RECEIVED: December 28, 2016

REVISED: March 20, 2017

ACCEPTED: April 13, 2017

PUBLISHED: April 28, 2017

On a realization of $\{\beta\}$ -expansion in QCD

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ABSTRACT: We suggest a simple algebraic approach to fix the elements of the $\{\beta\}$ -expansion for renormalization group invariant quantities, which uses additional degrees of freedom. The approach is discussed in detail for N²LO calculations in QCD with the MSSM gluino — an additional degree of freedom. We derive the formulae of the $\{\beta\}$ -expansion for the nonsinglet Adler D -function and Bjorken polarized sum rules in the actual N³LO within this quantum field theory scheme with the MSSM gluino and the scheme with the second additional degree of freedom. We discuss the properties of the $\{\beta\}$ -expansion for higher orders considering the N⁴LO as an example.

KEYWORDS: Renormalization Group, Renormalization Regularization and Renormalons

ARXIV EPRINT: [1610.01305](https://arxiv.org/abs/1610.01305)

Contents

1	Introduction	1
2	Algebraic approach for the $\{\beta\}$-expansion in N²LO	3
2.1	A simple illustration	3
2.2	The formalism of decomposition for D-function	4
2.3	How to obtain the $\{\beta\}$ -expansion for C^{Bjp} from one for D	5
3	The $\{\beta\}$-expansion for Bjorken polarized SR and D-function in N³LO	6
3.1	Decomposition with 2 degrees of freedom x, y	7
3.2	Relations between the elements of D and C^{Bjp}	7
3.3	What can we get at 3 degrees of freedom x, y, z	9
4	What can we expect for $\{\beta\}$-expansion in N⁴LO	10
5	Conclusion	11
A	Explicit formulas for the elements of D and C	12
B	The β-function coefficients	13

1 Introduction

The knowledge of the detailed structure of QCD perturbation expansions is rather important for a variety of tasks of which the renormalization group optimization of the series is the best known. The detailed structure looks as a double series (or a matrix representation) rather than a usual series [1, 2] even in the case of expansion of ‘physical’ quantities (having no anomalous dimension). We shall explore this structure for the QCD renormalization group invariant (RGI) one-scale dependent quantities as well as elaborate an algebraic approach to fix their elements within the $\{\beta\}$ -expansion [1]. The knowledge of these expansion elements makes it possible to put the task of perturbation series optimization and helps to relate different quantities. In this sense our paper continues the investigations of the perturbation expansions in [1–4]. There, we discussed in detail the optimization of the perturbation expansion in N²LO in [3] beyond the Brodsky-Lepage-Mackenzie approach (BLM) [5]. Moreover, in [2, 3], based on the $\{\beta\}$ -expansion in this order, we related the pair of different RGI quantities. Now let us introduce the appropriate physically important quantities whose perturbation expansions are the most advanced. We

take as patterns of the RGI quantities the phenomenological important Bjorken polarized sum rules $S^{\text{Bjp}}(Q^2, \mu^2)$,

$$S^{\text{Bjp}}(Q^2) = \frac{g_A}{6} \left[C_{\text{NS}}^{\text{Bjp}}(Q^2/\mu^2, a_s(\mu^2)) + \left(\sum_i q_i \right) C_{\text{S}}^{\text{Bjp}}(Q^2/\mu^2, a_s(\mu^2)) \right], \quad (1.1)$$

and the Adler function $D(Q^2, \mu^2)$,

$$D^{\text{EM}}\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right) = \left(\sum_i q_i^2 \right) d_R D_{\text{NS}}\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right) + \left(\sum_i q_i \right)^2 d_R D_{\text{S}}\left(\frac{Q^2}{\mu^2}, a_s(\mu^2)\right), \quad (1.2)$$

where q_i is the electric charge of the quark, g_A — nucleon axial charge, d_R — the dimension of the quark color representation. Perturbation expression for the nonsinglet (NS) coefficient functions of both the quantities at the renormalization scale $\mu^2 = Q^2$ can be written down as

$$D_{\text{NS}}(a_s(\mu^2)) = 1 + \sum_{n \geq 1} a_s^n(\mu^2) d_n, \quad C_{\text{NS}}^{\text{Bjp}}(a_s(\mu^2)) = 1 + \sum_{l \geq 1} a_s^l(\mu^2) c_l; \quad (1.3)$$

$a_s = \alpha_s/(4\pi)$, they are calculable in the $\overline{\text{MS}}$ -scheme and were obtained in order of $O(a_s^4)$ in [6]. We use here only the NS parts of these quantities, $D = D_{\text{NS}}$, $C^{\text{Bjp}} = C_{\text{NS}}^{\text{Bjp}}$, omitting the corresponding notation further in the text. The perturbation coefficients d_n (c_n) in eqs. (1.3) are the combinations of only the color coefficients.

Now, recall the structure of these perturbation coefficients. The $\{\beta\}$ -expansion representation introduced in [1] prescribes to decompose d_n or/and c_n or any other of RGI quantity in the following way:

$$d_1 = d_1[0], \quad (1.4a)$$

$$d_2 = \beta_0 d_2[1] + d_2[0], \quad (1.4b)$$

$$d_3 = \beta_0^2 d_3[2] + \beta_1 d_3[0, 1] + \beta_0 d_3[1] + d_3[0], \quad (1.4c)$$

$$d_4 = \beta_0^3 d_4[3] + \beta_1 \beta_0 d_4[1, 1] + \beta_2 d_4[0, 0, 1] + \beta_0^2 d_4[2] + \beta_1 d_4[0, 1] + \beta_0 d_4[1] + d_4[0], \quad (1.4d)$$

$\vdots$

$$d_n = \beta_0^{n-1} d_n[n-1] + \dots + d_n[0], \quad (1.4e)$$

where β_i are the coefficients of the QCD β -function

$$\mu^2 \frac{da_s(\mu^2)}{d\mu^2} = \beta(a_s) = -a_s^2(\mu^2) \sum_{i \geq 1} \beta_{i-1} a_s^{i-1}(\mu^2), \quad (1.5)$$

and the explicit expressions for β_{0-3} are presented in appendix A. The notation $i_0, i_1, \dots$ of the arguments of $d_n[i_0, i_1, \dots]$ denotes the powers of accompanying $\beta_0, \beta_1, \dots$. The elements $d_n[\cdot]$ of the decomposition do not depend on the number of active quarks n_f at least up to the actual order $O(a_s^3)$ ($n = 3$). This important property of the elements

