{9\mu f_\infty^2 - 6\lambda f_\infty x_n^2 + x_n^4}{x_n^2 (3\mu f_\infty^2 + 2\lambda f_\infty x_n^2 - x_n^4)}, \quad Q_n = x_n^4 \left(1 + \frac{\ell_*^2 \mu_c^2 f_\infty}{12\pi^2 L^2} \right) + \mu f_\infty^2 - \frac{2x_n^6}{f_\infty} \end{aligned} \quad (4.19)$$

Here we define x_n through $T(\mu_c, x) = \frac{T_0}{n}$, where with $n = 1$ we can obtain x_1 . Hence, the only remaining task here is to solve the following sixth order algebraic equation,

$$\frac{4x_n^6}{f_\infty} - \frac{2x_n^5}{n} - 2x_n^4 \left(1 + \frac{\ell_*^2 \mu_c^2 f_\infty}{12\pi^2 L^2} \right) + \frac{4\lambda f_\infty x_n^3}{n} + \frac{6\mu f_\infty^2 x_n}{n} - 2\mu f_\infty^2 = 0 \quad (4.20)$$

The largest real solution of the above equation determines the value of x_n . The analytical solution for x_n is very difficult to obtain, hence we would numerically solve this equation and compute the Rényi entropy. Here, by considering the limit $n \rightarrow 1$, one can obtain the entanglement entropy, $S_1(\mu_c)$. Like we did earlier, we will show here the variation of the Rényi entropy $S_n(\mu_c)$ normalized by the entanglement entropy $S_1(0)$, instead of plotting $S_n(\mu_c)$.

4.1 Numerical analysis and results

First, fixing the value of the ratio of central charges $\frac{c}{a}$, we analysed the behavior of Rényi entropy ratio $\frac{S_n(\mu_c)}{S_1(0)}$ with t_4 for different values of the chemical potential (μ_c). A typical case is exhibited in figure 4. We set $\frac{c}{a} = 1$, for which t_4 must be in the following physically allowed regime from eq. (4.14):

$$-0.00149 \leq t_4 \leq 0.00238.$$

We find a nearly linear behavior of $\frac{S_n(\mu_c)}{S_1(0)}$ as a function of t_4 , as can be seen from figure 4, where the red, orange, green, blue and purple lines correspond to $n = 1, 2, 3, 10$ and 100 , respectively. The same holds for other values of the chemical potential as well. However, we find that for a relatively small value of chemical potential (say $\frac{\mu_c \ell_*}{2\pi L} \sim 1$), the slopes of the lines are smaller than the ones seen here. In fact, the authors of [28] considered the uncharged quasi-topological gravity and concluded that the ratio $\frac{S_n}{S_1}$ is almost independent of t_4 for a fixed value of $\frac{c}{a}$ in the physically allowed regime and the dependence comes into play when we are well outside the physical regime. However, by coupling the quasi-topological gravity to a U(1) gauge field, it is evident in figure 4 that with a higher value of chemical potential, the Rényi entropy ratio increases linearly with t_4 . Notice that, as we increase chemical potential, the ratio $\frac{S_n(\mu_c)}{S_1(0)}$ reaches its maximum value for the maximum bound of t_4 (i.e., for $t_4 = 0.00238$ with $\frac{c}{a} = 1$). Another important observation is that, although the slope of the entropy ratio increases as the chemical potential increases, the slope is independent of n for a fixed value of chemical potential with fixed $\frac{c}{a}$.

Next, figure 5 shows the behavior of the entropy ratio with the chemical potential, with $t_4 = 0.00238$, where we have fixed $\frac{c}{a} = 1$. A similar graph (not shown here) is obtained for $t_4 = -0.00149$. These values of t_4 are the two extreme bounds (i.e., end points of the physically allowed parameter space for t_4) with $\frac{c}{a} = 1$. Here the entropy ratio increases monotonically and in a non-linear fashion, as μ_c increases. Here we follow the same color coding for n as in figure 4.

Now we go ahead to verify the Rényi entropy inequalities. First, we have verified that the first inequality of eq. (2.5) holds for any values of t_4 and chemical potential for a fixed value of $\frac{c}{a}$. The results become more interesting when we test the second inequality as

Figure 4. $\frac{S_n(\mu_c)}{S_1(0)}$ as a function of t_4 at fixed value of $\frac{c}{a} = 1$ and chemical potential $\frac{\mu_c \ell_*}{2\pi L} = 20$.

Figure 5. $\frac{S_n(\mu_c)}{S_1(0)}$ as a function of chemical potential at fixed value of $\frac{c}{a} = 1$ and $t_4 = 0.00238$.

Figure 6. With $\frac{c}{a} = 2$ the second inequality is violated with $t_4 = 0$, while the chemical potential is kept fixed at $\frac{\mu_c \ell_*}{2\pi L} = 0.45$.

Figure 7. With $\frac{c}{a} = 2$ the second inequality is obeyed with $t_4 = 0.0036$, while chemical potential is kept fixed at $\frac{\mu_c \ell_*}{2\pi L} = 0.45$.

shown in figures 6 and 7. These figures show the variation of $\frac{(n-1)}{n} \frac{S_n(\mu_c)}{S_1(0)}$ with n for fixed value of the chemical potential $\frac{\mu_c \ell_*}{2\pi L} = 0.45$ and the central charge ratio $\frac{c}{a} = 2$. For this chosen value of $\frac{c}{a}$, it is clear from eq. (4.14) that t_4 must be in the following physically allowed regime,

$$0 \leq t_4 \leq 0.0036.$$

Note that with the upper bound $t_4 = 0.0036$, the Rényi entropy obeys the second inequality, but with the lower bound $t_4 = 0$, it violates the inequality although $t_4 = 0$ is a well-accepted physical regime from causality considerations. This is interesting and let us discuss this further.

As we have mentioned, $t_4 = 0$ reduces our theory to charged Gauss-Bonnet gravity previously studied in [35]. In that paper, the authors showed that this violation of the Rényi entropy inequality can occur in CFTs dual to a hyperbolic black hole geometry when the

black hole possesses negative thermal entropy. Now, the negative thermal entropy of a topological black hole is a typical behavior of higher derivative gravity theories and is controlled by the parameters of the higher derivative terms [60, 61]. However, this does not necessarily mean that the black hole is thermally unstable. In fact, it is known that the Gauss-Bonnet coupling has to satisfy $\lambda > \frac{1}{4}$ for a thermally unstable black hole with negative specific heat in Gauss-Bonnet gravity. In our case, setting $\frac{c}{a} = 2$ and $t_4 = 0$, is equivalent to $\lambda = 0.09$ which is thermally stable.

