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IIB supergravity and the $E_{6(6)}$ covariant vector-tensor hierarchy

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Abstract: IIB supergravity is reformulated with a manifest local USp(8) invariance that makes the embedding of five-dimensional maximal supergravities transparent. In this formulation the ten-dimensional theory exhibits all the 27 one-form fields and 22 of the 27 two-form fields that are required by the vector-tensor hierarchy of the five-dimensional theory. The missing 5 two-form fields must transform in the same representation as a descendant of the ten-dimensional 'dual graviton'. The invariant $E_{6(6)}$ symmetric tensor that appears in the vector-tensor hierarchy is reproduced. Generalized vielbeine are derived from the supersymmetry transformations of the vector fields, as well as consistent expressions for the USp(8) covariant fermion fields. Implications are discussed for the consistency of the truncation of IIB supergravity compactified on the five-sphere to maximal gauged supergravity in five space-time dimensions with an SO(6) gauge group.

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1 Introduction

Maximal supergravity theories in various dimensions are known to possess intruiging duality symmetries which can optionally be broken by non-abelian gauge interactions. Many of these theories can be described as truncations from eleven-dimensional supergravity or from ten-dimensional IIB supergravity in the context of dimensional compactification on an internal manifold of appropriate dimensionality. Already at an early stage this raised the question whether the higher-dimensional supergravities might somehow reflect the exceptional duality symmetries that are present in their lower-dimensional 'descendants'. This question has a long history and is also relevant for proving the existence of consistent truncations to maximal supergravities, implying that any solution of the lower-dimensional maximal supergravity can be uplifted to the higher-dimensional one.

An early attempt to answer this question was based on a reformulation of the full eleven-dimensional supergravity obtained by performing a suitable Kaluza-Klein decomposition to four dimensions while retaining the full dependence on the seven internal coordinates [1]. The key element here was to ensure that the resulting theory was invariant under the four-dimensional R-symmetry group SU(8). This symmetry was *locally* realized with respect to all the eleven coordinates, and it was introduced by a gauge equivalent re-assembling of the original $\mathrm{Spin}(10,1)$ tangent space. The resulting supersymmetry transformation rules then took a form that was almost identical to the four-dimensional ones, which do indeed exhibit the typical characteristics of the $\mathrm{E}_{7(7)}$ dualities, but now with fields that still depend on all eleven space-time coordinates. Eventually this set-up made it possible to establish the consistency of the S^7 truncation, meaning that the whole field configuration of four-dimensional SO(8) gauged supergravity can be uplifted as a submanifold in the full eleven-dimensional theory by specifying the dependence of the fields on the seven internal coordinates [2, 3] .

Recently this approach was substantially extended by including the supersymmetry transformations of dual fields, which opened a new window to accessing the $\mathrm{E}_{7(7)}$ duality properties of the full eleven-dimensional supergravity [4–7]. Given these recent insights, it is a natural question whether similar structures can be derived for IIB supergravity in the context of a $5+5$ split of the coordinates. In the present paper we confirm that this is indeed the case and we present a detailed analysis to support this. Qualitatively the results turn out to be rather similar to the case of eleven-dimensional supergravity, but many new features arise. In this case the tangent space is re-assembled such that the theory is manifestly invariant under *local* USp(8). This group contains the USp(4) subgroup of the $10D$ tangent space group and the explicit U(1) of IIB supergravity as subgroups. Obviously the $\mathrm{SU}(1,1) \cong \mathrm{SL}(2)$ subgroup of $\mathrm{E}_{6(6)}$ is manifestly realized from the start. Another interesting aspect is that the five-dimensional gauged supergravity theories, when described in terms of the embedding tensor formalism [8], involve 27 vector and 27 two-rank tensor fields which constitute the beginning of an intricate vector-tensor hierarchy [9, 10]. As we will discover in this paper, these features are also present when one retains the dependence on the extra internal coordinates for IIB supergravity, so that this vector-tensor hierarchy does emerge in a ten-dimensional context. This is undoubtedly related to the fact that in the recent work on an $\mathrm{E}_{6(6)}$ *exceptional geometry* that incorporates both 11-dimensional and 10-dimensional IIB supergravity, the vector-tensor hierarchy also plays a key role [11]. Irrespective of these issues, the analysis presented in this paper has to address a number of subtle technical complications that are absent in the corresponding analysis of eleven-dimensional supergravity. Many of those are caused by the fact that the field representation of IIB supergravity is more reducible than that of the eleven-dimensional one, while the supersymmetry is an extended one (i.e. $N = 2$).

While it is clearly significant that the approach initiated in [1] can be applied successfully to IIB supergravity, we should also point out that a wider variety of alternative approaches has been developed meanwhile. These approaches are also aimed at understanding and/or exploiting the duality symmetries in the context of M-theory and string theory, and sometimes involve substantial extensions of the conventional supergravity framework. Some of them make use of additional space-time coordinates and extended geometrical structures or duality groups. One such approach is based on *generalized geometry* [12, 13], where one considers an extended tangent space that captures all the bosonic degrees of freedom, sometimes related to double field theory (see e.g. [14–21] and references quoted therein). There exists also a variety of extended duality groups that have been proposed in combination with a choice of an exceptional space-time, such as in [22–25]. The work in [11, 26–28] is based on extending the number of space-time coordinates subject to an 'exceptional geometry' so that the higher-dimensional theory is manifestly duality invariant.

It is worth stressing that the work described in this paper is exclusively based on the on-shell formulation of IIB supergravity, as originally constructed in [29–31]. As is well known

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the compactification of ten-dimensional type-IIB supergravity on a five-dimensional torus leads to five-dimensional maximal supergravity [32] with a non-linearly realized $E_{6(6)}$ symmetry, whose maximal compact subgroup USp(8) coincides with the R-symmetry group. Compactification on a curved internal manifold, such as the sphere S^5, will necessarily break some of the symmetries mentioned above. In the case of S^5 one expects to obtain the SO(6) gauging of maximal supergravity upon truncating the massive modes, because the isometry group of S^5 equals SO(6) [33]. Various results on the consistency of this truncation have already been reported in the literature (see, e.g. [21, 34]). From the five-dimensional viewpoint, the breaking of the $E_{6(6)}$ symmetry is understood as a result of the non-abelian gauge interactions, because the SO(6) gauge group is embedded into $E_{6(6)}$.

As we discussed above, it is possible to reformulate the higher-dimensional theory upon splitting the coordinates into 5 space-time and 5 internal coordinates, while retaining the full dependence on the two sets of coordinates. To ensure that the theory takes the form of the lower-dimensional theory with fields that depend in addition on the five internal coordinates, one adopts a gauge-equivalent version of the tangent space such that the tangent space group will be restricted to the product group $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$. Subsequently one combines the group SO(5) associated with the internal five-dimensional tangent space with the manifest local U(1) group of IIB supergravity. The crucial step is then to extend this product group to USp(8), which is the R-symmetry group for five-dimensional maximal supergravity. Hence we envisage

$$\begin{aligned}\mathrm{Spin}(9,1)\times\mathrm{U}(1) &\longrightarrow \mathrm{Spin}(4,1)\times\big[\mathrm{USp}(4)\times\mathrm{U}(1)\big]\\ &\longrightarrow \mathrm{Spin}(4,1)\times\mathrm{USp}(8)\,,\end{aligned}\tag{1.1}$$

where we now refer to the universal covering groups which are relevant for the fermions. Initially only the $\mathrm{USp}(4)\times\mathrm{U}(1)$ subgroup is realized as a *local* invariance that involves all ten coordinates. In order to realize the full *local* USp(8) invariance, it suffices to introduce a compensating $\mathrm{USp}(8)/[\mathrm{USp}(4)\times\mathrm{U}(1)]$ phase factor.

The ensuing analysis will be more subtle for IIB supergravity than for $11D$ supergravity. The latter contains a single fermion field corresponding to the gravitino that decomposes directly into $4D$ gravitini transforming in the $\mathbf{8}$ representation and $4D$ spin-1/2 fermions transforming in the $\mathbf{48}+\mathbf{8}$ representation of Spin(7). As was first demonstrated in [35], these fields can be reassembled upon extending the group Spin(7) to chiral SU(8), so that the gravitini transform in the $\mathbf{8}\oplus\overline{\mathbf{8}}$ representation and the spin-1/2 fields in the $\mathbf{56}\oplus\overline{\mathbf{56}}$ representation of chiral SU(8). The IIB fermion representation, on the other hand, is already reducible in 10 dimensions and consists of a complex gravitino and a complex dilatino field. The USp(4) tangent-space group can in principle be generalized for each of these fields to $\mathrm{SU}(4)\cong\mathrm{SO}(6)$. Furthermore, the fermions of the IIB theory transform under a *locally* realized U(1). Therefore, the R-symmetry group of the $5D$ fermions is extended from SU(4) to $\mathrm{SU}(4)\times\mathrm{U}(1)$. For the gravitini this group can be directly extended to the expected USp(8) R-symmetry group, under which the gravitini will transform in the $\mathbf{8}$ representation. However, for the spin-1/2 fermions one must combine the gravitino associated with the internal space, comprising 40 symplectic Majorana spinors, with the dilatino, comprising 8 such spinors, into an irreducible $\mathbf{48}$ representation of the group USp(8).

It is clear that assembling the different IIB fermions into a single irreducible spinor that transforms covariantly under USp(8), is a subtle task.

Therefore our strategy is to first identify the vector and tensor gauge fields and their supersymmetry transformations, subject to the vector-tensor hierarchy that is known from the embedding tensor formulation of $5D$ maximal supergravity [8]. Unlike in the case of 11-dimensional supergravity one must also include the tensor fields in the analysis, because in five dimensions the dynamical degrees of freedom for generic gaugings are always carried by a mixture of vector and tensor fields. Hence the vector-tensor hierarchy plays a key role here at a much earlier stage of the analysis and it is not sufficient to rely exclusively on a proper preparation of the target space as indicated in (1.1). As it turns out, five of the tensor fields are still unaccounted for, but even without these missing tensors we have sufficient information to determine the generalized vielbeine, the USp(8) covariant spinor fields, and the supersymmetry transformations of the generalized vielbeine. Using the vector-tensor hierarchy as a guide, one can incorporate the missing five tensor fields which turn out to transform in a representation that coincides precisely with that of a descendant of the $10D$ dual graviton [36–38]. Hence the dual graviton emerges in the form of tensor fields, unlike in the 11-dimensional situation [6] where the dual graviton resides in the vector sector. We present a basis for the vector and tensor fields which is manifestly in agreement with the $\mathrm{E}_{6(6)}$ assignments known from the $5D$ theory, which involves the invariant three-rank symmetric tensor of that group.

In spite of many subtle differences, the gross features of the present analysis are in agreement with those of 11-dimensional supergravity, implying that the approach that has been adopted is sufficiently robust to be applied to more complicated situations. The supersymmetry transformations of the fields are covariant under local USp(8) transformations. The results opens the way to study many other detailed questions, such as the consistency of the truncation to the SO(6) gauging of maximal five-dimensional supergravity or other consistent truncations along the lines followed in [39]. Also the precise relation with the consistent Kaluza-Klein truncations using exceptional field theory [40] is worth pursuing, as well as many other issues that have recently emerged.

This paper is organized as follows. In section 2 the relevant properties of IIB supergravity are summarized and the conventions are defined. Subsequently, in section 3, the Kaluza-Klein decompositions are carried out to ensure that the fields transform covariantly from the viewpoint of the $5D$ space-time. Also the conversion to $5D$ spinors and gamma matrices is discussed as well as the proper definitions of the $5D$ vector and tensor fields that emerge directly from the $10D$ boson fields. As it turns out, further redefinitions on the vector and tensor fields are required such that they transform under supersymmetry in a way that is consistent with the vector-tensor hierarchy. In section 4 the dual vector and tensor fields are introduced. Again their proper identification is based on covariance in the $5D$ space-time and on the vector-tensor hierarchy. As it turns out there are only 22 tensor fields at this stage. It is then demonstrated how the missing fields can emerge from a component of the $10D$ dual graviton. This enables one to obtain the symmetric $\mathrm{E}_{6(6)}$ tensor that appears in the transformation rules of the tensor fields. At this point the supersymmetry transformations of the bosonic vector and tensor fields clearly resemble the transformation rules encountered in the pure $5D$ theory as presented in [8], including those related to the vector-tensor hierarchy. By direct comparison between the supersymmetry transformations of the vector fields arising from ten dimensions and the five-dimensional ones, explicit expressions for the generalized vielbeine are derived in section 5. In addition the USp(8) covariant definitions of the spinor fields are obtained, as well as the supersymmetry transformations of the generalized vielbeine. A similar strategy is then applied to the tensor fields, which leads to a corresponding set of generalized vielbeine. Upon adopting suitable normalizations of the vector and tensor fields one can show that this new set of vielbeine constitutes the inverse of the generalized vielbeine determined in the vector sector. In section 6 the supersymmetry transformations of the fermions are considered and it is shown that they take a USp(8) covariant form. Finally, in section 7 the question of the consistent truncation to SO(6) gauged maximal $5D$ supergravity is adressed. We include two appendices, A and B, dealing with the definition and decomposition of gamma matrices and the spinor and R-symmetry representations associated with the various groups emerging upon decomposing the tangent-space into two separate $5D$ subspaces.

2 Summary of IIB supergravity

Here we summarize the relevant results for IIB supergravity in ten space-time dimensions [29–31]. The theory is described in terms of a zehnbein $E_M{}^A$, a gravitino field ψ_M, a spinor field λ, a complex three-rank tensor field strength, G_{MNP}, a five-rank field strength F_{MNPQR} subject to a duality constraint, a complex vector P_M and a U(1) gauge field Q_M. The fermions are complex and have opposite chirality,

$$\breve{\Gamma}_{11}\psi_M = \psi_M\,, \qquad \breve{\Gamma}_{11}\lambda = -\lambda\,, \tag{2.1}$$

where $\breve{\Gamma}_{11} = \mathrm{i}\breve{\Gamma}_1\breve{\Gamma}_2\cdots\breve{\Gamma}_{10}$, with $\breve{\Gamma}_A$ denoting the 10-dimensional gamma matrices. The fermions transform under local phase transformations according to

$$\psi_M \to \mathrm{e}^{\mathrm{i}\Lambda/2}\,\psi_M\,, \qquad \lambda \to \mathrm{e}^{3\mathrm{i}\Lambda/2}\,\lambda\,. \tag{2.2}$$

The zehnbein $E_M{}^A$ and the field strength F_{MNPQR} are invariant under U(1), unlike the other quantities, which transform as follows,

$$G_{MNP} \to \mathrm{e}^{\mathrm{i}\Lambda}\,G_{MNP}\,, \qquad P_M \to \mathrm{e}^{2\mathrm{i}\Lambda}\,P_M\,, \qquad Q_M \to Q_M + \partial_M\Lambda\,. \tag{2.3}$$

The vectors P_M and Q_M satisfy the Maurer-Cartan equations associated with the coset space $\mathrm{SU}(1,1)/\mathrm{U}(1)$, which is parametrized by the scalar fields of the theory,

$$\partial_{[M}Q_{N]} = -\mathrm{i}P_{[M}\,\bar{P}_{N]}\,, \qquad \mathcal{D}_{[M}P_{N]} = 0\,. \tag{2.4}$$

In this section the derivative $\mathcal{D}_M$ is covariant with respect to local Lorentz and local U(1) transformations.

The coset representative can be expressed in terms of an $\mathrm{SU}(1,1)$ doublet ϕ^α, ($\alpha = 1,2$), transforming under U(1) as

$$\phi^\alpha \to \mathrm{e}^{\mathrm{i}\Lambda}\,\phi^\alpha\,, \tag{2.5}$$

and subject to the SU(1,1) invariant constraint,

$$|\phi^1|^2 - |\phi^2|^2 = 1\,. \tag{2.6}$$

In what follows we use the convenient notation $\phi_\alpha \equiv \eta_{\alpha\beta}(\phi^\beta)^*$, with $\eta_{\alpha\beta} = \mathrm{diag}(+1,-1)$, so that the above constraint reads $\phi_\alpha\phi^\alpha = 1$. In this convention the vector fields take the following form,

$$\begin{aligned} Q_M &= -\,\mathrm{i}\phi_\alpha\,\partial_M\phi^\alpha\,,\\ P_M &= \varepsilon_{\alpha\beta}\,\phi^\alpha\,\mathcal{D}_M\phi^\beta\,,\\ \bar P_M &= -\,\varepsilon^{\alpha\beta}\,\phi_\alpha\,\mathcal{D}_M\phi_\beta\,, \end{aligned} \tag{2.7}$$

where the Levi-Civita symbol is normalized by $\varepsilon_{12} = \varepsilon^{12} = 1$. Note that $\eta_{\alpha\beta}\varepsilon^{\beta\gamma}\eta_{\gamma\delta} = -\varepsilon_{\alpha\delta}$. We note the following useful identities,

$$\phi_\alpha\,\mathcal{D}_M\phi^\alpha = 0\,,\qquad \phi_\alpha P_M = \varepsilon_{\alpha\beta}\,\mathcal{D}_M\phi^\beta\,,\qquad \phi^\alpha\bar P_M = -\varepsilon^{\alpha\beta}\,\mathcal{D}_M\phi_\beta\,. \tag{2.8}$$

Let us now turn to the tensor field strengths. The theory contains two tensor fields $A^\alpha{}_{MN}$ transforming under $\mathrm{SU}(1,1)\cong\mathrm{SL}(2)$. Here we use a pseudoreal basis with $A^\alpha{}_{MN} = \varepsilon^{\alpha\beta}(A_{MN})_\beta$, where the convention for lowering and raising of indices is the same as for ϕ^α. Their field strengths are defined as follows,

$$\begin{aligned} 3\,\partial_{[M}A^\alpha{}_{NP]} &= \phi^\alpha\,\bar G_{MNP} + \varepsilon^{\alpha\beta}\phi_\beta\,G_{MNP}\,,\\ G_{MNP} &= -\,3\,\varepsilon_{\alpha\beta}\,\phi^\alpha\,\partial_{[M}A^\beta{}_{NP]}\,,\\ \bar G_{MNP} &= 3\,\phi_\alpha\,\partial_{[M}A^\alpha{}_{NP]}\,. \end{aligned} \tag{2.9}$$

The tensor fields are subject to rigid SU(1,1) transformations, just as the scalar fields ϕ^α, and to tensor gauge transformations. The latter read

$$\delta A^\alpha{}_{MN} = 2\,\partial_{[M}\Xi^\alpha{}_{N]}\,. \tag{2.10}$$

Furthermore we have a 4-rank antisymmetric gauge potential A_{MNPQ}, which transforms under two types of gauge transformations

$$\delta A_{MNPQ} = 4\,\partial_{[M}\Lambda_{NPQ]} + \frac{3}{4}\mathrm{i}\,\varepsilon_{\alpha\beta}\,\Xi^\alpha{}_{[M}\,\partial_N A^\beta{}_{PQ]}\,. \tag{2.11}$$

The corresponding 5-form field strength is defined by

$$F_{MNPQR} = 5\,\partial_{[M}A_{NPQR]} - \frac{15}{8}\mathrm{i}\varepsilon_{\alpha\beta}\,A^\alpha{}_{[MN}\,\partial_P A^\beta{}_{QR]}\,. \tag{2.12}$$

The 3- and 5-rank field strengths satisfy the following Bianchi identities,

$$\begin{aligned} \mathcal{D}_{[M}G_{NPQ]} &= P_{[M}\,\bar G_{NPQ]}\,,\\ \partial_{[M}F_{NPQRS]} &= -\,\frac{5}{12}\mathrm{i}\,G_{[MNP}\,\bar G_{QRS]}\,. \end{aligned} \tag{2.13}$$

In addition there is a constraint on the 5-index field strength which involves the dual field strength,

$$\frac{1}{120}\mathrm{i}\,\varepsilon_{ABCDEFGHIJ}\,F^{FGHIJ} = F_{ABCDE} - \frac{1}{8}\mathrm{i}\,\bar\psi_M\breve\Gamma^{[M}\breve\Gamma_{ABCDE}\breve\Gamma^{N]}\psi_N + \frac{1}{16}\mathrm{i}\,\bar\lambda\,\breve\Gamma_{ABCDE}\,\lambda\,. \tag{2.14}$$

From the chirality of the fermion fields it follows that the fermionic bilinears in (2.14) are anti-selfdual, which is obviously required because otherwise (2.14) would decompose into two independent constraints that would overconstrain the system. Originally (2.14) was derived in superspace [31]. Suppressing the fermionic terms would imply that the bosonic field strength should be self-dual. Note that the constraint (2.14) is supersymmetric and it must transform into the fermionic field equations. Upon combining it with the Bianchi identity (2.13), one obtains the field equations for A_{MNPQ}.

Let us now turn to the fermions ψ_M and λ. The supersymmetry transformations for the spinor fields are as follows,

$$\begin{aligned}\delta\psi_M &= \mathcal{D}_M\epsilon - \frac{1}{480}\mathrm{i}F_{NPQRS}\,\breve\Gamma^{NPQRS}\,\breve\Gamma_M\epsilon - \frac{1}{96}G_{NPQ}\big(\breve\Gamma_M\,\breve\Gamma^{NPQ} + 2\,\breve\Gamma^{NPQ}\,\breve\Gamma_M\big)\epsilon^{\mathrm{c}}\,,\\ \delta\lambda &= -\,P_M\,\breve\Gamma^M\epsilon^{\mathrm{c}} - \frac{1}{24}G_{MNP}\breve\Gamma^{MNP}\epsilon\,,\end{aligned} \tag{2.15}$$

where the quantities $\breve\Gamma^{MN\cdots}$ denote anti-symmetrized products of $10D$ gamma matrices, and $\mathcal{D}_M\epsilon$ contains the spin-connection field $\omega_M{}^{AB}$ and the U(1) connection Q_M,

$$\mathcal{D}_M\epsilon = \left(\partial_M - \frac{1}{4}\omega_M{}^{AB}\breve\Gamma_{AB} - \frac{1}{2}\mathrm{i}Q_M\right)\epsilon\,. \tag{2.16}$$

Here ϵ is the space-time dependent spinor parameter of supersymmetry. In (2.16) we have introduced the Majorana conjugate of a $10D$ spinor ψ, which is defined by

$$\psi^{\mathrm{c}} = \breve C_\pm^{-1}\bar\psi^{\mathrm{T}}\,, \qquad \psi = \breve C_\pm^{-1}\bar\psi^{\mathrm{cT}}\,. \tag{2.17}$$

Here $\breve C_\pm$ denotes the charge conjugation matrix in 10 space-time dimensions which can be either symmetric or anti-symmetric. The gamma matrix conventions are discussed in detail in appendix A, but for the convenience of the reader we note

$$\breve C_\pm\,\breve\Gamma_A\breve C_\pm^{-1} = \pm\breve\Gamma_A{}^{\mathrm{T}}\,,\quad \breve C_\pm{}^{\mathrm{T}} = \pm\breve C_\pm\,,\quad \breve C_\pm{}^{\dagger} = \breve C_\pm^{-1}\,. \tag{2.18}$$

We also note the following equation for spinor bilinears with strings of gamma matrices,

$$\bar\chi\,\Gamma_{A_1}\cdots\Gamma_{A_n}\,\psi = -(\pm)^{n+1}\bar\psi^{\mathrm{c}}\,\Gamma_{A_n}\cdots\Gamma_{A_1}\,\chi^{\mathrm{c}}\,. \tag{2.19}$$

For type-IIB supergravity we have *chiral* spinors comprising 16 complex components. One can show that ψ and ψ^{c} have the same chirality (see appendix A for details) and since the spinors are complex (so that $\psi^{\mathrm{c}}\neq\psi$) one can adopt a pseudo-real representation by combining ψ and ψ^{c} into a 32-component chiral spinor $\Psi = (\psi,\psi^{\mathrm{c}})$, subject to

$$\Psi = \sigma_1\,\breve C_\pm^{-1}\bar\Psi^{\mathrm{T}}\,, \tag{2.20}$$

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where σ_1 denotes the standard 2×2 Pauli spin matrix. We need also the supersymmetry transformations for the bosons,

$$
\begin{aligned}
\delta E_M{}^A &= \frac{1}{2}(\bar\epsilon\,\breve\Gamma^A\psi_M + \bar\epsilon^{\rm c}\,\breve\Gamma^A\psi^{\rm c}_M)\,,\\
\delta\phi^\alpha &= \frac{1}{2}\varepsilon^{\alpha\beta}\phi_\beta\,\bar\epsilon^{\rm c}\lambda\,,\\
\delta A^\alpha{}_{MN} &= -\frac{1}{2}\phi^\alpha\big(\bar\lambda\,\breve\Gamma_{MN}\epsilon - 4\,\bar\epsilon\,\breve\Gamma_{[M}\psi_{N]}{}^{\rm c}\big) + \frac{1}{2}\varepsilon^{\alpha\beta}\phi_\beta\,\big(\bar\epsilon\,\breve\Gamma_{MN}\lambda + 4\,\bar\psi_{[M}{}^{\rm c}\,\breve\Gamma_{N]}\epsilon\big)\,,\\
\delta A_{MNPQ} &= \frac{1}{2}{\rm i}\bar\epsilon\,\breve\Gamma_{[MNP}\psi_{Q]} + \frac{1}{2}{\rm i}\bar\psi_{[M}\breve\Gamma_{NPQ]}\epsilon + \frac{3}{8}{\rm i}\,\varepsilon_{\alpha\beta}A^\alpha{}_{[MN}\,\delta A^\beta{}_{PQ]}\,.
\end{aligned}
\tag{2.21}
$$

The above transformation rules (2.15) and (2.21) have been derived by imposing the supersymmetry algebra,

$$
\big[\delta(\epsilon_1)\,,\delta(\epsilon_2)\big] = \xi^M D_M + \delta_\Xi\big(\Xi^\alpha{}_{MN}\big) + \delta_\Lambda(\Lambda_{MNP}) + \cdots\,,
\tag{2.22}
$$

where

$$
\begin{aligned}
\xi^M &= \frac{1}{2}\bar\epsilon_2\,\breve\Gamma^M\epsilon_1 + \frac{1}{2}\bar\epsilon_2^{\rm c}\,\breve\Gamma^M\epsilon_1{}^{\rm c}\,,\\
\Xi^\alpha{}_M &= -\,\phi^\alpha\,\bar\epsilon_2\breve\Gamma_M\epsilon_1{}^{\rm c} - \varepsilon^{\alpha\beta}\phi_\beta\,\bar\epsilon_2{}^{\rm c}\breve\Gamma_M\epsilon_1\,,\\
\Lambda_{MNP} &= \frac{1}{8}{\rm i}\big(\bar\epsilon_1\breve\Gamma_{MNP}\epsilon_2 - \bar\epsilon_2\breve\Gamma_{MNP}\epsilon_1\big) + \frac{3}{16}{\rm i}\big(\varepsilon_{\alpha\beta}\phi^\alpha A^\beta_{[MN}\,\bar\epsilon_2\breve\Gamma_{P]}\epsilon_1{}^{\rm c} + \phi_\alpha A^\alpha_{[MN}\,\bar\epsilon_2{}^{\rm c}\breve\Gamma_{P]}\epsilon_1\big)\,,
\end{aligned}
\tag{2.23}
$$

and where $\xi^M D_M$ denotes a fully covariantized space-time diffeomorphism.

For future use we also present the supersymmetry transformation rules for the Majorana conjugate spinors,

$$
\begin{aligned}
\delta\psi_M{}^{\rm c} &= \mathcal{D}_M\epsilon^{\rm c} + \frac{1}{480}{\rm i}F_{NPQRS}\,\breve\Gamma^{NPQRS}\,\breve\Gamma_M\epsilon^{\rm c} - \frac{1}{96}\bar G_{NPQ}\big(\breve\Gamma_M\,\breve\Gamma^{NPQ} + 2\,\breve\Gamma^{NPQ}\,\breve\Gamma_M\big)\epsilon\,,\\
\delta\lambda^{\rm c} &= \pm\,\bar P_M\breve\Gamma^M\epsilon \pm \frac{1}{24}\bar G_{MNP}\,\breve\Gamma^{MNP}\epsilon^{\rm c}\,.
\end{aligned}
\tag{2.24}
$$

To understand the various field equations it is convenient to first consider the following $10D$ Lagrangian of IIB supergravity up to terms of fourth-order in the fermion fields, ignoring for the moment the constraint (2.14),

$$
\begin{aligned}
\mathcal{L} = &-\frac{1}{2}E\,R - E\,\bar\psi_M\breve\Gamma^{MNP}\mathcal{D}_N\psi_P - \frac{1}{2}E\,\bar\lambda\breve{\not{\mathcal{D}}}\lambda - E\,|P_M|^2 - \frac{1}{24}E\,|G_{MNP}|^2\\
&- \frac{1}{60}E\,(F_{MNPQR})^2 + \frac{1}{384}\,\varepsilon^{MNPQRSTUVW}\,\varepsilon_{\alpha\beta}\,\partial_M A_{NPQR}\,A_{ST}{}^\alpha\,\partial_U A_{VW}{}^\beta\\
&- \frac{1}{2}E\,\big[\bar\psi_M{}^{\rm c}\,\breve\Gamma^N\breve\Gamma^M\lambda\,\bar P_N + \bar\lambda\,\breve\Gamma^M\breve\Gamma^N\psi_M{}^{\rm c}\,P_N\big]\\
&+ \frac{1}{240}{\rm i}E\,\bar\psi_M\breve\Gamma^{[M}\breve\Gamma^{ABCDE}\breve\Gamma^{N]}\psi_N\,F_{ABCDE}\\
&+ \frac{1}{48}E\,\big[\bar\psi_M\,\breve\Gamma^{[M}\breve\Gamma^{ABC}\,\breve\Gamma^{N]}\psi_N{}^{\rm c}\,G_{ABC} + \bar\psi_M^{\rm c}\,\breve\Gamma^{[M}\breve\Gamma^{ABC}\,\breve\Gamma^{N]}\psi_N\,\bar G_{ABC}\big]\\
&+ \frac{1}{48}E\,\big[\bar\psi_M\,\breve\Gamma^{ABC}\,\breve\Gamma^M\,\lambda\,\bar G_{ABC} - \bar\lambda\,\breve\Gamma^M\,\breve\Gamma^{ABC}\psi_M\,G^{ABC}\big]\\
&- \frac{1}{480}{\rm i}E\,\bar\lambda\,\breve\Gamma^{ABCDE}\,\lambda\,F_{ABCDE} + \cdots\,.
\end{aligned}
\tag{2.25}
$$

We have refrained from imposing the supersymmetric constraint (2.14) so that it makes sense to include a term proportional to $(F_{MNPQR})^2$, and furthermore we have included a Chern-Simons term that is invariant under tensor gauge transformations up to a total derivative. It is then straightforward to show that the field equation for the 4-form field that follows from this Lagrangian is consistent with the constraint (2.14) upon using the second Bianchi identity (2.13). Here we should remind the reader that there are extensive discussions in the literature about manifestly covariant Lagrangians that imply self-duality constraints for tensor fields (see, for instance, [41], where also the Chern-Simons terms is presented, and references cited therein). However, these features are not relevant for our purpose. We also recall that the field equations are already encoded in the supersymmetry transformations, as supersymmetry is only realized on-shell, so that one can determine most terms in (2.25) by imposing super-covariance of the field equations, just as was done in [30]. Our results are also consistent with [31] where an on-shell superspace treatment of IIB supergravity was presented.

