RECEIVED: December 7, 2017

REVISED: March 20, 2018

ACCEPTED: March 31, 2018

PUBLISHED: April 4, 2018

Logarithms and volumes of polytopes

Michael Enciso

*Mani L. Bhaumik Institute for Theoretical Physics, Department of Physics and Astronomy,
University of California at Los Angeles,
Los Angeles, CA 90095, U.S.A.*

E-mail: menciso@physics.ucla.edu

ABSTRACT: Describing the geometry of the dual amplituhedron without reference to a particular triangulation is an open problem. In this note we introduce a new way of determining the volume of the tree-level NMHV dual amplituhedron. We show that certain contour integrals of logarithms serve as natural building blocks for computing this volume as well as the volumes of general polytopes in any dimension. These building blocks encode the geometry of the underlying polytopes in a triangulation-independent way, and make identities between different representations of the amplitudes manifest.

KEYWORDS: 1/N Expansion, Scattering Amplitudes, Supersymmetric Gauge Theory

ARXIV EPRINT: [1612.07370](https://arxiv.org/abs/1612.07370)

Contents

1	Introduction	1
2	Polytopes in projective space	3
2.1	Projective geometry	3
2.2	Volumes of simplices	4
2.3	Volumes of general polytopes	6
2.4	The vertex formalism	7
2.5	Applications to NMHV amplitudes	10
3	Volumes and logarithms	12
3.1	One dimension	12
3.2	Two dimensions	13
3.3	Higher dimensions	13
4	Vertex objects from logarithms	15
4.1	Towards the vertex objects	15
4.2	Two-dimensional vertex objects	17
4.3	Higher-dimensional vertex objects	18
4.3.1	Three dimensions	18
4.3.2	Four dimensions	19
5	Conclusion and outlook	20

---

1 Introduction

Recent years have seen tremendous progress in understanding scattering amplitudes in both gauge and gravity theories. New mathematical structures that are not apparent in textbook formulations of quantum field theory have been uncovered, and many computations have been immensely streamlined in comparison with the standard Feynman diagram approach (see the recent reviews [1–3] and references therein). While many of these developments have applications in theories with various amounts of (including no) supersymmetry, the computational simplicity of maximally supersymmetric gauge and gravity theories make them ideal testing grounds for new ideas [4].

One of the major breakthroughs in the study of maximally supersymmetric gauge theories is the discovery of the amplituhedron, an object that encodes all tree-level amplitudes and loop-level integrands in planar $\mathcal{N} = 4$ super-Yang-Mills theory (sYM) [5, 6]. Schematically, and specializing to the case of tree amplitudes, the amplituhedron is a region of a particular positive Grassmannian [5, 7, 8]. This region encodes the amplitude via a volume

form with logarithmic singularities on its boundary, and after stripping off a canonical prefactor from this form what remains (up to some fermionic integrations) is the amplitude. For loop integrands the same is true but with the amplituhedron corresponding to a region of a particular generalization of the positive Grassmannian. In the rest of this note we restrict ourselves to the tree-level case.

For tree-level NMHV amplitudes, the amplitude obtained in this way is naturally interpreted as the volume of a polytope in a $\mathbb{CP}^4$ that is dual to the space in which the amplituhedron lives [5, 9]. N^k MHV tree amplitudes with $k \geq 1$ are therefore viewed as a type of “generalized volume” of a dual amplituhedron [9, 10]. For $k > 1$ a geometric understanding of the dual amplituhedron is unclear, though there are strong indications that such a picture should exist [10, 11].

In this note we introduce a new way of computing the volume of the tree-level NMHV (or $k = 1$) dual amplituhedron directly in the space in which the polytope lives. The basic objects in this method are contour integrals with simple, closed contours in the complex projective space containing the polytope. In ref. [9] the authors computed these volumes by integrating a particular volume form over the underlying polytope in the dual space, thus placing the information about the polytope in the contour (which has boundaries). As we will see in section 3, our method differs from that in ref. [9] by using contours that are closed (i.e., without boundary) and canonically specified by the integrands themselves. This is in contrast to, for example, “dlog” representations of amplitudes, where the contour is not specified by the integrand itself [7]. Additionally, the method we introduce is independent of any particular triangulation of the underlying polytope, and can be used to recover any such triangulation.

In ref. [12] we provided a definition of “combinatorial polytopes” which incorporates a general class of polytopes. For these polytopes neither convexity (and therefore positivity) nor even connectivity are necessary. We introduced a set of new objects that we denote by $F_{i_1 \dots i_n}$ and will now refer to as “vertex objects”. The reason for this naming convention is that the subscripts of these vertex objects correspond to the vertices of polytopes in a natural way that we will review shortly. In ref. [12] we showed that we obtain the volume of a polytope by summing these vertex objects over the vertices of the polytope. This way of expressing the volume of a polytope does not require any triangulation of the polytope to be known, and the volume of the polytope is uniquely expressed in terms of these vertex objects. These observations motivate us to view the vertex objects as basic building blocks for computing volumes of polytopes.

The vertex objects satisfy a simple relation that allows us to easily derive many non-trivial identities between different representations of the tree-level NMHV amplitude, as we will review in the next section. These identities and their more complex analogues for N^k MHV amplitudes with $k > 1$ can also be derived using global residue theorems (GRTs) on an auxiliary Grassmannian [13, 14]. In this picture, computing tree amplitudes and loop integrands is equivalent to specifying the correct contour for a particular integrand in the Grassmannian [7], and relations between different representations of the amplitude follow from the GRTs. Introducing this auxiliary space manifests the Yangian symmetry of the amplitudes [15], while the geometry of the underlying space whose volume corresponds

to the amplitude gets obscured. By showing that the vertex objects discussed above are naturally given by contour integrals in the dual space directly, we give a formalism that both manifests the relations between different representations of the amplitude while avoiding the introduction of an auxiliary space. This formalism has not been extended to $N^k\text{MHV}$ amplitudes with $k > 1$, but doing so will likely illuminate the underlying geometry of the dual amplituhedron.

The outline of this note is as follows: in the next section we briefly review some key properties of complex projective space and the standard generalization of volumes of polytopes to projective spaces. We will also briefly describe how NMHV tree amplitudes are expressed as volumes of polytopes and how the vertex objects are defined and used. In section 3 we show how contour integrals of logarithms naturally arise in computing the areas of quadrilaterals and their higher-dimensional analogues. In section 4 we show how the vertex objects correspond to a particular combination of these integrals.

2 Polytopes in projective space

In this section we review the ideas that will be needed in later sections. After discussing some key facts about (complex) projective spaces, we will review the standard generalization of volumes of polytopes in affine space to that of polytopes in projective space. We then briefly describe the formalism introduced in ref. [12], where the vertex objects encode the geometry of polytopes as well as give their volumes. Finally, we review how these vertex objects are used to manifest certain properties of the NMHV tree-amplitude. In the remaining sections of this note we show how these vertex objects are given as contour integrals in the space containing the polytope.

2.1 Projective geometry

In this brief review of projective geometry we follow ref. [16] and the first appendix of ref. [17], which provide more complete discussions of these ideas.

A point $Z^\alpha \in \mathbb{CP}^n$ is defined by $n + 1$ homogenous coordinates, one for each value of $\alpha = 0, \dots, n$. Each such point defines an $(n - 1)$ -dimensional hyperplane H_Z in the dual $\mathbb{CP}^{n*}$ by placing a single linear constraint on the homogenous coordinates of the dual elements. Namely, we have

$$H_Z \equiv \{A_\alpha \in \mathbb{CP}^{n*} \mid Z \cdot A \equiv Z^\alpha A_\alpha = 0\} \simeq \mathbb{CP}^{n-1} \subset \mathbb{CP}^{n*}. \quad (2.1)$$

The subspace H_Z is a linearly embedded $\mathbb{CP}^{n-1}$ in the dual $\mathbb{CP}^{n*}$. We will refer to linearly embedded $\mathbb{CP}^1$'s, $\mathbb{CP}^2$'s, and $\mathbb{CP}^k$'s with $k > 2$ respectively as lines, planes, and hyperplanes, even though the underlying topology of these spaces may be rather different. For example, a $\mathbb{CP}^1$ is a Riemann sphere though we will still refer to it as a line.