More specifically, it is clear from eq. (4.16) that the negative entropy black holes must satisfy,

$$x < \sqrt{3f_\infty(\lambda + \sqrt{\lambda^2 - \mu})}. \quad (4.21)$$

Now from eq. (4.15), the negative entropy black holes would appear when the chemical potential,

$$\left(\frac{\mu_c \ell_*}{2\pi \tilde{L}}\right)^2 < \frac{2}{3\mu} \left[\lambda \left(\sqrt{\lambda^2 - \mu} + 27\mu \right) + \mu \left(27\sqrt{\lambda^2 - \mu} - 4 \right) - \lambda^2 \right], \quad (4.22)$$

with

$$\lambda \left(\sqrt{\lambda^2 - \mu} + 27\mu \right) + \mu \left(27\sqrt{\lambda^2 - \mu} - 4 \right) - \lambda^2 > 0. \quad (4.23)$$

Eq. (4.22) can be rewritten in terms of t_4 and the central charges of the theory as,

$$\begin{aligned} \left(\frac{\mu_c \ell_*}{2\pi \tilde{L}}\right)^2 < \frac{1}{6\frac{c}{a} t_4 \alpha_2 \alpha_3} \left[-\alpha_1^2 \alpha_2 + \frac{c}{a} t_4 \frac{\alpha_2}{\alpha_3} (-16\alpha_3^2 + 27\alpha_2 \sqrt{\alpha_4}) \right. \\ \left. + \frac{\alpha_1}{\alpha_3} \left(\sqrt{\alpha_4} \alpha_2 \alpha_3 + 27\frac{c}{a} t_4 \alpha_2^2 \right) \right] \end{aligned} \quad (4.24)$$

where,

$$\begin{aligned} \alpha_1 &= -1 + \frac{c}{a} (1 + 6t_4), & \alpha_2 &= -1 + 5\frac{c}{a} (1 + 2t_4) \\ \alpha_3 &= -1 + 3\frac{c}{a} (1 + 3t_4), & \alpha_4 &= \left(-1 + \frac{c}{a}\right)^2 - 8\frac{c}{a} t_4. \end{aligned} \quad (4.25)$$

We see from here that negative entropy black holes appear below a certain value of the chemical potential determined by c , a and t_4 . For example, with $\frac{c}{a} = 2$ and $t_4 = 0$, from eq. (4.24), one can see that negative entropy black holes appear when $\frac{\mu_c \ell_*}{2\pi \tilde{L}} < 0.49$. In figure 6, since the chosen chemical potential (i.e., $\frac{\mu_c \ell_*}{2\pi \tilde{L}} = 0.45$) is less than the aforementioned value, negative entropy black holes would appear in this case and the CFT dual to this negative entropy hyperbolic black hole would exhibit a violation in the second inequality obeyed by Rényi entropy. However, with $t_4 = 0.0036$, negative entropy black holes would appear when $\frac{\mu_c \ell_*}{2\pi \tilde{L}} < 0.44$. Since the chosen potential is greater than this value, the black hole can not have negative entropy and hence, the corresponding CFT obeys the second inequality as shown in figure 7.

It is therefore evident that the physically justified condition of satisfying the second ERE inequality of eq. (2.5) constrains the parameter space of t_4 from the apparent condition

Figure 8. $\frac{S_n(\mu_c)}{S_1(0)}$ as a function of $\frac{c}{a}$ at fixed value of $t_4 = 0.001$ with chemical potential $\frac{\mu_c \ell_*}{2\pi \tilde{L}} = 1$.

Figure 9. $\frac{S_n(\mu_c)}{S_1(0)}$ as a function of chemical potential at fixed value of $t_4 = 0.001$ with $\frac{c}{a} = 2.183$.

given in eq. (4.14). In fact, it can be checked that, with $\frac{c}{a} = 2$ and $\frac{\mu_c \ell_*}{2\pi \tilde{L}} = 0.45$, the physically allowed regime of the t_4 parameter space is now

$$0.0029 \leq t_4 \leq 0.0036. \quad (4.26)$$

It is also straightforward to check how the second ERE inequality is controlled by the chemical potential for fixed values of $\frac{c}{a}$ and t_4 . We performed this analysis with $\frac{c}{a} = 2$ and $t_4 = 0.001$. With these CFT parameters, from our general analysis above, it can be checked that negative entropy black hole can appear only when $\frac{\mu_c \ell_*}{2\pi \tilde{L}} < 0.476$. Indeed, the second inequality of eq. (2.5) is violated below this value of the chemical potential, although the black hole is thermally stable in this region.

Next, we briefly describe the behavior of the entropy ratio $\frac{S_n(\mu_c)}{S_1(0)}$ as a function of $\frac{c}{a}$ and the chemical potential for a particular value of t_4 . The plots are similar to those obtained with Gauss-Bonnet gravity [35]. Figure 8 shows the variation of $\frac{S_n(\mu_c)}{S_1(0)}$ with the central charge ratio with $t_4 = 0.001$ for a typical value $\frac{\mu_c \ell_*}{2\pi \tilde{L}} = 1$. In this figure, the red, orange, green, blue and purple lines correspond to $n = 1, 2, 3, 10$ and 100 respectively. Like figure 4, this behavior is almost linear as long as we are inside the physical regime determined by eq. (4.14). But unlike the former, here the slopes of the lines are different for different n . It can be checked that with $\frac{\mu_c \ell_*}{2\pi \tilde{L}} = 0$, the lower bound of $\frac{c}{a}$ has the maximum value of $\frac{S_n(\mu_c)}{S_1(0)}$, while with $\frac{\mu_c \ell_*}{2\pi \tilde{L}} = 1$ i.e the case depicted in figure 8, the maximum value of $\frac{S_n(\mu_c)}{S_1(0)}$ is obtained for the upper bound of $\frac{c}{a}$. Hence, as we increase the chemical potential, the entropy ratio increases and there exists a critical value of the chemical potential beyond which $\frac{S_n(\mu_c)}{S_1(0)}$ attains its maximum value at the upper bound of $\frac{c}{a}$ for any value of n . For completeness, we also show a plot of the entropy ratio as a function of the chemical potential at fixed value of $t_4 = 0.001$ for different values of $\frac{c}{a} = 2.183$ in figure 9 where we have used the same color coding as in figure 8.

