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Momentum analyticity of the holographic electric polarizability in $2 + 1$ dimensions

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ABSTRACT: The static electric polarization of a holographic field theory dual to the Einstein-Maxwell theory in the background of AdS_4 with a Reissner-Nordström (AdS-RN) black hole is investigated. We prove that the holographic polarization is a meromorphic functions in complex momentum plane and locate analytically the asymptotic distribution of the poles along two straight lines parallel to the imaginary axis for a large momentum magnitude. The results are compared with the numerical result on Friedel-like poles of the same holographic model reported in the literature and with the momentum singularities of the one-loop polarization in weak-coupling spinor QED₃ and scalar QED₃ with the similarities and differences discussed.

KEYWORDS: AdS-CFT Correspondence, Gauge-gravity correspondence, Holography and condensed matter physics (AdS/CMT)

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1 Introduction

The gauge/gravity duality [1–5] has evolved into an important tool to explore the strongly interacting systems in high energy physics and condensed matter physics. By converting the path integral of a strongly interacting quantum system into the classical theory of a weakly coupled gravity-matter system with one more space dimension, the duality opens a new avenue towards a qualitative or even quantitative understanding of the non-perturbative effects [6, 7], especially when the problem cannot be tackled by numerical techniques. The holographic superconductor provides such an example, where the quantum effective action of a strongly coupled superconductor in 2+1 dimensions corresponds to a 3+1 dimensional classical action of an Abelian-Higgs theory coupled to the gravity with a AdS-Reissner-Nordström (AdS-RN) black hole [8], or with a AdS-Schwarzschild black hole in the probe limit [9, 10]. The ratio between the absorption threshold of the AC conductivity (the onset frequency of its real part) and the critical temperature extracted

from the holographic superconductor turns out to be close to the observed values from cuprates superconductors [11, 12].

Unlike the Type IIB superstring theory in $AdS_5 \times S^5$, which corresponds to $\mathcal{N} = 4$ super Yang-Mills in 3+1 dimensions, all holographic models used in condensed matter physics follow a bottom-up approach without the knowledge of the explicit Lagrangian underlying the quantum effective action implied by the gauge/gravity duality, as is reflected, for instance, in the lack of the evidences such as the Andreev reflection that link the order parameter in the holographic superconductivity to the Cooper pairing of the fermionic degrees of freedom if any. On top of all its applications, the duality itself remains a conjecture. Therefore in addition to conquer more strongly correlated systems with more sophisticated holographic models, it is equally important to collect more fundamental properties shared by existing holographic models and ordinary field theories, and thereby to accumulate more evidences supporting the conjectured duality. The work reported below serves the latter purpose and the holographic model to be addressed is the Einstein-Maxwell theory in 3+1 AdS-Reissner-Nordström (AdS-RN) black hole background, which describes the normal phase of the 2+1 dimensional holographic superconductor [13]. The analyticity of the electric component of the static polarization tensor $\Pi_{\mu\nu}(q)$ with respect to the spatial momentum q will be examined and the asymptotic distribution of the complex singularities underlying the Friedel oscillations of the dressed Coulomb potential will be obtained and compared with the both 2+1 spinor QED and scalar QED in weak coupling. In this regard, our result is of only theoretical values. Its phenomenological implications remain to be unveiled.

This research is the continuation of the previous one [13], where the momentum analyticity of the magnetic component of the polarization tensor (the spatially transverse component satisfying $q^i \Pi_{ij}(q) = 0$) of the same holographic model was investigated. There we proved that the static magnetic polarization function is a meromorphic function of the complex spatial momentum. Using the WKB approximation, we were able to show that the poles for large momentum magnitude are distributed asymptotically along two lines parallel to the imaginary axis of the complex momentum plane. The parallel analyses will be extended to the more interesting electric component of the polarization tensor, $\Pi_{00}(q)$ below. The Friedel oscillation in this case is speculated to support the Cooper pairing in some strongly correlated system [14]. Technically, the polarization tensor is extracted from the solutions of the linearized Einstein-Maxwell equations in the RN blackhole background. The metric tensor and gauge potential fluctuations can be divided to two groups according to the parity under the reflection with respect to the line perpendicular to the spatial momentum on the AdS boundary. The fluctuations underlying the magnetic component, treated in [13], belong to the odd parity group and they are the solutions of the two coupled Einstein-Maxwell equations in the static limit. The fluctuations underlying the electric component belong to the even parity group, which involve four coupled Einstein-Maxwell in the static limit, and their analytic treatment becomes more challenging. Nevertheless, after some twist and turns, we are able to reach similar conclusions as the magnetic component. The electric component $\Pi_{00}(q)$ is a meromorphic function of the spatial momentum q with poles distributed asymptotically along two lines parallel to the imaginary axis of

the complex q -plane for large $|q|$. The asymptotic locations of these poles match well with those extracted from the numerical solution of the Einstein-Maxwell equations for large $\text{Im } q$ [15].

The presence of the complex momentum singularities of the Green's function appears a common property of a field theory, either strongly coupled or weakly coupled, with a nonzero chemical potential as demonstrated in this work and the previous ones [13, 15, 16]. It is not, however, a sufficient evidence of fermionic degrees of freedom present in the boundary field theory of the gauge/gravity dual. The one-loop polarization tensor of scalar QED3 with a nonzero chemical potential displays those complex momentum singularities as well, similar to spinor QED.

This paper is organized as follows. The momentum analyticity of the polarization tensor of both spinor and scalar QED3 to one-loop order will be discussed in the next section with the result as a benchmark for comparison with its holographic counterpart. In section 3, the electric component of the holographic polarization tensor will be extracted from the even parity solutions of the linearized Einstein-Maxwell equations in AdS background, and the proof of its meromorphicity in a complex spatial momentum will be given. The WKB approximation will be employed in the section 4 to find out the asymptotic distribution of the poles on the complex momentum plane. In section 5, the analytic result obtained in this work will be compared with the numerical result of [15] and conclude the paper.

2 One-loop photon self-energy in 2 + 1-dimensional spacetime

To study the finite-temperature quantum electrodynamics, we start with the functional integral formalism:

$$\mathcal{Z} = \int \mathcal{D}(\phi_\alpha) e^{-S_E} = \int \mathcal{D}(\phi_\alpha) \exp \left\{ - \int_0^\beta d\tau \int d^3\vec{r} \mathcal{L}[\phi_\alpha] \right\} \quad (2.1)$$

where S_E is the Euclidean action and ϕ_α , $\alpha = 1, 2, \dots$ represents all fields under consideration.

2.1 Fermion case: spinor QED

We consider first the spinor QED, and the Lagrangian density in (2.1) is chosen as

$$\mathcal{L}[\psi, \bar{\psi}] = -\bar{\psi} \gamma_\lambda \left(\frac{\partial}{\partial x_\lambda} - ieA_\lambda \right) \psi + \mu \bar{\psi} \sigma_3 \psi \quad (2.2)$$

where ψ and A_λ are the massless fermion field and U(1) gauge potential in 2+1 dimensional space-time, respectively, e is the electric charge and μ is the chemical potential. The representation of γ matrices in 2 + 1 dimensional space-time reads

$$\gamma_0 = \sigma_3, \quad \gamma_1 = -i\sigma_1, \quad \gamma_2 = -i\sigma_2 \quad (2.3)$$

here σ 's are the Pauli matrices. With $\phi_\alpha = \psi$, A_λ and the Lagrangian density (2.2), the functional integral (2.1) becomes the grand partition function of spinor QED₂₊₁ at

a temperature $T = 1/\beta$ with the free fermion propagator in energy-momentum space given by:

$$S_F(p) = \frac{i}{(i\lambda + \mu)\sigma_3 - i\vec{\sigma} \cdot \vec{p}} \quad (2.4)$$

where $\vec{\sigma} = (\sigma_1, \sigma_2)$ and the Matsubara energy $\lambda = 2\pi T(n + \frac{1}{2})$, $n \in \mathbb{Z}$.

Our interest in this paper lies on the electric component, i.e. tt -component of the polarization tensor, $\Pi_{\mu\nu}(q)$, corresponding to the one-loop Feynman diagram in figure 1.

We have

$$\Pi_{tt}(q) \equiv e^2 \sigma^{\text{sc}}(q) \quad (2.5)$$

where the reduced polarization tensor $\sigma^{\text{sc}}(q)$ is given by

$$\sigma^{\text{sc}}(q) = \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \int \frac{d^2\vec{p}}{(2\pi)^2} \text{Tr} [\sigma_3 S_F(P) \sigma_3 S_F(P+Q)] \quad (2.6)$$

with $P = (i\lambda, \vec{p})$; $Q = (0, \vec{q})$ and $\vec{q} = (q, 0)$. To explore the Friedel oscillations, we have taken the static limit and aligned the spatial momentum along x-directions. The Matsubara frequency sum in (2.6) can be converted into a contour integral, and we obtain:

$$\begin{aligned} \sigma^{\text{sc}}(q) &= -2 \int \frac{d^2\vec{p}}{(2\pi)^2} \oint_{C_1} \frac{dz}{2\pi i} \frac{1}{e^{\beta z} + 1} \frac{(z + \mu)^2 + (p_1 + q)p_1 + p_2^2}{[(z + \mu)^2 + |\vec{p} + \vec{q}|^2] [(z + \mu)^2 - p^2]} \\ &= - \int \frac{d^2\vec{p}}{(2\pi)^2} \left\{ \left[\frac{1}{e^{\beta(|\vec{p} + \vec{q}| - \mu)} + 1} - \frac{1}{e^{-\beta(|\vec{p} + \vec{q}| + \mu)} + 1} \right] \frac{(\vec{p} + \vec{q})^2 + (p_1 + q)p_1 + p_2^2}{|\vec{p} + \vec{q}| [(\vec{p} + \vec{q})^2 - \vec{p}^2]} \right. \\ &\quad \left. - \left[\frac{1}{e^{\beta(p - \mu)} + 1} - \frac{1}{e^{-\beta(p + \mu)} + 1} \right] \frac{p^2 + (p_1 + q)p_1 + p_2^2}{p [p^2 - (\vec{p} + \vec{q})^2]} \right\} \\ &= \sigma_{\text{vac}}^{\text{sc}}(q) + \sigma_{\text{matt}}^{\text{sc}}(q) \end{aligned} \quad (2.7)$$

where $\sigma_{\text{vac}}^{\text{sc}}(q) = \sigma^{\text{sc}}(q)|_{T=0; \mu=0}$ and

$$\begin{aligned} \sigma_{\text{matt}}^{\text{sc}}(q) &= - \int \frac{d^2\vec{p}}{(2\pi)^2} \left\{ \frac{1}{e^{\beta(|\vec{p} + \vec{q}| - \mu)} + 1} \frac{(\vec{p} + \vec{q})^2 + (p_1 + q)p_1 + p_2^2}{|\vec{p} + \vec{q}| [(\vec{p} + \vec{q})^2 - \vec{p}^2]} \right. \\ &\quad \left. - \frac{1}{e^{\beta(p - \mu)} + 1} \frac{p^2 + (p_1 + q)p_1 - p_2^2}{p [(\vec{p} + \vec{q})^2 - p^2]} \right\} + (\mu \leftrightarrow -\mu) \end{aligned} \quad (2.8)$$

$$\begin{aligned} &= - \text{Re} \int \frac{d^2\vec{p}}{(2\pi)^2} \left\{ \frac{1}{e^{\beta(p - \mu)} + 1} \times \frac{1}{p} \right. \\ &\quad \left. \times \left[\frac{p^2 + p_1(p_1 - q) - p_2^2}{p^2 - (\vec{p} - \vec{q})^2 + i0^+} - \frac{p^2 + (p_1 + q)p_1 - p_2^2}{(\vec{p} + \vec{q})^2 - p^2 + i0^+} \right] + (\mu \leftrightarrow -\mu) \right\} \end{aligned} \quad (2.9)$$

$$= \frac{1}{2\pi} \int_0^\infty dp \left[1 + \frac{4p^2 - q^2}{q\sqrt{q^2 - 4p^2}} \theta(q - 2p) \right] \left(\frac{1}{e^{\beta(p - \mu)} + 1} + \frac{1}{e^{\beta(p + \mu)} + 1} \right) \quad (2.10)$$

