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Wilson lines, Grassmannians and gauge invariant off-shell amplitudes in $\mathcal{N} = 4$ SYM

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ABSTRACT: In this paper we consider tree-level gauge invariant off-shell amplitudes (Wilson line form factors) in $\mathcal{N} = 4$ SYM. For the off-shell amplitudes with one leg off-shell we present a conjecture for their Grassmannian integral representation in spinor helicity, twistor and momentum twistor parameterizations. The presented conjecture is successfully checked against BCFW results for MHV_n, NMHV₄ and NMHV₅ off-shell amplitudes. We have also verified that our Grassmannian integral representation correctly reproduces soft (on-shell) limit for the off-shell gluon momentum. It is shown that the (deformed) off-shell amplitude expressions could be also obtained using quantum inverse scattering method for auxiliary $\mathfrak{gl}(4|4)$ super spin chain.

KEYWORDS: Scattering Amplitudes, Extended Supersymmetry, Integrable Field Theories

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1 Introduction

$\mathcal{N} = 4$ SYM theory is an excellent playground for testing new computational methods for $D = 4$ dimensional gauge theories. In the last decade we have witnessed a serious progress in understanding the structure of amplitudes (S-matrix) in $\mathcal{N} = 4$ SYM as well as in other gauge theories (see for a review [1–5]). The main role in these achievements was played by a number of so called unitarity methods based on the exploration of the amplitude analytical structure [1–5]. The latter are given for example by BCFW recursion [6, 7] for tree amplitudes and generalized unitarity (see [1–4] and references therein) for loop amplitudes. We should especially mention the use of new variables, such as helicity spinors (see appendix A for details) and momentum twistors [8], together with superspace methods [9, 10] (see also recent review [11]). The application of these new methods provided us with the explicit answers for amplitudes both at high orders of perturbation theory and/or with large number of external legs (see [1–5] for a review and reference therein), which in their

turn lead to several important all-loop results as well as to the discovery of underlying integrable structure behind amplitudes of $\mathcal{N} = 4$ SYM [12–21].

Another novel way of studying scattering amplitudes is based on their Grassmannian integral representation [22–27]. It naturally unifies different BCFW representations for tree level amplitudes and loop level integrands [22, 23]. Moreover, it shed light on the integrable structure behind $\mathcal{N} = 4$ SYM amplitudes (S-matrix) [17, 28, 29]. This representation also naturally relates perturbative $\mathcal{N} = 4$ SYM and twistor string theories amplitudes [26]. In addition a possible geometrical interpretation of $\mathcal{N} = 4$ SYM (so called Amplituhedron) was discovered within Grassmannian picture [8, 30–36].

The unitary based methods were applied not only to study on-shell amplitudes, but also for partially off-shell objects, such as form factors in $\mathcal{N} = 4$ SYM theory [37–60]. The form factors are the matrix elements of the form¹

$$\langle p_1^{\lambda_1}, \dots, p_n^{\lambda_n} | \mathcal{O} | 0 \rangle, \quad (1.1)$$

where $\mathcal{O}$ is some gauge invariant operator, which when acting on the vacuum state of the theory produces multi-particle state $\langle p_1^{\lambda_1}, \dots, p_n^{\lambda_n} |$ with momenta $p_1, \dots, p_n$ and helicities $\lambda_1, \dots, \lambda_n$. One can view form factors as amplitudes of the processes where classical field coupled through gauge invariant operator $\mathcal{O}$ produces quantum state. Grassmannian representation is no exception and can be applied to form factors as well [47, 61–63].

Another interesting off-shell objects are gauge invariant off-shell amplitudes [64–72] (reggeon amplitudes within the context of Lipatov’s effective lagrangian), which typically appear in k_T — or high-energy factorization approach [74–77] as well as in the study of processes at multi-regge kinematics. We should mention, that there are also other studies of off-shell currents and amplitudes [78–82]. (see also [83] for a review and references to original papers). However, the off-shell objects studied there are either gauge dependent [78–80] or lack Lorentz invariance [81, 82]. Usually within the context of application of unitarity based methods to form factors $\mathcal{O}$ is local gauge invariant color singlet operator, for example operators from stress-tensor operator supermultiplet [37–41, 84–87]. However in general one can consider also gauge invariant non-local operators, such as Wilson loops (lines). This brings us to the following important observation. We can formulate *gauge invariant off-shell amplitudes* in Yang-Mills theory in terms of form factors of Wilson line operators [64–72]. See also [73] for the discussion of amplitudes with off-shell fermions. An insertion of Wilson line operator plays the role of off-shell or reggeized gluon in such formulation. Keeping in mind an extremely important role played by Wilson loops in $\mathcal{N} = 4$ SYM theory it is very interesting to study the possibility of Grassmannian integral representation for gauge invariant off-shell amplitudes (Wilson line form factors). The aim of this article is to investigate this question in detail.

This paper is organized as follows. In section 2 we discuss the definition and kinematics of gauge invariant off-shell amplitudes in terms of form factors of Wilson line operators corresponding to off-shell gluons in $\mathcal{N} = 4$ SYM theory. Here we also derive an expression

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¹The on-shell amplitudes in “all ingoing” notation may be viewed as a particular case of form factors of unity operator $\langle p_1^{\lambda_1}, \dots, p_n^{\lambda_n} | 0 \rangle$.

for the “minimal” 2+1 point tree level off-shell amplitude, such that two on-shell particles are treated in manifestly supersymmetric manner (using on-shell momentum superspace), while the off-shell gluon is left unsupersymmetrized. In section 3 we use generalization of BCFW recursion for off-shell amplitudes [64] to derive some explicit answers for gauge invariant amplitudes with one leg off-shell in MHV and NMHV sectors. In section 4 after reminding the reader the basic facts about Grassmannian integral representation for $\mathcal{N} = 4$ SYM on-shell amplitudes (on-shell diagram formalism, et.c.) we present our conjecture for Grassmannian integral representation for all tree-level gauge invariant amplitudes with one leg off-shell (Wilson-line form factors). We present our conjecture using spinor helicity, twistor and momentum twistor variables. In section 5 we verify our conjecture for the Grassmannian integral representation by reproducing the results obtained in section 2 together with the appropriate soft (on-shell) limit for off-shell momentum. We also derive a conjecture for NMHV $_{n+1}$ off-shell tree level amplitudes in terms of $[abcde]$ momentum twistor invariants. In section 6 we discuss integrability properties of tree level gauge invariant off-shell amplitudes in $\mathcal{N} = 4$ SYM theory. We show that off-shell amplitudes with one leg off-shell (Wilson-line form factor) are eigenvectors of the transfer matrix of $\mathfrak{gl}(4|4)$ super spin chain. In conclusion we summarize all the obtained results and discuss open questions. The appendixes contain details about spinor helicity formalism, $\mathcal{N} = 4$ SYM theory and evaluation of Grassmannian integrals by residues.

2 Wilson lines and off-shell amplitudes in $\mathcal{N} = 4$ SYM

Before discussing gauge invariant off-shell amplitudes in the context of $\mathcal{N} = 4$ SYM let us recall the corresponding description in the context of pure Yang-Mills theory.

It is convenient to define off-shell amplitudes we are interested in to be color ordered. For example, any tree diagram for n -gluon scattering could be reduced to a sum of single trace terms. Indeed, if we eliminate structure constants f^{abc} in favor of $SU(N_c)$ -generators t^a , using²

$$f^{abc} = -\frac{i}{\sqrt{2}} \left\{ \text{tr} (t^a t^b t^c) - \text{tr} (t^b t^a t^c) \right\} \quad (2.1)$$

together with Fierz rearrangement

$$(t^a)_{i_1}^{j_1} (t^a)_{i_2}^{j_2} = \delta_{i_1}^{j_2} \delta_{i_2}^{j_1} - \frac{1}{N_c} \delta_{i_1}^{j_1} \delta_{i_2}^{j_2} \quad (2.2)$$

to reduce number of traces in each diagram, it is easy to rewrite n -gluon tree amplitude in a color decomposed form:

$$\mathcal{A}_n^{\text{tree}}(p_i, h_i, a_i) = g^{n-2} \sum_{\sigma \in S_n/Z_n} \text{tr} (t^{a_{\sigma(1)}} \dots t^{a_{\sigma(n)}}) A_n^{\text{tree}}(\sigma(1^{h_1}), \dots, \sigma(n^{h_n})). \quad (2.3)$$

Here g is the gauge coupling $\frac{g^2}{4\pi} = \alpha_s$, p_i , h_i are the gluon momenta and helicities. $A_n^{\text{tree}}(1^{h_1}, \dots, n^{h_n})$ are the partial amplitudes, which contain all the kinematic information. S_n is the set of all permutations of n objects, while Z_n is the subset of cyclic permutations.

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²We use normalization $\text{tr} t^a t^b = \delta^{ab}$, so that $[t^a, t^b] = i\sqrt{2} f^{abc} t^c$.

Next, the kinematics of scattering amplitudes involving off-shell gluons could be conveniently described using off-shell momentum decomposition typically employed within high-energy factorization or k_T -factorization approach [74–77]. That is, the off-shell gluon momentum is written as

$$k^\mu = xp^\mu + k_T^\mu, \quad (2.4)$$

where p is the gluon direction (also known as the off-shell gluon polarization vector), such that $p^2 = 0$, $p \cdot k = 0$ and $x \in [0, 1]$. It is clear, that there is a freedom in such decomposition, which is typically parametrized by an auxiliary light-like four-vector q^μ , so that

$$k_T^\mu(q) = k^\mu - x(q)p^\mu \quad \text{with} \quad x(q) = \frac{q \cdot k}{q \cdot p} \quad \text{and} \quad q^2 = 0. \quad (2.5)$$

Using the fact, that now k_T^μ is transverse both with respect to p^μ and q^μ we can write down off-shell gluon transverse momentum k_T^μ in the basis of two “polarization” vectors as [64]:

$$k_T^\mu(q) = -\frac{\kappa}{2} \frac{\langle p | \gamma^\mu | q \rangle}{[pq]} - \frac{\kappa^*}{2} \frac{\langle q | \gamma^\mu | p \rangle}{\langle qp \rangle} \quad \text{with} \quad \kappa = \frac{\langle q | \not{k} | p \rangle}{\langle qp \rangle}, \quad \kappa^* = \frac{\langle p | \not{k} | q \rangle}{[pq]}. \quad (2.6)$$

It is easy to see, that $k^2 = -\kappa\kappa^*$. Moreover, using Schouten identities it could be shown, that both κ and κ^* are independent of auxiliary four-vector q^μ [64]. The on-shell states are described by their on-shell momenta and polarization vectors as usual. Having said that, let us continue with the gauge invariance of the off-shell scattering amplitudes.

It is well known, that off-shell scattering amplitudes are gauge dependent in general. To insure gauge invariance one typically needs to add additional non-standard contributions, that is those, which are not calculable from standard QCD Feynman rules. The first rigorous consideration of the gauge-invariant off-shell scattering amplitudes to our knowledge was performed within the context of Lipatov’s effective lagrangian [65–67] used to describe the QCD high-energy scattering in multi-regge kinematics.³ Within Lipatov’s effective lagrangian approach an off-shell gluon with additional contributions is interpreted as an effective reggeized gluon. Recently, a new manifestly gauge invariant definition of scattering amplitudes with an arbitrary number of off-shell external gluons appeared in [70], where off-shell gluons are described in terms of infinite Wilson lines.⁴ For example, the gauge-invariant scattering amplitude with one leg off-shell is given by the following matrix element⁵ [70]

$$\mathcal{A}_{n+1}(1, \dots, n, (n+1)^*) = \langle k_1, \epsilon_1, c_1; \dots; k_n, \epsilon_n, c_n | \mathcal{W}_p^{c_{n+1}}(k) | 0 \rangle \quad (2.7)$$

of Wilson line operator⁶

$$\mathcal{W}_p^c(k) = \int d^4x e^{ix \cdot k} \text{Tr} \left\{ \frac{1}{\pi g} t^c \mathcal{P} \exp \left[\frac{ig}{\sqrt{2}} \int_{-\infty}^{\infty} ds \, p \cdot A_b(x + sp) t^b \right] \right\}. \quad (2.8)$$

³See also [68, 69] for earlier studies of effective action for high-energy QCD scattering.

⁴Earlier studies preceding this construction employed eikonal quark lines and Slavnov-Taylor identities [71, 72].

⁵The asterisk denotes an off-shell gluon.

⁶The color generators are normalized as $\text{Tr}(t^a t^b) = \delta^{ab}$.

Here p is the direction of the off-shell gluon, k is its off-shell momentum and c — color index. Similarly $|k_i, \varepsilon_i, c_i\rangle$ describes on-shell gluon state with momentum k_i , polarization vector ε_i and color index c_i . Colour ordered version of this object can be obtained via (2.3). The helicities of on-shell gluons are not shown. We want to stress, the Wilson line we consider to describe off-shell gluon is colored. It is invariant $\delta\mathcal{W}_p^c(k) = 0$ under infinitesimal gauge transformations $\delta A_\mu = [D_\mu, \chi]$ with χ decreasing at $x \rightarrow \infty$, but at the same time it transform in the adjoint representation of global $SU(N_c)$ at constant χ as [65, 66]:

$$\delta\mathcal{W}_p(k) = g[\mathcal{W}_p(k), \chi]. \quad (2.9)$$

In the case of $\mathcal{N} = 4$ super Yang-Mills theory we have both scalars and gluons in the adjoint representation of gauge group in addition to the gluons of pure Yang-Mills theory.⁷ To keep track of the components of the on-shell states it is highly convenient to introduce an on-shell superspace [9]. For each external on-shell leg we introduce four Grassmann variables $\tilde{\eta}_A^i$ labeled by the $SU(4)$ index $A = 1, 2, 3, 4$ and leg index i . This allows us to collect the 16 states (creation/annihilation operators) into $\mathcal{N} = 4$ on-shell chiral superfield

$$\Omega_i = g_i^+ + \tilde{\eta}_A^i \psi_i^A + \frac{1}{2!} \tilde{\eta}_A^i \tilde{\eta}_B^i \phi_i^{AB} + \frac{1}{3!} \tilde{\eta}_A^i \tilde{\eta}_B^i \tilde{\eta}_C^i \epsilon^{ABCD} \bar{\psi}_{i,D} + \frac{1}{4!} \tilde{\eta}_A^i \tilde{\eta}_B^i \tilde{\eta}_C^i \tilde{\eta}_D^i \epsilon^{ABCD} g_i^-, \quad (2.10)$$

where $g^\pm$ denote two physical polarizations of gluon, n -particle on-shell state is then given by $|\Omega_{1\dots n}\rangle = \prod_{i=1}^n \Omega_i |0\rangle$.

The Wilson line operator above could be also supersymmetrized. First of all, there is both chiral [88–90] and non-chiral [90] versions of smooth supersymmetric Maldacena-Wilson loops [91, 92]. The supersymmetric chiral version of Maldacena-Wilson loop is given by [90]:

$$\mathcal{W}_{\text{MW}} = \frac{1}{N_c} \text{Tr} \mathcal{P} \exp \left(- \oint_Z d\tau (p^\mu A_\mu + \dot{\theta}^\alpha A_\alpha + q^i \Phi_i) \right), \quad (2.11)$$

where the superpath $Z = \{x^\mu, \theta^\alpha, q^i\}$ and q^i is constrained by $q^2 = q^i q^i = p^\mu p_\mu$, ($\mu = 1, \dots, 4$, $i = 1, \dots, 6$), so that $q^i = \sqrt{p^\mu p_\mu} n^i$ with n^i being a unit vector. In our case $p^\mu p_\mu = 0$ and the contribution of scalars could be neglected. In this case Maldacena-Wilson loop turns into a light-like supersymmetric Wilson loop [93–95]. Its chiral version is written as follows [93, 94]:

$$\mathcal{W}_{LL} = \frac{1}{N_c} \text{Tr} \mathcal{P} \exp \left(- \int \mathcal{A} \right), \quad (2.12)$$

where

$$\begin{aligned} \mathcal{A} = \frac{dx}{2} & \left(A + \bar{\psi}_A \theta^A + \frac{1}{2} D \phi_{AB} \theta^A \theta^B - \frac{1}{3!} \varepsilon_{ABCD} D \psi^A \theta^B \theta^C \theta^D \right. \\ & \left. + \frac{1}{4!} \varepsilon_{ABCD} D F \theta^A \theta^B \theta^C \theta^D + \dots \right) \\ & + d\theta^A \left(-\frac{1}{2} \phi_{AB} \theta^B + \frac{1}{3} \varepsilon_{ABCD} \psi^B \theta^C \theta^D + \frac{1}{6} \varepsilon_{ABCD} F \theta^B \theta^C \theta^D + \dots \right). \end{aligned} \quad (2.13)$$

Here the summation over dotted and undotted Weyl indexes is assumed. The factor $1/2$ comes from the substitution $dx^\mu A_\mu \rightarrow \frac{dx_{\alpha\dot{\alpha}}}{2} A^{\alpha\dot{\alpha}}$.

⁷See appendix A for $\mathcal{N} = 4$ SYM lagrangian and field content.

Figure 1. Color ordered Feynman rules from Wilson lines expansion together with $\mathcal{N} = 4$ SYM propagators and 3-point vertexes. A coil denotes ordinary gluon, solid line denotes fermion and dashed line stands for scalar. The Feynman rules coming from Wilson line expansion could be easily understood using the representation of the latter as eikonal quark line [70]. To denote eikonal quark line we use a coil crossed by a line.

In the present paper we however decided to keep off-shell gluon and corresponding light-like Wilson line non-supersymmetric, while treating the on-shell states in a supersymmetric fashion. The explicit calculations of different 3-point off-shell component amplitudes (interactions of the off-shell gluon with on-shell gluons, fermions and scalars) with the use of Feynman rules in figure 1 showed that the 3-point amplitude with one off-shell gluon and two other supersymmetric on-shell states could be written in the following form:

$$\begin{aligned}
 A_{2,2+1}^*(1^*, 2, 3) &= \frac{1}{\kappa_1^*} \prod_{A=1}^4 \frac{\partial}{\partial \tilde{\eta}_p^A} \left[\frac{\delta^4(k + \lambda_2 \tilde{\lambda}_2 + \lambda_3 \tilde{\lambda}_3) \delta^8(\lambda_p \tilde{\eta}_p + \lambda_2 \tilde{\eta}_2 + \lambda_3 \tilde{\eta}_3)}{\langle p2 \rangle \langle 23 \rangle \langle 3p \rangle} \right] \\
 &= \frac{\delta^4(k + \lambda_2 \tilde{\lambda}_2 + \lambda_3 \tilde{\lambda}_3)}{\kappa_1^*} \frac{\hat{\delta}^4(\langle p2 \rangle \tilde{\eta}_2 + \langle p3 \rangle \tilde{\eta}_3)}{\langle p2 \rangle \langle 23 \rangle \langle 3p \rangle}.
 \end{aligned} \tag{2.14}$$

The momenta of on-shell states are given by $k_i = \lambda_i \tilde{\lambda}_i$, while for the momentum of the off-shell gluon denoted by k we use decomposition (2.4). The off-shell gluon direction p and axillary “momentum” (four-vector) q are given by $p = \lambda_p \tilde{\lambda}_p$ and $q = \lambda_q \tilde{\lambda}_q$ light like four-vectors. We will label corresponding spinors as $\lambda_i \equiv |i\rangle$, $\tilde{\lambda}_i \equiv [i|$, $\lambda_p \equiv |p\rangle$, $\tilde{\lambda}_p \equiv [p|$ and $\lambda_q \equiv |q\rangle$, $\tilde{\lambda}_q \equiv [q|$. Also to denote helicities of the amplitudes the following notation will be used. We will call amplitude with n on-shell particles with overall helicity $\lambda_\Sigma = n + 2 - 2k$ as N^{k-2} MHV amplitude and refer to it as $A_{k,n+1}^*$. For on-shell amplitudes the standard notation will be used, that is $A_{k,n}$ will denote N^{k-2} MHV amplitude. In what follows to simplify notation we will however drop the derivatives over the Grassmann variables corresponding to the off-shell gluon and just write

$$A_{2,2+1}^*(1^*, 2, 3) = \frac{1}{\kappa_1^*} \frac{\delta^4(k + \lambda_2 \tilde{\lambda}_2 + \lambda_3 \tilde{\lambda}_3) \delta^8(\lambda_p \tilde{\eta}_p + \lambda_2 \tilde{\eta}_2 + \lambda_3 \tilde{\eta}_3)}{\langle p2 \rangle \langle 23 \rangle \langle 3p \rangle}. \tag{2.15}$$

Below we will also always drop the indexes of κ^* and κ , as the amplitudes we are going to consider in this paper will have only one off-shell gluon.

The component amplitudes could be extracted from (2.14) with the action of standard projection operators [5]. For example:

$$A_{2,2+1}^*(1^*, 2^-, 3^+) = \prod_{A=1}^4 \frac{\partial}{\partial \tilde{\eta}_p^A} [A_{2,2+1}^*(1^*, 2, 3)] = \frac{\delta^4(k + \lambda_2 \tilde{\lambda}_2 + \lambda_3 \tilde{\lambda}_3)}{\kappa_1^*} \frac{\langle p2 \rangle^4}{\langle p2 \rangle \langle 23 \rangle \langle 3p \rangle}. \quad (2.16)$$

It should be noted, that the coefficient in front of delta function in (2.16) could be equivalently rewritten in terms of $[ij]$ spinor products and κ factor [64] instead of $\langle ij \rangle$ spinor products and κ^* factor, similar to the case of four point on-shell amplitude. Note also that the off-shell amplitudes with equal on-shell gluon polarizations do vanish $A_{2,2+1}^*(1^*, 2^+, 3^+) = A_{2,2+1}^*(1^*, 2^-, 3^-) = 0$, what could be easily verified by explicit computations [64, 70].

