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Nested off-diagonal Bethe ansatz and exact solutions of the $su(n)$ spin chain with generic integrable boundaries

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Abstract: The nested off-diagonal Bethe ansatz method is proposed to diagonalize multi-component integrable models with generic integrable boundaries. As an example, the exact solutions of the $su(n)$-invariant spin chain model with both periodic and non-diagonal boundaries are derived by constructing the nested $T-Q$ relations based on the operator product identities among the fused transfer matrices and the asymptotic behavior of the transfer matrices.

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Contents

1 Introduction

The appearance of integrability in planar AdS/CFT [1] is a rather unexpected occurrence and has led to many remarkable results [2] (see also references therein) and even ultimately to the exact solution of planar $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory. The anomalous dimensions of single-trace operators of $\mathcal{N} = 4$ SYM are given by the eigenvalues of certain integrable closed spin chain Hamiltonians [2, 3]. Then it was shown [4, 5] that the computing of the anomalous dimensions of determinant-like operators of $\mathcal{N} = 4$ SYM can be mapped to the eigenvalue problem of certain integrable open spin chain (spin chain with boundary condition specified by reflection K-matrices or boundary scattering matrices) Hamiltonians [2, 6, 7], while by AdS/CFT the K-matrices of the open chain correspond to open strings attached to maximal giant gravitons [5, 8]. Therefore spin chain model has played an important role in understanding the physical contents of planar $\mathcal{N} = 4$ SYM theory and planar AdS/CFT. Moreover, it has already provided valuable insight into the important universality class of boundary quantum physical systems in condensed matter physics [9]. Motivated by the above great applications, in this paper, we develop the nested off-diagonal Bethe ansatz method, a generalization of the method proposed in [10–13], to solve the eigenvalue problem of multi-component spin chains with the most general integrable boundary terms.

So far, there have been several well-known methods for deriving the Bethe ansatz (BA) solutions of quantum integrable models: the coordinate BA [14–16], the T-Q approach [17–21], the algebraic BA [22–27], the analytic BA [28], the functional BA [29, 30] or the separation of variables method [31–34] and many others [35–55]. However, there exists a quite unusual class of integrable models which do not possess the U(1) symmetry (whose transfer matrices contain not only the diagonal elements but also some off-diagonal elements of the monodromy matrix and the usual U(1) symmetry is broken, i.e., the total spin is no longer conserved). Normally, most of the conventional methods do not work for these models even though their integrability has been proven for many years [25].

Recently, a systematic method [10–13] for dealing with such kind of models associated with $su(2)$ algebra was proposed by the present authors, which had been shown successfully to construct the exact solutions of the open Heisenberg spin chain with unparallel boundary fields, the XXZ spin torus, the closed XYZ chain with odd site number and other models with general boundary terms [56, 57]. With the help of the Hirota equation, Nepomechie [58] generalized the results of [10–13] to the arbitrary spin XXX open chain with general boundary terms. An expression for the corresponding eigenvectors was also proposed recently in [59].

The central idea of the method in [10–13] is to construct a proper $T-Q$ ansatz with an extra off-diagonal term (comparing with the ordinary ones [20]) based on the functional relations between the transfer matrix (the trace of the monodromy matrix) and the quantum determinant $\Delta_q(u)$, at some special points of the spectral parameter $u=\theta_j$, i.e.,[1]

$$t(\theta_j)t(\theta_j-\eta)\sim\Delta_q(\theta_j). \tag{1.1}$$

In this paper, we generalize the off-diagonal Bethe ansatz method to the multi-component integrable models (integrable spin chains associated with higher rank algebras). This generalization allows us to construct the nested $T-Q$ relations based on the recursive operator product identities and the asymptotic behavior of the transfer matrices for the systems with both periodic and arbitrary integrable open boundary conditions. We elucidate our method with the $su(n)$ spin chain (both periodic and open) model as an example. Our method might be used to the integrable systems associated with B_n, C_n and D_n algebras.