$d_n[\cdot]$ will be discussed at the beginning of section 3 and in section 4. The decompositions in eqs. (1.4) should contain the complete knowledge about strong charge renormalization by means of using there all of the possible β_i -terms [1–3] (see the beginning of section 4 for details). This kind of the expansion is the essential part of the procedures for the optimization of perturbation series, e.g., the decomposition (1.4b) was the starting point of the well-known BLM prescription [5] in NLO. Indeed, in the BLM the contribution $\beta_0 d_2[1]$ is transferred to the new normalization scale μ' of the coupling constant, $\mu^2 \rightarrow \mu'^2 = \mu^2 \exp(-d_2[1]/d_1)$; $a_s(\mu^2) \rightarrow a_s(\mu'^2) \equiv a'_s$; $d_2 \rightarrow d_2[0]$. Finally, $a_s^2 d_2 \rightarrow a_s'^2 d_2[0]$, that often improves the expansion for many cases (e.g., $d_2[0] \ll d_2$ for the Adler function). For further high order development [1] the principle of maximum conformality (PMC) was proposed [7, 8], which demands all of the β -terms in the decompositions eq. (1.4c), (1.4d), ... to be accumulated into the new scale; therefore, the n th term of expansion turns into $a_s^n d_n \xrightarrow{\text{PMC}} a_s'^n d_n[0]$ and one should know these proper elements $d_n[0]$. This PMC approach does not mandatory lead to the improvement of expansion, the latter depends on distribution and signs of different (β) contributions for each perturbation order. Different optimization conditions require the knowledge of different β -terms in the decompositions of eq. (1.4), which were discussed in detail in sections 5 and 6 in [3].

At NLO of QCD the decomposition in (1.4b) looks evident because the term proportional to $\frac{4}{3}T_R n_f$ unambiguously marks the contribution of the term proportional to $\beta_0 = \frac{11}{3}C_A - \frac{4}{3}T_R n_f$ in d_2 , see the discussion in section 3B in [3] and the result in eq. (A.1b) in appendix A. How to fix the elements of the decomposition in higher orders? The consideration of color coefficients content of the d_n is not enough for this, so one need to find additional conditions. To solve the problem, we introduce additional degrees of freedom (d.o.f.), new fields that interact following the universal gauge principle and enter only in intrinsic loops. Using the fermions in the adjoint representation (MSSM light gluino) as an additional d.o.f., we formulate a simple algebraic scheme to obtain the elements of the $\{\beta\}$ -expansion and demonstrate the results in N²LO, eq. (1.4c), in the following sections 2.1 and 2.2. Moreover, based on the Crewther relation [9] we derive the relation between C and D in section 2.3. This algebraic scheme is well algorithmized and appropriate to apply to high loop results. It is applied to the N³LO expansion in eq. (1.4d) in section 3, which fixes completely the elements of expansion. The required expressions for d_4 , c_4 with the additional d.o.f. are expected to be calculated in future. In section 4, we discuss the general structure and properties of $\{\beta\}$ -expansion for higher orders considering the N⁴LO as an example. The algebraic scheme fixes the expansion elements for this case too. Our main results are presented in Conclusion.

2 Algebraic approach for the $\{\beta\}$ -expansion in N²LO

2.1 A simple illustration

Let us consider the task to fix the decomposition elements in eq. (1.4) algebraically, taking eq. (1.4c) as an example. The “renormalon” term $d_3[2]$ (or $d_n[n-1]$ for any n) at the maximum power of β_0 can be identified by the maximum power of $\frac{4}{3}T_R n_f$ (here it is proportional to $C_F(\frac{4}{3}T_R n_f)^2$), or even calculated independently, see [10]. The corresponding

residual in the r.h.s. of eq. (1.4c) contains 5 Casimir coefficients C_F^3 , $C_F^2 C_A$, $C_F C_A^2$, $C_F^2 T_R n_f$, $C_F C_A T_R n_f$ that are distributed among three terms $d_3[\cdot]$ there. Finally we need to obtain these three unknown elements $d_3[0, 1]$, $d_3[1]$, $d_3[0]$ in the r.h.s. of eq. (2.1)

$$\bar{d}_3(x) \equiv d_3(x) - \beta_0^2(x) d_3[2] = \beta_1(x) d_3[0, 1] + \beta_0(x) d_3[1] + d_3[0], \quad (2.1)$$

where we put variable $x = \frac{4}{3} T_R n_f$. Taking eq. (2.1) at any three different values of x , $(x_1, x_2, x_3) = X$ and compiling the coupled system of linear equations we can obtain the unique solution of this system under the evident condition that the corresponding determinant Δ_3 ,

$$\Delta_3(X) = (\beta_0(x_2) - \beta_0(x_1))(\beta_1(x_1) - \beta_1(x_0)) - (\beta_0(x_1) - \beta_0(x_0))(\beta_1(x_2) - \beta_1(x_1)), \quad (2.2)$$

is not zero. The opposite condition $\Delta_3 = 0$ unambiguously means that the functions $\beta_0(x)$, $\beta_1(x)$ are linear in x ,

$$\frac{\beta_0(x_2) - \beta_0(x_1)}{\beta_0(x_1) - \beta_0(x_0)} = \frac{\beta_1(x_2) - \beta_1(x_1)}{\beta_1(x_1) - \beta_1(x_0)},$$

this is just the case of QCD with only quark degrees of freedom (see the explicit expressions in eq. (B.1)). Due to this reason one cannot untangle contributions from β_0 and β_1 in N²LO without an additional constraint (see the discussion in [3]). In the case of an additional degree of freedom that contributes to both sides of eq. (2.1), i.e., to the coefficient d_3 and to β_0 , β_1 , one can obtain the unique solution. The goal of this note is to elaborate an algebraic scheme to obtain the decompositions in eqs. (1.4) using additional d.o.f. like $n_{\tilde{g}}$ — the number of MSSM gluino (we use $y = \frac{4}{3} \frac{C_A}{2} n_{\tilde{g}}$) and, may be, other fields that interact following the universal gauge principle and appear only in intrinsic loops. The net effect of this field will be parameterized by means of the parameter z . Further, we shall suggest that the coefficients of perturbation expansion, like d_n (c_n) in the l.h.s. of (1.4), as well as the coefficients of the β -function in the r.h.s. of (1.4) are calculated within the $\overline{\text{MS}}$ scheme and are known functions on the arguments x, y, z . To be more exact we consider the Adler function $D(x, y)$ [11] as well as the β -coefficients $\beta_0(x, y)$, $\beta_1(x, y)$ presented in appendix B as functions on both the quark (x) and the MSSM gluinos (y) d.o.f. In this notation $\beta_i(x, 0) = \beta_i(x)$, $d_n(x, 0) = d_n(x), \dots$. The results for the decomposition presented below are valid also for the coefficient function $C^{\text{Bjp}}(x, y)$ up to the replacement of the notation and for any RGI one-scale quantities.

