# Equivalence of two contour prescriptions in superstring perturbation theory

**Ashoke Sen**

*Harish-Chandra Research Institute,*
*Chhatnag Road, Jhusi, Allahabad 211019, India*
*Homi Bhabha National Institute,*
*Training School Complex, Anushakti Nagar, Mumbai 400085, India*
*School of Physics, Korea Institute for Advanced Study,*
*Seoul 130-722, Korea*

*E-mail:* sen@mri.ernet.in

ABSTRACT: Conventional superstring perturbation theory based on the world-sheet approach gives divergent results for the S-matrix whenever the total center of mass energy of the incoming particles exceeds the threshold of production of any final state consistent with conservation laws. Two systematic approaches have been suggested for dealing with this difficulty. The first one involves deforming the integration cycles over the moduli space of punctured Riemann surfaces into complexified moduli space. The second one treats the amplitude as a sum of superstring field theory Feynman diagrams and deforms the integration contours over loop energies of the Feynman diagram into the complex plane. In this paper we establish the equivalence of the two prescriptions to all orders in perturbation theory. Since the second approach is known to lead to unitary amplitudes, this establishes the consistency of the first prescription with unitarity.

# Contents

## 1 Introduction

The Polyakov formalism gives an elegant description of closed string scattering amplitudes by expressing the $g$-loop, $N$-point scattering amplitude as a single term, given by an integral over the moduli space of genus $g$ Riemann surfaces with $N$ punctures, with integrand given by a correlation function of the vertex operators of external states, ghost fields and picture changing operators (for heterotic and type II strings) on the Riemann surface. However while actually computing the integral over the moduli one runs into difficulties: in most physical situations the result diverges. The divergence occurs whenever the total center of mass energy of the external states is above the threshold of production of any final state [1–3]. Since the initial state can always appear as the final state, this is true essentially in all cases with a few exceptions. One such exception is one loop two point function of stable string states needed for computing one loop renormalized mass.

There is actually a physical reason behind this. The naive expression based on Polyakov formalism is always real. But in any sensible quantum field theory we expect the amplitudes to possess an imaginary part so as to be consistent with unitarity. Therefore in order that such an imaginary part appears, the original amplitude must diverge, and must be defined by some sort of analytic continuation that could generate an imaginary part. Such analytic continuation has been carried out successfully for one loop four point function of massless fields [4–6]. This requires breaking up the original integral into several parts and analytic continuation of each part independently.

A general procedure for dealing with this problem was suggested in [7, 8]. In this approach one complexifies the moduli space of Riemann surfaces with punctures, and deforms the integration over the modular parameters into the complexified moduli space near the boundary of the moduli space following a suitable prescription. This in principle gives a procedure for extracting finite answers for the Polyakov amplitudes. Furthermore, due to the deformation of the integration contour into the complexified moduli space, the amplitude develops an imaginary part.

There is a second proposal for dealing with this problem based on superstring field theory [9]. In this approach one represents Polyakov amplitude as a sum of Feynman diagrams of superstring field theory, with each Feynman diagram given by usual integrals over loop momenta. The integrals over the spatial components of loop momenta are taken

to be along the real axes as usual, but integration over loop energies are chosen to be along specific contours in the complex plane, beginning at $-i\infty$ and ending at $i\infty$. This gives results free from divergences, but has imaginary part due to the loop energy integrals. The added advantage of this prescription is that this has been shown to give unitary S-matrix [9–11].

Our goal in the paper will be to show that the results obtained by these two apparently different procedures are the same. For one loop two point function this was shown in [12], but our result holds for all amplitudes to arbitrary order in perturbation theory. The unitarity of the S-matrix computed in the second approach then automatically establishes unitarity of the S-matrix computed in the first approach.

In section 2 we give a detailed description of these two prescriptions for getting finite results for string theory amplitudes. Section 3 proves the equivalence of these two procedures.