Intersections of lines, planes, and hyperplanes always exist in projective geometry. For example, three points Z_1^α , Z_2^α , and Z_3^α in $\mathbb{CP}^2$ give three lines in the dual $\mathbb{CP}^{2*}$ and each pair of lines intersects in a unique point. This is shown in figure 1, where the line dual to Z_i^α is labeled by i , and the intersection of lines i and j is labeled by $\{i, j\}$.

Figure 1. Three lines in $\mathbb{CP}^{2*}$ labeled by i corresponding to three points $\{Z_i^\alpha\}_{1 \leq i \leq 3}$ in $\mathbb{CP}^2$. The intersection of lines i and j is denoted by $\{i, j\}$. We note that $\{i, j\} = \{j, i\}$ implicitly.

Figure 2. A triangle in affine space defined by vertices (x_i, y_i) , with faces (edges) labeled by i corresponding to the points $\{Z_i^\alpha\}_{1 \leq i \leq 3}$ in the dual space that define them. The Z_i^α 's are explicitly defined in terms of the (x_i, y_i) coordinates in the text.

More generally, any two distinct $(n-1)$ -dimensional hyperplanes in $\mathbb{CP}^{n*}$ intersect in a unique $(n-2)$ -dimensional hyperplane. Namely, two points Z_1^α and Z_2^α in $\mathbb{CP}^n$ define two $(n-1)$ -dimensional hyperplanes H_{Z_1} and H_{Z_2} in $\mathbb{CP}^{n*}$, and we have that

$$H_{Z_1} \cap H_{Z_2} \simeq \mathbb{CP}^{n-2} \subset \mathbb{CP}^{n*}. \quad (2.2)$$

We therefore see that n distinct points in $\mathbb{CP}^n$ uniquely define a point in the dual $\mathbb{CP}^{n*}$ via the simultaneous intersection of their n dual hyperplanes.

2.2 Volumes of simplices

There is a natural generalization of the volume of a polytope to projective space. By first understanding this extension for the case of a simplex, the volume of more general polytopes follows immediately by considering sums of simplices. We will therefore follow ref. [9] and review how to express the volume of simplices in a projective way.

We begin by considering the area of a two-simplex, or a triangle, in real affine space with vertices located at (x_1, y_1) , (x_2, y_2) , (x_3, y_3) , as shown in figure 2. We can write its area A as [9]

$$A = \frac{1}{2} \frac{\langle Z_1 Z_2 Z_3 \rangle^2}{\langle Z_1 Z_2 P \rangle \langle Z_2 Z_3 P \rangle \langle Z_3 Z_1 P \rangle} \equiv [123], \quad (2.3)$$

where we have introduced the notation $\langle Z_1 \dots Z_n \rangle \equiv \varepsilon_{\alpha_1 \dots \alpha_n} Z_1^{\alpha_1} \dots Z_n^{\alpha_n}$, with the value of n taken from context. We have also defined

$$W_{i\alpha} \equiv \begin{pmatrix} x_i \\ y_i \\ 1 \end{pmatrix}, \quad P^\alpha = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad (2.4)$$

as well as

$$Z_1^\alpha \equiv \varepsilon^{\alpha\beta\gamma} W_{1\beta} W_{2\gamma}, \quad Z_2^\alpha \equiv \varepsilon^{\alpha\beta\gamma} W_{2\beta} W_{3\gamma}, \quad Z_3^\alpha \equiv \varepsilon^{\alpha\beta\gamma} W_{3\beta} W_{1\gamma}. \quad (2.5)$$

We note that the Z_i^α , $W_{i\alpha}$, and P^α all have three homogenous coordinates, in line with their being elements of $\mathbb{CP}^2$ (or its dual). We have simply “lifted” the affine coordinates into a particular coordinate patch of projective space by placing a 1 in the third component of the $W_{i\alpha}$ ’s.

Equation (2.3) is projectively well-defined in the Z_i^α ’s — which, according to the discussion in the previous subsection, determine the faces of the triangle — thus allowing their domain of definition to extend to $\mathbb{CP}^2$. We note that (2.3) is not projectively well-defined in P^α since it defines the line at infinity in $\mathbb{CP}^{2*}$ and therefore the scaling of the area — the scaling we choose here corresponds to the choice of placing 1 (as opposed to a different non-zero number) in the third component of the $W_{i\alpha}$ ’s. Equation (2.3) is also completely antisymmetric in the Z_i^α ’s, corresponding to the two possible orientations of the triangle.

It will be instructive to see explicitly how this works for one-dimensional simplices as well. A one-simplex is simply a line, and the distance L between two points x_1 and x_2 in $\mathbb{R}$ can be written as

$$L = \frac{\langle Z_1 Z_2 \rangle}{\langle Z_1 P \rangle \langle Z_2 P \rangle}. \quad (2.6)$$

Here we have defined

$$Z_1^\alpha \equiv \varepsilon^{\alpha\beta} W_{i\beta}, \quad W_{i\alpha} \equiv \begin{pmatrix} x_i \\ 1 \end{pmatrix}, \quad \text{and} \quad P^\alpha \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (2.7)$$

Equation (2.6) indeed reproduces $L = x_1 - x_2$, as expected, and it expresses the length of the line defined by the endpoints $W_{1\alpha}$ and $W_{2\alpha}$ in terms of their duals and the point at infinity defined by P^α . It is projective and antisymmetric in Z_1^α and Z_2^α , corresponding to the two different orientations of the line.

This generalizes to volumes of simplices in any dimension. For any $D + 1$ points $\{Z_i^\alpha\}_{1 \leq i \leq D+1}$ in $\mathbb{CP}^D$ there are $D + 1$ hyperplanes in the dual $\mathbb{CP}^{D*}$, and the volume of the simplex bounded by these hyperplanes is given by [9]

$$V = \frac{1}{D!} \frac{\langle Z_1 \dots Z_{D+1} \rangle^D}{\langle Z_1 \dots Z_D P \rangle \langle Z_2 \dots Z_{D+1} P \rangle \dots \langle Z_{D+1} \dots Z_{D-1} P \rangle} \equiv [12 \dots (D+1)]. \quad (2.8)$$

This expression is projective and totally antisymmetric in the Z_i^α ’s. The antisymmetry corresponds to the two possible orientations of the simplex.

The dimension most relevant for scattering amplitudes is four, so for completeness we will explicitly write the volume of a four-simplex, bounded by the five faces defined by $Z_1^\alpha, \dots, Z_5^\alpha$. Translating the above formula gives

$$V = \frac{1}{4!} \frac{\langle Z_1 Z_2 Z_3 Z_4 Z_5 \rangle^4}{\langle Z_1 Z_2 Z_3 Z_4 P \rangle \langle Z_2 Z_3 Z_4 Z_5 P \rangle \langle Z_3 Z_4 Z_5 Z_1 P \rangle \langle Z_4 Z_5 Z_1 Z_2 P \rangle \langle Z_5 Z_1 Z_2 Z_3 P \rangle} \equiv [12345]. \quad (2.9)$$

Figure 3. A quadrilateral in $\mathbb{CP}^{2*}$ defined by four lines labeled by i according to the points $\{Z_i^\alpha\}_{1 \leq i \leq 4}$ in $\mathbb{CP}^2$ that define them.

Figure 4. The quadrilateral shown in figure 3 viewed as the difference of two triangles.

2.3 Volumes of general polytopes

For a fixed dimension D , we can view any sum of simplices as the volume of a general polytope, expressed through some particular triangulation. For example, four points Z_1^α , Z_2^α , Z_3^α , and Z_4^α in $\mathbb{CP}^2$ define four lines in the dual $\mathbb{CP}^{2*}$. These four lines are depicted in figure 3 and are respectively labeled by 1, 2, 3, and 4.