For further convenience we list some of the field equations,

$$\begin{aligned}
\mathcal{D}^M P_M + \frac{1}{24} G_{MNP}\, G^{MNP} &= 0\,, \\
\mathcal{D}^M G_{MNP} + P^M\, \bar{G}_{MNP} - \frac{2}{3}\mathrm{i} F_{NPQRS}\, G^{QRS} &= 0\,, \\
R_{MN} + 2\, P_{(M}\, \bar{P}_{N)} + \frac{1}{4}\big(\bar{G}_{PQ(M}\, G^{PQ}{}_{N)} - \frac{1}{12} g_{MN}\, |G_{PQR}|^2\big) + \frac{1}{6} F_M{}^{PQRS}\, F_{NPQRS} &= 0\,, \\
\breve{\Gamma}^M \hat{D}_M \lambda + \frac{1}{240}\mathrm{i}\breve{\Gamma}^{NPQRS}\lambda\, F_{NPQRS} &= 0\,, \\
\breve{\Gamma}^{MNP}\, \widehat{D_N\psi_P} \mp \frac{1}{2}\breve{\Gamma}^Q\breve{\Gamma}^M\lambda^{\mathrm{c}} P_Q - \frac{1}{48}\breve{\Gamma}^{QRS}\breve{\Gamma}^M\lambda\, \bar{G}_{QRS} &= 0\,,
\end{aligned} \tag{2.26}$$

where $\hat{D}_M\lambda$ denotes the supercovariant derivative of the spinor λ and $\widehat{D_{[M}\psi_{N]}}$ the super-covariant curl of the gravitino. Here we suppressed higher-order fermion terms.

However, in section 4, we will need the field equations for the two-form fields including the terms quadratic in the fermions. They follow directly from the Lagrangian (2.25) and can be written as follows,

$$\partial_{[M} F_{NPQRSTU]\,\alpha} = 0\,, \tag{2.27}$$

where the seven-rank anti-symmetric tensors $F_{MNPQRST\,\alpha}$ are equal to

$$\begin{aligned}
F_{\alpha\, MNPQRST} = &-\frac{1}{7}\mathrm{i}\, E\, \varepsilon_{MNPQRSTUVW}\left(\varepsilon_{\alpha\gamma}\,\phi^\gamma\phi_\beta + \varepsilon_{\beta\gamma}\,\phi^\gamma\phi_\alpha\right)\partial^U A^{VW\,\beta} \\
&- 120\mathrm{i}\,\varepsilon_{\alpha\beta}\, A_{[MN}{}^\beta\left[\partial_P A_{QRST]} - \frac{1}{8}\mathrm{i}\varepsilon_{\gamma\delta}\, A_{PQ}{}^\gamma\, \partial_R A_{ST}{}^\delta\right] \\
&+ \frac{1}{7}\varepsilon_{\alpha\beta}\phi^\beta\left[\bar{\psi}_U\, \breve{\Gamma}^{[U}\breve{\Gamma}_{MNPQRST}\, \breve{\Gamma}^{V]}\psi_V{}^{\mathrm{c}} + \bar{\lambda}\,\breve{\Gamma}^U\, \breve{\Gamma}_{MNPQRST}\psi_U\right] \\
&+ \frac{1}{7}\phi_\alpha\left[\bar{\psi}_U{}^{\mathrm{c}}\,\breve{\Gamma}^{[U}\breve{\Gamma}_{MNPQRST}\,\breve{\Gamma}^{V]}\psi_V - \bar{\psi}_U\,\breve{\Gamma}_{MNPQRST}\breve{\Gamma}^U\,\lambda\right].
\end{aligned} \tag{2.28}$$

Note that the normalization of this tensor is arbitrary but the phase is dictated by the fact that its pseudo-reality condition is in line with that of the other pseudo-real fields.

3 Kaluza-Klein decompositions and additional field redefinitions

The strategy in this paper is to describe IIB supergravity as a field theory in a five-dimensional space-time, while still retaining the dependence on the five additional coordinates that describe an internal space. Hence the $10D$ coordinates are decomposed according to $x^M \to (x^\mu, y^m)$, where x^μ are regarded as the space-time coordinates and y^m as the coordinates of the internal manifold. Eventually, in a given background, the fields may be decomposed in terms of a complete basis of functions of the internal coordinates. For the T^5 background this is rather straightforward; the spectrum of the tower of Kaluza-Klein supermultiplets for S^5 has been studied in [42, 43]. However, at this stage we will not be assuming any particular space-time background and neither will we be truncating the theory in any way. We are only reformulating the theory in a form that emphasizes the five-dimensional space-time.

A crucial ingredient in this reformulation is provided by a change of the tangent-space group, which we have already indicated in (1.1). First we impose a gauge choice, reducing the $10D$ local Lorentz group to the product group $\mathrm{SO}(4,1)\times\mathrm{SO}(5)$, whose universal covering group equals $\mathrm{Spin}(4,1)\times\mathrm{USp}(4)$. The fermions then transform according to the product representation of this group, so that from a five-dimensional space-time perspective we are dealing with four complex $\mathrm{Spin}(4,1)$ spinors, each carrying four components. The fermions are subject to an extra local $\mathrm{U}(1)$ group, and the product group $\mathrm{USp}(4)\times\mathrm{U}(1)$ must be contained in the $5D$ R-symmetry group. Obviously we have to convert the $10D$ gamma matrices to those appropriate for five space-time dimensions, equiped with two sets of mutually commuting gamma matrices, one associated with space-time and the other one with the internal space. In due course we will also have to recombine the spin-1/2 fermion fields into an irreducible representation of the group $\mathrm{USp}(8)$, which is the R-symmetry group for eight symplectic Majorana supercharges in a $5D$ space-time. This last redefinition will be considered in section 5.

The next step is to redefine the fields such that they transform covariantly under the $5D$ space-time diffeomorphisms. These Kaluza-Klein decompositions were systematically discussed in the context of the T^7 reduction of $11D$ supergravity to $4D$ supergravity [35]. Furthermore, we will find that the vector and tensor fields require additional redefinitions beyond the Kaluza-Klein ones in order to generate transformations that reflect the vector-tensor hierarchy [8].

The standard Kaluza-Klein decompositions start with the vielbein field and its inverse, which we write in triangular form by exploiting the $10D$ local Lorentz transformations,

$$E_M{}^A = \begin{pmatrix} \Delta^{-1/3} e_\mu{}^\alpha & B_\mu{}^m\, e_m{}^a \\ & \\ 0 & e_m{}^a \end{pmatrix}, \qquad E_A{}^M = \begin{pmatrix} \Delta^{1/3} e_\alpha{}^\mu & -\Delta^{1/3} e_\alpha{}^\nu B_\nu{}^m \\ & \\ 0 & e_a{}^m \end{pmatrix}. \tag{3.1}$$

Here we used tangent-space indices $\alpha, \beta, \ldots$ associated with the $5D$ space-time and $a, b, \ldots$ associated with the $5D$ internal space.[1] The scalar factor Δ is defined by,

$$\Delta = \frac{\det[e_m{}^a(x,y)]}{\det[\mathring{e}_m{}^a(y)]}\,, \tag{3.2}$$

where $\mathring{e}_m{}^a$ is some reference frame for the internal space parametrized by the coordinates y^m. The rescaling of the fünfbein is such that the gravitational coupling constants in $10D$ and $5D$ are related by $\kappa^{-2}|_{10D} = \kappa^{-2}|_{5D} \int \mathrm{d}^5 y \, \det[\mathring{e}_m{}^a]$, so that we are in the $5D$ Einstein frame.

An important feature of the gauge choice made in (3.1) is that it must be preserved under supersymmetry. This requires to add to the $10D$ supersymmetry transformations a uniform field-dependent Lorentz transformation with a parameter equal to

$$\epsilon^{\alpha a} = -\epsilon^{a\alpha} = -\frac{1}{2} e_a{}^m \big(\bar\epsilon \, \breve\Gamma^\alpha \psi_m + \bar\epsilon^{\mathrm{c}} \, \breve\Gamma^\alpha \psi_m{}^{\mathrm{c}} \big)\,, \tag{3.3}$$

where $\psi_a = e_a{}^m \psi_m$. The supersymmetry transformation of $e_a{}^m$ is not affected by the compensating Lorentz transformations, so that we have

$$\delta\Delta = \frac{1}{2}\Delta\big(\bar\epsilon \, \breve\Gamma^a \psi_a + \bar\epsilon^{\mathrm{c}} \, \breve\Gamma^a \psi_a{}^{\mathrm{c}} \big)\,. \tag{3.4}$$

One can now determine the supersymmetry variation of the fünfbein $e_\mu{}^\alpha$, taking into account the compensating Lorentz transformation (3.3) and the effect of the factor Δ. Insisting on the fact that $e_\mu{}^\alpha$ transforms into the $5D$ gravitino field in the same way as before, one then derives a modified gravitino field,

$$\psi_\mu{}^{\mathrm{KK}} \equiv \Delta^{1/6} \big[\psi_\mu - B_\mu{}^m \psi_m \big] + \frac{1}{3} \Delta^{-1/6} e_\mu{}^\alpha \, \breve\Gamma_\alpha \breve\Gamma^a \psi_a\,, \tag{3.5}$$

and likewise for $\psi_\mu{}^{\mathrm{c}}$. This field transforms covariantly under $5D$ space-time diffeomorphisms by virtue of the presence of the field $B_\mu{}^m$. Accordingly we also perform field-dependent scale transformations on the supersymmetry parameter, the gravitino components ψ_a and the dilatino,

$$\epsilon^{\mathrm{KK}} = \Delta^{1/6} \epsilon\,, \qquad \psi_a{}^{\mathrm{KK}} = \Delta^{-1/6} \, e_a{}^m \, \psi_m\,, \qquad \lambda^{\mathrm{KK}} = \Delta^{-1/6} \lambda\,. \tag{3.6}$$

Subsequently we must convert to different gamma matrices that decompose into two commuting Clifford algebras corresponding to the 5-dimensional space-time and the 5-dimensional internal space, which must both commute with $\breve\Gamma_{11}$ so that they will be consistent with the $10D$ chirality restriction on the original spinors. As mentioned previously every $10D$ spinor decomposes into four complex $\mathrm{Spin}(4,1)$ spinors. The gamma matrix conversion is discussed in detail in appendix A and the results can be summarized as follows. The 32×32 gamma matrices $\breve\Gamma_A$ can be written as

$$\breve\Gamma_\alpha = -\mathrm{i}\big(\hat\gamma_\alpha \, \tilde\Gamma \big)\,, \qquad \breve\Gamma_{a+5} = -\mathrm{i}\big(\hat\Gamma_a \, \tilde\gamma \big)\,, \tag{3.7}$$

[1] Note that we are also using indices $\alpha, \beta \ldots$ for the $\mathrm{SU}(1,1)$ indices on the scalar doublet and the tensor fields. This should not cause any confusion.

where $\breve{\Gamma}_{11} = \mathrm{i}\tilde{\gamma}\tilde{\Gamma}$ with $\tilde{\gamma}$ and $\tilde{\Gamma}$ mutually anti-commuting hermitian matrices that square to $\mathbb{1}_{32}$. The tangent space indices in the $5+5$ split were already defined below (3.1).[2] Both $\hat{\gamma}^\alpha$ and $\hat{\Gamma}^a$ anti-commute with $\tilde{\gamma}$ and $\tilde{\Gamma}$ (and therefore commute with $\breve{\Gamma}_{11}$ as insisted on before). They generate two commuting five-dimensional Clifford algebras. Furthermore, we will insist on the Majorana condition $\hat{C}^{-1}\bar{\psi}^{\mathrm{T}} = \psi^{\mathrm{c}}$ for all the $5D$ spinor fields, where $\hat{C}$ is defined in terms of the $10D$ charge conjugation matrix in (A.13). For the gravitino fields and the supersymmetry parameters this leads to the following relations between $10D$ and $5D$ fields,

$$\psi\big|^{\mathrm{KK}}_{10D} = \psi\big|_{5D}\,, \qquad \psi^{\mathrm{c}}\big|^{\mathrm{KK}}_{10D} = \psi^{\mathrm{c}}\big|_{5D}\,, \qquad \bar{\psi}\big|^{\mathrm{KK}}_{10D} = -\mathrm{i}\bar{\psi}\big|_{5D}\tilde{\Gamma}\,, \tag{3.8}$$

where ψ denotes either ψ_M or ϵ.

For the dilatino field λ the situation is somewhat different in view of the fact that we wish to change its chirality by absorbing the matrix $\tilde{\Gamma}$. This conversion is of course no longer consistent with $10D$ Lorentz invariance, but it is convenient to define all the spinor fields with the same (positive) chirality.

$$\lambda\big|^{\mathrm{KK}}_{10D} = \tilde{\Gamma}\,\lambda\big|_{5D}\,, \qquad \lambda^{\mathrm{c}}\big|^{\mathrm{KK}}_{10D} = \mp\tilde{\Gamma}\,\lambda^{\mathrm{c}}\big|_{5D}\,, \qquad \bar{\lambda}\big|^{\mathrm{KK}}_{10D} = \mathrm{i}\,\bar{\lambda}\big|_{5D}\,, \tag{3.9}$$

Once these modifications have been performed, one can simply restrict oneself to the 16-dimensional subspace corresponding to the eigenspace of $\breve{\Gamma}_{11}$ with eigenvalue $+1$. After this one drops the carets on γ_α and Γ_a and thus obtains a description in term of 16-component complex spinors, with two mutually commuting sets of gamma matrices γ_α and Γ_a. Note that this is consistent with using the charge conjugation matrix $\hat{C}$, which was introduced as a 32-dimensional matrix but which commutes with the chirality operator (i.e. charge-conjugated fields carry the same chirality). With these conversions the relation (3.5) for the $10D$ gravitino field ψ_M with $M=\mu$ in terms of the $5D$ fields reads

$$\Delta^{1/6}\psi_\mu = \psi_\mu{}^{\mathrm{KK}} - \frac{1}{3}\mathrm{i}\gamma_\mu\Gamma^m\psi_m{}^{\mathrm{KK}} + \Delta^{1/3}B_\mu{}^m\,\psi_m{}^{\mathrm{KK}}\,. \tag{3.10}$$

Observe that here and henceforth $\gamma_\mu \equiv e_\mu{}^\alpha\,\gamma_\alpha$ and $\Gamma_m \equiv e_m{}^a\,\Gamma_a$, where the vielbein fields $e_\mu{}^\alpha$ and $e_m{}^a$ are defined in (3.1).

In this way one finds the following transformation rules for the $5D$ fields emerging from $E_M{}^A$ as defined in (3.5) and (3.6),

$$\begin{aligned}
\delta e_\mu{}^\alpha &= \frac{1}{2}\big[\bar{\epsilon}\,\gamma^\alpha\psi_\mu + \bar{\epsilon}^{\mathrm{c}}\gamma^\alpha\psi_\mu{}^{\mathrm{c}}\big]\,,\\
\delta B_\mu{}^m &= \frac{1}{2}\Delta^{-1/3}e_a{}^m\Big[\mathrm{i}\big(\bar{\epsilon}\,\Gamma^a\psi_\mu + \bar{\epsilon}^{\mathrm{c}}\,\Gamma^a\psi_\mu{}^{\mathrm{c}}\big)\\
&\qquad\qquad + \bar{\epsilon}\,\gamma_\mu\Big(\delta^a{}_b + \frac{1}{3}\Gamma^a\Gamma_b\Big)\psi^b + \bar{\epsilon}^{\mathrm{c}}\,\gamma_\mu\Big(\delta^a{}_b + \frac{1}{3}\Gamma^a\Gamma_b\Big)\psi^{b\mathrm{c}}\Big]\,,\\
\delta e_m{}^a &= \frac{1}{2}\mathrm{i}\big[\bar{\epsilon}\,\Gamma^a\psi_m + \bar{\epsilon}^{\mathrm{c}}\,\Gamma^a\psi_m{}^{\mathrm{c}}\big]\,,
\end{aligned} \tag{3.11}$$

[2]We employed Pauli-Källén conventions where x^α equals $\mathrm{i}x^0$ for $\alpha = 1$, so that all gamma matrices are hermitian.

up to an infinitesimal $5D$ local Lorentz transformation with a parameter proportional to $\Gamma^m\psi_m$. Since we will be suppressing terms of higher orders in the spinor fields, these transformations will not play a role when evaluating the fermion transformation rules later in this section. Here and in the following we are exclusively considering the $5D$ fields, so that we have dropped the additional labels.

We also evaluate the supersymmetry variations of the scalars and the dilatini,

$$
\begin{aligned}
\delta\phi^\alpha &= -\frac{1}{2}\mathrm{i}\,\varepsilon^{\alpha\beta}\phi_\beta\,\bar{\epsilon}^{\mathrm{c}}\lambda\,,\\
\delta\lambda &= \Delta^{-1/3}\big[-\mathrm{i}P_\alpha\gamma^\alpha + P_a\Gamma^a\big]\epsilon^{\mathrm{c}}\\
&\quad+\frac{1}{24}\Delta^{-1/3}\big[G_{abc}\,\Gamma^{abc} - 3\mathrm{i}\,G_{ab\alpha}\,\Gamma^{ab}\gamma^\alpha + 3\,G_{a\alpha\beta}\,\Gamma^a\gamma^{\alpha\beta} - \mathrm{i}\,G_{\alpha\beta\gamma}\,\gamma^{\alpha\beta\gamma}\big]\epsilon\,,\\
\delta\lambda^{\mathrm{c}} &= \Delta^{-1/3}\big[-\mathrm{i}\bar{P}_\alpha\gamma^\alpha + \bar{P}_a\Gamma^a\big]\epsilon\\
&\quad+\frac{1}{24}\Delta^{-1/3}\big[\bar{G}_{abc}\,\Gamma^{abc} - 3\mathrm{i}\,\bar{G}_{ab\alpha}\,\Gamma^{ab}\gamma^\alpha + 3\,\bar{G}_{a\alpha\beta}\,\Gamma^a\gamma^{\alpha\beta} - \mathrm{i}\,\bar{G}_{\alpha\beta\gamma}\,\gamma^{\alpha\beta\gamma}\big]\epsilon^{\mathrm{c}}\,,
\end{aligned}
\tag{3.12}
$$

where the tensors P and G refer to the components of P_A and G_{ABC}, which are defined with $10D$ tangent-space indices.

Subsequently we derive the expressions for the supersymmetry variation of the gravitino fields up to terms of higher order in the fermion fields, which will now also involve the components of the field strength F_{ABCDE} and the spin-connection fields written with $10D$ tangent-space indices. We first list the gravitino fields that carry a $5D$ space-time vector index,

$$
\begin{aligned}
\delta\psi_\mu &= \Big[\partial_\mu - \frac{1}{6}\partial_\mu\ln\Delta - \Delta^{-1/3}e_\mu{}^\alpha\Big(\frac{1}{4}\omega_\alpha{}^{\beta\gamma}\,\gamma_{\beta\gamma} + \frac{1}{2}\mathrm{i}\omega_\alpha{}^{\beta a}\,\Gamma_a\gamma_\beta + \frac{1}{4}\omega_\alpha{}^{ab}\,\Gamma_{ab} + \frac{1}{2}\mathrm{i}Q_\alpha\Big)\Big]\epsilon\\
&\quad - B_\mu{}^m\Big[\partial_m - \frac{1}{6}\partial_m\ln\Delta\Big]\epsilon\\
&\quad - \frac{1}{240}\mathrm{i}\Delta^{-1/3}\,\varepsilon^{abcde}\big[\mathrm{i}\,F_{abcde} - 5F_{\beta abcd}\gamma^\beta\Gamma_e - 5\mathrm{i}\,F_{\beta\gamma abc}\gamma^{\beta\gamma}\Gamma_{de}\big]\gamma_\mu\epsilon\\
&\quad - \frac{1}{96}\Delta^{-1/3}\big[-\mathrm{i}G_{bcd}\,\Gamma^{bcd}\gamma_\mu + 3G_{bc\alpha}\,\Gamma^{bc}\big(\gamma_\mu\gamma^\alpha + 2\,\gamma^\alpha\gamma_\mu\big)\\
&\qquad\qquad + 3\mathrm{i}G_{b\alpha\beta}\,\Gamma^b\big(\gamma_\mu\gamma^{\alpha\beta} - 2\,\gamma^{\alpha\beta}\gamma_\mu\big) + G_{\alpha\beta\gamma}\big(\gamma_\mu\gamma^{\alpha\beta\gamma} + 2\gamma^{\alpha\beta\gamma}\gamma_\mu\big)\big]\,\epsilon^{\mathrm{c}}\\
&\quad + \frac{1}{3}\mathrm{i}\Delta^{-1/6}\gamma_\mu\,\Gamma^a\delta\psi_a\,,\\
\delta\psi_\mu{}^{\mathrm{c}} &= \Big[\partial_\mu - \frac{1}{6}\partial_\mu\ln\Delta - \Delta^{-1/3}e_\mu{}^\alpha\Big(\frac{1}{4}\omega_\alpha{}^{\beta\gamma}\,\gamma_{\beta\gamma} + \frac{1}{2}\mathrm{i}\omega_\alpha{}^{\beta a}\,\Gamma_a\gamma_\beta + \frac{1}{4}\omega_\alpha{}^{ab}\,\Gamma_{ab} - \frac{1}{2}\mathrm{i}Q_\alpha\Big)\Big]\epsilon^{\mathrm{c}}\\
&\quad - B_\mu{}^m\big[\partial_m - \frac{1}{6}\partial_m\ln\Delta\big]\epsilon^{\mathrm{c}}\\
&\quad + \frac{1}{240}\mathrm{i}\Delta^{-1/3}\varepsilon^{abcde}\Big[\mathrm{i}\,F_{abcde} - 5F_{\beta abcd}\gamma^\beta\Gamma_e - 5\mathrm{i}\,F_{\beta\gamma abc}\gamma^{\beta\gamma}\Gamma_{de}\Big]\gamma_\mu\epsilon^{\mathrm{c}}\\
&\quad - \frac{1}{96}\Delta^{-1/3}\big[-\mathrm{i}\bar{G}_{bcd}\,\Gamma^{bcd}\gamma_\mu + 3\,\bar{G}_{bc\alpha}\,\Gamma^{bc}\big(\gamma_\mu\gamma^\alpha + 2\,\gamma^\alpha\gamma_\mu\big)\\
&\qquad\qquad + 3\mathrm{i}\bar{G}_{b\alpha\beta}\,\Gamma^b\big(\gamma_\mu\gamma^{\alpha\beta} - 2\,\gamma^{\alpha\beta}\gamma_\mu\big) + \bar{G}_{\alpha\beta\gamma}\big(\gamma_\mu\gamma^{\alpha\beta\gamma} + 2\gamma^{\alpha\beta\gamma}\gamma_\mu\big)\big]\,\epsilon\\
&\quad + \frac{1}{3}\mathrm{i}\Delta^{-1/6}\gamma_\mu\,\Gamma^a\delta\psi_a{}^{\mathrm{c}}\,.
\end{aligned}
\tag{3.13}
$$

where we made use of the self-duality condition on the field strength (2.14) and the gamma matrices defined in appendix A, and in particular of (A.17), to simplify the terms involving the various components of the field strength F_{ABCDE}.

The transformation rules for the gravitini that carry a vector index of the internal $5D$ space are given by

$$\begin{aligned}
\delta\psi_a =&\, \Delta^{-1/3} e_a{}^m\Big[\partial_m - \frac{1}{4}\omega_m{}^{\alpha\beta}\,\gamma_{\alpha\beta} - \frac{1}{2}\mathrm{i}\omega_m{}^{\alpha a}\,\Gamma_a\gamma_\alpha - \frac{1}{4}\omega_m{}^{ab}\,\Gamma_{ab} - \frac{1}{2}\mathrm{i}Q_m - \frac{1}{6}\partial_m\ln\Delta\Big]\epsilon \\
&+ \frac{1}{240}\mathrm{i}\Delta^{-1/3}\varepsilon^{bcdef}\big[F_{bcdef} + 5\mathrm{i}\,F_{\alpha bcde}\gamma^\alpha\Gamma_f - 5\,F_{\alpha\beta bcd}\gamma^{\alpha\beta}\Gamma_{ef}\big]\Gamma_a\epsilon \\
&- \frac{1}{96}\Delta^{-1/3}\big[G_{bcd}\,(\Gamma_a\Gamma^{bcd} + 2\,\Gamma^{bcd}\Gamma_a) - 3\mathrm{i}G_{bc\alpha}\,\gamma^\alpha\,(\Gamma_a\Gamma^{bc} - 2\,\Gamma^{bc}\Gamma_a) \\
&\qquad + 3G_{b\alpha\beta}\,\gamma^{\alpha\beta}\,(\Gamma_a\Gamma^b + 2\,\Gamma^b\Gamma_a) + \mathrm{i}G_{\alpha\beta\gamma}\gamma^{\alpha\beta\gamma}\Gamma_a\big]\,\epsilon^{\mathrm{c}}\,, \\
\delta\psi_a{}^{\mathrm{c}} =&\, \Delta^{-1/3} e_a{}^m\Big[\partial_m - \frac{1}{4}\omega_m{}^{\alpha\beta}\,\gamma_{\alpha\beta} - \frac{1}{2}\mathrm{i}\omega_m{}^{\alpha a}\,\Gamma_a\gamma_\alpha - \frac{1}{4}\omega_m{}^{ab}\,\Gamma_{ab} + \frac{1}{2}\mathrm{i}Q_m - \frac{1}{6}\partial_m\ln\Delta\Big]\epsilon^{\mathrm{c}} \\
&- \frac{1}{240}\mathrm{i}\Delta^{-1/3}\varepsilon^{bcdef}\big[F_{bcdef} + 5\mathrm{i}\,F_{\alpha bcde}\gamma^\alpha\Gamma_f - 5\,F_{\alpha\beta bcd}\gamma^{\alpha\beta}\Gamma^{ef}\big]\Gamma_a\epsilon^{\mathrm{c}} \\
&- \frac{1}{96}\Delta^{-1/3}\big[\bar G_{bcd}\,(\Gamma_a\Gamma^{bcd} + 2\,\Gamma^{bcd}\Gamma_a) - 3\mathrm{i}\bar G_{bc\alpha}\,\gamma^\alpha\,(\Gamma_a\Gamma^{bc} - 2\,\Gamma^{bc}\Gamma_a) \\
&\qquad + 3\bar G_{b\alpha\beta}\,\gamma^{\alpha\beta}\,(\Gamma_a\Gamma^b + 2\,\Gamma^b\Gamma_a) + \mathrm{i}\bar G_{\alpha\beta\gamma}\gamma^{\alpha\beta\gamma}\Gamma_a\big]\,\epsilon\,.
\end{aligned} \tag{3.14}$$

The next topic concerns the rank-2 tensor fields $A^\alpha{}_{MN}$, which decompose into twenty scalars $A^\alpha{}_{mn}$, ten $5D$ vectors $A^\alpha{}_{\mu m}$ and two $5D$ 2-rank tensors $A^\alpha{}_{\mu\nu}$. Their consistent Kaluza-Klein definitions are as follows,

$$\begin{aligned}
A^\alpha{}_{mn}{}^{\mathrm{KK}} &= A^\alpha{}_{mn}\,, \\
A^\alpha{}_{\mu m}{}^{\mathrm{KK}} &= A^\alpha{}_{\mu m} - B_\mu{}^p\,A^\alpha{}_{pm}\,, \\
A^\alpha{}_{\mu\nu}{}^{\mathrm{KK}} &= A^\alpha{}_{\mu\nu} + 2\,B_{[\mu}{}^p\,A^\alpha{}_{\nu]p} + B_\mu{}^p\,B_\nu{}^q A^\alpha{}_{pq}\,.
\end{aligned} \tag{3.15}$$

Their supersymmetry variations take the form,

$$\begin{aligned}
\delta A^\alpha{}_{mn} =&\, -\frac{1}{2}\mathrm{i}\,\phi^\alpha\big[\bar\epsilon^{\mathrm{c}}\,\Gamma_{mn}\lambda^{\mathrm{c}} - 4\,\bar\epsilon\,\Gamma_{[m}\psi_{n]}{}^{\mathrm{c}}\big] - \frac{1}{2}\mathrm{i}\,\varepsilon^{\alpha\beta}\phi_\beta\big[\bar\epsilon\,\Gamma_{mn}\lambda - 4\,\bar\epsilon^{\mathrm{c}}\,\Gamma_{[m}\psi_{n]}\big]\,, \\
\delta A^\alpha{}_{\mu m} =&\, -\frac{1}{2}\Delta^{-1/3}\phi^\alpha\Big[2\mathrm{i}\,\bar\epsilon\,\Gamma_m\psi_\mu{}^{\mathrm{c}} - 2\,\bar\epsilon\,\gamma_\mu\Big(\delta_m{}^n - \frac{1}{3}\Gamma_m\Gamma^n\Big)\psi_n{}^{\mathrm{c}} + \bar\epsilon^{\mathrm{c}}\,\Gamma_m\gamma_\mu\lambda^{\mathrm{c}}\Big] \\
&-\frac{1}{2}\Delta^{-1/3}\varepsilon^{\alpha\beta}\phi_\beta\Big[2\mathrm{i}\,\bar\epsilon^{\mathrm{c}}\,\Gamma_m\psi_\mu - 2\,\bar\epsilon^{\mathrm{c}}\gamma_\mu\Big(\delta_m{}^n - \frac{1}{3}\Gamma_m\Gamma^n\Big)\psi_n + \bar\epsilon\,\Gamma_m\gamma_\mu\lambda\Big] \\
&- \delta B_\mu{}^p\,A^\alpha{}_{pm}\,, \\
\delta A^\alpha{}_{\mu\nu} =&\, -\frac{1}{2}\Delta^{-2/3}\phi^\alpha\Big[-4\,\bar\epsilon\,\gamma_{[\mu}\psi_{\nu]}{}^{\mathrm{c}} + \frac{4}{3}\mathrm{i}\bar\epsilon\,\gamma_{\mu\nu}\Gamma^m\psi_m{}^{\mathrm{c}} + \mathrm{i}\,\bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\lambda^{\mathrm{c}}\Big] \\
&-\frac{1}{2}\Delta^{-2/3}\varepsilon^{\alpha\beta}\phi_\beta\Big[-4\,\bar\epsilon^{\mathrm{c}}\gamma_{[\mu}\psi_{\nu]} + \frac{4}{3}\mathrm{i}\,\bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\Gamma^m\psi_m + \mathrm{i}\,\bar\epsilon\,\gamma_{\mu\nu}\lambda\Big] \\
&+ 2\,\delta B_{[\mu}{}^p\,A^\alpha{}_{\nu]p}\,,
\end{aligned} \tag{3.16}$$

where we have suppressed the KK-label on both sides of the equations.