In passing, we have added an infinitesimal imaginary part to the two factors $(\vec{p} + \vec{q})^2 - p^2$ in the denominators of (2.9) in order to make each term convergent without modifying

Figure 1. The one-loop diagram in spinor QED.

the real part of the result. The dependence of the distribution function on $\vec{q}$ is thereby removed by shifting the loop momentum. Isolating out the finite part, we obtain:¹

$$\sigma_{\text{matt}}^{\text{sc}}(q) = -\frac{1}{2\pi q} \int_0^{\frac{q}{2}} dp \sqrt{q^2 - 4p^2} \left[T \sum_{\lambda'} \left(\frac{1}{i\lambda + \mu - p} - \frac{1}{i\lambda + \mu + p} \right) \right] - \frac{q}{16} + R(\mu, T) \quad (2.11)$$

here $R(\mu, T)$ stands for all of the terms independent of the momentum q . Taking the derivative with respect to q , we obtain that

$$\frac{d}{dq} \sigma_{\text{matt}}^{\text{sc}}(q) = -\frac{1}{2\pi} \int_0^{\frac{q}{2}} dp \frac{1}{\sqrt{q^2 - 4p^2}} \left[T \sum_{\lambda'} \left(\frac{1}{i\lambda + \mu - p} - \frac{1}{i\lambda + \mu + p} \right) \right] - \frac{1}{16} \quad (2.12)$$

Defining a small complex quantity ϵ :

$$\frac{q}{2} = \mu + i\lambda + \epsilon \quad (2.13)$$

substituting (2.13) into (2.12) and focusing on term $\lambda' = \lambda$, which makes the integral divergent in the limit $\epsilon \rightarrow 0$. After evaluating the integral over p , the final result reads:

$$\frac{d}{dq} \sigma_{\text{matt}}^{\text{sc}} = \frac{1}{2} \frac{d}{d\epsilon} \sigma_{\text{matt}}^{\text{sc}} = \frac{iT}{4\sqrt{\epsilon(q - \epsilon)}} + \text{non-singular terms} \quad (2.14)$$

we find that the derivative of $\sigma_{\text{matt}}^{\text{sc}}$ diverges as $O\left(\frac{1}{\sqrt{\epsilon}}\right)$ in the limit $\epsilon \rightarrow 0$. Besides, if starting with the definition:

$$\frac{q}{2} = -[\mu + i\lambda] + \epsilon \quad (2.15)$$

we will obtain the same conclusion. Therefore, the function $\sigma^{\text{sc}}(q)$ shows square root singularities at

$$q = \pm 2(\mu + i\lambda) \quad (2.16)$$

on the complex q -plane.

¹Here we employ the identity $\left(\frac{1}{e^{\beta(p-\mu)}+1} + \frac{1}{e^{\beta(p+\mu)}+1} \right) = T \sum_{\lambda} \left(\frac{1}{i\lambda+\mu-p} - \frac{1}{i\lambda+\mu+p} \right) + 1$, for its proof, one can consult appendix A of the paper [13] for its counterpart.

Figure 2. The one-loop diagram in scalar QED.

2.2 Boson case: scalar QED

In the above subsection, we have obtained all singularities of photon self-energy in spinor QED₂₊₁ with both nonzero temperature and nonzero chemical potential at the one-loop level. As a comparison, we will investigate its antithesis; the charged scalar QED₂₊₁. In this case, the Lagrangian in (2.1) becomes

$$\mathcal{L}[\phi, \phi^\dagger] = -(D^\mu \phi)^\dagger D_\mu \phi - m^2 \phi^\dagger \phi \quad (2.17)$$

where the co-variant derivative operator is $D_\mu \equiv \partial_\mu - ieA_\mu$. A nonzero mass term is included in order for the relativistic Bose distribution function to be well-defined. The free boson propagator of 4-momentum $(\nu_n, \vec{k})$ takes the form

$$\Delta_F(i\nu_n, \vec{k}) = \frac{1}{-(i\nu_n + \mu)^2 + |\vec{k}|^2 + m^2} \quad (2.18)$$

where the Matsubara frequency is $\nu_n \equiv 2\pi nT$ with $n \in \mathbb{Z}$ and μ is the chemical potential for boson system.

The one-loop approximation of the polarization function in scalar QED involves two diagrams, for our purpose, we're only interested in the one that the momentum of photon has a contribution to momentum integral, hence the polarization function (2.6) corresponds to the Feynman diagram in figure 2. and is given by

$$\begin{aligned} \sigma^{\text{sc}}(q) &= -\frac{1}{\beta} \sum_{s=-\infty}^{+\infty} \int \frac{d^2 \vec{k}}{(2\pi)^2} \Delta_F(i\nu_n + i\nu_s, k+q) \Delta_F(i\nu_s, \vec{k}) [i\nu_s + \mu + 2i\nu_n]^2 \\ &= \sigma_{\text{vac}}^{\text{sc}}(q) + \sigma_{\text{matt}}^{\text{sc}}(q) \end{aligned} \quad (2.19)$$

where $\sigma_{\text{vac}}^{\text{sc}}(q) \equiv \sigma^{\text{sc}}(q)|_{T=0; \mu=0}$ and the static limit is taken with the spatial photon momentum $q = |\vec{q}|$. Calculating the Matsubara frequency sum with the aid of a contour integral and shifting the loop momentum appropriately, we find that the last part in (2.19) is given by

$$\begin{aligned} \sigma_{\text{matt}}^{\text{sc}}(q) &= \frac{1}{2} \int \frac{d^2 \vec{l}}{(2\pi)^2} \frac{\sqrt{l^2 + m^2}}{e^{\beta(\sqrt{l^2 + m^2} - \mu)} - 1} \left[\frac{1}{|\vec{l}|^2 - |\vec{l} - \vec{q}|^2} - \frac{1}{|\vec{l} + \vec{q}|^2 - |\vec{l}|^2} \right] + (\mu \rightarrow -\mu) \\ &= -\frac{1}{q} \int_0^{\frac{k}{2}} \frac{l \cdot \sqrt{l^2 + m^2}}{\sqrt{q^2 - 4l^2}} \frac{dl}{2\pi} \left[\frac{1}{e^{\beta(\sqrt{l^2 + m^2} - \mu)} - 1} + \frac{1}{e^{\beta(\sqrt{l^2 + m^2} + \mu)} - 1} \right] \end{aligned} \quad (2.20)$$

$$= \frac{T}{4\pi q} \sum_{n'=-\infty}^{+\infty} \int_0^{\frac{q}{2}} dl \frac{l\sqrt{l^2+m^2}}{\sqrt{\frac{q^2}{4}-l^2}} \left(\frac{1}{i\nu_{n'}+\mu-\sqrt{l^2+m^2}} - \frac{1}{i\nu_{n'}+\mu+\sqrt{l^2+m^2}} \right) + R(\mu, T) \quad (2.21)$$

here $R(\mu, T)$ refers to the terms independent of the external momentum q . Introducing a new integration variable $x \equiv \frac{\sqrt{l^2+m^2}}{s_0}$ and new parameter, $s_0 \equiv \sqrt{m^2 + (\frac{q}{2})^2}$, we have $(\frac{q}{2})^2 - l^2 = s_0^2 [1 - x^2]$ hence

$$\sigma_{\text{matt}}^{\text{sc}}(q) = \frac{T}{4\pi q} \sum_{n'=-\infty}^{+\infty} \int_{m/s_0}^1 dx \frac{x^2}{\sqrt{1-x^2}} \left(\frac{1}{i\nu_{n'}+\mu-s_0x} - \frac{1}{i\nu_{n'}+\mu+s_0x} \right) \quad (2.22)$$

The singularity occurs when s_0 approaches to one of $i\nu_n + \mu$, or $-i\nu_n - \mu$. Denoting $s_0 = i\nu + \mu + \epsilon$ with ν one of ν_n and $z_0 \equiv \frac{i\nu+\mu}{i\nu+\mu+\epsilon}$, and isolating out the diverging term $n' = n$ in (2.22) from others in the limit $\epsilon \rightarrow 0$, we end up with

$$\sigma_{\text{matt}}^{\text{sc}}(q) = \frac{T}{4\pi q} \frac{1}{s_0} \int_{\frac{m}{s_0}}^1 dx \frac{1}{\sqrt{1-x^2}} \left[\frac{1}{z_0-x} - \frac{1}{z_0+x} \right] + \text{non-singular terms} \quad (2.23)$$

$$= -\frac{iT}{4q} \frac{1}{\sqrt{2(i\nu+\mu)\epsilon}} + \text{non-singular terms} \quad (2.24)$$

which indicates the divergent behavior

$$\sigma^{\text{sc}}(q) \sim O\left(\frac{1}{\sqrt{\epsilon}}\right), \quad \text{as} \quad \epsilon \rightarrow 0 \quad (2.25)$$

and the location of the singularities on the complex q -plane

$$q = \pm 2\sqrt{(i\nu+\mu)^2 - m^2} \quad (2.26)$$

From the previous results, we see that the locations of the singularities of the electric polarization function of spinor or scalar QED3 on the complex momentum plane stem from the poles of the fermionic or bosonic distribution functions and all the singularities are the branch points of square root. A difference between bosonic and fermionic cases is that we have to introduce a mass for boson with $|m| < |\mu|$ to make the integration of Bose-Einstein distribution $n_B(E) = 1/(e^{\beta(E\pm\mu)} - 1)$ well defined, while for the fermionic case, the mass is optional. Asymptotically, the singularities in either case are distributed along two straight lines $q = \pm\mu$ with equal spacing for large magnitude of q , independent of the mass. As we shall see, a similar asymptotic distribution emerges in strong coupling.

3 Static electric polarizability in gauge/gravity duality

The prototype of the gauge/gravity duality is the correspondence between the $\mathcal{N} = 4$ $SU(N_c)$ super Yang-Mills and the type IIB superstring in $AdS_5 \times S_5$ with the former residing on the boundary of AdS_5 . In particular, in the limit of large N_c and strong 't Hooft coupling,

the superstring side of the correspondence reduces to a classical supergravity and becomes tractable. Motivated by this relationship, the bottom-up approach to the gauge/gravity duality amounts to identify the classical action of the gravity-matter system in the bulk geometry of an asymptotically AdS boundary with the quantum effective action of some strongly interacting field theory on the boundary, i.e.

$$\Gamma[\dot{\phi}_\alpha] = S_{\text{cl.}}[\dot{\phi}_\alpha] \quad (3.1)$$

where $\dot{\phi}_\alpha$ represent the boundary value of the solution to the bulk equation of motion for the matter fields or metric tensor. Applying the linear response theory, we can obtain the one-particle irreducible correlator of the strongly-coupled system:

$$\langle \mathcal{O}_1(x) \mathcal{O}_2(x) \rangle_{\text{1PI}} = \frac{\delta^2 S_{\text{cl.}}}{\delta \dot{\phi}_i(x_1) \dot{\phi}_i(x_2)} \bigg|_{\dot{\phi}_j=0}, \quad j = 1, 2, \dots, \quad \text{but } j \neq i \quad (3.2)$$

with the operator $\mathcal{O}$ dual to $\dot{\phi}_\alpha$.