Now we verify the second inequality obeyed by Rényi entropy for fixed values of t_4 , and analyze the bound on the central charge ratio $\frac{c}{a}$. We set $t_4 = 0.002$ and the chemical

potential $\frac{\mu_c \ell_*}{2\pi \tilde{L}} = 0.45$. With this chosen value of t_4 , using eq. (4.14) one can easily determine the physically allowed regime for $\frac{c}{a}$ which is, $0.862 \leq \frac{c}{a} \leq 2.404$. Now with the lower bound of $\frac{c}{a}$, it can be checked that the second inequality of Rényi entropy is obeyed while with the upper bound, this inequality is violated. This is expected if we again connect this result with the negative entropy condition of the dual black hole geometry (although again the black hole is thermally stable in the negative entropy region). Following the same logic as we elaborated before, here one can explain these results with fixed values of t_4 using the same arguments based on eq. (4.24) and we will not go into the details. We just mention that for this example, we find that the bound on the central charge ratio imposed by the inequality of Rényi entropy is modified to

$$0.862 \leq \frac{c}{a} < 1.990. \quad (4.27)$$

We should point out here that the constraints on the parameters of charged QTG that we have presented here follow only from the apparent violation of eq. (2.5). We note here that the authors of [59] impose bounds on t_4 or $\frac{c}{a}$ by computing the expectation value of the energy, and demanding that it should be positive. The bounds on these quantities that follows from our analysis seem to be more stringent than [59] (the methods developed there are applicable in the presence of a chemical potential). The black holes that we consider do appear to be thermally stable in the ranges of parameters in which negative entropy solutions develop. The constraints on the parameters that we obtain should be checked from arguments of causality as advocated in the important work of [62]. This is an interesting question which we leave for a future analysis.

4.2 Scaling dimension of twist operators

Let us compute the scaling dimension $h_n(\mu_c)$ and the magnetic flux response $k_n(\mu_c)$ of the generalized twist operators in the four dimensional CFT dual to charged quasi-topological gravity. Recall that the scaling dimension is given by,

$$h_n(\mu_c) = \frac{n}{3} \frac{\mathcal{R}^3}{T_0} (\mathcal{E}(T_0, 0) - \mathcal{E}(\frac{T_0}{n}, \mu_c)) \quad (4.28)$$

where, $\mathcal{E}$ is the energy density given by,

$$\mathcal{E} = \frac{m}{2\ell_p^3 \mathcal{R}^3} \quad (4.29)$$

Here, we will use eq. (4.28) with the mass m being given in eq. (4.9),

$$m = 3\tilde{L}^2 \left(\frac{x^4}{f_\infty} - x^2 + \frac{\mu_c^2 \ell_*^2}{12\pi^2 \tilde{L}^2} x^2 + \lambda f_\infty + \frac{\mu}{x^2} f_\infty^2 \right) \quad (4.30)$$

Using eqs. (2.7), (4.29) and (4.30), it is straightforward to show that the scaling dimension h_n is given by,

$$h_n(\mu_c) = -\frac{\pi \tilde{L}^3 n}{\ell_p^3} \left(\frac{x_n^4}{f_\infty} - x_n^2 + \frac{\mu_c^2 \ell_*^2}{12\pi^2 \tilde{L}^2} x_n^2 + \lambda f_\infty + \frac{\mu}{x_n^2} f_\infty^2 \right) \quad (4.31)$$

Note that with $\mu_c = 0$ and $\mu = 0$ we recover the dimension of the twist operator in a CFT dual to Einstein Gauss-Bonnet gravity, while with $\mu_c = 0$ and $\mu = 0$ and $\lambda = 0$, the results match with the same in pure Einstein gravity, as expected. We also analyze the first order coefficients of the expansion of the scaling dimension h_{10} and h_{02} (see section 2) which are given by

$$h_{10} = \frac{2\pi}{3} \left(\frac{\tilde{L}}{\ell_p} \right)^3 (1 - 2\lambda f_\infty - 3\mu f_\infty^2) = \frac{2}{3\pi} c \quad (4.32)$$

and

$$h_{02} = -\frac{5}{18\pi} \left(\frac{\tilde{L}}{\ell_p} \right)^3 \left(\frac{\ell_*^2}{\tilde{L}^2} \right) \quad (4.33)$$

In appropriate limits, the results agree with the corresponding ones in pure Einstein and Gauss-Bonnet gravity.

On the other hand, the magnetic response can be calculated by computing the charge density,

$$\rho(x, \mu_c) = \frac{2}{\mathcal{R}^3} \left(\frac{\tilde{L}}{\ell_p} \right)^3 \left(\frac{\ell_*}{\tilde{L}} \right)^2 \frac{\mu_c}{4\pi} x^2 \quad (4.34)$$

This yields the result

$$k_n(\mu_c) = 2\pi n \mathcal{R}^3 \rho(x, \mu_c) = 2n \left(\frac{\tilde{L}}{\ell_p} \right)^3 \left(\frac{\ell_*}{\tilde{L}} \right)^2 \frac{\mu}{2} x_n^2 \quad (4.35)$$

Note that this result of magnetic response depends on the couplings λ and μ through x_n . One can also extract the expansion coefficients as,

$$k_{01}(\mu, \lambda) = \left(\frac{\tilde{L}}{\ell_p} \right)^3 \left(\frac{\ell_*}{\tilde{L}} \right)^2 \quad \text{and} \quad k_{11}(\mu, \lambda) = \frac{1}{3} \left(\frac{\tilde{L}}{\ell_p} \right)^3 \left(\frac{\ell_*}{\tilde{L}} \right)^2 \quad (4.36)$$

The expressions for magnetic response are again in agreement with the case of pure Einstein ($\lambda = 0$, $\mu = 0$, $\mu_c = 0$) and the Gauss-Bonnet gravity ($\mu = 0$, $\mu_c = 0$).

5 Charged Rényi entropy with Born-Infeld and Weyl corrected gravity

Finally, we briefly address the issue of charged Rényi entropy in two other interesting theories of modified gravity, namely Born-Infeld and Weyl corrected gravity. In the former case, exact hyperbolic black hole solution is known, and computation of the ERE is a straightforward numerical exercise which we detail below. For Weyl-corrected gravity, we obtain such a black hole solution first and then proceed to compute the ERE.