## 2    The two contour integration prescriptions

Even though ultimately we shall be interested in the computation of on-shell amplitudes, it will be useful to begin by discussing properties of general off-shell amplitudes using the language of superstring field theory. First consider an off-shell $n$-point interaction vertex of superstring field theory with external legs of masses $m_1, \cdots m_n$ and momenta $k_1, \cdots k_n$. The general form of the vertex is of the form

$$\int [dy] \exp\left[ -\sum_{i,j} g_{ij}(y) k_i \cdot k_j \right] P(y, \{k_i\}), \qquad (2.1)$$

where $y$ denote collectively the moduli of a Riemann surface with $n$ punctures, the integration runs over a region of the moduli space that does not include any degenerate Riemann surface, $g_{ij}(y)$ is some function of the moduli and $P(y, \{k_i\})$ is a function of the moduli and polynomial in the $k_i$'s and depends also on the quantum numbers of the external states of the vertex. The integrand in (2.1) is the correlation function of off-shell vertex operators inserted at the punctures using specific local coordinates appropriate to a particular version of the superstring field theory. $g_{ij}(y)$ can be made positive definite by 'adding stubs' to the vertices [13, 14] — the effect of which is to rescale the local coordinates at the punctures[1] by some number $\{\lambda_i(y)\}$ and consequently scale the correlation function by factors of $\prod_i (\lambda_i(y))^{k_i^2 + m_i^2}$. By taking $\lambda_i(y)$ to be sufficiently small we can add arbitrarily large diagonal components to $g_{ij}(y)$. We can now compute contributions from Feynman diagrams using these vertices. The propagator has the standard form $(k_i^2 + m_i^2)^{-1}$ possibly multiplied by some polynomial in $k_i$. If we denote by $\{\ell_s\}$ the independent loop momenta, by $\{p_\alpha\}$ the external momenta and by $\{k_i\}$ the momenta carried by individual internal propagators, given by linear combinations of $\{\ell_s\}$ and $\{p_\alpha\}$, then the contribution to a

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[1]This will have to be accompanied by a change in the region of integration of $y$ for higher order vertices.

Feynman diagram takes the general form

$$\int [dY] \int \prod_s d^D \ell_s \, \exp\left[-G_{rs}(Y)\, \ell_r \cdot \ell_s - 2H_{s\alpha}(Y)\, \ell_s \cdot p_\alpha - K_{\alpha\beta}(Y)\, p_\alpha \cdot p_\beta\right]$$
$$\times \prod_i (k_i^2 + m_i^2)^{-1}\, Q(Y, \ell, p)\,, \tag{2.2}$$

where $Y$ denotes collectively all the moduli from all the vertices, and $G_{rs}$, $H_{s\alpha}$ and $K_{\alpha\beta}$ are matrices that arise by combining the exponential factors (2.1) from all the vertices after expressing the momenta $k_i$ carried by various propagators in terms of independent loop momenta and external momenta. $Q(Y, \ell, p)$ is a function of the moduli $Y$ and a polynomial in the $\ell_i$'s and $p_\alpha$'s, arising from the products of the factors of $P$ from each vertex and the numerator factors in various propagators. Positive definiteness of $g_{ij}(y)$ in (2.1) ensures that the matrix $\begin{pmatrix} G & H \\ H^T & K \end{pmatrix}$ is positive definite and hence $G$ and $K$ themselves are positive definite. As we shall discuss shortly, (2.2) is ill defined at this stage since the integral over loop energies diverge if we take them to be along the real axes.

Before going on we note that adding stubs to vertices serves another purpose. By multiplying each vertex by a factor of $(\lambda_i(y))^{m_i^2}$ for small positive constants $\lambda_i(y)$, it suppresses contribution to the vertex from massive string states and makes the sum over intermediate states, whose number at some mass level $m$ grows as $e^{cm}$ for some positive constant $c$, converge. For this reason we shall not worry about the convergence of the sum over intermediate states and focus on the possible divergences in the contribution from fixed intermediate states.