3 Off-shell BCFW recursion in $\mathcal{N} = 4$ SYM

The off-shell BCFW recursion for gauge invariant gluon off-shell scattering amplitudes with an arbitrary number of off-shell gluons was worked out in [64]. Here, we will remind the reader the main results of [64], comment on supersymmetric extension of the off-shell BCFW and perform some example calculations, which will be later compared with the results obtained from our Grassmannian representation.

The BCFW recursion [6, 7] is based on the observation, that a contour integral of an analytical function f vanishing at infinity equals to zero, that is

$$\oint \frac{dz}{2\pi i} \frac{f(z)}{z} = 0. \quad (3.1)$$

and the integration contour expands to infinity. If f is a rational function of a complex variable we have

$$f(0) = - \sum_i \frac{\text{res}_i f(z)}{z_i}, \quad (3.2)$$

where the sum is over all poles of f and $\text{res}_i f(z)$ is a residue of f at pole z_i . In the original on-shell BCFW recursion the z -dependence of scattering amplitude is obtained by a z -dependent shift of particle's momenta. Similarly, the off-shell gluon BCFW recursion of [64] is formulated using a shift of momenta for two external gluons i and j with a vector

$$e^\mu = \frac{1}{2} \langle i | \gamma^\mu | j \rangle, \quad p_i \cdot e = p_j \cdot e = e \cdot e = 0, \quad (3.3)$$

so that

$$\hat{k}_i^\mu(z) \equiv k_i^\mu + z e^\mu = x_i(p_j) p_i^\mu - \frac{\kappa_i - [ij]z}{2} \frac{\langle i | \gamma^\mu | j \rangle}{[ij]} - \frac{\kappa_i^*}{2} \frac{\langle j | \gamma^\mu | i \rangle}{\langle ji \rangle}, \quad (3.4)$$

$$\hat{k}_j^\mu(z) \equiv k_j^\mu - z e^\mu = x_j(p_i) p_j^\mu - \frac{\kappa_j}{2} \frac{\langle j | \gamma^\mu | i \rangle}{[ji]} - \frac{\kappa_j^* + \langle ij \rangle z}{2} \frac{\langle i | \gamma^\mu | j \rangle}{\langle ij \rangle}. \quad (3.5)$$

This shift does not violate momentum conservation and we still have $p_i \cdot \hat{k}_i(z) = 0$ and $p_j \cdot \hat{k}_j(z) = 0$. We would like to note, that the overall effect of shifting momenta is that the values of κ_i and κ_j^* shift while κ_i^* and κ_j are not effected. In the on-shell limit this shift corresponds to the usual $[ij]$ BCFW shift. Note also, that we could also have chosen another shift vector $e^\mu = \frac{1}{2}\langle j|\gamma^\mu|i\rangle$ and shift κ_i^* and κ_j instead. The off-shell amplitudes we consider in this paper do also have a correct large z ($z \rightarrow \infty$) behavior [64], so that we should not worry about boundary terms at infinity.

The sum over the poles (3.2) for z -dependent off-shell gluon scattering amplitude is given by the following graphical representation⁸ [64]:

$$\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} = \sum_{i=2}^{n-2} \sum_h \mathbb{A}_{i,h} + \sum_{i=2}^{n-1} \mathbb{B}_i + \mathbb{C} + \mathbb{D}, \quad (3.6)$$

where

$$\begin{aligned} \mathbb{A}_{i,h} &= \begin{array}{c} i \\ \text{---} \\ \bullet \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} i+1 \\ \text{---} \\ \bullet \end{array} \frac{1}{k_{1,i}^2} \begin{array}{c} -h \\ \text{---} \\ \bullet \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \\ \mathbb{B}_i &= \begin{array}{c} i-1 \\ \text{---} \\ \bullet \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} i \\ \text{---} \\ \bullet \end{array} \frac{1}{2p_i \cdot k_{i,n}} \begin{array}{c} i \\ \text{---} \\ \bullet \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} i+1 \\ \text{---} \\ \bullet \end{array} \\ \mathbb{C} &= \frac{1}{\kappa_1} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \\ \mathbb{D} &= \frac{1}{\kappa_n^*} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \end{aligned} \quad (3.7)$$

$k_{i,j}^\mu \equiv k_i^\mu + k_{i+1}^\mu + \dots + k_j^\mu$ and h is an internal on-shell gluon helicity or a summation index over all on-shell states in the Nair on-shell supermultiplet in the supersymmetric case discussed later. Here and below we use the convention that double lines may stand both for off-shell and on-shell gluons unless the label of the particle contains asterisk. If it does, then the line is definitely off-shell. The thick solid lines stand for on-shell particles. The off-shell double lines can be bent apart to form a single eikonal quark lines represented by a coil crossed with a line in the notation of figure 1. The $\mathbb{A}_{i,h}$ terms are the usual on-shell BCFW terms, which correspond to the z -poles at which internal gluon propagator $\hat{k}_{1,i}^2(z)$ vanishes. The $\mathbb{B}_i$ terms refer to the situation when the denominators of eikonal propagators coming from Wilson line expansion vanish, that is $p_i \cdot \hat{k}_{i,n}(z) = 0$ and p_i^μ is the direction of Wilson line associated with the off-shell gluon. We want to stress, that this term is present only if i labels an off-shell external gluon. The $\mathbb{C}$ term is only present if the gluon number 1 is off-shell. It appears due to vanishing of the external momentum square $\hat{k}_1^2(z)$. Similarly, the $\mathbb{D}$ term is due to vanishing of the external momentum square $\hat{k}_n^2(z)$. It turns out that both these contributions could be calculated in terms of the same scattering diagrams with the off-shell gluons 1 or n exchanged for the on-shell ones. The

⁸We are considering color ordered scattering amplitudes and without loss of generality may use shift of two adjacent legs 1 and n .

helicity of the on-shell gluons depends on type of the term ($\mathbb{C}$ or $\mathbb{D}$) and the shift vector e^μ ($\frac{1}{2}\langle i|\gamma^\mu|j\rangle$ or $\frac{1}{2}\langle j|\gamma^\mu|i\rangle$) used. We refer the reader to [64] for further details. In what follows we will not see $\mathbb{C}$ or $\mathbb{D}$ contributions as the only shifts we are going to use involve only on-shell legs.

The use of shifts involving only on-shell legs also allows us easily perform the supersymmetrization of the off-shell BCFW recursion introduced in [64]. Indeed, it is easy to see, that the supersymmetric shifts of momenta and corresponding Grassmann variables are given by the on-shell BCFW $[ij]$ super-shifts:⁹

$$|\hat{i}\rangle = |1\rangle + z|j\rangle, \quad |\hat{j}\rangle = |j\rangle - z|i\rangle, \quad \hat{\eta}_A^i = \eta_A^i + z\eta_A^j. \quad (3.8)$$

No other spinors or Grassmann variables shift.

Now let us consider the solution of the described BCFW recursion for some particular cases of the scattering amplitudes with one leg off-shell. In the case of 4-point MHV off-shell amplitude $A_{2,3+1}^*(1^+2^+3^-4^*)$ (4-th leg is off-shell) the BCFW contributions relevant to $[12]$ -shift are depicted in figure 2. The $[12]$ -shift itself is given by the following expressions

$$|\hat{1}\rangle = |1\rangle + z|2\rangle, \quad |\hat{2}\rangle = |2\rangle, \quad |\hat{1}\rangle = |1\rangle, \quad |\hat{2}\rangle = |2\rangle - z|1\rangle, \quad (3.9)$$

that is¹⁰ $\hat{p}_1 = p_1 + ze$ and $\hat{p}_2 = p_2 - ze$, where $e^\mu = \frac{1}{2}\langle 1|\gamma^\mu|2\rangle$. The $A_{2,4}^{*(a)}$ contribution is given by

$$A_{2,4}^{*(a)} = \frac{1}{\kappa^*} \frac{\langle \hat{P}4 \rangle^3}{\langle p\hat{1} \rangle \langle \hat{1}\hat{P} \rangle} \times \frac{1}{(p_1 + k)^2} \times -\frac{[\hat{P}2]^4}{[23][3\hat{P}][\hat{P}2]}, \quad (3.10)$$

where $\hat{P} = \hat{p}_1 + k$. Now noticing, that $\langle \hat{P}4 \rangle [\hat{P}2] = \langle p3 \rangle [32]$ and $\langle 1\hat{P} \rangle [3\hat{P}] = \langle 12 \rangle [23]$ we get $A_{2,4}^{*(a)} = \frac{1}{\kappa^*} \frac{\langle 3p \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 3p \rangle \langle p1 \rangle}$. The $A_{2,4}^{*(b)}$ contribution on the other hand is given by

$$A_{2,4}^{*(b)} = \frac{\langle q|\gamma_\mu|\hat{1}\rangle}{\sqrt{2}\langle q\hat{1}\rangle} \sqrt{2}p^\mu \frac{-1}{2p \cdot p_1} \text{ (diagram)} = -\frac{\langle qp \rangle [p\hat{1}]}{\langle q1 \rangle \langle p1 \rangle [1p]} \text{ (diagram)} \quad (3.11)$$

where q is the polarization reference momentum for the first gluon. This contribution however turns out to be zero as ($z = -\frac{[1p]}{[2p]}$ from $p \cdot \hat{p}_1 = 0$):

$$|\hat{1}\rangle = |1\rangle - \frac{[1p]}{[2p]}|2\rangle = \frac{[21]}{[2p]} \left\{ \frac{[1][2]}{[21]} - \frac{[2][1]}{[21]} \right\} |p\rangle = \frac{[21]}{[2p]} |p\rangle. \quad (3.12)$$

In the last step we have used Schouten identity. So, finally (hereafter in this section we will drop the total momentum conservation δ -function $\delta^4(\lambda_1 \tilde{\lambda}_1 + \dots + \lambda_n \tilde{\lambda}_n + k)$)

$$A_{2,3+1}^*(1^+2^+3^-4^*) = \frac{1}{\kappa^*} \frac{\langle 3p \rangle^4}{\langle 12 \rangle \langle 23 \rangle \langle 3p \rangle \langle p1 \rangle}. \quad (3.13)$$

⁹These shifts respect both momentum and supermomentum conservation.

¹⁰For on-shell states in our convention $k_i = p_i$.

$$A_{2,4}^{*(a)} = \text{diagram} - \frac{1}{k_{2,3}^2} + \text{diagram} \quad A_{2,4}^{*(b)} = \text{diagram} \frac{-1}{2p \cdot k_{2,4}} + \text{diagram}$$

Figure 2. BCFW contributions for $A_{2,3+1}^{*}(1^{+}2^{+}3^{-}4^{*})$ for the case of $[1, 2]$ shift

$$A_{3,5}^{*(a)} = \text{diagram} - \frac{1}{k_{2,4}^2} + \text{diagram} \quad A_{3,5}^{*(b)} = 5^{*} = \text{diagram} - \frac{1}{k_{2,3}^2} + \text{diagram}$$

$$A_{3,5}^{*(c)} = 5^{*} = \text{diagram} + \frac{1}{k_{2,3}^2} - \text{diagram} \quad A_{3,5}^{*(d)} = \text{diagram} \frac{-1}{2p \cdot k_{2,4}} + \text{diagram}$$

Figure 3. BCFW contributions for $A_{3,4+1}^{*}(1^{+}2^{+}3^{-}4^{-}5^{*})$ for the case of $[1, 2]$ shift.

Using this result we can also immediately write down the answer for anti-MHV four point amplitude

$$A_{2,3+1}^{*}(1^{-}2^{-}3^{+}4^{*}) = \frac{1}{\kappa} \frac{[3p]^4}{[12][23][3p][p1]}. \quad (3.14)$$

In the case of NMHV off-shell amplitude with 5 legs $A_{3,4+1}^{*}(1^{+}2^{+}3^{-}4^{-}5^{*})$ corresponding BCFW contributions could be found in figure 3. The contribution $A_{3,5}^{*(c)}$ is zero for the same reason the similar contribution turns out to be zero in the on-shell BCFW recursion. On the other hand, contribution $A_{3,5}^{*(d)}$ is zero for the same reason as $A_{2,4}^{*(b)}$ contribution in previous example. The $A_{3,4+1}^{*(a)}$ contribution is given by

$$A_{3,4+1}^{*(a)} = \frac{1}{\kappa^{*}} \frac{\langle \widehat{P}p \rangle^3}{\langle p\widehat{1} \rangle \langle \widehat{1}\widehat{P} \rangle} \times \frac{1}{(p_2 + p_3 + p_4)^2} \times \frac{\langle 34 \rangle^4}{\langle \widehat{2}3 \rangle \langle 34 \rangle \langle 4\widehat{P} \rangle \langle \widehat{P}2 \rangle}, \quad (3.15)$$

where $\widehat{P} = \widehat{p}_1 + k$ and k is off-shell momentum of 5-th leg. To simplify this expression it is convenient to multiply both its numerator and denominator by $[\widehat{P}2]^3$. Now using easily derived relations ($z = \frac{p_{2,4}^2}{\langle 1|3+4|2 \rangle}$ is determined from the condition $(\widehat{p}_1 + k)^2 = 0$)

$$\begin{aligned} \langle \widehat{1}\widehat{P} \rangle [\widehat{P}2] &= -\langle 1|3+4|2 \rangle, & \langle 4\widehat{P} \rangle [\widehat{P}2] &= \langle 34 \rangle [32], & \langle \widehat{P}2 \rangle [\widehat{P}2] &= \langle 34 \rangle [34], \\ \langle \widehat{P}p \rangle [\widehat{P}2] &= \langle p|3+4|2 \rangle, & \langle 5\widehat{1} \rangle &= \langle p1 \rangle, \\ \langle \widehat{2}3 \rangle &= \langle 23 \rangle - z\langle 13 \rangle = \langle 23 \rangle - \frac{p_{2,4}^2}{\langle 1|3+4|2 \rangle} \langle 13 \rangle \end{aligned} \quad (3.16)$$

the expression for $A_{3,4+1}^{*(a)}$ contribution could be written in the following form

$$A_{3,4+1}^{*(a)} = \frac{1}{\kappa^*} \frac{\langle p|3+4|2]^3 \langle 34 \rangle}{\langle p1 \rangle [23] [34] p_{2,4}^2 (\langle 1|3+4|2 \rangle \langle 23 \rangle - p_{2,4}^2 \langle 13 \rangle)}. \quad (3.17)$$

This expression can be further simplified to the form

$$A_{3,4+1}^{*(a)} = \frac{1}{\kappa^*} \frac{\langle p|3+4|2]^3}{\langle p1 \rangle [23] [34] p_{2,4}^2 \langle 1|2+3|4 \rangle}. \quad (3.18)$$

Analogously for the $A_{3,4+1}^{*(b)}$ contribution we have

$$A_{3,4+1}^{*(b)} = \frac{1}{\kappa} \frac{[\widehat{p1}]^4}{[p4][4\widehat{P}][\widehat{P1}][1p]} \times \frac{1}{(p_2 + p_3)^2} \times \frac{[\widehat{2P}]^4}{[\widehat{P3}][32][\widehat{2P}]}, \quad (3.19)$$

where $\widehat{P} = \widehat{p}_1 + p_4 + k$. Again, to simplify this expression, it is convenient to multiply both its numerator and denominator by $\langle \widehat{P1} \rangle^3$. Using the relations ($z = \frac{\langle 23 \rangle}{\langle 13 \rangle}$ is determined from the condition $(\widehat{p}_2 + p_3)^2 = 0$)

$$\begin{aligned} [2\widehat{P}]\langle \widehat{P1} \rangle &= -[23]\langle 31 \rangle, & [4\widehat{P}]\langle \widehat{P1} \rangle &= \langle 1|2+3|4 \rangle, & [\widehat{P3}]\langle \widehat{P1} \rangle &= -\langle 12 \rangle [23], \\ [\widehat{1p}] &= [1p] + z[2p] = -\frac{\langle 3|1+2|p \rangle}{\langle 13 \rangle}, & [3\widehat{2}] &= [32], \\ [\widehat{P1}]\langle \widehat{P1} \rangle &= \langle 1|2+3|\widehat{1} \rangle = \langle 1|2+3|1 \rangle + p_{2,3}^2 = p_{1,3}^2, \end{aligned} \quad (3.20)$$

we get

$$A_{3,5}^{*(b)} = \frac{1}{\kappa} \frac{\langle 3|1+2|p \rangle^3}{[4p]\langle 1|2+3|4 \rangle \langle 12 \rangle \langle 23 \rangle (\langle 1|2+3|1 \rangle + p_{2,3}^2)}, \quad (3.21)$$

which after additional simplifications can be written as

$$A_{3,5}^{*(b)} = \frac{1}{\kappa} \frac{\langle 3|1+2|p \rangle^3}{\langle 12 \rangle \langle 23 \rangle [4p] p_{1,3}^2 \langle 1|2+3|4 \rangle}. \quad (3.22)$$

Combining all terms together we finally obtain:

$$A_{3,4+1}^*(1^+2^+3^-4^-5^*) = \frac{1}{\kappa^*} \frac{\langle p|3+4|2]^3}{\langle p1 \rangle [23] [34] p_{2,4}^2 \langle 1|2+3|4 \rangle} + \frac{1}{\kappa} \frac{\langle 3|1+2|p \rangle^3}{\langle 12 \rangle \langle 23 \rangle [4p] p_{1,3}^2 \langle 1|2+3|4 \rangle}. \quad (3.23)$$

It is interesting to compare this result with $[1, 2]$ BCFW shift representation of 6 - point on-shell amplitude $A_{3,6}(1^+2^+3^-4^-5^-6^+)$:

$$A_{3,6}(1^+2^+3^-4^-5^-6^+) = \frac{\langle 5|3+4|2]^3}{\langle 56 \rangle \langle 61 \rangle [23] [34] p_{2,4}^2 \langle 1|2+3|4 \rangle} + \frac{\langle 3|1+2|6]^3}{\langle 12 \rangle \langle 23 \rangle [45] [56] p_{1,3}^2 \langle 1|2+3|4 \rangle}. \quad (3.24)$$

This expression could be further supersymmetrized and written as

$$A_{3,6} = A_{2,6} R_{142} + A_{2,6} R_{153} + A_{2,6} R_{152}, \quad (3.25)$$

where

$$A_{2,6}R_{152} = \frac{\delta^8(q_{1\dots 6})\hat{\delta}^4(126)}{\langle 34\rangle\langle 45\rangle[12][16]\langle 5|3+4|2\rangle\langle 3|4+5|6\rangle p_{3,5}^2}, \quad (3.26)$$

$$A_{2,6}R_{142} = \frac{\delta^8(q_{1\dots 6})\hat{\delta}^4(234)}{\langle 56\rangle\langle 16\rangle[43][23]\langle 1|5+6|4\rangle\langle 5|4+3|2\rangle p_{2,4}^2}, \quad (3.27)$$

$$A_{2,6}R_{153} = \frac{\delta^8(q_{1\dots 6})\hat{\delta}^4(456)}{\langle 12\rangle\langle 23\rangle[45][65]\langle 1|3+2|4\rangle\langle 3|5+4|6\rangle p_{1,3}^2}. \quad (3.28)$$

Here to simplify notation we used abbreviation $\hat{\delta}^4(ijk) \equiv \hat{\delta}^4(\eta_i[jk] + \text{perm.})$. Component expression obtained before can be extracted from supersymmetric version as coefficient in front of $\tilde{\eta}_3^4\tilde{\eta}_4^4\tilde{\eta}_5^4$. The third term vanishes for this particular component. Next, each term in (3.25) could be associated with a particular residue in the Grassmannian integral considered in next section and will be discussed in more detail later on.

4 Grassmannian representation for off-shell amplitudes in $\mathcal{N} = 4$ SYM

The Grassmannian representation for the off-shell amplitudes with one leg off-shell could be obtained in the same way as the Grassmannian representation for the form factors [61–63], see also [47]. It was noticed in [62], that the top-cell diagrams¹¹ for the form factors could be obtained from the top-cell diagrams for amplitudes by applying to them square and merge/unmerge moves until a box appears on the boundary of the corresponding on-shell diagram, which should be then replaced with the corresponding minimal form factor. Graphically, this relation reads¹²

(4.1)

where the box at the legs $n+1$ and $n+2$ was replaced for the sake of concreteness. A similar relation of form factor on-shell diagrams to the amplitude on-shell diagrams was obtained in [61, 63] based on soft limit procedure. There the corresponding box diagram is deformed by extra soft factor, what makes the box equivalent to the corresponding minimal form factor. One should check, that the obtained this way top-cell diagrams for off-shell amplitudes do have correct color ordering compared to the case of form factors of color singlet operators. In the latter case we always have correct color ordering. In what follows,

¹¹In general, there will be several top-cell diagrams for a particular form factor compared to single top-cell diagram in the case of off-shell amplitudes.