Subsequently we consider the 4-rank tensor A_{MNPQ} which decomposes into five $5D$ scalars A_{mnpq}, ten $5D$ vectors $A_{\mu mnp}$, ten $5D$ 2-rank tensors $A_{\mu\nu mn}$, five $5D$ 3-rank tensors $A_{\mu\nu\rho p}$ and one $5D$ 4-rank tensor $A_{\mu\nu\rho\sigma}$. Their consistent definition is

$$
\begin{aligned}
A_{mnpq}{}^{\mathrm{KK}} &= A_{mnpq}\,, \\
A_{\mu mnp}{}^{\mathrm{KK}} &= A_{\mu mnp} - B_\mu{}^q\, A_{qmnp}\,, \\
A_{\mu\nu mn}{}^{\mathrm{KK}} &= A_{\mu\nu mn} + 2\, B_{[\mu}{}^q\, A_{\nu]qmn} + B_\mu{}^p\, B_\nu{}^q\, A_{pqmn}\,, \\
A_{\mu\nu\rho m}{}^{\mathrm{KK}} &= A_{\mu\nu\rho m} + 3\, B_{[\mu}{}^p\, A_{\nu\rho]mp} + 3\, B_{[\mu}{}^p\, B_\nu{}^q\, A_{\rho]mpq} - B_\mu{}^p\, B_\nu{}^q\, B_\rho{}^r\, A_{pqrm}\,, \\
A_{\mu\nu\rho\sigma}{}^{\mathrm{KK}} &= A_{\mu\nu\rho\sigma} + 4\, B_{[\mu}{}^p\, A_{\nu\rho\sigma]p} + 6\, B_{[\mu}{}^p\, B_\nu{}^q\, A_{\rho\sigma]pq} + 4\, B_{[\mu}{}^p\, B_\nu{}^q\, B_\rho{}^r\, A_{\sigma]pqr} \\
&\qquad + B_\mu{}^p\, B_\nu{}^q\, B_\rho{}^r\, B_\sigma{}^s\, A_{pqrs}\,.
\end{aligned}
\tag{3.17}
$$

The supersymmetry variations for these fields then take the following form,

$$
\begin{aligned}
\delta A_{mnpq} =& -\frac{1}{2}\bar\epsilon\,\Gamma_{[mnp}\psi_{q]} + \frac{1}{2}\bar\epsilon^{\mathrm{c}}\Gamma_{[mnp}\psi_{q]}{}^{\mathrm{c}} + \frac{3}{8}\mathrm{i}\,\varepsilon_{\alpha\beta}A^\alpha{}_{[mn}\,\delta A^\beta{}_{pq]}\,, \\
\delta A_{\mu mnp} =& \frac{1}{8}\Delta^{-1/3}\Big[\bar\epsilon\,\Gamma_{mnp}\psi_\mu + 3\mathrm{i}\,\bar\epsilon\gamma_\mu\Gamma_{[mn}\Big(\delta_{p]}{}^q - \frac{1}{9}\Gamma_{p]}\Gamma^q\Big)\psi_q\Big] \\
&+ \frac{1}{8}\Delta^{-1/3}\Big[-\bar\epsilon^{\mathrm{c}}\Gamma_{mnp}\psi_\mu{}^{\mathrm{c}} - 3\mathrm{i}\,\bar\epsilon^{\mathrm{c}}\gamma_\mu\Gamma_{[mn}\Big(\delta_{p]}{}^q - \frac{1}{9}\Gamma_{p]}\Gamma^q\Big)\psi_q{}^{\mathrm{c}}\Big] \\
&+ \frac{3}{16}\mathrm{i}\varepsilon_{\alpha\beta}\big[A^\alpha{}_{\mu[m}\,\delta A^\beta{}_{np]} - \delta A^\alpha{}_{\mu[m}\,A^\beta{}_{np]} - \delta B_\mu{}^q A^\alpha{}_{q[m}\,A^\beta{}_{np]}\big] \\
&- \delta B_\mu{}^q\, A_{qmnp}\,, \\
\delta A_{\mu\nu mn} =& \frac{1}{4}\Delta^{-2/3}\Big[\mathrm{i}\bar\epsilon\,\Gamma_{mn}\gamma_{[\mu}\psi_{\nu]} - \bar\epsilon\,\gamma_{\mu\nu}\Gamma_{[m}\Big(\delta_{n]}{}^p - \frac{1}{3}\Gamma_{n]}\Gamma^p\Big)\psi_p\Big] \\
&+ \frac{1}{4}\Delta^{-2/3}\Big[-\mathrm{i}\bar\epsilon^{\mathrm{c}}\,\Gamma_{mn}\,\gamma_{[\mu}\psi_{\nu]}{}^{\mathrm{c}} + \bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\Gamma_{[m}\Big(\delta_{n]}{}^p - \frac{1}{3}\Gamma_{n]}\Gamma^p\Big)\psi_p{}^{\mathrm{c}}\Big] \\
&+ \frac{1}{16}\mathrm{i}\varepsilon_{\alpha\beta}\big[A^\alpha{}_{\mu\nu}\,\delta A^\beta{}_{mn} + A^\alpha{}_{mn}\,\delta A^\beta{}_{\mu\nu} - 4\,A^\alpha{}_{[\mu[m}\,\delta A^\beta{}_{\nu]n]}\big] \\
&+ \frac{1}{8}\mathrm{i}\varepsilon_{\alpha\beta}\,\delta B_{[\mu}{}^p\big[A^\alpha{}_{\nu]p}\,A^\beta{}_{mn} - 2\,A^\alpha{}_{\nu][m}\,A^\beta{}_{n]p}\big] \\
&+ 2\,\delta B_{[\mu}{}^p\,A_{\nu]pmn}\,, \\
\delta A_{\mu\nu\rho m} =& \frac{1}{8}\Delta^{-1}\big[3\,\bar\epsilon\,\Gamma_m\gamma_{[\mu\nu}\psi_{\rho]} + \mathrm{i}\bar\epsilon\,\gamma_{\mu\nu\rho}(\delta_m{}^p - \Gamma_m\Gamma^p)\psi_p\big] \\
&+ \frac{1}{8}\Delta^{-1}\big[-3\,\bar\epsilon^{\mathrm{c}}\,\Gamma_m\gamma_{[\mu\nu}\psi_{\rho]}{}^{\mathrm{c}} - \mathrm{i}\bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu\rho}(\delta_m{}^p - \Gamma_m\Gamma^p)\psi_p{}^{\mathrm{c}}\big] \\
&+ \frac{3}{16}\mathrm{i}\,\varepsilon_{\alpha\beta}\,\big[A^\alpha{}_{[\mu\nu}\,\delta A^\beta{}_{\rho]m} - \delta A^\alpha{}_{[\mu\nu}\,A^\beta{}_{\rho]m}\big] \\
&+ \frac{3}{16}\mathrm{i}\,\varepsilon_{\alpha\beta}\,\delta B_{[\mu}{}^p\,\big[A^\alpha{}_{\nu\rho]}\,A^\beta_{pm} + 2A^\alpha{}_{\nu m}\,A^\beta{}_{\rho]p}\big] \\
&+ 3\,\delta B_{[\mu}{}^p\,A_{\nu\rho]mp}\,, \\
\delta A_{\mu\nu\rho\sigma} =& \frac{1}{2}\Delta^{-4/3}\Big[\mathrm{i}\bar\epsilon\,\gamma_{[\mu\nu\rho}\psi_{\sigma]} + \frac{1}{3}\bar\epsilon\,\gamma_{\mu\nu\rho\sigma}\Gamma^p\psi_p\Big] \\
&+ \frac{1}{2}\Delta^{-4/3}\Big[-\mathrm{i}\bar\epsilon^{\mathrm{c}}\gamma_{[\mu\nu\rho}\psi_{\sigma]}{}^{\mathrm{c}} - \frac{1}{3}\bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu\rho\sigma}\Gamma^p\psi_p{}^{\mathrm{c}}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{3}{8}\mathrm{i}\varepsilon_{\alpha\beta}\, A^{\alpha}{}_{[\mu\nu}\,\delta A^{\beta}{}_{\rho\sigma]} - \frac{3}{4}\mathrm{i}\varepsilon_{\alpha\beta}\,\delta B_{[\mu}{}^{p}\, A^{\alpha}{}_{\nu\rho}\, A^{\beta}{}_{\sigma]p} \\
&+ 4\,\delta B_{[\mu}{}^{p}\, A_{\nu\rho\sigma]p}\,,
\end{aligned}
\tag{3.18}
$$

where again we suppressed the KK-label on both sides of these equations.

Let us review the various fields that we have obtained and compare them with the fields that are generically contained in maximal $5D$ supergravity. First of all we have the fünfbein field $e_\mu{}^\alpha$ and the eight independent gravitini fields consisting of the fields $(\psi_\mu, \psi_\mu{}^{\mathrm{c}})$. Furthermore there are 48 spin-1/2 fields consisting of $(\psi_a, \psi_a{}^{\mathrm{c}})$, and $(\lambda, \lambda^{\mathrm{c}})$.

Then there are 42 scalar fields, consisting of $e_m{}^a$, ϕ^α, $A^\alpha{}_{mn}$ and A_{mnpq}. The field $e_m{}^a$ corresponds to 15 scalars and the fields ϕ^α to 2 scalars upon subtracting the degrees of freedom associated with tangent space transformations of the internal space and local U(1) transformations. The fields $A^\alpha{}_{mn}$ and A_{mnpq} describe 20 and 5 scalars, respectively. The total number of scalars is thus equal to the dimension of the $\mathrm{E}_{6(6)}/\mathrm{USp}(8)$ coset space that parametrizes the scalars in $5D$ maximal supergravity.

To appreciate the systematics of the vector and tensor fields we introduce the following (re)definitions. The 25 vector fields that we have obtained at this stage will be denoted by

$$
\begin{aligned}
C_\mu{}^m &= B_\mu{}^m\,, \\
C_\mu{}^\alpha{}_m &= A^\alpha{}_{\mu m}{}^{\mathrm{KK}}\,, \\
C_{\mu\, mnp} &= A_{\mu mnp}{}^{\mathrm{KK}} - \frac{3}{16}\mathrm{i}\varepsilon_{\alpha\beta} A^\alpha{}_{\mu[m}{}^{\mathrm{KK}}\, A^\beta{}_{np]}\,,
\end{aligned}
\tag{3.19}
$$

where the extra term in the definition of $C_{\mu\, mnp}$ has been included such that its supersymmetry variation will not contain the vector field. Observe also that in the above result we have suppressed the KK-label for the scalar field $A^\beta{}_{np}$; henceforth we will do this consistently for both $A^\beta{}_{np}$ and A_{mnpq}. The fields $C_\mu{}^m$ and $C_{\mu\, mnp}$ can be combined into the 15-dimensional anti-symmetric representation of SL(6). The remaining vector fields $C_\mu{}^\alpha{}_m$ transform as five doublets under $\mathrm{SU}(1,1) \cong \mathrm{SL}(2)$. As compared to the vector fields of $5D$ maximal supergravity, we should expect six such doublets. As we will show in the next section, the extra doublet will emerge from a dual tensor field, $A^\alpha{}_{MNPQRS}$, which leads to the fields $A^\alpha{}_{\mu mnpqr}$. In view of the self-duality constraint (2.14), we do not expect any tensor fields dual to A_{MNPQ}.

The 25 vector fields (3.19) transform as follows,

$$
\begin{aligned}
\delta C_\mu{}^m = {}& \frac{1}{2}\Delta^{-1/3} e_a{}^m \Big[\mathrm{i}\big(\bar\epsilon\,\Gamma^a\psi_\mu + \bar\epsilon^{\mathrm{c}}\,\Gamma^a\psi_\mu{}^{\mathrm{c}}\big) \\
&\qquad + \bar\epsilon\,\gamma_\mu\Big(\delta^a{}_b + \frac{1}{3}\Gamma^a\Gamma_b\Big)\psi^b + \bar\epsilon^{\mathrm{c}}\,\gamma_\mu\Big(\delta^a{}_b + \frac{1}{3}\Gamma^a\Gamma_b\Big)\psi^{b\mathrm{c}}\Big]\,, \\
\delta C_\mu{}^\alpha{}_m = {}& -\frac{1}{2}\Delta^{-1/3}\phi^\alpha\Big[2\mathrm{i}\,\bar\epsilon\,\Gamma_m\psi_\mu{}^{\mathrm{c}} - 2\,\bar\epsilon\,\gamma_\mu\Big(\delta_m{}^n - \frac{1}{3}\Gamma_m\Gamma^n\Big)\psi_n{}^{\mathrm{c}} + \bar\epsilon^{\mathrm{c}}\,\Gamma_m\gamma_\mu\lambda^{\mathrm{c}}\Big] \\
& -\frac{1}{2}\Delta^{-1/3}\varepsilon^{\alpha\beta}\phi_\beta\Big[2\mathrm{i}\,\bar\epsilon^{\mathrm{c}}\,\Gamma_m\psi_\mu - 2\,\bar\epsilon^{\mathrm{c}}\gamma_\mu\Big(\delta_m{}^n - \frac{1}{3}\Gamma_m\Gamma^n\Big)\psi_n + \bar\epsilon\,\Gamma_m\gamma_\mu\lambda\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{2}\mathrm{i}\,\Delta^{-1/3}A^{\alpha}{}_{mp}\big[\bar\epsilon\,\Gamma^{p}\psi_{\mu}+\bar\epsilon^{\mathrm{c}}\,\Gamma^{p}\psi_{\mu}{}^{\mathrm{c}}\big]\\
&+\frac{1}{2}\Delta^{-1/3}A^{\alpha}{}_{mp}\Big[\bar\epsilon\,\gamma_{\mu}(e_a{}^{p}+\frac{1}{3}\Gamma^{p}\Gamma_a)\psi^{a}+\bar\epsilon^{\mathrm{c}}\,\gamma_{\mu}(e_a{}^{p}+\frac{1}{3}\Gamma^{p}\Gamma_a)\psi^{ac}\Big]\,,
\end{aligned}
$$

$$
\begin{aligned}
\delta C_{\mu\,mnp}=&\frac{1}{8}\Delta^{-1/3}\Big[\bar\epsilon\,\Gamma_{mnp}\psi_{\mu}+3\mathrm{i}\,\bar\epsilon\gamma_{\mu}\Gamma_{[mn}\Big(\delta_{p]}{}^{q}-\frac{1}{9}\Gamma_{p]}\Gamma^{q}\Big)\psi_{q}\Big]\\
&+\frac{1}{8}\Delta^{-1/3}\Big[-\bar\epsilon^{\mathrm{c}}\,\Gamma_{mnp}\psi_{\mu}{}^{\mathrm{c}}-3\mathrm{i}\,\bar\epsilon^{\mathrm{c}}\gamma_{\mu}\Gamma_{[mn}\Big(\delta_{p]}{}^{q}-\frac{1}{9}\Gamma_{p]}\Gamma^{q}\Big)\psi_{q}{}^{\mathrm{c}}\Big]\\
&-\frac{3}{16}\mathrm{i}\,\Delta^{-1/3}\varepsilon_{\alpha\beta}\,A^{\alpha}{}_{[mn}\phi^{\beta}\Big[2\mathrm{i}\,\bar\epsilon\,\Gamma_{p]}\psi_{\mu}{}^{\mathrm{c}}-2\,\bar\epsilon\,\gamma_{\mu}\Big(\delta_{p]}{}^{n}-\frac{1}{3}\Gamma_{p]}\Gamma^{n}\Big)\psi_{n}{}^{\mathrm{c}}+\bar\epsilon^{\mathrm{c}}\,\Gamma_{p]}\gamma_{\mu}\lambda^{\mathrm{c}}\Big]\\
&+\frac{3}{16}\mathrm{i}\,\Delta^{-1/3}\,A^{\alpha}{}_{[mn}\,\phi_{\alpha}\Big[2\mathrm{i}\,\bar\epsilon^{\mathrm{c}}\,\Gamma_{p]}\psi_{\mu}-2\,\bar\epsilon^{\mathrm{c}}\gamma_{\mu}\Big(\delta_{p]}{}^{n}-\frac{1}{3}\Gamma_{p]}\Gamma^{n}\Big)\psi_{n}+\bar\epsilon\,\Gamma_{p]}\gamma_{\mu}\lambda\Big]\\
&+\frac{1}{2}\mathrm{i}\,\Delta^{-1/3}\Big[A_{mnpq}+\frac{3}{16}\mathrm{i}\varepsilon_{\alpha\beta}A^{\alpha}{}_{[mn}\,A^{\beta}{}_{p]q}\Big]\big[\bar\epsilon\,\Gamma^{q}\psi_{\mu}+\bar\epsilon^{\mathrm{c}}\,\Gamma^{q}\psi_{\mu}{}^{\mathrm{c}}\big]\\
&+\frac{1}{2}\Delta^{-1/3}\Big[A_{mnpq}+\frac{3}{16}\mathrm{i}\varepsilon_{\alpha\beta}A^{\alpha}{}_{[mn}\,A^{\beta}{}_{p]q}\Big]\\
&\qquad\times\Big[\bar\epsilon\,\gamma_{\mu}\Big(e_b{}^{q}+\frac{1}{3}\Gamma^{q}\Gamma_b\Big)\psi^{b}+\bar\epsilon^{\mathrm{c}}\,\gamma_{\mu}\Big(e_b{}^{q}+\frac{1}{3}\Gamma^{q}\Gamma_b\Big)\psi^{bc}\Big]\,.
\end{aligned}\tag{3.20}
$$

Furthermore we have identified 12 two-rank tensor fields, which we define by

$$
\begin{aligned}
C_{\mu\nu}{}^{\alpha}&=A^{\alpha}_{\mu\nu}{}^{\mathrm{KK}}-C_{[\mu}{}^{p}\,C_{\nu]}{}^{\alpha}{}_{p}\,,\\
C_{\mu\nu\,mn}&=A_{\mu\nu mn}{}^{\mathrm{KK}}-\frac{1}{16}\mathrm{i}\varepsilon_{\alpha\beta}A^{\alpha}_{\mu\nu}{}^{\mathrm{KK}}\,A^{\beta}{}_{mn}-C_{[\mu}{}^{p}\,C_{\nu]pmn}\,.
\end{aligned}\tag{3.21}
$$

The supersymmetry transformations of these tensors are expressed by

$$
\begin{aligned}
&\delta C_{\mu\nu}{}^{\alpha}+C_{[\mu}{}^{p}\,\delta C_{\nu]}{}^{\alpha}{}_{p}+C_{[\mu}{}^{\alpha}{}_{p}\,\delta C_{\nu]}{}^{p}\\
&\quad=-\frac{1}{2}\Delta^{-2/3}\phi^{\alpha}\Big[-4\,\bar\epsilon\,\gamma_{[\mu}\psi_{\nu]}{}^{\mathrm{c}}+\frac{4}{3}\mathrm{i}\bar\epsilon\,\gamma_{\mu\nu}\Gamma^{m}\psi_{m}{}^{\mathrm{c}}+\mathrm{i}\,\bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\lambda^{\mathrm{c}}\Big]\\
&\qquad-\frac{1}{2}\Delta^{-2/3}\varepsilon^{\alpha\beta}\phi_{\beta}\Big[-4\,\bar\epsilon^{\mathrm{c}}\gamma_{[\mu}\psi_{\nu]}+\frac{4}{3}\mathrm{i}\,\bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\Gamma^{m}\psi_{m}+\mathrm{i}\,\bar\epsilon\,\gamma_{\mu\nu}\lambda\Big]\,,\\
&\delta C_{\mu\nu\,mn}+C_{[\mu}{}^{p}\,\delta C_{\nu]pmn}+C_{[\mu\,pmn}\,\delta C_{\nu]}{}^{p}+\frac{1}{4}\mathrm{i}\varepsilon_{\alpha\beta}\,C_{[\mu}{}^{\alpha}{}_{[m}\,\delta C_{\nu]}{}^{\beta}{}_{n]}\\
&\quad=\frac{1}{4}\Delta^{-2/3}\Big[\mathrm{i}\bar\epsilon\,\Gamma_{mn}\gamma_{[\mu}\psi_{\nu]}-\bar\epsilon\,\gamma_{\mu\nu}\Gamma_{[m}\Big(\delta^{p}_{n]}-\frac{1}{3}\Gamma_{n]}\Gamma^{p}\Big)\psi_{p}\Big]\\
&\qquad+\frac{1}{4}\Delta^{-2/3}\Big[-\mathrm{i}\bar\epsilon^{\mathrm{c}}\,\Gamma_{mn}\,\gamma_{[\mu}\psi_{\nu]}{}^{\mathrm{c}}+\bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\Gamma_{[m}\Big(\delta_{n]}{}^{p}-\frac{1}{3}\Gamma_{n]}\Gamma^{p}\Big)\psi_{p}{}^{\mathrm{c}}\Big]\\
&\qquad-\frac{1}{16}\mathrm{i}\,\Delta^{-2/3}\varepsilon_{\alpha\beta}\,A^{\alpha}{}_{mn}\,\phi^{\beta}\Big[-4\,\bar\epsilon\,\gamma_{[\mu}\psi_{\nu]}{}^{\mathrm{c}}+\frac{4}{3}\mathrm{i}\bar\epsilon\,\gamma_{\mu\nu}\Gamma^{m}\psi_{m}{}^{\mathrm{c}}+\mathrm{i}\,\bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\lambda^{\mathrm{c}}\Big]\\
&\qquad+\frac{1}{16}\mathrm{i}\,\Delta^{-2/3}\,A^{\alpha}{}_{mn}\,\phi_{\alpha}\Big[-4\,\bar\epsilon^{\mathrm{c}}\gamma_{[\mu}\psi_{\nu]}+\frac{4}{3}\mathrm{i}\,\bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\Gamma^{m}\psi_{m}+\mathrm{i}\,\bar\epsilon\,\gamma_{\mu\nu}\lambda\Big]\,.
\end{aligned}\tag{3.22}
$$

These transformation rules are in line with what is known from the vector-tensor hierarchy that appears in the context of the embedding tensor formalism [8, 10]. We have actually verified that also the variation of the three-rank tensor fields, $A_{\mu\nu\rho m}{}^{\rm KK}$ listed in (3.17) will exhibit the same structure upon introducing a suitable modification. Since we will not be considering tensors of rank higer than two, we refrain from giving further details.

At this point the number of tensor fields is less than the 27 fields that one expects on the basis of $5D$ maximal supergravity in the context of the embedding tensor formalism. Ten extra 2-rank tensors $A^{\alpha}{}_{\mu\nu mnpq}$ will be provided by the dual field, $A^{\alpha}{}_{MNPQRS}$, which will bring the total of 2-rank tensors to 22. The dual vectors and tensors are evaluated in the next section.

4 Dual fields and the vector-tensor hierarchy

In (2.27) we presented the field equation for the tensor fields $A^{\alpha}{}_{MN}$ written as a Bianchi identity of the seven-rank field strength $F_{\alpha\,MNPQRST}$ defined in (2.28). The field equation thus implies that this field strength can be written in terms of a dual six-form field $A_{\alpha\,MNPQRS}$ according to

$$F_{\alpha\,MNPQRST} = 6\,\partial_{[M}A_{\alpha\,NPQRST]}\,. \tag{4.1}$$

It is not possible to derive an expression for $A_{\alpha\,MNPQRS}$ in closed form, but it is possible to determine how this field transforms under supersymmetry. Obviously, the Bianchi identity (2.27) should transform under supersymmetry into fermionic equations which are of at most first order in derivatives. Therefore one expects that $F_{\alpha\,MNPQRST}$ transforms into fermionic field equations and into terms that carry explicit space-time derivatives such that they can be identified as the result of the supersymmetry variation of the dual six-form. Because the field equations are supercovariant all the contributions of the variation of the six-form can be identified from the terms that are proportional to the derivative of the supersymmetry parameters. The consistency of this approach can easily be verified and it leads to the following result,

$$\begin{aligned}
\delta A_{\alpha\,MNPQRS} =&\, \varepsilon_{\alpha\beta}\phi^{\beta}\left(\frac{1}{6}\bar\lambda\,\breve\Gamma_{MNPQRS}\epsilon + 2\,\bar\epsilon\,\breve\Gamma_{[MNPQR}\psi_{S]}{}^{\rm c}\right)\\
&- \phi_{\alpha}\left(\frac{1}{6}\bar\epsilon\,\breve\Gamma_{MNPQRS}\lambda - 2\,\bar\psi_{[M}{}^{\rm c}\,\breve\Gamma_{NPQRS]}\epsilon\right)\\
&- 20{\rm i}\,\varepsilon_{\alpha\beta}A^{\beta}{}_{[MN}\left(\delta A_{PQRS]} - \frac{1}{8}{\rm i}\,\varepsilon_{\gamma\delta}A^{\gamma}{}_{PQ}\,\delta A^{\delta}{}_{RS]}\right)\,.
\end{aligned} \tag{4.2}$$

In particular we note the dual fields $A^{\alpha}{}_{\mu mnpqr}$ and $A^{\alpha}{}_{\mu\nu mnpq}$, which constitute two $5D$ vector fields and twelve $5D$ tensor fields transforming under SU$(1,1)$. We first consider the transformation rule of the vector field $A^{\alpha}{}_{\mu mnpqr}$, which takes the following form,

$$\begin{aligned}
&\delta A_{\alpha\,\mu mnpqr} =\\
&-\frac{1}{3}{\rm i}\,\Delta^{-1/3}\varepsilon_{\alpha\beta}\phi^{\beta}\left[\bar\epsilon\,\Gamma_{mnpqr}\psi_{\mu}{}^{\rm c} + 5{\rm i}\,\bar\epsilon\,\gamma_{\mu}\left(\Gamma_{[mnpq}\delta_{r]}{}^{s} - \frac{1}{15}\Gamma_{r]}\Gamma^{s}\right)\psi_{s}{}^{\rm c} + \frac{1}{2}{\rm i}\bar\epsilon^{\rm c}\,\gamma_{\mu}\Gamma_{mnpqr}\lambda^{\rm c}\right]\\
&-\frac{1}{3}{\rm i}\,\Delta^{-1/3}\phi_{\alpha}\left[\bar\epsilon^{\rm c}\,\Gamma_{mnpqr}\psi_{\mu} + 5{\rm i}\,\bar\epsilon^{\rm c}\,\gamma_{\mu}\Gamma_{[mnpq}\left(\delta_{r]}{}^{s} - \frac{1}{15}\Gamma_{r]}\Gamma^{s}\right)\psi_{s} + \frac{1}{2}{\rm i}\bar\epsilon\,\gamma_{\mu}\Gamma_{mnpqr}\lambda\right]
\end{aligned}$$

$$
\begin{aligned}
&-\frac{20}{3}\mathrm{i}\,\varepsilon_{\alpha\beta}\big(A^{\beta}{}_{\mu[m}\,\delta A_{npqr]} - 2\,A^{\beta}{}_{[mn}\,\delta A_{pqr]\mu}\big)\\
&-\frac{5}{6}\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta}\big(2\,A^{(\beta}{}_{\mu[m}A^{\gamma)}{}_{np}\,\delta A^{\delta}{}_{qr]} - A^{\beta}{}_{[mn}A^{\gamma}{}_{pq}\,\delta A^{\delta}{}_{r]\mu}\big)\\
&+\frac{40}{3}\mathrm{i}\,\varepsilon_{\alpha\beta}\;\delta B_{\mu}{}^{s}A^{\beta}{}_{[mn}\left(A_{pqr]s} - \frac{1}{16}\mathrm{i}\varepsilon_{\gamma\delta}A^{\gamma}_{pq}\,A^{\delta}_{r]s}\right),
\end{aligned}
\tag{4.3}
$$

where on the right-hand side all the fields have been subject to Kaluza-Klein redefinitions. The field $A_{\alpha\,\mu mnpqr}$ already transforms consistently as a vector in the $5D$ space-time because tensors anti-symmetric in more than five internal-space indices must vanish. The consistency of the above result is confirmed by the fact that no terms are generated proportional to the Kaluza-Klein vector field $B_{\mu}{}^{m}$, simply because the corresponding terms are fully anti-symmetric in six internal-space indices and therefore vanish.

However, from the perspective of the vector-tensor hierarchy further redefinitions are required, as the supersymmetry variations should not contain any vector fields, but at most variations of vector fields. A preliminary analysis suggests to add modifications that are quadratic and cubic terms in the four- and two-form fields but here we have to make sure that also the modification itself transforms consistently as a vector in the $5D$ space-time. This leads us to the following redefinition,

$$
C_{\mu\,\alpha mnpqr} = A_{\alpha\,\mu mnpqr} + \frac{20}{3}\mathrm{i}\,\varepsilon_{\alpha\beta}\,C_{\mu}{}^{\beta}{}_{[m}\,A_{npqr]} - \frac{5}{6}\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta}\,A^{\beta}{}_{[mn}\,C_{\mu}{}^{\gamma}{}_{p}\,A^{\delta}{}_{qr]}\,,
\tag{4.4}
$$

where $C_{\mu}{}^{\alpha}{}_{m}$ is a proper vector field defined in (3.19). Under supersymmetry the field $C_{\mu\,\alpha mnpqr}$ transforms in the required way,

$$
\begin{aligned}
\delta C_{\mu\,\alpha mnpqr} = &-\frac{1}{3}\mathrm{i}\,\Delta^{-1/3}\varepsilon_{\alpha\beta}\phi^{\beta}\bigg[\bar\epsilon\,\Gamma_{mnpqr}\psi_{\mu}{}^{\mathrm{c}} + 5\mathrm{i}\,\bar\epsilon\,\gamma_{\mu}\left(\Gamma_{[mnpq}\delta_{r]}{}^{s} - \frac{1}{15}\Gamma_{r]}\Gamma^{s}\right)\psi_{s}{}^{\mathrm{c}}\\
&\qquad\qquad\qquad\qquad + \frac{1}{2}\mathrm{i}\bar\epsilon^{\mathrm{c}}\,\gamma_{\mu}\Gamma_{mnpqr}\lambda^{\mathrm{c}}\bigg]\\
&-\frac{1}{3}\mathrm{i}\,\Delta^{-1/3}\phi_{\alpha}\bigg[\bar\epsilon^{\mathrm{c}}\,\Gamma_{mnpqr}\psi_{\mu} + 5\mathrm{i}\,\bar\epsilon^{\mathrm{c}}\,\gamma_{\mu}\Gamma_{[mnpq}\left(\delta_{r]}{}^{s} - \frac{1}{15}\Gamma_{r]}\Gamma^{s}\right)\psi_{s}\\
&\qquad\qquad\qquad\qquad + \frac{1}{2}\mathrm{i}\bar\epsilon\,\gamma_{\mu}\Gamma_{mnpqr}\lambda\bigg]\\
&+\frac{20}{3}\mathrm{i}\,\varepsilon_{\alpha\beta}\big[\delta C_{\mu}{}^{\beta}{}_{[m}\,A_{npqr]} - 2\,\delta C_{\mu[mnp}\,A^{\beta}{}_{qr]} - 2\;\delta C_{\mu}{}^{s}A_{s[mnp}\,A^{\beta}{}_{qr]}\big]\\
&+\frac{5}{2}\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta}\left[\delta C_{\mu}{}^{\gamma}{}_{[m}\,A^{\delta}{}_{np}\,A^{\beta}{}_{qr]} + \frac{1}{3}\delta C_{\mu}{}^{s}\,A^{\gamma}{}_{s[m}\,A^{\beta}{}_{np}\,A^{\delta}{}_{qr]}\right],
\end{aligned}
\tag{4.5}
$$

where, for conciseness, we refrained from substituting the explicit expressions for C_{μ}^{m}, $\delta C_{\mu}^{\alpha}{}_{m}$ and $\delta C_{\mu\,mnp}$ in the right-hand of the last equation.