The following formal manipulation converts (2.2) into the usual expression for amplitudes as integrals over the moduli space of Riemann surfaces. We first replace each propagator by

$$(k_i^2 + m_i^2)^{-1} = \int_0^\infty dt_i \, \exp[-t_i(k_i^2 + m_i^2)]\,, \tag{2.3}$$

ignoring the fact that $k_i^2 + m_i^2$ may have negative real part and hence the above relation may not be valid. For each propagator we also introduce an angular variable $\theta_i$ whose integral imposes the $L_0 = \bar{L}_0$ constraint that the propagating states must satisfy.[2] In the second step we carry out integration over all loop momenta by gaussian integration rules, pretending that the integrals converge even though the integral over loop energies may not converge. This leads to an expression of the form

$$\int dY \prod_j \int_0^\infty dt_j \int_0^{2\pi} d\theta_j \, \mathcal{F}(Y, \{t_i\}, \{\theta_i\}, \{p_\alpha\})\,, \tag{2.4}$$

for some function $\mathcal{F}$. Together the integral over $Y$, $\{\theta_i\}$ and $\{t_i\}$ can be identified as integration over the moduli space of Riemann surfaces. A given Feynman diagram of

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[2]Superstring field theory actually only has states with $L_0 = \bar{L}_0$ as propagating states, but in order to make contact with moduli space integral representation we need to temporarily relax the constraint and allow other states in the conformal field theory, not satisfying this constraint, to propagate. We compensate for it by introducing the projection operator $(2\pi)^{-1} \int_0^{2\pi} d\theta_i e^{i\theta_i(L_0 - \bar{L}_0)}$.

course covers only part of the full moduli space, but when we add the contribution from all Feynman diagrams we recover the integral over the full moduli space. For our analysis it will be sufficient to work with an expression where integration over the angular variables $\theta_i$ have already been performed, yielding

$$\int dY \prod_j \int_0^\infty dt_j \, F(Y, \{t_i\}, \{p_\alpha\}), \tag{2.5}$$

where $F$ is obtained from $\mathcal{F}$ after performing integration over the $\theta_j$'s.

Let us now examine the original expression (2.2) in little more detail. If we regard the integration over all loop momenta to be running along the real axis then this integral diverges due to the following reason. The $\ell_i^0$ dependent quadratic term in the exponent is given by $\exp[G_{rs}(Y) \, \ell_s^0 \, \ell_r^0]$. Since $G_{rs}$ is positive definite, this diverges exponentially for large $\ell_s^0$, making the integral divergent. There are also additional divergences from the subspaces on which one or more of the $k_i^2 + m_i^2$ factor vanishes. A different problem exists in (2.5) in that the integrals over $t_i$'s typically diverge for large $t_i$. This reflects that the replacement (2.3) may not be a valid one due to the real part of $k_i^2 + m_i^2$ being negative.

There are two approaches to this problem which we now review. In both approaches we shall multiply all the external energies by a common complex number $u$ and define the integral as the limit $u \to 1$ *from the first quadrant*.[3] Therefore if the physical external momenta of the $\alpha$-th particle is $(E_\alpha, \vec{p}_\alpha)$, we set $p_\alpha = (uE_\alpha, \vec{p}_\alpha)$. Since all the $E_\alpha$'s are non-zero in a physical situation, the $p_\alpha^0$'s are also non-zero as long as $u \neq 0$.

1. *Moduli space contour integral:* in this approach we replace the integral (2.5) by [7, 8]

$$\int dY \prod_j \int_0^{i\infty} dt_j \, F(Y, \{t_i\}, \{p_\alpha\}). \tag{2.6}$$

We shall argue in section 3 that this gives a finite result for all $u$ in the first quadrant. This will be called the moduli space contour integral representation since from the point of view of the world-sheet theory, $t_i$ are part of the coordinates of the moduli space of Riemann surfaces. For given $\{(E_\alpha, \vec{p}_\alpha)\}$ we shall denote the result of this computation for general $u$ by the function $F_{\text{moduli}}(u)$.