The area of the shaded quadrilateral can be written as

$$A = [123] - [124], \quad (2.10)$$

which is the area of the triangle bounded by the faces 1, 2, and 3 minus the area of the triangle bounded by the faces 1, 2, and 4. This is depicted in figure 4.

By inspection of figure 3 we also see that we can write the area of the same region as

$$A = [431] - [432], \quad (2.11)$$

by viewing this area as the difference between the area of the triangle bounded by the faces 4, 3, and 1, and the triangle bounded by the faces 4, 3, and 2. We therefore see that we have

$$[123] - [124] = [431] - [432], \quad (2.12)$$

which, when one unravels the definition of these 3-brackets, is a non-trivial relation.

Proving this relation through repeated application of Schouten identities on the $\langle \dots \rangle$ brackets quickly shows that this geometric proof is more convenient, especially for analogous relations in higher dimensions. However, this geometric proof is not very precise, for a few reasons. For one, we have not been careful to keep track of the orientation of the quadrilateral in our two different triangulations. A second and more serious ambiguity is that our notion of a polytope itself is rather tenuous. Namely, once we extend our underlying space from a real affine space to a complex projective space, any notion of “inside” or “outside” is lost. Moreover, one generally thinks of a D -dimensional polytope in a D -dimensional space as being some full-dimensional region carved out by a finite number of hyperplanes. However, by complexifying our compact space, we end up talking about D -dimensional polytopes in $\mathbb{CP}^D$, which is a space of $2D$ real dimensions. A third issue with trying to define a polytope as a sum of volumes of simplices is that there are (infinitely) many triangulations that correspond to the same polytope. Some triangulations may make apparent certain geometric qualities of the underlying polytope while masking others.

The amplituhedron makes precise sense of these polytopes as a region in a positive Grassmannian, and for the NMHV case under consideration, this Grassmannian is simply a projective space [5]. In this program one considers convex polytopes, which places positivity constraints on the external kinematics. One then analytically continues to consider general kinematics. In ref. [12] we instead focused solely on the combinatorial structure of polytopes. We then gave a precise definition of a general type of polytope that is not necessarily convex or even connected. In the next subsection we will briefly review these ideas in two dimensions, as well as introduce the two-dimensional vertex objects $\{F_{ij}\}$. We refer to ref. [12] for details and the higher-dimensional cases.

2.4 The vertex formalism

We consider again the quadrilateral in figure 3 and our goal will be to give it a precise definition. While this figure does not correctly depict the topology of the objects involved — as mentioned above, the lines are actually Riemann spheres — it does correctly depict the intersection structure of these objects. We therefore define this polytope by its intersection structure, saying that this is the “quadrilateral” defined by starting at the vertex $\{1, 4\}$ and walking along line 4 to arrive at the vertex $\{2, 4\}$, then walking along line 2 to arrive at the vertex $\{2, 3\}$, then walking along line 3 to arrive at the vertex $\{3, 1\}$, and then walking along line 1 to arrive back at the vertex $\{1, 4\}$. This is depicted in figure 5.

This set of instructions can be succinctly summarized by the list (1423), which we define to be shorthand for

$$\{1, 4\} \rightarrow \{4, 2\} \rightarrow \{2, 3\} \rightarrow \{3, 1\} \rightarrow \{1, 4\} \quad (2.13)$$

where each “ $\rightarrow$ ” means to travel along the line whose label is common to the vertex on either side of the arrow.

In ref. [12] we introduced a collection $\{F_{ij}\}$ of vertex objects defined as a particular sum of volumes of simplices. These objects are referred to as vertex objects because a vertex of a two-dimensional polytope is labeled by two lines, as is each F_{ij} . We found that

Figure 5. The quadrilateral depicted in figure 3 defined solely through the intersection of its faces.

these vertex objects are antisymmetric, so that $F_{ij} = -F_{ji}$, and that they satisfy¹

$$F_{ij} + F_{jk} + F_{ki} = [ijk] \quad (2.14)$$

for any choice of i , j , and k , where we recall that $[ijk]$ is the volume of the two-simplex bounded by the three lines i , j , and k .

We consider the sum $F_{14} + F_{42} + F_{23} + F_{31}$ over the vertices of this quadrilateral. Using the antisymmetry of each F_{ij} and equation (2.14), we find

$$\begin{aligned} F_{14} + F_{42} + F_{23} + F_{31} &= F_{14} + F_{42} + ([231] - F_{12}) \\ &= [123] - (F_{12} + F_{24} + F_{41}) \\ &= [123] - [124], \end{aligned} \quad (2.15)$$

which is precisely the volume of the quadrilateral that the list of vertices in (2.13) defines. Applying equation (2.14) to the left hand side of (2.15) in a different order also shows that

$$F_{14} + F_{42} + F_{23} + F_{31} = [431] - [432]. \quad (2.16)$$

This gives a quick and rigorous proof of the non-trivial identity (2.12). Indeed, all possible triangulations of the quadrilateral can be obtained by applying (2.14) to the left hand side of (2.15), giving a simple algebraic method for proving many non-trivial identities amongst sums of simplices [12].

This example is a special case of a more general phenomenon — given any set of vertex-connecting instructions defining any polygon, summing the corresponding F_{ij} for each vertex yields the area of that polygon. This process works for general polygons, even disconnected ones. For example, suppose we have six elements $\{Z_i^\alpha\}$, $1 \leq i \leq 6$, defining six lines, as shown on the left hand side of figure 6. We can then define the disconnected polygon shown on the right hand side of this figure by the instructions

$$\{5, 1\} \rightarrow \{1, 6\} \rightarrow \{6, 2\} \rightarrow \{2, 4\} \rightarrow \{4, 5\} \rightarrow \{5, 6\} \rightarrow \{6, 3\} \rightarrow \{3, 5\} \rightarrow \{5, 1\}. \quad (2.17)$$

¹These vertex objects differ from those introduced in ref. [12] by a factor of 2.

Figure 6. A general polygon (right) defined solely through a set of instructions for traversing the intersections of six lines (left). Lines are labeled by i according to the Z_i^α that defines them, and the intersection of two lines i and j is denoted by $\{i, j\}$. It is implied that $\{i, j\} = \{j, i\}$.

Analogously to the case of the quadrilateral, this set of instructions corresponds to the list (51624563). It is then the case, rather surprisingly, that the area A of this polygon can be written simply as

$$A = F_{51} + F_{16} + F_{62} + F_{24} + F_{45} + F_{56} + F_{63} + F_{35}. \quad (2.18)$$

This can be checked against any particular triangulation of this polygon. Additionally, any triangulation of this polygon can be obtained from this expression through repeated use of (2.14).

The general result can be stated as follows. Let $\{Z_i^\alpha\}$ be a collection of N elements in $\mathbb{CP}^2$ defining N lines in the dual $\mathbb{CP}^{2*}$. The most general polygon in this dual $\mathbb{CP}^{2*}$ is given by a list $(i_1 i_2 \dots i_n)$, corresponding to the instructions

$$\{i_1, i_2\} \rightarrow \{i_2, i_3\} \rightarrow \dots \rightarrow \{i_{n-1}, i_n\} \rightarrow \{i_n, i_1\} \rightarrow \{i_1, i_2\}. \quad (2.19)$$

The area A of this polygon is then given by the following sum over the vertices:

$$A = \sum_{k=1}^n F_{i_k i_{k+1}}, \quad (2.20)$$

and from this expression any particular triangulation can be obtained [12]. This form of the area is independent of any particular triangulation and is inherently tied to the data of the polygon itself — its vertices and how we traverse them.

We note that many different lists give rise to the same polygon. For example, any cyclic permutation of a list gives the same polygon. More trivially, the list (1234) is identical to the list (12121234), since the latter corresponds to staying on the vertex $\{1, 2\}$ over and over again before moving on. However, the final result in terms of the vertex objects (up

to trivial cancellations) is identical. For example, the sum of these objects corresponding to the list (12121234) is simply

$$F_{12} + F_{21} + F_{12} + F_{21} + F_{12} + F_{23} + F_{34} + F_{41}, \quad (2.21)$$

which, after using the antisymmetry of F_{ij} yields the same result as the list (1234). Indeed, the sum in (2.20) is dependent only on the equivalence class of lists, where equivalence of lists is defined by their determining the same polygon. In ref. [12] we show how to extend this definition of polygon to arbitrary higher-dimensional polytopes.