3.1 Gravity preliminaries in the bulk theory

In order to generate the strong coupling counterpart of the polarization tensor discussed in section 2 via gauge/gravity duality, we consider an Einstein-Maxwell theory in the bulk with a Reissner-Nordström black hole and an asymptotically AdS_4 boundary. The action on r.h.s. of (3.1) takes the form

$$S = \int d^4x \sqrt{-g} \{ G_4 (R - 2\Lambda) - K_4 (F_{\mu\nu} F^{\mu\nu}) \} \quad (3.3)$$

where R is the scalar curvature, Λ is the negative cosmological constant, $\Lambda = -\frac{3}{L^2}$, L is the AdS radius, and $F_{\mu\nu}$ is a 2-form field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. For the clarity of notations, we shall scale the gauge potential such that $\frac{K_4}{G_4} = L^2$.

The action (3.3) leads to a solution of equations of motion with a RN-AdS black hole metric and a background gauge potential and its on-shell value corresponds to the thermodynamic potential of the normal phase of a holographic superconductor where the charged scalar field in the bulk vanishes [8–12]. In terms of the Poincaré frame, the background black hole metric reads

$$d\bar{s}^2 = \bar{g}_{\mu\nu} dx^\mu dx^\nu = \frac{L^2}{z^2} \left(-f(z) dt^2 + \frac{dz^2}{f(z)} + dx^2 + dy^2 \right) \quad (3.4)$$

where the metric function

$$f(z) = 1 - (1 + Q^2) \left(\frac{z}{z_+} \right)^3 + Q^2 \left(\frac{z}{z_+} \right)^4 \quad (3.5)$$

and the background gauge potential

$$\bar{A} = \bar{A}_t dt = \mu \left(1 - \frac{z}{z_+} \right) dt \quad (3.6)$$

where z_+ refers to the horizon in this coordinates, Q is the charge of the black hole and μ corresponds to the chemical potential of the boundary field theory. The Hawking temperature of the RN-AdS black hole is given by

$$T = \frac{1}{4\pi z_+} (3 - Q^2) = \frac{\mu}{Q} \frac{3 - Q^2}{4\pi} \quad (3.7)$$

which is also the temperature of the boundary field theory in equilibrium and the charge is related to the chemical potential via $Q = \mu z_+$

Because of (3.7), the positivity of the temperature requires $Q^2 \in (0, 3)$, which makes $z = z_+$ the closest zero to the AdS boundary $z = 0$ for real z and thereby the horizon of the RN-AdS black hole. The physical domain of the radial coordinate is therefore $0 \leq z \leq z_+$.

3.2 Fluctuations of gauge fields and metric fields

According to the holographic dictionary, the electric current operator and the energy-momentum tensor on the boundary field theory are dual to the fluctuations of the gauge fields and metric tensor in the bulk theory, respectively. The static electric Green function in the strong coupling limit we are focusing on will be obtained from the solution of the bulk equations of motion for these fluctuations.

Starting with the definitions

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \quad (3.8)$$

$$A_\mu = \bar{A}_\mu + a_\mu \quad (3.9)$$

where the background fields $(\bar{g}_{\mu\nu}, \bar{A}_\mu)$ refer to the background solutions (3.4) and (3.6), and $(h_{\mu\nu}, a_\mu)$ represent the corresponding fluctuations, respectively.² The two-point Green's functions are extracted from the quadratic terms of the on-shell bulk action in the boundary values of the fluctuations $(h_{\mu\nu}, a_\mu)$, thus we need only to solve the Einstein-Maxwell equations up to the 1st-order in $h_{\mu\nu}$ and a_μ . We work in the radial gauge

$$h_{z\nu} = 0, \quad a_z = 0, \quad \nu = \{t, x, y, x\} \quad (3.10)$$

and in frequency-momentum space

$$\begin{aligned} h_{\mu\nu}(t, z, x, y) &\sim e^{i(-\omega t + qx)} h_{\mu\nu}(z|\omega, q) \\ a_\mu(t, z, x, y) &\sim e^{i(-\omega t + qx)} a_\mu(z|\omega, q) \end{aligned} \quad (3.11)$$

where the fluctuations turn into the functions of the radial variable z , energy ω and momentum q . We have aligned the momentum along the x -axis by taking advantages of the SO(2) symmetry in $x - y$ plane, such that the fluctuation fields are classified into two categories according to the parity under $y \rightarrow -y$

$$\text{Odd Parity:} \quad h_{ty}, \quad h_{xy}, \quad a_y \quad (3.12)$$

$$\text{Even Parity:} \quad h_{tt}, \quad h_{tx}, \quad h_{xx}, \quad h_{yy}, \quad a_t, \quad a_x \quad (3.13)$$

²To maintain the basic property of a metric tensor $g_{\mu\rho}g^{\rho\nu} = \delta_\mu^\nu$, we have $g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu}$, and $\sqrt{-g} = \sqrt{-\bar{g}} (1 + \frac{1}{2}\bar{g}^{\mu\nu}h_{\mu\nu} + O(h^2))$.

In [13], we have studied the fluctuation underlying the transverse polarization, a_y , which belongs to the odd parity group. The fluctuation underlying the electric polarization we are interested here, a_0 belongs to the even parity group. As we shall see, in the static limit, a_x and h_{tx} are decoupled from the other components of even parity [22].

In the static limit, $\omega \rightarrow 0$, the linearized Einstein-Maxwell for $h_\mu(z, q)$ and $a_t(z, q)$ read

$$0 = h_t'' + \frac{3}{2} \frac{u^2}{f} \left(\frac{f}{u^2} \right)' h_t' + \frac{u^2}{2f} \left(\frac{f}{u^2} \right)' [h_x' + h_y'] + \frac{Q^2}{f} (2u^2 - q^2) h_t + 4 \frac{Q^2}{\mu} \frac{u^2}{f} a_t' \quad (3.14)$$

$$0 = [h_t'' + h_x'' + h_y''] + \frac{u^2}{2f} \left(\frac{f}{u^2} \right)' [h_t' + h_x' + h_y'] + \frac{f'}{f} h_t' + 2Q^2 \frac{u^2}{f} h_t + 4 \frac{Q^2}{\mu} \frac{u^2}{f} a_t' \quad (3.15)$$

$$0 = h_x'' + \frac{1}{f} \left(Q^2 u^3 - \frac{3}{u} \right) h_x' - \frac{1}{u} [h_t' + h_y'] - Q^2 (q^2 + 2u^2) h_t - Q^2 q^2 h_y - 4 \frac{Q^2}{\mu} u^2 a_t' \quad (3.16)$$

$$0 = h_y'' + \frac{1}{f} \left(Q^2 u^3 - \frac{3}{u} \right) h_y' - \frac{1}{u} [h_t' + h_x'] - Q^2 q^2 h_y - 2Q^2 u^2 h_t - 4 \frac{Q^2}{\mu} u^2 a_t' \quad (3.17)$$

$$0 = [h_t' + h_y'] + \frac{f'}{2f} h_t + 4 \frac{Q^2}{\mu} u^2 a_t \quad (3.18)$$

$$0 = a_t'' - \frac{Q^2}{f} q^2 a_t + \frac{\mu}{2} (h_t' - h_x' - h_y') \quad (3.19)$$

$$0 = h_t'' - \frac{2}{u} h_t' - 4 \frac{Q^2}{\mu} \frac{u^2}{f} a_x' \quad (3.20)$$

$$0 = q(fh_t' - f'h_t) \quad (3.21)$$

$$0 = (fa_x')' - \mu h_t' \quad (3.22)$$

where we have introduced two dimensionless quantities

$$q = \frac{q}{\mu}, \quad u = \frac{z}{z_+} \quad (3.23)$$

the prime in those equations means the derivative with respect to u , and the AdS radius $L = 1$ is set to one. Then the metric function becomes $f = 1 - (1 + Q^2)u^3 + Q^2u^4$.

It's easy to see from (3.14) to (3.22) that in the static limit, the even parity is splitted into two independent subsets:

$$\{h_t^t, h_x^x, h_y^y, a_t\} \quad \text{and} \quad \{h_t^x, a_x\} \quad (3.24)$$

and our interest lies in the first one because the electric polarization is extracted from the fluctuation a_t evaluated on the boundary. To disentangle the coupling between a_t and metric fluctuations h_t^t, h_x^x and h_y^y in (3.14)–(3.19) we employ the master fields method [20–22] that converts those coupled equations into two decoupled master field equations. The two gauge invariant master fields, $\Phi_\pm(u|q)$, are defined by

$$\Phi_\pm \equiv \alpha_\pm \Phi_1 + \frac{Q^2}{\mu} \Phi_2 \quad (3.25)$$

and satisfy the following two ordinary differential equations:

$$\Phi_{\pm}'' + \frac{f'}{f} \Phi_{\pm}' - \frac{U_{\pm}}{f^2} \Phi_{\pm} = 0 \quad (3.26)$$

where

$$\Phi_1 \equiv \frac{1}{u} h_y^y + \frac{f}{Q^2[k^2 - Z^2]u^2 - f'u} (h_x^{x'} + h_y^{y'}) \quad (3.27)$$

$$\Phi_2 \equiv -a_t' + \mu h_y^y - \frac{\mu}{2} h_t^t \quad (3.28)$$

and both $\alpha_{\pm}$ and $U_{\pm}$ are the functions of $u, \mathbf{q}$. We left the detailed forms of $\alpha_{\pm}$ and $U_{\pm}$ to the appendix A.

The master field equations (3.26) can be transformed into a Schrödinger-like equations

$$\Psi_{\pm}''(u|\mathbf{q}) + V_{\pm}(u|\mathbf{q}) \Psi_{\pm}(u|\mathbf{q}) = 0 \quad (3.29)$$

via the definition

$$\Phi_{\pm}(u|\mathbf{q}) \equiv \frac{1}{\sqrt{f(u)}} \Psi_{\pm}(u|\mathbf{q}) \quad (3.30)$$

and

$$V_{\pm}(u|\mathbf{q}) = V_{\pm}(u|k) = - \left[\frac{U_{\pm}}{f^2} + \frac{1}{4} \left(\frac{f'}{f} \right)^2 + \frac{1}{2} \left(\frac{f'}{f} \right)' \right] \quad (3.31)$$

where we define the modified momentum k as

$$k^2 \equiv \mathbf{q}^2 + \left[\frac{3}{4} \left(1 + \frac{1}{Q^2} \right) \right]^2 \quad (3.32)$$

According to the correspondence (3.1) between the quantum effective action of the boundary field theory and the bulk action, the on-shell action (3.3) corresponding to the linearized Einstein-Maxwell equations is a quadratic functional of the boundary values of all h_{μ}^{ν} and a_{μ} with the coefficients corresponding to various two point 1PI Green's functions in strong coupling limit. We denote $\mathcal{C}_{tt}$ as the electric polarization, $\mathcal{C}_{tt}(\omega = 0, \mathbf{q}) = \sigma^{\text{sc}}(\mathbf{q})$. After imposing the asymptotic conditions

$$\lim_{u \rightarrow 0} a_x \equiv \mathring{a}_x = 0, \quad \lim_{u \rightarrow 0} a_y \equiv \mathring{a}_y = 0 \quad \text{and} \quad \lim_{u \rightarrow 0} h_{\mu\nu} \equiv \mathring{h}_{\mu\nu} = 0 \quad (3.33)$$

$\mathcal{C}_{tt}$, extracted from the functional derivative in (3.2), reads

$$\begin{aligned} \mathcal{C}_{tt}(\omega, \mathbf{q}) &= K_4 \lim_{u \rightarrow 0} \sqrt{-\bar{g}} \bar{g}^{uu} \bar{g}^{tt} \frac{a_t'}{a_t} \\ &= -\frac{K_4}{z_+} \lim_{u \rightarrow 0} \frac{a_t'}{a_t} \end{aligned} \quad (3.34)$$

Owing to (3.25) and (3.30), we obtain

$$a_t'(u|k) \equiv \frac{1}{\sqrt{f}} \frac{\mu}{2Q^2 k} \{ (Z - 2u)[\Psi_+ - \Psi_-] - k[\Psi_+ + \Psi_-] \} + \mu h_y^y - \frac{\mu}{2} h_t^t \quad (3.35)$$

where the dimensionless quantity

$$Z \equiv \frac{3}{4} \left(1 + \frac{1}{Q^2} \right) \quad (3.36)$$

The situation to tackle the solution for (3.34) is different from the case in odd parity [13] since the equation (3.35) connects the master field to the first derivative of gauge fluctuation.