2.2 The formalism of decomposition for D-function

To simplify the system of equations (SE) based on eq. (2.3) (the extended by the y d.o.f. eq. (2.1)),

$$\bar{d}_3(x, y) \equiv d_3(x, y) - \beta_0^2(x, y) d_3[2] = \beta_1(x, y) d_3[0, 1] + \beta_0(x, y) d_3[1] + d_3[0], \quad (2.3)$$

we take for the components of X : $x_0, x_1, (x_{01}, y_{01})$, the special values — the roots of equations

$$\beta_0(x_0) = 0, \quad \beta_1(x_1) = 0, \quad \{\beta_0(x_{01}, y_{01}) = 0, \beta_1(x_{01}, y_{01}) = 0\}. \quad (2.4)$$

For this X_3 the SE_3 looks like

$$\begin{cases} d_3(x_{01}, y_{01}) = d_3[0] \\ d_3(x_0, 0) = \beta_1(x_0)d_3[0, 1] + d_3[0] \\ \bar{d}_3(x_1, 0) = \beta_0(x_1)d_3[1] + d_3[0]. \end{cases} \quad (2.5)$$

Now the value of the determinant $\Delta_3(X_3) = \beta_0(x_1)\beta_1(x_0) = -\frac{C_A^2(7C_A+11C_F)^2}{(5C_A+3C_F)} \neq 0$ that follows from eq. (2.2) or, can be obtained from the SE_3 (2.5) directly. Therefore the unique solution of the SE_3 is

$$d_3[0] = d_3(x_{01}, y_{01}), \quad (2.6a)$$

$$d_3[0, 1] = (d_3(x_0) - d_3(x_{01}, y_{01}))/\beta_1(x_0), \quad (2.6b)$$

$$d_3[1] = (\bar{d}_3(x_1) - d_3(x_{01}, y_{01}))/\beta_0(x_1). \quad (2.6c)$$

These values were obtained first in [1] using another trick; here they are presented explicitly in eq. (A.1) in appendix A.

2.3 How to obtain the $\{\beta\}$ -expansion for C^{Bjp} from one for D

To relate the already known structure of d_3 (the solutions in (2.6)) to the corresponding $\{\beta\}$ -expansion of c_3 , we use the generalized Crewther relation (CR) [2, 10],

$$D(a_s) \cdot C^{\text{Bjp}}(a_s) = \mathbb{1} + \beta(a_s) \cdot K(a_s), \quad (2.7)$$

where $K(a_s) = \sum_{n=1} a_s^{n-1} K_n$ is a polynomial in a_s . In the case of the β -function having identically zero coefficients $\beta_i = 0$, the generalized CR (2.7) returns to its initial form [9] with only $\mathbb{1}$ in its r.h.s. that expresses the unbroken conformal symmetry. The later condition relates the $d_n[0]$, $c_n[0]$ elements in every order (see definition (4.1) and eq. (4.2) in [3])

$$c_n[0] = -d_n[0] - \sum_{l=1}^{n-1} d_l[0]c_{n-l}[0]. \quad (2.8)$$

The explicit closed solution of the relation (2.8) with respect to $c_k[0]$ is

$$c_k[0] = (-)^k \det[D_0^{(k)}] \equiv (-)^k \begin{vmatrix} d_1 & 1 & 0 & \dots & 0 \\ d_2 & d_1 & 1 & \dots & 0 \\ d_3 & d_2 & d_1 & \dots & 0 \\ \dots & & & \dots & 0 \\ d_{k-1} & \dots & \dots & d_1 & 1 \\ d_k & d_{k-1} & d_{k-2} & \dots & d_2 & d_1 \end{vmatrix}, \quad (2.9)$$

here $D_0^{(k)}$ -matrix, which consists of $d_i \equiv d_i[0]$ elements. The general relation (2.9) can be also treated as a prediction for C^{Bjp} by means of D that is based on the $\{\beta\}$ -expansion and CR. In the third order of a_s the knowledge of the element $c_3[0] = -d_3[0] + 2d_1d_2[0] - (d_1)^3$, followed from (2.8), allows us to fix all the other elements of the expansion in this order,

i.e. to disentangle the contributions from $c_3[1]\beta_0$ and $c_3[0,1]\beta_1$, which were discussed in detail in section IVB in [3].

From another side one can use in the r.h.s. of eq. (2.7) the second term proportional to $\beta(a_s)$ that expresses the conformal symmetry-breaking. This leads to the series of relations [2, 3] for the elements of the different orders n at β_{n-1} , e.g.,

$$d_2[1] + c_2[1] = d_3[0, 1] + c_3[0, 1] = \dots = d_n[\underbrace{0, 0, \dots, 1}_{n-1}] + c_n[\underbrace{0, 0, \dots, 1}_{n-1}] = 3C_F \left(\frac{7}{2} - 4\zeta_3 \right). \quad (2.10)$$

In the third order this gives the equation $c_3[0, 1] = d_2[1] + c_2[1] - d_3[0, 1]$, which fixes $c_3[0, 1]$ and also admits restoration of two other terms $c_3[1]$, $c_3[0]$.

Both of the ways provide the same results for the elements $c_3[0, 1]$, $c_3[1]$, $c_3[0]$ of the $\{\beta\}$ -expansion [3]. As a byproduct of the procedure we predicted the c_3 of C^{Bjp} , see eq. (4.11) in [3], if the $\{\beta\}$ -expansion for d_3 of D is already known and vice versa, $C^{\text{Bjp}} \hookrightarrow D$. This demonstrates that the elements of $\{\beta\}$ -expansion provide appropriate “bricks” for complete determination of RGI quantities.