¹²We have borrowed this nice picture from [62].

after a brief remainder of the on-shell diagrams and Grassmannians, we will present the details of the derivation of the Grassmannian representation for N^k MHV amplitudes with one leg off-shell.

4.1 On-shell diagrams and Grassmannians

The on-shell N^{k-2} MHV tree level scattering amplitudes or leading singularities of their loop counterparts in the planar $\mathcal{N} = 4$ SYM could be written in terms of contour integrals over the Grassmannians $G(k, n)$ [22]. The Grassmannian $G(k, n)$ is defined as a space of k -dimensional planes in $\mathbb{C}^n$, passing through origin, so that its points are given by $k \times n$ matrices C modulo $GL(k)$ transformation related to the basis choice. Thus, for N^{k-2} MHV on-shell amplitude we have

$$A_{k,n} = \int_{\Gamma_{k,n}} \frac{d^{k \times n} C}{\text{Vol}[GL(k)]} \frac{\delta^{k \times 2}(C \cdot \tilde{\lambda}) \delta^{k \times 4}(C \cdot \tilde{\eta}) \delta^{(n-k) \times 2}(C^\perp \cdot \lambda)}{(1 \cdots k)(2 \cdots k+1) \cdots (n \cdots k-1)}, \quad (4.2)$$

where $\Gamma_{k,n}$ is the integration contour,¹³ that is the prescription describing which particular combinations of consecutive minors of matrix C should vanish when computing residues. Here, $(i_1, \dots, i_k)$ denotes minor corresponding to columns $i_1, \dots, i_k$ and $C^\perp$ is the orthogonal complement of C fulfilling $C(C^\perp)^T = 0$. The δ -functions in the formula above encode momentum and supermomentum conservation, that is, for example

$$(C \cdot \tilde{\lambda})_I^\dot{\alpha} = \sum_{i=1}^n C_{Ii} \tilde{\lambda}_i^\dot{\alpha} = 0 \quad \text{and} \quad (C^\perp \cdot \lambda)_J^\alpha = \sum_{i=1}^n C_{Ji}^\perp \lambda_i^\alpha = 0 \quad \text{imply} \quad \lambda \cdot \tilde{\lambda} = 0. \quad (4.3)$$

and similarly for supermomentum. The appearance of the Grassmannians in the description of on-shell scattering amplitudes was fully understood with the introduction of *on-shell diagrams* [23]. These diagrams (graphs) are built though gluing two basic trivalent vertices — “black” and “white”, corresponding to three-point MHV $A_{2,3}$ and anti-MHV $A_{1,3}$ amplitudes:

$$\begin{aligned} \text{Black vertex} &= A_{2,3} = \frac{\delta^4(\lambda_1 \tilde{\lambda}_1 + \lambda_2 \tilde{\lambda}_2 + \lambda_3 \tilde{\lambda}_3) \delta^8(\lambda_1 \tilde{\eta}_1 + \lambda_2 \tilde{\eta}_2 + \lambda_3 \tilde{\eta}_3)}{\langle 12 \rangle \langle 23 \rangle \langle 31 \rangle}, \\ \text{White vertex} &= A_{1,3} = \frac{\delta^4(\lambda_1 \tilde{\lambda}_1 + \lambda_2 \tilde{\lambda}_2 + \lambda_3 \tilde{\lambda}_3) \delta^4([12] \tilde{\eta}_3 + [23] \tilde{\eta}_1 + [31] \tilde{\eta}_2)}{[12] [23] [31]}. \end{aligned} \quad (4.4)$$

For a research preceding on-shell diagram formalism see [22, 24–26]. The on-shell diagrams are used to describe different BCFW terms (Yangian invariants) in the BCFW decomposition of the tree level scattering amplitudes or integrands in the case of loop amplitudes. The MHV level k and the number of legs (multiplicity) of the amplitude the on-shell graph corresponds to could be related to the numbers of white n_w and black n_b vertexes together with the number of internal lines n_I as

$$n = 3(n_w + n_b) - 2n_I, \quad k = n_w + 2n_b - n_I. \quad (4.5)$$

¹³The integration goes not over the all points of the Grassmannian, but only those, which belong to the so called positive Grassmannian $G_+(k, n)$ [23]. The latter is a submanifold of $G(k, n)$, such that all consecutive minors for its points are positive. In what follows we will always understand positive Grassmannian $G_+(k, n)$ when referring to Grassmannian $G(k, n)$.

In the Grassmannian representation $A_{2,3}$ and $A_{1,3}$ amplitudes are given by an integral over Grassmannians $G(2,3)$ and $G(1,3)$ correspondingly. The gluing procedure then give rise to a larger Grassmannian $G(k,n)$. The number of degrees of freedom d of a general on-shell diagram is given by the number of its edges minus number of its internal nodes (we subtract $GL(1)$ gauge redundancy associated to every internal node)

$$d = n_I - (n_b + n_w). \quad (4.6)$$

For a planar¹⁴ on-shell diagram with F -faces this is equal to $d = F - 1$. The corresponding Grassmannian integral written in terms of graph's degrees of freedom takes so called “ $d \log$ ” form [23]:

$$\oint \frac{df_1}{f_1} \oint \frac{df_2}{f_2} \dots \oint \frac{df_d}{f_d} \delta^{k \times 2}(C(f_i) \cdot \tilde{\lambda}) \delta^{k \times 4}(C(f_i) \cdot \tilde{\eta}) \delta^{(n-k) \times 2}(C^\perp(f_i) \cdot \lambda), \quad (4.7)$$

where we used face variables f_i ,¹⁵ however similar form could be also obtained using d independent edge variables. The $k \times n$ matrix C is expressed in terms of faces or edge variables using so called *boundary measurement* operation [99]. To do so, one first introduces a *perfect matching* P , which is a subset of edges in the graph, such that every internal node is the endpoint of exactly one edge in P and external nodes belong to one or no edge in P . In one-to-one correspondence to perfect matching is a *perfect orientation*. A perfect orientation is an assignment of special orientation to edges, such that each white vertex has a single incoming arrow and each black vertex has a single outgoing arrow. The edge with a special orientation at each internal node (directed from black to white vertex in our case) is precisely the edge belonging to the perfect matching subset [99, 100]. Given a perfect orientation all external nodes are divided into sources and sinks. Then entries of the matrix C are given by [99]:

$$C_{ij}(\alpha) = \sum_{\Gamma \in \{i \rightarrow j\}} (-1)^{s_\Gamma} \prod_{e \in \Gamma} \alpha_e^{\{-1,1\}}, \quad (4.8)$$

where i runs over sources, j runs over all external nodes and Γ is an oriented path from i to j consistent with perfect orientation. If the edge is traversed in the direction from white to black node,¹⁶ then the power of edge variable is 1, and -1 when traversing in opposite direction. The s_Γ in the formula above is the number of sources strictly between nodes i and j .

The encodings of the scattering amplitudes in terms of on-shell graphs is not unique [23]. On-shell diagrams form equivalence classes. Equivalent diagrams are related by a sequence of equivalence moves, such as square move, merge/unmerge move and bubble reduction, see figure 4. It should be noted, that while the bubble reduction decreases the number of degrees of freedom in the diagram by one, the region of Grassmannian parametrized by the diagram (cell) stays the same. It turns out that there is a one-to-one

¹⁴For nonplanar on-shell diagrams see [98] and references therein.

¹⁵The face variables are given by products of all edge variables around the faces and are subject to constraint $\prod_{i=1}^F f_i = 1$.

¹⁶It is just a convention for assigning edge variables, which could have been chosen differently.

Figure 4. Equivalence moves for on-shell diagrams: a) square move, b) merge/unmerge move for black nodes (similar for white nodes), c) bubble reduction.

correspondence between the reduced modulo equivalence transformations on-shell diagrams and decorated permutations [23]. A decorated permutation is an injective map

$$\sigma : \{1, \dots, n\} \rightarrow \{1, \dots, 2n\}, \quad (4.9)$$

such that $1 \leq \sigma(i) \leq i + n$ and σ modulo n is an ordinary permutation. The permutation is constructed from the on-shell diagram as follows: starting from i -th leg one follows internal edges of the graph turning right at each black vertex and left at each white vertex, the external leg j where this path ends is given by the image $\sigma(i)$. There is also a correspondence between submanifolds of $G(k, n)$ (*positroid cells*) and on-shell diagrams labeled by decorated permutations [23]. The permutation in this case encodes a linear dependence between columns of C -matrix describing points of the Grassmannian: $\sigma(i) \geq i$ labels the first column $c_{\sigma(i)}$ such that $c_i \in \text{span}\{c_{i+1}, \dots, c_{\sigma(i)}\}$. It is possible also to construct in a systematic way an on-shell diagram starting from a corresponding permutation [23]. The procedure is known as a BCFW bridge addition construction. First, the permutation is decomposed into a chain of consequent transpositions. Then each transposition (i, j) is interpreted as a BCFW bridge. And finally these BCFW bridges are applied to a corresponding empty vacuum diagram with the prescribed values of k and n .¹⁷ The BCFW bridge addition operation is given by

$$R_{ij}f(\lambda_i, \tilde{\lambda}_i, \tilde{\eta}_i, \lambda_j, \tilde{\lambda}_j, \tilde{\eta}_j) = \int \frac{d\alpha}{\alpha} f(\lambda_i - \alpha\lambda_j, \tilde{\lambda}_i, \tilde{\eta}_i, \lambda_j, \tilde{\lambda}_j + \alpha\tilde{\lambda}_i, \tilde{\eta}_j + \alpha\tilde{\eta}_i) \quad (4.10)$$

All BCFW terms in the BCFW decomposition of the on-shell scattering amplitude $A_{k,n}$ could be obtained starting from a single on-shell diagram (the *top-cell* diagram) corresponding to a permutation which is a cyclic shift by k

$$A_{k,n} : \quad \sigma = (k+1, \dots, n, 1, \dots, k). \quad (4.11)$$

¹⁷See [23] for more details.

Figure 5. Top-cell on-shell diagram for $A_{k,n}$ on-shell amplitude.

A representative on-shell top-cell diagram could be easily constructed as follows¹⁸ [99]: draw k horizontal lines, $(n-k)$ vertical lines so that the left most and topmost are boundaries and substitute three and four-crossings according to the rules in figure 5. The on-shell diagrams corresponding to BCFW channels are then obtained by removing $(k-2)(n-k-2)$ edges from top cell diagram. It should be noted that not all edges are removable, but only those which removal lowers the dimension of the on-shell diagram by exactly one. The corresponding positroid cells are given by submanifolds with extra linear dependencies between k consecutive columns of the points C of the Grassmannian. The positroid cells with larger number of linear dependent columns are boundaries of positroid cells with smaller number of linear dependent columns. In the case of top-cell there are no k linear dependent consecutive columns.

The on-shell diagrams for scattering amplitudes with one leg off-shell are given by the corresponding on-shell diagrams for on-shell scattering amplitudes with one of the vertexes exchanged for the off-shell vertex introduced in section 2. The cutting off off-shell vertex (for certainty we will assume that the number of the off-shell leg is $n+1$) from a diagram with $n+1$ legs results in the on-shell diagram with $n+2$ legs containing only black and white on-shell vertexes. The same on-shell diagram could also be obtained starting from the on-shell diagram for on-shell scattering amplitude with $n+2$ legs and cutting off the box¹⁹ at the boundary of the diagram containing legs $n+1$ and $n+2$. We will need the latter on-shell diagram as a building block later in this section in the gluing procedure [62] used to derive a Grassmannian representation for the scattering amplitudes with one leg off-shell. It is not hard to see, that the permutation for this diagram is given by (the exchanged legs $n+1$ and $n+2$ in the top-cell permutation for $A_{k,n+2}$) [62]:

$$\tilde{\sigma} = (k+1, \dots, n, n+2, n+1, 1, 2, \dots, k, k-1). \quad (4.12)$$

4.2 Grassmannian representation for amplitudes with one leg off-shell

Now we are ready to proceed with the derivation of Grassmannian representation for scattering amplitudes with one leg off-shell. First, let us derive the Grassmannian represen-

¹⁸See also [15, 20] for review.

¹⁹One generally needs to perform a series of square and merge/unmerge moves to get a box at a prescribed position.

tation for 3-point off-shell vertex presented in section 2. The easiest way²⁰ to get it — is through the action of R -matrix operators (bridge addition operators) on the deformed three particle vacuum state analogues to the case of form factors [62] ($\lambda_1 = \lambda_p$):

$$\frac{1}{\kappa^*} \delta^2(\lambda_1) \delta^2(\tilde{\lambda}_2) \delta^4(\tilde{\eta}_2) \delta^2(\tilde{\lambda}_3) \delta^4(\tilde{\eta}_3), \quad (4.13)$$

where (k is the off-shell gluon momentum and p is its direction)

$$\tilde{\lambda}_2 = \tilde{\lambda}_2 + \frac{\langle 3|q}{\langle 32 \rangle}, \quad \tilde{\lambda}_3 = \tilde{\lambda}_3 + \frac{\langle 2|q}{\langle 23 \rangle} \quad \text{and} \quad q = k - p \quad (4.14)$$

so that the sum of particle momenta are

$$\begin{aligned} \lambda_1^\alpha \tilde{\lambda}_1^{\dot{\alpha}} + \lambda_2^\alpha \tilde{\lambda}_2^{\dot{\alpha}} + \lambda_3^\alpha \tilde{\lambda}_3^{\dot{\alpha}} &= \lambda_1^\alpha \tilde{\lambda}_1^{\dot{\alpha}} + \lambda_2^\alpha \tilde{\lambda}_2^{\dot{\alpha}} + \lambda_3^\alpha \tilde{\lambda}_3^{\dot{\alpha}} - \frac{\varepsilon_{cb}}{\langle 23 \rangle} (\lambda_2^\alpha \lambda_3^c - \lambda_2^c \lambda_3^\alpha) q^{b\dot{\alpha}} \\ &= \lambda_1^\alpha \tilde{\lambda}_1^{\dot{\alpha}} + \lambda_2^\alpha \tilde{\lambda}_2^{\dot{\alpha}} + \lambda_3^\alpha \tilde{\lambda}_3^{\dot{\alpha}} + q^{\alpha\dot{\alpha}}. \end{aligned} \quad (4.15)$$

The deformed (the case of non-zero spectral parameters u_i) off-shell three-point amplitude is then given by

$$A_{2,2+1}^*(1^*, 2, 3) = \frac{1}{\kappa^*} R_{23}(u_{32}) R_{12}(u_{31}) \delta^2(\lambda_1) \delta^2(\tilde{\lambda}_2) \delta^4(\tilde{\eta}_2) \delta^2(\tilde{\lambda}_3) \delta^4(\tilde{\eta}_3), \quad (4.16)$$

where as before (u is the spectral parameter)

$$R_{ij}(u) f(\lambda_i, \tilde{\lambda}_i, \tilde{\eta}_i, \lambda_j, \tilde{\lambda}_j, \tilde{\eta}_j) = \int \frac{d\alpha}{\alpha^{1+u}} f(\lambda_i - \alpha \lambda_j, \tilde{\lambda}_i, \tilde{\eta}_i, \lambda_j, \tilde{\lambda}_j + \alpha \tilde{\lambda}_i, \tilde{\eta}_j + \alpha \tilde{\eta}_i). \quad (4.17)$$

and $u_{ij} = u_i - u_j$. So, we have:

$$\begin{aligned} A_{2,2+1}^*(1^*, 2, 3) &= \frac{1}{\kappa^*} R_{23}(u_{23}) \int \frac{d\alpha_1}{\alpha_1^{1+u_{31}}} \delta^2(\lambda_1 - \alpha_1 \lambda_2) \delta^2(\tilde{\lambda}_2 + \alpha_1 \tilde{\lambda}_1) \delta^4(\tilde{\eta}_2 + \alpha_1 \tilde{\eta}_1) \delta^2(\tilde{\lambda}_3) \delta^4(\tilde{\eta}_3) \\ &= \frac{1}{\kappa^*} \int \frac{d\alpha_2}{\alpha_2^{1+u_{32}}} \int \frac{d\alpha_1}{\alpha_1^{1+u_{31}}} \delta^2(\lambda_1 - \alpha_1 \lambda_2 + \alpha_1 \alpha_2 \lambda_3) \times \\ &\quad \times \delta^2(\tilde{\lambda}_2 + \alpha_1 \tilde{\lambda}_1) \delta^4(\tilde{\eta}_2 + \alpha_1 \tilde{\eta}_1) \delta^2(\tilde{\lambda}_3 + \alpha_2 \tilde{\lambda}_2) \delta^4(\tilde{\eta}_3 + \alpha_2 \tilde{\eta}_2) \\ &= \frac{1}{\kappa^*} \int \frac{d\alpha_2}{\alpha_2^{1+u_{32}}} \int \frac{d\alpha_1}{\alpha_1^{1+u_{31}}} \delta^4(C(\alpha_1, \alpha_2) \cdot \tilde{\lambda}) \delta^8(C(\alpha_1, \alpha_2) \cdot \tilde{\eta}) \delta^2(C^\perp(\alpha_1, \alpha_2) \cdot \lambda), \end{aligned} \quad (4.18)$$

where

$$C(\alpha_1, \alpha_2) = \begin{pmatrix} \alpha_1 & 1 & 0 \\ 0 & \alpha_2 & 1 \end{pmatrix} \quad \text{and} \quad C^\perp = \begin{pmatrix} 1 & -\alpha_1 & \alpha_1 \alpha_2 \end{pmatrix} \quad (4.19)$$

The same formula could be written as an integral over $G(2, 3)$ Grassmanian as

$$A_{2,2+1}^*(1^*, 2, 3) = \frac{1}{\kappa^*} \int \frac{d^{2 \times 3} C}{\text{Vol}[\text{GL}(2)]} \frac{1}{(12)^{1-u_{23}} (23)^{1-u_{31}} (31)^{1-u_{12}}} \delta^4(C \cdot \tilde{\lambda}) \delta^8(C \cdot \tilde{\eta}) \delta^4(C^\perp \cdot \lambda), \quad (4.20)$$

²⁰Here we should note, that this is not a valid spin chain description for 3-point off-shell amplitude, but just a technical trick.

where $(ii + 1)$ denote consecutive minors of C -matrix. In the following when deriving Grassmannian integral representation for off-shell amplitudes we will however need only undeformed 3-point off-shell amplitude at zero values of spectral parameters, which is given by

$$A_{2,2+1}^*(1^*, 2, 3) = \frac{1}{\kappa^*} \int \frac{d^{2 \times 3} C}{\text{Vol}[\text{GL}(2)]} \frac{1}{(12)(23)(31)} \delta^4(C \cdot \tilde{\lambda}) \delta^8(C \cdot \tilde{\eta}) \delta^4(C^\perp \cdot \lambda), \quad (4.21)$$

4.2.1 Spinor helicity representation

As we already mentioned in order to write down $N^k\text{MHV}$ amplitudes as the integrals over the Grassmannians we will use the gluing procedure as in [62], that is we break the corresponding top cell on-shell diagram for amplitude into two pieces: the mentioned before box, which we replace with minimal 3-point off-shell amplitude (4.21) and the remaining purely on-shell piece with $n + 2$ legs, for which a Grassmannian integral representation is known. After that we glue these two pieces together, i.e. we perform the on-shell phase space integration. The on-shell piece is given by Grassmannian integral representation of corresponding on-shell amplitude and can be written as²¹ [23]:

$$I_{n+2,k} = \int \frac{d\alpha_1}{\alpha_1} \dots \frac{d\alpha_m}{\alpha_m} \delta^{k \times 2}(C \cdot \tilde{\lambda}) \delta^{k \times 4}(C \cdot \eta) \delta^{(n+2-k) \times 2}(C^\perp \cdot \lambda), \quad (4.22)$$

where the matrix C depends on the Grassmannian coordinates α_i 's, $C = C(\alpha_i) \in G(k, n + 2)$ and m is the dimension of the corresponding cell in the Grassmannian. The concrete parametrization of matrix C corresponding to on-shell diagram labeled by permutation σ could be obtained with the help of Mathematica package positroid.m [101]. So, gluing the minimal 3-point off-shell amplitude (4.21) to the legs $n + 1$ and $n + 2$ and accounting for all helicity configurations in the gluing channel we get²²

$$A_{k,n+1}^* = \int \prod_{i=n+1}^{n+2} \frac{d^2 \lambda_i d^2 \tilde{\lambda}_i}{\text{Vol}[\text{GL}(1)]} d^4 \tilde{\eta}_i A_{2,2+1}^*(p^*, n + 1, n + 2) \Big|_{\lambda \rightarrow -\lambda} I_{n+2,k} \quad (4.23)$$

That is

$$\begin{aligned} A_{k,n+1}^* &= \frac{1}{\kappa^*} \int \frac{d\beta_2}{\beta_2} \int \frac{d\beta_1}{\beta_1} \delta^2(\lambda_p + \beta_1 \lambda_{n+1} - \beta_1 \beta_2 \lambda_{n+2}) \delta^2(\tilde{\lambda}_{n+1} + \beta_1 \tilde{\lambda}_p) \\ &\quad \times \delta^4(\tilde{\eta}_{n+1} + \beta_1 \tilde{\eta}_p) \delta^2(\tilde{\lambda}_{n+2} + \beta_2 \tilde{\lambda}_{n+1}) \delta^4(\tilde{\eta}_{n+2} + \beta_2 \tilde{\eta}_{n+1}) \\ &\quad \times \int \prod_{i=n+1}^{n+2} \frac{d^2 \lambda_i d^2 \tilde{\lambda}_i}{\text{Vol}[\text{GL}(1)]} d^4 \tilde{\eta}_i \\ &\quad \times \int \frac{d\alpha_1}{\alpha_1} \dots \frac{d\alpha_m}{\alpha_m} \delta^{k \times 2}(C \cdot \tilde{\lambda}) \delta^{k \times 4}(C \cdot \eta) \delta^{(n+2-k) \times 2}(C^\perp \cdot \lambda), \end{aligned} \quad (4.24)$$

²¹We hope that here and later from the context it will be clear, that k here is $N^{k-2}\text{MHV}$ degree and not the off-shell gluon momentum.