3.3 Analyticity of the electric polarization with respect to the momentum

Unlike the odd parity case, the master field equations are far more complicated here. Also the intricate links from the master field to a_t , shown in (3.29), (3.34) and (3.35), prohibit a straightforward way to read off the analyticity of $\mathcal{C}_{tt}(0|\mathbf{q})$ with respect to the momentum q from the master field equations. The alternative strategy we are following is to investigate the coupled Einstein-Maxwell equations directly. Our main concern is the analytic property of the fluctuation $a_t(u|q)$ and its derivative $a'_t(u|q)$ in the limit $u \rightarrow 0$. We shall prove below that $\mathcal{C}_{tt}(0, \mathbf{q})$ is a meromorphic function of q .

It is convenient to work with the new variable $\zeta \equiv 1 - u$, hence $\zeta = 0$ represents the horizon, and the prime in this section refers to the derivative with respect to ζ , e.g. $a'_t \equiv \frac{da_t}{d\zeta}$. Following ref. [15], equations (3.14), (3.15), (3.16) and (3.17) can be combined to cancel the 2nd order derivatives and to yield an expression for $h_x^{x'}$

$$h_x^{x'} = -h_y^{y'} + \frac{2u}{f' + 4f} \left\{ -\frac{2f}{u} h_t^{t'} + Q^2 [q^2 + 2u^2] h_t^t + Q^2 q^2 h_y^y - 4 \frac{Q^2}{\mu} u^2 a'_t \right\} \quad (3.37)$$

Substituting (3.37) into (3.17) and (3.19), we are left with the three coupled Einstein-Maxwell equations (3.17), (3.18) and (3.19) for a_t , h_t^t and h_y^y , respectively

$$\begin{aligned} 0 &= h_y^{y''} + \frac{C_3}{\mu\zeta} a'_t + C_4 h_t^{t'} + \frac{C_5}{\zeta} h_y^{y'} + \frac{D_6}{\zeta} h_t^t + \frac{D_7}{\zeta} h_y^y \\ 0 &= h_t^{t'} + h_y^{y'} + \frac{D_1}{\mu\zeta} a_t + \frac{D_2}{\zeta} h_t^t \\ 0 &= a_t'' + C_1 a'_t + \mu C_2 h_t^{t'} + \frac{D_3}{\zeta} a_t + \mu D_4 h_t^t + \mu D_5 h_y^y \end{aligned} \quad (3.38)$$

where the coefficients are given by

$$\begin{aligned} C_1 &= -\frac{4Q^2 u^2}{f' + \frac{4f}{u}} & C_2 &= -\frac{1}{2} \frac{u f' + 8f}{u f' + 4f} \\ C_3 &= \frac{4Q^2 \zeta u^2}{f} \frac{f' + \frac{2f}{u}}{f' + \frac{4f}{u}} & C_4 &= \frac{f'}{u f' + 4f} \\ C_5 &= \left(\frac{f'}{f} + \frac{2}{u} \right) \zeta \\ D_1 &= -\frac{4Q^2 \zeta}{f} u^2 & D_2 &= \frac{\zeta f'}{2f} \end{aligned}$$

$$\begin{aligned}
 D_3 &= -\frac{Q^2 \mathfrak{q}^2}{f} \zeta & D_4 &= \frac{Q^2 u(2u^2 + \mathfrak{q}^2)}{uf' + 4f} \\
 D_5 &= \frac{Q^2 \mathfrak{q}^2 u}{uf' + 4f} & D_6 &= -\zeta \frac{2Q^2 u^2 (uf' + 2f) - f \mathfrak{q}^2}{uf' + 4f} \\
 D_7 &= -\zeta Q^2 \frac{\mathfrak{q}^2}{f} \frac{uf' + 2f}{uf' + 4f}
 \end{aligned} \tag{3.39}$$

and they are all analytic and nonvanishing at the horizon $\zeta = 0$. Making Taylor expansions, we have

$$C_i = \sum_{m=0}^{\infty} c_i^{(m)} \zeta^m \tag{3.40}$$

$$D_i = \sum_{m=0}^{\infty} d_i^{(m)} \zeta^m \tag{3.41}$$

where all coefficients are polynomials in momentum q .

Now we are ready to explore the solution that is analytic at the horizon. It follows from the third equation of (3.38) that $a_t = O(\zeta)$ in the limit $\zeta \rightarrow 0$. Then the second equation of (3.38) implies that $h_t^t = O(\zeta)$ in the limit $\zeta \rightarrow 0$ as well. Therefore the power series solution of (3.38) takes the form

$$a_t = \mu \zeta \sum_{n=0}^{\infty} \alpha_n \zeta^n \equiv a_t(u|\mathfrak{q}) \tag{3.42}$$

$$h_t^t = \sum_{n=0}^{\infty} \beta_n \zeta^n \equiv h_t^t(u|\mathfrak{q}) \tag{3.43}$$

$$h_y^y = \sum_{n=0}^{\infty} \gamma_n \zeta^n \equiv h_y^y(u|\mathfrak{q}) \tag{3.44}$$

For the later convenience, we use the same format for the series of h_t^t as that for h_y^y but set $\beta_0 = 0$. Substituting series (3.40)–(3.44) into equations (3.38), we find the following recursion relations:

$$\begin{aligned}
 (n+1)(n + c_5^{(0)})\gamma_{n+1} + \sum_{k=0}^n (k+1) c_3^{(n-k)} \alpha_k + \sum_{k=0}^n k c_4^{(n-k)} \beta_k + \sum_{k=0}^n k c_5^{(n-k+1)} \gamma_k \\
 + \sum_{k=0}^n d_6^{(n-k)} \beta_k + \sum_{k=0}^n d_7^{(n-k)} \gamma_k = 0
 \end{aligned} \tag{3.45}$$

$$(n+1 + d_2^{(0)})\beta_{n+1} + (n+1)\gamma_{n+1} + \sum_{k=0}^n d_1^{(n-k)} \alpha_k + \sum_{k=0}^n d_2^{(n+1-k)} \beta_k = 0 \tag{3.46}$$

$$\begin{aligned}
 (n+1)(n+2)\alpha_{n+1} + (n+1)c_2^{(0)}\beta_{n+1} + \sum_{k=0}^n (k+1)c_1^{(n-k)}\alpha_k + \sum_{k=0}^n k c_2^{(n+1-k)}\beta_k \\
 + \sum_{k=0}^n d_3^{(n-k)}\alpha_k + \sum_{k=0}^n d_4^{(n-k)}\beta_k + \sum_{k=0}^n d_5^{(n-k)}\gamma_k = 0
 \end{aligned}
 \tag{3.47}$$

Given α_k , β_k and γ_k for $k \leq n$, explicit formulas for α_{n+1} , β_{n+1} and γ_{n+1} can be obtained from (3.45), (3.46) and (3.47). Indeed, the formula for γ_{n+1} follows from (3.45). Substituting it into (3.46) yields the formula for β_{n+1} . With explicit expressions of β_{n+1} , the formula for α_{n+1} is obtained from (3.47). These recursion formulas are displayed in the appendix B, which show that each order of a series is a polynomial in q and is an analytic function of q . Using an inductive method analogous to that used for the analyticity of the solution of a second order ordinary differential equation around a canonical singularity [23, 24], we are able to prove that the series solution of (3.42), (3.43) and (3.44) converges uniformly with respect to a finite q in a circle $|\zeta| < r$ on the complex ζ -plane around the horizon $\zeta = 0$. It follows from Weierstrass M-test theorem [23, 24] that the solutions of (3.42), (3.43) and (3.44) also define an analytic function with respect to q for a fixed ζ , so does the derivative of the solution with respect to ζ . Because of the absence of singularities of the coefficients C_i 's and D_i 's in the segment $0 < \zeta < 1$ ($0 < u < 1$), the solutions can be analytically continued to the outside of the convergence circle. For example, we may choose a point ζ_0 , such that $0 < \zeta_0 < r$ and its distance, d , to the nearest singularity of the coefficients satisfies $d > r - \zeta_0$. Around the point ζ_0 , the three Einstein-Maxwell equations (3.17), (3.18) and (3.19) can be written as

$$\begin{aligned}
 0 &= h_y'' + \frac{\bar{C}_3}{\mu} a_t' + \bar{C}_4 h_t^{t'} + \bar{C}_5 h_y^{y'} + \bar{D}_6 h_t^t + \bar{D}_7 h_y^y \\
 0 &= h_t^{t'} + h_y^{y'} + \frac{\bar{D}_1}{\mu} a_t + \bar{D}_2 h_t^t \\
 0 &= a_t'' + \bar{C}_1 a_t' + \mu \bar{C}_2 h_t^{t'} + \bar{D}_3 a_t + \mu \bar{D}_4 h_t^t + \mu \bar{D}_5 h_y^y
 \end{aligned}
 \tag{3.48}$$

where all coefficients are analytic for $|\zeta - \zeta_0| < d$. Substituting the ansatz

$$\begin{aligned}
 a_t(u|\mathbf{q}) &= \sum_{n=0}^{\infty} \bar{\alpha}_n (\zeta - \zeta_0)^n \\
 h_t^t(u|\mathbf{q}) &= \sum_{n=0}^{\infty} \bar{\beta}_n (\zeta - \zeta_0)^n \\
 h_y^y(u|\mathbf{q}) &= \sum_{n=0}^{\infty} \bar{\gamma}_n (\zeta - \zeta_0)^n
 \end{aligned}
 \tag{3.49}$$

into (3.48), one obtains the recursion relations

$$\begin{aligned}
 0 = & (n+1)(n+2) \bar{\gamma}_{n+2} + \sum_{k=0}^n (k+1) \bar{c}_3^{(n-k)} \bar{\alpha}_{k+1} + \sum_{k=0}^n (k+1) \bar{c}_4^{(n-k)} \bar{\beta}_{k+1} \\
 & + \sum_{k=0}^n (k+1) \bar{c}_5^{(n-k)} \gamma_{k+1} + \sum_{k=0}^n \bar{d}_6^{(n-k)} \bar{\beta}_k + \sum_{k=0}^n \bar{d}_7^{(n-k)} \bar{\gamma}_k
 \end{aligned} \tag{3.50}$$

$$0 = (n+2)(\bar{\gamma}_{n+2} + \bar{\beta}_{n+2}) + \sum_{k=0}^{n+1} \bar{d}_1^{(n-k+1)} \bar{\alpha}_k + \sum_{k=0}^{n+1} \bar{d}_2^{(n-k+1)} \bar{\beta}_k \tag{3.51}$$