Subsequently we consider the tensor field $A_{\alpha\,\mu\nu mnpq}$. To ensure that this field transforms as a proper $5D$ tensor one performs the standard Kaluza-Klein redefinition,

$$
A_{\alpha\,\mu\nu mnpq}{}^{\mathrm{KK}} = A_{\alpha\,\mu\nu mnpq} + 2B_{[\mu}{}^{r}\,A_{\alpha\nu]mnpqr}\,.
\tag{4.6}
$$

This modified tensor field transforms as

$$
\begin{aligned}
&\delta A_{\alpha\,\mu\nu mnpq} = \\
&\quad -\frac{2}{3}\mathrm{i}\,\Delta^{-2/3}\varepsilon_{\alpha\beta}\phi^{\beta}\Big[\mathrm{i}\bar\epsilon\,\Gamma_{mnpq}\gamma_{[\mu}\psi_{\nu]}{}^{\mathrm{c}} - 2\,\bar\epsilon\,\gamma_{\mu\nu}\Gamma_{[mnp}\Big(\delta_{q]}{}^{r} - \frac{1}{6}\Gamma_{q]}\Gamma^{r}\Big)\psi_{r}{}^{\mathrm{c}} - \frac{1}{4}\bar\epsilon^{\mathrm{c}}\,\gamma_{\mu\nu}\Gamma_{mnpq}\lambda^{\mathrm{c}}\Big] \\
&\quad -\frac{2}{3}\mathrm{i}\,\Delta^{-2/3}\phi_{\alpha}\Big[\mathrm{i}\bar\epsilon^{\mathrm{c}}\,\Gamma_{mnpq}\gamma_{[\mu}\psi_{\nu]} - 2\,\bar\epsilon^{\mathrm{c}}\,\gamma_{\mu\nu}\Gamma_{[mnp}\Big(\delta_{q]}{}^{r} - \frac{1}{6}\Gamma_{q]}\Gamma^{r}\Big)\psi_{r} - \frac{1}{4}\bar\epsilon\,\gamma_{\mu\nu}\Gamma_{mnpq}\lambda\Big] \\
&\quad -\frac{4}{3}\mathrm{i}\,\varepsilon_{\alpha\beta}\big[A^{\beta}{}_{\mu\nu}\,\delta A_{mnpq} - 8\,A^{\beta}{}_{[\mu[m}\,\delta A_{\nu]npq]} + 6\,A^{\beta}{}_{[mn}\,\delta A_{pq]\mu\nu}\big] \\
&\quad -\frac{1}{6}\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta}\big(2\,A^{(\beta}{}_{\mu\nu}\,A^{\gamma)}{}_{[mn}\,\delta A^{\delta}{}_{pq]} + A^{\beta}{}_{[mn}\,A^{\gamma}{}_{pq]}\,\delta A^{\delta}{}_{\mu\nu}\big) \\
&\quad +\frac{2}{3}\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta}\big(A^{\beta}{}_{[\mu[m}\,A^{\gamma}{}_{\nu]n}\,\delta A^{\delta}{}_{pq]} + 2\,A^{(\beta}{}_{[\mu[m}\,A^{\gamma)}{}_{np}\,\delta A^{\delta}{}_{\nu]q]}\big) \\
&\quad +\frac{16}{3}\mathrm{i}\,\varepsilon_{\alpha\beta}\,\delta B_{[\mu}{}^{r}\,\big(2A^{\beta}{}_{\nu][m}\,A_{npq]r} + 3\,A_{\nu]r[mn}\,A^{\beta}{}_{pq]}\big) \\
&\quad +\frac{1}{3}\,\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta}\,\delta B_{[\mu}{}^{r}\,\big(4\,A^{(\beta}{}_{\nu][m}A^{\gamma)}{}_{np}\,A^{\delta}{}_{q]r} + A^{\delta}{}_{\nu]r}A^{\beta}{}_{[mn}\,A^{\gamma}{}_{pq]}\big) \\
&\quad +2\,\delta B_{[\mu}{}^{r}\,A_{\alpha\nu]mnpqr}\,,
\end{aligned}
\tag{4.7}
$$

where we again dropped KK-label on both sides of the equation.

Again this result is not consistent with regard to the vector-tensor hierarchy so that further redefinitions of the tensor field are required. As it turns out, they take the following form,

$$
\begin{aligned}
C_{\mu\nu\,\alpha mnpq} = A_{\mu\nu\,\alpha mnpq} &+ \frac{4}{3}\mathrm{i}\,\varepsilon_{\alpha\beta}\,A^{\beta}{}_{\mu\nu}\,A_{mnpq} \\
&- \frac{1}{6}\varepsilon_{\alpha\beta}\,\varepsilon_{\gamma\delta}\,\big[A^{\gamma}{}_{\mu\nu}\,A^{\beta}{}_{[mn}\,A^{\delta}{}_{pq]} - 8\,C_{[\mu}{}^{\beta}{}_{[m}\,C_{\nu]}{}^{\gamma}{}_{n}\,A^{\delta}{}_{pq]}\big] \\
&- \frac{16}{3}\mathrm{i}\,\varepsilon_{\alpha\beta}\,C_{[\mu}{}^{\beta}{}_{[m}\,C_{\nu]npq]} - C_{[\mu}{}^{r}C_{\nu]\,\alpha mnpqr}\,,
\end{aligned}
\tag{4.8}
$$

where on the the right-hand side the KK-labels have again been suppressed. The transformation rule of $C_{\mu\nu\,\alpha mnpq}$ takes the form

$$
\begin{aligned}
&\delta C_{\mu\nu\,\alpha mnpq} - \frac{16}{3}\mathrm{i}\,\varepsilon_{\alpha\beta}\,\big[C_{[\mu}{}^{\beta}{}_{[m}\,\delta C_{\nu]npq]} + C_{[\mu[npq}\,\delta C_{\nu]}{}^{\beta}{}_{m]}\big] \\
&\qquad + C_{[\mu}{}^{r}\,\delta C_{\nu]\,\alpha mnpqr} + C_{[\mu\,\alpha mnpqr}\,\delta C_{\nu]}{}^{r} \\
&\quad = -\frac{2}{3}\mathrm{i}\,\Delta^{-2/3}\varepsilon_{\alpha\beta}\phi^{\beta}\bigg[\Big[\mathrm{i}\bar\epsilon\,\Gamma_{mnpq}\gamma_{[\mu}\psi_{\nu]}{}^{\mathrm{c}} - 2\,\bar\epsilon\,\gamma_{\mu\nu}\Gamma_{[mnp}\Big(\delta_{q]}{}^{r} - \frac{1}{6}\Gamma_{q]}\Gamma^{r}\Big)\psi_{r}{}^{\mathrm{c}} \\
&\qquad\qquad - \frac{1}{4}\bar\epsilon^{\mathrm{c}}\,\gamma_{\mu\nu}\Gamma_{mnpq}\lambda^{\mathrm{c}}\Big] + A_{mnpq}\Big[-4\,\bar\epsilon\,\gamma_{[\mu}\psi_{\nu]}{}^{\mathrm{c}} + \frac{4}{3}\mathrm{i}\bar\epsilon\,\gamma_{\mu\nu}\Gamma^{m}\psi_{m}{}^{\mathrm{c}} + \mathrm{i}\,\bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\lambda^{\mathrm{c}}\Big]\bigg] \\
&\qquad - \frac{2}{3}\mathrm{i}\,\Delta^{-2/3}\phi_{\alpha}\bigg[\Big[\mathrm{i}\bar\epsilon^{\mathrm{c}}\,\Gamma_{mnpq}\gamma_{[\mu}\psi_{\nu]} - 2\,\bar\epsilon^{\mathrm{c}}\,\gamma_{\mu\nu}\Gamma_{[mnp}\Big(\delta_{q]}{}^{r} - \frac{1}{6}\Gamma_{q]}\Gamma^{r}\Big)\psi_{r} - \frac{1}{4}\bar\epsilon\,\gamma_{\mu\nu}\Gamma_{mnpq}\lambda\Big] \\
&\qquad\qquad - A_{mnpq}\Big[-4\,\bar\epsilon^{\mathrm{c}}\gamma_{[\mu}\psi_{\nu]} + \frac{4}{3}\mathrm{i}\,\bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\Gamma^{m}\psi_{m} + \mathrm{i}\,\bar\epsilon\,\gamma_{\mu\nu}\lambda\Big]\bigg] \\
&\qquad - 2\mathrm{i}\,\Delta^{-2/3}\,\varepsilon_{\alpha\beta}\,A^{\beta}{}_{[mn}\bigg[\Big[\mathrm{i}\bar\epsilon\,\Gamma_{pq]}\gamma_{[\mu}\psi_{\nu]} - \bar\epsilon\,\gamma_{\mu\nu}\Gamma_{p}\Big(\delta_{q]}{}^{r} - \frac{1}{3}\Gamma_{q]}\Gamma^{r}\Big)\psi_{r}\Big]
\end{aligned}
$$

$$
\begin{aligned}
&-\left[\mathrm{i}\bar\epsilon^{\mathrm{c}}\,\Gamma_{pq]}\,\gamma_{[\mu}\psi_{\nu]}{}^{\mathrm{c}}-\bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\Gamma_{p}\left(\delta_{q]}^{\ r}-\frac{1}{3}\Gamma_{q]}\Gamma^{r}\right)\psi_{r}{}^{\mathrm{c}}\right]\Bigg]\\
-\,\mathrm{i}\,\Delta^{-2/3}\varepsilon_{\alpha\beta}\,A^{\beta}{}_{[mn}&\left[\varepsilon_{\gamma\delta}\,A^{\gamma}{}_{pq]}\,\phi^{\delta}\left[\mathrm{i}\,\bar\epsilon\,\gamma_{[\mu}\psi_{\nu]}{}^{\mathrm{c}}+\frac{1}{3}\bar\epsilon\,\gamma_{\mu\nu}\Gamma^{r}\psi_{r}^{\ \mathrm{c}}+\frac{1}{4}\bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\lambda^{\mathrm{c}}\right]\right.\\
&\left.-\,A^{\gamma}{}_{pq]}\,\phi_{\gamma}\left[\mathrm{i}\,\bar\epsilon^{\mathrm{c}}\gamma_{[\mu}\psi_{\nu]}+\frac{1}{3}\,\bar\epsilon^{\mathrm{c}}\gamma_{\mu\nu}\Gamma^{r}\psi_{r}+\frac{1}{4}\bar\epsilon\,\gamma_{\mu\nu}\lambda\right]\right]. \qquad (4.9)
\end{aligned}
$$

To conclude this section let us summarize the situation regarding the vector and tensor fields. We have identified precisely 27 vector fields, namely,

$$
C_\mu{}^M=\left\{C_\mu^{\ m}\,,\,C_{\mu\,mnp}\,,\,C_{\mu\ m}^{\ \alpha}\,,\,C_{\mu\,\alpha mnpqr}\right\}. \qquad (4.10)
$$

For the tensor fields the situation is somewhat different. First of all, we expect 27 tensor fields whereas previously we found only 22 fields. Secondly, we note that the tensor fields, which we will denote by $C_{\mu\nu\,Q}$, carry different indices. The vector-tensor hierarchy implies that there must be 5 additional tensor fields and furthermore requires the existence of a constant tensor $d_{Q,MN}$, symmetric in (M,N), in order to obtain the characteristic term $d_{Q,MN}\,C_{[\mu}{}^M\,\delta C_{\nu]}{}^N$ in $\delta C_{\mu\nu\,Q}$. Assuming that the overall covariance of this expression must be preserved and that precisely five additional fields are needed, one deduces that these five fields can be precisely represented by new fields $C_{\mu\nu\,m;npqrs}$, where the array $[npqrs]$ is fully antisymmetric. Hence the decomposition of the 27 tensors takes the following form, in direct analogy with (4.10),

$$
C_{\mu\nu\,Q}=\left\{C_{\mu\nu\,m;npqrs}\,,\,C_{\mu\nu\,mn}\,,\,C_{\mu\nu\,\alpha mnpq}\,,\,C_{\mu\nu}{}^{\alpha}\right\}. \qquad (4.11)
$$

The new field $C_{\mu\nu\,m;npqrs}$ indeed has the representation that is expected from the dualization of $10D$ gravity [36, 37] (although this dualization can not be fully understood at the non-linear level in $10D$ [38]).

The systematics of the vector and tensor fields can be improved upon converting to dual representations by extracting the anti-symmetric tensors $\mathring{e}\,\varepsilon_{mnpqr}$ and/or $\varepsilon_{\alpha\beta}$. Note that the first tensor depends only on the reference background of the internal space, because of the definition $\mathring{e}(y)\equiv\det[\,\mathring{e}_m{}^a(y)]$, and not on the space-time coordinates x^μ. Hence these conversions have no bearing on the supersymmetry transformations nor the vector and tensor gauge transformations. Now consider the following redefinitions for the vector fields,

$$
\begin{aligned}
C_\mu^{\ m}&=C_\mu^{\ m}\,, & C_{\mu\,mnp}&=\frac{1}{128}\sqrt{5}\,\mathring{e}\,\varepsilon_{mnpqr}\,C_\mu^{\ qr}\,,\\
C_{\mu\ m}^{\ \alpha}&=\mathrm{i}\,\varepsilon^{\alpha\beta}\,C_{\mu\,\beta m}\,, & C_{\mu\,\alpha mnpqr}&=-\frac{1}{6}\sqrt{5}\mathring{e}\,\varepsilon_{mnpqr}\,C_{\mu\,\alpha}\,.
\end{aligned} \qquad (4.12)
$$

For the tensor fields the corresponding redefinitions are

$$
\begin{aligned}
C_{\mu\nu\,m;npqrs}&\propto\mathring{e}\,\varepsilon_{npqrs}\,C_{\mu\nu\,m}\,, & C_{\mu\nu\,mn}&=C_{\mu\nu\,mn}\,,\\
C_{\mu\nu\,\alpha mnpq}&=\frac{1}{6}\sqrt{5}\mathrm{i}\mathring{e}\,\varepsilon_{mnpqr}\,\varepsilon_{\alpha\beta}\,C_{\mu\nu}{}^{\beta r}\,, & C_{\mu\nu}{}^{\alpha}&=C_{\mu\nu}{}^{\alpha}\,.
\end{aligned} \qquad (4.13)
$$

Now the vector and tensor fields can be written as $C_\mu{}^M$ and $C_{\mu\nu\,M}$, respectively, where the indices M decompose according to ${}^M=\left\{{}^m\,,\,{}^{mn}\,,\,{}_{\alpha m}\,,\,{}_{\alpha}\right\}$ and ${}_M=\left\{{}_m\,,\,{}_{mn}\,,\,{}^{\alpha m}\,,\,{}^{\alpha}\right\}$, respectively. Here we observe that the normalization of the vector and tensor fields is at this point completely arbitrary. Nevertheless, identifying the (upper) index M on $C_\mu{}^M$ with the $\overline{\mathbf{27}}$ representation of $\mathrm{E}_{6(6)}$ and the (lower) index M on $C_{\mu\nu M}$ as the $\mathbf{27}$ representation, then the decompositions (4.12) and (4.13) correspond to the branchings

$$\begin{aligned}
\overline{\mathbf{27}} &\stackrel{\mathrm{SL}(2)\times\mathrm{SL}(6)}{\longrightarrow} (\mathbf{1},\overline{\mathbf{15}})+(\mathbf{2},\mathbf{6}) \stackrel{\mathrm{SL}(2)\times\mathrm{SO}(5)}{\longrightarrow} (\mathbf{1},\mathbf{5})+(\mathbf{1},\mathbf{10})+(\mathbf{2},\mathbf{5})+(\mathbf{2},\mathbf{1})\,,\\
\mathbf{27} &\stackrel{\mathrm{SL}(2)\times\mathrm{SL}(6)}{\longrightarrow} (\mathbf{1},\mathbf{15})+(\mathbf{2},\overline{\mathbf{6}}) \stackrel{\mathrm{SL}(2)\times\mathrm{SO}(5)}{\longrightarrow} (\mathbf{1},\mathbf{5})+(\mathbf{1},\mathbf{10})+(\mathbf{2},\mathbf{5})+(\mathbf{2},\mathbf{1})\,.
\end{aligned} \tag{4.14}$$

At this point it makes sense to compare our results for variations of the tensor fields to the corresponding expressions known from maximal $5D$ supergravity [8]. In the latter case these variations are encoded in the symmetric three-rank $\mathrm{E}_{6(6)}$ invariant tensor d_{MNP},

$$\delta C_{\mu\nu\,M} - 2\,d_{MNP}\,C_{[\mu}{}^N\,\delta C_{\nu]}{}^P\,. \tag{4.15}$$

Expressions such as these are characteristic for the vector-tensor hierarchy. Obviously the tensor d_{MNP} decomposes into three $\mathrm{SL}(2)\times\mathrm{SO}(5)$ invariant components,

$$d_{MNP} \propto \begin{cases} d(_{mn}|^{\alpha p}|^{\beta q}) = \delta_{mn}{}^{pq}\,\varepsilon^{\alpha\beta}\,,\\ d(_{mn}|_{pq}|_r) = \mathring{e}\,\varepsilon_{mnpqr}\,,\\ d(_m|^{\alpha n}|^\beta) = \delta_m{}^n\,\varepsilon^{\alpha\beta}\,, \end{cases} \tag{4.16}$$

where normalization factors are not specified because they can be changed by rescaling the normalization of the vector and tensor fields. Nevertheless the fact that a single symmetric tensor d_{MNP} must encode the variations above for all the fields does pose certain restrictions on the relative normalizations of vectors and tensor fields, especially because the product of the normalization of a tensor and its corresponding dual vector is constrained, just as in the maximal $5D$ theory [8]. We return to this issue in the next section, but note that this normalization condition has been incorporated when adopting the rescalings of the vector and tensor fields in (4.12) and (4.13), repectively. It then turns out that the following expressions for the independent components of the combined variations (4.15) must be equivalent to the following,

$$\begin{aligned}
&\delta C_{\mu\nu}{}^{\alpha m} - \frac{1}{8}\mathrm{i}\,\varepsilon^{\alpha\beta}\big[C_{[\mu\,\beta n}\,\delta C_{\nu]}{}^{mn} + C_{[\mu}{}^{mn}\,\delta C_{\nu]\,\beta n}\big] - \mathrm{i}\,\varepsilon^{\alpha\beta}\big[C_{[\mu}{}^m\,\delta C_{\nu]\beta} + C_{[\mu\beta}\,\delta C_{\nu]}{}^m\big]\,,\\
&\delta C_{\mu\nu}{}^\alpha + \mathrm{i}\,\varepsilon^{\alpha\beta}\big[C_{[\mu}{}^m\,\delta C_{\nu]\,\beta m} + C_{[\mu\,\beta m}\,\delta C_{\nu]}{}^m\big]\,,\\
&\delta C_{\mu\nu\,mn} + \frac{1}{128}\sqrt{5}\,\mathring{e}\,\varepsilon_{mnpqr}\big[C_{[\mu}{}^p\,\delta C_{\nu]}{}^{qr} + C_{[\mu}{}^{qr}\,\delta C_{\nu]}{}^p\big] - \frac{1}{4}\mathrm{i}\,\varepsilon^{\alpha\beta}\,C_{[\mu\alpha[m}\,\delta C_{\nu]\beta n]}\,,\\
&\delta C_{\mu\nu\,m} - \mathrm{i}\,\varepsilon^{\alpha\beta}\big[C_{[\mu\,\alpha m}\,\delta C_{\nu]\,\beta} - C_{[\mu\,\alpha}\,\delta C_{\nu]\,\beta m}\big] + \frac{1}{256}\sqrt{5}\mathring{e}\,\varepsilon_{mnpqr}\,C_{[\mu}{}^{np}\,\delta C_{\nu]}{}^{qr}\,,
\end{aligned} \tag{4.17}$$

where the last line is not derived directly from the $10D$ supergravity as the tensor field $C_{\mu\nu\,m}$ is associated with the elusive dual graviton. Nethertheless it is remarkable that one can also derive the coefficients in the variation of $C_{\mu\nu\,m}$ by comparing to the $5D$ vector-tensor hierarchy.

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5 Generalized vielbeine and USp(8) covariant spinors

The spinor fields ψ_μ, ${\psi_\mu}^{\rm c}$, ψ_a, ${\psi_a}^{\rm c}$, λ and $\lambda^{\rm c}$, which were defined in section 3, obviously transform under the $\mathrm{Spin}(4,1)\times\mathrm{USp}(4)$ subgroup of the $10D$ tangent space group $\mathrm{Spin}(9,1)$. Hence every $10D$ spinor consists of four complex $\mathrm{Spin}(4,1)$ spinors which rotate among each other under $\mathrm{USp}(4)$ transformations. In the following we will not consider the $\mathrm{Spin}(4,1)$ aspects but concentrate on the extension of the $\mathrm{USp}(4)$ transformations to the full automorphism group of the $5D$ space-time Clifford algebra. This so-called R-symmetry group contains also the $\mathrm{U}(1)$ group of IIB supergravity (which can be regarded as the $10D$ R-symmetry group) and it can be further extended by realizing that the spinors can actually transform under $\mathrm{SU}(4)\cong\mathrm{SO}(6)$ (for instance, by regarding them as chiral spinors of $\mathrm{SO}(6)$). It is then convenient to introduce corresponding $\mathrm{SO}(6)$ gamma matrices as well, which requires to combine the spinors with their charge conjugates, i.e. ψ_μ with ${\psi_\mu}^{\rm c}$, and likewise, ψ_a with ${\psi_a}^{\rm c}$, and λ with $\lambda^{\rm c}$. This is described in detail in appendix B. The $\mathrm{SO}(6)$ gamma matrices will be denoted by $\mathbf{\Gamma}_{\hat a}$, with $\hat a=1,\ldots,6$, and act on the eight-component pseudo-real spinors. We may then introduce the chirality operator $\mathbf{\Gamma}_7\equiv \mathrm{i}\mathbf{\Gamma}_1\mathbf{\Gamma}_2\cdots\mathbf{\Gamma}_6$, which decomposes as $\mathbf{\Gamma}_7=\mathbb{1}_4\otimes\sigma_3$, so that the $\mathrm{SO}(6)$ chirality of the charge conjugate fermions is opposite to the original ones. Here we are using a basis where the positive-chirality (negative-chirality) components carry positive (negative) $\mathrm{U}(1)$ charge. In this section and henceforth we will be using these 8-component spinor arrays whenever possible (labeled by indices $A=1,\ldots,8$) and they will simply be denoted by ${\psi_\mu}^A$, ${\psi_a}^A$ and λ^A. Each of these spinors are then $5D$ symplectic Majorana spinors, i.e.,

$$C^{-1}{\bar\psi_A}{}^{\rm T}=\Omega_{AB}\,\psi^B\,, \tag{5.1}$$

where C is the charge conjugation matrix in five space-time dimensions and Ω is the anti-symmetric $\mathrm{USp}(8)$ invariant tensor.

The appearance of Ω indicates that the full R-symmetry group is equal to $\mathrm{USp}(8)$, as is to be expected for $5D$ spinors. Indeed, the gravitini ${\psi_\mu}^A$ transform consistently in the $\mathbf{8}$ representation of this extended R-symmetry group. However, the fields ψ_a and λ cannot possibly transform in the $\mathbf{8}$ representation, in view of the fact that the $\mathrm{U}(1)$ charges of the fields ${\psi_a}^A$ and λ^A are equal to $\pm1/2$ and $\pm3/2$, respectively. Therefore those fields must transform in a different representation of the $\mathrm{USp}(8)$ group. In view of the values for the $\mathrm{U}(1)$ charges and the fact that ${\psi_a}^A$ and λ^A define precisely 48 $5D$ symplectic Majorana spinors, these fields must combine into the $\mathbf{48}$ representation of the group $\mathrm{USp}(8)$. At this point we should recall that only the $\mathrm{USp}(4)\times\mathrm{U}(1)$ subgroup is realized as a local gauge invariance, as they originate from the symmetries of $10D$ IIB supergravity that were already realized as local ones. As we have stressed in the introduction, the full $\mathrm{USp}(8)$ R-symmetry group can be realized locally upon introducing a compensating phase factor belonging to $\mathrm{USp}(8)/[\mathrm{USp}(4)\times\mathrm{U}(1)]$. We will postpone the introduction of this phase factor till later, so that the present calculations will describe the results subject to a gauge condition that sets the compensating phase factor equal to unity. However, it is important to realize that the local transformations depend on both sets of coordinates, x^μ and y^m. This is the reason why we adopted the indices $A,B,\ldots$ for the spinors in this case, while in the maximal $5D$ supergravity, the spinors will carry indices $i, j, \ldots$ with local R-symmetry transformations that depend only on the space-time coordinates x^μ. This issue will be important in section 7, when considering the *truncation* of $10D$ supergravity to $5D$,

In the previous section we have identified 27 vector fields $C_\mu{}^M$ as listed in (4.12), which transform under supersymmetry into the symplectic Majorana spinors $\psi_\mu{}^A$, $\psi_a{}^A$ and λ^A. As it turns out the supersymmetry variations of these fields can be written in the same way as the variations of the vector fields in $5D$ maximal supergravity [8],

$$\delta C_\mu{}^M = 2\left[\mathrm{i}\bar{\Omega}^{AC}\,\bar{\epsilon}_C\,\psi_\mu{}^B + \bar{\epsilon}_C\,\gamma_\mu\chi^{ABC}\right]\mathcal{V}_{AB}{}^M\,, \tag{5.2}$$

except that, as explained above, we changed the USp(8) indices from $i, j, \ldots$ to $A, B, \ldots$. Here Ω^{AB} is the symplectic USp(8) invariant tensor introduced aboved and the $\mathcal{V}_{AB}{}^M$ depend on the 42 scalar fields. All these fields depend on coordinates x^μ and y^m. In the pure $5D$ theory the corresponding quantities $\mathcal{V}_{ij}{}^M$ are defined in terms of the $\mathrm{E}_{6(6)}/\mathrm{USp}(8)$ coset representative. The transformations (5.2) are consistent with the USp(8) R-symmetry group and the anti-symmetric traceless spinors χ^{ABC} are symplectic Majorana spinors, satisfying

$$C^{-1}\bar{\chi}_{ABC}{}^{\mathrm{T}} = \Omega_{AD}\,\Omega_{BE}\,\Omega_{CF}\,\chi^{DEF}\,, \tag{5.3}$$

in direct correspondence with the $5D$ theory [8]. Because of the anti-symmetry in $[ABC]$ and the condition $\Omega_{AB}\,\chi^{ABC} = 0$, this representation is irreducible. Hence the spinor χ^{ABC} should be linearly related to the spinors $\psi_a{}^A$ and λ^A. Indeed, as we demonstrate in appendix B (cf. (B.4)) the branching of the **8** and **48** USp(8) representations of the fermions with respect to the $\mathrm{SU}(4)\times\mathrm{U}(1)$ subgroup accounts precisely for the fermion fields $\psi_\mu{}^A$, $\psi_a{}^A$ and λ^A including their U(1) charge assignments.

The supersymmetry transformation rules for the vector fields $C_\mu{}^M$ in terms of the spinors $\psi_\mu{}^A$, $\psi_a{}^A$, λ^A based on IIB supergravity follow from (3.20) and (4.5) upon taking into account the redefinitions (4.12). By comparing these expressions to (5.2) we obtain explicit representations of the so-called *generalized vielbeine* $\mathcal{V}_{AB}{}^M$, which depend on all $10D$ coordinates. Furthermore we can deduce the explicit relation between the USp(8) covariant spinor field χ^{ABC} and the fields $\psi_a{}^A$ and λ^A. In the same fashion one can evaluate the supersymmetry transformations of the tensor fields, a topic that will be dealt with at the end of this section.

Matrices in spinor space can be decomposed into direct products of the $5D$ gamma matrices γ^μ and the SO(6) gamma matrices. The latter products can be conveniently decomposed into 28 anti-symmetric matrices Ω, $\Omega\,\mathbf{\Gamma}_{\hat{a}}$, $\Omega\,\mathbf{\Gamma}_{\hat{a}}\mathbf{\Gamma}_7$ and $\Omega\,\mathbf{\Gamma}_{\hat{a}\hat{b}}\mathbf{\Gamma}_7$, and 36 symmetric matrices $\Omega\,\mathbf{\Gamma}_7$, $\Omega\,\mathbf{\Gamma}_{\hat{a}\hat{b}}$ and $\Omega\,\mathbf{\Gamma}_{\hat{a}\hat{b}\hat{c}}$. The latter are proportional to the anti-hermitian generators of USp(8) (note that the matrices $\mathbf{\Gamma}_{\hat{a}\hat{b}}$ are the generators of the group $\mathrm{SU}(4)\cong\mathrm{SO}(6)$). Before obtaining a representation of the generalized vielbeine $\mathcal{V}_{AB}{}^M$ we note that the USp(8) transformations of the spinors $\psi_\mu{}^A$ and ϵ^A have been defined in appendix B, and they imply that the bilinears $\Omega^{AC}\,\bar{\epsilon}_C\psi_\mu{}^B$ transform in the **27** representation of USp(8). Since the vector fields are not subject to the R-symmetry, it follows that the generalized vielbeine $\mathcal{V}_{AB}{}^M$ transform in the same representation, so that they can be expanded in the corresponding gamma matrix combinations,

$$\begin{aligned}\mathcal{V}_{AB}{}^{M} =& \mathcal{V}_a{}^M\big(\Omega\,\boldsymbol{\Gamma}^a\big)_{AB} + \mathcal{V}_6{}^M\big(\Omega\,\boldsymbol{\Gamma}^6\big)_{AB} + \tilde{\mathcal{V}}_a{}^M\big(\Omega\,\boldsymbol{\Gamma}^a\boldsymbol{\Gamma}_7\big)_{AB} + \tilde{\mathcal{V}}_6{}^M\big(\Omega\,\boldsymbol{\Gamma}^6\boldsymbol{\Gamma}_7\big)_{AB}\\ &+ \mathcal{V}_{ab}{}^M\big(\Omega\,\boldsymbol{\Gamma}^{ab}\boldsymbol{\Gamma}_7\big)_{AB} + 2\,\mathcal{V}_{a6}{}^M\big(\Omega\,\boldsymbol{\Gamma}^{a6}\boldsymbol{\Gamma}_7\big)_{AB}\,,\end{aligned} \tag{5.4}$$

which defines the branching of the **27** representation of USp(8) with respect to SO(5) (which directly follows via the branching with respect to SO(6)),

$$\mathbf{27} \;\stackrel{\mathrm{SO}(6)}{\longrightarrow}\; \mathbf{6} + \bar{\mathbf{6}} + \mathbf{15} \;\stackrel{\mathrm{SO}(5)}{\longrightarrow}\; \mathbf{1} + \mathbf{5} + \mathbf{1} + \mathbf{5} + \mathbf{10} + \mathbf{5}\,. \tag{5.5}$$

The generalized vielbeine can now be directly determined from the supersymmetry transformations of the vector fields, which leads to

$$\begin{aligned}
\mathcal{V}_{AB}{}^{m} =& -\frac{1}{4}\mathrm{i}\,\Delta^{-1/3}\,e_a{}^m\,\big(\Phi^{\mathrm{T}}\Omega\,\boldsymbol{\Gamma}^{a6}\boldsymbol{\Gamma}_7\,\Phi\big)_{AB}\,,\\
\mathcal{V}_{AB}{}^{mn} =& -\frac{4}{5}\sqrt{5}\mathrm{i}\,\Delta^{2/3}\,\big(\Phi^{\mathrm{T}}\Omega\,\boldsymbol{\Gamma}^{mn}\boldsymbol{\Gamma}_7\,\Phi\big)_{AB}\\
&+\frac{4}{5}\sqrt{5}\,\mathring{e}^{-1}\varepsilon^{mnpqr}A^{\alpha}{}_{pq}\,\mathcal{V}_{AB\,\alpha r}\\
&+\frac{32}{15}\sqrt{5}\,\mathring{e}^{-1}\varepsilon^{mnpqr}\Big[A_{pqrs}-\frac{3}{16}\mathrm{i}\varepsilon_{\alpha\beta}A^{\alpha}{}_{pq}A^{\beta}{}_{rs}\Big]\,\mathcal{V}_{AB}{}^{s}\,,\\
\mathcal{V}_{AB\,\alpha m} =& \frac{1}{4}\mathrm{i}\,\Delta^{-1/3}\Big[(\phi_\alpha-\varepsilon_{\alpha\beta}\phi^\beta)\,\big(\Phi^{\mathrm{T}}\Omega\,\boldsymbol{\Gamma}_m\,\Phi\big)_{AB}+(\phi_\alpha+\varepsilon_{\alpha\beta}\phi^\beta)\,\big(\Phi^{\mathrm{T}}\Omega\,\boldsymbol{\Gamma}_m\boldsymbol{\Gamma}_7\,\Phi\big)_{AB}\Big]\\
&+\mathrm{i}\,\varepsilon_{\alpha\beta}A^{\beta}{}_{mn}\,\mathcal{V}_{AB}{}^{n}\,,\\
\mathcal{V}_{AB\,\alpha} =& \frac{1}{10}\sqrt{5}\mathrm{i}\,\Delta^{2/3}\Big[(\phi_\alpha-\varepsilon_{\alpha\beta}\phi^\beta)\big(\Phi^{\mathrm{T}}\Omega\,\boldsymbol{\Gamma}_6\,\Phi\big)_{AB}+(\phi_\alpha+\varepsilon_{\alpha\beta}\phi^\beta)\big(\Phi^{\mathrm{T}}\Omega\,\boldsymbol{\Gamma}_6\boldsymbol{\Gamma}_7\,\Phi\big)_{AB}\Big]\\
&+\frac{1}{16}\mathrm{i}\,\varepsilon_{\alpha\beta}A^{\beta}{}_{mn}\,\mathcal{V}_{AB}{}^{mn}\\
&-\frac{1}{15}\sqrt{5}\,\mathring{e}^{-1}\varepsilon^{mnpqr}\Big[A_{mnpq}\,\mathcal{V}_{AB\alpha r}+2\mathrm{i}\,\varepsilon_{\alpha\beta}\,A^{\beta}{}_{mn}\,A_{pqrs}\,\mathcal{V}_{AB}{}^{s}\Big]\\
&-\frac{1}{40}\sqrt{5}\mathrm{i}\,\varepsilon_{\alpha\beta}\,\mathring{e}^{-1}\varepsilon^{mnpqr}\Big[A^{\beta}{}_{mn}\,A^{\gamma}{}_{pq}\,\mathcal{V}_{AB\,\gamma r}-\frac{1}{3}\mathrm{i}\,\varepsilon_{\gamma\delta}\,A^{\gamma}{}_{sm}\,A^{\delta}{}_{np}\,A^{\beta}{}_{qr}\,\mathcal{V}_{AB}{}^{s}\Big]\,.
\end{aligned} \tag{5.6}$$

In the above equations we have now included the compensating phase factors $\Phi^A{}_B$ that were discussed earlier, which enable the USp(8) R-symmetry group to be realized locally. The phase factors are simply generated by a redefinition of the fermion fields, as $\Phi \in \mathrm{USp}(8)$ is assumed to transform under the action of USp(8) from the right and under $\mathrm{USp}(4)\times\mathrm{U}(1)$ from the left, so that fermion fields $\Phi^\dagger\Psi$, where Ψ denotes the original fields in a proper basis, transform indeed under this local group. Previously we have assumed the gauge condition $\Phi = \mathbb{1}$ which suffices to carry out most of the various calculations. In fact, we will continue to use this gauge condition in most of what follows. The phase factors can always be introduced later to elevate the R-symmetry group to a local invariance group, just as what was done long ago for $11D$ supergravity [1].