²²Without loss of generality we may choose off-shell leg to lie between legs 1 and n .

were $\tilde{\lambda}_{n+1} = \tilde{\lambda}_{n+1} + \frac{\langle n+2|q}{\langle n+2|n+1\rangle}$ and $\tilde{\lambda}_{n+2} = \tilde{\lambda}_{n+2} + \frac{\langle n+1|q}{\langle n+1|n+2\rangle}$. Performing integrations over $\tilde{\lambda}_{n+1}$, $\tilde{\lambda}_{n+2}$, $\tilde{\eta}_{n+1}$ and $\tilde{\eta}_{n+2}$ we get

$$\tilde{\lambda}_{n+1} = \frac{\langle n+2|q}{\langle n+2|n+1\rangle} - \beta_1 \tilde{\lambda}_p, \quad \tilde{\lambda}_{n+2} = \frac{\langle n+1|q}{\langle n+1|n+2\rangle} + \beta_1 \beta_2 \tilde{\lambda}_p, \quad (4.25)$$

$$\tilde{\eta}_{n+1} = -\beta_1 \tilde{\eta}_p, \quad \tilde{\eta}_{n+2} = \beta_1 \beta_2 \tilde{\eta}_p. \quad (4.26)$$

To remove the $\text{Vol}[\text{GL}(1)]^2$ redundancy in the remaining integrations over λ , we use their parametrization as in [62]:

$$\lambda_{n+1} = \xi_A - \beta_3 \xi_B, \quad \lambda_{n+2} = \xi_B - \beta_4 \xi_A. \quad (4.27)$$

Where ξ_A and ξ_B are two arbitrary but linearly independent reference spinors. Then $\langle n+1 \ n+2 \rangle = (\beta_3 \beta_4 - 1) \langle \xi_B \ \xi_A \rangle$,

$$\int \frac{d^2 \lambda_{n+1}}{\text{Vol}[\text{GL}(1)]} \frac{d^2 \lambda_{n+2}}{\text{Vol}[\text{GL}(1)]} = -\langle \xi_A \ \xi_B \rangle^2 \int d\beta_3 d\beta_4 \quad (4.28)$$

and²³

$$\begin{aligned} A_{k,n+1}^* &= \frac{1}{\kappa^*} \langle \xi_A \ \xi_B \rangle^2 \int \frac{d\alpha_1}{\alpha_1} \dots \frac{d\alpha_m}{\alpha_m} \frac{d\beta_1}{\beta_1} \frac{d\beta_2}{\beta_2} \frac{d\beta_3 d\beta_4}{(1 - \beta_3 \beta_4)^2} \\ &\quad \times \delta^2(\lambda_p + \beta_1(1 + \beta_2 \beta_4) \xi_A - \beta_1(\beta_2 + \beta_3) \xi_B) \\ &\quad \times \delta^{k \times 2} (C'(\alpha_i, \beta_i) \cdot \underline{\tilde{\lambda}}) \delta^{k \times 4} (C'(\alpha_i, \beta_i) \cdot \underline{\tilde{\eta}}) \delta^{(n+2-k) \times 2} (C'^{\perp}(\alpha_i, \beta_i) \cdot \underline{\lambda}). \end{aligned} \quad (4.29)$$

Here, the following notation was introduced

$$\begin{aligned} C'_{n+1} &= \frac{1}{1 - \beta_3 \beta_4} C_{n+1} + \frac{\beta_3}{1 - \beta_3 \beta_4} C_{n+2}, & C'^{\perp}_{n+1} &= C_{n+1}^{\perp} - \beta_4 C_{n+2}^{\perp}, \\ C'_{n+2} &= \frac{1}{1 - \beta_3 \beta_4} C_{n+2} + \frac{\beta_4}{1 - \beta_3 \beta_4} C_{n+1}, & C'^{\perp}_{n+2} &= C_{n+2}^{\perp} - \beta_3 C_{n+1}^{\perp}, \end{aligned} \quad (4.30)$$

and

$$\begin{aligned} \underline{\lambda}_i &= \lambda_i, & i &= 1, \dots, n, & \underline{\lambda}_{n+1} &= \xi_A, & \underline{\lambda}_{n+2} &= \xi_B \\ \underline{\tilde{\lambda}}_i &= \tilde{\lambda}_i, & i &= 1, \dots, n, & \underline{\tilde{\lambda}}_{n+1} &= \frac{\langle \xi_B | k \rangle}{\langle \xi_B \ \xi_A \rangle}, & \underline{\tilde{\lambda}}_{n+2} &= -\frac{\langle \xi_A | k \rangle}{\langle \xi_B \ \xi_A \rangle}, \\ \underline{\tilde{\eta}}_i &= \tilde{\eta}_i, & i &= 1, \dots, n, & \underline{\tilde{\eta}}_{n+1} &= \tilde{\eta}_p, & \underline{\tilde{\eta}}_{n+2} &= 0. \end{aligned} \quad (4.31)$$

The factor of $1/(1 - \beta_3 \beta_4)^2$ in (4.29) is a Jacobian from reorganizing the $C^{\perp} \cdot \lambda$ δ -functions (see [62] for further details). Now, rewriting first δ -function in (4.29) as

$$\begin{aligned} &\delta^2(\lambda_p + \beta_1(1 + \beta_2 \beta_4) \xi_A - \beta_1(\beta_2 + \beta_3) \xi_B) \\ &= \frac{1}{\beta_1^2 \beta_2 \langle \xi_A \ \xi_B \rangle} \delta\left(\beta_3 - \frac{\langle \xi_{AP} \rangle}{\beta_1 \langle \xi_A \ \xi_B \rangle} + \beta_2\right) \cdot \delta\left(\beta_4 - \frac{\langle \xi_{BP} \rangle}{\beta_1 \beta_2 \langle \xi_A \ \xi_B \rangle} + \frac{1}{\beta_2}\right) \end{aligned} \quad (4.32)$$

²³In deriving this formula it was convenient to put $\tilde{\lambda}_p$ to zero, so that $q = k$. Here $k = q + p$ and q should not be confused with the off-shell gluon polarization vector.

$$I = \text{[diagram]}, \quad \tilde{\sigma} = (3, 5, 4, 2, 1), \quad C = \begin{pmatrix} 1 & 0 & -\alpha_2 & -\alpha_2\alpha_4 & -\alpha_1 \\ 0 & 1 & \alpha_3 & \alpha_3\alpha_4 & 0 \end{pmatrix}.$$

Figure 6. On-shell sub-diagram I obtained by removing off-shell vertex from $A_{2,3+1}^*$ diagram, corresponding permutation $\tilde{\sigma}$ and C matrix.

choosing $\xi_A = \lambda_p$, $\xi_B = \xi$ and performing integrations over β_3, β_4 we get

$$A_{k,n+1}^* = \frac{1}{\kappa^*} \langle \xi p \rangle \int \frac{d\alpha_1}{\alpha_1} \dots \frac{\alpha_m}{\alpha_m} \frac{d\beta_1 d\beta_2}{\beta_1 \beta_2^2} \times \\ \times \delta^{k \times 2} (C'(\alpha_i, \beta_i) \cdot \underline{\tilde{\lambda}}) \delta^{k \times 4} (C'(\alpha_i, \beta_i) \cdot \underline{\tilde{\eta}}) \delta^{(n+2-k) \times 2} (C'^{\perp}(\alpha_i, \beta_i) \cdot \underline{\lambda}), \quad (4.33)$$

where now

$$\begin{aligned} C'_{n+1} &= -\beta_1 C_{n+1} + \beta_1 \beta_2 C_{n+2}, & C'^{\perp}_{n+1} &= C_{n+1}^{\perp} + \frac{1 + \beta_1}{\beta_1 \beta_2} C_{n+2}^{\perp}, \\ C'_{n+2} &= -\beta_1 C_{n+2} + \frac{1 + \beta_1}{\beta_2} C_{n+1}, & C'^{\perp}_{n+2} &= C_{n+2}^{\perp} + \beta_2 C_{n+1}^{\perp}, \end{aligned} \quad (4.34)$$

and

$$\begin{aligned} \underline{\lambda}_i &= \lambda_i, & i &= 1, \dots, n, & \underline{\lambda}_{n+1} &= \lambda_p, & \underline{\lambda}_{n+2} &= \xi \\ \underline{\tilde{\lambda}}_i &= \tilde{\lambda}_i, & i &= 1, \dots, n, & \underline{\tilde{\lambda}}_{n+1} &= \frac{\langle \xi | k \rangle}{\langle \xi p \rangle}, & \underline{\tilde{\lambda}}_{n+2} &= -\frac{\langle p | k \rangle}{\langle \xi p \rangle}, \\ \underline{\tilde{\eta}}_i &= \tilde{\eta}_i, & i &= 1, \dots, n, & \underline{\tilde{\eta}}_{n+1} &= \tilde{\eta}_p, & \underline{\tilde{\eta}}_{n+2} &= 0. \end{aligned} \quad (4.35)$$

To write down (4.33) in terms of the integral over the points of the $G(k, n+2)$ Grassmannian parametrized by the elements of C' matrix we have considered a number of particular examples. For example, in the case of $A_{2,3+1}^*$ scattering amplitude the corresponding permutation and C -matrix are shown in figure 6. The C' matrix in this case is given by

$$C' = \begin{pmatrix} 1 & 0 & -\alpha_2 & \beta_1(\alpha_2\alpha_4 - \alpha_1\beta_2) & \alpha_1\beta_1 - \frac{\alpha_2\alpha_4(1+\beta_1)}{\beta_2} \\ 0 & 1 & \alpha_3 & -\alpha_3\alpha_4\beta_1 & \frac{\alpha_3\alpha_4(1+\beta_1)}{\beta_2} \end{pmatrix} \quad (4.36)$$

Taking into account the Jacobian of transformation from the Grassmannian C_{ij} matrix elements to edge variables α_i, β_j , that is $J = \frac{\alpha_1\alpha_3^2\alpha_4\beta_1^2}{\beta_2}$, one can easily see that the result for $A_{2,3+1}^*$ off-shell MHV₃₊₁ scattering amplitude could be written in the following form

$$A_{2,3+1}^* = \frac{\langle \xi p \rangle}{\kappa^*} \int \frac{d^{2 \times 5} C'}{\text{Vol}[\text{GL}(2)]} \frac{\delta^{2 \times 2} (C' \cdot \underline{\tilde{\lambda}}) \delta^{2 \times 4} (C' \cdot \underline{\tilde{\eta}}) \delta^{3 \times 2} (C'^{\perp} \cdot \underline{\lambda})}{(12)(23)(34)(45)(41)}. \quad (4.37)$$

Similar consideration of the other examples allows us to write down a general conjecture

for the integral over Grassmannian for a general set of values for n and k . Lets consider the following integral over Grassmannian:

$$\Omega_{n+2}^k[\Gamma] = \int_{\Gamma} \frac{d^{k \times (n+2)} C'}{\text{Vol}[\text{GL}(k)]} \text{Reg.} \frac{\delta^{k \times 2} (C' \cdot \underline{\tilde{\lambda}}) \delta^{k \times 4} (C' \cdot \underline{\tilde{\eta}}) \delta^{(n+2-k) \times 2} (C'^{\perp} \cdot \underline{\lambda})}{(1 \cdots k) \cdots (n+1 \cdots k-2)(n+2 \ 1 \cdots k-1)}, \quad (4.38)$$

with

$$\text{Reg.} = \frac{\langle \xi p \rangle}{\kappa^*} \frac{(n+2 \ 1 \cdots k-1)}{(n+1 \ 1 \cdots k-1)}. \quad (4.39)$$

The choice of notation Reg. will be clarified in section 4.

We claim that for appropriate choice of the integration contour $\Gamma = \Gamma_{k,n+2}$ the following identity holds

$$A_{k,n+1}^* = \Omega_{n+2}^k[\Gamma_{k,n+2}]. \quad (4.40)$$

4.2.2 twistor representation

The transformation of the obtained result to twistor space could be performed in full analogy to the case of on-shell amplitudes [22] and form factors [62]. Super twistors are defined as

$$\mathcal{W}_i = (\underline{\tilde{\mu}}_i, \underline{\tilde{\lambda}}_i, \underline{\tilde{\eta}}_i), \quad (4.41)$$

where $\underline{\tilde{\mu}}_i$ is related to $\underline{\lambda}_i$ via Witten's half Fourier transformation [102]

$$\int d^2 \underline{\lambda}_j \exp(-i \underline{\tilde{\mu}}_j^\alpha \underline{\lambda}_{j\alpha}). \quad (4.42)$$

Now recalling, that $\underline{\lambda}_{n+1} = \lambda_p$, $\underline{\lambda}_{n+2} = \xi$ and using (4.42) $\langle \xi p \rangle$ could be written as

$$\langle \xi p \rangle = \left\langle \frac{\partial}{\partial \underline{\tilde{\mu}}_{n+1}} \frac{\partial}{\partial \underline{\tilde{\mu}}_{n+2}} \right\rangle. \quad (4.43)$$

Rewriting δ -function $\delta^{2 \times (n+2-k)}(C'^{\perp} \cdot \underline{\lambda})$ as [62]:

$$\prod_{K=1}^k \int d^2 \rho_K \delta^{2 \times (n+2)} \left(\underline{\lambda}_i - \sum_{L=1}^k \rho_L C'_{Li} \right), \quad (4.44)$$

applying half Fourier transformation (4.42) to this representation and performing the integrals over $\underline{\lambda}_i$ via δ -functions, we find

$$\prod_{K=1}^k \int d^2 \rho_K \exp \left(-i \sum_{j=1}^{n+2} \sum_{L=1}^k \rho_L^\alpha C'_{Lj} \underline{\tilde{\mu}}_{\alpha j} \right) = \delta^{2k}(C' \cdot \underline{\tilde{\mu}}). \quad (4.45)$$

So, we can rewrite (4.38) and (4.40) as

$$\begin{aligned} \Omega_{n+2}^k[\Gamma] &= \frac{1}{\kappa^*} \left\langle \frac{\partial}{\partial \underline{\tilde{\mu}}_{n+1}} \frac{\partial}{\partial \underline{\tilde{\mu}}_{n+2}} \right\rangle \\ &\times \int_{\Gamma} \frac{d^{k \times (n+2)} C'}{\text{Vol}[\text{GL}(k)]} \frac{\delta^{4k|4k}(C' \cdot \mathcal{W})}{(1 \cdots k) \cdots (n+1 \cdots k-2)(n+1 \ 1 \cdots k-1)}. \\ A_{k,n+1}^* &= \Omega_{n+2}^k[\Gamma_{k,n+2}] \end{aligned} \quad (4.46)$$

Figure 7. Momenta and dual coordinates in the case of amplitude with one off-shell and $n = 4$ on-shell legs. In contrast to the case of off-shell amplitudes, the n on-shell momenta do not add up to zero but to the off-shell gluon momentum k .

4.2.3 Momentum twistor representation

The transformation to momentum twistor space could be performed using the strategy of [25, 62, 103]. The momentum super twistor variables $\mathcal{Z}_i = (\underline{\lambda}_i, \underline{\mu}_i, \underline{\eta}_i)$ [8] are defined through introduction of dual super coordinates

$$\underline{\lambda}_i \tilde{\lambda}_i = y_i - y_{i+1}, \quad \underline{\lambda}_i \tilde{\eta}_i = \vartheta_i - \vartheta_{i+1}. \quad (4.47)$$

In figure 7 we have shown a contour in the dual space formed by on-shell particles momenta together with momenta of two auxiliary states $n + 1$ and $n + 2$. The components of momentum twistors are defined then through the following incidence relations

$$\underline{\mu}_i = \underline{\lambda}_i y_i = \underline{\lambda}_i y_{i+1}, \quad \underline{\eta}_i = \underline{\lambda}_i \vartheta_i = \underline{\lambda}_i \vartheta_{i+1}. \quad (4.48)$$

Inverting these relations we get

$$\begin{aligned} \tilde{\lambda}_i &= \frac{\langle i+1 \ i \rangle \underline{\mu}_{i-1} + \langle i \ i-1 \rangle \underline{\mu}_{i+1} + \langle i-1 \ i+1 \rangle \underline{\mu}_i}{\langle i-1 \ i \rangle \langle i \ i+1 \rangle}, \\ \tilde{\eta}_i &= \frac{\langle i+1 \ i \rangle \underline{\eta}_{i-1} + \langle i \ i-1 \rangle \underline{\eta}_{i+1} + \langle i-1 \ i+1 \rangle \underline{\eta}_i}{\langle i-1 \ i \rangle \langle i \ i+1 \rangle}. \end{aligned} \quad (4.49)$$

Now following [62] we use representation of $\delta^{2 \times (n+2-k)}(C'^{\perp} \cdot \underline{\lambda})$ as in (4.44). Fixing part of $\text{GL}(k)$ redundancy as

$$\rho = \begin{pmatrix} 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & 0 & 1 \end{pmatrix}. \quad (4.50)$$

we get

$$C'_{k-1 \ i} = \underline{\lambda}_i^1, \quad C'_{ki} = \underline{\lambda}_i^2. \quad (4.51)$$

and

$$\begin{aligned} \Omega_{n+2}^k[\Gamma] &= \frac{\langle \xi p \rangle}{\kappa^*} \delta^4(\underline{\lambda} \cdot \tilde{\lambda}) \delta^8(\underline{\lambda} \cdot \tilde{\eta}) \int_{\Gamma} \frac{d^{(k-2) \times (n+2)} C'}{\text{Vol}[\text{GL}(k-2) \times T_{k-2}]} \times \\ &\quad \times \frac{\delta^{2 \times (k-2)}(C' \cdot \tilde{\lambda}) \delta^{4 \times (k-2)}(C' \cdot \tilde{\underline{\lambda}})}{(1 \cdots k) \cdots (n+1 \cdots k-2)(n+1 \ 1 \cdots k-1)}, \end{aligned} \quad (4.52)$$

where the integral and δ -function contains only the first $k - 2$ rows of C' . The action of the shift symmetry T_{k-2} on these rows is given by

$$C'_{Ii} \rightarrow C'_{Ii} + r_{1I} \underline{\lambda}_i^1 + r_{2I} \underline{\lambda}_i^2, \quad I = 1, \dots, k-2, \quad (4.53)$$

with arbitrary r_{1I} and r_{2I} . According to [103], (4.49) leads to

$$\sum_{i=1}^{n+2} C'_{Ii} \tilde{\lambda}_i = - \sum_{i=1}^{n+2} D_{Ii} \underline{\lambda}_i, \quad \sum_{i=1}^{n+2} C'_{Ii} \tilde{\eta}_i = - \sum_{i=1}^{n+2} D_{Ii} \underline{\eta}_i, \quad I = 1, \dots, k-2, \quad (4.54)$$

where the matrix D is given by

$$D_{Ii} = \frac{\langle i \ i+1 \rangle C'_{I \ i-1} + \langle i-1 \ i \rangle C'_{I \ i+1} + \langle i+1 \ i-1 \rangle C'_{I \ i}}{\langle i-1 \ i \rangle \langle i \ i+1 \rangle}. \quad (4.55)$$

Next, we rewrite the minors of C' in terms of minors of D . The consecutive minors could be rewritten as [25, 103]:

$$(C'_1 \dots C'_k) = -\langle 1 \ 2 \rangle \dots \langle k-1 \ k \rangle (D_2 \dots D_{k-1}), \quad (4.56)$$

while the only non-consecutive minor we need is given by [62]:

$$\begin{aligned} (C'_{n+1} C'_1 \dots C'_{k-1}) = & -\langle n+1 \ 1 \rangle \langle 1 \ 2 \rangle \dots \langle k-2 \ k-1 \rangle (D_1 \dots D_{k-2}) \\ & -\langle n+1 \ n+2 \rangle \langle 1 \ 2 \rangle \dots \langle k-2 \ k-1 \rangle (D_{n+2} D_2 \dots D_{k-2}). \end{aligned} \quad (4.57)$$

Using (4.56) and (4.57) we have

$$(1 \dots k)_{C'} \dots (n+2 \dots k-1)_{C'} = (-1)^{n+2} (\langle 1 \ 2 \rangle \dots \langle n+2 \ 1 \rangle)^{k-1} (1 \dots k)_D \dots (n+2 \dots k-1)_D. \quad (4.58)$$

and

$$\frac{(n+2 \ 1 \dots k-1)_{C'}}{(n+1 \ 1 \dots k-1)_{C'}} = \frac{\langle n+2 \ 1 \rangle (1 \dots k-2)_D}{\langle n+1 \ 1 \rangle (1 \dots k-2)_D + \langle n+1 \ n+2 \rangle (n+2 \ 2 \dots k-2)_D} \quad (4.59)$$