$$\begin{aligned}
 0 = & (n+1)(n+2) \bar{\alpha}_{n+2} + \sum_{k=0}^n (k+1) \bar{c}_1^{(n-k)} \bar{\alpha}_{k+1} + \sum_{k=0}^n (k+1) \bar{c}_2^{(n-k)} \bar{\beta}_{k+1} \\
 & + \sum_{k=0}^n \bar{d}_3^{(n-k)} \bar{\alpha}_k + \sum_{k=0}^n \bar{d}_4^{(n-k)} \bar{\beta}_k + \sum_{k=0}^n \bar{d}_5^{(n-k)} \bar{\gamma}_k
 \end{aligned} \tag{3.52}$$

where $\bar{c}_i^{(m)}$'s and $\bar{d}_i^{(m)}$'s stand for the coefficients of the Taylor expansion of $\bar{C}_i$'s and $\bar{D}_i$'s around ζ_0 and are polynomials in q . With a_t , h_t^t , h_y^y and their derivatives at ζ_0 , provided by the series solution of (3.42), (3.43) and (3.44), all coefficients of (3.48) can be determined and are analytic in q . It follows that the series solution (3.48) and its derivative, being convergent for $|\zeta - \zeta_0| < d$, are analytic functions of q for a fixed ζ . The procedure can be repeated until we reach a series solution whose convergence circle pass through the AdS boundary $\zeta = 1$, which is another singularity of the coefficients C_i 's and D_i 's. The power series solution in the neighborhood of the AdS boundary

$$\begin{aligned}
 a_t(u|q) &= \sum_{n=0}^{\infty} a_n (1-\zeta)^n = \sum_{n=0}^{\infty} a_n u^n \\
 h_t^t(u|q) &= \sum_{n=0}^{\infty} b_n (1-\zeta)^n = \sum_{n=0}^{\infty} b_n u^n \\
 h_y^y(u|q) &= \sum_{n=0}^{\infty} c_n (1-\zeta)^n = \sum_{n=0}^{\infty} c_n u^n
 \end{aligned} \tag{3.53}$$

shows no diverging behavior in itself and its derivative as $\zeta \rightarrow 1$. Consequently, the series solution and its derivatives with respect to u remain convergent in the limit $\zeta \rightarrow 1$. It follows from the Weierstrass theorem that $a_t(0|q)$, $h_t^t(0|q)$ and $h_y^y(0|q)$ together with their derivatives are all analytic in q . The details behind each step of the proof outlined above can be found in appendix B.

A subtlety arises: the series solution of a_t is developed from two arbitrary constants, α_0 and γ_0 , which are not sufficient to make $h_y^y(0|q)$ and $h_t^t(0|q)$ vanish while maintaining a nonzero value of $a_t(0|q)$. This issue was noted in [15, 25] and the reason is that we take the radial gauge and demand the analyticity of $a_t(u|q)$, $h_y^y(u|q)$ and $h_t^t(u|q)$ in u at the horizon. Unlike the master fields, $a_t(u|q)$, $h_y^y(u|q)$ and $h_t^t(u|q)$ are not gauge invariant, a

coordinate transformation

$$\begin{aligned} z &\rightarrow z + e^{iqx} \phi(u) \\ x &\rightarrow z + e^{iqx} \chi(u) \end{aligned} \quad (3.54)$$

with

$$\begin{aligned} \phi(u) &= Au\sqrt{f(u)} \\ \chi(u) &= -iqA \int_0^u d\xi \frac{\xi}{\sqrt{f(u)}} \end{aligned} \quad (3.55)$$

can be made *within* the radial gauge at a cost of introducing a square root singularity $\sim O(\sqrt{1-u})$ at the horizon. Under this transformation, we find that

$$\begin{aligned} \delta h_y^y|_{u=0} &= \delta h_t^t|_{u=0} = -2A \\ \delta a_t|_{u=0} &= 0 \\ \delta a_t'|_{u=0} &= \mu A \end{aligned} \quad (3.56)$$

Therefore, we may choose the constants α_0 and γ_0 such that

$$h_y^y(0|\mathbf{q}) = h_t^t(0|\mathbf{q}) \equiv H(\mathbf{q}) \quad (3.57)$$

and use the gauge transformation (3.54) with $A = -H(\mathbf{q})/2$ to cancel the transformed metric fluctuations at the boundary. Consequently

$$a_t(0|\mathbf{q}) \rightarrow a_t(0|\mathbf{q}) \quad a_t'(0|\mathbf{q}) \rightarrow a_t'(0|\mathbf{q}) + \frac{1}{2}\mu H(\mathbf{q}) \quad (3.58)$$

and the polarization function $\mathcal{C}_{00}$ reads

$$\mathcal{C}_{00}(0, \mathbf{q}) = \frac{a_t'(0|\mathbf{q}) + \frac{1}{2}\mu H(\mathbf{q})}{a_t(0|\mathbf{q})} \quad (3.59)$$

which is evidently a meromorphic function of $\mathbf{q}$. The theme is proved.

4 The WKB approximation at large momentum magnitude

4.1 The WKB solutions

In the previous section, we obtained the general formula of the electric polarization tensor, $\mathcal{C}_{tt}(\omega, \mathbf{q})$, in a strongly coupled field theory through the gauge/gravity duality and linked it to the modified master fields, $\Psi_{\pm}$. The expression of $\mathcal{C}_{tt}$, (3.34), in the static limit and its analyticity imply that the spectrum of singularities of electric polarization corresponds to the nontrivial zeros of a_t on the complex momentum-plane. While the exact locations of these singularities can only be determined numerically [15], the asymptotic distribution of them for a large magnitude of momentum can be located analytically through WKB approximation like what we did in our last work for the transverse component of the

polarization tensor [13]. As we shall see below, the complex momentum singularities of the electric component share the same Friedel like asymptotic distribution as the transverse component. Because of the reality of $\mathcal{C}_{tt}(0, \mathbf{q})$ for real $\mathbf{q}$, the distribution of the complex singularities is symmetric with respect to the real axis and we need only to examine the upper half-plane of the complex $\mathbf{q}$.

In the master field framework, all physical quantities involved are taken as the functions of the modified momentum k through (3.32) for convenience. We denote the real and imaginary parts of k as

$$k = w + ip \quad (4.1)$$

Considering the case where $p \gg |w|$. The “potential” of the Schrödinger-like equation (3.29) can be approximated as

$$V_{\pm}(u|k) = -\frac{1}{f^2} U_{\pm} \equiv \frac{Q^2}{f} [-ik \pm iu]^2 \quad (4.2)$$

and the WKB solution reads³

$$(\Psi)_{\pm} = \frac{f^{1/4}}{Q^{1/2}} \left(C_1^{\pm} \exp \left\{ Q \int_u^1 \frac{-k \pm v}{\sqrt{f}} dv \right\} + C_2^{\pm} \exp \left\{ -Q \int_u^1 \frac{-k \pm v}{\sqrt{f}} dv \right\} \right) \quad (4.3)$$

where $C_1^{\pm} = C_1^{\pm}(k)$ and $C_2^{\pm} = C_2^{\pm}(k)$ are four constants to be determined. The difference between (4.3) and the exact solutions are of the order $O(1/p)$, and can be ignored for a large q . Matching the WKB solution with the near horizon solution below, the four constants can be reduced to two of them.

In the limit $u \rightarrow 1$, $f(u) \simeq (3 - Q^2)(1 - u)$ as $u \rightarrow 1$, the integral in the exponential parts of the WKB solution can be written as

$$\int_u^1 du \frac{Q[-k \pm v]}{\sqrt{f}} \simeq 2\lambda_{\pm} \sqrt{1 - u}, \quad \text{as } u \rightarrow 0 \quad (4.4)$$

where

$$\lambda_{\pm} \equiv \frac{Q[-k \pm 1]}{\sqrt{3 - Q^2}} \quad (4.5)$$

The WKB solution remains approximate in this region as long as $\lambda_{\pm} \sqrt{1 - u} \gg 1$ following from the condition of the approximation: $|V'_{\pm}| \ll |V_{\pm}|^{\frac{3}{2}}$ and its asymptotic form there is given by

$$(\Psi_{\pm})_{\text{WKB}} \Big|_{u \rightarrow 1} = \frac{(3 - Q^2)^{1/4}}{Q^{1/2}} [1 - u]^{1/4} \cdot \left(C_1^{\pm} e^{2\lambda_{\pm} \sqrt{1 - u}} + C_2^{\pm} e^{-2\lambda_{\pm} \sqrt{1 - u}} \right) \quad (4.6)$$

On the other hand, the Schrödinger-like equation (3.29) can be transformed into the modified Bessel equations of the zeroth order, of which the general solution is given by

$$\Psi_{\pm} = \sqrt{1 - u} \cdot [b_{\pm} I_0(2\lambda_{\pm} \sqrt{1 - u}) + c_{\pm} K_0(2\lambda_{\pm} \sqrt{1 - u})] \quad (4.7)$$

with $b_{\pm}$ and $c_{\pm}$ constant coefficients.

³A brief discussion on the WKB approximation and its validity is included in the appendix C for self-containedness.

For a similar reason in the case of *equation (4.10)* in [13], the second term associated to $K_0(2\lambda_\pm\sqrt{1-u})$, with a logarithmic singularity should be dropped out of (4.7), which means $c_\pm \equiv 0$, in order to prevent the on-shell action from divergence. Furthermore, we are working with the gauge-invariant master field, it's physically legitimate to demand the regularity conditions for the master fields at the horizon.

In the region $\lambda_\pm\sqrt{1-u} \gg 1$, where both the WKB solutions and the Bessel function solutions approximate, we have two forms of the same solution, (4.6) and the asymptotic form of (4.7) at $c_\pm = 0$, i.e.

$$\Psi_\pm = \frac{b_\pm}{2\sqrt{\pi}} \lambda_\pm^{-\frac{1}{2}} (1-u)^{\frac{1}{4}} \left(e^{+2\lambda_\pm\sqrt{1-u}} + i e^{-2\lambda_\pm\sqrt{1-u}} \right) \quad (4.8)$$

where we have substituted the asymptotic form of $I_0(z)$ for large $|z|$,

$$I_0(z) \simeq \frac{1}{\sqrt{2\pi z}} [i e^{-z} + e^z] \quad (4.9)$$

Note that for the limit of $z \rightarrow \infty$ along a line parallel to the imaginary axis, both terms inside the bracket of (4.9) have to be retained. Matching (4.7) and (4.8), we obtain that

$$iC_1^\pm = C_2^\pm \equiv iC_\pm \quad (4.10)$$

thus the WKB solutions of modified master fields (4.6) become

$$\begin{aligned} (\Psi_\pm)_{\text{WKB}} = \frac{f^{1/4}}{Q^{1/4}} C_\pm \left[\exp \left\{ Q \left(-k \int_u^1 \frac{1}{f(v)} dv + (\pm) \int_u^1 \frac{v}{f(v)} dv \right) \right\} \right. \\ \left. + i \exp \left\{ Q \left(k \int_u^1 \frac{1}{f(v)} dv - (\pm) \int_u^1 \frac{v}{f(v)} dv \right) \right\} \right] \end{aligned} \quad (4.11)$$

4.2 Constraints on the AdS boundary for fluctuations

The WKB solutions obtained in the previous sub-section can be extended all the way to the AdS boundary since $u = 0$ ($\zeta = 1$) is an ordinary point of the master field equation. The background metric near the boundary can be approximated by a pure AdS one. In order to extract the electric component of the holographic polarization tensor (3.2), we demand that the metric tensor fluctuations vanish on the AdS boundary, i.e. $(h_t^t, h_x^x, h_y^y) \rightarrow 0$ as $u \rightarrow 0$. In this sub-section, we will investigate the asymptotic form of Einstein equations (3.14)–(3.18) to find a simple approach that leads from the WKB approximation of the master field to the solution for a_t and its derivative on the AdS boundary satisfying the above constraints.