The paper is organized as follows. Section 2 serves as an introduction of our notations and some basic ingredients. We briefly describe the inhomogeneous $su(n)$-invariant spin chain with periodic boundary condition. Based on some operator product relations for the antisymmetric fused transfer matrices and their asymptotic behaviors, the nested $T-Q$ ansatz of their eigenvalues and the corresponding Bethe ansatz equations (BAEs) are constructed. In section 3, we study the $su(n)$-invariant open spin chains with general

[1]These kind relations for the eigenvalues of the transfer matrices were previously obtained by the separation of variables method for the open XXX spin chain [31], for the XXZ spin chain with antiperiodic boundary condition and were used to determine the eigenvalues [32, 33]. Then this idea was generalized to the open XXZ chain with one general non-diagonal and one diagonal or triangular boundary K-matrices [34].

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open boundary integrable conditions. Based on some properties of the R-matrix and K-matrices, we obtain the important operator product identities among the fused transfer matrices of the open chains and their asymptotic behaviors when $u \longrightarrow \infty$. In section 4, we focus on the $su(3)$-invariant open spin chain with the most general non-diagonal boundary terms. The nested Bethe ansatz solution for the eigenvalues of the transfer matrix and the corresponding Bethe ansatz equations (BAEs) are given in detail based on the operator product identities of the transfer matrix and their asymptotic behaviors and values of the transfer matrices at some special points. The results for the $su(n)$-invariant case is given in section 5. We summarize our results and give some discussions in section 6. Some detailed technical proof is given in appendices A and B.

2 $su(n)$-invariant spin chain with periodic boundary conditions

2.1 Transfer matrix

Let $\mathbf{V}$ denote an n-dimensional linear space. The Hamiltonian of $su(n)$-invariant quantum spin system with periodic boundary condition is given by [60, 61]

$$H = \sum_{j=1}^{N} P_{j,j+1}, \tag{2.1}$$

where N is the number of sites, $P_{j,j+1}$ is permutation operator, $P_{ac}^{bd} = \delta_{ad}\delta_{bc}$ with $a, b, c, d = 1, \cdots, n$. The integrability of the system (2.1) is guaranteed by the $su(n)$-invariant R-matrix $R(u) \in \mathrm{End}(\mathbf{V} \otimes \mathbf{V})$ [62, 63]

$$R_{12}(u) = u + \eta P_{1,2}, \tag{2.2}$$

where u is the spectral parameter and η is the crossing parameter. The R-matrix satisfies the quantum Yang-Baxter equation (QYBE)

$$R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2), \tag{2.3}$$

and possesses the following properties:

$$\begin{aligned}
&\text{Initial condition:} && R_{12}(0) = \eta P_{1,2}, && (2.4)\\
&\text{Unitarity:} && R_{12}(u)R_{21}(-u) = \rho_1(u)\,\mathrm{id}, \quad \rho_1(u) = -(u+\eta)(u-\eta), && (2.5)\\
&\text{Crossing-unitarity:} && R_{12}^{t_1}(u)R_{21}^{t_1}(-u-n\eta) = \rho_2(u)\,\mathrm{id}, \quad \rho_2(u) = -u(u+n\eta), && (2.6)\\
&\text{Fusion conditions:} && R_{12}(-\eta) = -2\eta P_{1,2}^{(-)}, \quad R_{12}(\eta) = 2\eta P_{1,2}^{(+)}. && (2.7)
\end{aligned}$$

Here $R_{21}(u) = P_{1,2}R_{12}(u)P_{1,2}$, $P_{1,2}^{(\mp)} = \frac{1}{2}\{1 \mp P_{1,2}\}$ is anti-symmetric (symmetric) project operator in the tensor product space $\mathbf{V} \otimes \mathbf{V}$, and t_i denotes the transposition in the i-th space. Here and below we adopt the standard notation: for any matrix $A \in \mathrm{End}(\mathbf{V})$, A_j is an embedding operator in the tensor space $\mathbf{V} \otimes \mathbf{V} \otimes \cdots$, which acts as A on the j-th space and as an identity on the other factor spaces; $R_{ij}(u)$ is an embedding operator of R-matrix in the tensor space, which acts as an identity on the factor spaces except for the i-th and j-th ones.