3 The $\{\beta\}$ -expansion for Bjorken polarized SR and D-function in N³LO

In the 5 loop case, $d_4(x)$ was first obtained in [12] as the polynomial with numerical coefficients, then all the color coefficients in decomposition for $d_4(x)$ and $c_4(x)$ were presented in [6]. Following the $\{\beta\}$ -expansion we propose for these coefficients the decomposition,

$$\begin{aligned} \bar{d}_4(x, y) &\equiv d_4(x, y) - \beta_0^3 d_4[3] \\ &= \beta_1 \beta_0 d_4[1, 1] + \beta_2 d_4[0, 0, 1] + \beta_0^2 d_4[2] + \beta_1 d_4[0, 1] + \beta_0 d_4[1] + d_4[0]. \end{aligned} \quad (3.1)$$

Here one has six unknown elements $d_4[1, 1]$, $d_4[0, 0, 1]$, $d_4[2]$, $d_4[0, 1]$, $d_4[1]$, $d_4[0]$, while the seventh element $d_4[3]$ can be directly identified. The ten Casimirs (here $T_f \equiv T_R n_f$) C_F^4 , $C_F^3 T_f$, $C_F^2 T_f^2$, $C_F T_f^3$, $C_F^3 C_A$, $C_F^2 T_f C_A$, $C_F T_f^2 C_A$, $C_F^2 C_A^2$, $C_F T_f C_A^2$, $C_F C_A^3$ are distributed among all of the $d_4[\cdot]$ elements, while the abelian elements of the box subgraphs with four gluon legs, related to color coefficients $n_f d_F^{abcd} d_F^{abcd} / d_R$ (quark box inside), $d_F^{abcd} d_A^{abcd} / d_R$ (gluon box inside), enter into $d_4[0]$. These terms do not contribute to the renormalization of the charge¹ a_s , see also the discussion of the subject in [13]. Although the corresponding 5-loop diagrams contain these one-loop boxes, further contraction of the subgraphs (see the discussion in [4]) do not contribute to β_0 . Due to this reason $d_4[0]$ get (x, y) -dependence $d_4[0] \rightarrow d_4[0](x, y)$. We decompose it as $d_4[0](x, y) = \tilde{d}_4[0] + \delta d_4(x, y)$, where the (x, y) -dependent part $\delta d(x, y)$ is well recognized, while $\delta d_4(x, 0)(\delta c_4(x, 0))$ is already known from the result in [6] (see eq. (A.7) in appendix A). Therefore, the $n_f(n_{\bar{g}})$ -dependence becomes partly separated from the charge renormalization for the first time in N³LO.

¹I thank A. Grozin for clarifying this subject.

3.1 Decomposition with 2 degrees of freedom x, y

To obtain these $d_4[\cdot]$ elements, one can take six points X in the plane (x, y) . Then one takes the set X as the arguments of $d_4(x, y)$ and $\beta_i(x, y)$ and compiles the system of linear equations (SE_6), based on eq. (3.1), with respect to these six unknown elements of the β -expansion. Again, to simplify the calculation, we take for these six components of X_6 : $x_0, x_1, x_2, (x_{01}, y_{01}), (x_{02}, y_{02}), (x_{12}, y_{12})$ the roots of the equations and the systems of equations

$$\beta_0(x_0, 0) = 0, \quad \beta_1(x_1, 0) = 0, \quad \beta_2(x_{2m(p)}, 0) = 0, \quad \{\beta_0(x_{01}, y_{01}) = 0, \beta_1(x_{01}, y_{01}) = 0\}, \quad (3.2a)$$

$$\{\beta_0(x_{02}, y_{02}) = 0, \beta_2(x_{02}, y_{02}) = 0\}, \quad \{\beta_1(x_{12}, y_{12}) = 0, \beta_2(x_{12}, y_{12}) = 0\}. \quad (3.2b)$$

We shall supply the solutions in eqs. (3.2b) the subscripts: $x_{02m(p)}, y_{02m(p)}, x_{12m(p)}, y_{12m(p)}$, to separate the different roots $m(-), p(+)$ of quadratic equations for the cases where $\beta_2(x, y)$ are involved (see the expression in (B.1c)). The determinant $\Delta_6(X_6)$ of the corresponding SE_6 is

$$\Delta_6(X_6) = \beta_0(x_1)\beta_0(x_{2m})\beta_1(x_{2m})\text{Re}\beta_0(x_{12m}, y_{12m})[\beta_0(x_1) - \text{Re}\beta_0(x_{12m}, y_{12m})]\delta_6, \quad (3.3)$$

$$\delta_6 = [\beta_1(x_{02m}, y_{02m})\beta_2(x_0) - \beta_1(x_{02m}, y_{02m})\beta_2(x_{01}, y_{01}) + \beta_1(x_0)\beta_2(x_{01}, y_{01})]. \quad (3.4)$$

This value $\Delta_6(X_6) \neq 0$, the solution of this SE_6 exists and unique, and can be obtained like the solution of eq. (2.6) for the N^2LO in section 2.2. Therefore, to derive the $\{\beta\}$ -expansion for d_4 , it is *enough to obtain one at an additional single d.o.f. y* , $d_4 \rightarrow d_4(x, y)$ together with the coefficients $\beta_{0,1,2}(x, y)$ (see appendix B).

We present the solutions of SE_6 for a number of elements in the explicit form, taking the notation for the arguments (x_{ij}, y_{ij}) and the function Y_4 for shortness,

$$r_{ij} = (x_{ij}, y_{ij}), \quad Y_4(X_6) \equiv \bar{d}_4(X_6) - \delta d_4(X_6), \quad (3.5)$$

$$d_4[0, 0, 1] = [Y_4(r_{01})(\beta_1(x_0) - \beta_1(r_{02})) - Y_4(r_{02})\beta_1(x_0) + Y_4(x_0)\beta_1(r_{02})]/\delta_6, \quad (3.6a)$$

$$d_4[0, 1] = [Y_4(r_{02})(\beta_2(x_0) - \beta_2(r_{01})) - Y_4(r_{01})\beta_2(x_0) + Y_4(x_0)\beta_2(r_{01})]/\delta_6, \quad (3.6b)$$

$$\tilde{d}_4[0] = [Y_4(r_{02})\beta_2(r_{01})\beta_1(x_0) + Y_4(r_{01})\beta_1(r_{02})\beta_2(x_0) - Y_4(x_0)\beta_2(r_{01})\beta_1(r_{02})]/\delta_6. \quad (3.6c)$$

Just these elements will be used for the relation with similar elements in C^{Bjp} .

Of course, one can take another set X'_6 and construct the corresponding SE'_6 . In any case the solution for the elements $d_4[\cdot]$ should be the same. The usage of the roots in eqs. (3.2) to construct X_6 leads to the simplification of the final SE.