In ref. [12] we also defined the corresponding vertex objects in higher dimensions. For example, in three dimensions we defined a collection of vertex objects $\{F_{ijk}\}$ completely antisymmetric in their subscripts and satisfying

$$F_{ijk} - F_{jkl} + F_{kli} - F_{lij} = [ijkl] \quad (2.22)$$

for any choice of i, j, k, l . We continue to use the term “vertex objects” because for a three-dimensional polytope a vertex is defined by the intersection of three planes, each defined by a Z_i^α , and these planes determine the subscripts of a given F_{ijk} . In four dimensions we defined a collection $\{F_{ijkl}\}$ of vertex objects that are totally antisymmetric in their subscripts and that satisfy

$$F_{ijkl} + F_{jklm} + F_{klmi} + F_{lmij} + F_{mijk} = [ijklm] \quad (2.23)$$

for any choice of i, j, k, l, m .

The volume of any polytope is given by the sum over its vertices of these vertex objects. This expression of the volume is unique, and any triangulation of the polytope can be recovered from this expression using (2.22), (2.23), and their higher-dimensional analogues. Additionally, the expression of the volume of a polytope in terms of the vertex objects also encodes the geometry of all lower-dimensional boundary polytopes and readily gives their volumes as well [12].

We note that equation (2.23) is reminiscent of the formula

$$\partial[ijklm] = [ijkl] + [jklm] + [klmi] + [lmij] + [mijk] \quad (2.24)$$

given in [9], describing the boundary $\partial[ijklm]$ of the simplex and encoding where the poles of $[ijklm]$ are. Equation (2.23) does the same, and is also a genuine equality between the volume of the simplex and objects that correspond to its vertices. Thus the objects on the left of equation (2.23) are fundamentally different than those on the right of equation (2.24). Similar statements can be made about equations (2.14) and (2.22) and the lower-dimensional analogues of equation (2.24).

2.5 Applications to NMHV amplitudes

Quite surprisingly, the n -point NMHV tree-level superamplitude M_{NMHV}^n in $\mathcal{N} = 4$ planar sYM can be written as the volume of a polytope in $\mathbb{CP}^{4*}$ [9]. Indeed, M_{NMHV}^n can be represented as

$$M_{\text{NMHV}}^n = \sum_{i,j}^n [*i(i+1)j(j+1)] \quad (2.25)$$

where the $\{Z_i^\alpha\}$ implicitly inside the five-brackets in the sum are n points in $\mathbb{CP}^4$ encoding the external kinematics and Z_*^α is a reference vector in $\mathbb{CP}^4$. The sum on i, j is understood modulo n , and polytopes of this form are known as *cyclic polytopes* [5].

For any given n , M_{NMHV}^n has many different expressions depending on our choice of Z_*^α . For example, if we choose $Z_*^\alpha = Z_1^\alpha$, then for $n = 6$ we have

$$M_{\text{NMHV}}^6 = [12345] + [12356] + [13456], \quad (2.26)$$

while if we choose $Z_*^\alpha = Z_2^\alpha$, then we have

$$M_{\text{NMHV}}^6 = [23456] + [23461] + [24561]. \quad (2.27)$$

Just as the relation (2.12) is not obvious at the level of Schouten identities on the $\langle \dots \rangle$ brackets, the equivalence of the right hand sides of (2.26) and (2.27) is non-trivial. These two representations of M_{NMHV}^6 were initially found by performing two different BCFW shifts on the amplitude [9, 18, 19]. The geometric interpretation is that they correspond to two different triangulations of the same underlying polytope. As discussed in the introduction, their equality can also be understood by using a global residue theorem in an auxiliary Grassmannian [13, 14]. Part of the utility of the vertex objects is to show that the right hand sides of (2.26) and (2.27) are equal directly — namely, they are identical when expressed in term of these objects. By using equation (2.23) on each simplex in either (2.26) or (2.27), we find

$$M_{\text{NMHV}}^6 = F_{1234} + F_{1245} + F_{1256} + F_{2345} + F_{2356} + F_{2361} + F_{3456} + F_{3461} + F_{4561}. \quad (2.28)$$

The amplitude is therefore uniquely expressed in terms of the vertex objects. From this expression and equation (2.23), any triangulation of M_{NMHV}^6 can be obtained.

For general n , we have

$$\begin{aligned} M_{\text{NMHV}}^n &= \sum_{i,j}^n [*i(i+1)j(j+1)] \\ &= \sum_{i,j}^n F_{*i(i+1)j} + F_{i(i+1)j(j+1)} + F_{(i+1)j(j+1)*} + F_{j(j+1)*i} + F_{(j+1)*i(i+1)} \\ &= \sum_{i,j}^n F_{i(i+1)j(j+1)}, \end{aligned} \quad (2.29)$$

where in the second equality we used equation (2.23) and in the last equality we used the cyclicity of the sum and antisymmetry of the vertex objects to cancel in pairs any terms with $*$ as a subscript. This shows manifestly that the amplitude is independent of Z_*^α and that the underlying polytope has vertices only where the four hyperplanes defined by Z_i^α , Z_{i+1}^α , Z_j^α , and Z_{j+1}^α intersect.

We refer to ref. [12] for further discussion of this vertex formalism. In the next two sections we show that these vertex objects are naturally defined as contour integrals of logarithms.

3 Volumes and logarithms

In [12] the vertex objects are defined as a particular sum of simplices. Thus, in some sense, writing the volume of a polytope in terms of these objects may be viewed as simply choosing a particular triangulation. However, we will now show that these objects are naturally defined in terms of contour integrals of logarithms, thus giving them an existence independent of simplices. This further motivates the view that the vertex objects are basic building blocks for computing volumes of polytopes.

As mentioned in the introduction, our integrals differ from those discussed in ref. [9] in that the latter involve contours with boundaries on the underlying polytope. Evaluating volumes in this way leads to the presence of spurious vertices (which correspond physically to spurious poles) associated to a particular triangulation. For example, the vertex $\{1, 2\}$ is a spurious vertex in the triangulation depicted in figure 4, since it is not present in the underlying polytope but shows up in individual terms in the triangulation. As we will see, the integrals we use have closed contours, so evaluating them corresponds to a straightforward application of Cauchy’s residue theorem. Moreover, they give rise to the vertex objects used in the vertex formalism discussed above, in which only the genuine (i.e., non-spurious) vertices of the polytope play a role.

3.1 One dimension

As a warmup, we begin our discussion in one dimension. Another way of writing the length L of a line from x_1 to x_2 is as

$$L = x_1 - x_2 = \int_{x_1 \leq x \leq x_2} dx = \frac{1}{2\pi i} \int_{x_1 \leq x \leq x_2} 2\pi i \, dx. \quad (3.1)$$

By allowing the x variables to be complex, we can define the complex logarithm function $\log\left(\frac{x-x_1}{x-x_2}\right)$ with its branch cut connecting the point x_1 to the point x_2 along the real axis. We can then rewrite $2\pi i$ as $\text{Disc}\left(\log\left(\frac{x-x_1}{x-x_2}\right)\right)$ — the discontinuity of the logarithm across its branch cut — giving

$$L = \frac{1}{2\pi i} \int_{x_1 \leq x \leq x_2} \text{Disc}\left(\log\left(\frac{x-x_1}{x-x_2}\right)\right) dx. \quad (3.2)$$

Unwrapping the contour allows one to drop the “Disc” from the integrand and obtain

$$L = \frac{1}{2\pi i} \oint \log\left(\frac{x-x_1}{x-x_2}\right) dx, \quad (3.3)$$

where the contour surrounds the cut. Evaluating this explicitly (for example, by going around the pole at infinity) recovers $L = x_1 - x_2$, as expected.