Let us introduce the "row-to-row" (or one-row) monodromy matrix $T(u)$, which is an $n \times n$ matrix with operator-valued elements acting on $\mathbf{V}^{\otimes N}$,

$$T_0(u) = R_{0N}(u-\theta_N)R_{0\,N-1}(u-\theta_{N-1})\cdots R_{01}(u-\theta_1). \tag{2.8}$$

Here $\{\theta_j|j=1,\cdots,N\}$ are arbitrary free complex parameters which are usually called as inhomogeneous parameters.

The transfer matrix $t^{(p)}(u)$ of the spin chain with periodic boundary condition (or closed chain) is given by [24]

$$t^{(p)}(u) = tr_0 T_0(u). \tag{2.9}$$

The QYBE implies that one-row monodromy matrix $T(u)$ satisfies the following relation

$$R_{00'}(u-v)T_0(u)T_{0'}(v) = T_{0'}(v)T_0(u)R_{00'}(u-v). \tag{2.10}$$

The above equation leads to the fact that the transfer matrices with different spectral parameters commute with each other: $[t^{(p)}(u), t^{(p)}(v)] = 0$. Then $t^{(p)}(u)$ serves as the generating functional of the conserved quantities, which ensures the integrability of the closed spin chain. The Hamiltonian (2.1) can be obtained from the transfer matrix as following

$$H = \eta \frac{\partial \ln t(u)}{\partial u}|_{u=0,\theta_j=0}. \tag{2.11}$$

2.2 Operator product identities

Our main tool is the so-called fusion technique [65–69]. We shall only consider the anti-symmetric fusion procedure which leads to the desired operator identities to determine the spectrum of the transfer matrix $t^{(p)}(u)$ given by (2.9).

For this purpose, let us introduce the anti-symmetric projectors which are determined by the following induction relations

$$P^{(-)}_{1,2,\cdots,m+1} = \frac{1}{m+1}\left(1 - P_{1,2} - P_{1,3} - \ldots - P_{1,m+1}\right) P^{(-)}_{2,3,\cdots,m+1}, \quad m = 1,\ldots,n-1.$$

We introduce further the fused one-row monodromy matrices $T_{\langle 1,\ldots,m\rangle}(u)$ (cf. (2.8))

$$T_{\langle 1,\ldots,m\rangle}(u) = P^{(-)}_{1,2,\ldots,m}\, T_1(u)T_2(u-\eta)\ldots T_m(u-(m-1)\eta)\, P^{(-)}_{1,2,\ldots,m}, \tag{2.12}$$

and the associated fused transfer matrices $t^{(p)}_m(u)$

$$t^{(p)}_m(u) = tr_{12\cdots m}\{T_{\langle 1,\ldots,m\rangle}(u)\}, \quad m = 1,\cdots,n, \tag{2.13}$$

which includes the fundamental transfer matrix $t^{(p)}(u)$ given by (2.9) as the first one, i.e., $t^{(p)}(u) = t^{(p)}_1(u)$. It follows from the fusion of the R-matrix [65–69] that the fused transfer matrices constitute commutative families

$$[t^{(p)}_i(u),\, t^{(p)}_j(v)] = 0, \quad i,j = 1,\ldots,n. \tag{2.14}$$

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We note that $t_n^{(p)}(u)$ is the quantum determinant (proportional to the identity operator for generic u and $\{\theta_j\}$),

$$t_n^{(p)}(u) = \Delta_q^{(p)}(u) \times \mathrm{id} = \prod_{l=1}^{N}(u-\theta_l+\eta)\prod_{j=1}^{N}\prod_{k=1}^{n-1}(u-\theta_j-k\eta) \times \mathrm{id}. \tag{2.15}$$