Next, we use the T_{k-2} shift symmetry to set $C'_{I1} = C'_{I2} = 0$ [62, 103], so that

$$\frac{d^{(k-2) \times (n+2)} C'}{\text{Vol}[\text{GL}(k-2) \ltimes T_{k-2}]} = \langle 1 \ 2 \rangle^{k-2} \frac{d^{(k-2) \times (n)} C'}{\text{Vol}[\text{GL}(k-2)]}. \quad (4.60)$$

Changing the integration variables from C' to D

$$\frac{d^{(k-2) \times (n)} C'}{\text{Vol}[\text{GL}(k-2)]} = \left(\frac{\langle 1 \ 2 \rangle \dots \langle n+2 \ 1 \rangle}{\langle 1 \ 2 \rangle^2} \right)^{k-2} \frac{d^{(k-2) \times (n)} D}{\text{Vol}[\text{GL}(k-2)]}, \quad (4.61)$$

and undoing gauge fixing of first two columns of C' -matrix yields factors of

$$\langle 1 \ 2 \rangle \delta^2(D_{Ii} \underline{\lambda}_i) \quad (4.62)$$

for $I = 1, \dots, k-2$. The details of these steps could be found in [103]. The final expression for scattering amplitude with one leg off-shell for appropriate choice of integration contour $\Gamma_{k,n+2}$ can be represented as

$$\frac{A_{k,n+1}^*}{A_{2,n+1}^*} = \omega_{n+2}^k[\Gamma_{k,n+2}],$$

$$\omega_{n+2}^k[\Gamma] = \int_{\Gamma} \frac{d^{(k-2) \times (n+2)} D}{\text{Vol}[\text{GL}(k-2)]} \frac{1}{1 + \frac{\langle p\xi \rangle}{\langle p1 \rangle} \frac{(n+2 \ 2 \dots k-2)}{(1 \dots k-2)}} \frac{\delta^{4(k-2)|4(k-2)}(D \cdot \mathcal{Z})}{(1 \dots k-2) \dots (n+2 \dots k-3)}. \quad (4.63)$$

4.3 MHV_{n+1}, NMHV_{4,5} and NMHV_{n+1} amplitudes

Let us now perform some checks of the formula derived in previous subsections. First we start with off-shell MHV_n case. Using (4.38) for $k = 2$ and setting $\xi = \lambda_1$ the off-shell MHV_n amplitude could be written as

$$A_{k,n+1}^* = \frac{\langle \xi p \rangle}{\kappa^*} \int \frac{d^{2 \times (n+2)} C'}{\text{Vol}[\text{GL}(2)]} \frac{\delta^4(C' \cdot \underline{\tilde{\lambda}}) \delta^8(C' \cdot \underline{\tilde{\eta}}) \delta^{2n}(C'^{\perp} \cdot \underline{\lambda})}{(12)(23) \dots (n+1 \ n+2)(n+1 \ 1)}, \quad (4.64)$$

where

$$\begin{aligned} \underline{\lambda}_i &= \lambda_i, & i &= 1, \dots, n, & \underline{\lambda}_{n+1} &= \lambda_p, & \underline{\lambda}_{n+2} &= \lambda_1 \\ \underline{\tilde{\lambda}}_i &= \tilde{\lambda}_i, & i &= 1, \dots, n, & \underline{\tilde{\lambda}}_{n+1} &= \frac{\langle 1|k \rangle}{\langle 1p \rangle}, & \underline{\tilde{\lambda}}_{n+2} &= -\frac{\langle p|k \rangle}{\langle 1p \rangle}, \\ \underline{\tilde{\eta}}_i &= \tilde{\eta}_i, & i &= 1, \dots, n, & \underline{\tilde{\eta}}_{n+1} &= \tilde{\eta}_p, & \underline{\tilde{\eta}}_{n+2} &= 0. \end{aligned} \quad (4.65)$$

Using the standard gauge fixing for C' -matrix

$$C' = \begin{pmatrix} 1 & 0 & c'_{13} & \dots & c'_{1n+2} \\ 0 & 1 & c'_{23} & \dots & c'_{2n+2} \end{pmatrix}, \quad C'^{\perp} = \begin{pmatrix} -c'_{13} & -c'_{23} & 1 & 0 & \dots & 0 \\ -c'_{14} & -c'_{24} & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ -c'_{1n+2} & -c'_{2n+2} & 0 & 0 & \dots & 1 \end{pmatrix} \quad (4.66)$$

and integrating out the last $2n$ δ -functions we get

$$C' = \begin{pmatrix} 1 & 0 & -\frac{\langle 23 \rangle}{\langle 12 \rangle} & \dots & -\frac{\langle 2n \rangle}{\langle 12 \rangle} & -\frac{\langle 2p \rangle}{\langle 12 \rangle} & 1 \\ 0 & 1 & \frac{\langle 13 \rangle}{\langle 12 \rangle} & \dots & \frac{\langle 1n \rangle}{\langle 12 \rangle} & \frac{\langle 1p \rangle}{\langle 12 \rangle} & 0 \end{pmatrix} \quad (4.67)$$

and

$$A_{k,n+1}^* = \frac{1}{\kappa^*} \frac{1}{\langle 12 \rangle \langle 23 \rangle \dots \langle np \rangle \langle p1 \rangle} \delta^4 \left(\sum_{i=1}^n p_i + k \right) \delta^8 \left(\sum_{i=1}^n \lambda_i \tilde{\eta}_i + \lambda_p \tilde{\eta}_p \right). \quad (4.68)$$

In the case of NMHV₄ amplitude with one leg off-shell the corresponding Grassmannian integral representation is given by (here we will not choose any specific value of axillary spinor ξ)

$$A_{3,3+1}^* = \frac{\langle \xi p \rangle}{\kappa^*} \int \frac{d^{3 \times 5} C'}{\text{Vol}[\text{GL}(3)]} \frac{(512)}{(412)} \frac{\delta^6(C' \cdot \underline{\tilde{\lambda}}) \delta^8(C \cdot \underline{\tilde{\eta}}) \delta^4(C'^{\perp} \cdot \underline{\lambda})}{(123)(234)(345)(451)(512)}, \quad (4.69)$$

where $\tilde{\lambda}_i, \underline{\lambda}_i$ and $\tilde{\eta}_i$ are given by (4.65) with $n = 3$. Again using the standard gauge fixing for C' -matrix

$$C' = \begin{pmatrix} 1 & 0 & 0 & c'_{14} & c'_{15} \\ 0 & 1 & 0 & c'_{24} & c'_{25} \\ 0 & 0 & 1 & c'_{34} & c'_{35} \end{pmatrix}, \quad C'^{\perp} = \begin{pmatrix} -c'_{14} & -c'_{24} & -c'_{34} & 1 & 0 \\ -c'_{15} & -c'_{25} & -c'_{35} & 0 & 1 \end{pmatrix} \quad (4.70)$$

and integrating over the first 6 δ -functions we get

$$C' = \begin{pmatrix} 1 & 0 & 0 & c'_{14} = \frac{[15]}{[45]} & c'_{15} = \frac{[14]}{[45]} \\ 0 & 1 & 0 & c'_{24} = \frac{[25]}{[45]} & c'_{25} = \frac{[24]}{[45]} \\ 0 & 0 & 1 & c'_{34} = \frac{[35]}{[45]} & c'_{35} = \frac{[34]}{[45]} \end{pmatrix}, \quad (4.71)$$

so that

$$A_{3,3+1}^* = \delta^4(p_1 + p_2 + p_3 + k) \prod_{i=1}^3 \delta^4 \left(\tilde{\eta}_i + \tilde{\eta}_4 \frac{[i5]}{[45]} + \tilde{\eta}_5 \frac{[i4]}{[45]} \right) \times \\ \times \frac{\langle \xi p \rangle}{\kappa^*} \frac{(512)}{(412)} \times [45]^2 \times \frac{1}{[45]^3} \times \frac{[45]^5}{[12][23][34][45][51]}, \quad (4.72)$$

where using (4.71) we have

$$\frac{(512)}{(412)} = \frac{[43]}{[53]}. \quad (4.73)$$

In the expression above factor $[45]^2$ comes from $\delta^4(C'^{\perp} \cdot \underline{\lambda})$, which will give rise to the total momentum conservation δ -function, while $1/[45]^3$ is the contribution from integrating over $\delta^6(C' \cdot \tilde{\lambda})$. Combining all terms together we see that

$$A_{3,3+1}^* = \frac{\langle \xi p \rangle}{\kappa^*} \frac{[45]^3}{[12][23][35][51]} \delta^4(p_1 + p_2 + p_3 + k) \prod_{i=1}^3 \delta^4 \left(\tilde{\eta}_i + \tilde{\eta}_4 \frac{[i5]}{[45]} + \tilde{\eta}_5 \frac{[i4]}{[45]} \right). \quad (4.74)$$

Now one can use decomposition (2.4) of off-shell momentum k in terms of a pair on-shell momenta²⁴ p and $q = \xi \tilde{\xi}$ together with the definition (4.35) of spinors with numbers 4 and 5 to obtain the following relations

$$\begin{aligned} [i5] &= \frac{[i|k|p]}{\langle \xi p \rangle}, & [i4] &= \frac{[i|k|\xi]}{\langle \xi p \rangle}, & [45] &= \frac{k^2}{\langle \xi p \rangle}, \\ k^2 &= \kappa^* \kappa, & k|p] &= |p]\kappa^*. \end{aligned} \quad (4.75)$$

Using these relations $A_{3,3+1}^*$ could be rewritten in the following way

$$A_{3,3+1}^* = \delta^4(p_1 + p_2 + p_3 + k) \frac{\kappa^3}{[1p][p3][32][21]} \prod_{i=1}^3 \delta^4 \left(\tilde{\eta}_i + \frac{[pi]}{\kappa} \tilde{\eta}_p \right), \quad (4.76)$$

²⁴Here we choose q equal to $q = \xi \tilde{\xi}$, so that $\kappa = \langle \xi | k | p \rangle / \langle \xi p \rangle$ and $\kappa^* = \langle p | k | \xi \rangle / [p \xi]$.

which is simplified further to the form

$$A_{3,3+1}^* = \frac{\delta^4(p_1 + p_2 + p_3 + k)}{[12][23][3p][p1]} \frac{1}{\kappa} \frac{\delta^8(q_{1\dots 3} + \lambda_p \eta_p) \delta^4([p2]\tilde{\eta}_1 - [p1]\tilde{\eta}_2)}{\langle 3p \rangle^4}. \quad (4.77)$$

This result is in complete agreement with the result obtained previously from BCFW recursion. Indeed, taking for example component proportional to $\tilde{\eta}_1^4 \tilde{\eta}_2^4 \tilde{\eta}_4^4$ we reproduce (3.14).

Similar checks could be performed using momentum twistors. In the case of $k = 2$ the matrix D is zero-dimensional and all consecutive minors of D equal to one, while the nonconsecutive minor is zero. Thus, the integral in (4.63) is zero-dimensional while the integrand is 1, so that the result is given by prefactor $A_{2,n+1}^*$.

For $k = 3$ we have

$$D = (d_1 \ d_2 \ \dots \ d_{n+2}). \quad (4.78)$$

and

$$\frac{A_{3,n+1}^*}{A_{2,n+1}^*} = \omega_{n+2}^3[\Gamma_{3,n+2}] = \int_{\Gamma_{3,n+2}} \frac{d^{1 \times (n+2)} D}{\text{Vol}[\text{GL}(1)]} \frac{1}{1 + \frac{\langle p\xi \rangle}{\langle p1 \rangle} \frac{d_{n+2}}{d_1}} \frac{1}{d_1 d_2 \dots d_{n+2}} \delta^{4|4}(D \cdot \mathcal{Z}). \quad (4.79)$$

Let us first check this formula for $n = 3$. Choosing $\xi = \lambda_2$ (The choice $\xi = \lambda_1$ leads to divergences in denominator) we get

$$\frac{A_{3,3+1}^*}{A_{2,3+1}^*} = \int \frac{d^{1 \times 5} D}{\text{Vol}[\text{GL}(1)]} \frac{1}{1 + \frac{\langle p2 \rangle}{\langle p1 \rangle} \frac{d_5}{d_1}} \frac{1}{d_1 d_2 d_3 d_4 d_5} \delta^{4|4}(d_1 \mathcal{Z}_1 + d_2 \mathcal{Z}_2 + d_3 \mathcal{Z}_3 + d_4 \mathcal{Z}_4 + d_5 \mathcal{Z}_5). \quad (4.80)$$

The $\text{GL}(1)$ redundancy could be used to fix $d_5 = \langle 1 \ 2 \ 3 \ 4 \rangle$, where the four-bracket is defined as

$$\langle i \ j \ k \ l \rangle = \varepsilon_{ABCD} \mathcal{Z}_i^A \mathcal{Z}_j^B \mathcal{Z}_k^C \mathcal{Z}_l^D \quad (4.81)$$

The remaining four integration variables are then completely determined by δ -functions:

$$d_1 = \langle 2 \ 3 \ 4 \ 5 \rangle, \quad d_2 = \langle 3 \ 4 \ 5 \ 1 \rangle, \quad d_3 = \langle 4 \ 5 \ 1 \ 2 \rangle, \quad d_4 = \langle 5 \ 1 \ 2 \ 3 \rangle \quad (4.82)$$

and $A_{3,3+1}^*$ is given by

$$A_{3,3+1}^* = A_{2,3+1}^* \frac{[1 \ 2 \ 3 \ 4 \ 5]}{1 + \frac{\langle p2 \rangle}{\langle p1 \rangle} \frac{\langle 1 \ 2 \ 3 \ 4 \rangle}{\langle 2 \ 3 \ 4 \ 5 \rangle}}, \quad (4.83)$$

where the five-bracket is defined as

$$[i \ j \ k \ l \ m] = \frac{\delta^4(\langle i \ j \ k \ l \rangle \eta_m + \text{cyclic permutation})}{\langle i \ j \ k \ l \rangle \langle j \ k \ l \ m \rangle \langle k \ l \ m \ i \rangle \langle l \ m \ i \ j \rangle \langle m \ i \ j \ k \rangle}. \quad (4.84)$$

To compare (4.83) with previously obtained formula (4.76) it is convenient to use the following representation for the five-bracket

$$\begin{aligned} & [n \ j - 1 \ j \ k - 1 \ k] \\ &= R_{njk} \\ &= \frac{\langle j - 1 \ j \rangle^4 \langle k - 1 \ k \rangle^4 \delta^4(\Xi_{njk})}{\langle n \ j - 1 \ j \ k - 1 \rangle \langle j - 1 \ j \ k - 1 \rangle \langle j \ k - 1 \ k \ n \rangle \langle k - 1 \ k \ n \ j - 1 \rangle \langle k \ n \ j - 1 \ j \rangle}, \end{aligned} \quad (4.85)$$

where

$$\Xi_{njk} = \langle n | y_{nk} y_{kj} | j \rangle \tilde{\eta}_j + \langle n | y_{nj} y_{jk} | k \rangle \tilde{\eta}_k - \langle n | y_{nk} y_{kj} + y_{nj} y_{jk} | n \rangle \tilde{\eta}_n. \quad (4.86)$$

and $y_{jk} = y_j - y_k = p_j + \dots + p_{k-1}$. The four-brackets are easily calculated using the identity

$$\langle j-1 \ j \ k-1 \ k \rangle = \langle j-1 \ j \rangle \langle k-1 \ k \rangle y_{jk}^2. \quad (4.87)$$

This way we get

$$[1 \ 2 \ 3 \ 4 \ 5] = \frac{\langle 23 \rangle^4 \langle 45 \rangle^4 \delta^4(\Xi_{135})}{\langle 1 \ 2 \ 3 \ 4 \rangle \langle 2 \ 3 \ 4 \ 5 \rangle \langle 3 \ 4 \ 5 \ 1 \rangle \langle 4 \ 5 \ 1 \ 2 \rangle \langle 5 \ 1 \ 2 \ 3 \rangle}, \quad (4.88)$$

where

$$\Xi_{135} = \frac{\langle 12 \rangle^2 \kappa^*}{\langle 2p \rangle} \left\{ [p2] \tilde{\eta}_1 - [p1] \tilde{\eta}_2 \right\}, \quad (4.89)$$

and

$$\begin{aligned} \langle 1 \ 2 \ 3 \ 4 \rangle &= \langle 12 \rangle \langle 3p \rangle \langle 23 \rangle [32], & \langle 2 \ 3 \ 4 \ 5 \rangle &= -\langle 12 \rangle \langle 3p \rangle \langle 23 \rangle [13], & \langle 3 \ 4 \ 5 \ 1 \rangle &= -\langle 12 \rangle \langle 3p \rangle k^2, \\ \langle 4 \ 5 \ 1 \ 2 \rangle &= \langle 12 \rangle^2 \kappa^* [p1], & \langle 5 \ 1 \ 2 \ 3 \rangle &= \langle 12 \rangle^2 \langle 23 \rangle [12]. \end{aligned} \quad (4.90)$$

Finally the expression for $A_{3,3+1}^*$ is given by

$$A_{3,3+1}^* = \frac{1}{\kappa} \frac{\delta^4(p_1 + p_2 + p_3 + k)}{[12][23][3p][p1]} \frac{\delta^8(\lambda_1 \tilde{\eta}_1 + \lambda_2 \tilde{\eta}_2 + \lambda_3 \tilde{\eta}_3 + \lambda_p \tilde{\eta}_p) \delta^4([p2] \tilde{\eta}_1 - [p1] \tilde{\eta}_2)}{\langle 3p \rangle^4}. \quad (4.91)$$

Now let us consider NMHV₄₊₁ off-shell amplitude. In this case the Grassmannian integral Ω is no longer localized on δ -functions and the result of integration depends on the choice of integration contour Γ :

$$\Omega_{4+1}^k[\Gamma] = \frac{\langle \xi p \rangle}{\kappa^*} \int_{\Gamma} \frac{d^{3 \times 6} C'}{\text{Vol}[\text{GL}(3)]} \frac{(612)}{(512)} \frac{\delta^6(C' \cdot \tilde{\lambda}) \delta^8(C \cdot \tilde{\eta}) \delta^4(C'^{\perp} \cdot \underline{\lambda})}{(123)(234)(345)(456)(561)(612)}, \quad (4.92)$$

Using integration technique described in appendix B (see also [22]) the Grassmannian integral may be reduced to the integral over one complex parameter τ . The minors $(ii + 1i + 2)$ in the denominator of integrand are linear functions of τ in this case and the integral over τ can be easily evaluated by taking residues. Next, we choose integration contour Γ_{135} to encircle poles of inverse minors $1/(123)$, $1/(345)$ and $1/(561)$. The corresponding residues will be labeled as $\{1\}$, $\{3\}$ and $\{5\}$. In fact this particular integration contour is the same as the one in the case of NMHV₆ on-shell amplitude. The ratios of minors $(612)/(512)$ evaluated at the mentioned residues are given by

$$\left. \frac{(612)}{(512)} \right|_{\{1\}} = \frac{[45]}{[64]}, \quad \left. \frac{(612)}{(512)} \right|_{\{5\}} = \frac{\langle 16 \rangle}{\langle 15 \rangle}. \quad (4.93)$$

Now, let us consider a particular Grassmann component proportional to $\tilde{\eta}_3^4 \tilde{\eta}_4^4 \tilde{\eta}_5^4$, which corresponds to the helicities of on-shell particles $(1^+ 2^+ 3^- 4^-)$, considered in the section 3.

Using decomposition (2.4) of off-shell momentum k in terms of a pair of on-shell momenta p and $q = \xi\tilde{\xi}$ together with the definition (4.35) of spinors with numbers 5 and 6 (which are similar to NMHV₄ case considered before) the results of evaluation of residues could be written as (hereafter we will drop total momentum conservation δ -function)

$$\begin{aligned}\{1\} &= \frac{\langle \xi p \rangle}{\kappa^*} \frac{[45]}{[64]} \times \frac{\langle 3|1+2|6 \rangle^3}{\langle 12 \rangle \langle 23 \rangle [45] [56] p_{1,3}^2 \langle 1|2+3|4 \rangle} \\ &= \frac{\langle \xi p \rangle}{\kappa^*} \frac{\langle \xi | k | 4 \rangle}{\kappa^* [p4]} \times \frac{\langle 3|(1+2)k|p \rangle^3 \langle \xi p \rangle^{-3}}{\langle 12 \rangle \langle 23 \rangle \langle \xi | k | 4 \rangle \langle \xi p \rangle^{-1} k^2 \langle \xi p \rangle^{-1} p_{1,3}^2 \langle 1|2+3|4 \rangle} \\ &= \frac{1}{\kappa} \frac{\langle 3|1+2|p \rangle^3}{\langle 12 \rangle \langle 23 \rangle [p4] p_{1,3}^2 \langle 1|2+3|4 \rangle},\end{aligned}\tag{4.94}$$

$$\begin{aligned}\{5\} &= \frac{\langle \xi p \rangle}{\kappa^*} \frac{\langle 16 \rangle}{\langle 15 \rangle} \times \frac{\langle 5|3+4|2 \rangle^3}{\langle 56 \rangle \langle 61 \rangle [23] [34] p_{2,4}^2 \langle 1|2+3|4 \rangle} \\ &= \frac{1}{\kappa^*} \frac{\langle p|3+4|2 \rangle^3}{\langle p1 \rangle [23] [34] p_{2,4}^2 \langle 1|2+3|4 \rangle},\end{aligned}\tag{4.95}$$

and $\{3\} = 0$. So that, finally we get

$$\Omega_{4+2}^3[\Gamma_{135}] \Big|_{\tilde{\eta}_3^4 \tilde{\eta}_4^4 \tilde{\eta}_5^4} = \{1\} + \{3\} + \{5\} = A_{3,4+1}^*(1^+ 2^+ 3^- 4^- 5^*).\tag{4.96}$$

Other helicity configurations as well as supersymmetric (with respect to on-shell particles) result can be obtained in similar fashion.