To begin with, the asymptotic form of (3.15) reads

$$[h_t^t + h_x^x + h_y^y]'' - \frac{1}{u} [h_t^t + h_x^x + h_y^y]' = 0 \quad (4.12)$$

which suggests two asymptotic behaviors in the limit $u \rightarrow 0$, $[h_t^t + h_x^x + h_y^y] \sim 1$ or $[h_t^t + h_x^x + h_y^y] \sim u^2$. The vanishing limits of the metric fluctuations as $u \rightarrow 0$ rules out the first one and we have

$$h_x^x + h_t^t + h_y^y = O(u^2) \quad (4.13)$$

Employing this relationship in (3.14), (3.16) and (3.17), we find their identical asymptotic forms,

$$h_x^{x''} - \frac{2}{u} h_x^{x'} = 0 \quad (4.14)$$

and the ones with the index x replaced by y and t . Together with the vanishing boundary conditions as $u \rightarrow 0$, we end up with

$$h_t^t, h_x^x, h_y^y \sim O(u^3) \quad (4.15)$$

It follows from (4.15) and the power series expansion of the eq. (3.19) in u that

$$\lim_{u \rightarrow 0} a_t'' = Q^2 q^2 \lim_{u \rightarrow 0} a_t \quad (4.16)$$

$$\lim_{u \rightarrow 0} a_t''' = Q^2 q^2 \lim_{u \rightarrow 0} a_t' \quad (4.17)$$

The eq. (3.35) and its derivatives link the quantities in (4.16) and (4.17) to the master fields as follows

$$\dot{a}_t' \equiv \lim_{u \rightarrow 0} a_t'(u|k) = \frac{\mu}{2Q^2} \frac{1}{k} \left\{ -(Z+k) \dot{\Psi}_- + (Z-k) \dot{\Psi}_+ \right\} \quad (4.18)$$

$$\dot{a}_t'' \equiv \lim_{u \rightarrow 0} a_t''(u|k) = \frac{\mu}{2Q^2} \frac{1}{k} \left\{ 2 \dot{\Psi}_- - (Z+k) \dot{\Psi}'_- - 2 \dot{\Psi}_+ + (Z-k) \dot{\Psi}'_+ \right\} \quad (4.19)$$

$$\dot{a}_t''' \equiv \lim_{u \rightarrow 0} a_t'''(u|k) = \frac{\mu}{2Q^2} \frac{1}{k} \left\{ 4 \dot{\Psi}_- - (Z+k) \dot{\Psi}''_- - 4 \dot{\Psi}_+ + (Z-k) \dot{\Psi}''_+ \right\} \quad (4.20)$$

where a ring over a quantity denotes the limit: $u \rightarrow 0$. Now we are fully equipped to locate the singularities of $\mathcal{C}_{tt}(0, q)$ under the WKB approximation.

At this point, it is worth comparing the solutions of the master field equations discussed in this section and the solutions of the Einstein-Maxwell equations analyzed in the last section. Starting with the gauge potential and metric fluctuations that are analytic at the horizon, we are unable to make all metric fluctuation vanishing on the AdS boundary. Employing the residual gauge degrees of freedom, we are able to get rid of all nonzero metric fluctuations on the boundary at the cost of introducing non-analytic behavior of the transformed fluctuations. This non-analytic behavior will not show up in the master fields because of their gauge invariance.

4.3 Singularities of electric polarization from WKB approximation

Based on the knowledge acquired in previous sub-sections, we are ready to find the spectrum of singularities for the electric polarization (3.34) at large momentum magnitude, which corresponds to the zeros of a_t on the boundary.

Starting with the WKB solutions (4.11), we find its first derivative

$$\begin{aligned} (\Psi_{\pm})'_{\text{WKB}} = f^{1/4} \frac{C_{\pm}}{Q^{1/2}} \left[-Q \frac{-k \pm u}{\sqrt{f}} \exp \left\{ Q \left(-k \int_u^1 \frac{1}{f(v)} dv + (\pm) \int_u^1 \frac{v}{f(v)} dv \right) \right\} \right. \\ \left. + i Q \frac{-k \pm u}{\sqrt{f}} \exp \left\{ Q \left(k \int_u^1 \frac{1}{f(v)} dv - (\pm) \int_u^1 \frac{v}{f(v)} dv \right) \right\} \right] + \text{etc.} \end{aligned} \quad (4.21)$$

where the etc. represents all terms that vanish as u^2 or faster in the limit $u \rightarrow 0$. The second order derivative can be expressed in terms of the modified master fields itself through the Schrödinger-like equations (3.29). It follows then that,

$$\left(\mathring{\Psi}_{\pm}\right)_{\text{WKB}} \equiv \lim_{u \rightarrow 0} (\Psi_{\pm})_{\text{WKB}} \simeq C_{\pm} Q^{-1/2} H_{+}^{(\pm)} \quad (4.22)$$

$$\left(\mathring{\Psi}'_{\pm}\right)_{\text{WKB}} \equiv \lim_{u \rightarrow 0} (\Psi'_{\pm})_{\text{WKB}} \simeq C_{\pm} k Q^{1/2} H_{-}^{(\pm)} \quad (4.23)$$

$$\left(\mathring{\Psi}''_{\pm}\right)_{\text{WKB}} \equiv \lim_{u \rightarrow 0} (\Psi''_{\pm})_{\text{WKB}} \simeq C_{\pm} k^2 Q^{3/2} H_{+}^{(\pm)} \quad (4.24)$$

where we retain only the principal term of the “potential” $V_{\pm}(u|\mathbf{q})$ (3.31) and (A.5) in accordance with the WKB approximation,

$$\left(\lim_{u \rightarrow 0} V_{\pm} + Q^2 \mathbf{q}^2\right)_{\text{WKB}} = 0 \quad (4.25)$$

and introduce the notations:

$$\begin{aligned} H_{-}^{(\pm)} &\equiv e^{Q(-k L_1 \pm L_2)} - i e^{Q(k L_1 - (\pm) L_2)} \\ H_{+}^{(\pm)} &\equiv e^{Q(-k L_1 \pm L_2)} + i e^{Q(k L_1 - (\pm) L_2)} \end{aligned} \quad (4.26)$$

for brevity with L_1 and L_2 denoting the two elliptic integrals

$$\begin{aligned} L_1 &\equiv \int_0^1 \frac{1}{f(v)} dv \\ L_2 &\equiv \int_0^1 \frac{v}{f(v)} dv \end{aligned} \quad (4.27)$$

According to condition (4.16) together with (4.19), (4.22) and (4.23), we obtain $\lim_{u \rightarrow 0} a_t$ in WKB approximation:

$$(\mathring{a}_t)_{\text{WKB}} \simeq \frac{1}{Q^2 k^2} (\mathring{a}_t'')_{\text{WKB}} = -\frac{\mu}{2Q^2} \frac{Q^{-3/2}}{k} \left[H_{-}^{(-)} C_{-} + H_{-}^{(+)} C_{+} \right] \quad (4.28)$$

As its zeros give rise to the poles of $(\mathcal{C}_{tt})_{\text{WKB}}$, we find one of the equations for their locations:

$$(\mathring{a}_t)_{\text{WKB}} \sim H_{-}^{(-)} C_{-} + H_{-}^{(+)} C_{+} = 0 \quad (4.29)$$

Substituting (4.18) and (4.20) into (4.17) with $\mathring{\Psi}_{\pm}$, $\mathring{\Psi}'_{\pm}$ and $\mathring{\Psi}''_{\pm}$ given by their WKB approximations (4.22), (4.23) and (4.24), we find another equation for the poles,

$$H_{-}^{(-)} C_{-} - H_{-}^{(+)} C_{+} = 0 \quad (4.30)$$

Equations (4.29) and (4.30), form a system of linear homogeneous equations in two unknowns coefficients, C_{-} and C_{+} , the existence of nontrivial solutions for them implies that

$$0 = \begin{vmatrix} H_{-}^{(-)} & H_{-}^{(+)} \\ H_{-}^{(-)} & -H_{-}^{(+)} \end{vmatrix} = -2 H_{-}^{(-)} H_{-}^{(+)} \quad (4.31)$$

Substituting the explicit expressions of $H_-^{(-)}$ and $H_-^{(+)}$ in (4.26) and (4.27), we obtain

$$e^{-2QL_1k} - ie^{-2QL_2} - ie^{2QL_2} - e^{2QL_1k} = 0 \quad (4.32)$$

the solution of (4.32) in the complex-plane is given by

$$(e^{-2QL_1w} - e^{2QL_1w}) \cos[2QL_1p] = 0 \quad (4.33)$$

and

$$-\frac{e^{-2QL_2} + e^{2QL_2}}{e^{-2QL_1w} + e^{2QL_1w}} = \sin[2QL_1p] \quad (4.34)$$

Finally, from (4.33) and (4.34), we obtain the positions of the singularities of the modified momentum $k = w + ip$ for $p \gg |w|$, i.e.,

$$w = \pm \frac{L_2}{L_1} \quad (4.35)$$

$$p = \frac{\pi}{QL_1} \left[n - \frac{1}{4} \right] \quad (4.36)$$

It follows from (3.32) that $\mathbf{q} \simeq k$ for $k \gg |w|$ and the asymptotic locations of the poles of $\mathcal{C}(0, \mathbf{q})$ on the upper complex $\mathbf{q}$ plane are thereby

$$\mathbf{q} \simeq \pm w + i \frac{\pi}{QL_1} \left[n - \frac{1}{4} \right] \quad (4.37)$$

The asymptotic locations of the poles on the lower complex $\mathbf{q}$ plane are obtained from (4.37) by a reflection with respect to the real axis.

5 Concluding remarks

Let us recapitulate what we did in this work. We started with an one-loop calculation of the static electric component of the polarization tensor of a spinor QED3 and a scalar QED3 with chemical potential μ and explored its analyticity on the complex momentum plane. We found an infinite number of branch points along two straight-lines parallel to the imaginary axis with real part equal to $\pm\mu$. The nature of these singularities and their distribution serve a benchmark of the holographic polarization tensor discussed in the subsequent sections and the ones close to the real axis are responsible to the Friedel oscillations observed in some materials. Then we explored the analyticity of the static electric component of the holographic polarization tensor and proved that it is a meromorphic function of the complex momentum. Employing the WKB approximation, we are able to locate analytically the asymptotic distributions of the poles along two lines parallel to the imaginary axis of the momentum plane, i.e.

$$\frac{q}{\mu} = \pm \frac{L_2}{L_1} \pm i \frac{\pi}{QL_1} \left| n - \frac{1}{4} \right| \quad (5.1)$$

for $n \gg 1$.