3.2 Relations between the elements of D and C^{Bjp}

Suppose that the coefficient functions for the Adler D-function $d_4(x, y)$ and the Bjorken SR $c_4(x, y)$ are known. Then, based on two terms in the r.h.s. of the Crewther relation, eq. (2.7), one can obtain for the sum of these functions $d_4(x, y) + c_4(x, y)$ and their elements (see [2]) a series of the relations. In part, one can obtain from (2.8) the sum of “zero” elements,

$$d_4[0](x, y) + c_4[0](x, y) = \tilde{d}_4[0] + \tilde{c}_4[0] = 2d_1d_3[0] - 3d_1^2d_2[0] + d_2[0]^2 + d_1^4, \quad (3.7)$$

the term $\delta d_4(x, y)$ is cancel in the sum $d_4[0] + c_4[0]$. Generally speaking, for n -order case

$$c_n[0] + d_n[0] = - \sum_{l=1}^{n-1} d_{n-l}[0] (-)^l \det[D_0^{(l)}] = (-)^n \begin{vmatrix} d_1 & 1 & 0 & \dots & 0 \\ d_2 & d_1 & 1 & \dots & 0 \\ d_3 & d_2 & d_1 & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 \\ d_{k-1} & \dots & \dots & d_1 & 1 \\ \mathbf{0} & d_{k-1} & d_{k-2} & \dots & d_2 & d_1 \end{vmatrix}, \quad (3.8)$$

where $D_0^{(l)}$ is defined in (2.9). The l.h.s. of eq. (3.8) is of n -order, while its r.h.s. depends on the $d_l[0]$ elements of less orders $l \leq n-1$, therefore this equation can serve a good check for next order results. The others relations are:

$$(d_4 + c_4)(x_{01}, y_{01}) = \beta_2(x_{01}, y_{01}) (\underline{d_4[0, 0, 1] + c_4[0, 0, 1]}) + \underline{d_4[0] + c_4[0]} \quad (3.9a)$$

$$= \beta_2(x_{01}, y_{01}) \underline{3C_F \left(\frac{7}{2} - 4\zeta_3 \right)} + \underline{(2d_1 d_3[0] - 3d_1^2 d_2[0] + d_2[0]^2 + d_1^4)} \quad (3.9b)$$

$$= \beta_2(x_{01}, y_{01}) \underline{3C_F \left(\frac{7}{2} - 4\zeta_3 \right)} + \underline{(3C_F)^2 \left[\left(\frac{175}{6} - 144\zeta_3 \right) C_A^2 + 44C_F C_A - \frac{37}{4} C_F^2 \right]}$$

$$(d_4 + c_4)(x_{02}, y_{02}) = \beta_1(x_{02}, y_{02}) (\underline{d_4[0, 1] + c_4[0, 1]}) + \underline{d_4[0] + c_4[0]} \quad (3.9c)$$

$$= \beta_1(x_{02}, y_{02}) \underline{3C_F \left[C_A \left(\frac{47}{9} - \frac{16}{3} \zeta_3 \right) + C_F \left(-\frac{397}{18} - \frac{136}{3} \zeta_3 + 80\zeta_5 \right) \right]} \\ + \underline{3C_F \left(\frac{40}{3} - 12\zeta_3 \right)} + \underline{(3C_F)^2 \left[\left(\frac{175}{6} - 144\zeta_3 \right) C_A^2 + 44C_F C_A - \frac{37}{4} C_F^2 \right]}, \quad (3.9d)$$

$$(d_4 + c_4)(x_0) = \beta_2(x_0) (\underline{d_4[0, 0, 1] + c_4[0, 0, 1]}) + \beta_1(x_0) (\underline{d_4[0, 1] + c_4[0, 1]}) \\ + \underline{d_4[0] + c_4[0]} \quad (3.9e)$$

$$= 3C_F \left[-\frac{111}{4} C_F^3 + C_A C_F^2 \left(-\frac{1661}{36} + \frac{2618}{3} \zeta_3 - 880\zeta_5 \right) \right. \\ \left. + C_A^2 C_F \left(-\frac{3337}{18} + \frac{896}{3} \zeta_3 - 3516\zeta_5 \right) + C_A^3 \left(-\frac{28931}{144} + \frac{1351}{6} \zeta_3 \right) \right]. \quad (3.9f)$$

The r.h.s. s of (3.9a), (3.9c), (3.9e) are presented by mean of the already known results for d_3 and c_3 — (3.9b), (3.9d), see [3] and appendix A here. The last eq. (3.9e), suggested in [2], has already been verified and is put here for illustration and comparison with the two previous equations. Let us conclude,

- (i) For the values of $d_4 + c_4$ on (x_{01}, y_{01}) and (x_{02}, y_{02}) eqs. (3.9) provide the simple check — the r.h.s. of (3.9b) and (3.9d), respectively.
- (ii) The equalities of the underlined terms in eqs. (3.9) are realized independently following eqs. (2.10) and (2.8). Therefore, the second equalities in (3.9a) and (3.9c) allow one to get $d_4[0, 0, 1], d_4[0, 1], d_4[0]$ through their partners $c[\cdot]$ and vice versa using

the solutions in eq. (3.6), see eqs. (30), (31)² in [2]. But, one cannot restore all the elements of C in N³LO based only on CR and the known D opposite to the case of N²LO, see section 2.3.

Let us mention thereupon an alternative approach to fix $d_4[\cdot]$, $c_4[\cdot]$ without additional d.o.f., which was suggested in [4] and was inspired by the structure of the r.h.s. of CR (2.7). The idea is based on the specific proposition that the perturbation series for D and C^{Bjp} can be expanded in powers $(\beta(a_s)/a_s)^n$ similar to that had been proposed for the “conformal symmetry braking term” $\beta(a_s)K(a_s)$ in the r.h.s. of CR, see the presentation in eq. (6) in [2]. The results for the elements obtained within this approach differ from ours.