The next task is to establish the relation between the USp(8) covariant spinors χ^{ABC} and the spinors originating from $10D$, $\psi_a{}^A$ and λ^A. Comparing the terms proportional to these fields in the supersymmetry variations of the vector fields, one finds the following set of equations,

$$
\begin{aligned}
\psi_a{}^A =&\; -\mathrm{i}\Big[\chi^{ABC}\,\delta_a^b - \frac{1}{8}(\mathbf{\Gamma}_a\mathbf{\Gamma}^b)^A{}_D\,\chi^{DBC}\Big]\,\big[\Omega\,\mathbf{\Gamma}_{b6}\mathbf{\Gamma}_7\big]_{BC}\,,\\
\big[(\mathbb{1}\pm\mathbf{\Gamma}_7)\mathbf{\Gamma}_{a6}\big]^A{}_D\,\lambda^D =&\; \pm\,\mathrm{i}\,\big[\Omega\,\mathbf{\Gamma}_a(\mathbb{1}\pm\mathbf{\Gamma}_7)\big]_{BC}\,(\mathbb{1}\pm\mathbf{\Gamma}_7)^A{}_D\,\chi^{DBC}\,,\\
(\mathbb{1}\pm\mathbf{\Gamma}_7)^A{}_D\,\lambda^D =&\; \pm\,\mathrm{i}\,\big(\Omega\,\mathbf{\Gamma}_6(\mathbb{1}\pm\mathbf{\Gamma}_7)\big)_{BC}\,(\mathbb{1}\pm\mathbf{\Gamma}_7)^A{}_D\,\chi^{DBC}\,,\\
\Big[\Big(\mathbf{\Gamma}_{[ab}(\delta_{c]}{}^d\mathbb{1} - \frac{1}{9}\mathbf{\Gamma}_{c]}\mathbf{\Gamma}^d\Big)\mathbf{\Gamma}_7\Big]^A{}_D\,\psi_d{}^D =&\; -\frac{1}{6}\varepsilon_{abcde}\big(\Omega\,\mathbf{\Gamma}^{de}\mathbf{\Gamma}_7\big)_{BC}\,\chi^{ABC}\,,\\
\big[(\Omega\,\mathbf{\Gamma}_a(\mathbb{1}\pm\mathbf{\Gamma}_7)\big]_{BC}\,(\mathbb{1}\mp\mathbf{\Gamma}_7)^A{}_D\,\chi^{DBC} =&\; \pm\,2\mathrm{i}\,\Big[(\mathbb{1}\mp\mathbf{\Gamma}_7)\mathbf{\Gamma}_6\,\Big(\delta_a{}^b\mathbb{1} - \frac{1}{3}\mathbf{\Gamma}_a\mathbf{\Gamma}^b\Big)\Big]^A{}_D\,\psi_b{}^D\,,\\
\big[(\mathbb{1}\pm\mathbf{\Gamma}_7)\mathbf{\Gamma}^a\big]^A{}_D\,\psi_a{}^D =&\; \mp\,\frac{3}{4}\mathrm{i}\,\big[\Omega\,\mathbf{\Gamma}_6(\mathbb{1}\mp\mathbf{\Gamma}_7)\big]_{BC}\,(\mathbb{1}\pm\mathbf{\Gamma}_7)^A{}_D\,\chi^{DBC}\,.
\end{aligned}
\tag{5.7}
$$

These are the relations that determine the (linear) relation between the spinors $\psi_a{}^A$ and λ^A and the USp(8) covariant spinors χ^{ABC}. Just as in $11D$ supergravity, where the expression for the $4D$ spinors χ^{ABC} as first given in [35] is only unique up to Fierz reordering, there are various different ways to express the solution for χ^{ABC}. One solution follows by substituting the SO(6) covariant parametrization derived in appendix B into (5.7), which then leads to (B.21). However, given that the ansatz for χ^{ABC} is not unique, one might wonder whether there exists an alternative version of this solution that may be even more concise. Indeed we have found such a solution taking the form

$$
\begin{aligned}
\chi^{ABC} =&\; -\frac{3}{8}\mathrm{i}\big[(\mathbf{\Gamma}_6\,\bar\Omega)^{[AB}\,(\mathbf{\Gamma}_7\lambda)^{C]} + (\mathbf{\Gamma}_7\mathbf{\Gamma}_6\,\bar\Omega)^{[AB}\,\lambda^{C]}\big]\\
&\; -\frac{3}{4}\mathrm{i}\,(\mathbf{\Gamma}^a\mathbf{\Gamma}_6\mathbf{\Gamma}_7\,\bar\Omega)^{[AB}\,\psi_a{}^{C]} - \frac{1}{4}\mathrm{i}\,\bar\Omega^{[AB}\,(\mathbf{\Gamma}_6\mathbf{\Gamma}_7\mathbf{\Gamma}^a\psi_a)^{C]}\,.
\end{aligned}
\tag{5.8}
$$

which also satisfies (5.7). Its equivalence to (B.21) has been confirmed by demonstrating that both solutions are related by Fierz reordering to a single expression that involves eight different structures. This result satisfies the reality condition (5.3) and vanishes upon contraction with Ω_{AB}. Note also that the above expression should in principle have been contracted with three different phase factors $\Phi^\dagger$ as was discussed above. For clarity of the presentation we have set $\Phi = \mathbb{1}$.

Subsequently we derive a formula for the supersymmetry transformations of the generalized vielbeine $\mathcal{V}_{AB}{}^M$. For maximal $5D$ supergravity [8] there exists the following expression (with indices $i,j,\ldots$ replaced again by $A,B,\ldots$),

$$
\begin{aligned}
\delta\mathcal{V}_{AB}{}^M =&\; -\mathrm{i}\big[4\,\Omega_{G[A}\,\bar\chi_{BCD]}\,\epsilon^G + 3\,\Omega_{[AB}\,\bar\chi_{CD]G}\,\epsilon^G\big]\,\bar\Omega^{CE}\bar\Omega^{DF}\,\mathcal{V}_{EF}{}^M\\
=&\;\mathrm{i}\,\Omega_{AC}\Omega_{BD}\,\big[4\,\bar\Omega^{G[C}\,\bar\epsilon_G\,\chi^{DEF]} + 3\,\bar\Omega^{[CD}\,\bar\epsilon_G\,\chi^{EF]G}\big]\,\mathcal{V}_{EF}{}^M\,.
\end{aligned}
\tag{5.9}
$$

This result is expected to be identical to the result that one obtains by calculating the variations of the generalized vielbeine (5.6) induced by the supersymmetry transformations of the scalar fields,

$$
\begin{aligned}
\delta e_m{}^a &= \frac{1}{2} e_m{}^b\, \bar\epsilon\, \boldsymbol{\Gamma}^{a6} \boldsymbol{\Gamma}_7 \psi_b\,,\\
\delta\phi^\alpha &= -\frac{1}{4}\varepsilon^{\alpha\beta}\phi_\beta\, \bar\epsilon\, \boldsymbol{\Gamma}_6(\mathbb{1}+\boldsymbol{\Gamma}_7)\lambda\,,\\
\delta\phi_\alpha &= -\frac{1}{4}\varepsilon_{\alpha\beta}\phi^\beta\, \bar\epsilon\, \boldsymbol{\Gamma}_6(\mathbb{1}-\boldsymbol{\Gamma}_7)\lambda\,,\\
\delta A^\alpha{}_{mn} &= -\frac{1}{4}\mathrm{i}\, e_m{}^a\, e_n{}^b\, (\phi^\alpha + \varepsilon^{\alpha\beta}\phi_\beta)\, \bar\epsilon\, (\boldsymbol{\Gamma}_{ab}\lambda - 4\,\boldsymbol{\Gamma}_{[a}\psi_{b]})\\
&\quad + \frac{1}{4}\mathrm{i}\, e_m{}^a\, e_n{}^b\, (\phi^\alpha - \varepsilon^{\alpha\beta}\phi_\beta)\, \bar\epsilon\, (\boldsymbol{\Gamma}_{ab}\boldsymbol{\Gamma}_7\lambda - 4\,\boldsymbol{\Gamma}_{[a}\boldsymbol{\Gamma}_7\psi_{b]})\\
\delta A_{mnpq} &= -\frac{1}{2}\mathrm{i}\, e_m{}^a\, e_n{}^b\, e_p{}^c\, e_q{}^d\, \bar\epsilon\, \boldsymbol{\Gamma}_6 \boldsymbol{\Gamma}_{[abc}\psi_{d]} + \frac{3}{8}\mathrm{i}\varepsilon_{\alpha\beta} A^\alpha{}_{[mn}\, \delta A^\beta{}_{pq]}\,.
\end{aligned}
\tag{5.10}
$$

Based on the similar construction for $11D$ supergravity [1], we expect the supersymmetry transformations of the vielbeine induced by the variations (5.10) to coincide with (5.9) up to a uniform USp(8) transformation. By very laborious calculations we have been able to demonstrate that this expectation is correct so that (5.9) can be regarded as the supersymmetry transformation rule for the vielbeine. More precisely, the results induced by (5.10) take the form

$$
\delta\mathcal{V}_{AB}{}^M = \delta\mathcal{V}_{AB}{}^M\big|_{(5.9)} - \Lambda^C{}_{[A}\, \mathcal{V}_{B]C}{}^M \tag{5.11}
$$

where $\Lambda^A{}_B$ is the field-dependent infinitesimal USp(8) transformation given by

$$
\begin{aligned}
\Lambda^A{}_B &= -\frac{1}{16}\bar\epsilon\, \boldsymbol{\Gamma}_7[\boldsymbol{\Gamma}_{ab}\lambda + 4\,\boldsymbol{\Gamma}_{[a}\psi_{b]}]\, \big(\boldsymbol{\Gamma}^{ab6}\big)^A{}_B\\
&\quad + \frac{1}{48}\bar\epsilon\, \boldsymbol{\Gamma}_7[\boldsymbol{\Gamma}_{abc6}\lambda + 2\,\boldsymbol{\Gamma}_{abcd6}\psi^d]\, \big(\boldsymbol{\Gamma}^{abc}\big)^A{}_B\\
&\quad + \frac{1}{4}\bar\epsilon\, \boldsymbol{\Gamma}_7\boldsymbol{\Gamma}_{ac}\psi^c\, \big(\boldsymbol{\Gamma}^{a6}\big)^A{}_B + \frac{1}{4}\bar\epsilon\, \boldsymbol{\Gamma}_7\boldsymbol{\Gamma}_{6[a}\psi_{b]}\, \big(\boldsymbol{\Gamma}^{ab}\big)^A{}_B\,.
\end{aligned}
\tag{5.12}
$$

We now proceed with the supersymmetry transformations of the tensor fields $C_{\mu\nu\, mn}$, $C_{\mu\nu}{}^{\alpha m}$ and $C_{\mu\nu}{}^\alpha$ that were defined in (4.11), following the same approach as for the vector fields. Their supersymmetry transformations follow upon substituting the results specified in (3.22) and (4.9). Subsequently we compare them to the five-dimensional transformation rules for the tensor fields [8] with the indices adjusted as before,

$$
\begin{aligned}
&\delta C_{\mu\nu\, M} - 2\, d_{MNP}\, C_{[\mu}{}^N\, \delta C_{\nu]}{}^P\\
&\quad = \frac{4}{5}\sqrt{5}\, \mathcal{V}_M{}^{AB}\big[2\,\bar\psi_{[\mu A}\,\gamma_{\nu]}\epsilon^C\, \Omega_{BC} - \mathrm{i}\bar\chi_{ABC}\,\gamma_{\mu\nu}\epsilon^C\big]\\
&\quad = -\frac{4}{5}\sqrt{5}\, \mathcal{V}_M{}^{AB}\big[2\,\Omega_{AC}\,\bar\epsilon_B\,\gamma_{[\mu}\psi_{\nu]}{}^C + \mathrm{i}\Omega_{AD}\Omega_{BE}\,\bar\epsilon_C\,\gamma_{\mu\nu}\chi^{DEC}\big]\,.
\end{aligned}
\tag{5.13}
$$

In $5D$ maximal gauged supergravity the tensor fields constitute a **27** representation of $\mathrm{E}_{6(6)}$. From IIB supergravity we have initially identified only 22 different tensor fields. The missing five tensors $C_{\mu\nu\, m}$ have been identified as originating from a component of the $10D$ dual graviton. The second term on the left-hand side of (5.13) has already been specified in (4.17).

From the terms in (5.13) proportional to $\psi_\mu{}^C$ one can directly obtain the following expressions for the 22 components of $\mathcal{V}_M{}^{ij}$, by making use of the supersymmetry transformations of the corresponding tensors derived in the previous sections,

$$
\begin{aligned}
\mathcal{V}_{mn}{}^{AB} &= -\frac{1}{32}\sqrt{5}\mathrm{i}\,\Delta^{-2/3} e_m{}^a e_n{}^b \big(\Phi^\dagger\,\boldsymbol{\Gamma}_{ab}\boldsymbol{\Gamma}_7\bar{\Omega}\,\bar{\Phi}\big)^{AB} + \frac{1}{8}\mathrm{i}\,\varepsilon_{\alpha\beta}\,A^\alpha{}_{mn}\,\mathcal{V}^{\beta\,AB}\,,\\
\mathcal{V}^{\alpha m\,AB} &= -\frac{1}{4}\mathrm{i}\Delta^{1/3}\,e_a{}^m\left[\left(\phi^\alpha-\varepsilon^{\alpha\beta}\phi^\beta\right)\big(\Phi^\dagger\,\boldsymbol{\Gamma}^e\bar{\Omega}\,\bar{\Phi}\big)^{AB} - \left(\phi^\alpha+\varepsilon^{\alpha\beta}\phi_\beta\right)\big(\Phi^\dagger\boldsymbol{\Gamma}^e\boldsymbol{\Gamma}_7\bar{\Omega}\,\bar{\Phi}\big)^{AB}\right]\\
&\quad+\frac{1}{15}\sqrt{5}\,\mathring{e}^{-1}\varepsilon^{mnpqr}\left[A_{npqr}\,\mathcal{V}^{\alpha\,AB}+\frac{3}{8}\mathrm{i}\,A^\alpha{}_{np}A^\beta{}_{qr}\,\varepsilon_{\beta\gamma}\mathcal{V}^{\gamma\,AB}-6\,A^\alpha{}_{np}\mathcal{V}_{qr}{}^{AB}\right],\\
\mathcal{V}^{\alpha\,AB} &= -\frac{1}{8}\sqrt{5}\mathrm{i}\,\Delta^{-2/3}\left[\left(\phi^\alpha-\varepsilon^{\alpha\beta}\phi_\beta\right)\big(\Phi^\dagger\,\boldsymbol{\Gamma}_6\bar{\Omega}\,\bar{\Phi}\big)^{AB}-\left(\phi^\alpha+\varepsilon^{\alpha\beta}\phi_\beta\right)\big(\Phi^\dagger\,\boldsymbol{\Gamma}_6\boldsymbol{\Gamma}_7\bar{\Omega}\,\bar{\Phi}\big)^{AB}\right],
\end{aligned}
\tag{5.14}
$$

where we have again included the phase factors Φ. Before discussing how to obtain the missing components of $\mathcal{V}_M{}^{AB}$ that are associated with the dual graviton, we first consider the contractions of the form $\mathcal{V}_M{}^{AB}\,\mathcal{V}_{AB}{}^N$ making use of the expressions (5.6) and (5.14). As it turns out the only non-zero contractions are given by

$$
\begin{aligned}
\mathcal{V}_{mn}{}^{AB}\,\mathcal{V}_{AB}{}^{pq} &= 2\,\delta_{mn}{}^{pq}\,,\\
\mathcal{V}^{\alpha\,AB}\,\mathcal{V}_{AB\,\beta} &= \delta^\alpha{}_\beta\,,\\
\mathcal{V}^{\alpha m\,AB}\,\mathcal{V}_{AB\,\beta n} &= \delta^\alpha{}_\beta\,\delta^m{}_n\,,
\end{aligned}
\tag{5.15}
$$

suggesting that

$$
\mathcal{V}_M{}^{AB}\,\mathcal{V}_{AB}{}^N = \delta_M{}^N\,. \tag{5.16}
$$

This condition is actually identical to the one that holds in $5D$ maximal gauged supergravity. In the same spirit as before, we may assume that (5.16) holds in this case as well, and this then enables us to also determine the five missing components $\mathcal{V}_m{}^{AB}$,

$$
\begin{aligned}
\mathcal{V}_m{}^{AB} &= -\frac{1}{2}\mathrm{i}\,\Delta^{1/3}e_m{}^a\big(\Phi^\dagger\boldsymbol{\Gamma}_{a6}\boldsymbol{\Gamma}_7\bar{\Omega}\bar{\Phi}\big)^{AB}\\
&\quad+\frac{16}{15}\sqrt{5}\,\mathring{e}^{-1}\varepsilon^{npqrs}\left[A_{mqrs}-\frac{3}{16}\mathrm{i}\,\varepsilon_{\alpha\beta}A^\alpha{}_{qr}A^\beta{}_{sm}\right]\mathcal{V}_{np}{}^{ij}-\mathrm{i}A^\alpha{}_{mn}\,\varepsilon_{\alpha\beta}\,\mathcal{V}^{\beta n\,AB}\\
&\quad+\frac{1}{15}\sqrt{5}\mathrm{i}\,\mathring{e}^{-1}\varepsilon^{npqrs}\,\varepsilon_{\alpha\beta}\left[A_{npqr}\,A^\beta{}_{sm}-\frac{1}{8}\mathrm{i}\,\varepsilon_{\gamma\delta}A^\beta{}_{np}A^\gamma{}_{qr}A^\delta{}_{sm}\right]\mathcal{V}^{\alpha\,AB}\,.
\end{aligned}
\tag{5.17}
$$

Note that the conditions (5.16) implies that also the supersymmetry transformations of the $\mathcal{V}_M{}^{AB}$ are determined and take the same form as the corresponding supersymmetry transformations in $5D$ maximal supergravity. Needless to say, the results obtained from the vector fields on the covariant spinors χ^{ABC} can be verified also from the perpective of the transformations of the tensor fields. The results turn out to be mutually consistent.

This completes the evaluation of the bosons and their supersymmetry transformations. We have succeeded in identifying these fields from IIB supergravity such that the results resemble as closely as possible the structure of the $5D$ maximal gauged supergravities [8] while retaining the full dependence on all ten coordinates. For the fields associated with the dual graviton, we obtained their supersymmetry transformations by requiring them to be consistent with the global structure exhibited for the other fields. In this way the results exhibit covariance with respect to the duality group $E_{6(6)}$, although the IIB theory is not in any way invariant under this group. This is further confirmed by the fact that the following representation of the invariant tensor d_{MNP} which was noted for maximal $5D$ supergravity [8],

$$d_{MNP} = \frac{2}{5}\sqrt{5}\,\mathcal{V}_M{}^{AB}\,\mathcal{V}_M{}^{CD}\,\mathcal{V}_M{}^{EF}\,\Omega_{BC}\,\Omega_{DE}\,\Omega_{FA}\,, \tag{5.18}$$

is also satisfied here, as this expression precisely reproduces the tensor d_{MNP} as specified in (4.17).

As a final comment we note that the generalized vielbeine are pseudo-real. This property is inherited form the (pseudo-)reality of the tensors and the fermionic bilinears. We remind the reader that taking complex conjugates of vielbeine that carry the SU(1,1) requires the contraction with a two-dimensional metric $\eta_{\alpha\beta} = \mathrm{diag}(+1,-1)$ in order to obtain a covariant quantity (see section 2).

6 The fermion transformation rules

In the previous sections we concentrated mostly on the supersymmetry transformations of the vector and tensor fields. Their supersymmetry transformations take the form of USp(8) invariant contractions between covariant spinor bilinears with the generalized vielbeine. This is consistent with the fact that the vector and tensor fields are invariant under the R-symmetry. Also the space-time fünfbein is invariant under USp(8), and so is its supersymmetry transformations. The scalar fields do not transform covariantly (cf. (5.10)), but indirectly they do respect the USp(8) symmetry as their supersymmetry transformations induce covariant variations on the generalized vielbeine. In view of the above it is therefore of interest to consider the supersymmetry transformations of the fermion fields, $\psi_\mu{}^A$ and χ^{ABC}, to verify whether they will also take a USp(8) covariant form. These results will not only complement the previous results, but they will enable one to properly identify various bosonic USp(8) tensors. Here we follow the same strategy as was applied to $11D$ supergravity [1]. As it will turn out, the global structure of the results of the ensuing analysis is rather similar.

The analysis starts with the fermionic transformation rules given in (3.12)–(3.14), but now written with eight-component symplectic Majorana spinors and SO(6) gamma matrices. We start by presenting the spin-1/2 fields, $\psi_a{}^A$ and λ^A, which transform as follows under supersymmetry (up to terms of higher order in the fermions),

$$\begin{aligned}
\delta\psi_a =\,& \Delta^{-1/3} e_a{}^m \Big[\partial_m - \frac{1}{4}\omega_m{}^{\alpha\beta}\,\gamma_{\alpha\beta} - \frac{1}{6}\partial_m \ln\Delta\Big]\epsilon \\
& - \frac{1}{2}\Delta^{-1/3}\Big[\omega_a{}^{\alpha b}\,\gamma_\alpha \boldsymbol{\Gamma}_{b6}\boldsymbol{\Gamma}_7 + \frac{1}{2}\omega_a{}^{bc}\boldsymbol{\Gamma}_{bc} + \mathrm{i}Q_a\boldsymbol{\Gamma}_7\Big]\epsilon \\
& + \frac{1}{240}\Delta^{-1/3}\varepsilon^{bcdef}\big[F_{bcdef}\boldsymbol{\Gamma}_{a6} - 5\gamma^\alpha F_{\alpha bcde}\boldsymbol{\Gamma}_f\boldsymbol{\Gamma}_a\boldsymbol{\Gamma}_7 - 5\gamma^{\alpha\beta}F_{\alpha\beta bcd}\boldsymbol{\Gamma}_{ef}\boldsymbol{\Gamma}_{a6}\big]\epsilon \\
& - \frac{1}{96}\mathrm{i}\Delta^{-1/3}\big[(G_{bcd}\mathbb{P}_+ - \bar{G}_{bcd}\mathbb{P}_-)(\boldsymbol{\Gamma}_a\boldsymbol{\Gamma}^{bcd} + 2\,\boldsymbol{\Gamma}^{bcd}\boldsymbol{\Gamma}_a)\boldsymbol{\Gamma}_6
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad - 3\,\gamma^{\alpha}(G_{bc\alpha}\mathbb{P}_+ + \bar{G}_{bc\alpha}\mathbb{P}_-)(\boldsymbol{\Gamma}_a\boldsymbol{\Gamma}^{bc} - 2\,\boldsymbol{\Gamma}^{bc}\boldsymbol{\Gamma}_a)\Big]\epsilon \\
&- \frac{1}{96}\mathrm{i}\Delta^{-1/3}\gamma^{\alpha\beta}\Big[3(G_{\alpha\beta b}\mathbb{P}_+ - \bar{G}_{\alpha\beta b}\mathbb{P}_-)(\boldsymbol{\Gamma}_a\boldsymbol{\Gamma}^b + 2\,\boldsymbol{\Gamma}^b\boldsymbol{\Gamma}_a)\boldsymbol{\Gamma}_6 \\
&\qquad\qquad - \frac{1}{2}\varepsilon_{\alpha\beta\gamma\delta\tau}(G^{\gamma\delta\tau}\mathbb{P}_+ + \bar{G}^{\gamma\delta\tau}\mathbb{P}_-)\boldsymbol{\Gamma}_a\Big]\epsilon\,, \\
\delta\lambda = {}& \Delta^{-1/3}\Big[\gamma^{\alpha}(P_\alpha\,\mathbb{P}_+ - \bar{P}_\alpha\,\mathbb{P}_-)\boldsymbol{\Gamma}_6 + (P_a\,\mathbb{P}_+ + \bar{P}_a\,\mathbb{P}_-)\boldsymbol{\Gamma}^a\Big]\epsilon \\
&+ \frac{1}{8}\Delta^{-1/3}\Big[-\frac{1}{3}\mathrm{i}(G_{abc}\,\mathbb{P}_+ - \bar{G}_{abc}\,\mathbb{P}_-)\boldsymbol{\Gamma}^{abc6} - \mathrm{i}\,\gamma^{\alpha}(G_{\alpha ab}\,\mathbb{P}_+ + \bar{G}_{\alpha ab}\,\mathbb{P}_-)\boldsymbol{\Gamma}^{ab} \\
&\qquad\qquad - \mathrm{i}\,\gamma^{\alpha\beta}(G_{\alpha\beta a}\,\mathbb{P}_+ - \bar{G}_{\alpha\beta a}\,\mathbb{P}_-)\boldsymbol{\Gamma}^{a6} \\
&\qquad\qquad + \frac{1}{6}\mathrm{i}\,\gamma^{\alpha\beta}\varepsilon_{\alpha\beta\gamma\delta\tau}(G^{\gamma\delta\tau}\,\mathbb{P}_+ + \bar{G}^{\gamma\delta\tau}\,\mathbb{P}_-)\Big]\epsilon\,,
\end{aligned} \tag{6.1}
$$

where we employed the SO(6) chiral projection operators $\mathbb{P}_\pm = \frac{1}{2}\big(\mathbb{1} \pm \boldsymbol{\Gamma}_7\big)$. To verify that these results are consistent with USp(8) R-symmetry is subtle and requires us to first combine the two equations (6.1) into the covariant tri-spinor variation $\delta\chi^{ABC}$. For this one makes use of (5.8). Since this is rather involved, let us first proceed to the gravitino variation and return to the spin-1/2 variations at the end of the section.

The supersymmetry transformations of the gravitino fields ${\psi_\mu}^A$ take the following form, where we have ordered the various terms in a particular way in view of what will follow,

$$
\begin{aligned}
\delta\psi_\mu = {}& \Big[\partial_\mu - {B_\mu}^m\partial_m - \frac{1}{6}(\partial_\mu - {B_\mu}^m\partial_m)\ln\Delta \\
&\qquad\qquad - \frac{1}{4}\Delta^{-1/3}{e_\mu}^\alpha\Big({\omega_\alpha}^{\beta\gamma}\gamma_{\beta\gamma} + \frac{2}{3}\gamma_\alpha\gamma_\beta\,{\omega_a}^{a\beta}\Big)\Big]\epsilon \\
&- \frac{1}{2}\Delta^{-1/3}{e_\mu}^\alpha\Big[\mathrm{i}Q_\alpha\boldsymbol{\Gamma}_7 + \frac{1}{2}{\omega_\alpha}^{ab}\boldsymbol{\Gamma}_{ab} - \frac{1}{12}\varepsilon^{abcde}F_{\alpha bcde}\boldsymbol{\Gamma}_{a6} \\
&\qquad\qquad + \frac{1}{4}\mathrm{i}(G_{\alpha ab}\mathbb{P}_+ - \bar{G}_{\alpha ab}\mathbb{P}_-)\boldsymbol{\Gamma}^{ab6}\Big]\epsilon \\
&+ \frac{1}{24}\Delta^{-1/3}{e_\mu}^\alpha({\gamma_\alpha}^{\beta\gamma} - 4{\delta_\alpha}^\beta\gamma^\gamma)\Big[\mathrm{i}(G_{\beta\gamma a}\mathbb{P}_+ + \bar{G}_{\beta\gamma a}\mathbb{P}_-)\boldsymbol{\Gamma}^a \\
&\qquad\qquad + \frac{1}{6}\mathrm{i}\varepsilon_{\beta\gamma\delta\tau\lambda}(G^{\delta\tau\lambda}\mathbb{P}_+ - \bar{G}^{\delta\tau\lambda}\mathbb{P}_-)\boldsymbol{\Gamma}^6 \\
&\qquad\qquad - 2\,\omega_{a\,\beta\gamma}\,\boldsymbol{\Gamma}^{a6}\boldsymbol{\Gamma}_7 - \frac{1}{3}\varepsilon^{abcde}F_{\beta\gamma abc}\boldsymbol{\Gamma}_{de}\boldsymbol{\Gamma}_7\Big]\epsilon \\
&+ \frac{1}{3}\Delta^{-1/3}{e_\mu}^\alpha\gamma_\alpha\,\boldsymbol{\Gamma}^{m6}\boldsymbol{\Gamma}_7\Big[\partial_m - \frac{1}{6}\partial_m\ln\Delta - \frac{1}{2}\mathrm{i}Q_m\boldsymbol{\Gamma}_7 - \frac{1}{4}{\omega_m}^{bc}\boldsymbol{\Gamma}_{bc} \\
&\qquad\qquad + \frac{1}{600}\varepsilon^{bcdef}F_{bcdef}\boldsymbol{\Gamma}_{m6} - \frac{1}{24}\mathrm{i}(G_{mbc}\mathbb{P}_+ - \bar{G}_{mbc}\mathbb{P}_-)\boldsymbol{\Gamma}^{bc6}\Big]\epsilon \\
&+ \frac{1}{2}\Delta^{-1/3}{e_\mu}^\alpha\Big[(\omega_{\alpha\,a\beta} - \omega_{a\,\alpha\beta})\gamma^\beta\,\boldsymbol{\Gamma}^{a6}\boldsymbol{\Gamma}_7 + \frac{1}{3}\gamma_\alpha\gamma^\beta\,\omega_{a\,\beta b}\,\boldsymbol{\Gamma}^{ab}\Big]\epsilon\,,
\end{aligned} \tag{6.2}
$$

where we haved used the same notation as in (6.1) and have suppressed terms of higher-order in the fermion fields. It is worth noting at this point that some terms already combine into representations of USp(8). In particular, the terms in the second bracket span the **36** and thus take values in $\mathfrak{usp}(8)$ and those in the third bracket span the $\overline{\mathbf{27}}$ representation of USp(8). The structure of the last two brackets is more subtle and will be discussed momentarily.