It would be interesting to compare the asymptotic locations of the poles (4.35) and (4.36) with the those extracted from the numerical solutions of the Einstein-Maxwell

Our notation	The notation of [15]
a_t	$\delta A_t/2$
h_t^t	δg_t^t
h_x^x	δg_x^x
h_y^y	δg_y^y
μ	$\mu_0/2$
q	k
u	z/z_+

Table 1. The Translation of Notations.

equations reported in [15]. Comparing our Einstein-Maxwell equations with those in the appendix A of [15], we noted the following relationships between our notations and theirs.⁴

It follows from table 1 that

$$\frac{k}{\mu_0} = \frac{q}{2\mu} \quad (5.2)$$

According to eqs. (4.35) and (4.36), we find the poles at

$$\frac{k}{\mu_0} = \pm 0.309558 + 2.12399 \left(n - \frac{1}{4} \right) i \quad (5.3)$$

for $\frac{T}{\mu_0} = 0.21$ and

$$\frac{k}{\mu_0} = \pm 0.399722 + 0.288434 \left(n - \frac{1}{4} \right) i \quad (5.4)$$

for $\frac{T}{\mu_0} = 0.0006$, here $n \in \mathbb{Z}$. Moreover, the equations (5.3) and (5.4) indicate that the separation between any two nearest poles is 2.12399 and 0.288434 for the two cases. These results are very close to the numerically determined locations in figure 7 of [15]. We can see that our WKB-type formalism holds in a large imaginary part of complex momentum in principle, which means integer parameters n in (5.3) and (5.4) should be large enough as to the coefficient $\frac{1}{4}$ can be neglected asymptotically. However, comparing with these numerical simulation, even in a regime not too far away from the real axis, our WKB approximations works reasonably well.

Our study on the analyticity of the electric component of the static polarization tensor is consistent with the previous knowledge on the zero chemical potential case [19]. In the absence of the chemical potential, all of the Friedel-like poles migrate to the imaginary axis, and it was also proved that the static polarization tensor is a meromorphic function. The existence of these poles can be inferred from the Matsubara formulation of the boundary field theory, whose Lagrangian density is $O(3)$ invariant at $\mu = 0$. Mathematically, it

⁴Throughout this paper, we have scaled the U(1) gauge potential in eq. (3.3) such that $\frac{K_4}{G_4} = L^2 = 1$. An arbitrary ratio $\frac{K_4}{G_4} \equiv \eta^2$ amounts to the transformations $a_t \rightarrow \eta a_t$ and $\mu \rightarrow \eta \mu$ in the Einstein-Maxwell equations (3.14)–(3.22). While eq. (5.1) is $\frac{q}{\mu} = \pm \frac{L^2}{L_1^2} \eta \pm i \frac{\pi}{Q L_1} \eta |n - \frac{1}{4}|$. The notation in [15] corresponds to $\eta = \frac{1}{2}$.

is optional to interpret any dimension as Euclidean time with the associated momentum as the imaginary energy. Therefore the poles on the imaginary q -axis corresponds to the excitation of another boundary field theory with the original compact Euclidean time and one of the spatial dimension interpreted as a compact space dimension and Euclidean time.

The WKB approximation can be readily applied when the momentum q is real and large. Up to the leading order, the WKB solution (4.3) takes the form

$$(\Psi)_{\pm} \simeq \text{const.} \exp \left(|q| z_+ \int_u^1 \frac{1}{\sqrt{f}} dv \right) \quad (5.5)$$

in this case, where we have dropped the exponentially small term, identified k with q/μ and substituted $Q = \mu z_+$. It follows from (3.34) and (4.16) together with the leading order of (4.18) and (4.20) that

$$\mathcal{C}_{00}(0, q) = K_4 q \quad (5.6)$$

The UV behavior is not impacted by the nonzero temperature and chemical potential as expected. Notice that because of the low dimensionality and gauge invariance, the polarization tensor is free from UV divergence.

The presence of the Friedel-like singularities appears a common property of a quantum field theory (strongly or weakly coupled) with a nonzero chemical potential. It is not yet a sufficient evidence of the fermionic degrees of freedom in the boundary field theory of the gravity dual. More investigations are warranted to probe the fermionic degrees of freedom in a holography implied field theory, for example by adding a spinor field in a bulk theory [26–31].

A The master field equations in even parity

The function $\alpha_{\pm}$ in the definition of the master fields (3.25) is given by

$$\alpha_{\pm} \equiv \frac{Q^2}{2} \left[Z \pm \sqrt{Z^2 + \mathfrak{q}} \right] - Q^2 u \quad (A.1)$$

with Z defined in (3.36).

The explicit expression of $U_{\pm}$ in the “potential” of the Schrödinger-like equation (3.31) reads

$$U_{\pm} \equiv \pm f \left\{ \frac{Z - 2u \pm k}{2k} \cdot V + \frac{Q^2 k^4 - \left[\frac{f'}{u} + 2Z \right] k^2 + Q^2 Z^4 + \frac{f'}{u} Z^2 + 8f}{2k(Z - 2u \pm k)} \right. \\ \left. - \frac{2u \left(Q^2 k^2 - Q^2 Z + \frac{f'}{u} \right)}{k} + \frac{4f[Q^2(8u - 3) - 3]}{k \left(Q^2 k^2 - Q^2 Z^2 - \frac{f'}{u} \right)} \right\} \quad (A.2)$$

where

$$\begin{aligned}
 V \equiv & \frac{1}{[Q^2 u k^2 - Q^2 Z^2 u - f']^2} \{ Q^6 u^2 \cdot \mathfrak{q}^6 + 3 Q^4 (1 + Q^2) u^3 \cdot \mathfrak{q}^4 \\
 & + [8 Q^2 u^2 - 153 Q^2 (1 + Q^2)^2 u^4 + 400 Q^4 (1 + Q^2) u^5 + 264 Q^6 u^6] \cdot \mathfrak{q}^2 \\
 & + u^2 [-738 (1 + Q^2)^2 + 1944 Q^2 (1 + Q^2) u - 1248 Q^4 u^2 + 1197 (1 + Q^2)^3 u^3 \\
 & - 4482 Q^2 (1 + Q^2)^2 u^4 + 5544 Q^4 (1 + Q^2) u^5 - 2272 Q^6 u^6] \}
 \end{aligned} \tag{A.3}$$

From the explicit expressions of $U_{\pm}$, (A.2) and (A.3), it's easy to verify that the AdS boundary $u = 0$ is an ordinary point of the master field equations (3.26) because none of its coefficients diverges there.

For a large momentum magnitude, the dimensionless momentum is approximately equal to modified momentum: i.e. $\mathfrak{q} \sim k$, and

$$\begin{aligned}
 V &= \frac{Q^4 \mathfrak{q}^4 + 4 u^2 Q^4 \mathfrak{q}^2}{Q^2 \mathfrak{q}^2 + 3 u (1 - u Q^2)} + O\left(\frac{1}{\mathfrak{q}^2}\right) \\
 &= Q^2 k^2 + Q^2 (4 u^2 - Z^2) + O\left(\frac{1}{\mathfrak{q}^2}\right)
 \end{aligned} \tag{A.4}$$

Substituting (A.4) into (A.3), after some algebra, we obtain

$$U_{\pm} = f \left\{ Q^2 [k \mp u]^2 + u^2 Q^2 - \frac{1}{2} Q^2 Z + O\left(\frac{1}{k}\right) \right\} \tag{A.5}$$

Only the leading and sub-leading terms of (A.5) in large k are required to carry out the WKB approximation for our purpose and we may write

$$U_{\pm} \simeq f Q^2 [k \mp u]^2 = -f Q^2 [-i k \pm i u]^2 \tag{A.6}$$

which turns into (4.2), and $O\left(\frac{1}{\mathfrak{q}}\right) = O\left(\frac{1}{k}\right)$ holds in the WKB approximation.

B The proof of the convergence of the power series solutions for a_t, h_t^t, h_y^y

The explicit recursion formulas for the coefficients of the power series solution of (3.42), (3.43) and (3.44), following from the eqs. (3.45), (3.46) and (3.47) read

$$\begin{aligned}
 \gamma_{n+1} = & -\frac{1}{(n+1)(n+c_5^{(0)})} \sum_{k=0}^n \left[(k+1) c_3^{(n-k)} \alpha_k + \left(k c_4^{(n-k)} + d_6^{(n-k)} \right) \beta_k \right. \\
 & \left. + \left(k c_5^{(n-k+1)} + d_7^{(n-k)} \right) \gamma_k \right]
 \end{aligned} \tag{B.1}$$

$$\begin{aligned}
 \beta_{n+1} = & \frac{1}{(n+c_5^{(0)})(n+1d_2^{(0)})} \sum_{k=0}^n \left[(k+1) c_3^{(n-k)} \alpha_k + \left(k c_4^{(n-k)} + d_6^{(n-k)} \right) \beta_k \right. \\
 & \left. + \left(k c_5^{(n-k+1)} + d_7^{(n-k)} \right) \gamma_k \right] - \frac{1}{n+1+d_2^{(0)}} \sum_{k=0}^{\infty} \left[d_1^{(n-k)} a_k + d_2^{(n-k+1)} \beta_k \right]
 \end{aligned} \tag{B.2}$$

$$\begin{aligned}
 \alpha_{n+1} = & \sum_{k=0}^n \left\{ \left[\frac{c_2^{(0)} \cdot d_1^{(n-k)}}{(n+1+d_2^{(0)})(n+2)} - \frac{c_2^{(0)} \cdot (k+1) c_3^{(n-k)}}{(n+c_5^{(0)})(n+1+d_2^{(0)})(n+2)} \right. \right. \\
 & \left. \left. - \frac{(k+1) c_1^{(n-k)} + d_3^{(n-k)}}{(n+1)(n+3)} \right] \cdot \alpha_k \right. \\
 & + \left[\frac{c_2^{(0)}}{(n+1+d_2^{(0)})(n+2)} \left(d_2^{(n-k+1)} - \frac{k c_4^{(n-k)} + c_6^{(n-k)}}{n+c_5^{(0)}} \right) - \frac{k c_2^{(n-k+1)} + d_4^{(n-k)}}{(n+1)(n+2)} \right] \cdot \beta_k \\
 & \left. + \left[- \frac{c_2^{(0)} \cdot (k c_5^{(n-k+1)} + d_2^{(n-k)})}{(n+c_5^{(0)})(n+1+d_2^{(0)})} - \frac{d_5^{(n-k)}}{(n+1)(n+2)} \right] \cdot \gamma_k \right\}
 \end{aligned} \tag{B.3}$$

where $c_i^{(m)}$'s and $d_i^{(m)}$'s are defined by the Taylor expansions of the coefficient functions C_i 's and D_i 's in (3.39). We have

$$c_1^{(0)} = \frac{4Q^2}{3-Q^2} \quad c_2^{(0)} = -\frac{1}{2} \tag{B.4}$$

$$c_3^{(0)} = \frac{4}{3-Q^2} \quad c_4^{(0)} = 1 \tag{B.5}$$

$$c_5^{(0)} = 1 \tag{B.6}$$

$$d_1^{(0)} = -\frac{4Q^2}{3-Q^2} \quad d_2^{(0)} = \frac{1}{2} \tag{B.7}$$

$$d_3^{(0)} = -\frac{Q^2}{3-Q^2} \mathfrak{q}^2 \quad d_4^{(0)} = -\frac{Q^2 (\mathfrak{q}^2 + 2)}{3-Q^2} \tag{B.8}$$

$$d_5^{(0)} = -\frac{Q^2 \mathfrak{q}^2}{3-Q^2} \quad d_6^{(0)} = -\frac{2Q^2}{3-Q^2} \tag{B.9}$$

$$d_7^{(0)} = -\frac{Q^2}{3-Q^2} \tag{B.10}$$

The coefficients α_n , β_n and γ_n can be bounded following the induction method employed in [23, 24] for the power series solution of an ordinary differential equation around a regular point. The analyticity of C_i and D_i at the horizon $\zeta = 0$ implies that there exists a circular domain: $|\zeta| < r$, where all functions of C_i , D_i are bound and satisfy the Cauchy inequality

$$|c_i^{(k)}| < \frac{M}{8r^k} \quad |d_i^{(k)}| < \frac{M}{8r^k} \quad |c_i^{(k)} + d_j^{(k)}| < \frac{M}{8r^k} \tag{B.11}$$

$$|\alpha_0| \leq \frac{M}{r} \quad |\beta_0| = 0 \leq \frac{M}{r} \quad |\gamma_0| \leq \frac{M}{r} \tag{B.12}$$

for a finite constant M . We choose $r < 1$ and $M > 1$ and start with the inequalities

$$|\alpha_0| \leq \frac{M}{r} \quad |\beta_0| = 0 \leq \frac{M}{r} \quad |\gamma_0| \leq \frac{M}{r} \tag{B.13}$$

for given α_0 and γ_0 . Assuming that all α_k , β_k and γ_k with $k \leq n$ are known and satisfy the inequalities

$$|\alpha_k| \leq \left(\frac{M}{r}\right)^k \quad |\beta_k| \leq \left(\frac{M}{r}\right)^k \quad |\gamma_k| \leq \left(\frac{M}{r}\right)^k \quad (\text{B.14})$$

we shall prove that

$$|\alpha_{n+1}| \leq \left(\frac{M}{r}\right)^{n+1} \quad |\beta_{n+1}| \leq \left(\frac{M}{r}\right)^{n+1} \quad |\gamma_{n+1}| \leq \left(\frac{M}{r}\right)^{n+1} \quad (\text{B.15})$$