Making the same definitions as in (2.7) we can rewrite (3.3) as a contour integral in $\mathbb{CP}^{1*}$ as

$$L = \frac{1}{2\pi i} \oint \log\left(\frac{Z_1 \cdot X}{Z_2 \cdot X}\right) \frac{DX}{(P \cdot X)^2}, \quad (3.4)$$

where $DX \equiv \varepsilon^{\alpha\beta} X_\alpha dX_\beta$ is the canonical volume form (of weight two) on $\mathbb{CP}^{1*}$ and $X_\alpha \equiv \begin{pmatrix} x \\ 1 \end{pmatrix}$. By explicitly evaluating this integral we find

$$L = \frac{\langle Z_1 Z_2 \rangle}{\langle Z_1 P \rangle \langle Z_2 P \rangle}, \quad (3.5)$$

in agreement with equation (2.6). In this way, the length of a line is naturally represented as a contour integral of a logarithm.

3.2 Two dimensions

Motivated by the one-dimensional result, we consider the $\mathbb{CP}^{2*}$ integral

$$A = \frac{1}{(2\pi i)^2} \oint \log \left(\frac{Z_1 \cdot X}{Z_2 \cdot X} \right) \log \left(\frac{Z_3 \cdot X}{Z_4 \cdot X} \right) \frac{DX}{(P \cdot X)^3}, \quad (3.6)$$

where $DX \equiv \varepsilon^{\alpha\beta\gamma} X_\alpha dX_\beta dX_\gamma$ is the canonical volume form on $\mathbb{CP}^{2*}$ of weight three. The contour is again defined by the integrand in a canonical way: first go around the branch cut of $\log \left(\frac{Z_3 \cdot X}{Z_4 \cdot X} \right)$ and then go around the branch cut of $\log \left(\frac{Z_1 \cdot X}{Z_2 \cdot X} \right)$. This gives

$$A = [123] - [124], \quad (3.7)$$

which is precisely the area of the quadrilateral given in equation (2.10). If we swap Z_3^α , and Z_4^α with Z_1^α and Z_2^α in equation (3.6) and pick up a minus sign from the change in orientation of the contour, one readily sees that

$$A = -([341] - [342]), \quad (3.8)$$

thus proving the identity $[123] - [123] = [431] - [432]$ that we obtained in section 2.3. This identity is now made manifest by the integrand of (3.6).

We have expressed a two-dimensional area as a closed contour integral whose contour specification comes naturally with the integrand itself. The objects whose area we compute in this way are quadrilaterals, defined by four lines. Before describing how the vertex objects are obtained from these kinds of integrals, we quickly discuss how we can use these integrals to compute the volume of three- and D -dimensional “quadrilaterals”, or hypercubes.

3.3 Higher dimensions

Consider the following contour integral in $\mathbb{CP}^{3*}$:

$$V = \frac{1}{(2\pi i)^3} \oint \log \left(\frac{Z_1 \cdot X}{Z_2 \cdot X} \right) \log \left(\frac{Z_3 \cdot X}{Z_4 \cdot X} \right) \log \left(\frac{Z_5 \cdot X}{Z_6 \cdot X} \right) \frac{DX}{(P \cdot X)^4}, \quad (3.9)$$

where $DX \equiv \varepsilon^{\alpha\beta\gamma\delta} X_\alpha dX_\beta dX_\gamma dX_\delta$. The contour is a three-torus $(S^1)^3$ that goes around the branch cut of each logarithm. We find that

$$V = [1235] - [1236] - [1245] + [1246]. \quad (3.10)$$

Figure 7. A triangulation of the cube using four simplices.

This corresponds to the volume of a three-dimensional “cube”, where we simply mean a polytope bounded by 3 pairs of faces. One way to see that equation (3.10) is triangulating a “cube” with faces 1 and 2 opposite each other, 3 and 4 opposite each other, and 5 and 6 opposite each other is by examining figure 7, which shows the superposition of the four simplices in (3.10) leaving the volume of a “cube”.

As in the two-dimensional case, there is more than one expression for the volume of this cube. Namely, just as we could get two different expressions for the area of a quadrilateral by viewing it as the difference between two different pairs of triangles, we can get three expressions for the volume of the cube as a superposition of four simplices. In particular, we also have

$$V = -([3415] - [3416] - [3425] - [3426]) \text{ and } V = -([5631] - [5632] - [5641] - [5642]), \quad (3.11)$$

which correspond to the different ways of decomposing the cube analogously to figure 7 corresponding to figure 8. Figure 8 is the three-dimensional analog of figure 4. As in the two-dimensional case, these identities are manifest from the integrand in (3.9) by swapping, for example, Z_1^α and Z_2^α with Z_3^α and Z_4^α , or with Z_5^α and Z_6^α , and picking up a minus sign from the change in orientation of the contour.

For completeness we write down the contour integral that gives the volume of a D -dimensional “hypercube” bounded by $2D$ faces in “pairs”. With $Z_1^\alpha, \dots, Z_{2D}^\alpha$ defining the $2D$ faces, we have a generalization of the lower-dimensional cases:

$$V = \frac{1}{(2\pi i)^D} \oint \log \left(\frac{Z_1 \cdot X}{Z_2 \cdot X} \right) \dots \log \left(\frac{Z_{2D-1} \cdot X}{Z_{2D} \cdot X} \right) \frac{DX}{(P \cdot X)^{D+1}}, \quad (3.12)$$

where DX is the natural generalization of the lower-dimensional volume forms and the contour goes around the branch cut of each logarithm.

We note that these (hyper-)cubes are not directly related to the polytopes that are relevant for scattering amplitudes: equation (3.12) applied to $D = 4$ gives the volume of a

Figure 8. The three-dimensional analogue of figure 4, showing the three possible ways of forming a triangulation analogous to that shown in figure 7.

four-dimensional hypercube, which has 8 codimension-1 faces and 16 vertices, whereas the four-dimensional cyclic polytope whose volume is given by equation (2.25) corresponding to the scattering of $n = 8$ particles has 8 codimension-1 faces and 20 vertices. However, as we will show in the next section, these volumes of hypercubes can be used to obtain the vertex objects and thus to compute the volumes of general polytopes, including the cyclic polytopes relevant for scattering amplitudes.

4 Vertex objects from logarithms

4.1 Towards the vertex objects

We motivate the vertex objects by first seeing how to recover the volume of a simplex from integrals of logarithms. We define

$$T_{12} \equiv \frac{1}{(2\pi i)^2} \oint_{\gamma_{12}} \log \left(\frac{Z_1 \cdot X}{Z_2 \cdot X} \right) \log \left(\frac{Z_3 \cdot X}{Q \cdot X} \right) \frac{DX}{(P \cdot X)^3} = [123] - [12Q], \quad (4.1)$$

where γ_{12} is the same contour that we have described before, only now we are making it explicit. We have also introduced a fixed reference vector Q^α , defining a reference line in $\mathbb{CP}^{2*}$. Cyclicly permuting 1, 2, and 3, we define

$$T_{23} \equiv \frac{1}{(2\pi i)^2} \oint_{\gamma_{23}} \log \left(\frac{Z_2 \cdot X}{Z_3 \cdot X} \right) \log \left(\frac{Z_1 \cdot X}{Q \cdot X} \right) \frac{DX}{(P \cdot X)^3} = [231] - [23Q], \quad (4.2)$$

as well as

$$T_{31} \equiv \frac{1}{(2\pi i)^2} \oint_{\gamma_{31}} \log \left(\frac{Z_3 \cdot X}{Z_1 \cdot X} \right) \log \left(\frac{Z_2 \cdot X}{Q \cdot X} \right) \frac{DX}{(P \cdot X)^3} = [312] - [31Q]. \quad (4.3)$$

It is important to note that γ_{12} , γ_{23} , and γ_{31} are all different contours, each being the contour defined by the integrand of the corresponding integral — namely, the contour that goes around the branch cut of each logarithm. Performing these integrations and summing them up, we find that

$$T_{12} + T_{23} + T_{31} = 2[123]. \quad (4.4)$$

The dependence on Q^α , while present in each T_{ij} , drops out of this sum of integrals and we are left with twice the volume of a single simplex. At the level of the integrated results $T_{ij} = [ijk] - [ijQ]$, the Q^α -independence of this sum results from using the four-term identity in equation (2.12), which itself is the result of non-trivial algebra using Schouten identities. At the level of the integrands, however, these cancellations become more manifest, and we will explore them here in some detail.