3.3 What can we get at 3 degrees of freedom x, y, z

Let us imagine that we have an additional third “intrinsic” d.o.f. that manifests itself as the parameter z . In this case $d_n = d_n(x, y, z)$, $\beta_i = \beta_i(x, y, z)$; therefore, one can use the points in (x, y, z) space to construct the set X :

$$\{\beta_0(x_{012}, y_{012}, z_{012}) = 0, \beta_1(x_{012}, y_{012}, z_{012}) = 0, \beta_2(x_{012}, y_{012}, z_{012}) = 0\} \quad (3.10)$$

instead of the later constraint in eq. (3.2b) (if the solution of SE (3.10) exists). Let us call this solution $r_{012} = (x_{012}, y_{012}, z_{012})$ and $\beta_i(x) = \beta_i(x, 0, 0)$, $\beta_i(x, y) = \beta_i(x, y, 0)$, $d_n(x) = d_n(x, 0, 0)$, $d_n(x, y) = d_n(x, y, 0), \dots$ for shortness. From eq. (3.1) and eq. (3.10) it immediately follows that

$$d_4(r_{012}) = \tilde{d}_4[0] + \delta d(r_{012}), \quad (3.11)$$

$$c_4(r_{012}) + d_4(r_{012}) = \tilde{d}_4[0] + \tilde{c}_4[0] = 2d_1d_3[0] - 3d_1^2d_2[0] + d_2[0]^2 + d_1^4. \quad (3.12)$$

Equation (3.12) provides an independent test for the c_4, d_4 results. In the case of constraint (3.10), the procedure for obtaining the $\{\beta\}$ -expansion is simplified significantly. Let us show the list of the evident solutions of SE₆ taking into account the definition of Y_4 in (3.5)

$$d_4[0, 0, 1] = (Y_4(x_{01}, y_{01}) - Y_4(r_{012}))/\beta_2(x_{01}, y_{01}), \quad (3.13a)$$

$$d_4[0, 1] = (Y_4(x_{02}, y_{02}) - Y_4(r_{012}))/\beta_1(x_{02}, y_{02}), \quad (3.13b)$$

$$d_4[0, 1] = (Y_4(x_0) - Y_4(r_{012}) - \beta_2(x_0)d_4[0, 0, 1])/\beta_1(x_0). \quad (3.13c)$$

Here eq. (3.13c) together with eq. (3.13a) admit checking of eq. (3.13b). The solutions in (3.11), (3.13) for $d_4[0], d_4[0, 1], d_4[0, 0, 1]$ look evidently easier than ones in (3.6) obtained with only single additional d.o.f. The solutions for the elements $d_4[1], d_4[2]$ can also be easily obtained, but they look rather cumbersome and we do not show them. The solutions presented here exhaust the problem of fixing the elements of the $\{\beta\}$ -expansion in N³LO.

²There in the r.h.s. of eq. (31) was missed the term $+(-47/48 + \zeta_3)C_F C_A$.

4 What can we expect for $\{\beta\}$ -expansion in $N^4\text{LO}$

Let us consider the structure of a 6 loop result in order a_s^5 , i.e., at $n = 5$,

$$\begin{aligned} d_5(x, \dots) = & \beta_0^4 d_5[4] + \beta_2 \beta_0 d_1[1, 0, 1] + \beta_1^2 d_5[0, 2] + \beta_1 \beta_0^2 d_5[2, 1] + \beta_3 d_5[0, 0, 0, 1] \\ & + \beta_0^3 d_5[3] + \beta_1 \beta_0 d_5[1, 1] + \beta_2 d_5[0, 0, 1] \\ & + \beta_0^2 d_5[2] + \beta_1 d_5[0, 1] + \beta_0 d_5[1] + d_5[0]. \end{aligned} \quad (4.1)$$

The number of new elements in this order, counting the elements starting with β_0^4 up to β_3 in the first line of eq. (4.1), coincides with the number of partitions $p(5-1) = 5$. The other terms in (4.1) repeat the structure of the result in previous order at $n = 4$. In general, for the term of the order n , d_n , one should count new terms from $\beta_0^{(n-1)}$ up to $\beta_{(n-1)-1}$ that gives their number $p(n-1)$, while the complete number $N(n)$ of all the terms is the sum $N(n) = \sum_{l=0}^{(n-1)} p(l)$ that leads to series $\{1, 2, 4, 7, \mathbf{12}$ (here), $19, 30, 45, \dots\}$ sequentially in each order, see, e.g., [15],³ and the terms in eqs. (1.4), for an example.

The coefficient $d_5(x, y)$ is formed by the variety of 6-loop diagrams that get contributions from the intrinsic box- and pentagon-subgraphs with gluon legs that introduce into d_5 a specific $(n_f, n_{\bar{g}})$ -dependence that does not relate to the charge renormalization. Indeed, the new color coefficients $\frac{n_{\bar{g}}}{2} d_F^{abcde} d_A^{abcde} / d_R$ (gluino pentagon inside), $n_f d_F^{abcde} d_F^{abcde} / d_R$ (quark pentagon inside) enter into $d_5[0](x, y)$ together with the contributions from the box-graphs, which was already mentioned in section 3. The contributions from the latter box-graphs, $n_{\bar{g}} d_F^{abcd} d_A^{abcd} / d_R$, $n_f d_F^{abcd} d_F^{abcd} / d_R$ enter⁴ now into the element $d_5[1] \rightarrow d_5[1](x, y)$. All these contributions, proportional to $n_f, n_{\bar{g}}$, are well recognized and can be accumulated in the specific term $\delta d_5(x, y)$, like it was done for $\delta d_4(x, y)$ in section 3. The element $\beta_0^4 d_5[4]$ should also be well recognized; therefore, one has $11 = \mathbf{12} - 1$ unknown elements $d_5[\cdot]$. By analogy with the previous lower orders procedure one can compile SE_{11} based on eq. (4.1) with the rearranged l.h.s.

$$Y_5(X) = \bar{d}_5(X) - \delta d_5(X) = d_5(X) - \beta_0^4(X) d_5[4] - \delta d_5(X),$$

take the equation with the arguments at 11 points (X_{11}) on the plane (x, y) . The SE_{11} constructed in this way has unique solution with respect to the $d_5[\cdot]$ elements under the condition the corresponding determinant of the system $\Delta_{11}(X_{11}) \neq 0$.

Let us take for these 11 components of X_{11} the roots of the equations

$$\beta_0(x_0) = 0, \quad \beta_1(x_1) = 0, \quad \beta_2(x_{2m}) = 0, \quad \beta_2(x_{2p}) = 0, \quad (4.2a)$$

$$\{\beta_0(x_{01}, y_{01}) = 0, \quad \beta_1(x_{01}, y_{01}) = 0\}, \quad \{\beta_0(x_{02m}, y_{02m}) = 0, \quad \beta_2(x_{02m}, y_{02m}) = 0\}, \quad (4.2b)$$

$$\{\beta_0(x_{02p}, y_{02p}) = 0, \quad \beta_2(x_{02p}, y_{02p}) = 0\}, \quad (4.2c)$$

$$\{\beta_1(x_{12}, y_{12}) = 0, \quad \beta_2(x_{12}, y_{12}) = 0\}, \quad \{\beta_0(x_{03}, y_{03}) = 0, \quad \beta_3(x_{03}, y_{03}) = 0\}, \quad (4.2d)$$

$$\{\beta_1(x_{13}, y_{13}) = 0, \quad \beta_3(x_{13}, y_{13}) = 0\}, \quad \{\beta_2(x_{23}, y_{23}) = 0, \quad \beta_3(x_{23}, y_{23}) = 0\}. \quad (4.2e)$$

³I thank N. Volchanskiy who paid my attention to this ref.