In the original prescription of [8] the integration contour over $t_j$ was taken to be along the real axis up to some constant value $t_0$ and then turned towards $t_0 + i\infty$. We can absorb the integration up to $t_0$ into the definition of the vertices by extending the region of integration over $y$ in (2.1) by adding stubs to lower order vertices. Therefore we do not suffer from any loss of generality in taking the $t_j$ integral from 0 to $i\infty$. The prescription of [8] also had a damping factor $e^{i\epsilon t_j}$ in the integrand for each $t_j$. As we shall see, we can dispense with this damping factor as long as we define the amplitude as a result of taking $u \to 1$ limit from the first quadrant.

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[3]In our notation first quadrant will not include the real axis where the amplitude is expected to encounter poles and branch cuts and have to be defined as the $u \to 1$ limit of the amplitude in the first quadrant.

2. *Loop energy contour integral:* in this approach we work directly with the momentum space integrals (2.2) without using the Schwinger parameter representation (2.3). We let the contours of integration over loop energies begin at $-i\infty$ and end at $i\infty$ in order to make the integral converge. In the interior of the complex plane the integration contours are chosen as follows [9]. If we take $u$ to be on the imaginary axis and take the loop energy integration contours to be along the imaginary axis then the energies carried by all the internal lines are imaginary and therefore the $(k_i^2 + m_i^2)^{-1}$ factors in (2.2) do not have any poles on the integration contours. Therefore for this configuration the position of each pole relative to the loop energy integration contours is well defined. As we deform $u$ towards 1, the integration contours must be deformed to maintain this relative position, i.e. no pole should cross the integration contour during this deformation. It was shown in [9] that this is always possible, i.e. we never encounter a situation where during the deformation two poles approach each other from opposite sides of the contour preventing us from deforming the contour away from the poles.

In the following we shall use a slight variation of this prescription where we take the ends of the loop energy integration contours to approach $\pm e^{i\phi_0}\infty$, for some fixed angle $\phi_0$ in the range

$$\pi/4 < \phi_0 < \pi/2\,. \tag{2.7}$$

$(\ell_i^0)^2$ will still have negative real part for $\ell_i^0 \to \pm e^{i\phi_0}\infty$, making the integrals convergent. To fix the positions of the poles relative to the contour, we can start with $u = re^{i\phi_0}$ for $0 < r < \infty$. In this case, if we take the loop energy integrals to lie along the axis $\ell_i^0 = |\ell_i^0|e^{i\phi_0}$, then the energies carried by all the propagators have phase proportional to $e^{i\phi_0}$ and $k_i^2 + m_i^2$ is non-zero on the contour. As we now deform $u$ to 1, we deform the contours accordingly with the ends fixed at $\pm e^{i\phi_0}\infty$ so that the poles lie on same side of the integration contour as for $u = re^{i\phi_0}$.

Since the integral converges for all $\phi_0$ in the range (2.7), we can deform $\phi_0$ to $\pi/2$ without affecting the result of the integral. Therefore the new contours give the same result as the old contours corresponding to the choice $\phi_0 = \pi/2$.

The above prescription can be reexpressed by saying that we define the amplitude at $u = 1$ by analytic continuation from the line $u = re^{i\phi_0}$ for $0 < r < \infty$. Since during this deformation we never encounter a situation where the momentum integration contours are pinched by poles approaching each other from opposite sides of the contour [9], the integral has no singularity in the first quadrant and gives an explicit representation of the analytically continued function. For fixed $\{(E_\alpha, \vec{p}_\alpha)\}$ we shall denote the result of this computation for general $u$ by the function $F_{\text{energy}}(u)$.

Each of these prescriptions has its own advantages. The moduli space contour integral prescription has the advantage of preserving the original form of the Polyakov amplitude except for the deformation of the integration contour over the moduli into the complexified moduli space. The approach based on loop energy contour integral has the advantage of being formulated in the language of superstring field theory which is needed for addressing

issues of mass renormalization and shift of vacuum under quantum correction when they occur. Furthermore this approach leads to a proof of unitarity of perturbative superstring theory. Clearly it will be beneficial to determine if these two approaches are equivalent. This is the question to which we now turn.