We begin by rewriting T_{ij} , where $i, j \in \{1, 2, 3\}$ and $k \in \{1, 2, 3\} \setminus \{i, j\}$, as follows:

$$\begin{aligned} T_{ij} &= \frac{1}{(2\pi i)^2} \oint_{\gamma_{ij}} \log\left(\frac{Z_i \cdot X}{Z_j \cdot X}\right) \log\left(\frac{Z_k \cdot X}{Q \cdot X}\right) \frac{DX}{(P \cdot X)^3}, \\ &= \frac{1}{(2\pi i)^2} \oint_{\gamma_{ij}} \log\left(\frac{Z_i \cdot X}{Z_j \cdot X}\right) \log(Z_k \cdot X) \frac{DX}{(P \cdot X)^3} \\ &\quad - \frac{1}{(2\pi i)^2} \oint_{\gamma_{ij}} \log\left(\frac{Z_i \cdot X}{Z_j \cdot X}\right) \log(Q \cdot X) \frac{DX}{(P \cdot X)^3}. \end{aligned} \quad (4.5)$$

The contour γ_{ij} now goes around the branch cut of $\log(\frac{Z_i \cdot X}{Z_j \cdot X})$ and that of either $\log(Z_k \cdot X)$ or $\log(Q \cdot X)$, depending on which term we are considering. If we now consider only the Q^α -dependent terms in the sum $T_{12} + T_{23} + T_{31}$, we find

$$\begin{aligned} (T_{12} + T_{23} + T_{31})|_{Q^\alpha\text{-dependent}} &= -\frac{1}{(2\pi i)^2} \left(\oint_{\gamma_{12}} \log\left(\frac{Z_1 \cdot X}{Z_2 \cdot X}\right) \log(Q \cdot X) \frac{DX}{(P \cdot X)^3} \right. \\ &\quad + \oint_{\gamma_{23}} \log\left(\frac{Z_2 \cdot X}{Z_3 \cdot X}\right) \log(Q \cdot X) \frac{DX}{(P \cdot X)^3} \\ &\quad \left. + \oint_{\gamma_{31}} \log\left(\frac{Z_3 \cdot X}{Z_1 \cdot X}\right) \log(Q \cdot X) \frac{DX}{(P \cdot X)^3} \right). \end{aligned} \quad (4.6)$$

By deforming each γ_{ij} to go around the $P \cdot X = 0$ pole as opposed to the $\log(\frac{Z_i \cdot X}{Z_j \cdot X})$ branch cut, and thus picking up an overall minus sign, we can bring all of these integrands under the same integral and the integrand vanishes:

$$\begin{aligned} (T_{12} + T_{23} + T_{31})|_{Q^\alpha\text{-dependent}} &= \frac{1}{(2\pi i)^2} \left(\oint_{\gamma} \left(\log\left(\frac{Z_1 \cdot X}{Z_2 \cdot X}\right) + \log\left(\frac{Z_2 \cdot X}{Z_3 \cdot X}\right) + \log\left(\frac{Z_3 \cdot X}{Z_1 \cdot X}\right) \right) \right. \\ &\quad \left. \times \log(Q \cdot X) \frac{DX}{(P \cdot X)^3} \right) \\ &= 0, \end{aligned} \quad (4.7)$$

where the contour γ goes around the $P \cdot X = 0$ pole and the $\log(Q \cdot X)$ branch cut. We can therefore write the sum of T_{ij} 's as

$$\begin{aligned} T_{12} + T_{23} + T_{31} &= \frac{1}{(2\pi i)^2} \left(\oint_{\gamma_{12}} \log\left(\frac{Z_1 \cdot X}{Z_2 \cdot X}\right) \log(Z_3 \cdot X) \frac{DX}{(P \cdot X)^3} \right. \\ &\quad + \oint_{\gamma_{23}} \log\left(\frac{Z_2 \cdot X}{Z_3 \cdot X}\right) \log(Z_1 \cdot X) \frac{DX}{(P \cdot X)^3} \\ &\quad \left. + \oint_{\gamma_{31}} \log\left(\frac{Z_3 \cdot X}{Z_1 \cdot X}\right) \log(Z_2 \cdot X) \frac{DX}{(P \cdot X)^3} \right), \end{aligned} \quad (4.8)$$

which is manifestly independent of Q^α . However, this representation depends on an implicit choice of line at infinity as this defines the branch of, for example, $\log(Z_3 \cdot X)$ that γ_{12}

circles. In the following we will therefore keep Q^α and its higher-dimensional analogues in our expressions and note the independence of our expressions on these reference boundaries wherever necessary. We also note that a naive summation of the integrands of the T_{ij} 's would give a vanishing result, but that performing the integrations along the contours as specified by the branch cuts of their respective integrands gives non-vanishing results.

The Q^α -independence of this sum of integrals can also be directly checked by differentiating with respect to Q^α . In particular, by again using the fact that $\log(\frac{Z_i \cdot X}{Z_j \cdot X}) + \log(\frac{Z_j \cdot X}{Z_k \cdot X}) + \log(\frac{Z_k \cdot X}{Z_i \cdot X}) = 0$, it can be easily shown — at the integrand level — that

$$K^\alpha \frac{\partial}{\partial Q^\alpha} (T_{12} + T_{23} + T_{31}) = 0, \quad (4.9)$$

and therefore that this sum is independent of Q^α . In the following we use integrals similar to those defining the T_{ij} 's to define the vertex objects, and we will keep the reference boundaries in these expressions. The sums of these objects that we will be interested in will be independent of these boundaries, and the integrand-level proofs of these statements are similar to those found here.

4.2 Two-dimensional vertex objects

Suppose that we have N points $\{Z_i^\alpha\}_{1 \leq i \leq N}$ in $\mathbb{CP}^2$, each defining a line in $\mathbb{CP}^{2*}$. We define the following collection of $\binom{N}{2}$ integrals:

$$F_{ij} \equiv \frac{1}{2} \frac{1}{(2\pi i)^2} \oint_{\gamma_{ij}} \log\left(\frac{Z_i \cdot X}{Z_j \cdot X}\right) \left(\sum_{k \neq i,j} \log\left(\frac{Z_k \cdot X}{Q \cdot X}\right) \right) \frac{DX}{(P \cdot X)^3} \equiv \frac{1}{(2\pi i)^2} \oint_{\gamma_{ij}} f_{ij}(X) \frac{DX}{(P \cdot X)^3}, \quad (4.10)$$

where the contour γ_{ij} first goes around all of the branch cuts from $Z_k \cdot X = 0$ to $Q \cdot X = 0$ and then around the branch cut from $Z_i \cdot X = 0$ to $Z_j \cdot X = 0$. The factor of $\frac{1}{2}$ is conventional. Integrating this gives

$$F_{ij} = \frac{1}{2} \sum_{k \neq i,j}^N ([ijk] - [ijQ]) = \frac{1}{2} \left(\sum_{k \neq i,j}^N [ijk] \right) - \frac{1}{2} (N-2) [ijQ]. \quad (4.11)$$

These are (up to a factor of 2) the vertex objects of ref. [12] and so in particular we have, for any $i, j, k \in \{1, \dots, N\}$, that

$$F_{ij} + F_{jk} + F_{ki} = [ijk]. \quad (4.12)$$

Each individual vertex object depends on Q^α as well as all N of the Z_i^α 's, but the dependence on Q^α and all other Z_l^α 's (i.e., for $l \neq i, j, k$) drops out in the above sum, for reasons identical to those discussed in section 4.1.