⁴The definition of the color elements $d_R^{a_1 a_2 \dots a_n}$ is presented, e.g., in [16].

This choice of X_{11} simplifies the set of equations, as we made sure in the previous cases of the constructing sets $X_{3,6}$ in eqs. (2.4), (3.2), respectively. The determinant corresponding to $\text{SE}_{11} \Delta_{11}(X_{11}) \neq 0$ but looks too cumbersome to show it here. It is clear that our algebraic scheme works further with the increasing perturbation order.

5 Conclusion

In this paper we have considered the important task of obtaining the elements of the detailed structure of the QCD perturbation expansion — the $\{\beta\}$ -expansion for the renormalization group invariant quantities. The explicit knowledge of the elements of this expansion (i) gives a possibility to perform various kinds of optimization of the perturbation series; (ii) taken together with the Crewther relation it allows one to establish nontrivial relations between different physical quantities. We suggest an algebraic approach to fix the elements of the $\{\beta\}$ -expansion for these quantities using additional degrees of freedom, and demonstrate that for the resolution of the detailed structure it is enough to use a *single additional* degree of freedom to the quark one.

This approach is discussed in detail for N²LO calculations of the nonsinglet Adler D -function and for the Bjorken polarized sum rules C^{Bjp} within QCD with the $n_{\tilde{g}}$ of MSSM gluinos — the additional degree of freedom. We derive the explicit formulae for the elements of the $\{\beta\}$ -expansion for these quantities, named $d_n[\cdot]$ and $c_n[\cdot]$ respectively, see eq. (1.4d) in the actual case of N³LO within the aforementioned quantum field theory scheme. This $\{\beta\}$ -expansion together with the explicit expressions for the elements $d_4[\cdot]$ ($c_4[\cdot]$) can be considered as a prediction for any additional degrees of freedom that can be taken into consideration. Indeed, these degrees of freedom enter into either the well-known coefficients of the β -function, β_i , or the well-recognized terms of the structure.

Another kind of predictions is provided by the relation between the elements $d_n[\cdot]$ and $c_n[\cdot]$ in virtue of the Crewther relation. We constructed the fixation procedure also for the case of two additional degrees of freedom. Finally, we discussed the structure and properties of $\{\beta\}$ -expansion for higher orders considering the N⁴LO with the $n_{\tilde{g}}$ of MSSM gluinos as an example, where the expansion elements can be also fixed following to our algebraic procedure.

The next natural step in the development of this investigation would be the calculation of D or C^{Bjp} with the additional degrees of freedom in N³LO. These results allow one to obtain relations between their elements that leads to the new predictions for one of them. Moreover, this provides the basis of optimization of the approximation for the physically important $R_{e^+e^- \rightarrow h}(s)$ -ratio or for the Bjorken polarized sum rules.

Acknowledgments

I would like to thank A. Bednyakov, K. Chetyrkin, A. Grozin, L. Lipatov for the fruitful discussions, to A. Kataev for the discussion and constructive criticism, and to the reviewer of this paper for the reasonable, clarifying criticism. The work was supported in part by the BelRFFR-JINR, grant F16D-004 and by the Russian Foundation for Basic Research, Grant No. 14-01-00647.

A Explicit formulas for the elements of D and C

For the Adler function D^{NS} the corresponding elements read [2, 3]⁵

$$d_1 = 3C_F; \quad (\text{A.1a})$$

$$d_2[1] = d_1 \left(\frac{11}{2} - 4\zeta_3 \right); \quad d_2[0] = d_1 \left(\frac{C_A}{3} - \frac{C_F}{2} \right); \quad (\text{A.1b})$$

$$d_3[2] = d_1 \left(\frac{302}{9} - \frac{76}{3}\zeta_3 \right); \quad d_3[0, 1] = d_1 \left(\frac{101}{12} - 8\zeta_3 \right); \quad (\text{A.1c})$$

$$d_3[1] = d_1 \left(C_A \left(-\frac{3}{4} + \frac{80}{3}\zeta_3 - \frac{40}{3}\zeta_5 \right) - C_F \left(18 + 52\zeta_3 - 80\zeta_5 \right) \right); \quad (\text{A.1d})$$

$$d_3[0] = d_1 \left(\left(\frac{523}{36} - 72\zeta_3 \right) C_A^2 + \frac{71}{3} C_A C_F - \frac{23}{2} C_F^2 \right). \quad (\text{A.1e})$$

$$c_1 = -3 C_F; \quad (\text{A.2a})$$

$$c_2[1] = 2c_1; \quad c_2[0] = \left(\frac{1}{3} C_A - \frac{7}{2} C_F \right) c_1; \quad (\text{A.2b})$$

$$c_3[2] = \frac{115}{18} c_1; \quad c_3[0, 1] = c_1 \left(\frac{59}{12} - 4\zeta_3 \right); \quad (\text{A.2c})$$

$$c_3[1] = -c_1 \left(\left(\frac{166}{9} - \frac{16}{3}\zeta_3 \right) C_F + \left(\frac{215}{36} - 32\zeta_3 + \frac{40}{3}\zeta_5 \right) C_A \right); \quad (\text{A.2d})$$

$$c_3[0] = c_1 \left(\left(\frac{523}{36} - 72\zeta_3 \right) C_A^2 + \frac{65}{3} C_F C_A + \frac{C_F^2}{2} \right); \quad (\text{A.2e})$$

$$d_3[0, 1] - c_3[0, 1] = d_1 \left(\frac{40}{3} - 12\zeta_3 \right); \quad (\text{A.3})$$

$$\begin{aligned} (d_4[0] + c_4[0])(x, y) &= \tilde{d}_4[0] + \tilde{c}_4[0] = 2d_1 d_3[0] - 3d_1^2 d_2[0] + d_2[0]^2 + d_1^4 \\ &= d_1^2 \left[\left(\frac{175}{6} - 144\zeta_3 \right) C_A^2 + 44C_F C_A - \frac{37}{4} C_F^2 \right]. \end{aligned} \quad (\text{A.4})$$

⁵We had a missprint in the expression for $d_3[1]$ in articles [2, 3]: in the first parenthesis at C_A should be $-\frac{3}{4}$, see eq. (A.1d), instead of $+\frac{3}{4}$ there.