At the end of this section let us reproduce the result for $A_{3,4+1}^*$ off-shell amplitude in a manifestly supersymmetric way (with respect to on-shell particles) using momentum twistor representation. Considering Grassmannian integral

$$\omega_{4+2}^3[\Gamma] = \int_{\Gamma} \frac{d^{1 \times 6} D}{\text{Vol}[\text{GL}(1)]} \frac{1}{1 + \frac{\langle p\xi \rangle}{\langle p1 \rangle} \frac{d_6}{d_1}} \frac{1}{d_1 d_2 d_3 d_4 d_5 d_6} \delta^{4|4} \left(\sum_{i=1}^6 d_i \mathcal{Z}_i \right),\tag{4.97}$$

and using integration method suggested in [103] (see also appendix B) the result for integration contour Γ_{246} encircling poles in d_2 , d_4 and d_6 reads:

$$\omega_{4+2}^3[\Gamma_{246}] = \frac{1}{1 + \frac{\langle p\xi \rangle}{\langle p1 \rangle} \frac{\langle 1345 \rangle}{\langle 3456 \rangle}} [13456] + \frac{1}{1 + \frac{\langle p\xi \rangle}{\langle p1 \rangle} \frac{\langle 1235 \rangle}{\langle 2356 \rangle}} [12356] + [12345] = \frac{A_{3,4+1}^*}{A_{2,4+1}^*}.\tag{4.98}$$

We verified that this expression is free from spurious poles of the form $1/\langle abcd \rangle$.²⁵ To simplify comparison with the results of BCFW recursion lets us rewrite answer for $A_{3,6}$ (3.25) in terms of momentum twistor variables:

$$A_{3,6}/A_{2,6} = [12345] + [13456] + [12356].\tag{4.99}$$

In fact it is not difficult to consider a generalization of this result for an arbitrary number of on-shell legs n . Choosing integration contour $\Gamma_{3,n+2}$ similar to the case of [12]

²⁵We are going to discuss this in more detail in a separate publication.

BCFW shift representation of NMHV_{n+2} the amplitude (with an additional condition to avoid pole $\langle p\xi\rangle d_{n+2} = -\langle p1\rangle d_1$) and following along the same lines as before (see also discussion in appendix B) the Grassmannian integral:

$$\omega_{n+2}^3[\Gamma] = \int_{\Gamma} \frac{d^{1 \times (n+2)} D}{\text{Vol}[\text{GL}(1)]} \frac{1}{1 + \frac{\langle p\xi\rangle}{\langle p1\rangle} \frac{d_{n+2}}{d_1}} \frac{1}{d_1 d_2 \dots d_{n+2}} \delta^{4|4}(D \cdot \mathcal{Z}), \quad (4.100)$$

is evaluated to

$$\omega_{n+2}^3[\Gamma_{3,n+2}] = \sum_{i < j}^{n+2} c_j [1 \ i - 1 \ i \ j - 1 \ j]. \quad (4.101)$$

with

$$c_{n+2} = \frac{1}{1 + \frac{\langle p\xi\rangle}{\langle p1\rangle} \frac{\langle 1 \ i-1 \ i \ n+1 \rangle}{\langle i-1 \ i \ n+1 \ n+2 \rangle}}, \text{ and } c_j = 1 \text{ if } j < n+2. \quad (4.102)$$

It is natural to conjecture that

$$\omega_{n+2}^3[\Gamma_{3,n+2}] = \frac{A_{3,n+1}^*}{A_{2,n+1}^*}. \quad (4.103)$$

4.4 Regularization of soft limit and soft theorems for deformed Grassmannian integral

As was noted in [61, 63], in a similar case of Grassmannian description of form factors of operators from stress tensor operator supermultiplet the deformation of Grassmannian integral (the combination of non-consecutive minors in addition to the string of consecutive minors in the denominator of integrands in (4.38) and (4.63)) can be considered as IR regulator of some sort. Namely, it regulates soft limit behavior with respect to the momentum carried by operator $q \rightarrow 0$.

Here we want to argue that similar behavior holds also in the case of deformations of the Grassmannian integrals considered here. Let us show, that the Grassmannian integral (4.38)

$$\Omega_{n+2}^k[\Gamma] = \int_{\Gamma} \frac{d^{k \times (n+2)} C'}{\text{Vol}[\text{GL}(k)]} \text{Reg.} \frac{\delta^{k \times 2} (C' \cdot \underline{\tilde{\lambda}}) \delta^{k \times 4} (C' \cdot \underline{\tilde{\eta}}) \delta^{(n+2-k) \times 2} (C'^{\perp} \cdot \underline{\lambda})}{(1 \dots k) \dots (n+1 \dots k-2)(n+2 \ 1 \dots k-1)}, \quad (4.104)$$

with

$$\text{Reg.} = \frac{\langle \xi p \rangle}{\kappa^*} \frac{(n+2 \ 1 \dots k-1)}{(n+1 \ 1 \dots k-1)}, \quad (4.105)$$

is regular with respect to the holomorphic soft limit of the axillary four-vector q . Here we are using spinorial decomposition of q and $\underline{\lambda}_{n+2}$ from (4.38): $q \equiv \xi \tilde{\xi}$, $\underline{\lambda}_{n+2} \equiv \xi$. So, we are expecting to obtain finite limit $\epsilon \rightarrow 0$ for the following expression

$$\Omega_{n+2}^k[\Gamma] \Big|_{\xi \mapsto \epsilon \xi}. \quad (4.106)$$

The behavior of $\Omega_{n+2}^k[\Gamma]$ with respect to holomorphic soft limit of $\underline{\lambda}_i$, $i \leq n$ is controlled by standard soft theorems and could be obtained for our Grassmannian integrals using the method presented in [104] (here to simplify notation we used $\tilde{\lambda} \equiv \underline{\lambda}$):

$$\Omega_{n+2}^k[\Gamma_{k,n+2}] \Big|_{\lambda_i \mapsto \epsilon \lambda_i} = \left(\frac{\hat{S}_1}{\epsilon^2} + \frac{\hat{S}_2}{\epsilon} \right) \Omega_{n+2}^k[\Gamma_{k,n+1}] + \text{fin.} + O(\epsilon), \quad \epsilon \rightarrow 0, \quad (4.107)$$

with

$$\begin{aligned} \hat{S}_1 &= \frac{\langle i-1 i+1 \rangle}{\langle ii+1 \rangle \langle ii-1 \rangle}, \\ \hat{S}_2 &= \frac{\tilde{\lambda}_i^\alpha}{\langle ii-1 \rangle} \frac{\partial}{\partial \tilde{\lambda}_{i-1}^\alpha} + \frac{\tilde{\lambda}_i^\alpha}{\langle ii+1 \rangle} \frac{\partial}{\partial \tilde{\lambda}_{i+1}^\alpha} + \frac{\eta_{A,i}}{\langle ii-1 \rangle} \frac{\partial}{\partial \eta_{A,i-1}} + \frac{\eta_{A,i}}{\langle ii+1 \rangle} \frac{\partial}{\partial \eta_{A,i+1}}. \end{aligned} \quad (4.108)$$

Now let's consider behavior of $\Omega_{n+2}^k[\Gamma]$ with respect to soft limit of ξ . We will consider $k=3$ as an example. It is convenient to fix $GL(3)$ gauge and parametrize C' -matrix as in [104] (the columns are numerated as $(1, 2, \dots, n, n+1, n+2)$):

$$C' = \begin{pmatrix} 0 & c_{n2} & \dots & 1 & 0 & c_{nn+2} \\ 0 & c_{n+12} & \dots & 0 & 1 & c_{n+1n+2} \\ 1 & c_{12} & \dots & 0 & 0 & c_{1n+2} \end{pmatrix}. \quad (4.109)$$

In such parametrization the minors in Reg. are given by

$$(n+212) = c_{n2}c_{n+1n+2} - c_{n+12}c_{nn+2}, \quad (n+112) = c_{n2}. \quad (4.110)$$

Then Ω_{n+2}^3 could be rewritten as (here as in [104] we suppress explicit dependence on Grassmann variables)

$$\begin{aligned} \Omega_{n+2}^3[\Gamma_{3,n+2}] \Big|_{\xi \mapsto \epsilon \xi} &= \int d^3 c_{In+2} \delta^2(\epsilon \xi - \underline{\lambda}_I c_{In+2}) \frac{\epsilon \text{Reg.}(nn+11)'(n+112)'}{(nn+1n+2)(n+1n+21)(n+212)} \\ &\quad \times \hat{\Omega}_{n+1}^3[\Gamma_{3,n+1}] \Big|_{\xi \mapsto \epsilon \xi}, \end{aligned} \quad (4.111)$$

where (index I runs over $1, n, n+1$)

$$\int d^3 c_{In+2} = \int dc_{1n+2} dc_{n+1n+2} dc_{nn+2}, \quad (4.112)$$

and minors with primes like $(nn+11)'$ are evaluated in $Gr(3, n+1)$ Grassmannian in contrast to other minors evaluated in $Gr(3, n+2)$ Grassmannian, $\Gamma_{3,n+2}$ contour contains the same poles as $\Gamma_{3,n+1}$ together with additional pole $(n+1 \ n+2 \ 1)$. Here, we have also used the fact that $\langle \xi p \rangle / \kappa^* \Big|_{\xi \mapsto \epsilon \xi} = \epsilon \langle \xi p \rangle / \kappa^*$. The hat in $\hat{\Omega}_{n+1}^3$ denotes the absence of Reg. factor and that $\tilde{\lambda}_1$ and $\tilde{\lambda}_{n+1}$ spinors are shifted as

$$\begin{aligned} \tilde{\lambda}_1 &\mapsto \tilde{\lambda}_1 + c_{1n+2} \tilde{\lambda}_{n+2}, \\ \tilde{\lambda}_{n+1} &\mapsto \tilde{\lambda}_{n+1} + c_{n+1n+2} \tilde{\lambda}_{n+2}. \end{aligned} \quad (4.113)$$

The integral $\int d^3 c_{In}$ can be evaluated by taking residue at pole $1/(n+1n+21)$, which fixes the values of c_{nn+2} and c_{n+1n+2} , c_{1n+2} coefficients to be (we use spinor definitions from (4.35))

$$c_{nn+2} = 0, \quad c_{n+1n+2} = \frac{\langle 1\xi \rangle \epsilon}{\langle 1p \rangle}, \quad c_{1n+2} = \frac{\langle p\xi \rangle \epsilon}{\langle 1p \rangle}. \quad (4.114)$$

Then the result of integration could be written as

$$\int d^3 c_{In+2} \delta^2(\epsilon \xi - \underline{\lambda}_I c_{In+2}) \frac{\epsilon \text{Reg.}(nn+11)'(n+112)'}{(nn+1n+2)(n+1n+21)(n+212)} = \frac{\langle 1p \rangle}{\epsilon^2 \langle 1\xi \rangle \langle \xi p \rangle} \epsilon \text{Reg.},$$

with Reg. function being evaluated at $(n+1n+21)$ residue, which is given by

$$\epsilon \text{Reg.} \Big|_{(n+1n+21)} = \epsilon^2 \frac{\langle \xi p \rangle}{\kappa^*} \frac{\langle 1\xi \rangle}{\langle 1p \rangle}. \quad (4.115)$$

So, taking $\epsilon \rightarrow 0$ limit and accounting for (4.113) and (4.114) we get

$$\Omega_{n+2}^3[\Gamma_{3,n+2}] \Big|_{\xi \mapsto \epsilon \xi} = \frac{1}{\kappa^*} \hat{\Omega}_{n+1}^3[\Gamma_{3,n+1}] \Big|_{\epsilon=0} + O(\epsilon). \quad (4.116)$$

Thus, we see that presented here deformation of Grassmannian integral could be also considered as a regularization of soft limit behavior. The case of general k is more involved, but we should get similar result, that is

$$\Omega_{n+2}^k[\Gamma_{k,n+2}] \Big|_{\xi \mapsto \epsilon \xi} = \text{fin.} + O(\epsilon). \quad (4.117)$$

The soft limit considered could also serve as a prescription of how to obtain corresponding on-shell amplitudes from our off-shell expressions. Indeed, $\hat{\Omega}_{n+1}^3[\Gamma_{3,n+1}] \Big|_{\epsilon=0}$ in the above expressions is nothing else but the Grassmannian representation of NMHV $_{n+1}$ amplitude with $n+1$ on-shell particles, where momentum of i 'th particle p_{n+1} is given by

$$p_{n+1} = \lambda_p \frac{k|\xi\rangle}{\langle \xi p \rangle}. \quad (4.118)$$

This observation is in agreement with the on-shell limit prescription of [64]. In addition we would like to note, that double soft limit with respect to $\underline{\lambda}_{n+2}$ and $\underline{\lambda}_{n+1}$ will be singular and controlled by soft theorems.

5 Amplitudes with one leg off-shell and integrability

Yangian symmetry, which combines invariance under superconformal and dual superconformal transformations [10], for on-shell tree-level amplitudes was proven in [12]. It was further claimed [28, 29], that the Grassmannian integral representation for amplitudes (4.2) is the most general form of rational Yangian invariant making all symmetries of the theory manifest. The existence of Yangian symmetry allows us to reformulate the problem of finding expressions for the scattering amplitudes in the language of integrable systems, in particular in the language of Quantum Inverse Scattering Method (QISM).

The study of tree-level scattering amplitudes within the context of QISM was started in [15, 16], where the notion of spectral parameter was introduced. The latter was interpreted as a deformed particle helicity. Later the authors of [17, 18] proposed to study certain auxiliary spin chain monodromies build from local Lax operators. The introduced monodromies depended on an auxiliary spectral parameter, while the spectral parameters of [15, 16] played the role of inhomogeneities of Lax operators. Yangian invariants and thus on-shell amplitudes are then found as the eigenstates of these monodromies. Further, in [19, 20] a systematic classification of Yangian invariants obtained within QISM was provided. Yangian invariance can be defined in a very compact form as a system of eigenvalue equations for the elements of a suitable monodromy matrix $M(u)$ [17–19]:

$$M_{ab}(u)|\Psi\rangle = C_{ab}|\Psi\rangle, \quad (5.1)$$

where u is the auxiliary spectral parameter mentioned above, C_{ab} are monodromy eigenvalues and monodromy eigenvectors $|\Psi\rangle$ are elements of the Hilbert space $V = V_1 \otimes \dots \otimes V_n$ with V_i being a representation space of a particular $\mathfrak{gl}(N|M)$ representation. To describe tree-level scattering amplitudes one will need to specialize to the case of $N|M = 4|4$ and its non-compact representations build using a single family of Jordan-Schwinger harmonic superoscillators $\bar{\mathbf{w}}^{\mathcal{A}}, \mathbf{w}^{\mathcal{B}}, \mathcal{A}, \mathcal{B} = 1 \dots 8$. The latter could be conveniently realized in terms of Heisenberg pairs

$$J^{\mathcal{AB}} = \bar{\mathbf{w}}^{\mathcal{A}} \mathbf{w}^{\mathcal{B}} = x^{\mathcal{A}} p^{\mathcal{B}}, \quad x^{\mathcal{A}} = \left(\lambda^{\alpha}, -\frac{\partial}{\partial \tilde{\lambda}_{\dot{\alpha}}}, \frac{\partial}{\partial \tilde{\eta}^{\dot{A}}} \right), \quad p^{\mathcal{A}} = \left(\frac{\partial}{\partial \lambda^{\alpha}}, \tilde{\lambda}^{\dot{\alpha}}, \tilde{\eta}^{\dot{A}} \right), \quad (5.2)$$

such that $[x^{\mathcal{A}}, p^{\mathcal{B}}] = (-1)^{|\mathcal{A}|} \delta^{\mathcal{AB}}$. Here $[\cdot, \cdot]$ denotes graded commutator and $|\cdot|$ -grading. A vacuum state for the Hilbert space V used to construct Yangian invariants $|\Psi\rangle_{n,k}$ corresponding to the on-shell $N^{k-2}\text{MHV}$ n -point tree-level amplitudes $A_{n,k}$ is given by

$$|\mathbf{0}\rangle_{k,n} = \delta_1^+ \dots \delta_{n-k}^+ \delta_{n-k+1}^- \dots \delta_n^-, \quad (5.3)$$

where $\delta_i^+ \equiv \delta^2(\lambda_i)$ is the vacuum for positive helicity state at position i and $\delta_i^- \equiv \delta^2(\tilde{\lambda}_i) \delta^4(\tilde{\eta}_i)$ is the corresponding vacuum for negative helicity state at the same position. The monodromy matrix of the auxiliary spin chain reads

$$M(u, \{v_i\}) = \mathcal{L}_1(u, v_1) \dots \mathcal{L}_k(u, v_k) \mathcal{L}_{k+1}(u, v_{k+1}) \dots \mathcal{L}_n(u, v_n), \quad (5.4)$$

where u is again the auxiliary spectral parameter, v_i are spin chain inhomogeneities and Lax operators $\mathcal{L}_i(u, v)$ are given by

$$\mathcal{L}(u, v) = u - v + \sum_{a,b} e_{ab} J_{ba}, \quad (5.5)$$

where matrix e_{ab} acting in the auxiliary space is given by $(e_{ab})_{cd} = \delta_{ac} \delta_{bd}$. It is easy to see, that the action of Lax operators on particles vacuums is given by

$$\mathcal{L}_i(u) \delta_i^+ = (u - 1) \mathbb{I} \delta_i^+, \quad \mathcal{L}_i(u) \delta_i^- = u \mathbb{I} \delta_i^-. \quad (5.6)$$

The solution of the eigenvalue equation²⁶ for monodromy matrix (5.1) leads to the expressions for Yangian invariants labeled by the permutations σ with minimal²⁷ decomposition $\sigma = (i_1, j_1) \dots (i_P, j_P)$ [17, 19, 20]:

$$|\Psi\rangle = \mathcal{R}_{i_1, j_1}(\bar{u}_1) \dots \mathcal{R}_{i_P, j_P}(\bar{u}_P) |\mathbf{0}\rangle_{k, n} \quad (5.7)$$

with [17] (see also [18])

$$\mathcal{R}_{ij}(u) = \Gamma(-u)(x_j \cdot p_i)^u = \int_0^\infty \frac{d\alpha}{\alpha^{1+u}} e^{-\alpha(x_j \cdot p_i)}, \quad (5.8)$$

where Γ is the Euler gamma function and

$$\bar{u}_p = v_{\tau_p(i_p)} - v_{\tau_p(j_p)}, \quad \tau_p = \tau_{p-1} \circ (i_p, j_p) = (i_1, j_1) \dots (i_p, j_p). \quad (5.9)$$

As we already mentioned, amplitudes with one leg off-shell can be considered as form factors of Wilson line operator (2.8) corresponding to the off-shell leg. It turns out, that QISM machinery could be also used in the case of form factors [62], see also [61]. The only new ingredient needed is the spin chain vacuum state corresponding to the minimal form factor. The latter in the case of form factors of stress-tensor operator supermultiplet is given by [62]:

$$\mathcal{F}_{2,2}(1, 2) = \delta^2(\tilde{\lambda}_1) \delta^4(\tilde{\eta}_1) \delta^2(\tilde{\lambda}_2) \delta^4(\tilde{\eta}_2), \quad (5.10)$$

where²⁸

$$\begin{aligned} \tilde{\lambda}_1 &= \tilde{\lambda}_1 - \frac{\langle 2|q}{\langle 2|1\rangle} & \underline{\eta}_1^- &= \eta_1^- - \frac{\langle 2|\gamma^-}{\langle 2|1\rangle} & \underline{\eta}_1^+ &= \eta_1^+ \\ \tilde{\lambda}_2 &= \tilde{\lambda}_2 - \frac{\langle 1|q}{\langle 1|2\rangle} & \underline{\eta}_2^- &= \eta_2^- - \frac{\langle 1|\gamma^-}{\langle 1|2\rangle} & \underline{\eta}_2^+ &= \eta_2^+ \end{aligned} \quad (5.11)$$

The vacuum state for the deformed minimal amplitude (vertex) with one leg off-shell could be easily obtained by performing integrations in (4.18). This way we get:

$$A_{2,2+1}^*(1^*, 2, 3) = \frac{1}{\kappa^*} \frac{\langle 23 \rangle}{\langle p2 \rangle \langle p3 \rangle} \left(\frac{\langle p3 \rangle}{\langle p2 \rangle} \right)^{u_{32}} \left(\frac{\langle 23 \rangle}{\langle p3 \rangle} \right)^{u_{31}} \delta^2(\tilde{\lambda}_2) \delta^2(\tilde{\lambda}_3) \delta^4(\tilde{\eta}_2) \delta^4(\tilde{\eta}_3), \quad (5.12)$$

where (k is the off-shell gluon momentum and p is its direction as before)

$$\tilde{\lambda}_2 = \tilde{\lambda}_2 + \frac{\langle 3|k}{\langle 32 \rangle}, \quad \tilde{\lambda}_3 = \tilde{\lambda}_3 + \frac{\langle 2|k}{\langle 23 \rangle} \quad \tilde{\eta}_2 = \tilde{\eta}_2 + \frac{\langle p3 \rangle}{\langle 23 \rangle} \tilde{\eta}_p, \quad \tilde{\eta}_3 = \tilde{\eta}_3 + \frac{\langle p2 \rangle}{\langle 32 \rangle} \tilde{\eta}_p. \quad (5.13)$$

Here we will however restrict ourselves to the case of undeformed minimal off-shell amplitude given by

$$A_{2,2+1}^*(2, 3) = A_{2,2+1}^*(1^*, 2, 3) = \frac{1}{\kappa^*} \frac{\langle 23 \rangle}{\langle p2 \rangle \langle p3 \rangle} \delta^2(\tilde{\lambda}_2) \delta^2(\tilde{\lambda}_3) \delta^4(\tilde{\eta}_2) \delta^4(\tilde{\eta}_3). \quad (5.14)$$

²⁶It should be noted that this eigenvalue equation is different from the usual Bethe ansatz equations, which diagonalize only the trace of monodromy matrix.