5.1 Charged Rényi entropy with Born-Infeld gravity

We will start with the standard Einstein-Born-Infeld action in five dimensional AdS space which is given by,

$$S = \frac{1}{2\ell_p^3} \int d^5x \sqrt{-g} \left[R + \frac{12}{L^2} + b^2 \ell_*^2 \left(1 - \sqrt{1 + \frac{F^{\mu\nu} F_{\mu\nu}}{2b^2}} \right) \right], \quad (5.1)$$

where, b is the Born-Infeld parameter. L is the AdS length, and is related to the cosmological constant Λ by $\Lambda = -\frac{6}{L^2}$. Here again in terms of the five dimensional gauge coupling g_5 we have, $g_5^2 = \frac{2\ell_*^3}{\ell_*^2}$. As is well known, the Born-Infeld term reduces to the standard Maxwell term $(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu})$, when we take the limit $b \rightarrow \infty$. On the other hand, with the limit $b \rightarrow 0$, the Born-Infeld term vanishes and we are left with Einstein's gravity with no gauge potential.

The authors of [36] and particularly [37] studied the black hole solutions in these theories, having horizons with positive(elliptic), zero (planar) and negative (hyperbolic) constant curvatures. Since we are interested here computing the charged Rényi entropy holographically, we only need to consider the charged hyperbolic black holes. The hyperbolic black hole solution of the Einstein-Born-Infeld gravity is given by,

$$ds^2 = -\left(\frac{r^2}{L^2}f(r) - 1\right)\frac{L^2}{\mathcal{R}^2}dt^2 + \frac{dr^2}{\left(\frac{r^2}{L^2}f(r) - 1\right)} + r^2 d\Sigma_3^2, \quad (5.2)$$

with the gauge field given by the expression

$$A = \phi(r) dt. \quad (5.3)$$

Here, $d\Sigma_3^2$ is the line element for the three dimensional hyperbolic plane H^3 . The function $f(r)$ has the form

$$f(r) = 1 - \frac{L^2 m}{3r^4} + \frac{b^2 \ell_*^2 r^2}{12} \left(1 - \sqrt{1 + \frac{q^2}{b^2 \ell_*^2 r^6}}\right) + \frac{L^2 q^2}{8r^6} {}_2\mathcal{F}_1\left(\frac{1}{3}, \frac{1}{2}, \frac{4}{3}, -\frac{q^2}{b^2 \ell_*^2 r^6}\right) \quad (5.4)$$

where m and q are constants of integration and ${}_2\mathcal{F}_1$ is the standard hypergeometric function. The constants m and q are related to the black hole mass M and electric charge Q , respectively. Also, the position of the horizon r_h is given by $f(r_h) = \frac{L^2}{r_h^2}$. Further, the gauge field $\phi(r)$ is given by,

$$\phi(r) = \frac{L q}{2\ell_* \mathcal{R} r^2} {}_2\mathcal{F}_1\left(\frac{1}{3}, \frac{1}{2}, \frac{4}{3}, -\frac{q^2}{b^2 \ell_*^2 r^6}\right) - \frac{\mu_c}{2\pi \mathcal{R}}, \quad (5.5)$$

where, μ_c is the chemical potential, given by

$$\mu_c = \frac{\pi L q}{\ell_* r_h^2} {}_2\mathcal{F}_1\left(\frac{1}{3}, \frac{1}{2}, \frac{4}{3}, -\frac{q^2}{b^2 \ell_*^2 r_h^6}\right). \quad (5.6)$$

Notice that μ_c is chosen in such a way that the gauge field $\phi(r)$ vanishes at the horizon r_h in order to prevent the appearance of a conical singularity. Another important point to note is that, compared to [37], we have inserted an extra factor of $\frac{L^2}{\mathcal{R}^2}$ in the metric component g_{tt} . With this inclusion, we make sure that the boundary CFT dual to this black hole geometry resides on $R \times H^3$ where the hyperbolic spatial slice H^3 has curvature $\mathcal{R}$,

$$ds_\infty^2 = -dt^2 + \mathcal{R}^2 d\Sigma_3^2. \quad (5.7)$$

One can also compute the mass parameter m in terms of the horizon radius r_h from, $f(r_h) = \frac{L^2}{r_h^2}$ as

$$m = \frac{3r_h^4}{L^2} - 3r_h^2 + \frac{b^2 \ell_*^2 r_h^4}{4} \left(1 - \sqrt{1 + \frac{q^2}{b^2 \ell_*^2 r_h^6}} \right) + \frac{3q^2}{8r_h^2} {}_2F_1 \left(\frac{1}{3}, \frac{1}{2}, \frac{4}{3}, -\frac{q^2}{b^2 \ell_*^2 r_h^6} \right). \quad (5.8)$$

The Hawking temperature of the black hole can be computed as,

$$T = \frac{1}{2\pi r_h} \left(\frac{L}{\mathcal{R}} + \frac{r_h^3 f'(r_h)}{2 L \mathcal{R}} \right). \quad (5.9)$$

The black hole entropy computed using the Bekenstein-Hawking formula reads

$$S = 2\pi \left(\frac{r_h}{\ell_p} \right)^3 V_\Sigma \quad (5.10)$$

As was done in the previous section, we now express the chemical potential, temperature and entropy of the black hole in terms of the dimensionless variable $x = r_h/L$ as,

$$\mu_c = \frac{\pi L q}{\ell_* L^2 x^2} {}_2F_1 \left(\frac{1}{3}, \frac{1}{2}, \frac{4}{3}, -\frac{q^2}{b^2 \ell_*^2 L^6 x^6} \right), \quad (5.11)$$

$$T = \frac{2x^2 - 1}{2\pi x \mathcal{R}} + \frac{b^2 \ell_*^2 L^2}{12\pi \mathcal{R}} x \left[1 - \sqrt{1 + \frac{q^2}{b^2 \ell_*^2 L^2 x^6}} \right], \quad (5.12)$$

$$S = 2\pi \left(\frac{L}{\ell_p} \right)^3 x^3 V_\Sigma. \quad (5.13)$$

Here we should mention that since we want to compute the Rényi entropy in the grand canonical ensemble, we should fix the chemical potential μ_c . For this purpose, we need to express q in terms of μ_c and substitute that in eq. (5.12) to have an expression of temperature in terms of μ_c . But from eq. (5.11), it is difficult to analytically express q in terms of μ_c (this is because of the factor of $\frac{q^2}{x^6}$ inside the hypergeometric function). Hence, we numerically evaluate the Rényi entropy at a fixed chemical potential.