Following [1], the next step is to expand the components of the $10D$ spin connection about the reference background of the internal $5D$ space characterized by the fünfbein $\mathring{e}_m{}^a(y)$. To this purpose, we write the spin connection in terms of the anholonomity coefficients, which depend on the zehnbein and its derivatives,

$$\begin{aligned}\omega_{MAB} &= \frac{1}{2} E_M{}^C (\Omega_{ABC} - \Omega_{BCA} - \Omega_{CAB})\,,\\ \Omega_{AB}{}^C &= 2\, E_{[A}{}^M E_{B]}{}^N\, \partial_M E_N{}^C\,.\end{aligned} \tag{6.3}$$

Writing the internal fünfbein as

$$e_m{}^a(x,y) = \mathring{e}_m{}^b(y)\, S_b{}^a(x,y)\,, \qquad e^m{}_a(x,y) = S^{-1}{}_a{}^b(x,y)\,\mathring{e}^m{}_b(y)\,, \tag{6.4}$$

such that $\Delta = \det[S_a{}^b]$, one can evaluate the components of $\Omega_{AB}{}^C$ making use of (3.1),

$$\begin{aligned}\Omega_{\alpha\beta}{}^\gamma &= 2\,\Delta^{1/3}\Big[e_{[\alpha}{}^\mu e_{\beta]}{}^\nu\, \mathcal{D}_\mu e_\nu{}^\gamma - \frac{1}{3} e_{[\alpha}{}^\mu\, \delta_{\beta]}{}^\gamma\, \mathcal{D}_\mu \ln\Delta\Big]\,,\\ \Omega_{\alpha\beta}{}^c &= 2\,\Delta^{2/3} e_{[\alpha}{}^\mu e_{\beta]}{}^\nu\, \mathring{e}_m{}^b S_b{}^c\, \mathcal{D}_\mu B_\nu{}^m\,,\\ \Omega_{a\beta}{}^\gamma &= S^{-1}{}_a{}^b\, \mathring{e}_b{}^m \Big[e_\beta{}^\nu\, \partial_m e_\nu{}^\gamma - \frac{1}{3}\delta_\beta{}^\gamma\, \partial_m \ln\Delta\Big]\,,\\ \Omega_{ab}{}^\gamma &= 0\,,\\ \Omega_{a\beta}{}^c &= \Delta^{1/3} S^{-1}{}_a{}^b\, e_\beta{}^\mu \big[\mathring{e}_b{}^m \mathring{e}_n{}^d\, S_d{}^c \mathring{D}_m B_\mu{}^n - \mathcal{D}_\mu S_b{}^c\big] + \Delta^{1/3} e_\beta{}^\mu B_\mu{}^m\, \mathring{\omega}_m{}^c{}_a\,,\\ \Omega_{ab}{}^c &= -\,2\, S^{-1}{}_{[a}{}^d\, S^{-1}{}_{b]}{}^e\, \mathring{e}_e{}^m\, \mathring{D}_m S_d{}^c - 2\,\mathring{\omega}_m{}^c{}_{[a}\, S^{-1}{}_{b]}{}^d\, \mathring{e}_d{}^m\,.\end{aligned} \tag{6.5}$$

Here we have defined $\mathcal{D}_\mu = \partial_\mu - B_\mu{}^m \mathring{D}_m$, where $\mathring{D}_m$ is the derivative that is covariant with respect to tangent-space transformations of the background. Hence it contains the spin connection $\mathring{\omega}_m{}^{ab}(y)$,

$$\mathring{\omega}_{mab} = \frac{1}{2}\mathring{e}_m{}^c(\mathring{\Omega}_{abc} - \mathring{\Omega}_{bca} - \mathring{\Omega}_{cab})\,, \qquad \mathring{\Omega}_{ab}{}^c = 2\,\mathring{e}_{[a}{}^m \mathring{e}_{b]}{}^n\, \partial_m \mathring{e}_n{}^c\,, \tag{6.6}$$

and possibly the corresponding Christoffel connection, depending on the tensor it acts on. These results exhibit, up to dimension dependent coefficients, the same structure as in the $11D$ case and we refer to [1] for further details.

After substitution of the expressions (6.5) into (6.2) and some rearrangements, one obtains the following result,

$$\begin{aligned}\delta\psi_\mu{}^A = {}& D_\mu \epsilon^A - \frac{1}{6}(\mathring{D}_m C_\mu{}^m)\epsilon^A + \frac{1}{12}\mathrm{i}\,(\gamma_\mu{}^{\beta\gamma} - 4\, e_\mu{}^\beta \gamma^\gamma)\mathcal{H}_{\beta\gamma}{}^{AB}\,\Omega_{BC}\,\epsilon^C\\ &- \frac{4}{3}\mathrm{i}\,\bar{\Omega}^{AC}\,\mathcal{V}_{CB}{}^m\, D_m\big(\gamma_\mu\,\epsilon^B\big) - \frac{2}{3}\mathrm{i}\,\bar{\Omega}^{AC} D_m\big(\gamma_\mu\, \mathcal{V}_{CB}{}^m\big)\epsilon^B\,,\end{aligned} \tag{6.7}$$

where we have now written the field $B_\mu{}^m$ as $C_\mu{}^m$, as the above expression is the final result. Here D_μ and D_m denote the full $\mathrm{Spin}(4,1)\times\mathrm{USp}(8)$ covariant derivatives with $\mathrm{USp}(8)$ connections $\mathcal{Q}_\mu$ and $\mathcal{Q}_m$, such that

$$
\begin{aligned}
D_\mu\epsilon^A &= \mathcal{D}_\mu\epsilon^A - \frac{1}{4}\hat{\omega}_\mu{}^{\alpha\beta}\gamma_{\alpha\beta}\epsilon^A - \mathcal{Q}_\mu{}^A{}_B\,\epsilon^B\,,\\
D_m\epsilon^A &= \mathring{D}_m\epsilon^A - \mathcal{Q}_m{}^A{}_B\,\epsilon^B\,.
\end{aligned}
\tag{6.8}
$$

Here the modified spin connection $\hat{\omega}_\mu{}^{\alpha\beta}$ is defined by

$$
\hat{\omega}_\mu{}^{\alpha\beta} = \omega_\mu{}^{\alpha\beta} + \frac{2}{3}\mathring{D}_m B_\nu{}^m\, e_\mu{}^{[\alpha}e^{\beta]\nu}\,,
\tag{6.9}
$$

where $\omega_\mu{}^{\alpha\beta}$ is the regular torsion-free spin connection expressed in terms of the fünfbein $e_\mu{}^\alpha$. The two $\mathrm{USp}(8)$ connections, $\mathcal{Q}_\mu{}^A{}_B$ and $\mathcal{Q}_m{}^A{}_B$, are equal to

$$
\begin{aligned}
\mathcal{Q}_\mu{}^A{}_B =& \frac{1}{4}\Big[S^{-1\,ac}\,S_d{}^b\,\mathring{e}_c{}^m\,\mathring{e}_n{}^d\,\mathring{D}_m B_\mu{}^n - (S^{-1}\mathcal{D}_\mu S)^{ab}\Big]\,(\Phi^\dagger\mathbf{\Gamma}_{ab}\Phi)^A{}_B\\
&+\frac{1}{2}\mathrm{i}\Delta^{-1/3}e_\mu{}^\alpha Q_\alpha\,(\Phi^\dagger\mathbf{\Gamma}_7\Phi)^A{}_B\\
&-\frac{1}{24}\Delta^{-1/3}e_\mu{}^\alpha\varepsilon^{abcde}F_{\alpha abcd}\,(\Phi^\dagger\mathbf{\Gamma}_{e6}\Phi)^A{}_B\\
&+\frac{1}{8}\mathrm{i}\Delta^{-1/3}e_\mu{}^\alpha\big(G_{\alpha ab}\,(\Phi^\dagger\mathbb{P}_+\mathbf{\Gamma}^{ab6}\Phi)^A{}_B - \bar{G}_{\alpha ab}\,(\Phi^\dagger\mathbb{P}_-\mathbf{\Gamma}^{ab6}\Phi)^A{}_B\big)\\
&-(\Phi^\dagger\,\partial_\mu\Phi)^A{}_B\,,\\
\mathcal{Q}_m{}^A{}_B =& -\frac{1}{4}(S^{-1}\mathring{D}_m S)^{ab}\,(\Phi^\dagger\mathbf{\Gamma}_{ab}\Phi)^A{}_B + \frac{1}{2}\mathrm{i}Q_m\,(\Phi^\dagger\mathbf{\Gamma}_7\Phi)^A{}_B\\
&+\frac{1}{24}\mathrm{i}\big(G_{mbc}\,(\Phi^\dagger\mathbb{P}_+\mathbf{\Gamma}^{bc6}\Phi)^A{}_B - \bar{G}_{mbc}\,(\Phi^\dagger\mathbb{P}_-\mathbf{\Gamma}^{bc6}\Phi)^A{}_B\big)\\
&-\frac{1}{600}\varepsilon^{abcde}F_{abcde}\,(\Phi^\dagger\mathbf{\Gamma}_{m6}\Phi)^A{}_B - (\Phi^\dagger\,\partial_m\Phi)^A{}_B\,.
\end{aligned}
\tag{6.10}
$$

The field strength $\mathcal{H}_{\alpha\beta}{}^{AB}$ spans the $\overline{\mathbf{27}}$ of $\mathrm{USp}(8)$ and reads

$$
\begin{aligned}
\mathcal{H}_{\alpha\beta}{}^{AB} =& \mathrm{i}\Delta^{-1/3}\Big[(S^{-1})_a{}^b\,\mathring{e}_b{}^m e_{[\alpha}{}^\mu\,\partial_m e_{\mu\beta]} - \Delta^{2/3}\,e_{[\alpha}{}^\mu e_{\beta]}{}^\nu\,\mathring{e}_m{}^b\,S_b{}^a\,\mathcal{D}_\mu B_\nu{}^m\Big]\\
&\quad\times(\Phi^\dagger\mathbf{\Gamma}_{a6}\mathbf{\Gamma}_7\bar{\Omega}\bar{\Phi})^{AB}\\
&-\frac{1}{2}\Delta^{-1/3}\big(G_{a\alpha\beta}\,(\Phi^\dagger\mathbb{P}_+\mathbf{\Gamma}^a\bar{\Omega}\bar{\Phi})^{AB} + \bar{G}_{a\alpha\beta}\,(\Phi^\dagger\mathbb{P}_-\mathbf{\Gamma}^a\bar{\Omega}\bar{\Phi})^{AB}\big)\\
&-\frac{1}{12}\Delta^{-1/3}\varepsilon_{\alpha\beta\gamma\delta\lambda}\big(G^{\gamma\delta\lambda}\,(\Phi^\dagger\mathbb{P}_+\mathbf{\Gamma}_6\bar{\Omega}\bar{\Phi})^{AB} - \bar{G}^{\gamma\delta\lambda}\,(\Phi^\dagger\mathbb{P}_-\mathbf{\Gamma}_6\bar{\Omega}\bar{\Phi})^{AB}\big)\\
&-\frac{1}{6}\mathrm{i}\Delta^{-1/3}\varepsilon^{abcde}F_{\alpha\beta abc}\big(\Phi^\dagger\mathbf{\Gamma}_{de}\mathbf{\Gamma}_7\bar{\Omega}\bar{\Phi}\big)^{AB}\,,
\end{aligned}
\tag{6.11}
$$

Finally we have used the identity

$$
D_m\mathcal{V}_{AB}{}^m = -\Big[(S^{-1}\mathring{D}_m S)^{(ab)}\,e_a{}^m\,e_{nb} + \frac{1}{3}\partial_n\ln\Delta\Big]\,\mathcal{V}_{AB}{}^n\,.
\tag{6.12}
$$

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With these definitions the local USp(8) covariance of the gravitino supersymmetry variations has been established.

Now we return to the supersymmetry transformations of the spin-1/2 fields. Upon combining the results (6.1) into the covariant form $\delta\chi^{ABC}$, one obtains, after some rearrangements similar to those used in $\delta\psi_\mu{}^A$,

$$
\begin{aligned}
\delta\chi^{ABC} =& \frac{1}{2}\mathrm{i}\,\gamma^\mu \mathcal{P}_\mu{}^{ABCD}\,\Omega_{DE}\,\epsilon^E \\
&- \frac{3}{16}\,\gamma^{\alpha\beta}\Big[\mathcal{H}_{\alpha\beta}{}^{[AB}\,\epsilon^{C]} - \frac{1}{3}\,\bar\Omega^{[AB}\mathcal{H}_{\alpha\beta}{}^{C]D}\Omega_{DE}\,\epsilon^E\Big] \\
&- 3\,\bar\Omega^{D[A}\Big[\bar\Omega^{B|E}\,\mathcal{V}_{DE}{}^m\,D_m\epsilon^{C]} - \frac{1}{3}\bar\Omega^{BC]}\,\mathcal{V}_{DE}{}^m\,D_m\epsilon^E\Big] \\
&- \frac{3}{2}\bar\Omega^{D[A}\Big[\bar\Omega^{B|E}\,D_m\mathcal{V}_{DE}{}^m\,\epsilon^{C]} - \frac{1}{3}\bar\Omega^{BC]}\,D_m\mathcal{V}_{DE}{}^m\,\epsilon^E\Big] \\
&- 2\,\mathcal{P}_m{}^{ABCD}\,\mathcal{V}_{DE}{}^m\epsilon^E\,.
\end{aligned}
\tag{6.13}
$$

In this expression two new tensors appear, $\mathcal{P}_\mu{}^{ABCD}$ and $\mathcal{P}_m{}^{ABCD}$, which transform in the **42** representation of USp(8). These tensors also appear in the so-called vielbein postulates,

$$
\begin{aligned}
&\mathring{D}_m\mathcal{V}_{AB}{}^n - 2\,\mathcal{Q}_m{}^C{}_{[A}\,\mathcal{V}_{B]C}{}^n + \Omega_{AC}\,\Omega_{BD}\,\mathcal{P}_m{}^{CDEF}\,\mathcal{V}_{EF}{}^n = 0\,, \\
&\mathcal{D}_\mu\mathcal{V}_{AB}{}^m + \frac{1}{3}\mathring{D}_nC_\mu{}^n\,\mathcal{V}_{AB}{}^m + \mathring{D}_nC_\mu{}^m\,\mathcal{V}_{AB}{}^n \\
&\qquad\qquad -2\,\mathcal{Q}_\mu{}^C{}_{[A}\,\mathcal{V}_{B]C}{}^m + \Omega_{AC}\,\Omega_{BD}\,\mathcal{P}_\mu{}^{CDEF}\,\mathcal{V}_{EF}{}^m = 0\,.
\end{aligned}
\tag{6.14}
$$

Note that these expressions are similar to the corresponding postulates in $11D$ [1]. Such equations will apply to all the generalized vielbeine, but we refrain from presenting further results. Note that we have again written $B_\mu{}^m$ as $C_\mu{}^m$.

Both the supersymmetry transformations (6.7) and (6.13) thus take a manifestly USp(8) covariant form. The two new tensors, $\mathcal{P}_\mu{}^{ABCD}$ and $\mathcal{P}_m{}^{ABCD}$, are defined by (in the gauge where the phase factor Φ is set to unity)

$$
\begin{aligned}
\mathcal{P}_\mu{}^{ABCD} =& \frac{1}{8}\Delta^{-1/3}(P_\mu + \bar P_\mu)\Big[(\mathbf{\Gamma}_a\bar\Omega)^{[AB}(\mathbf{\Gamma}^a\bar\Omega)^{CD]} + (\mathbf{\Gamma}_6\bar\Omega)^{[AB}(\mathbf{\Gamma}_6\bar\Omega)^{CD]} + 2\,\bar\Omega^{[AB}\bar\Omega^{CD]}\Big] \\
&- \frac{1}{8}\Delta^{-1/3}(P_\mu - \bar P_\mu)\Big[(\mathbf{\Gamma}_a\bar\Omega)^{[AB}(\mathbf{\Gamma}^a\mathbf{\Gamma}_7\bar\Omega)^{CD]} + (\mathbf{\Gamma}_6\bar\Omega)^{[AB}(\mathbf{\Gamma}_6\mathbf{\Gamma}_7\bar\Omega)^{CD]}\Big] \\
&- \frac{3}{16}\Big((S^{-1}\mathcal{D}_\mu S)_{(ab)} - \delta_{c(a}e_{b)}{}^m e_n{}^c\mathring{D}_mB_\mu{}^n\Big) \\
&\qquad \times\Big[(\mathbf{\Gamma}^{ac}\mathbf{\Gamma}_7\bar\Omega)^{[AB}(\mathbf{\Gamma}^b{}_c\mathbf{\Gamma}_7\bar\Omega)^{CD]} + (\mathbf{\Gamma}^a\mathbf{\Gamma}_6\mathbf{\Gamma}_7\bar\Omega)^{[AB}(\mathbf{\Gamma}^b\mathbf{\Gamma}_6\mathbf{\Gamma}_7\bar\Omega)^{CD]}\Big] \\
&- \frac{3}{16}\Big(\partial_\mu\ln\Delta - \mathring{D}_mB_\mu{}^m\Big)\Big[(\mathbf{\Gamma}^c\mathbf{\Gamma}_6\mathbf{\Gamma}_7\bar\Omega)^{[AB}(\mathbf{\Gamma}_c\mathbf{\Gamma}_6\mathbf{\Gamma}_7\bar\Omega)^{CD]} - \frac{10}{3}\,\bar\Omega^{[AB}\bar\Omega^{CD]}\Big] \\
&- \frac{1}{32}\,\Delta^{-1/3}e_\mu{}^\alpha F_{\alpha cdef}\varepsilon_a{}^{cdef}(\mathbf{\Gamma}_{ab}\mathbf{\Gamma}_7\bar\Omega)^{[AB}(\mathbf{\Gamma}^b\mathbf{\Gamma}_6\mathbf{\Gamma}_7\bar\Omega)^{CD]} \\
&- \frac{1}{32}\Delta^{-1/3}e_\mu{}^\alpha\varepsilon_{abcde}(G_\alpha{}^{de} + \bar G_\alpha{}^{de})(\mathbf{\Gamma}^a\mathbf{\Gamma}_7\bar\Omega)^{[AB}(\mathbf{\Gamma}^{bc}\mathbf{\Gamma}_7\bar\Omega)^{CD]}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{16}\mathrm{i}\Delta^{-1/3}e_\mu{}^\alpha(G_{\alpha ab}-\bar{G}_{\alpha ab})\\
&\quad\times\Big[(\mathbf{\Gamma}^{ab}\mathbf{\Gamma}_7\bar{\Omega})^{[AB}(\mathbf{\Gamma}_6\mathbf{\Gamma}_7\bar{\Omega})^{CD]}+2(\mathbf{\Gamma}^{a}\mathbf{\Gamma}_7\bar{\Omega})^{[AB}(\mathbf{\Gamma}^b\mathbf{\Gamma}_6\mathbf{\Gamma}_7\bar{\Omega})^{CD]}\Big]\,, \qquad (6.15)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{P}_m{}^{ABCD}=&\,\frac{1}{8}(P_m+\bar{P}_m)\Big[(\mathbf{\Gamma}_a\bar{\Omega})^{[AB}(\mathbf{\Gamma}^a\bar{\Omega})^{CD]}+(\mathbf{\Gamma}_6\bar{\Omega})^{[AB}(\mathbf{\Gamma}_6\bar{\Omega})^{CD]}+2\,\bar{\Omega}^{[AB}\bar{\Omega}^{CD]}\Big]\\
&-\frac{1}{8}(P_m-\bar{P}_m)\Big[(\mathbf{\Gamma}_a\bar{\Omega})^{[AB}(\mathbf{\Gamma}^a\mathbf{\Gamma}_7\bar{\Omega})^{CD]}+(\mathbf{\Gamma}_6\bar{\Omega})^{[AB}(\mathbf{\Gamma}_6\mathbf{\Gamma}_7\bar{\Omega})^{CD]}\Big]\\
&-\frac{3}{16}(S^{-1}\mathring{D}_mS)_{(ab)}\\
&\qquad\times\Big[(\mathbf{\Gamma}^{ac}\mathbf{\Gamma}_7\bar{\Omega})^{[AB}(\mathbf{\Gamma}^b{}_c\mathbf{\Gamma}_7\bar{\Omega})^{CD]}+(\mathbf{\Gamma}^a\mathbf{\Gamma}_6\mathbf{\Gamma}_7\bar{\Omega})^{[AB}(\mathbf{\Gamma}^b\mathbf{\Gamma}_6\mathbf{\Gamma}_7\bar{\Omega})^{CD]}\Big]\\
&-\frac{3}{16}\partial_m\ln\Delta\Big[(\mathbf{\Gamma}^c\mathbf{\Gamma}_6\mathbf{\Gamma}_7\bar{\Omega})^{[AB}(\mathbf{\Gamma}_c\mathbf{\Gamma}_6\mathbf{\Gamma}_7\bar{\Omega})^{CD]}-\frac{10}{3}\,\bar{\Omega}^{[AB}\bar{\Omega}^{CD]}\Big]\\
&-\frac{1}{800}\,e_m{}^aF_{cdefg}\varepsilon^{cdefg}(\mathbf{\Gamma}_{ab}\mathbf{\Gamma}_7\bar{\Omega})^{[AB}(\mathbf{\Gamma}^b\mathbf{\Gamma}_6\mathbf{\Gamma}_7\bar{\Omega})^{CD]}\\
&-\frac{1}{96}e_m{}^f\varepsilon_{abcde}(G^{de}{}_f+\bar{G}^{de}{}_f)(\mathbf{\Gamma}^a\mathbf{\Gamma}_7\bar{\Omega})^{[AB}(\mathbf{\Gamma}^{bc}\mathbf{\Gamma}_7\bar{\Omega})^{CD]}\\
&+\frac{1}{48}\mathrm{i}\,e_m{}^c\,(G_{abc}-\bar{G}_{abc})\\
&\qquad\times\Big[(\mathbf{\Gamma}^{ab}\mathbf{\Gamma}_7\bar{\Omega})^{[AB}(\mathbf{\Gamma}_6\mathbf{\Gamma}_7\bar{\Omega})^{CD]}+2(\mathbf{\Gamma}^{a}\mathbf{\Gamma}_7\bar{\Omega})^{[AB}(\mathbf{\Gamma}^b\mathbf{\Gamma}_6\mathbf{\Gamma}_7\bar{\Omega})^{CD]}\Big]\,. \qquad (6.16)
\end{aligned}
$$

Note that the above formulae (6.15) and (6.16) are unique up to Fierz reordering.

7 On the consistent truncation to 5D SO(6) gauged supergravity

The results of this paper can be used to establish the full consistency of the truncation of IIB supergravity compactified on the sphere S^5 to 5D SO(6) gauged supergravity [33], along the same lines that were followed originally for the truncation of 11D supergravity compactified on the sphere S^7 to 4D SO(8) gauged supergravity [2–4, 44–46]. For IIB supergravity some partial results have already appeared in the literature [21, 34] and they will be confirmed below from the results of this paper. It is clear that additional results can be obtained by a more complete analysis, but a full treatment is outside the scope of this paper. The same holds for a study of more general truncations along the lines pursued in [39] for 11D supergravity.

It is worth stressing that this concept of a consistent truncation goes beyond proving that solutions of 5D maximal SO(6) gauged supergravity can be uplifted to the IIB theory. Rather, starting from the fully supersymmetric solution with $\mathrm{AdS}_5\times S^5$, one sweeps out the full field configuration space of 5D maximal supergravity in the ten-dimensional field configuration space. This is done by writing the 10D fields as functions of the 5D fields, involving y-dependent functions, mostly constructed from the S^5 Killing spinors, in such a way that the 10D supersymmetry transformations remain consistent upon extracting these y-dependent factors. In the case at hand these eight independent, pseudo-real, Killing spinors, $\eta^A(y)$, satisfy

$$
\Big(\mathring{D}_m+\frac{1}{2}m_5\,\mathring{e}_m{}^a\,\mathbf{\Gamma}_{a6}\Big)\eta(y)=0\,. \qquad (7.1)
$$

Here m_5 denotes the inverse S^5 radius which is related to the background value of the field strength F_{mnpqr} by $m_5 = \frac{1}{120}\,\mathring{e}\,\varepsilon^{mnpqr}\mathring{F}_{mnpqr}$. Furthermore $\mathring{D}_m$ equals the S^5 background covariant derivative and $\mathring{e}_m{}^a$ is the globally defined fünfbein on S^5. The Killing spinor equation (7.1) is motivated by the fact that it characterizes the supersymmetry of the $\mathrm{AdS}_5 \times S^5$ solution of IIB supergravity. Note that all Killing spinors in this section will be commuting.

In view of what follows it is useful to first discuss these Killing spinors in more detail. Since (7.1) is a first-order differential equation it allows for eight independent solutions. However, in five Euclidean dimensions, the Clifford algebra associated with the SO(5) gamma matrices has an automorphism group equal to $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$. Consequently one can choose six independent spinors that are not related by the action of the automorphism group, so that the orbit that is then swept out under the action of the SU(2) automorphism group will yield the two remaining independent spinors. Bilinears constructed from the Killing spinors that involve only the original SO(5) gamma matrices will necessarily be invariant under the automorphism group and therefore the number of independent spinor bilinears of this type will constitute $6 \otimes 6$ independent bilinears which decompose into 15 anti-symmetric and 21 symmetric components. This argument, which incidentally also plays a role when analyzing the number of degrees of freedom of the generalized vielbeine in section 5, explains why the bilinears produce precisely 15 independent Killing vectors. More specifically it follows that

$$\mathring{e}_a{}^m\,\bar{\eta}_1(y)\,\mathbf{\Gamma}^{a6}\mathbf{\Gamma}_7\,\eta^2(y) = \sum_{[\hat{a}\hat{b}]} C^{\hat{a}\hat{b}}\,K^m{}_{\hat{a}\hat{b}}(y)\,, \tag{7.2}$$

where $\eta^{1,2}$ are two possible Killing spinors (with $\bar{\eta} \equiv \eta^\dagger$) and the indices $\hat{a}, \hat{b}, \ldots$ denote the components of the defining representation of the SO(6); in this background this SO(6) corresponds to the isometry group of the sphere S^5. The fifteen Killing vectors are labeled with anti-symmetric pairs $[\hat{a}\hat{b}]$, and the $C^{\hat{a}\hat{b}}$ are constants. To prove this relation one can write the gamma matrices in terms of the original SO(5) gamma matrices and/or one can prove directly that the left-hand side of (7.2) satisfies the Killing equation by virtue of (7.1).

Taking the derivative of the Killing vectors one finds another tensor that is also anti-symmetric in $[\hat{a}\hat{b}]$ (note that indices are lowered/raised with the S^5 metric $\mathring{g}_{mn}$ and its inverse),

$$\mathring{D}_m K_{n\,\hat{a}\hat{b}} = m_5\,K_{mn\,\hat{a}\hat{b}}\,. \tag{7.3}$$

In five dimensions this tensor is known as a Killing tensor. It satisfies the equation

$$\mathring{D}_m K_{np\,\hat{a}\hat{b}} = -2\,m_5\,\mathring{g}_{m[n}\,K_{p]\,\hat{a}\hat{b}}\,. \tag{7.4}$$

From the previous results one then derives

$$\mathring{e}_m{}^a\,\mathring{e}_n{}^b\,\bar{\eta}_1(y)\,\mathbf{\Gamma}_{ab}\mathbf{\Gamma}_7\,\eta^2(y) = \sum_{[\hat{a}\hat{b}]} C^{\hat{a}\hat{b}}\,K_{mn\,\hat{a}\hat{b}}(y)\,. \tag{7.5}$$

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After these observations we turn to the consistent truncation ansätze for the $10D$ fields. We start from eight independent Killing spinors, now labeled by indices $i, j, \ldots = 1, 2, \ldots, 8$, such that these spinors form an orthonormal basis in the USp(8) spinor space and are subject to a pseudo-reality condition,

$$\bar{\eta}^i(y)\,\eta_j(y) = \delta^i{}_j\,, \qquad \bar{\eta}^i{}_A = \bar{\Omega}^{ij}\,\Omega_{AB}\,\eta_j{}^B\,, \tag{7.6}$$

where $\bar{\Omega}^{ij}$ and Ω_{AB} are the symplectic matrices used before. The truncation for the fermions, the supersymmetry parameters and the space-time vielbein $e_\mu{}^\alpha$ are then assumed to take the form,[3]

$$\begin{aligned}
\psi_\mu{}^A(x,y) &= \psi_\mu{}^i(x)\,\eta_i^A(y)\,,\\
\epsilon^A(x,y) &= \epsilon^i(x)\,\eta_i^A(y)\,,\\
\chi^{ABC}(x,y) &= \chi^{ijk}(x)\,\eta_i^A(y)\,\eta_j{}^B(y)\,\eta_k{}^C(y)\,,\\
e_\mu{}^\alpha(x,y) &= e_\mu{}^\alpha(x)\,.
\end{aligned} \tag{7.7}$$

Making this assumption will obviously restrict the USp(8) R-symmetry transformations to

$$U^A{}_B(x,y) = U^i{}_j(x)\,\eta_i^A(y)\,\bar{\eta}^j{}_B(y)\,, \tag{7.8}$$

and leaves the group structure intact by virtue of the conditions (7.6). Observe that the supersymmetry transformations for $e_\mu{}^\alpha$ are consistent under this truncation. However, for the other bosons the truncation ansatz is more subtle.

To derive the truncation ansätze for the remaining bosons one first considers their supersymmetry variations into the fermions, defined according to (7.7). For instance, consider (5.2), which will now take the form,

$$\delta C_\mu{}^M(x,y) = 2\left[\mathrm{i}\bar{\Omega}^{ik}\,\bar{\epsilon}_k(x)\,\psi_\mu^j(x) + \bar{\epsilon}_k(x)\,\gamma_\mu\chi^{ijk}(x)\right]\mathcal{V}_{ij}{}^M(x,y)\,, \tag{7.9}$$

where

$$\mathcal{V}_{ij}{}^M(x,y) = \eta_i^A(y)\,\eta_j^B(y)\,\mathcal{V}_{AB}{}^M(x,y)\,. \tag{7.10}$$

The consistency of the truncation now requires that the y-dependence of $C_\mu{}^M$ and $\mathcal{V}_{ij}{}^M$ will match.

Before deriving some of the additional truncation results, let us first compare the situation regarding the compactification on the torus T^5 and the sphere S^5. In the torus truncation all the fields $C_\mu{}^M$ will appear and will be independent of the torus coordinates y^m. Consequently the generalized vielbeine $\mathcal{V}_{ij}{}^M$ will also be y-independent and they will be precisely equal to the corresponding quantities $U_{ij}{}^M(x)$ that are a representative of the $\mathrm{E}_{6(6)}$/USp(8) coset space.[4] The tensor fields $C_{\mu\nu\,M}$ can be gauged away in the torus truncation where they carry no additional information and they are simply dual to the vector fields.