In equation (4.10) we wrote F_{ij} as an integral over a function $f_{ij}(X)$ on the dual space. We readily see that $f_{ij} = -f_{ji}$, and can also show that for any choice of i, j , and k ,

$$f_{ij} + f_{jk} + f_{ki} = 0. \quad (4.13)$$

The antisymmetry of the f_{ij} 's as well as property (4.13) imply that the collection of functions $\{f_{ij}\}$ form a representative of a Čech cohomology class on a subspace of $\mathbb{CP}^{2*}$.

In twistor theory, Čech cohomology is a natural setting in which to discuss the Penrose transform, which takes a cohomology class on (a subspace of) twistor space to a finite-normed on-shell field configuration on space-time [16]. The appearance of Čech cohomology here is of a different nature, and the role it is playing in this discussion is still unclear. For the remainder of this note we will not explore this issue. Instead, we simply note this curious connection to cohomology, as it may be important for generalizing these ideas to the $N^k\text{MHV}$ amplituhedron with $k > 1$. For now, we simply move on to describing how to construct the higher-dimensional vertex objects in terms of integrals of logarithms.

4.3 Higher-dimensional vertex objects

Analogous vertex objects can be defined in any dimension. Namely, in D dimensions there exist objects $F_{i_1 \dots i_D}$ such that for any choice of $D+1$ hyperplanes defined by $\{Z_{i_k}\}_{1 \leq k \leq D+1}$, one has the identity

$$F_{i_1 i_2 \dots i_D} + (-1)^D F_{i_2 i_3 \dots i_{D+1}} + F_{i_3 i_4 \dots i_1} + \dots + (-1)^D F_{i_{D+1} i_1 \dots i_{D-1}} = [i_1 i_2 \dots i_{D+1}]. \quad (4.14)$$

Given any polytope in $\mathbb{CP}^D$, one obtains its volume by summing the vertex objects over the vertices of the polytope. In particular, any vertex of the polytope is defined (as reviewed in section 2) by the intersection of D hyperplanes corresponding to $Z_{i_1}, \dots, Z_{i_D}$, and for this vertex one simply includes an $F_{i_1 \dots i_D}$. The precise definition of higher-dimensional polytopes in $\mathbb{CP}^D$ is described in ref. [12], as is the precise way of summing the vertex objects over the vertices. In this subsection, we will see how these higher-dimensional vertex objects arise as contour integrals of logarithms. We will explicitly show this only for dimensions three and four.

4.3.1 Three dimensions

Let $\{Z_i^\alpha\}_{1 \leq i \leq N}$ be N points in $\mathbb{CP}^3$ defining N planes in the dual $\mathbb{CP}^{3*}$. Motivated by the two-dimensional case, we define

$$F_{ij;k} \equiv \frac{1}{(2\pi i)^3} \oint_{\gamma_{ij;k}} \log \left(\frac{Z_i \cdot X}{Z_j \cdot X} \right) \log \left(\frac{Z_k \cdot X}{Q_2 \cdot X} \right) \sum_{l \neq i,j,k} \log \left(\frac{Z_l \cdot X}{Q_1 \cdot X} \right) \frac{DX}{(P \cdot X)^4}, \quad (4.15)$$

where Q_1^α and Q_2^α are fixed reference points in $\mathbb{CP}^3$ defining fixed reference planes in $\mathbb{CP}^{3*}$. The contour $\gamma_{ij;k}$ is an $(S^1)^3$ contour going around the branch cuts of the logarithms in the natural way. Antisymmetrizing over i, j , and k , and noting that each $F_{ij;k}$ is antisymmetric in its first two indices, we then define

$$F_{ijk} \equiv \frac{1}{2 \cdot 3!} F_{[ij;k]} = \frac{1}{3!} (F_{ij;k} + F_{jk;i} + F_{ki;j}). \quad (4.16)$$

Each F_{ijk} depends on Q_1^α and all N of the Z_i^α 's, although it is independent of Q_2^α . To see this Q_2^α -independence, we note that the sum $F_{ij;k} + F_{jk;i} + F_{ki;j}$ gives the exact same

cyclic sum of the $\log(\frac{Z_i \cdot X}{Z_j \cdot X}) \log(\frac{Z_k \cdot X}{Q_2 \cdot X})$ terms in equation (4.15) that appear in the two-dimensional case discussed in section 4.1, and thus the arguments used there to prove the Q^α -independence of this sum directly apply.

Having established that each F_{ijk} depends only the reference boundary Q_1^α , it is worth comparing the meaning of this boundary to that of the dummy boundary Z_*^α that appears² in the BCFW/CSW triangulation of equation (2.25). In the latter, the boundary Z_*^α defines a particular triangulation of the underlying polytope, and the independence of the volume of the polytope on this boundary follows from the general independence of the volume on triangulation. In contrast, Q_1^α determines a boundary that is used to define the branch cuts of the logarithms that appear in the definition of F_{ijk} . The boundary Q_2^α plays a similar role. As discussed in section 4.1, a manifestly Q -independent representation of F_{ijk} can be given, at the cost of making an implicit choice of a plane at infinity that determines the branches of the logarithms. Thus, the Q 's that make an appearance here can be viewed as generalized “planes at infinity”, the intersection of which gives a line at infinity. These boundaries therefore do not play a direct role in defining any triangulation — such a boundary would enter into the subscripts of F_{ijk} itself, as in the middle line of equation (2.29). This is in line with the fact that the vertex objects encode triangulation-independent data about the underlying polytope [12]. Moreover, with an explicit choice of endpoints for the branch cuts of the logarithms that define F_{ijk} , these Q^α 's can be removed entirely.

It is straightforward to show that the F_{ijk} functions are (up to a factor of $2 \times 3!$) the vertex objects defined in ref. [12]. It then follows that for any choice of $i, j, k, l \in \{1, \dots, N\}$, one has

$$F_{ijk} - F_{jkl} + F_{kli} - F_{lij} = [ijkl], \quad (4.17)$$

where $[ijkl]$ is the volume of the three-simplex bounded by the four faces defined by Z_i^α , Z_j^α , Z_k^α , and Z_l^α . The dependence on Q_1^α and all other Z_m^α 's drops out in this sum.

4.3.2 Four dimensions

The definition of the four-dimensional vertex objects is similar. Let $\{Z_i^\alpha\}_{1 \leq i \leq N}$ be N points in $\mathbb{CP}^4$ defining N hyperplanes in the dual $\mathbb{CP}^{4*}$. Define

$$F_{ij;k;l} = \frac{1}{(2\pi i)^4} \oint_{\gamma_{ij;k;l}} \log\left(\frac{Z_i \cdot X}{Z_j \cdot X}\right) \log\left(\frac{Z_k \cdot X}{Q_3 \cdot X}\right) \log\left(\frac{Z_l \cdot X}{Q_2 \cdot X}\right) \sum_{m \neq i,j,k,l} \log\left(\frac{Z_m \cdot X}{Q_1 \cdot X}\right) \frac{DX}{(P \cdot X)^5}, \quad (4.18)$$

where Q_1^α , Q_2^α , and Q_3^α are fixed reference points in $\mathbb{CP}^4$ defining reference hyperplanes in $\mathbb{CP}^{4*}$. The contour $\gamma_{ij;k;l}$ is an $(S^1)^4$ contour going around the branch cuts of the logarithms in the natural way. We define

$$F_{ijkl} \equiv \frac{1}{2 \cdot 4!} F_{[ij;k;l]} = \frac{1}{4!} (F_{ij;k;l} - F_{ij;l;k} + F_{ik;l;j} - F_{ik;j;l} + F_{il;j;k} - F_{il;k;j} \\ + F_{jk;i;l} - F_{jk;l;i} + F_{jl;k;i} - F_{jl;i;k} + F_{kl;i;j} - F_{kl;j;i}). \quad (4.19)$$

²We note that we are currently working in three dimensions, whereas the BCFW/CSW triangulation triangulates a four-dimensional polytope. However, the statements we make here directly carry over to the four dimensional case discussed in the next section.