At this point, we briefly describe the numerical routine for computing the Rényi entropy. First, we fix the chemical potential μ_c to a certain value and numerically solve eq. (5.11) to generate $q(x)$. Note that the entropy S does not explicitly depend on q , while the temperature T does. But once we numerically generate the function $q(x)$, we can rewrite the function $T(q, x)$ as $T(x)$.

Since we already have the function $q(x)$ we can straightforwardly determine x_1 and x_n (as in the earlier case) by numerically solving the following equations,

$$T_0 = \frac{1}{2\pi \mathcal{R}} = \frac{2x_1^2 - 1}{2\pi x_1 \mathcal{R}} + \frac{b^2 \ell_*^2 L^2}{12\pi \mathcal{R}} x_1 \left[1 - \sqrt{1 + \frac{q^2}{b^2 \ell_*^2 L^2 x_1^6}} \right], \quad (5.14)$$

$$\frac{T_0}{n} = \frac{1}{2\pi \mathcal{R} n} = \frac{2x_n^2 - 1}{2\pi x_n \mathcal{R}} + \frac{b^2 \ell_*^2 L^2}{12\pi \mathcal{R}} x_n \left[1 - \sqrt{1 + \frac{q^2}{b^2 \ell_*^2 L^2 x_n^6}} \right]. \quad (5.15)$$

Figure 10. First Inequality Test for Rényi entropy for Born-Infeld gravity with $b = 0.05$.

Figure 11. Second Inequality Test for Rényi entropy for Born-Infeld gravity with $b = 0.05$.

Now we can compute the Rényi entropy S_n . Since, S_n contains the volume factor V_Σ involving the short-distance cut-off of the boundary CFT, we should evaluate the ratio of Rényi entropy S_n to the entanglement entropy S_1 , as before. Broadly, we find here the following behaviour. Firstly, $\frac{S_n(\mu_c)}{S_1(0)}$ always increases in a non-linear fashion as the chemical potential μ_c increases. Second, for any value of the Born-Infeld parameter b , the Rényi entropy decreases as one increases the value of n . Finally, we find that for a particular value of n and μ_c , $\frac{S_n(\mu_c)}{S_1(0)}$ decreases as b increases. To end this subsection, we mention that the inequalities of eq. (2.5) are satisfied. We show this result for fixed $b = 0.05$ in figures 10 and 11. In both these figures, the red, orange, green, blue and purple lines (from bottom to top) correspond to $\frac{\mu_c \ell_*}{2\pi L} = 0.01, 0.25, 0.5, 0.75$ and 1 respectively. An entirely similar analysis follows for a fixed chemical potential.

Here, the mass m is given by,

$$m\left(\frac{T_0}{n}, \mu_c\right) = \frac{r_h^4}{L^2} - 3r_h^2 + \frac{b^2 \ell_*^2 r_h^4}{4} \left(1 - \sqrt{1 + \frac{q^2}{b^2 \ell_*^2 r_h^6}}\right) + \frac{3q^2}{8r_h^2} {}_2F_1\left(\frac{1}{3}, \frac{1}{2}, \frac{4}{3}, -\frac{q^2}{b^2 \ell_*^2 r_h^6}\right) \quad (5.16)$$

Since $m(T_0, 0) = 0$, finally we have the expression of the scaling dimension,

$$h_n = -\frac{n\pi L}{3\ell_p^3} m\left(\frac{T_0}{n}, \mu_c\right) \quad (5.17)$$

However, from eq. (5.11) it is evident that expressing q in terms of μ_c is difficult because of the presence of the hypergeometric function. Hence, it is difficult to obtain an analytical expression of the expansion of the scaling dimension, i.e h_{10} or h_{02} are difficult to calculate. The same difficulty also holds for the calculation of the coefficients of magnetic response. Of course one could have considered various limits of the parameters in order to (perturbatively) expand the hypergeometric function, but such an analysis may be of limited interest and we will not belabor upon this here. Alternatively, it is possible to do the computation in a canonical (fixed charge) ensemble. However we will not undertake such an analysis here.

5.2 Charged Rényi entropy in Weyl corrected gravity

Next, we briefly consider the Einstein-Maxwell action along with a Weyl coupling. We will start with the action

$$S = \frac{1}{2\ell_p^3} \int d^5x \sqrt{-g} \left(R + \frac{12}{L^2} - \frac{\ell_*^2}{4} F_{\mu\nu} F^{\mu\nu} + \gamma \ell_*^2 L^2 C_{\mu\nu\rho\lambda} F^{\mu\nu} F^{\rho\lambda} \right), \quad (5.18)$$

where $C_{\mu\nu\rho\lambda}$ is the five dimensional Weyl-tensor and γ is the Weyl coupling. The motivation for this action has been provided in [39, 40] (in which black holes in this theory with planar and spherical horizon were studied), to which we refer the reader for further details.

The hyperbolic black hole solutions here can be obtained by choosing the following metric and gauge field ansatz,

$$ds^2 = - \left(\frac{r^2}{L^2} f_0(r) - 1 \right) (1 + \gamma \mathcal{F}_1(r)) \mathcal{N}(r)^2 dt^2 + \frac{dr^2}{\left(\frac{r^2}{L^2} f_0(r) - 1 \right) (1 + \gamma \mathcal{F}_2(r))} + r^2 d\Sigma_3^2, \quad (5.19)$$

and

$$A = (\phi_0(r) + \gamma \mathcal{H}(r)) dt, \quad (5.20)$$

where $f_0(r)$ and $\phi_0(r)$ are the solutions to linear order in γ representing a hyperbolic Reissner-Nordström black hole solution in five dimensional AdS space, with

$$\begin{aligned} f_0(r) &= 1 - \frac{L^2 m}{3r^4} + \frac{L^2 q^2}{12r^6}, \\ \phi_0(r) &= \frac{q\tilde{L}}{2r^2\ell_*\mathcal{R}} - \frac{q\tilde{L}}{2\ell_*\mathcal{R}r_h^2}. \end{aligned} \quad (5.21)$$