²⁷The decomposition is minimal in a sense, that there exists no other decomposition of σ into a smaller number of transpositions.

²⁸See appendix A of [61] for the notation used in harmonic superspace description of form factors of operators from stress-tensor operator supermultiplet.

Next, similar to the case of form factors of operators from stress-tensor operator supermultiplet [62] let us consider off-shell amplitudes defined as²⁹

$$\hat{\mathcal{A}} = \mathcal{R}_{i_1 j_1}(\bar{u}_1) \cdots \mathcal{R}_{i_P j_P} A_{2,2+1}^{*,\delta}(m-1, m) \quad (5.15)$$

with

$$A_{2,2+1}^{*,\delta}(m-1, m) = \delta_1^+ \cdots \delta_{m-2}^+ A_{2,2+1}^*(m-1, m) \delta_{m+1}^- \cdots \delta_n^-. \quad (5.16)$$

As was shown in [17, 19, 20, 62], the monodromy matrix (5.4) (as a consequence of Yang-Baxter equation) satisfies the following relation

$$M(u, \{v_i\}) \mathcal{R}_{i_1 j_1}(\bar{u}_1) \cdots \mathcal{R}_{i_P j_P}(\bar{u}_P) = \mathcal{R}_{i_1 j_1}(\bar{u}_1) \cdots \mathcal{R}_{i_P j_P}(\bar{u}_P) M(u, \{v_{\sigma(i)}\}), \quad (5.17)$$

where $M(u, \{v_{\sigma(i)}\})$ is the monodromy matrix with inhomogeneities v_i replaced with $v_{\sigma(i)}$. Now, commuting monodromy matrix through $\mathcal{R}$ -chain in (5.15) we get

$$M_n(u, \{v_i\}) \hat{\mathcal{A}} = \mathcal{R}_{i_1 j_1}(\bar{u}_1) \cdots \mathcal{R}_{i_P j_P}(\bar{u}_P) M_n(u, \{v_{\sigma(i)}\}) A_{2,2+1}^{*,\delta}(m-1, m). \quad (5.18)$$

Taking into account, that Lax operators act diagonally on vacua (5.6) we can write:

$$\begin{aligned} M_n(u, \{v_{\sigma(i)}\}) A_{2,2+1}^{*,\delta}(m-1, m) &= \prod_{i=1}^{m-2} (u - v_{\sigma(i)} - 1) \prod_{j=m+1}^n (u - v_{\sigma(j)}) \\ &\times \delta_1^+ \cdots \delta_{m-2}^+ \left[M_2(u, \{v_{\sigma(i)}\}) A_{2,2+1}^*(m-1, m) \right] \delta_{m+1}^- \cdots \delta_n^-, \end{aligned} \quad (5.19)$$

where length 2 monodromy matrix is given by

$$M_2(u, \{v_{\sigma(i)}\}) = \mathcal{L}_{m-1}(u - v_{\sigma(m-1)}) \mathcal{L}_m(u - v_{\sigma(m)}). \quad (5.20)$$

The minimal off-shell vertex $A_{2,2+1}^*(m-1, m)$ is not an eigenstate of monodromy matrix $M_2(u, \{v_{\sigma(i)}\})$ and Yangian invariance for off-shell amplitudes is broken similar to the case of form factors with $q^2 \neq 0$ [62]. This conclusion easily follows from the fact, that the momentum-like generators do not contain off-shell momentum and the action of monodromy matrix on minimal off-shell vertex produces [62]

$$(\lambda_{m-1} \tilde{\lambda}_{m-1} + \lambda_m \tilde{\lambda}_m) \delta^4(\lambda_{m-1} \tilde{\lambda}_{m-1} + \lambda_m \tilde{\lambda}_m + k), \quad (5.21)$$

which does not vanish. On the other hand, again similar to the case of form factors with $q^2 \neq 0$ the amplitudes with one leg off-shell turn out to be eigenvectors of transfer matrix. The later is defined as the super trace of monodromy matrix over auxiliary space:

$$\mathcal{T}_n(u, \{v_i\}) = \text{str} M_n(u, \{v_i\}). \quad (5.22)$$

One can use Yang-Baxter equation to show, that this transfer matrix is $\mathfrak{gl}(4|4)$ invariant, that is

$$\left[\mathcal{T}_n(u, \{v_i\}), \sum_{i=1}^n x_i^{\mathcal{A}} p_i^{\mathcal{B}} \right] = 0. \quad (5.23)$$

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²⁹Actually it is just one of BCFW contributions and the expression for amplitude is obtained as linear combination of such terms.

To show, that amplitudes with one leg off-shell are annihilated by transfer matrix, let us consider first the minimal off-shell vertex and length 2 transfer matrix with equal inhomogeneities

$$\mathcal{T}_2(u-v) = \text{str} \mathcal{L}_m(u-v) \mathcal{L}_{m-1}(u-v). \quad (5.24)$$

Next, we can exploit $\mathfrak{gl}(4|4)$ invariance of transfer matrix and consider the particular component of the minimal off-shell amplitude $A_{2,2+1}^*(m-1, m)$, for example:

$$\frac{1}{\kappa^*} \frac{\langle p \ m-1 \rangle \langle p \ m \rangle}{\langle m-1 \ m \rangle} \tilde{\eta}_{m-1}^2 \tilde{\eta}_m^2 \tilde{\eta}_p^4 \delta^4(\lambda_{m-1} \tilde{\lambda}_{m-1} + \lambda_m \tilde{\lambda}_m + k), \quad (5.25)$$

that is scalar-scalar-off-shell gluon vertex. Here, superscripts next to Grassmann variables are the numbers of copies of corresponding Grassmann variables and not $SU(4)_R$ indexes. Similar to [62] we have

$$\mathcal{T}_2(u-v) \delta^4(\lambda_{m-1} \tilde{\lambda}_{m-1} + \lambda_m \tilde{\lambda}_m + k) = 0, \quad (5.26)$$

$$\mathcal{T}_2(u-v) \tilde{\eta}_{m-1}^2 \tilde{\eta}_m^2 \tilde{\eta}_p^4 = \left((u-v-1) x_{m-1}^{\mathcal{A}} p_{m-1}^{\mathcal{A}} + (-1)^{\mathcal{A}} p_{m-1}^{\mathcal{A}} x_{m-1}^{\mathcal{B}} p_m^{\mathcal{B}} x_m^{\mathcal{A}} \right) \tilde{\eta}_{m-1}^2 \tilde{\eta}_m^2 \tilde{\eta}_p^4 = 0 \quad (5.27)$$

and we have also checked, that

$$\mathcal{T}_2(u-v) \frac{\langle p \ m-1 \rangle \langle p \ m \rangle}{\langle m-1 \ m \rangle} = 0 \quad (5.28)$$

This shows, that the minimal off-shell amplitude $A_{2,2+1}^*(m-1, m)$ is an eigenstate of the transfer matrix, that is

$$\mathcal{T}_2(u-v) A_{2,2+1}^*(m-1, m) = 0. \quad (5.29)$$

Moreover, as the transfer matrix satisfies a relation similar to (5.19) the same statement also applies to any planar on-shell diagram with a minimal off-shell vertex insertion:

$$\mathcal{T}_n(u, \{v_i\}) \hat{\mathcal{A}} = 0. \quad (5.30)$$

with $v_i = v$. This property is in fact a consequence of multiplicative renormalizability of minimal off-shell vertex, similar to multiplicative renormalizability of stress-tensor operator supermultiplet in [62].

At the end of this section let us show on a simple example of $A_{2,3+1}^*$ off-shell amplitude how one can use QISM technique to get explicit expressions for spectral deformations of off-shell amplitudes and Yangian invariants they are build from. First we perform the minimal decomposition of corresponding permutation $\sigma = (3, 1, 2) = (2, 3)(1, 2)$. Then $A_{2,3+1}^*(\bar{u}_1, \bar{u}_2)$ is given by

$$A_{2,3+1}^*(\bar{u}_1, \bar{u}_2) = \mathcal{R}_{23}(\bar{u}_1) \mathcal{R}_{12}(\bar{u}_2) \delta^2(\lambda_1) \frac{1}{\kappa^*} \frac{\langle 23 \rangle}{\langle p2 \rangle \langle p3 \rangle} \delta^2(\tilde{\lambda}_2) \delta^2(\tilde{\lambda}_3) \delta^4(\tilde{\eta}_2) \delta^4(\tilde{\eta}_3), \quad (5.31)$$

where $\lambda_i, \tilde{\eta}_i$ are defined in (5.13) and $\bar{u}_1 = v_{32} = v_3 - v_2$, $\bar{u}_2 = v_{31} = v_3 - v_1$. Using definition of $\mathcal{R}$ operators (5.8) we get:

$$\begin{aligned}
& A_{2,3+1}^*(v_1, v_2, v_3) \\
&= \frac{1}{\kappa^*} \int \frac{d\alpha_2}{\alpha_2} \int \frac{d\alpha_1}{\alpha_1} \frac{\langle 23 \rangle}{\langle p2 \rangle \langle p3 \rangle \left(1 - \alpha_2 \frac{\langle p3 \rangle}{\langle p2 \rangle}\right)} \times \\
&\quad \times \delta^2(\lambda_1 - \alpha_1 \lambda_2 + \alpha_1 \alpha_2 \lambda_3) \delta^2(\tilde{\lambda}_2 + \alpha_1 \tilde{\lambda}_1) \delta^4(\tilde{\eta}_2 + \alpha_1 \tilde{\eta}_1) \delta^2(\tilde{\lambda}_3 + \alpha_2 \tilde{\lambda}_2) \delta^4(\tilde{\eta}_3 + \alpha_2 \tilde{\eta}_2) \\
&= \frac{1}{\kappa^*} \frac{1}{\langle p1 \rangle \langle 12 \rangle \langle 23 \rangle \langle 3p \rangle} \left(\frac{\langle 23 \rangle}{\langle 13 \rangle} \right)^{v_{31}} \left(\frac{\langle 13 \rangle}{\langle 12 \rangle} \right)^{v_{32}} \delta^4 \left(\sum_{i=1}^3 \lambda_i \tilde{\lambda}_i + k \right) \delta^8 \left(\sum_{i=1}^3 \lambda_i \tilde{\eta}_i + \lambda_p \tilde{\eta}_p \right).
\end{aligned} \tag{5.32}$$

The off-shell amplitude $A_{2,3+1}^*$ is then recovered by setting to zero deformation parameters $v_i = 0$.

6 Conclusion

In this article we considered Grassmannian integral representation for tree level gauge invariant $\mathcal{N} = 4$ SYM off-shell amplitudes with one leg off-shell (Wilson line form factors). We presented a conjecture for Grassmannian integral representation for amplitudes with all on-shell particles treated in a manifestly supersymmetric manner and the only off-shell gluon (Wilson line insertion) treated in non-supersymmetric way. We have considered spinor helicity, twistor and momentum twistor versions of Grassmannian integral representation and successfully checked them by reproducing BCFW results for MHV_{n+1} , NMHV_{3+1} and NMHV_{4+1} gauge invariant off-shell amplitudes. In addition from our Grassmannian integral representation we reproduced appropriate soft (on-shell) limit. Using Grassmannian integral representation we have also derived a conjecture for NMHV_{n+1} gauge invariant off-shell amplitudes. We have investigated integrability properties of gauge invariant off-shell amplitudes and showed that similar to the case of form factors of local gauge invariant operators gauge invariant off-shell amplitudes are no longer eigenvectors of monodromy matrix of the auxiliary $\mathfrak{gl}(4|4)$ spin chain. The latter, however, turn out to be eigenvectors of corresponding transfer matrix.

There are several possible generalizations of the construction and ideas presented in this article. First, it would be interesting to investigate loop corrections to the gauge invariant off-shell amplitudes. Off-shellness of the gluon should play the role of natural IR regularization.³⁰ This will help to clarify the relation between Grassmannian integrals considered here and leading singularities of off-shell loop amplitudes. The rigorous supersymmetric consideration of the off-shell gluon (Wilson line) is another direction to follow. Finally, it would be extremely interesting to consider generalization of the Grassmannian integral representation and spin chain construction considered here to the case of gauge

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³⁰This brings up the question of existence of IR regularization preserving all symmetries of tree level amplitudes in $\mathcal{N} = 4$ SYM.

invariant off-shell amplitudes with an arbitrary number of off-shell gluons (Wilson line insertions).

At the end we want to mention an important conceptual question, which arises in this and similar studies [38, 39, 44, 62]. We have seen, that structures such as Grassmannian integrals and spin chains considered here, which (at least naively) could be considered as a purely on-shell objects, also appear for partially off-shell objects, such as form factors and off-shell amplitudes. Next, the natural question is how far on-shell techniques could be extended in the case of off-shell kinematics.

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A $\mathcal{N} = 4$ SYM and spinor helicity formalism

$\mathcal{N} = 4$ supersymmetric gauge theory is a maximally supersymmetric gauge theory in four spacetime dimensions. The field content of $\mathcal{N} = 4$ SYM consists from six scalars ϕ^{AB} (antisymmetric in the $SU(4)_R$ indices $A, B = 1 \dots 4$), four Weyl fermion fields ψ_α^A and gauge field strength tensor $F^{\mu\nu}$, all transforming in the adjoint representation of the $SU(N_c)$ gauge group. The lagrangian of $\mathcal{N} = 4$ SYM is given by [105, 106]:

$$\mathcal{L}_{\mathcal{N}=4 \text{ SYM}} = \text{tr} \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{4} (D_\mu \phi^{AB}) (D^\mu \bar{\phi}_{AB}) + \frac{1}{32} g^2 [\phi^{AB}, \phi^{CD}] [\bar{\phi}_{AB}, \bar{\phi}_{CD}] \right. \\ \left. + i \bar{\psi}_{\dot{\alpha}A} \sigma_\mu^{\dot{\alpha}\beta} D^\mu \psi_\beta^A - \frac{1}{2} g \psi^{\alpha A} [\bar{\phi}_{AB}, \psi_\alpha^B] + \frac{1}{2} g \bar{\psi}_{\dot{\alpha}A} [\phi^{AB}, \bar{\psi}_{\dot{\alpha}B}] \right\}, \quad (\text{A.1})$$

where the field strength is $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{ig}{\sqrt{2}} [A_\mu, A_\nu]$ and the covariant derivative is $D_\mu = \partial_\mu - \frac{ig}{\sqrt{2}} [A_\mu, \cdot]$. All fields are matrix valued in $SU(N_c)$ gauge group, i.e. $\Phi \equiv \Phi^a t^a$ with generators normalized as $\text{tr } t^a t^b = \delta^{ab}$. $\bar{\psi}_{\dot{\alpha}A} = (\psi_\alpha^A)^*$ and

$$\bar{\phi}_{AB} = (\phi^{AB})^* = \frac{1}{2} \epsilon_{ABCD} \phi^{CD}, \quad \epsilon_{ABCD} = \epsilon^{ABCD} \quad (\text{A.2})$$

The lagrangian of $\mathcal{N} = 4$ is invariant under the following supersymmetry transformations

$$\delta A^\mu = -i \xi^{\alpha A} \bar{\sigma}^\mu_{\dot{\alpha}\beta} \bar{\psi}_A^{\dot{\beta}} - i \bar{\xi}_{\dot{\alpha}A} \sigma^\mu^{\dot{\alpha}\beta} \psi_\beta^A, \\ \delta \phi^{AB} = -i \sqrt{2} \{ \xi^{\alpha A} \psi_\alpha^B - \xi^{\alpha B} \psi_\alpha^A - \epsilon^{ABCD} \bar{\xi}_{\dot{\alpha}C} \bar{\psi}_D^{\dot{\alpha}} \}, \\ \delta \psi_\alpha^A = \frac{i}{2} F_{\mu\nu} \sigma^{\mu\nu}{}_\alpha{}^\beta \xi_\beta^A - \sqrt{2} (D_\mu \phi^{AB}) \bar{\sigma}^\mu_{\dot{\alpha}\beta} \bar{\xi}_B^{\dot{\alpha}} + \frac{ig}{\sqrt{2}} [\phi^{AB}, \bar{\phi}_{BC}] \xi_\alpha^C, \\ \delta \bar{\psi}_A^{\dot{\alpha}} = \frac{i}{2} F_{\mu\nu} \bar{\sigma}^{\mu\nu\dot{\alpha}}{}_{\dot{\beta}} \bar{\xi}_A^{\dot{\beta}} + \sqrt{2} (D_\mu \bar{\phi}_{AB}) \sigma^{\mu\dot{\alpha}\beta} \xi_\beta^B + \frac{ig}{\sqrt{2}} [\bar{\phi}_{AB}, \phi^{BC}] \bar{\xi}_C^{\dot{\alpha}}, \quad (\text{A.3})$$

where

$$\sigma^\mu{}^{\dot{\alpha}\beta} = \epsilon^{\beta\gamma} \bar{\sigma}^\mu{}_{\dot{\gamma}\delta} \epsilon^{\delta\dot{\alpha}}, \quad \bar{\sigma}^\mu{}_{\alpha\dot{\beta}} = \epsilon_{\dot{\beta}\dot{\gamma}} \sigma^\mu{}^{\dot{\gamma}\delta} \epsilon_{\delta\alpha}, \quad (\text{A.4})$$

and

$$\sigma^{\mu\nu}{}_{\alpha}{}^{\beta} \equiv \frac{i}{2} \left[\bar{\sigma}^\mu{}_{\alpha\dot{\gamma}} \sigma^\nu{}^{\dot{\gamma}\beta} - \bar{\sigma}^\nu{}_{\alpha\dot{\gamma}} \sigma^\mu{}^{\dot{\gamma}\beta} \right], \quad \bar{\sigma}^{\mu\nu}{}^{\dot{\alpha}}{}_{\dot{\gamma}} \equiv \frac{i}{2} \left[\sigma^\mu{}^{\dot{\alpha}\gamma} \bar{\sigma}^\nu{}_{\gamma\dot{\beta}} - \sigma^\nu{}^{\dot{\alpha}\gamma} \bar{\sigma}^\mu{}_{\gamma\dot{\beta}} \right] \quad (\text{A.5})$$

To rewrite scattering amplitudes in color ordered form one uses a representation of color factors for adjoint $\text{SU}(N_c)$ states f^{abc} in terms of color factors t^a associated with smaller fundamental $\text{SU}(N_c)$ representation (2.1). The idea behind the spinor helicity formalism is similar, that is one consider trading Lorentz vector p_i^μ for kinematical quantities, that transform under a smaller representation of Lorentz group. The latter are given by two-dimensional (Weyl) spinors. So, the four momentum p_i^μ is exchanged with a pair of spinors:

$$p_i^\mu \rightarrow u_+(p_i) = \frac{1}{2}(1 + \gamma_5)u(p_i) \equiv |i\rangle \equiv \lambda_{i,\alpha}, \quad u_-(p_i) = \frac{1}{2}(1 - \gamma_5)u(p_i) \equiv |i] \equiv \tilde{\lambda}_i^{\dot{\alpha}} \quad (\text{A.6})$$

These spinors satisfy the massless Dirac equation

$$\hat{p}_i u_\pm(p_i) = \hat{p}_i |i\rangle = \hat{p}_i |i] = 0. \quad (\text{A.7})$$