Let us evaluate the product of the fundamental transfer matrix and the fused ones at some special points θ_j and $\theta_j - \eta$. According to the definition (2.13), we thus have the following functional relations among the transfer matrices

$$t^{(p)}(\theta_j)t_m^{(p)}(\theta_j-\eta) = t_{m+1}^{(p)}(\theta_j), \quad m = 1,\ldots,n-1, \quad j = 1,\cdots,N. \tag{2.16}$$

The initial condition (2.4), the properties (2.7) of the R-matrix (e.g, $R_{ij}(\eta)R_{ij}(-\eta) = 0$) and the properties (A.7) (see below) imply that the fused monodromy matrices $T_{\langle 1,\ldots,m\rangle}(u)$ given by (2.12) vanishes at some special points,

$$T_{\langle 1,\ldots,m\rangle}(\theta_j+k\eta) = 0, \quad \text{for } k = 1,\ldots m-1, \quad j = 1,\ldots,N. \tag{2.17}$$

This fact allows us to introduce some commutative operators $\{\tau_m^{(p)}(u)\}$ associated with the fused transfer matrices $\{t_m^{(p)}(u)\}$

$$t_m^{(p)}(u) = \prod_{l=1}^{N}\prod_{k=1}^{m-1}(u-\theta_l-k\eta)\tau_m^{(p)}(u), \quad [\tau_l^{(p)}(u), \tau_m^{(p)}(v)] = 0, \quad l,m = 1,\ldots,n. \tag{2.18}$$

We use the convention: $\tau^{(p)}(u) = \tau_1^{(p)}(u)$. From the above equations and the definitions (2.13) of the fused transfer matrices, we conclude that the operators $\{\tau_m^{(p)}(u)\}$, as functions of u, are polynomials of degree N with the following asymptotic behaviors

$$\tau_m^{(p)}(u) = \frac{n!}{m!(n-m)!}u^N + \ldots, \quad u \to \infty. \tag{2.19}$$

The operator identities (2.16) implies that these operators satisfy the following functional relations

$$\tau^{(p)}(\theta_j)\tau_m^{(p)}(\theta_j-\eta) = \prod_{l=1}^{N}(\theta_j-\theta_l-\eta)\tau_{m+1}^{(p)}(\theta_j), \quad j = 1,\ldots,N, \quad m = 1,\ldots,n-1. \tag{2.20}$$

2.3 Nested T-Q relation

The explicit expression (2.15) of the quantum determinant, the asymptotic behaviors (2.19) and the functional relations (2.20) allow one to determine the eigenvalues of all the operators $\{\tau_m^{(p)}(u)\}$ and consequently those of $\{t_m^{(p)}(u)\}$ completely with the help of the relation (2.18) as follows. The commutativity of the transfer matrices with different spectral parameters implies that they have common eigenstates. Let $|\Psi\rangle$ be a common eigenstate of $\{t_m^{(p)}(u)\}$, which does not depend upon u, with the eigenvalue $\Lambda_m^{(p)}(u)$, i.e.,

$$t_m^{(p)}(u)|\Psi\rangle = \Lambda_m^{(p)}(u)|\Psi\rangle, \quad m = 1,\ldots n.$$

The analyticity of the R-matrix implies that the eigenvalues $\Lambda_m^{(p)}(u)$ are polynomials of u with a degree of mN. The relations (2.18)–(2.20) give rise to some similar relations of $\{\Lambda_m^{(p)}(u)\}$ which allow us to determine $\{\Lambda_m^{(p)}(u)\}$ completely. Here we give the final result. The proof can be obtained by simple checking the solution satisfying the resulting relations.