From the results in [6] it follows that

$$d_4[3] = d_1 \left(\frac{6131}{27} - \frac{406}{3} \zeta_3 - 60 \zeta_5 \right), \quad c_4[3] = c_1 \left(\frac{2}{27} \right), \quad (\text{A.5})$$

$$\delta d_4(x, y) = \left[\frac{y}{2C_A} \frac{d_A^{abcd} d_F^{abcd}}{d_R} + x \frac{d_F^{abcd} d_F^{abcd}}{d_R} \right] 3 \cdot (-104 - 128 \zeta_3 + 320 \zeta_5), \quad (\text{A.6})$$

$$\delta c_4(x, y) = -\delta d_4(x, y). \quad (\text{A.7})$$

B The β -function coefficients

The required β -function coefficients with the Minimal Supersymmetric Model (MSSM) light gluinos $n_{\tilde{g}}$ [14], and the number n_f of quark flavors, calculated in the $\overline{\text{MS}}$ scheme are

$$\beta_0(x, y) = \frac{11}{3} C_A - x - y = \frac{11}{3} C_A - \frac{4}{3} \left(T_R n_f + \frac{n_{\tilde{g}} C_A}{2} \right); \quad (\text{B.1a})$$

$$\begin{aligned} \beta_1(x, y) &= \frac{34}{3} C_A^2 - (5C_A + 3C_F)x - 8C_A y \\ &= \frac{34}{3} C_A^2 - \frac{20}{3} C_A \left(T_R n_f + \frac{n_{\tilde{g}} C_A}{2} \right) - 4 \left(T_R n_f C_F + \frac{n_{\tilde{g}} C_A}{2} C_A \right); \end{aligned} \quad (\text{B.1b})$$

$$\begin{aligned} \beta_2(x, y) &= \frac{2857}{54} C_A^3 - x \left(\frac{1415}{36} C_A^2 + \frac{205}{12} C_A C_F - \frac{3}{2} C_F^2 \right) + x^2 \left(\frac{11}{4} C_F + \frac{79}{24} C_A \right) \\ &\quad - y \frac{494}{9} C_A^2 + 2xy \left(\frac{11}{8} C_F + \frac{14}{3} C_A \right) C_A + y^2 \frac{145}{24} C_A \\ &= \frac{2857}{54} C_A^3 - n_f T_R \left(\frac{1415}{27} C_A^2 + \frac{205}{9} C_A C_F - 2C_F^2 \right) + (n_f T_R)^2 \left(\frac{44}{9} C_F + \frac{158}{27} C_A \right) \\ &\quad - \frac{988}{27} n_{\tilde{g}} C_A (C_A^2) + n_{\tilde{g}} C_A n_f T_R \left(\frac{22}{9} C_A C_F + \frac{224}{27} C_A^2 \right) + (n_{\tilde{g}} C_A)^2 \frac{145}{54} C_A, \end{aligned} \quad (\text{B.1c})$$

where we have introduced appropriate rescaled variables $x = \frac{4}{3} T_R n_f$ and $y = \frac{4}{3} \frac{C_A}{2} n_{\tilde{g}}$ after the first equality to simplify the expressions. The N³LO coefficient $\beta_3(n_f, n_{\tilde{g}})$ has been

obtained recently in [16, 17],

$$\begin{aligned}
 \beta_3(n_f, n_{\bar{g}}) = & -\left(\frac{150653}{486} - \frac{44}{9}\zeta_3\right)C_A^4 + \left(\frac{80}{9} - \frac{704}{3}\zeta_3\right)\frac{d_A^{abcd}d_A^{abcd}}{N_A} \\
 & + n_{\bar{g}}\left[\left(\frac{68507}{243} - \frac{52}{9}\zeta_3\right)C_A^4 - \left(\frac{256}{9} - \frac{832}{3}\zeta_3\right)\frac{d_A^{abcd}d_A^{abcd}}{N_A}\right] \\
 & - n_{\bar{g}}^2\left[\left(\frac{26555}{486} - \frac{8}{9}\zeta_3\right)C_A^4 + \left(\frac{176}{9} - \frac{128}{3}\zeta_3\right)\frac{d_A^{abcd}d_A^{abcd}}{N_A}\right] - \frac{23}{27}C_A(n_{\bar{g}}C_A)^3 \\
 & + n_f T_R\left[-46C_F^3 + \left(\frac{4204}{27} - \frac{352}{9}\zeta_3\right)C_A C_F^2 - \left(\frac{7073}{243} - \frac{656}{9}\zeta_3\right)C_A^2 C_F\right. \\
 & \quad \left.+ \left(\frac{39143}{81} - \frac{136}{3}\zeta_3\right)C_A^3\right] - n_f\left(\frac{512}{9} - \frac{1664}{3}\zeta_3\right)\frac{d_F^{abcd}d_A^{abcd}}{N_A} \\
 & + (n_f T_R)^2\left[-\left(\frac{1352}{27} - \frac{704}{9}\zeta_3\right)C_F^2 - \left(\frac{17152}{243} + \frac{448}{9}\zeta_3\right)C_A C_F\right. \\
 & \quad \left.- \left(\frac{7930}{81} + \frac{224}{9}\zeta_3\right)C_A^2\right] + n_f^2\left(\frac{704}{9} - \frac{512}{3}\zeta_3\right)\frac{d_F^{abcd}d_F^{abcd}}{N_A} \\
 & - (n_f T_R)^3\left[\frac{1232}{243}C_F + \frac{424}{243}C_A\right] \\
 & + n_{\bar{g}}C_A(n_f T_R)\left[\left(\frac{152}{27} + \frac{64}{9}\zeta_3\right)C_F^2 - \left(\frac{23480}{243} - \frac{352}{9}\zeta_3\right)C_A C_F\right. \\
 & \quad \left.- \left(\frac{30998}{243} + \frac{128}{3}\zeta_3\right)C_A^2\right] + n_f n_{\bar{g}}\left(\frac{704}{9} - \frac{512}{3}\zeta_3\right)\frac{d_F^{abcd}d_A^{abcd}}{N_A} \\
 & - (n_{\bar{g}}C_A)^2 n_f T_R\left[\frac{308}{243}C_F + \frac{934}{243}C_A\right] - n_{\bar{g}}C_A(n_f T_R)^2\left[\frac{1232}{243}C_F + \frac{1252}{243}C_A\right],
 \end{aligned} \tag{B.2}$$

where $T_R = \frac{1}{2}$, $C_F = \frac{N_c^2-1}{2N_c}$, $C_A = N_c$, $N_A = 2C_F C_A = N_c^2 - 1$.

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