[3] The phase factor Φ is only implicit in the formulae below, but it actually plays a crucial role to ensure that consistency is achieved (see e.g. [45]).

[4] Here we deviate from the notation used in [8] where the $U_{ij}{}^M(x)$ are denoted also by $\mathcal{V}_{ij}{}^M$.

The situation for the S^5 compactification is different, as in this case the various 'physical' fields reside in both the $C_\mu{}^M$ and $C_{\mu\nu\, M}$ [33]. More precisely, in this case there are fifteen vector fields transforming in the adjoint representation of the SO(6) subgroup of $\mathrm{E}_{6(6)}$ and twelve tensor fields transforming as a direct product of the vector representation of the same SO(6) subgroup and the doublet represention of the SU(1,1) subgroup of $\mathrm{E}_{6(6)}$. The remaining vector and tensor fields in the sphere truncation are the duals of these $15 \oplus 12$ fields, which can be gauged away in the embedding tensor approach. This decomposition in terms of the expected vector and tensor fields must be reflected in the truncation ansätze for the vectors and tensors.

It is important to realize that the fields $C_\mu{}^M$ and $C_{\mu\nu\, M}$ are gauge fields, which excludes field-dependent multiplicative redefinitions. Given that the y-dependence should be extracted in the form of the geometric quantities associated with the sphere, it is rather obvious what the truncation ansätze should be. Let us first demonstrate this for the $\mathrm{SU}(1,1)$ invariant vector and tensor fields, $C_\mu{}^m$, $C_\mu{}^{mn}$, $C_{\mu\nu\, m}$ and $C_{\mu\nu\, mn}$, each of which can be decomposed into the fifteen Killing vectors or tensors according to

$$
\begin{aligned}
C_\mu{}^m(x,y) &= K^m{}_{\hat a\hat b}(y)\, A_\mu{}^{\hat a\hat b}(x)\,, \\
C_\mu{}^{mn}(x,y) &= K^{mn}{}_{\hat a\hat b}(y)\, \tilde A_\mu{}^{\hat a\hat b}(x)\,, \\
C_{\mu\nu\, m}(x,y) &= K_m{}^{\hat a\hat b}(y)\, B_{\mu\nu\,\hat a\hat b}(x)\,, \\
C_{\mu\nu\, mn}(x,y) &= K_{mn}{}^{\hat a\hat b}(y)\, \tilde B_{\mu\nu\,\hat a\hat b}(x)\,.
\end{aligned} \tag{7.11}
$$

However, as explained above, in $5D$ one has only fifteen vector and fifteen tensor fields in the $\mathrm{SU}(1,1)$ invariant sector, so that one must assume that $A_\mu{}^{\hat a\hat b}(x)$ and $\tilde A_\mu{}^{\hat a\hat b}(x)$ are identical up to a possible multiplicative constant; the same holds for the tensor fields $B_{\mu\nu\,\hat a\hat b}(x)$ and $\tilde B_{\mu\nu\,\hat a\hat b}(x)$. However, here and in the following we will not be concerned about numerical factors, also in view of the fact that we have not adopted specific normalizations for the Killing vectors and tensors.

A similar decomposition applies to the generalized vielbeine $\mathcal{V}_{ij}{}^m$, $\mathcal{V}_{ij}{}^{mn}$, $\mathcal{V}_m{}^{ij}$ and $\mathcal{V}_{mn}{}^{ij}$ which appear in the variation of the above fields,

$$
\begin{aligned}
\mathcal{V}_{ij}{}^m(x,y) &= U_{ij}{}^{\hat a\hat b}(x)\, K^m{}_{\hat a\hat b}(y)\,, \\
\mathcal{V}_{ij}{}^{mn}(x,y) &= U_{ij}{}^{\hat a\hat b}(x)\, K^{mn}{}_{\hat a\hat b}(y)\,, \\
\mathcal{V}_m{}^{ij}(x,y) &= U_{\hat a\hat b}{}^{ij}(x)\, K_m{}^{\hat a\hat b}(y)\,, \\
\mathcal{V}_{mn}{}^{ij}(x,y) &= U_{\hat a\hat b}{}^{ij}(x)\, K_{mn}{}^{\hat a\hat b}(y)\,,
\end{aligned} \tag{7.12}
$$

where, as we have explained above, $U_{ij}{}^{\hat a\hat b}(x)$ and $U_{\hat a\hat b}{}^{ij}(x)$ are the (unique) components of the $\mathrm{E}_{6(6)}/\mathrm{USp}(8)$ coset space satisfying

$$
U_{\hat a\hat b}{}^{ij}(x)\, U_{ij}{}^{\hat c\hat d}(x) = 2\,\delta_{\hat a\hat b}{}^{\hat c\hat d}\,. \tag{7.13}
$$

Note that the combined equations (7.11) and (7.12) ensure that the corresponding supersymmetry transformations are consistent under the truncation.

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Subsequently we consider the following identities that follow from direct calculation using the generalized vielbeine presented in section 5, after converting the USp(8) indices according to (7.10),

$$\begin{aligned}
\bar{\mathcal{V}}^{ik\,m}\,\mathcal{V}_{kj}{}^{n} + \bar{\mathcal{V}}^{ik\,n}\,\mathcal{V}_{kj}{}^{m} &= -\,\frac{1}{4}\delta^{i}{}_{j}\,\bar{\mathcal{V}}^{kl\,m}\,\mathcal{V}_{kl}{}^{n}\,, \\
\bar{\Omega}^{ik}\,\bar{\Omega}^{jl}\,\mathcal{V}_{ij}{}^{m}\,\mathcal{V}_{kl}{}^{n} &= \frac{1}{2}\Delta^{-2/3}g^{mn}\,, \\
\bar{\Omega}^{ik}\,\bar{\Omega}^{jl}\,\mathcal{V}_{ij}{}^{m}\,\mathcal{V}_{kl}{}^{np} &= \frac{32}{15}\sqrt{5}\,\mathring{e}^{-1}\varepsilon^{npqrs}\Big[A_{qrst} + \frac{3}{16}\mathrm{i}\varepsilon_{\alpha\beta}\,A^{\alpha}{}_{qr}A^{\beta}{}_{st}\Big]\,\bar{\mathcal{V}}^{ij\,m}\,\mathcal{V}_{ij}{}^{t}\,,
\end{aligned} \tag{7.14}$$

where g^{mn} is the full (inverse) internal metric, which depends on the scalar fields. It is therefore different from the S^5 inverse metric $\mathring{g}^{mn}(y)$, unless the scalar fields take their background values. We remind the reader that the generalized vielbeine are pseudo-real so that the complex conjugate equals $\bar{\mathcal{V}}^{ij\,m} \equiv \big(\mathcal{V}_{ij}{}^{m}\big)^{*} = \bar{\Omega}^{ik}\,\bar{\Omega}^{jk}\,\mathcal{V}_{kl}{}^{m}$. Hence it follows that

$$\Delta^{-2/3}\,g^{mn}(x,y) = 2\,\bar{\Omega}^{ik}\,\bar{\Omega}^{jl}\,U_{ij}{}^{\hat{a}\hat{b}}(x)\,U_{kl}{}^{\hat{c}\hat{d}}(x)\;K^{m}{}_{\hat{a}\hat{b}}(y)\,K^{n}{}_{\hat{c}\hat{d}}(y)\,. \tag{7.15}$$

with $\Delta^2 = \det[g(x,y)]/\det[\mathring{g}(y)]$. This result is rather generic and was first found for $11D$ supergravity compactified on S^7 [46] with the prefactor Δ^{-1}. For IIB supergravity compactified on S^5 the above result was established in [21, 34].

The next step is to study the consequences of the third identity (7.14). Substitution of the generalized vielbeine leads to the equation

$$\begin{aligned}
\Delta^{-2/3}\Big[A_{mnpq} + \frac{3}{16}\mathrm{i}\varepsilon_{\alpha\beta}\,A^{\alpha}{}_{[mn}A^{\beta}{}_{p]q}\Big] &= \frac{1}{64}\sqrt{5}\,\bar{\Omega}^{ik}\,\bar{\Omega}^{jl}\,U_{ij}{}^{\hat{a}\hat{b}}(x)\,U_{kl}{}^{\hat{c}\hat{d}}(x)\,g_{qr}(x,y) \\
&\qquad \times\,\mathring{e}\,\varepsilon_{mnptu}K^{r}{}_{\hat{a}\hat{b}}(y)\,K^{tu}{}_{\hat{c}\hat{d}}(y)\,.
\end{aligned} \tag{7.16}$$

This identity has been derived in the context of generalized geometry [21] where the corresponding reduction manifold admits a generalized parallelization. The derivation above follows the same approach as the one followed in the context of $11D$ supergravity [4], where it gave rise to the non-linear ansatz of the internal tensor A_{mnp}. One term on the left-hand side should be modified in view of the fact that there is a non-zero background four-form potential $\mathring{A}_{mnpq}$ because the five-form field strength is non-vanishing in this background. The term A_{mnpq} on the left-hand side should therefore be replaced by $A_{mnpq} - \mathring{A}_{mnpq}$.

Subsequently we continue to the twelve vector and twelve tensor fields that transform under SU(1,1), namely $C_{\mu\,\alpha}$, $C_{\mu\,\alpha m}$, $C_{\mu\nu}{}^{\alpha}$ and $C_{\mu\nu}{}^{\alpha m}$, which should be decomposed into the twelve vector and and twelve tensor fields that one expects in $5D$. However, in view of their number, it is not possible to expand these fields in terms of Killing vectors or tensors. Therefore we introduce the SO(6) vector fields $Y^{\hat{a}}(y)$ that satisfy $Y^{\hat{a}}(y)\,Y_{\hat{a}}(y) = 1$, whose parametrization in terms of the y^m is based on the same SO(6)/SO(5) coset representative as all other geometric quantities of S^5, such as the metric and the Killing vectors and tensors (see, e.g. [46]). In that case one can parametrize the remaining vector and tensor fields in terms of the twelve expected $5D$ fields,

$$\begin{aligned}
C_{\mu\,\alpha}(x,y) &= Y^{\hat{a}}(y)\,A_{\mu\,\alpha\hat{a}}(x)\,, \\
C_{\mu\,\alpha m}(x,y) &= \partial_m Y^{\hat{a}}(y)\,A_{\mu\,\alpha\hat{a}}(x)\,,
\end{aligned}$$

$$\begin{aligned} C_{\mu\nu}{}^{\alpha}(x,y) &= Y_{\hat{a}}(y)\, B_{\mu\nu}{}^{\alpha\hat{a}}(x)\,, \\ C_{\mu\nu}{}^{\alpha m}(x,y) &= \mathring{g}^{mn}\, \partial_n Y_{\hat{a}}(y)\, B_{\mu\nu}{}^{\alpha\hat{a}}(x)\,. \end{aligned} \tag{7.17}$$

A similar decomposition now applies to the generalized vielbeine $\mathcal{V}_{ij\,\alpha}$, $\mathcal{V}_{ij\,\alpha m}$, $\mathcal{V}^{\alpha\, ij}$ and $\mathcal{V}^{\alpha m\, ij}$ which appear in the variation of the above fields,

$$\begin{aligned} \mathcal{V}_{ij\,\alpha}(x,y) &= U_{ij\,\alpha\hat{a}}(x)\, Y^{\hat{a}}(y)\,, \\ \mathcal{V}_{ij\,\alpha m}(x,y) &= U_{ij\,\alpha\hat{a}}(x)\, \partial_m Y^{\hat{a}}(y)\,, \\ \mathcal{V}^{\alpha\, ij}(x,y) &= U^{\alpha\hat{a}\, ij}(x)\, Y_{\hat{a}}(y)\,, \\ \mathcal{V}^{\alpha m\, ij}(x,y) &= U^{\alpha\hat{a}\, ij}(x)\, \mathring{g}^{mn}\, \partial_n Y_{\hat{a}}(y)\,, \end{aligned} \tag{7.18}$$

where $U_{ij\,\alpha\hat{a}}(x)$ and $U^{\alpha\hat{a}\, ij}(x)$ are again related to specific components of the $\mathrm{E}_{6(6)}/\mathrm{USp}(8)$ coset space that appear in the $5D$ theory. They satisfy

$$U^{\alpha\hat{a}\, ij}(x)\, U_{ij\,\beta\hat{b}}(x) = \delta^{\alpha}{}_{\beta}\, \delta^{\hat{a}}{}_{\hat{b}}\,. \tag{7.19}$$

Now we consider the following identities that can be derived for the generalized vielbeine,

$$\begin{aligned} \bar{\Omega}^{ik}\, \bar{\Omega}^{jl}\, \mathcal{V}_{ij}{}^{m}\, \mathcal{V}_{kl\,\alpha n} &= \mathrm{i}\varepsilon_{\alpha\beta} A^{\beta}{}_{np}\, \bar{\mathcal{V}}^{ij\, m}\, \mathcal{V}_{ij}{}^{p}\,, \\ \varepsilon_{\alpha\gamma}\, \Omega_{ik}\, \Omega_{jl}\, \mathcal{V}^{\gamma ij}\, \mathcal{V}^{\beta\, kl} &= \frac{5}{4}\Delta^{-4/3}\big(\delta_{\alpha}{}^{\beta} - 2\, \phi_{\alpha}\phi^{\beta}\big)\,. \end{aligned} \tag{7.20}$$

From these identities we can derive the following results upon substituting the above truncation ansätze,

$$\begin{aligned} \Delta^{-2/3}\, A^{\alpha}{}_{mn} &= 2\mathrm{i}\, \varepsilon^{\alpha\beta}\, \bar{\Omega}^{ik}\, \bar{\Omega}^{jl}\, U_{ij}{}^{\hat{a}\hat{b}}(x)\, U_{kl\,\beta\hat{c}}(x)\, K^{p}{}_{\hat{a}\hat{b}}(y)\, g_{p[m}(x,y)\, \partial_{n]} Y^{\hat{c}}(y)\,, \\ \Delta^{-4/3}\big(\delta_{\alpha}{}^{\beta} - 2\, \phi_{\alpha}\phi^{\beta}\big) &= \frac{4}{5}\varepsilon_{\alpha\gamma}\, \Omega_{ik}\, \Omega_{jl}\, U^{\gamma\hat{a}\, ij}(x)\, U^{\beta\hat{b}\, kl}(x)\, Y_{\hat{a}}(y)\, Y_{\hat{b}}(y)\,. \end{aligned} \tag{7.21}$$

The first result has recently been derived based on generalized geometry [21], while the second result has been obtained long ago (under some mild assumptions) in [34] by using the same strategy as in this section.

It is clear that so far we have probed only part of the possible identities that can be derived based on the results of this paper. At the same time, the mutual consistency of the various implications should also be carefully investigated, in the same way as this was done for $11D$ supergravity. It should be interesting to pursue these questions further.

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A Decomposition of gamma matrices and spinors

We start from 32×32 hermitian gamma matrices $\breve{\Gamma}_A$, where $A = 1, 2, \ldots 10$, satisfying the Clifford algebra anti-commutation relation, $\{\breve{\Gamma}_A, \breve{\Gamma}_B\} = 2\,\delta_{AB}\,\mathbb{1}_{32}$, and proceed in a way that is independent of a specific representation for these gamma matrices. The hermitian chirality operator, $\breve{\Gamma}_{11}$, is defined by

$$\breve{\Gamma}_{11} = \mathrm{i}\breve{\Gamma}_1\,\breve{\Gamma}_2 \cdots \breve{\Gamma}_{10}\,, \tag{A.1}$$

and satisfies

$$\breve{\Gamma}_{11}{}^2 = \mathbb{1}_{32}\,, \qquad \{\breve{\Gamma}_A, \breve{\Gamma}_{11}\} = 0\,. \tag{A.2}$$

Moreover we note the identity,

$$\breve{\Gamma}^{ABCDE} = \frac{1}{120}\mathrm{i}\varepsilon^{ABCDEFGHIJ}\,\breve{\Gamma}_{FGHIJ}\,\breve{\Gamma}_{11}\,. \tag{A.3}$$

When considering compactifications from ten- to five-dimensional space-times, the $10D$ tangent space is decomposed accordingly into a direct product of two five-dimensional spaces, one corresponding to a five-dimensional space-time and one corresponding to a five-dimensional internal space. Since we are dealing with spinor fields, it is then important to identify the gamma matrices appropriate to this product space in terms of the original $10D$ gamma matrices.

To do so one first decomposes the gamma matrices into two sets, $\breve{\Gamma}_\alpha$ with $\alpha = 1, 2, \ldots, 5$ and $\breve{\Gamma}_{a+5}$ with $a = 1, 2, \ldots, 5$.[5] Subsequently one introduces hermitian matrices associated with the two five-dimensional sectors,

$$\tilde{\gamma} = \breve{\Gamma}_1\,\breve{\Gamma}_2\,\breve{\Gamma}_3\,\breve{\Gamma}_4\,\breve{\Gamma}_5\,, \qquad \tilde{\Gamma} = \breve{\Gamma}_6\,\breve{\Gamma}_7\,\breve{\Gamma}_8\,\breve{\Gamma}_9\,\breve{\Gamma}_{10}\,. \tag{A.4}$$

which satisfy the following properties,

$$\tilde{\gamma}^2 = \mathbb{1}_{32}\,, \quad \tilde{\Gamma}^2 = \mathbb{1}_{32}\,, \quad \{\tilde{\gamma}, \tilde{\Gamma}\} = 0\,, \quad \breve{\Gamma}_{11} = \mathrm{i}\tilde{\gamma}\,\tilde{\Gamma}\,. \tag{A.5}$$

Subsequently one defines two sets of mutually commuting *hermitian* gamma matrices,

$$\hat{\gamma}_\alpha = \mathrm{i}\breve{\Gamma}_\alpha\,\tilde{\Gamma}\,, \qquad \hat{\Gamma}_a = \mathrm{i}\breve{\Gamma}_{a+5}\,\tilde{\gamma}\,, \tag{A.6}$$

so that $\{\hat{\gamma}_\alpha, \hat{\gamma}_\beta\} = 2\,\delta_{\alpha\beta}\mathbb{1}_{32}$, $\{\hat{\Gamma}_a, \hat{\Gamma}_b\} = 2\,\delta_{ab}\mathbb{1}_{32}$, and $[\hat{\gamma}_\alpha, \hat{\Gamma}_a] = 0$. The matrices $\hat{\gamma}_\alpha$ will refer to the five-dimensional space-time (to account for the signature one may write one of the five gamma matrices, say $\hat{\gamma}^1$ as $\mathrm{i}\hat{\gamma}^0$) and the matrices $\hat{\Gamma}_a$ to the five-dimensional internal space. The matrices $\hat{\gamma}_\alpha$ and $\hat{\Gamma}_a$ commute with $\breve{\Gamma}_{11}$, as one can easily verify from the above equations. It is important to note that

$$\begin{aligned} \hat{\gamma}_{[\alpha}\hat{\gamma}_\beta\hat{\gamma}_\gamma\hat{\gamma}_\delta\hat{\gamma}_{\tau]} &= \varepsilon_{\alpha\beta\gamma\delta\tau}\,\Gamma_{11}\,, \\ \hat{\Gamma}_{[a}\hat{\Gamma}_b\hat{\Gamma}_c\hat{\Gamma}_d\hat{\Gamma}_{e]} &= -\,\varepsilon_{abcde}\,\Gamma_{11}\,, \end{aligned} \tag{A.7}$$

[5]At this stage there is no difference between upper and lower indices, so that we are dealing with a positive Euclidean metric.

where $\varepsilon_{12345} = +1$. Obviously, by choosing an explicit representation for the $10D$ gamma matrices, one obtains explicit expressions for the various matrices that we have defined above which will reflect their properties.

Let us now consider the charge conjugation matrix. In ten dimensions there exist two possible options for the charge conjugation matrix, denoted by $\breve{C}_\pm$, satisfying

$$\breve{C}_\pm \, \breve{\Gamma}_A \breve{C}_\pm^{-1} = \pm \breve{\Gamma}_A{}^{\mathrm{T}} , \quad \breve{C}_\pm{}^{\mathrm{T}} = \pm \breve{C}_\pm , \quad \breve{C}_\pm{}^\dagger = \breve{C}_\pm^{-1} , \tag{A.8}$$

which lead to the following results,

$$\breve{C}_\pm \breve{\Gamma}_{11} \breve{C}_\pm^{-1} = -\breve{\Gamma}_{11}{}^{\mathrm{T}} , \quad \breve{C}_\pm \tilde{\gamma} \breve{C}_\pm^{-1} = \pm \tilde{\gamma}^{\mathrm{T}} , \quad \breve{C}_\pm \tilde{\Gamma} \breve{C}_\pm^{-1} = \pm \tilde{\Gamma}^{\mathrm{T}} . \tag{A.9}$$

From the first equation (A.9), it follows that $\breve{C}_\pm$ satisfy

$$(\breve{C}_\pm \breve{\Gamma}_{11})^{\mathrm{T}} = \breve{\Gamma}_{11}{}^{\mathrm{T}} \, \breve{C}_\pm{}^{\mathrm{T}} = \mp (\breve{C}_\pm \, \breve{\Gamma}_{11}) , \tag{A.10}$$

so that the two options for the charge conjugation matrix can simply be related by multiplication with Γ_{11}. Furthermore we note that both $\breve{C}_\pm \tilde{\Gamma}$ and $\breve{C}_\pm \tilde{\gamma}$ are symmetric and unitary matrices. Up to a phase factor, these can act as the charge conjugation matrices in the $5D$ context, as is demonstrated by

$$(\breve{C}_\pm \tilde{\Gamma}) \hat{\gamma}_\alpha (\breve{C}_\pm \tilde{\Gamma})^{-1} = \hat{\gamma}_\alpha{}^{\mathrm{T}} , \quad (\breve{C}_\pm \tilde{\Gamma}) \hat{\Gamma}_a (\breve{C}_\pm \tilde{\Gamma})^{-1} = \hat{\Gamma}_a{}^{\mathrm{T}} . \tag{A.11}$$

Similar relations hold for $(\breve{C}_\pm \tilde{\gamma})$.

To appreciate the significance of this result, let us consider the definition of the Dirac conjugate in the $5D$ context, defined by $\psi^\dagger \mathrm{i} \hat{\gamma}^0$, where $\hat{\gamma}^0$ was related to $\hat{\gamma}^1$ as explained below (A.6). From these relations it follows straightforwardly that the $5D$ Dirac conjugate $\bar{\psi}\big|_{5D}$ is related to the $10D$ conjugate according to

$$\bar{\psi}\big|_{5D} = \mathrm{i} \bar{\psi}\big|_{10D} \, \tilde{\Gamma} . \tag{A.12}$$

Consequently, identifying the Majorana conjugate defined in (2.17) in the $10D$ context with the one in the $5D$ context, one concludes that the charge conjugation matrix in the $5D$ context equals

$$\hat{C} = \mathrm{i} \tilde{\Gamma}^{\mathrm{T}} \breve{C}_\pm = \pm \mathrm{i} \breve{C}_\pm \, \tilde{\Gamma} , \tag{A.13}$$

so that $\hat{C}^{-1} \big[\bar{\psi}|_{5\mathrm{D}} \big]^{\mathrm{T}} = \psi^{\mathrm{c}}$, and likewise $\psi^{\mathrm{T}} = \bar{\psi}^{\mathrm{c}}|_{5\mathrm{D}} \, \hat{C}^{-1}$. As a consequence the two commuting sets of 32×32 gamma matrices, $\hat{\gamma}_\alpha$ and $\hat{\Gamma}_a$, satisfy the relations known from five dimensions,

$$\hat{C} \hat{\gamma}_\alpha \hat{C}^{-1} = \hat{\gamma}_\alpha{}^{\mathrm{T}} , \qquad \hat{C} \hat{\Gamma}_a \hat{C}^{-1} = \hat{\Gamma}_a{}^{\mathrm{T}} , \qquad \hat{C}^{\mathrm{T}} = \hat{C} , \qquad \hat{C}^\dagger = \hat{C}^{-1} . \tag{A.14}$$

This leads to the rearrangement formula,

$$\bar{\chi} \, \Gamma \psi = -\bar{\psi}^{\mathrm{c}} \, \hat{C}^{-1} \Gamma^{\mathrm{T}} \hat{C} \, \chi^{\mathrm{c}} , \tag{A.15}$$

where Γ denotes any matrix in the spinor space, which in all cases of interest takes the form a product of gamma matrices $\hat{\Gamma}^a$ and $\hat{\gamma}_\alpha$. Observe that the new charge conjugation matrix is not anti-symmetric, as one might expect on the basis of a single irreducible $5D$ Clifford algebra representation. We return to this issue shortly.

In this paper we discuss the type-IIB theory where the spinor fields are chiral and complex. Therefore the above formulae have to be projected on an eigenspace of Γ_{11} and the effective $5D$ gamma matrices defined in (A.6) are consistent with the $10D$ chirality constraint on the spinor fields, because they are proportional to an even number of the original $10D$ gamma matrices. However, it is important to realize that IIB supergravity contains independent spinor fields of *opposite* chirality, namely ψ_M and λ. This leads to a subtlety in view of (A.7), which indicates that different chirality spinors involve inequivalent gamma matrix representations in $5D$. However, one has to keep in mind that the chirality assignment can easily be changed in the $5D$ context by redefining the spinors by multiplication with one of the matrices (A.4).

Let us now assume that we are starting from $10D$ with fermion fields of *positive* chirality. Hence we can choose a Weyl basis where $\breve{\Gamma}_{11}$ is diagonal and make use of the fact that it commutes with the mutually commuting gamma matrices $\hat{\gamma}_\alpha$ and $\hat{\Gamma}_a$. Hence we write

$$\hat{\gamma}_\alpha = \sigma_3 \otimes \gamma_\alpha \otimes \mathbb{1}_4\,, \qquad \hat{\Gamma}_a = \sigma_3 \otimes \mathbb{1}_4 \otimes \Gamma_a\,, \tag{A.16}$$

where $\breve{\Gamma}_{11} = \sigma_3 \otimes \mathbb{1}_{16}$ and γ_α and Γ_a are 4×4 matrices. It then follows from (A.7) that they define irreducible representations of the respective Clifford algebras, as

$$\gamma_{[\alpha}\gamma_\beta\gamma_\gamma\gamma_\delta\gamma_{\tau]} = \varepsilon_{\alpha\beta\gamma\delta\tau}\,\mathbb{1}_4\,, \qquad \Gamma_{[a}\Gamma_b\Gamma_c\Gamma_d\Gamma_{e]} = -\varepsilon_{abcde}\,\mathbb{1}_4\,. \tag{A.17}$$

The $10D$ chiral spinors thus transform under the direct product group $\mathrm{Spin}(1,4)\times\mathrm{USp}(4)$, whose generators are provided by the anti-symmetrized products of gamma matrices, $\gamma_{\alpha\beta}$ and Γ_{ab}, respectively. Correspondingly the charge conjugation matrix $\hat{C}$ can be written (adjusting possible phase factors) as the direct product of the two $5D$ *anti-symmetric* charge conjugation matrices,

$$\hat{C}_{(16)} = C \otimes \Omega_{(4)}\,, \tag{A.18}$$

where C denotes the anti-symmetric charge conjugation matrix for a $5D$ space-time spinor and $\Omega_{(4)}$ is the symplectic matrix that is invariant under the USp(4) R-symmetry. In this case we may write (A.14) as

$$C\,\gamma_\alpha C^{-1} = {\gamma_\alpha}^{\mathrm{T}}\,, \qquad \Omega_{(4)}\,\Gamma_a \Omega_{(4)}^{-1} = {\Gamma_a}^{\mathrm{T}}\,. \tag{A.19}$$

However, the chiral spinors are complex which implies that the fields (ψ,ψ^{c}), which constitute the 32-component spinor Ψ, can again be rearranged in a pseudo-real form as in (2.20). The doubling of field components enables one to realize the extension of the R-symmetry group from $\mathrm{USp}(4)\times\mathrm{U}(1)$ to USp(8). It then follows from (2.20) that the extended USp(8) invariant tensor must take the form

$$\Omega = \Omega_{(4)} \otimes \sigma_1\,. \tag{A.20}$$

Consequently, (2.20) and (A.18) imply the symplectic Majorana condition,

$$C^{-1}\bar{\Psi}^{\mathrm{T}} = \Omega\,\Psi\,, \tag{A.21}$$

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where Ω is an 8×8 anti-symmetric matrix. Both matrices C and Ω are anti-symmetric and unitary.

We close this appendix with some additional definitions that will be useful in the next appendix B. First of all we write the anti-symmetric tensor $\Omega_{(4)}$ as $\Omega_{(4)IJ}$ and its complex conjugate as $\bar{\Omega}_{(4)}{}^{IJ}$, so that $\Omega_{(4)IJ}\,\bar{\Omega}_{(4)}{}^{JK} = -\delta_I{}^K$, where $I, J, K = 1, \ldots, 4$. The gamma matrices Γ_a are then written as $\Gamma_a{}^I{}_J$, so that

$$\Omega_{(4)}{}^{\mathrm{T}} = -\Omega_{(4)}\,, \quad (\Omega_{(4)}\,\Gamma_a)^{\mathrm{T}} = -(\Omega_{(4)}\,\Gamma_a)\,, \quad (\Omega_{(4)}\,\Gamma_{ab})^{\mathrm{T}} = (\Omega_{(4)}\,\Gamma_{ab})\,, \tag{A.22}$$

with similar relations for $(\Gamma_a\,\bar{\Omega}_{(4)})^{IJ}$ and $(\Gamma_{ab}\,\bar{\Omega}_{(4)})^{IJ}$. The six matrices $\Omega_{(4)IJ}$ and $(\Omega_{(4)}\,\Gamma_a)_{IJ}$ form a complete set of 4×4 anti-symmetric matrices, and the ten matrices $(\Omega_{(4)}\,\Gamma_{ab})_{IJ}$ a complete set of 4×4 symmetric matrices This leads to the completeness relations

$$\begin{aligned} \Omega_{(4)IJ}\,\bar{\Omega}_{(4)}{}^{KL} + (\Omega_{(4)}\,\Gamma_a)_{IJ}\,(\Gamma^a\,\bar{\Omega}_{(4)})^{KL} &= 4\,\delta_{[I}{}^K\,\delta_{J]}{}^L\,, \\ (\Omega_{(4)}\,\Gamma_{ab})_{IJ}\,(\Gamma^{ab}\,\bar{\Omega}_{(4)})^{KL} &= 8\,\delta_{(I}{}^K\,\delta_{J)}{}^L\,. \end{aligned} \tag{A.23}$$

B The R-symmetry group and the fermion representations

In the previous appendix we considered a $10D$ chiral spinor and described its properties in the context of a product of a five-dimensional space-time and a five-dimensional internal space. The gamma matrices and the charge conjugation matrices were decomposed accordingly. The $10D$ spinors then transform under a subgroup of the original $\mathrm{Spin}(1,9)$ transformations consisting of the $\mathrm{Spin}(1,4)$ group associated with the $5D$ space-time and the group $\mathrm{USp}(4)$ associated with the internal space.