Following (B.1), we obtain that

$$\begin{aligned} |\gamma_{n+1}| &\leq \frac{1}{(n+1)(n+c_5^{(0)})} \left\{ \sum_{k=0}^n (k+1) |c_3|^{n-k} |\alpha_k| + \sum_{k=0}^n \left(k |c_4^{n-k}| + |d_6^{(n-k)}| \right) |\beta_k| \right. \\ &\quad \left. + \sum_{k=0}^n \left(k |c_5^{n-k+1}| + |d_7^{(n-k)}| \right) |\gamma_k| \right\} \\ &\leq \frac{M}{8(n+1)(n+c_5^{(0)}) r^{n+1}} \left[r \sum_{k=0}^n (k+1) M^k + r \sum_{k=0}^n (k+1) M^k + \sum_{k=0}^n (k+1) M^k \right] \\ &\leq \frac{3(n+2)}{16(n+c_5^{(0)})} \left(\frac{M}{r}\right)^{n+1} \\ &\leq \frac{3}{8} \left(\frac{M}{r}\right)^{n+1} \leq \left(\frac{M}{r}\right)^{n+1} \end{aligned} \quad (\text{B.16})$$

where we have used $c_5^{(0)} = 1$ in the last step. Then (3.46) and the $d_2^{(0)} = \frac{1}{2}$ yield

$$\begin{aligned} |\beta_{n+1}| &\leq |\gamma_{n+1}| + \frac{1}{n+1} \sum_{k=0}^n \left(|d_1^{(n-k)}| |\alpha_k| + |d_2^{(n-k+1)}| |\beta_k| \right) \\ &\leq \frac{3}{8} \left(\frac{M}{r}\right)^{n+1} + \frac{1}{8} (r+1) \left(\frac{M}{r}\right)^{n+1} \\ &\leq \frac{5}{8} \left(\frac{M}{r}\right)^{n+1} \leq \left(\frac{M}{r}\right)^{n+1} \end{aligned} \quad (\text{B.17})$$

Finally, it follows from (3.47) that

$$\begin{aligned} |\alpha_{n+1}| &\leq \frac{|c_2^{(0)}|}{n+2} |\beta_{n+1}| + \frac{1}{(n+1)(n+2)} \sum_{k=0}^n \left\{ \left[k |c_1^{(n-k)}| + |c_1^{(n-k)}| + |d_3^{(n-k)}| \right] |\alpha_k| \right. \\ &\quad \left. + \left[k |c_2^{(n-k+1)}| + |d_4^{(n-k)}| \right] |\beta_k| + |d_5^{(n-k)}| |\gamma_k| \right\} \\ &\leq \frac{5}{8} \left(\frac{M}{r}\right)^{n+1} + \frac{(r+2)}{8} \left(\frac{M}{r}\right)^{n+1} \leq \left(\frac{M}{r}\right)^{n+1} \end{aligned} \quad (\text{B.18})$$

The theme is then proved and the inequalities (B.14) hold for any k . It follows from eq. (3.53) that

$$\begin{aligned}
 |a_t| &\leq \mu|\zeta| \sum_{n=0}^{\infty} |\alpha_n| |\zeta|^n \leq \mu|\zeta| \sum_{n=0}^{\infty} \left(\frac{M|\zeta|}{r} \right)^n = \frac{\mu|\zeta|}{1 - \frac{M|\zeta|}{r}} \\
 |h_t^t| &\leq \sum_{n=0}^{\infty} |\beta_n| |\zeta|^n \leq \sum_{n=0}^{\infty} \left(\frac{M|\zeta|}{r} \right)^n = \frac{1}{1 - \frac{M|\zeta|}{r}} \\
 |h_y^y| &\leq \sum_{n=0}^{\infty} |\gamma_n| |\zeta|^n \leq \sum_{n=0}^{\infty} \left(\frac{M|\zeta|}{r} \right)^n = \frac{1}{1 - \frac{M|\zeta|}{r}}
 \end{aligned} \tag{B.19}$$

The series solution (3.42)–(3.44) is therefore convergent for $|\zeta| < \frac{r}{M}$ and is analytic there. Since the coefficients of the recursion formulas (B.1), (B.2) and (B.3) are all polynomials of q , eqs. (3.42)–(3.44) also define three analytic functions of q for ζ within the convergence circle.

Next we prove the absence of diverging solutions at AdS boundary, $u = 0$ ($\zeta = 1$). To see this, we write down the asymptotic form of the Einstein-Maxwell equations (3.38) near the boundary by retaining only the leading term of the power series of the coefficient functions C_i and D_i in u .

$$0 = h_t^{t'} + h_y^{y'} + p_1 u^2 a_t + p_2 u^2 h_t^t \tag{B.20}$$

$$0 = a_t'' + p_3 u^3 a_t' + p_4 h_t^{t'} + p_5 a_t + p_6 u h_t^t + p_7 u h_y^y \tag{B.21}$$

$$0 = h_y^{y''} - \frac{2}{u} h_y^{y'} + p_8 u^2 a_t' + p_9 u^2 h_t^{t'} + p_{10} h_t^t + p_{11} h_y^y \tag{B.22}$$

with $p_1, \dots, p_{11}$ numerical constants. The terms dropped in (B.22) will not contribute to the analysis below. Substituting the power series solution

$$\begin{aligned}
 a_t &= a_0 + a_1 u + a_2 u^2 + a_3 u^3 + \dots \\
 h_t^t &= b_0 + b_1 u + b_2 u^2 + b_3 u^3 + \dots \\
 h_y^y &= c_0 + c_1 u + c_2 u^2 + c_3 u^3 + \dots
 \end{aligned} \tag{B.23}$$

into (B.20), (B.21) and (B.22), we obtain

$$\begin{aligned}
 0 &= b_1 + c_1 \\
 0 &= b_2 + c_2 \\
 0 &= 3(b_3 + c_3) + p_1 a_0 + p_2 b_0
 \end{aligned} \tag{B.24}$$

from (B.20); and

$$\begin{aligned}
 0 &= 2a_2 + p_4 b_1 + p_5 a_0 \\
 0 &= 6a_3 + 2p_4 b_2 + p_5 a_1 + p_6 b_0 + p_7 c_0
 \end{aligned} \tag{B.25}$$

from (B.21), and

$$\begin{aligned}
 0 &= c_1 \\
 0 &= -2c_2 + p_{10} b_0 + p_{11} c_0 \\
 0 &= p_{10} b_1 + p_{11} c_1
 \end{aligned} \tag{B.26}$$

from (B.22). Consequently, we find that

$$\begin{aligned}
b_1 &= c_1 = 0 \\
a_2 &= -\frac{1}{2}p_5a_0 \\
c_2 &= \frac{1}{2}(p_{10}b_0 + p_{11}c_0) \\
b_2 &= -c_2 = -\frac{1}{2}(p_{10}b_0 + p_{11}c_0) \\
a_3 &= \frac{1}{6}[(p_4p_{10} - p_6)b_0 + (p_4p_{11} - p_7)c_0 + p_5a_1] \\
c_3 &= -b_3 - \frac{1}{3}(p_1a_0 + p_2b_0)
\end{aligned} \tag{B.27}$$

Iteratively, the five free parameters on r.h.s., a_0 , a_1 , b_0 , c_0 and b_3 , covers all solutions near the boundary. None of them and their derivatives diverge at $u = 0$. It follows that the series solution (3.42)–(3.44) or the analytic continuation in (3.53) and its derivative with respect to u , remains convergent on the AdS boundary, and therefore are analytic functions of q at $u = 0$ following the Weierstrass theorem.

C Validity domain for the WKB approximation

Consider a Schrödinger-like equation

$$\frac{d^2\Psi}{du^2} - V\Psi = 0 \tag{C.1}$$

where the complex potential is parametrized as

$$V(u) = (\lambda a + b)^2 + c \tag{C.2}$$

where $\lambda \gg 1$, (a, b) are functions of u , independent of λ , and c is a function of u and λ but is of the $O(1)$ in λ . To the order of approximation made in this work, the function c may be dropped.

On writing

$$\Psi(u) = e^{S(u)} \tag{C.3}$$

the differential equation satisfied by $S(u)$ reads

$$\frac{d^2S}{du^2} + \left(\frac{dS}{du}\right)^2 - (\lambda a + b)^2 - c = 0 \tag{C.4}$$

Let

$$S = \lambda S_0 + S_1 + O\left(\frac{1}{\lambda}\right) + \dots \tag{C.5}$$

the eq. (C.4) takes the form

$$\lambda^2 \left[\left(\frac{dS_0}{du}\right)^2 - a^2 \right] + \lambda \left(\frac{d^2S_0}{du^2} - 2i \frac{dS_0}{du} \frac{dS_1}{du} + 2iab \right) = O(1) \tag{C.6}$$

upto $O(1)$ in λ , which boils down to the following equations:

$$\begin{aligned} \left(\frac{dS_0}{du}\right)^2 - a^2 &= 0 \\ i\frac{d^2S_0}{du^2} + 2\frac{dS_0}{du}\frac{dS_1}{du} - 2ab &= 0 \end{aligned} \quad (\text{C.7})$$

It follows that

$$\begin{aligned} \frac{dS_0}{du} &= \pm a \\ \frac{dS_1}{du} &= -\frac{1}{2}\frac{d}{du}\ln a + b \end{aligned} \quad (\text{C.8})$$

The WKB solution is then

$$\Psi(u) \simeq \frac{1}{\sqrt{a(u)}} e^{\pm \int_0^u du' [\lambda a(u') + b(u')]} \quad (\text{C.9})$$

and its difference from the exact solution is $O(1/\lambda)$.

In this work, λ corresponds to the imaginary part of the momentum. It is well-known that the WKB approximation fails near the zero of the potential (turning point), which does not happen here. The approximation also breaks down near the singularity of the potential where the $O(\lambda)$ terms in (C.4) diverges faster than $O(\lambda^2)$ terms. Therefore the validity of the WKB requires that $\lambda a \gg \frac{da}{du}$, which, in our case with $a = \frac{1}{\sqrt{f}}$, excludes the region arbitrarily close to the horizon.

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