There are also negative-energy solutions $v_\pm$, but for $p_i^2 = 0$ they are not distinct from $u_\mp(p_i)$. The undotted and dotted spinor indices correspond to two different spinor representations of Lorentz group $L_+^\uparrow = \text{SO}(3,1) = \text{SO}(4, \mathbb{C})_{\downarrow R} \approx (\text{SL}(2, \mathbb{C}) \otimes \text{SL}(2, \mathbb{C}))_{\downarrow R}$, which are labeled by a pair (j_1, j_2) of eigenvalues $j_i(j_i + 1)$ of the $\text{SL}(2)$ Casimir operators $\mathbf{J}_i^2$: $\lambda_\alpha \sim (\frac{1}{2}, 0)$ and $\tilde{\lambda}^{\dot{\alpha}} \sim (0, \frac{1}{2})$. The raising and lowering of Weyl spinor indices is done with help of antisymmetric tensors $\varepsilon^{\alpha\beta}$ and $\varepsilon^{\dot{\alpha}\dot{\beta}}$:

$$\lambda^\alpha = \varepsilon^{\alpha\beta} \lambda_\beta, \quad \tilde{\lambda}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}} \tilde{\lambda}^{\dot{\beta}}, \quad \lambda_\alpha = \lambda^\beta \varepsilon_{\beta\alpha}, \quad \tilde{\lambda}^{\dot{\alpha}} = \tilde{\lambda}_{\dot{\beta}} \varepsilon^{\dot{\beta}\dot{\alpha}}, \quad (\text{A.8})$$

where $\varepsilon^{\alpha\beta} = -\varepsilon_{\dot{\alpha}\dot{\beta}}$ and $\varepsilon^{12} = \varepsilon_{12} = -\varepsilon_{\dot{1}\dot{2}} = -\varepsilon^{\dot{1}\dot{2}} = 1$. Within spinor helicity formalism one defines the spinor products as:

$$\begin{aligned} \langle ij \rangle &\equiv \varepsilon^{\alpha\beta} \lambda_{i,\alpha} \lambda_{j,\beta} = \bar{u}_-(p_i) u_+(p_j), \\ [ij] &\equiv \varepsilon^{\dot{\alpha}\dot{\beta}} \tilde{\lambda}_{i,\dot{\alpha}} \tilde{\lambda}_{j,\dot{\beta}} = \bar{u}_+(p_i) u_-(p_j). \end{aligned} \quad (\text{A.9})$$

Rewriting the massless positive energy projector

$$u_+(p_i) \bar{u}_+(p_i) = |i\rangle [i] = \frac{1}{2}(1 + \gamma_5) \hat{p}_i \frac{1}{2}(1 - \gamma_5) \quad (\text{A.10})$$

in two-component notation we get

$$\lambda_{i,\alpha} \tilde{\lambda}_{i,\dot{\alpha}} = p_{i,\mu} \sigma^\mu_{\alpha\dot{\alpha}} = \hat{p}_{i,\alpha\dot{\alpha}} = \begin{pmatrix} p_i^t + p_i^z & p_i^x - ip_i^y \\ p_i^x + ip_i^y & p_i^t - p_i^z \end{pmatrix} \quad (\text{A.11})$$

It should be noted, that the determinant of this 2×2 matrix vanishes, $\det(\hat{p}_i) = p_i^2 = 0$, which is consistent with its factorization into a column vector $\lambda_{i,\alpha}$ times a row vector $\tilde{\lambda}_{i,\dot{\alpha}}$. Contracting (A.11) with $\sigma^\nu{}^{\dot{\alpha}\alpha}$ it is possible to reconstruct the four-momenta p_i^μ from the spinors:

$$2p_i^\mu = [i|\gamma^\mu|i] \equiv \tilde{\lambda}_{i,\dot{\alpha}}\sigma^{\mu\dot{\alpha}\alpha}\lambda_{i,\alpha}. \quad (\text{A.12})$$

There are also the following useful in calculations properties of the spinor products:

$$\begin{aligned} \text{anti-symmetry: } & \langle ij \rangle = -\langle ji \rangle, \quad [ij] = -[ji], \quad \langle ii \rangle = [ii] = 0, \\ \text{squaring: } & \langle ij \rangle [ji] = s_{ij}, \\ \text{momentum conservation: } & \sum_{j=1}^n \langle ij \rangle [jk] = 0, \\ \text{Schouten identity: } & \langle ij \rangle \langle kl \rangle - \langle ik \rangle \langle jl \rangle = \langle il \rangle \langle kj \rangle. \end{aligned} \quad (\text{A.13})$$

where $s_{ij} = (p_i + p_j)^2$. One could also use spinor helicity formalism to write down the *physical* polarization vectors for massless vector particles with definite helicity in terms of a pair of massless spinors:

$$\varepsilon_{i,\mu}^+ = \varepsilon_\mu^+(p_i, r_i) = \frac{\langle r_i | \gamma_\mu | i \rangle}{\sqrt{2} \langle r_i i \rangle}, \quad \varepsilon_{i,\mu}^- = \varepsilon_\mu^-(p_i, r_i) = -\frac{[r_i | \gamma_\mu | i]}{\sqrt{2} [r_i i]}, \quad (\text{A.14})$$

$$\hat{\varepsilon}_{\alpha\dot{\alpha}}^+ = \hat{\varepsilon}_{\alpha\dot{\alpha}}^+(p_i, r_i) = \sqrt{2} \frac{\lambda_{r_i,\alpha} \tilde{\lambda}_{i,\dot{\alpha}}}{\langle r_i i \rangle}, \quad \hat{\varepsilon}_{i,\alpha\dot{\alpha}}^- = \hat{\varepsilon}_{\alpha\dot{\alpha}}^-(p_i, r_i) = -\sqrt{2} \frac{\tilde{\lambda}_{r_i,\dot{\alpha}} \lambda_{i,\alpha}}{[r_i i]}, \quad (\text{A.15})$$

where we have also written 2×2 matrix version by contracting polarization vectors with σ matrix and using Fierz identity for Pauli matrices $\sigma_{\alpha\dot{\alpha}}^\mu \sigma_\mu^{\beta\dot{\beta}} = 2\delta_\alpha^\beta \delta_{\dot{\alpha}}^{\dot{\beta}}$. Here p_i^μ is particle momentum and r_i^μ is the null reference momentum accounting for gauge dependence of polarization vectors (local gauge invariance allows an independent choice of reference momenta for different particles). λ_{r_i} and $\tilde{\lambda}_{r_i}$ are two-component left and right-handed spinors associated to reference momentum. It is easy to see, that polarization vectors (A.14) obey the required transversality conditions with respect to particle momentum:

$$\varepsilon_i^\pm \cdot p_i = 0. \quad (\text{A.16})$$

They are also transverse with respect to reference momentum r_i : $\varepsilon_i^\pm \cdot r_i = 0$. Besides the physical polarization vectors defined above, there are two other *unphysical* polarizations vectors:

$$\varepsilon_{i,\mu}^L = p_i^\mu, \quad \varepsilon_{i,\mu}^T = \frac{\langle r_i | \gamma_\mu | r_i \rangle}{2p_i \cdot r_i} = \frac{r_i^\mu}{p_i \cdot r_i}. \quad (\text{A.17})$$

It is easy to check, that polarization vectors satisfy the following orthogonality

$$0 = \varepsilon^+ \cdot \varepsilon^+ = \varepsilon^+ \cdot \varepsilon^L = \varepsilon^+ \cdot \varepsilon^T = \varepsilon^- \cdot \varepsilon^- = \varepsilon^- \cdot \varepsilon^L = \varepsilon^- \cdot \varepsilon^T = \varepsilon^T \cdot \varepsilon^T = \varepsilon^L \cdot \varepsilon^L$$

and normalization

$$1 = \varepsilon^+ \cdot \varepsilon^- = \varepsilon^L \cdot \varepsilon^T.$$

conditions.

B Grassmannian integral evaluation

For the evaluation of Grassmannian integrals we used the strategy of [22]. First, consider bosonic δ -functions in Grassmannian integral

$$\delta^{k \times 2} \left(C \cdot \tilde{\lambda} \right) \delta^{(n+2-k) \times 2} \left(C^\perp \cdot \lambda \right). \quad (\text{B.1})$$

Fixing $\text{GL}(k)$ gauge so that first k columns of C_{al} form unity matrix we get

$$\delta^{k \times 2} \left(C \cdot \tilde{\lambda} \right) \delta^{(n+2-k) \times 2} \left(C^\perp \cdot \lambda \right) = \prod_{a=1}^k \delta^2 \left(\tilde{\lambda}_a + \sum_{i=k+1}^n c_{ai} \tilde{\lambda}_i \right) \prod_{i=k+1}^n \delta^2 \left(\lambda_i + \sum_{a=1}^k c_{ai} \lambda_a \right). \quad (\text{B.2})$$

Next, δ -function constraints lead to the following underdetermined system of linear equations

$$\begin{aligned} c_{ai} \lambda_a &= -\lambda_i, \\ c_{ai} \tilde{\lambda}_i &= -\tilde{\lambda}_a, \end{aligned} \quad (\text{B.3})$$

where $a = 1 \dots k$, $i = k+1 \dots n$. For other $\text{GL}(k)$ gauges the structure of the above equations will be identical, the only difference are the values taken by indexes a and i . The general solution of this system of equations could be parametrized by $(k-2)(n-k-2)$ complex parameters τ_A :

$$c_{ai}(\tau) = c_{ai}^* + d_{aiA} \tau_A, \quad (\text{B.4})$$

where d_{aiA} are some rational functions of $\lambda, \tilde{\lambda}$'s and c_{ai}^* is some particular solution of (B.3). Using these solutions the bosonic δ -functions could be rewritten as

$$\begin{aligned} & \prod_{a=1}^k \delta^2 \left(\tilde{\lambda}_a + \sum_{i=k+1}^n c_{ai} \tilde{\lambda}_i \right) \prod_{i=k+1}^n \delta^2 \left(\lambda_i + \sum_{a=1}^k c_{ai} \lambda_a \right) \\ &= \delta^4 \left(\sum_{j=1}^n \lambda_j \tilde{\lambda}_j \right) J(\lambda, \tilde{\lambda}) \int d^{(k-2)(n-k-2)} \tau_A \prod_{a=1}^k \prod_{i=k+1}^n \delta(c_{ai} - c_{ai}(\tau)), \end{aligned} \quad (\text{B.5})$$

where $J(\lambda, \tilde{\lambda})$ is Jacobian of transformation. Note that the number of δ -functions in l.h.s. and r.h.s. of the above equation is the same. In l.h.s. we have $2n$, while in r.h.s. — $k(n-k) + 4 - (k-2)(n-k-2)$ of them. Now, the integration $\int \frac{d^{n \times k} C_{al}}{\text{Vol}[\text{GL}(k)]}$ could be removed using δ -functions and the only integrations remained will be with respect to τ_A variables ($A = 1, (k-2)(n-k-2)$). Expressing minors of C_{al} -matrix and the Grassmann δ -functions in terms of τ_A using (B.4) the integrand of the original Grassmannian integral becomes a rational function of τ_A variables and the corresponding integral over τ_A could be evaluated using (multidimensional) residue theorem.

In the case of $Gr(3, 6)$ Grassmannian integral considered in the paper it is convenient to choose $\text{GL}(3)$ gauge as

$$C = \begin{pmatrix} 1 & c_{12} & 0 & c_{14} & 0 & c_{16} \\ 0 & c_{32} & 1 & c_{34} & 0 & c_{36} \\ 0 & c_{52} & 0 & c_{54} & 1 & c_{56} \end{pmatrix}. \quad (\text{B.6})$$

Then (B.5) reduces to

$$\begin{aligned} & \prod_{i'=1,3,5} \delta^2 \left(\tilde{\lambda}_{i'} + \sum_{j=2,4,6} c_{i'j} \tilde{\lambda}_j \right) \prod_{j=2,4,6} \delta^2 \left(\lambda_j + \sum_{i'=1,3,5} c_{i'j} \lambda_{i'} \right) \\ &= \delta^4 \left(\sum_{i=1}^6 \lambda_i \tilde{\lambda}_i \right) \int d\tau \prod_{i'=1,3,5} \prod_{j=2,4,6} \delta(c_{i'j} - c_{i'j}(\tau)), \end{aligned} \quad (\text{B.7})$$

with

$$c_{i'j}(\tau) = c_{ij}^* + \epsilon_{i'k'p'} \epsilon_{jlm} \langle k'p' \rangle [lm] \tau. \quad (\text{B.8})$$

The minors $M_1 = (123), \dots, M_6 = (612)$ of C_{al} matrix are linear functions of τ and the corresponding integral over τ could be evaluated using residues (we assume that we are considering particular component of $\delta^{k \times 2}(C \cdot \tilde{\eta})$ expansion in Grassmann variables such that the overall behavior of the integrand is no worse than $1/\tau^2$ at infinity. At the end we may supersymmetrize the result if necessary). We were interested in residues at poles $1/M_1, 1/M_3$ and $1/M_5$. In the gauge chosen the corresponding minors are given by $M_1 = c_{52}(\tau)$, $M_3 = c_{14}(\tau)$ and $M_5 = c_{36}(\tau)$. To simplify the evaluation of residues even further one can notice that for each of the residues the particular solution $c_{i'j}^*$ could be chosen independently, so that $c_{52}^* = 0$ for M_1 , $c_{14}^* = 0$ for M_3 , and $c_{36}^* = 0$ for M_5 . Then the residue theorem fixes $\tau = 0$ at each residue and all other $c_{i'j}(\tau = 0)$ matrix elements are easily evaluated. This way the coefficients of C_{al} matrix for residues $\{1\}, \{3\}, \{5\}$ at poles $1/M_{1,3,5}$ are given by

$$C|_{\{1\}} = \begin{pmatrix} 1 & c_{12} = \frac{\langle 23 \rangle}{\langle 13 \rangle} & 0 & c_{14} = \frac{\langle 3|1+2|6]}{\langle 13 \rangle [46]} & 0 & c_{16} = \frac{\langle 3|1+2|4]}{\langle 13 \rangle [46]} \\ 0 & c_{32} = \frac{\langle 12 \rangle}{\langle 13 \rangle} & 1 & c_{34} = \frac{\langle 1|2+3|6]}{\langle 13 \rangle [46]} & 0 & c_{36} = \frac{\langle 1|2+3|4]}{\langle 13 \rangle [46]} \\ 0 & c_{52} = 0 & 0 & c_{54} = \frac{[56]}{[46]} & 1 & c_{56} = \frac{[54]}{[46]} \end{pmatrix}, \quad (\text{B.9})$$

$$C|_{\{3\}} = \begin{pmatrix} 1 & c_{12} = \frac{[16]}{[16]} & 0 & c_{14} = 0 & 0 & c_{16} = \frac{[12]}{[62]} \\ 0 & c_{32} = \frac{\langle 5|3+4|6]}{\langle 35 \rangle [26]} & 1 & c_{34} = \frac{\langle 45 \rangle}{\langle 35 \rangle} & 0 & c_{36} = \frac{\langle 5|3+4|2]}{\langle 35 \rangle [26]} \\ 0 & c_{52} = \frac{\langle 3|4+5|6]}{\langle 53 \rangle [26]} & 0 & c_{54} = \frac{\langle 43 \rangle}{\langle 53 \rangle} & 1 & c_{56} = \frac{\langle 3|4+5|2]}{\langle 53 \rangle [62]} \end{pmatrix}, \quad (\text{B.10})$$

$$C|_{\{5\}} = \begin{pmatrix} 1 & c_{12} = \frac{\langle 5|1+6|4]}{\langle 15 \rangle [24]} & 0 & c_{14} = \frac{\langle 5|1+6|2]}{\langle 15 \rangle [24]} & 0 & c_{16} = \frac{\langle 56 \rangle}{\langle 15 \rangle} \\ 0 & c_{32} = \frac{[34]}{[24]} & 1 & c_{34} = \frac{[32]}{[42]} & 0 & c_{36} = 0 \\ 0 & c_{52} = \frac{\langle 1|5+6|4]}{\langle 15 \rangle [24]} & 0 & c_{54} = \frac{\langle 1|5+6|2]}{\langle 15 \rangle [24]} & 1 & c_{56} = \frac{\langle 61 \rangle}{\langle 51 \rangle} \end{pmatrix}. \quad (\text{B.11})$$

The general case of $Gr(k, n)$ Grassmannian is more involved since one have to consider integral over multiple complex parameters and multidimensional generalization of residue

theorems. In the NMHV_n case, which we were discussing in the main body of the paper, the situation is simplified for $n > 6$ if we are considering Grassmannian integral in momentum twistor representation (here we follow [103]):

$$\omega_{n+2}^3[\Gamma] = \int_{\Gamma} \frac{d^{1 \times (n+2)} D}{\text{Vol}[\text{GL}(1)]} \frac{\text{Reg.}}{d_1 d_2 d_3 \dots d_{n+2}} \delta^{4|4} \left(\sum_{i=1}^{n+2} d_i Z_i \right),$$

$$\text{Reg.} = \frac{1}{1 + \frac{\langle p\xi \rangle}{\langle p1 \rangle} \frac{d_{n+2}}{d_1}}, \quad (\text{B.12})$$

which is integral over $Gr(1, n+2)$ Grassmannian. Here, $d^{1 \times (n+2)} D$ is $n+2$ -dimensional volume form in $\mathbb{C}^{n+2}$ $d^{1 \times (n+2)} D = \bigwedge_{i=1}^{n+2} dd_i$. One of the variables may be set to a prescribed value by fixing $\text{GL}(1)$ gauge, while four other variables may be fixed by solving bosonic part of δ -function constraints $\delta^{4|0} \left(\sum_{i=1}^{n+2} d_i Z_i \right)$. So, we are left with $(n+2) - 5$ dimensional integral. Now, let us rearrange integrations in ω_{n+2}^3 in the following way. First, we fix $\text{GL}(1)$ gauge by imposing δ -function constraint for some d_a , from $d_1, \dots, d_{n+2}$:

$$\int_{\Gamma} \frac{d^{1 \times n+2} D}{\text{Vol}[\text{GL}(1)]} = \int_{\Gamma} d^{1 \times n+2} D \, d_a^{(0)} \delta \left(d_a^{(0)} - d_a \right), \quad (\text{B.13})$$

where $d_a^{(0)}$ is some number, for example we can choose $d_a^{(0)} = 1$. Next, we can use the bosonic part of δ -functions $\delta^{4|0} \left(\sum_{i=1}^{n+2} d_i Z_i \right)$ to solve these constraints for four arbitrary (but different from d_a) d_b, d_c, d_d, d_e variables and rewrite the above δ -functions as:

$$\delta^{4|0} \left(\sum_{i=1}^{n+2} d_i Z_i \right) = \frac{1}{\langle bcde \rangle} \prod_{i=b,c,d,e} \delta \left(d_i^{(0)} - d_i \right), \quad (\text{B.14})$$

with

$$d_b^{(0)} = \sum_{j \neq b,c,d,e} \frac{\langle cdej \rangle}{\langle bcde \rangle} d_j, \quad d_c^{(0)} = \sum_{j \neq b,c,d,e} \frac{\langle bejb \rangle}{\langle bcde \rangle} d_j,$$

$$d_d^{(0)} = \sum_{j \neq b,c,d,e} \frac{\langle ejbc \rangle}{\langle bcde \rangle} d_j, \quad d_e^{(0)} = \sum_{j \neq b,c,d,e} \frac{\langle jbcd \rangle}{\langle bcde \rangle} d_j. \quad (\text{B.15})$$

Finally we rewrite the expression for ω_{n+2}^3 as

$$\omega_{n+2}^3[\Gamma] = \frac{d_a^{(0)}}{\langle bcde \rangle} \int_{\Gamma} d^{1 \times n+2} D \prod_{i=a,b,c,d,e} \delta \left(d_i^{(0)} - d_i \right) \frac{\text{Reg.}}{d_1 \dots d_{n+2}} \delta^{0|4} \left(\sum_{i=1}^{n+2} d_i \chi_i \right). \quad (\text{B.16})$$

Now we can choose contour $\Gamma = \Gamma_{abcde}$ in $\mathbb{C}^{n+2}$ to encircle points $d_i^{(0)}$ for $i = a, b, c, d, e$ and $d_i = 0$ for all other i . This allows us to replace first four δ -functions left with $1/(d_i^{(0)} - d_i)$ and rewrite the above integral as

$$\omega_{n+2}^3[\Gamma_{abcde}] = \frac{d_a^{(0)}}{\langle bcde \rangle} \int_{\Gamma_{abcde}} d^{1 \times n+2} D \prod_{i=a,b,c,d,e} \frac{1}{d_i^{(0)} - d_i} \prod_{j \neq a,b,c,d,e} \frac{1}{d_j} \frac{\text{Reg.} \delta^{0|4} \left(\sum_{i=1}^{n+2} d_i \chi_i \right)}{d_a d_b d_c d_d d_e}. \quad (\text{B.17})$$

Using multidimensional generalization of residue theorem we get

$$\omega_{n+2}^3[\Gamma_{abcde}] = \text{Reg.}[abcde], \quad (\text{B.18})$$

where $\text{Reg.} = 0$ if the set a, b, c, d, e does not contain 1 and in opposite case (*in addition* to the requirement $1 \in (a, b, c, d, e)$):

$$\text{Reg.} = \frac{1}{1 + \frac{\langle p\xi \rangle \langle a b c d \rangle}{\langle p1 \rangle \langle b c d e \rangle}}, \text{ if one of } a, b, c, d, e \text{ equals to } n+2, \quad (\text{B.19})$$

and

$$\text{Reg.} = 1 \text{ in other cases.} \quad (\text{B.20})$$

Choosing combinations of Γ_{abcde} contours we will get linear combinations of the above terms $\text{Reg.}[abcde]$. In the main text we choose contour similar to the case of [12]-shift BCFW representation of NMHV_{n+1} on-shell amplitude. In particular, in the case of $n+2 = 6$ this choice provides us with the local expression (free from spurious poles) and we hope that similar pattern will hold for $n+2 > 6$. Cancellation of spurious poles is a little tricky question and we will consider it in more detail in separate publication.

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