## 3   Equivalence of the two prescriptions

Our goal will be to show that the two prescriptions give the same result. We shall show this by establishing the following two properties of $F_{\text{moduli}}(u)$:

1. The integration over the $t_j$'s in (2.6) converges for all $u$ in the first quadrant. This shows that $F_{\text{moduli}}(u)$ given by (2.6) is an analytic function of $u$ in the first quadrant.

2. When $u = re^{i\phi_0}$, $\phi_0$ being the same angle in the range (2.7) that was used in defining the loop energy contour integral prescription, $F_{\text{moduli}}(u)$ and $F_{\text{energy}}(u)$ coincide.

$F_{\text{energy}}(u)$ was already shown to be an analytic function of $u$ in the first quadrant of the complex $u$ plane [9]. Since both $F_{\text{moduli}}(u)$ and $F_{\text{energy}}(u)$ are analytic functions of $u$ in the first quadrant and coincide on the $u = re^{i\phi_0}$ line, they must agree everywhere in the first quadrant and in particular in the $u \to 1$ limit. This is the desired result.

Therefore it remains to prove the two assertions made above. For this analysis, while defining $F_{\text{moduli}}(u)$ we shall skip the steps involved in eqs. (2.3)–(2.6) by directly making the replacement

$$(k_i^2 + m_i^2)^{-1} = \int_0^{i\infty} dt_i e^{-t_i(k_i^2 + m_i^2)} = \int_0^{i\infty} dt_i e^{t_i (k_i^0)^2 - t_i (\vec{k}_i^2 + m_i^2)}, \qquad (3.1)$$

and then carrying out integration over loop momenta by treating them as gaussian integration. The main reason for going through step (2.3) was to display the relation to the conventional expression (2.4) for the amplitudes as integrals over the moduli space of Riemann surfaces, but this will not play any role in the analysis below since we shall be working with the deformed integration contours. Substituting (3.1) into (2.2) we get

$$\int [dY] \prod_j \int_0^{i\infty} dt_j \int \prod_s d^D \ell_s \left[ Q(Y, \ell, p) \right.$$

$$\times \exp\left\{ -G_{rs}(Y)\ell_r \cdot \ell_s - 2H_{s\alpha}(Y)\ell_s \cdot p_\alpha - K_{\alpha\beta}(Y)p_\alpha \cdot p_\beta - \sum_i t_i(k_i^2 + m_i^2) \right\} \right]. \quad (3.2)$$

The moduli space contour integral prescription tells us that we should now carry out the integration over the $\ell_s$'s formally regarding them as Gaussian integrals (after rotating $\ell_s^0$ to $i\ell_s^E$). This gives an expression for $F_{\text{moduli}}(u)$ of the form

$$F_{\text{moduli}}(u) = \int [dY] \prod_j \int_0^{i\infty} dt_j \, R(Y, p, t) \, \exp\left[ -J_{\alpha\beta}(Y, t)p_\alpha \cdot p_\beta - \sum_i t_i m_i^2 \right]. \quad (3.3)$$

The exponent $J_{\alpha\beta}$ is obtained by 'completing the squares' in the exponent of (3.2) after expressing $k_i$'s in terms of $\{\ell_s\}$ and $\{p_\alpha\}$ and then picking up the left-over $\ell_s$ independent

terms. $R(Y, p, t)$ is determined by the $Q(Y, \ell, p)$ factor and is a polynomial in the $p_\alpha$'s and rational function of the $t_i$'s.