However, $\mathrm{USp}(4)$ is not the full automorphism group (or R-symmetry group) of the eight symplectic Majorana spinors. This group is actually equal to $\mathrm{USp}(8)$, which consists of the unitary transformations that leave the symplectic and unitary tensor Ω, invariant. The generators of this group can be easily identified in terms of direct products of the 4×4 gamma matrices Γ_a, defined in (A.16), their anti-symmetrized products Γ_{ab} and the unit matrix $\mathbb{1}_4$, and the 2×2 matrices $(\mathbb{1}_2, \sigma_1, \sigma_2, \sigma_3)$. As a result one derives all the 36 generators of the Lie algebra $\mathfrak{usp}(8) = \mathfrak{su}(8) \cap \mathfrak{sp}(8, \mathbb{R})$, by constructing the complete set of traceless and anti-hermitian matrices that preserve the symplectic form Ω,

$$\begin{aligned} &T \equiv \mathrm{i}\mathbb{1}_4 \otimes \sigma_3\,, && T_a \equiv \mathrm{i}\Gamma_a \otimes \sigma_3\,, && \\ &T_{ab}{}^0 \equiv \Gamma_{ab} \otimes \mathbb{1}_2\,, && T_{ab}{}^1 \equiv \Gamma_{ab} \otimes \sigma_1\,, && T_{ab}{}^2 \equiv \Gamma_{ab} \otimes \sigma_2\,. \end{aligned} \tag{B.1}$$

As expected these matrices close under commutation,

$$\begin{aligned} &[T, T_{ab}{}^1] = -2\,T_{ab}{}^2\,, && [T, T_{ab}{}^2] = 2\,T_{ab}{}^1\,, && [T_a, T_b] = -2\,T_{ab}{}^0\,, \\ &[T_a, T_{bc}{}^0] = 4\,\delta_{a[b}\,T_{c]}\,, && [T_a, T_{bc}{}^1] = \varepsilon_{abcde}\,T^{de\,2}\,, && [T_a, T_{bc}{}^2] = -\varepsilon_{abcde}\,T^{de\,1}\,, \\ &[T_{ab}{}^0, T^{cd\,0}] = -8\,\delta_{[a}{}^{[c}\,T_{b]}{}^{d]0}\,, && [T_{ab}{}^0, T^{cd\,1}] = -8\,\delta_{[a}{}^{[c}\,T_{b]}{}^{d]1}\,, && [T_{ab}{}^0, T^{cd\,2}] = -8\,\delta_{[a}{}^{[c}\,T_{b]}{}^{d]2}\,, \end{aligned}$$

$$
\begin{aligned}
&[T_{ab}{}^1, T^{cd\,1}] = -8\,\delta_{[a}{}^{[c}\,T_{b]}{}^{d]0}, \quad [T_{ab}{}^2, T^{cd\,2}] = -8\,\delta_{[a}{}^{[c}\,T_{b]}{}^{d]0}, \\
&[T_{ab}{}^1, T^{cd\,2}] = -2\,\varepsilon_{abcde}\,T^e.
\end{aligned} \tag{B.2}
$$

Observe that the generators are anti-hermitian and the structure constants are real, in agreement with $\mathfrak{usp}(8)$ being a real form. The $T_{ab}{}^0$ are the generators of the group USp(4) $\cong$ SO(5). When extended with the generators T_a one obtains the group SU(4) $\cong$ SO(6) which obviously commutes with the generator T. As we will exhibit later, T corresponds to the SO(6) chirality operator. The latter commutes with the U(1) transformations of the original $10D$ theory. Clearly SU(4) $\times$ U(1) is a maximal subgroup of USp(8).

A chiral $10D$ spinor Ψ can be decomposed into eight $5D$ symplectic Majorana spinors ψ^A, where $A = 1, \ldots, 8$. Note that from now on we employ indices $A, B, \ldots$ to label the symplectic Majorana spinors. The same indices were previously used in the $10D$ theory (in particular in section 2 and appendix A) to denote the $10D$ tangent-space components. This should not give rise to confusion in view of the fact that the $10D$ tangent space will no longer play a role in what follows. In view of the direct-product structure indicated in (B.1) the indices A can be written as index pairs $A = (I\alpha)$, where $I = 1, \ldots, 4$ are USp(4) indices and $\alpha = +, -$. Here $\alpha = +$ ($\alpha = -$) indicates that we are dealing with a chiral (anti-chiral) SO(6) spinor with positive (negative) U(1) charge.[6]. Based on this direct-product structure the eight $5D$ gravitini $\psi_\mu{}^A$ transform under the USp(8) R-symmetry group with generators that can be read off directly from (B.1). It is thus clear that that each of the $\psi_\mu{}^A$ decomposes into two components of opposite SO(6) chirality which therefore carry opposite values of the U(1) charge. This fact enables us to unambiguously identify the various chiral fermionic components on the basis of this charge. Furthermore we note that the symplectic Majorana constraint (A.21) relates fermion fields of opposite U(1) charges, which is consistent with the form of the symplectic matrix Ω defined in (A.20). For instance, for the gravitini we have

$$
C^{-1}\bar{\psi}_{\mu\,I+}{}^{\mathrm{T}} = (\Omega_{(4)})_{IJ}\,\psi_\mu{}^{J-}\,, \tag{B.3}
$$

where C denotes the charge conjugation matrix associated with the five-dimensional space-time.

Let us now turn to the spin-1/2 fermions which originate from the fields $(\psi_a, \psi_a{}^{\mathrm{c}})$ and $\lambda, \lambda^{\mathrm{c}}$ and constitute 48 independent $5D$ symplectic Majorana spinors. From $5D$ maximal supergravity we know that these spinors can be written as a symplectic traceless, fully anti-symmetric three-rank USp(8) tensor χ^{ABC}. This is consistent with the fact that the spin-1/2 fields carry U(1) charges $\pm 1/2$ and $\pm 3/2$. We intend to determine the (linear) relation between the components of χ^{ABC} and the fields $\psi_a{}^A$ and λ^A by making use of the fact that these fields do all transform consistently under the action of the maximal subgroup SU(4) $\times$ U(1) of USp(8). To see how this works let us present the branching of

[6]We ignore the various redefinitions of the spinors that are considered in section 3. These redefinitions should be performed before making the decompositions described in this appendix, but their precise details are not relevant here.

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${\psi_\mu}^A$ and χ^{ABC} under the SU(4) $\times$ U(1) subgroup,

$$
\begin{aligned}
\mathbf{8} &\overset{\mathrm{SU(4)\times U(1)}}{\longrightarrow} \left(\mathbf{4}, \frac{1}{2}\right) \oplus \left(\overline{\mathbf{4}}, -\frac{1}{2}\right), \\
\mathbf{48} &\overset{\mathrm{SU(4)\times U(1)}}{\longrightarrow} \left(\overline{\mathbf{4}}, \frac{3}{2}\right) \oplus \left(\mathbf{4}, -\frac{3}{2}\right) \oplus \left(\mathbf{20}, \frac{1}{2}\right) \oplus \left(\overline{\mathbf{20}}, -\frac{1}{2}\right).
\end{aligned} \tag{B.4}
$$

The chiral representations on the right-hand side are now unambiguously identified by the corresponding U(1) charge, so that they must correspond to the fields ψ_μ, ${\psi_\mu}^{\mathrm{c}}$, and λ, λ^{c}, ψ_a and ${\psi_a}^{\mathrm{c}}$, respectively.[7] To determine the precise relation, we again write the indices of the symplectic Majorana field χ^{ABC} by employing the direct-product representation introduced before, with $A = I\alpha$, $B = J\beta$ and $C = K\gamma$. Since α, β, γ take only two possible index values, at least two of them must always be equal. Hence we may distinguish the fields $\chi^{I\pm\, J\pm\, K\pm}$, which must be fully anti-symmetric in the indices I, J, K, and thus correspond to $4+4$ symplectic Majorana fields, and the fields $\chi^{I\pm\, J\pm\, K\mp}$, which are anti-symmetric in the indices I, J, and thus define $24+24$ fields. The remaining fields $\chi^{I\alpha\, J\beta\, K\gamma}$ follow then from imposing the overall anti-symmetry. However, unlike the fields $\chi^{I\pm\, J\pm\, K\pm}$, the fields $\chi^{I\pm\, J\pm\, K\mp}$ are not manifestly traceless with respect to contractions with the symplectic matrix Ω. This implies that one must impose the additional condition

$$
\chi^{I\pm\, J\pm\, K\mp}\,(\Omega_{(4)})_{JK} = 0\,, \tag{B.5}
$$

which reduces the number of independent spinors in this sector to $20+20$, as it should.

Let us first analyze the correspondence for the spinors χ^{ABC} with positive U(1) charge $+\frac{3}{2}$, which must be linearly related to the $10D$ spinor λ. The former must be given by $\chi^{I+\, J+\, K+}$, which must necessarily be fully anti-symmetric in USp(4) indices. From (A.22) one then concludes that $\chi^{I+\, J+\, K+}$ can be decomposed into two terms, namely $(\bar\Omega_{(4)})^{[IJ}\,(\lambda^{\mathrm{c}})^{K]}$ and $(\Gamma^a\bar\Omega_{(4)})^{[IJ}(\Gamma_a\lambda^{\mathrm{c}})^{K]}$. However, the first completeness relation (A.23) leads to

$$
(\Gamma^a\bar\Omega_{(4)})^{IJ}(\Gamma_a\psi)^K = -4\,(\bar\Omega_{(4)})^{K[I}\,\psi^{J]} - (\bar\Omega_{(4)})^{IJ}\,\psi^K\,, \tag{B.6}
$$

for an arbitrary USp(4) spinor ψ, so that the two terms are in fact related. Hence we may adopt the following ansatz,

$$
\chi^{I+\, J+\, K+} = c_{3/2}\,(\bar\Omega_{(4)})^{[IJ}\,\lambda^{K]}\,, \tag{B.7}
$$

where $c_{3/2}$ is a complex proportionality factor which is undetermined at this stage. The fields with charge $-\frac{3}{2}$ are then defined through the symplectic Majorana condition,

$$
\begin{aligned}
\chi^{I-\, J-\, K-} &\equiv -\,(\bar\Omega_{(4)})^{IL}\,(\bar\Omega_{(4)})^{JM}\,(\bar\Omega_{(4)})^{KN}\,C^{-1}\,{\bar\chi_{L+\, M+\, N+}}^{\mathrm{T}} \\
&= \bar c_{3/2}\,(\bar\Omega_{(4)})^{[IJ}\,(\lambda^{\mathrm{c}})^{K]}\,.
\end{aligned} \tag{B.8}
$$

The relation between the spinors $\chi^{I+\, J+\, K-}$ and ψ_a with U(1) charge $+\frac{1}{2}$ is more subtle. First consider the following ansatz,

$$
\chi^{I+\, J+\, K-} = c_{1/2}\left[(\Gamma^a\bar\Omega_{(4)})^{IJ}\,(\hat\psi_a)^K - (\bar\Omega_{(4)})^{IJ}\,(\Gamma^a\,\hat\psi_a)^K\right], \tag{B.9}
$$

[7] A vector-spinor in odd dimension d can consistently transform under SO$(d+1)$ by describing it as an irreducible chiral vector-spinor in $d+1$ dimensions.

where $\hat\psi_a = \psi_a + \alpha\,\Gamma_a\Gamma^b\psi_b$ with α an undetermined parameter, so that we are now dealing with two new parameters, $c_{1/2}$ and α. The linear combination in (B.9) is chosen such that the USp(8) constraint (B.5) is satisfied. An alternative version of (B.9), which is the one that we will actually use, is

$$\begin{aligned}\chi^{I+J+K-} =& c_{1/2}\left[(\Gamma^a\bar\Omega_{(4)})^{IJ}\,(\psi_a)^K - (\bar\Omega_{(4)})^{IJ}\,(\Gamma^a\,\psi_a)^K\right]\\ &+ c'_{1/2}\left[(\bar\Omega_{(4)})^{IJ}\,(\Gamma^a\psi_a)^K + \frac{2}{3}(\bar\Omega_{(4)})^{K[I}\,(\Gamma^a\psi_a)^{J]}\right],\end{aligned} \tag{B.10}$$

but also this expression can be rewritten by making use of the identity

$$(\Gamma^a\,\bar\Omega_{(4)})^{[IJ}\,(\psi_a)^{K]} = -(\bar\Omega_{(4)})^{[IJ}(\Gamma^a\,\psi_a)^{K]}\,. \tag{B.11}$$

As before we define the spinor components with U(1) charge $-\frac{1}{2}$ by

$$\begin{aligned}\chi^{I-J-K+} \equiv& -(\bar\Omega_{(4)})^{IL}\,(\bar\Omega_{(4)})^{JM}\,(\bar\Omega_{(4)})^{KN}\,C^{-1}\,\bar\chi_{L+M+N-}{}^{\mathrm{T}}\\ =& \bar c_{1/2}\left[(\Gamma^a\bar\Omega_{(4)})^{IJ}\,(\psi_a{}^{\mathrm{c}})^K - (\bar\Omega_{(4)})^{IJ}\,(\Gamma^a\,\psi_a{}^{\mathrm{c}})^K\right]\\ &+ \bar c'_{1/2}\left[(\bar\Omega_{(4)})^{IJ}\,(\Gamma^a\psi_a{}^{\mathrm{c}})^K + \frac{2}{3}(\bar\Omega_{(4)})^{K[I}\,(\Gamma^a\psi_a{}^{\mathrm{c}})^{J]}\right].\end{aligned} \tag{B.12}$$

Hence we have obtained the linear relation between χ^{ABC} and the original $10D$ spinors, depending on three unknown complex constants, $c_{3/2}$, $c_{1/2}$ $c'_{1/2}$. Their values are determined in section 5, as we will be discussing at the end of this appendix.

We will now merge the chiral and anti-chiral spinors with opposite U(1) charges into eight-component symplectic Majorana spinors. In that case it is convenient to introduce SO(6) gamma matrices and chiral projection operators. The 8×8 gamma matrices $(\boldsymbol{\Gamma}_{\hat a})^A{}_B$, where $\hat a = 1,\ldots,6$, are defined in terms of direct products of 4×4 and 2×2 matrices, just as in (B.1),

$$\boldsymbol{\Gamma}_a \equiv \Gamma_a\otimes\sigma_1\,,\qquad \boldsymbol{\Gamma}_6 \equiv \mathbb{1}_4\otimes\sigma_2\,. \tag{B.13}$$

These (hermitian) gamma matrices satisfy the Clifford property

$$\left\{\boldsymbol{\Gamma}_{\hat a}\,,\boldsymbol{\Gamma}_{\hat b}\right\} = 2\,\delta_{\hat a\hat b}\,\mathbb{1}_8\,. \tag{B.14}$$

and satisfy the following charge-conjugation properties,

$$\Omega\,\boldsymbol{\Gamma}_{\hat a}\,\Omega^{-1} = \boldsymbol{\Gamma}_{\hat a}{}^{\mathrm{T}}\,,\ \text{with}\ \Omega^{\mathrm{T}} = -\Omega\,,\quad \Omega^{-1} = -\bar\Omega\,, \tag{B.15}$$

where the anti-symmetric charge conjugation matrix Ω_{AB} was defined in (A.20). The chirality operator $\boldsymbol{\Gamma}_7$ is obtained in the standard way,

$$\boldsymbol{\Gamma}_{[\hat a}\,\boldsymbol{\Gamma}_{\hat b}\cdots\boldsymbol{\Gamma}_{\hat f]} = -\mathrm{i}\varepsilon_{\hat a\hat b\hat c\hat d\hat e\hat f}\,\boldsymbol{\Gamma}_7\,,\quad \text{where}\ \boldsymbol{\Gamma}_7 = \mathbb{1}_4\otimes\sigma_3\,. \tag{B.16}$$

Observe that $\boldsymbol{\Gamma}_7$ is hermitian and behaves under charge conjugation as $\Omega\,\boldsymbol{\Gamma}_7\,\Omega^{-1} = -\boldsymbol{\Gamma}_7{}^{\mathrm{T}}$. Furthermore $\boldsymbol{\Gamma}_7$ coincides with the U(1) charge that was already present in the original $10D$ theory.

The gamma matrices $\mathbf{\Gamma}_{\hat a}$ and their multiple anti-symmetrized products define a complete basis for matrices in the 8-dimensional spinor space. They can conveniently be decomposed into 28 anti-symmetric matrices Ω, $\Omega\mathbf{\Gamma}_{\hat a}$, $\Omega\mathbf{\Gamma}_{\hat a}\mathbf{\Gamma}_7$ and $\Omega\mathbf{\Gamma}_{\hat a\hat b}\mathbf{\Gamma}_7$, and 36 symmetric matrices $\Omega\mathbf{\Gamma}_7$, $\Omega\mathbf{\Gamma}_{\hat a\hat b}$ and $\Omega\mathbf{\Gamma}_{\hat a\hat b\hat c}$. The latter are related to the anti-hermitian generators of USp(8) that were already defined in (B.1),

$$\begin{aligned} T &= \mathrm{i}\mathbf{\Gamma}_7\,, \qquad & T_a &= \mathbf{\Gamma}_{a6}\,, \qquad & T_{ab}{}^1 &= \frac{1}{6}\varepsilon_{abcde6}\,\mathbf{\Gamma}^{cde}\,, \\ & & T_{ab}{}^0 &= \mathbf{\Gamma}_{ab}\,, & T_{ab}{}^2 &= \mathbf{\Gamma}_{ab6}\,. \end{aligned} \tag{B.17}$$

We have now obtained a parametrization of the relation between the fields χ^{ABC} and the fields λ, λ^{c}, ψ_a and $\psi_a{}^{\mathrm{c}}$ originating from the $10D$ theory in terms of (anti-)chiral components. This relation is in accordance with the $\mathrm{SU}(4)\times\mathrm{U}(1)$ branching of the spinor fields presented in (B.4). The resulting expressions for given charges were given in (B.7), (B.8), (B.10), (B.12), which can be converted in terms of the SO(6) gamma matrices $\mathbf{\Gamma}_{\hat a}$. Since we have established this relation for chiral and anti-chiral components separately, it is convenient to introduce chiral projection operators

$$\mathbb{P}_\pm = \frac{1}{2}\big(\mathbb{1}\pm\mathbf{\Gamma}_7\big)\,. \tag{B.18}$$

The spinor χ^{ABC} is subsequently decomposed in tri-spinors with all possible chiralities,

$$\chi^{ABC} = \chi^{ABC}_{+++} + \chi^{ABC}_{---} + \chi^{ABC}_{++-} + \chi^{ABC}_{+-+} + \chi^{ABC}_{-++} + \chi^{ABC}_{--+} + \chi^{ABC}_{-+-} + \chi^{ABC}_{+--}\,. \tag{B.19}$$

For the spinors with U(1) charge equal to $+3/2$ and $+1/2$ we derive, respectively,

$$\begin{aligned} \chi^{ABC}{}_{(+++)} =&\, \mathrm{i}\,c_{3/2}\,\mathbb{P}_+{}^A{}_D\,\mathbb{P}_+{}^B{}_E\,\mathbb{P}_+{}^C{}_F\,\big[\mathbf{\Gamma}_7\mathbf{\Gamma}_6\,\bar\Omega\big]^{[DE}\,\lambda^{F]}\,, \\ \chi^{ABC}{}_{(++-)} =&\, \mathrm{i}\,c_{1/2}\,\mathbb{P}_+{}^A{}_D\,\mathbb{P}_+{}^B{}_E\,\mathbb{P}_-{}^C{}_F\big[\big[\mathbf{\Gamma}^a\,\bar\Omega\big]^{DE}\,\big(\mathbf{\Gamma}_7\mathbf{\Gamma}_6\,\psi_a\big)^F - \big[\mathbf{\Gamma}_7\mathbf{\Gamma}_6\,\bar\Omega\big]^{DE}\,\big(\mathbf{\Gamma}^a\,\psi_a\big)^F\big] \\ &+ \mathrm{i}\,c'_{1/2}\,\mathbb{P}_+{}^A{}_D\,\mathbb{P}_+{}^B{}_E\,\mathbb{P}_-{}^C{}_F\Big[\big[\mathbf{\Gamma}_7\mathbf{\Gamma}_6\,\bar\Omega\big]^{DE}\,\big(\mathbf{\Gamma}^a\,\psi_a\big)^F - \frac{2}{3}\,\bar\Omega^{F[D}\,\big[\mathbf{\Gamma}_7\mathbf{\Gamma}^{a6}\psi_a\big]^{E]}\Big]\,, \end{aligned} \tag{B.20}$$

where the spinors λ and ψ_a are now 8-component spinors consisting of $(\lambda,\lambda^{\mathrm{c}})$ and $(\psi_a,\psi_a{}^{\mathrm{c}})$. The labels $(+++)$ and $(++-)$ on the left-hand side indicate how the indices are contracted with the chiral projectors. Note that the combinations $(+-+)$ and $(-++)$ are related upon interchanging the indices A, B, C correspondingly. The corresponding spinors with charges $-3/2$ and $-1/2$ read the same with $c_{3/2}$, $c_{1/2}$ and $c'_{1/2}$ replaced by their complex conjugates and with opposite projectors.

Confronting the above decompositons to the equations (5.7) uniquely determines the three constants to $c_{3/2} = -\frac{3}{4}$, $c_{1/2} = -\frac{1}{4}$ and $c'_{1/2} = -\frac{1}{2}$. The corresponding expression for χ^{ABC} equals

$$\begin{aligned} \chi^{ABC} =& -\frac{3}{8}\mathrm{i}\big[\big(\mathbf{\Gamma}_6\,\bar\Omega\big)^{[AB}\,\big(\mathbf{\Gamma}_7\lambda\big)^{C]} + \big(\mathbf{\Gamma}_7\mathbf{\Gamma}_6\,\bar\Omega\big)^{[AB}\,\lambda^{C]}\big] \\ & -\frac{3}{8}\mathrm{i}\big[\big(\mathbf{\Gamma}^a\,\bar\Omega\big)^{[AB}\,\big(\mathbf{\Gamma}_7\mathbf{\Gamma}_6\psi_a\big)^{C]} - \big(\mathbf{\Gamma}_7\mathbf{\Gamma}^a\,\bar\Omega\big)^{[AB}\,\big(\mathbf{\Gamma}_6\psi_a\big)^{C]}\big] \end{aligned}$$

$$
\begin{aligned}
&-\frac{3}{8}\mathrm{i}\big[(\mathbf{\Gamma}_7\mathbf{\Gamma}_6\,\bar{\Omega})^{[AB}\,(\mathbf{\Gamma}^a\psi_a)^{C]}-(\mathbf{\Gamma}_6\,\bar{\Omega})^{[AB}\,(\mathbf{\Gamma}_7\mathbf{\Gamma}^a\psi_a)^{C]}\big]\\
&-\frac{1}{2}\mathrm{i}\,\bar{\Omega}^{[AB}\,(\mathbf{\Gamma}_7\mathbf{\Gamma}_6\mathbf{\Gamma}^a\psi_a)^{C]}\,. \qquad \text{(B.21)}
\end{aligned}
$$

Here we shoud stress that this form of the solution is not unique as it can be rewritten by Fierz reordering. In section 5 we have presented an equivalent but shorter expression.

References

[1] B. de Wit and H. Nicolai, $d = 11$ *supergravity with local* SU(8) *invariance*, *Nucl. Phys.* **B 274** (1986) 363 [INSPIRE].

[2] B. de Wit and H. Nicolai, *The consistency of the* S^7 *truncation in* $D = 11$ *supergravity*, *Nucl. Phys.* **B 281** (1987) 211 [INSPIRE].

[3] H. Nicolai and K. Pilch, *Consistent truncation of* $D = 11$ *supergravity on* $AdS_4 \times S^7$, *JHEP* **03** (2012) 099 [arXiv:1112.6131] [INSPIRE].

[4] B. de Wit and H. Nicolai, *Deformations of gauged* SO(8) *supergravity and supergravity in eleven dimensions*, *JHEP* **05** (2013) 077 [arXiv:1302.6219] [INSPIRE].

[5] H. Godazgar, M. Godazgar and H. Nicolai, *Testing the non-linear flux ansatz for maximal supergravity*, *Phys. Rev.* **D 87** (2013) 085038 [arXiv:1303.1013] [INSPIRE].

[6] H. Godazgar, M. Godazgar and H. Nicolai, *Generalised geometry from the ground up*, *JHEP* **02** (2014) 075 [arXiv:1307.8295] [INSPIRE].

[7] H. Godazgar, M. Godazgar and H. Nicolai, *Nonlinear Kaluza-Klein theory for dual fields*, *Phys. Rev.* **D 88** (2013) 125002 [arXiv:1309.0266] [INSPIRE].

[8] B. de Wit, H. Samtleben and M. Trigiante, *The maximal* $D = 5$ *supergravities*, *Nucl. Phys.* **B 716** (2005) 215 [hep-th/0412173] [INSPIRE].

[9] B. de Wit and H. Samtleben, *Gauged maximal supergravities and hierarchies of nonAbelian vector-tensor systems*, *Fortsch. Phys.* **53** (2005) 442 [hep-th/0501243] [INSPIRE].

[10] B. de Wit, H. Nicolai and H. Samtleben, *Gauged supergravities, tensor hierarchies and M-theory*, *JHEP* **02** (2008) 044 [arXiv:0801.1294] [INSPIRE].

[11] O. Hohm and H. Samtleben, *Exceptional field theory I:* $E_{6(6)}$ *covariant form of M-theory and type IIB*, *Phys. Rev.* **D 89** (2014) 066016 [arXiv:1312.0614] [INSPIRE].

[12] N. Hitchin, *Generalized Calabi-Yau manifolds*, *Quart. J. Math. Oxford Ser.* **54** (2003) 281 [math/0209099] [INSPIRE].

[13] M. Gualtieri, *Generalized complex geometry*, math/0401221 [INSPIRE].

[14] W. Siegel, *Two vierbein formalism for string inspired axionic gravity*, *Phys. Rev.* **D 47** (1993) 5453 [hep-th/9302036] [INSPIRE].

[15] C.M. Hull, *Generalised geometry for M-theory*, *JHEP* **07** (2007) 079 [hep-th/0701203] [INSPIRE].

[16] C. Hull and B. Zwiebach, *Double field theory*, *JHEP* **09** (2009) 099 [arXiv:0904.4664] [INSPIRE].

[17] D.S. Berman and M.J. Perry, *Generalized geometry and M-theory*, *JHEP* **06** (2011) 074 [arXiv:1008.1763] [INSPIRE].

[18] O. Hohm and S.K. Kwak, *Frame-like geometry of double field theory*, *J. Phys.* **A 44** (2011) 085404 [arXiv:1011.4101] [INSPIRE].

[19] A. Coimbra, C. Strickland-Constable and D. Waldram, *Supergravity as generalised geometry I: type II theories*, *JHEP* **11** (2011) 091 [arXiv:1107.1733] [INSPIRE].

[20] D.S. Berman, H. Godazgar, M.J. Perry and P. West, *Duality invariant actions and generalised geometry*, *JHEP* **02** (2012) 108 [arXiv:1111.0459] [INSPIRE].

[21] K. Lee, C. Strickland-Constable and D. Waldram, *Spheres, generalised parallelisability and consistent truncations*, arXiv:1401.3360 [INSPIRE].

[22] P.C. West, *E_{11} and M-theory*, *Class. Quant. Grav.* **18** (2001) 4443 [hep-th/0104081] [INSPIRE].

[23] T. Damour, M. Henneaux and H. Nicolai, *E_{10} and a 'small tension expansion' of M-theory*, *Phys. Rev. Lett.* **89** (2002) 221601 [hep-th/0207267] [INSPIRE].

[24] T. Damour, A. Kleinschmidt and H. Nicolai, *$K(E_{10})$, supergravity and fermions*, *JHEP* **08** (2006) 046 [hep-th/0606105] [INSPIRE].

[25] P. West, *Generalised geometry, eleven dimensions and E_{11}*, *JHEP* **02** (2012) 018 [arXiv:1111.1642] [INSPIRE].

[26] C. Hillmann, *Generalized $E_{7(7)}$ coset dynamics and $D = 11$ supergravity*, *JHEP* **03** (2009) 135 [arXiv:0901.1581] [INSPIRE].

[27] O. Hohm and H. Samtleben, *Exceptional form of $D = 11$ supergravity*, *Phys. Rev. Lett.* **111** (2013) 231601 [arXiv:1308.1673] [INSPIRE].

[28] O. Hohm and H. Samtleben, *Exceptional field theory. II. $E_{7(7)}$*, *Phys. Rev.* **D 89** (2014) 066017 [arXiv:1312.4542] [INSPIRE].

[29] J.H. Schwarz and P.C. West, *Symmetries and transformations of chiral $N = 2$ $D = 10$ supergravity*, *Phys. Lett.* **B 126** (1983) 301 [INSPIRE].

[30] J.H. Schwarz, *Covariant field equations of chiral $N = 2$ $D = 10$ supergravity*, *Nucl. Phys.* **B 226** (1983) 269 [INSPIRE].

[31] P.S. Howe and P.C. West, *The complete $N = 2$, $D = 10$ supergravity*, *Nucl. Phys.* **B 238** (1984) 181 [INSPIRE].

[32] E. Cremmer, *Supergravities in five dimensions*, in *Superspace and supergravity*, S.W. Hawking and M. Roček eds., Cambridge University Press, Cambridge U.K. (1981), reprinted in *Supergravities in diverse dimensions*, A. Salam and E. Sezgin eds., North-Holland/World Scientific (1989) 422.

[33] M. Günaydin, L.J. Romans and N.P. Warner, *Compact and noncompact gauged supergravity theories in five dimensions*, *Nucl. Phys.* **B 272** (1986) 598 [INSPIRE].

[34] K. Pilch and N.P. Warner, *$N = 2$ supersymmetric RG flows and the IIB dilaton*, *Nucl. Phys.* **B 594** (2001) 209 [hep-th/0004063] [INSPIRE].

[35] E. Cremmer and B. Julia, *The* SO(8) *supergravity*, *Nucl. Phys.* **B 159** (1979) 141 [INSPIRE].

[36] T. Curtright, *Generalized gauge fields*, *Phys. Lett.* **B 165** (1985) 304 [INSPIRE].

[37] C.M. Hull, *Duality in gravity and higher spin gauge fields*, JHEP **09** (2001) 027 [hep-th/0107149] [INSPIRE].

[38] X. Bekaert, N. Boulanger and M. Henneaux, *Consistent deformations of dual formulations of linearized gravity: a no go result*, Phys. Rev. **D 67** (2003) 044010 [hep-th/0210278] [INSPIRE].

[39] H. Godazgar, M. Godazgar and H. Nicolai, *Embedding tensor of Scherk-Schwarz flux compactifications from eleven dimensions*, Phys. Rev. **D 89** (2014) 045009 [arXiv:1312.1061] [INSPIRE].

[40] O. Hohm and H. Samtleben, *Consistent Kaluza-Klein truncations via exceptional field theory*, JHEP **01** (2015) 131 [arXiv:1410.8145] [INSPIRE].

[41] G. Dall'Agata, K. Lechner and D.P. Sorokin, *Covariant actions for the bosonic sector of $D = 10$ IIB supergravity*, Class. Quant. Grav. **14** (1997) L195 [hep-th/9707044] [INSPIRE].

[42] H.J. Kim, L.J. Romans and P. van Nieuwenhuizen, *The mass spectrum of chiral $N = 2$ $D = 10$ supergravity on S^5*, Phys. Rev. **D 32** (1985) 389 [INSPIRE].

[43] M. Günaydin and N. Marcus, *The spectrum of the S^5 compactification of the chiral $N = 2$, $D = 10$ supergravity and the unitary supermultiplets of* $\mathrm{U}(2, 2/4)$, Class. Quant. Grav. **2** (1985) L11 [INSPIRE].

[44] B. de Wit and H. Nicolai, *$N = 8$ supergravity*, Nucl. Phys. **B 208** (1982) 323 [INSPIRE].

[45] B. de Wit and H. Nicolai, *On the relation between $D = 4$ and $D = 11$ supergravity*, Nucl. Phys. **B 243** (1984) 91 [INSPIRE].

[46] B. de Wit, H. Nicolai and N.P. Warner, *The embedding of gauged $N = 8$ supergravity into $D = 11$ supergravity*, Nucl. Phys. **B 255** (1985) 29 [INSPIRE].