Similarly to the two- and three-dimensional cases, each individual F_{ijkl} is independent of Q_2^α and Q_3^α , though it is dependent on Q_1^α and all N of the Z_i^α 's.

Up to a factor of $2 \times 4!$, the F_{ijkl} functions are precisely the vertex objects defined in ref. [12]. It therefore follows that for any choice of i, j, k, l , and m , we have

$$F_{ijkl} + F_{jklm} + F_{klmi} + F_{lmij} + F_{mijk} = [ijklm], \quad (4.20)$$

where $[ijklm]$ is the volume of a four-simplex bounded by the five faces defined by Z_i^α , Z_j^α , Z_k^α , Z_l^α , and Z_m^α . Again, the dependence on Q_1^α and all other Z_n^α 's drops out in this sum. This completes the proof that the vertex objects of ref. [12], which can be used as basic building blocks for computing volumes of general polytopes as described in that reference, have a natural definition as simple contour integrals in the same space in which those polytopes live.

5 Conclusion and outlook

In this paper we showed that volumes of general polytopes can be computed using contour integrals of logarithms directly in the space in which the polytopes live. The contours of these integrals are canonically specified by the integrands themselves, and the organizing principle for combining these integrals comes directly from the geometry of the polytope — the intersections of its faces — and thus does not rely on any particular triangulation. We also found a surprising connection between the integrands of the two-dimensional vertex objects and Čech cohomology. It would be interesting to further explore this connection.

The vertex objects that we have defined are useful for computing NMHV tree-level amplitudes in the planar limit of $\mathcal{N} = 4$ super-Yang-Mills, and we have seen logarithms appear naturally. It would be interesting to see how these ideas might generalize to loop level. Additionally, since our discussion has been limited to tree-level amplitudes, these results readily apply at tree level to Yang-Mills theories with less (and no) supersymmetry. It would therefore be interesting to see if similar ideas can be used for less supersymmetric theories beyond tree level. Taking the planar limit appears to be crucial in this discussion, as momentum (super-)twistors play a fundamental role and these cease to exist in non-planar theories. Nonetheless, it is worth exploring if and to what extent this discussion can be extended to the non-planar sector of the theory.

The vertex objects we defined can be used to obtain identities amongst sums of simplices, and these identities can therefore now be viewed as being obtained from contour integrals of logarithms directly in the space containing the polytope. This differs dramatically from the Grassmannian picture discussed in the introduction. Understanding the relation between these two approaches will help extend the method introduced in this note to N^k MHV tree amplitudes for $k > 1$, since the Grassmannian picture is already well-understood for these more complicated cases. Expressing volumes in terms of the vertex objects naturally encodes the geometry of the underlying polytope. If the analogous objects can be found for the $k > 1$ cases, likely by first making a connection to the Grassmannian picture, then this should shed light on the geometry of the dual amplituhedron directly, without a need for any auxiliary spaces.

Acknowledgments

I thank Zvi Bern for his help. I would also like to thank Andrew Hodges, and I am grateful to Julio Parra Martinez and Enrico Herrmann for many interesting discussions. I am grateful to the Mani L. Bhaumik Institute for Theoretical Physics for its support.

Open Access. This article is distributed under the terms of the Creative Commons Attribution License ([CC-BY 4.0](https://creativecommons.org/licenses/by/4.0/)), which permits any use, distribution and reproduction in any medium, provided the original author(s) and source are credited.

References

- [1] H. Elvang and Y.-t. Huang, *Scattering Amplitudes*, [arXiv:1308.1697](https://arxiv.org/abs/1308.1697) [[INSPIRE](#)].
- [2] L.J. Dixon, *Calculating scattering amplitudes efficiently, in QCD and beyond. Proceedings, Theoretical Advanced Study Institute in Elementary Particle Physics, TASI-95, Boulder, U.S.A., June 4-30, 1995*, pp. 539–584, [hep-ph/9601359](https://arxiv.org/abs/hep-ph/9601359) [[INSPIRE](#)].
- [3] J.M. Henn and J.C. Plefka, *Scattering Amplitudes in Gauge Theories*, in *Lecture Notes in Physics* **883**, Springer (2014).
- [4] N. Arkani-Hamed, F. Cachazo and J. Kaplan, *What is the Simplest Quantum Field Theory?*, *JHEP* **09** (2010) 016 [[arXiv:0808.1446](https://arxiv.org/abs/0808.1446)] [[INSPIRE](#)].
- [5] N. Arkani-Hamed and J. Trnka, *The Amplituhedron*, *JHEP* **10** (2014) 030 [[arXiv:1312.2007](https://arxiv.org/abs/1312.2007)] [[INSPIRE](#)].
- [6] N. Arkani-Hamed and J. Trnka, *Into the Amplituhedron*, *JHEP* **12** (2014) 182 [[arXiv:1312.7878](https://arxiv.org/abs/1312.7878)] [[INSPIRE](#)].
- [7] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo, A.B. Goncharov, A. Postnikov and J. Trnka, *Grassmannian Geometry of Scattering Amplitudes*, Cambridge University Press, (2016), [[arXiv:1212.5605](https://arxiv.org/abs/1212.5605)] [[INSPIRE](#)].
- [8] A. Postnikov, *Total positivity, Grassmannians and networks*, [math/0609764](https://arxiv.org/abs/math/0609764) [[INSPIRE](#)].
- [9] N. Arkani-Hamed, J.L. Bourjaily, F. Cachazo, A. Hodges and J. Trnka, *A Note on Polytopes for Scattering Amplitudes*, *JHEP* **04** (2012) 081 [[arXiv:1012.6030](https://arxiv.org/abs/1012.6030)] [[INSPIRE](#)].
- [10] N. Arkani-Hamed, A. Hodges and J. Trnka, *Positive Amplitudes In The Amplituhedron*, *JHEP* **08** (2015) 030 [[arXiv:1412.8478](https://arxiv.org/abs/1412.8478)] [[INSPIRE](#)].
- [11] L. Ferro, T. Lukowski, A. Orta and M. Parisi, *Towards the Amplituhedron Volume*, *JHEP* **03** (2016) 014 [[arXiv:1512.04954](https://arxiv.org/abs/1512.04954)] [[INSPIRE](#)].
- [12] M. Enciso, *Volumes of Polytopes Without Triangulations*, *JHEP* **10** (2017) 071 [[arXiv:1408.0932](https://arxiv.org/abs/1408.0932)] [[INSPIRE](#)].
- [13] L.J. Mason and D. Skinner, *Dual Superconformal Invariance, Momentum Twistors and Grassmannians*, *JHEP* **11** (2009) 045 [[arXiv:0909.0250](https://arxiv.org/abs/0909.0250)] [[INSPIRE](#)].
- [14] N. Arkani-Hamed, F. Cachazo, C. Cheung and J. Kaplan, *A Duality For The S Matrix*, *JHEP* **03** (2010) 020 [[arXiv:0907.5418](https://arxiv.org/abs/0907.5418)] [[INSPIRE](#)].
- [15] J.M. Drummond, J.M. Henn and J. Plefka, *Yangian symmetry of scattering amplitudes in $N = 4$ super Yang-Mills theory*, *JHEP* **05** (2009) 046 [[arXiv:0902.2987](https://arxiv.org/abs/0902.2987)] [[INSPIRE](#)].

- [16] S. Huggett and K. Tod, *An Introduction to Twistor Theory*, Cambridge University Press, (1994).
- [17] L. Hughston and T. Hurd, *A CP^5 Calculus For Space-time Fields*, *Phys. Rept.* **100** (1983) 275.
- [18] A. Hodges, *Eliminating spurious poles from gauge-theoretic amplitudes*, *JHEP* **05** (2013) 135 [[arXiv:0905.1473](#)] [[INSPIRE](#)].
- [19] R. Britto, F. Cachazo, B. Feng and E. Witten, *Direct proof of tree-level recursion relation in Yang-Mills theory*, *Phys. Rev. Lett.* **94** (2005) 181602 [[hep-th/0501052](#)] [[INSPIRE](#)].