Solving Einstein's and Maxwell's equation up to linear order in γ , one can determine the form of $\mathcal{F}(r)$, $\mathcal{H}(r)$ and $\mathcal{N}(r)$ as

$$\begin{aligned} \mathcal{F}_1(r) &= \frac{L^2 q^2 (L^2 (32r^2 r_h^8 (m + 3r^2) - q^2 (7r_h^8 + r^8)) + 96r^6 (r^2 - 2r_h^2) r_h^6)}{12r^6 r_h^8 (L^2 (q^2 - 4r^2 (m + 3r^2)) + 12r^6)}, \\ \mathcal{F}_2(r) &= \frac{L^2 q^2 (L^2 (64r^2 r_h^8 (m + 3r^2) - q^2 (15r_h^8 + r^8)) + 96r^6 (r^2 - 3r_h^2) r_h^6)}{12r^6 r_h^8 (L^2 (q^2 - 4r^2 (m + 3r^2)) + 12r^6)}, \\ \mathcal{H}(r) &= \frac{Lq (L^2 (r_h^8 (16mr^2 - 7q^2) + 48r^8 r_h^4 + 3q^2 r^8) - 48r^8 r_h^6)}{12r^8 \ell_* \mathcal{R} r_h^8}, \\ \mathcal{N}(r) &= \frac{\tilde{L}}{\mathcal{R}} \end{aligned} \quad (5.22)$$

In the above equations, $\tilde{L}$ is the effective AdS curvature scale. It is straightforward to show that for Weyl-corrected gravity, $\tilde{L} = L$. The Hawking temperature of the Weyl-corrected black hole can be computed up to linear order in γ as,

$$T = \frac{2x^2 - 1}{2\pi x \mathcal{R}} - \frac{q^2}{24\pi L^4 x^5 \mathcal{R}} + \gamma \left(\frac{q^4}{36\pi L^8 x^{11} \mathcal{R}} + \frac{q^2}{2\pi L^4 x^7 \mathcal{R}} - \frac{2q^2}{3\pi L^4 x^5 \mathcal{R}} \right) \quad (5.23)$$

where again we have set $r_h = Lx$. Note that, in the limit $\gamma \rightarrow 0$, we get back the temperature of a hyperbolic RNAdS black hole. Using Wald's prescription one can compute the linear order correction in entropy as [40],

$$S = 2\pi \left(\frac{L}{\ell_p}\right)^3 V_\Sigma \left(x^3 - \gamma \frac{q^2}{L^4 x^3}\right). \quad (5.24)$$

The Rényi entropy can be calculated at this stage by exactly the same methods and numerical routines given in the previous sections, and we do not repeat the description of the procedures here. We just mention here that we find that the ERE increases in an almost linear fashion with respect to the Weyl coupling, and that the inequalities of eq. (2.5) are indeed satisfied for small values of γ . We have checked this up to $|\gamma| \sim 0.005$.

A word regarding the upper limit of the Weyl coupling γ is in order. Ideally, the maximum magnitude of γ for which our results are trustworthy should be determined by computing the corrections to the Rényi entropy that arise by expanding the metric perturbatively to second order in γ and by ensuring that changes in the value of the Rényi entropy are small for the value of the Weyl coupling chosen. This is what we had done for the case of Einstein cubic gravity in section 3. In this case, however, due to the complicated nature of the expressions involved (details can be found in appendix B), such an analysis could not be performed. This is a caveat in our analysis. Specifically, to compute the contribution of the second order correction terms to Rényi entropy, we need to evaluate the lower limit (x_n) and upper limit (x_1) (see eq. (4.17)) numerically, using $T(x_n) = \frac{1}{2\pi\mathcal{R}n}$, where the temperature $T(x)$ is given in eq. (B.6). For this computation, one needs to generate $q(x)$ at a certain value of the chemical potential μ_c . Unfortunately, unlike the case with first order correction, we were unable to determine $q(x)$ precisely, due to some issues with numerical stability. However, we should mention that such an analysis was performed in [40] for black holes with spherical horizons, where it was checked that small values of γ like the ones chosen in this paper were indeed trustable.

For completeness, using the methods described earlier, we also computed the scaling dimension $h_n(\mu_c)$ of the twist operators. Since the form of h_n is not particularly illuminating we do not write it here. However, we should mention that the expansion coefficient $h_{10} = \frac{2}{3}\pi\left(\frac{L}{\ell_p}\right)^3$ turns out to be the same as with pure Einstein gravity. It is not difficult to convince oneself that this is due to the fact that in this case the AdS curvature scale $\tilde{L} = L$.

6 Summary and conclusions

In this paper, we have undertaken a detailed study of holographic entanglement Rényi entropies in various extended theories of gravity. The results of this paper complements the ones currently available in the literature, and provides novel examples of the computation of EREs in strongly coupled quantum field theories in four dimensions. As mentioned in the introduction, all the theories that we consider have tuneable parameters, which might be of interest in understanding field theories for realistic systems. Let us now summarise the main findings of this paper.

We initiated a holographic study of Einsteinian cubic gravity recently proposed in [30]. Here, we constructed a black hole solution of the theory perturbatively, up to the second

order in a particular parameter, in a five dimensional bulk theory. We computed the ERE for this case and verified the relevant inequalities satisfied by the same. We then computed the central charges of the dual field theory, and checked that it matched with a corresponding calculation of the Weyl anomaly on the gravity side. The validity of our perturbative analysis was also justified. To the best of our knowledge, ours is the first holographic study of the ECG, and establishes its physicality as far as the ERE is concerned. This is the first main contribution of this paper.

We next considered examples of extended theories of gravity with a chemical potential. In this context, we studied the ERE for charged quasi-topological theories of gravity in five dimensions. We showed here that there exists more stringent bounds on the parameters of the theory than one would expect from those arising out of ruling out negative excitations in the boundary CFT, when the inequalities involving the ERE are imposed. Specifically, we found that such violations are induced by negative entropy black holes in the bulk theory, as was found in [35] for CFTs dual to charged Gauss-Bonnet gravity. We established such bounds on the cubic coupling parameter and the ratio of the central charges, in some specific examples. This is the second contribution of this work.

We remark here that interestingly, we saw that negative entropy in Gauss-Bonnet black holes that arose at a certain value of the central charge ratio and the chemical potential was removed when the coupling to cubic interaction was turned on. This might indicate that in a top-down string theoretic approach, the negative entropy problem in QTG might as well be cured by possible higher order terms, although this is a mere speculation. As we have mentioned in the main text, it will be useful to consider the constraints on the CFT parameters from causality arguments advocated in [62], although we have not attempted it here.