The second assertion made at the beginning of this section is easier to prove; so we shall begin with this one. We start with $F_{\text{energy}}(u)$. As mentioned below (2.7), for $u = re^{i\phi_0}$, in the computation of $F_{\text{energy}}(u)$ we can take all the loop energies to have phase $e^{i\phi_0}$. In this case $(k_i^0)^2$ has phase $e^{2i\phi_0}$ for each propagator making each propagator non-singular. The strategy will be to show that the formal steps used in going from $F_{\text{energy}}(u)$ to $F_{\text{moduli}}(u)$ now hold as true identities. We shall begin with (3.1). First note that

$$t_i (k_i^0)^2 = i e^{2i\phi_0} |t_i| \, |(k_i^0)^2| \tag{3.4}$$

has negative real part proportional to $-\sin(2\phi_0)$ for $\phi_0$ in the range (2.7). On the other hand $-t_i(\vec{k}_i^2 + m_i^2)$ is purely imaginary and therefore just contributes a phase to the integrand of (3.1). Therefore the integration over $t_i$ in (3.1) converges and the use of (3.1) is justified. The only case where the integral may fail to converge is if $k_i^0 = 0$. But these are subspaces of measure zero in the loop energy integration space, and except in low dimensions where infrared divergences are severe, the contribution from such subspaces can be ignored.

After the replacement (3.1) the loop energy integration of (3.2), taken along the constant phase lines with phase $e^{i\phi_0}$ with $\phi_0$ in the range (2.7), also converges since the $G_{rs}(Y)\ell_r^0\ell_s^0$ term in the exponent has negative real part proportional to $\cos(2\phi_0)$ due to positive definiteness of $G_{rs}(Y)$ and the $t_i(k_i^0)^2$ factors in the exponent have negative real part proportional to $-\sin(2\phi_0)$. Integration over spatial components of loop momenta converges due to the factor of $e^{-G_{rs}(Y)\vec{\ell}_r \cdot \vec{\ell}_s}$, the $e^{-t_i \vec{k}_i^2}$ factors giving pure phase. Therefore we can carry out the integration over loop momenta treating them as gaussian integrals, leading to the expression for $F_{\text{moduli}}(u)$ given in (3.3). This shows that for $u = re^{i\phi_0}$, $F_{\text{moduli}}(u)$ and $F_{\text{energy}}(u)$ agree for all $r$ in the range $0 < r < \infty$.

We now turn to the first assertion, i.e. showing that $F_{\text{moduli}}(u)$ given in (3.3) is an analytic function of $u$ in the first quadrant. For this we need to show that the integrations over $\{t_i\}$ in the expression (3.3) converge as long as $u$ lies in the first quadrant. This in turn requires determining the behavior of $J_{\alpha\beta}$ in the limit when some of the $|t_i|$'s become large. When all $|t_i|$ are large, say $t_i = i\lambda v_i$ with large positive $\lambda$ and real positive $v_i$, then the effect of the $G_{rs}$, $H_{s\alpha}$ and $K_{\alpha\beta}$ terms in the exponent of (3.2) can be neglected. In this limit $\lambda$ can be absorbed into a rescaling of $\{p_\alpha\}$ and the integration variables $\{\ell_s\}$, and we have

$$J_{\alpha\beta}(Y, t) \simeq i\lambda \, L_{\alpha\beta}(\{v_i\}). \tag{3.5}$$

We shall now argue that the matrix $L_{\alpha\beta}$ is positive semi-definite. For this we note that if we consider the integration over euclidean momenta then the same analysis that leads us from (3.2) to (3.3), (3.5) tells us that for a real positive parameter $\sigma$

$$\int \prod_{s=1}^N d^{D-1}\ell_s \exp\left[-\sigma \sum_i v_i \vec{k}_i^2\right] \propto \sigma^{-N(D-1)/2} \exp[-\sigma L_{\alpha\beta}(\{v_i\})\vec{p}_\alpha \cdot \vec{p}_\beta]. \tag{3.6}$$