Finally, we briefly studied ERE in Born-Infeld and Weyl-corrected gravity, with a perturbative analysis to establish black hole solutions in the latter. We found that the ERE inequalities are satisfied to the limit of approximation that we consider. We also commented upon the scaling dimension of the twist operators.

It might be interesting to investigate the Rényi entropy with an imaginary chemical potential in the higher derivative charged modified gravity theories considered in this paper, following the line of [34, 35]. It will also be interesting to study the charged ERE in canonical ensembles where the charge parameter is kept fixed, instead of the chemical potential. Further, instead of a perturbative analysis, one could numerically solve the Einstein equations to obtain numerical black hole solutions in Einsteinian cubic gravity and Einstein gravity with a Weyl-corrected gauge field. This way of studying the ERE would help understand better the entropy bounds, and hence the bounds on the coupling constants of the theories, from the validation of the Rényi entropy inequalities.

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A Correction up to $\mathcal{O}(\beta^2)$ in Einsteinian cubic gravity

Here we note down the hyperbolic black hole solution up to $\mathcal{O}(\beta^2)$ in Einsteinian cubic gravity, considering the following metric ansatz,

$$ds^2 = - \left(\frac{r^2}{L^2} f_0(r) - 1 \right) (1 + \beta \mathcal{F}_1(r) + \beta^2 \mathcal{G}_1(r)) \mathcal{N}(r)^2 dt^2 + \frac{1}{\left(\frac{r^2}{L^2} f_0(r) - 1 \right) (1 + \beta \mathcal{F}_2(r) + \beta^2 \mathcal{G}_2(r))} dr^2 + r^2 d\Sigma_3^2. \quad (\text{A.1})$$

Here, $f_0(r)$ is the zeroth order solution while $\mathcal{F}_1(r)$ and $\mathcal{F}_2(r)$ are the $\mathcal{O}(\beta)$ terms given in section 3. $\mathcal{G}_1(r)$ and $\mathcal{G}_2(r)$ are the $\mathcal{O}(\beta^2)$ corrections to be determined. Since, the recipe to find out the solution is already discussed in detail in section 3, we just mention the functions second order in β ,

$$\mathcal{G}_1(r) = \frac{3 \text{ A}}{7r^{16}r_h^6(-L^2r^2 + r^4 - \omega^4)}, \quad \mathcal{G}_2(r) = \frac{3 \text{ B}}{7r^{16}r_h^6(-L^2r^2 + r^4 - \omega^4)} \quad (\text{A.2})$$

where,

$$\begin{aligned} \text{A} &= 7r^2\omega^{16}r_h^6(14447r^2 - 13314L^2) + r^4\omega^{12}r_h^6(-48000L^4 + 104336L^2r^2 - 57883r^4) \\ &\quad + 14r^{10}\omega^4r_h^4(12L^2 - 19r^2)(-54L^4r_h^2 + 75L^2r_h^4 - 31r_h^6 + 11L^6) - 44926\omega^{20}r_h^6 \\ &\quad + 7r^8\omega^8r_h^4(1875L^2r_h^4 - 2r_h^2(675L^4 - 18L^2r^2 + 38r^4) - 775r_h^6 + 275L^6) + r^{16} \\ &\quad (-1501L^8r_h^2 + 4313L^6r_h^4 - 5610L^4r_h^6 + 3212L^2r_h^8 + 7r_h^6(r^4 - 88r_h^4) + 195L^{10}) \\ \text{B} &= 27r^2\omega^{16}r_h^6(5285r^2 - 5104L^2) - 7r^4\omega^{12}r_h^6(9600L^4 - 19872L^2r^2 + 10481r^4) \\ &\quad + 14r^{10}\omega^4r_h^4(16L^2 - 23r^2)(-54L^4r_h^2 + 75L^2r_h^4 - 31r_h^6 + 11L^6) - 70028\omega^{20}r_h^6 \\ &\quad + 7r^8\omega^8r_h^4(2775L^2r_h^4 - 2r_h^2(999L^4 - 24L^2r^2 + 46r^4) - 1147r_h^6 + 407L^6) + r^{16} \\ &\quad (-1501L^8r_h^2 + 4313L^6r_h^4 - 5610L^4r_h^6 + 3212L^2r_h^8 + 7r_h^6(r^4 - 88r_h^4) + 195L^{10}) \end{aligned}$$

Also, the function $\mathcal{N}(r)$ at $\mathcal{O}(\beta^2)$ is given by,

$$\mathcal{N}(r) = \frac{\tilde{L}}{\mathcal{R}} = \frac{L}{\mathcal{R}\sqrt{f_\infty}} \quad (\text{A.3})$$

where f_∞ is determined as, $f_\infty = 1 - \beta + 3\beta^2$.

For completeness, here we mention the Hawking temperature and the Wald entropy at $\mathcal{O}(\beta^2)$,

$$T = \frac{2x^2 - 1}{2\pi x \mathcal{R}} - \beta \frac{(x^2 - 1)^3}{\pi x^5 \mathcal{R}} + \beta^2 \frac{(x^2 - 1)^3 (1834x^4 - 459x^2 - 201)}{14\pi x^9 \mathcal{R}} \quad (\text{A.4})$$

and

$$S_{\text{Wald}} = 2\pi \left(\frac{L}{\ell_p}\right)^3 V_\Sigma \left[x^3 + \frac{3}{2}\beta \left(25x^3 - 48x + \frac{18}{x}\right) + \frac{3}{8}\beta^2 \left(905x^3 - 3168x + \frac{3996}{x} - \frac{2112}{x^3} + \frac{384}{x^5}\right) \right]. \quad (\text{A.5})$$

We also computed the Rényi entropy correction at $\mathcal{O}(\beta^2)$, but do not write it explicitly here, since the expression is unwieldy. Instead we give the expression for the entanglement entropy which is obtained by taking the $n \rightarrow 1$ limit.

$$S_1 = 2\pi \left(\frac{L}{\ell_p}\right)^3 V_\Sigma \left(1 - \frac{15}{2}\beta + \frac{15}{8}\beta^2\right). \quad (\text{A.6})$$

We also calculated the scaling dimension of the twist operator at $\mathcal{O}(\beta^2)$ and computed the quantity $\partial_n h_n|_{n=1}$, in a similar way as we did with the leading order solution,

$$\partial_n h_n|_{n=1} = \frac{2}{3}\pi \left(\frac{L}{\ell_p}\right)^3 \left(1 + \frac{9}{2}\beta - \frac{33}{8}\beta^2\right). \quad (\text{A.7})$$

As a further check for our computations with the second order solution, we calculated the $\mathcal{O}(\beta^2)$ correction to the Weyl anomaly, i.e., the correction to the central charges c and a using the methods outlined in section 3.3. We find that the coefficients match exactly with those that can be read off from the above expressions (A.6) and (A.7),

$$\begin{aligned} c &= \pi^2 \left(\frac{L}{\ell_p}\right)^3 \left(1 + \frac{9}{2}\beta - \frac{33}{8}\beta^2\right) \\ a &= \pi^2 \left(\frac{L}{\ell_p}\right)^3 \left(1 - \frac{15}{2}\beta + \frac{15}{8}\beta^2\right). \end{aligned} \quad (\text{A.8})$$

B Correction up to $\mathcal{O}(\gamma^2)$ in Weyl-corrected gravity

Here we note down the Weyl-corrected hyperbolic black hole solution up to $\mathcal{O}(\gamma^2)$, considering the following ansatz for the metric and gauge field ansatz,

$$\begin{aligned} ds^2 &= - \left(\frac{r^2}{L^2} f_0(r) - 1\right) (1 + \gamma \mathcal{F}_1(r) + \gamma^2 \mathcal{G}_1(r)) \mathcal{N}(r)^2 dt^2 \\ &\quad + \frac{dr^2}{\left(\frac{r^2}{L^2} f_0(r) - 1\right) (1 + \gamma \mathcal{F}_2(r) + \gamma^2 \mathcal{G}_2(r))} + r^2 d\Sigma_3^2, \end{aligned} \quad (\text{B.1})$$

and

$$A = \phi_0(r) + \gamma \mathcal{H}(r) + \gamma^2 \mathcal{J}(r) dt, \quad (\text{B.2})$$

where $f_0(r)$ is the zeroth order solution, $\mathcal{F}_1(r)$ and $\mathcal{F}_2(r)$ are the first order solutions of the metric, while $\mathcal{G}_1(r)$ and $\mathcal{G}_2(r)$ represent the $\mathcal{O}(\gamma^2)$ corrections to the metric. On the other hand, $\phi_0(r)$ and $\mathcal{H}(r)$ are respectively the zeroth order and first order corrections to the gauge field, while $\mathcal{J}(r)$ stands for the second order correction to the gauge field. Since,

the procedure to find the solution has been described earlier with $\mathcal{O}(\gamma)$ correction, here we simply write down the functions at $\mathcal{O}(\gamma^2)$,

$$\mathcal{G}_1(r) = \frac{1}{90r^{12}r_h^{14}(12r^6 - 4L^2mr^2 + L^2q^2 - 12L^2r^4)} \left[L^2q^2(L^4(8q^2r^2r_h^4(672r^{12} - 11r_h^{10}(41m + 171r^2)) + 384r^4r_h^8(mr_h^6(11m + 54r^2) + 63r^{10}) + q^4(5r^8r_h^6 + 365r_h^{14} + 268r^{14})) - 12L^2r^6r_h^6(384(7mr^2r_h^8 + 18r^8r_h^4) + q^2(40r^2r_h^6 - 1895r_h^8 + 809r^8)) + 58752r^{14}r_h^{12}) \right], \quad (\text{B.3})$$

$$\mathcal{G}_2(r) = \frac{1}{45r^{12}r_h^{14}(12r^6 - 4L^2mr^2 + L^2q^2 - 12L^2r^4)} \left[L^2q^2(L^4(32q^2(84r^{14}r_h^4 - r^2r_h^{14}(350m + 927r^2)) + 192r^4r_h^8(mr_h^6(53m + 180r^2) + 63r^{10}) + q^4(5r^8r_h^6 + 2025r_h^{14} + 134r^{14})) - 6L^2r^6r_h^6(192r^2r_h^4(35mr_h^4 + 36r^6) + q^2(80r^2r_h^6 - 5625r_h^8 + 809r^8)) + 29376r^{14}r_h^{12}) \right], \quad (\text{B.4})$$

and

$$\mathcal{J}(r) = \frac{1}{180\ell_*\mathcal{R}r^{14}r_h^{14}} \left[Lq(L^4(1152(m^2r^4r_h^{14} - 9r^{14}r_h^8) - 48q^2(r^2r_h^{14}(33m + 32r^2) + 31r^{14}r_h^4) + q^4(5r^8r_h^6 + 275r_h^{14} + 44r^{14})) + 24L^2r^6r_h^6(q^2(-20r^2r_h^6 + 75r_h^8 + 71r^8) + 864r^8r_h^4) - 10368r^{14}r_h^{12}) \right]. \quad (\text{B.5})$$

Also, $\mathcal{N}(r)$ is simply given by, $\mathcal{N}(r) = \frac{L}{\mathcal{R}}$.

For completeness, we also note down the Hawking temperature and Wald entropy computed at $\mathcal{O}(\gamma^2)$,

$$T(x) = \frac{2x^2 - 1}{2\pi x\mathcal{R}} - \frac{q^2}{24\pi L^4 x^5 \mathcal{R}} + \gamma \left(\frac{q^4}{36\pi L^8 x^{11} \mathcal{R}} + \frac{q^2}{2\pi L^4 x^7 \mathcal{R}} - \frac{2q^2}{3\pi L^4 x^5 \mathcal{R}} \right) + \gamma^2 \left(\frac{49q^6}{540\pi L^{12} x^{17} \mathcal{R}} - \frac{7q^4}{18\pi L^8 x^{13} \mathcal{R}} + \frac{121q^4}{45\pi L^8 x^{11} \mathcal{R}} - \frac{32q^2}{5\pi L^4 x^9 \mathcal{R}} + \frac{136q^2}{5\pi L^4 x^7 \mathcal{R}} - \frac{104q^2}{5\pi L^4 x^5 \mathcal{R}} \right), \quad (\text{B.6})$$

and

$$S_{\text{Wald}} = 2\pi \left(\frac{L}{\ell_p} \right)^3 V_{\Sigma} \left(x^3 - \gamma \frac{q^2}{L^4 x^3} + \gamma^2 \frac{6q^4 - 48L^4 q^2 x^6 + 48L^4 q^2 x^4}{L^8 x^9} \right). \quad (\text{B.7})$$

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