This result is no longer formal, it holds as a true identity. Now the integrand on the left hand side is real and is a monotonically decreasing function of $\sigma$ for all $\{\vec{p}_\alpha\}$. Therefore

the right hand side must also be a monotonically decreasing function of $\sigma$ for all $\{\vec{p}_\alpha\}$. This shows that $L_{\alpha\beta}\,\vec{p}_\alpha \cdot \vec{p}_\beta \geq 0$ for any $\{\vec{p}_\alpha\}$, i.e. $L_{\alpha\beta}$ is a positive semi-definite matrix. Substituting (3.5) into (3.3) we can now see that for generic $\{E_\alpha\}$ the $J_{\alpha\beta}(Y,t)\,p_\alpha^0 p_\beta^0 \simeq i\,\lambda\,u^2\,L_{\alpha\beta}\,E_\alpha E_\beta$ term has negative real part proportional to $-\sin(2\,\text{phase}(u))$ in the range $0 < \text{phase}(u) < \pi/2$. Rest of the terms in the exponent of (3.3) are either purely imaginary or subleading in the large $\lambda$ limit. Therefore the integrand is exponentially suppressed in this limit and the integral converges.

Next we have to consider the case where a subset of the $t_i$'s become large keeping the others finite. First suppose that the $k_i$'s corresponding to these $t_i$'s can be taken as independent loop momenta, i.e. there is no relation between these $k_i$'s. In that case in the $t_i \to \infty$ limit, the (formal) integration over these $k_i$'s in (3.2) using the rules of gaussian integration effectively sets them to zero. The remaining integrand is then independent of these large $t_i$'s except for the oscillatory factors proportional to $e^{-t_i m_i^2}$, $t_i^{-D/2}$ multiplicative factors coming from the integration over the $k_i$'s and possibly further negative powers of $t_i$ due to the polynomials in the $k_i$'s coming from the $Q(Y,\ell,p)$ factor in the integrand of (3.2). After the (formal) integration over the rest of the loop momenta are performed, we get a finite $J_{\alpha\beta}$ independent of the large $t_i$'s. Therefore the $-J_{\alpha\beta}\,p_\alpha \cdot p_\beta$ term in the exponent of (3.3) does not provide any suppression factor. However the $t_i^{-D/2}$ factors suppress the integrand in the large $t_i$ limit and give a finite integral for $D > 2$. For $D \leq 2$ we anyway expect infrared divergences in the presence of massless states since a single massless propagator can give divergent contributions of the form $\int d^D k/k^2$. Therefore we shall not worry about this case.

Let us now consider the case where a subset of the $t_i$'s become large and the corresponding $k_i$'s satisfy one of more constraints relating linear combinations of these $k_i$'s to linear combinations of some external momenta. Let us denote by $\hat{t}_a$ the $t_i$'s which become large and by $\tilde{t}_m$ the $t_i$'s which remain finite. We now parametrize the $\hat{t}_a$'s as $\hat{t}_a = i\,\lambda\,\hat{v}_a$ where $\lambda$ is large and positive and $\hat{v}_a$'s are finite and positive. Furthermore let us denote by $\{P_s\}$ the linear combinations of the $\{p_\alpha\}$'s which enter the constraints involving the $k_i$'s whose Schwinger parameters are large. In this case we can first carry out the formal integration over the $k_i$'s associated with the large $t_i$'s subject to these constraints and generate a term in the exponent of the form $-i\,\lambda\,\widehat{L}_{st}(\{\hat{v}_a\})P_s \cdot P_t$ in addition to a finite term, and then carry out formal integration over the rest of the loop momenta, generating a finite term in the exponent. Following the same logic as the one given before one can argue that $\widehat{L}$ is positive semi-definite. The previous arguments can now be repeated to show that for generic $\{E_\alpha\}$, $i\,\lambda\,\widehat{L}_{st}\,P_s^0 P_t^0$ has negative real part proportional to $-\sin(2\,\text{Phase}(u))$, making the integration over $\hat{t}_a$'s converge.

This establishes that $F_{\text{moduli}}(u)$ is an analytic function of $u$ in the first quadrant. As argued before, this in turn proves that $F_{\text{moduli}}(u) = F_{\text{energy}}(u)$.

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