Let us introduce n functions $\{z_p^{(l)}(u)|l=1,\ldots,n\}$,

$$z_p^{(l)}(u) = Q_p^{(0)}(u)\frac{Q_p^{(l-1)}(u+\eta)Q_p^{(l)}(u-\eta)}{Q_p^{(l-1)}(u)Q_p^{(l)}(u)}, \quad l=1,\ldots n, \tag{2.21}$$

where the functions $Q_p^{(l)}(u)$ are given by

$$Q_p^{(0)}(u) = \prod_{j=1}^{N}(u-\theta_j), \tag{2.22}$$

$$Q_p^{(r)}(u) = \prod_{l=1}^{L_r}(u-\lambda_l^{(r)}), \quad r=1,\ldots,n-1, \tag{2.23}$$

$$Q_p^{(n)}(u) = 1, \tag{2.24}$$

where $\{L_r|r=1,\ldots n-1\}$ are some non-negative integers and the parameters $\{\lambda_l^{(r)}|l=1,\ldots L_r,\ r=1,\ldots n-1\}$ will be determined by the Bethe ansatz equations (2.26) (see below). The eigenvalues $\Lambda_m^{(p)}(u)$ of the m-th fused transfer matrix $t_m^{(p)}(u)$ is then given by

$$\Lambda_m^{(p)}(u) = \sum_{1\le i_1<i_2<\ldots<i_m\le n} z_p^{(i_1)}(u)z_p^{(i_2)}(u-\eta)\ldots z_p^{(i_m)}(u-(m-1)\eta), \quad m=1,\ldots,n. \tag{2.25}$$

The regular property of $\Lambda^{(p)}(u)$ implies that the residues of $\Lambda^{(p)}(u)$ at each apparent simple pole $\lambda_l^{(r)}$ have to vanish. This leads to the associated BAEs,

$$\prod_{j=1,\neq l}^{L_r}\frac{\lambda_l^{(r)}-\lambda_j^{(r)}-\eta}{\lambda_l^{(r)}-\lambda_j^{(r)}+\eta} = \prod_{k=1}^{L_{r-1}}\frac{\lambda_l^{(r)}-\lambda_k^{(r-1)}}{\lambda_l^{(r)}-\lambda_k^{(r-1)}+\eta}\prod_{m=1}^{L_{r+1}}\frac{\lambda_l^{(r)}-\lambda_m^{(r+1)}-\eta}{\lambda_l^{(r)}-\lambda_m^{(r+1)}}, \tag{2.26}$$
$$l=1,\ldots L_r, \quad r=1,2,\ldots,n-1, \quad L_0=N, \quad L_N=0, \quad \lambda_l^{(0)}=\theta_l.$$

By taking the limit $\theta_j=0$, the above BAEs are readily reduced to those previously obtained by other Bethe ansatz methods [62–64].

3 $su(n)$-invariant spin chain with general open boundary conditions

3.1 Transfer matrix

Integrable open chain can be constructed as follows [15, 25]. Let us introduce a pair of K-matrices $K^-(u)$ and $K^+(u)$. The former satisfies the reflection equation (RE)

$$\begin{aligned}&R_{12}(u_1-u_2)K_1^-(u_1)R_{21}(u_1+u_2)K_2^-(u_2)\\&\quad= K_2^-(u_2)R_{12}(u_1+u_2)K_1^-(u_1)R_{21}(u_1-u_2),\end{aligned} \tag{3.1}$$

and the latter satisfies the dual RE

$$\begin{aligned}
&R_{12}(u_2-u_1)K_1^+(u_1)R_{21}(-u_1-u_2-n\eta)K_2^+(u_2) \\
&\quad = K_2^+(u_2)R_{12}(-u_1-u_2-n\eta)K_1^+(u_1)R_{21}(u_2-u_1). \qquad (3.2)
\end{aligned}$$

For open spin-chains, instead of the standard "row-to-row" monodromy matrix $T(u)$ (2.8), one needs to consider the "double-row" monodromy matrix $\mathcal{J}(u)$

$$\mathcal{J}_0(u) = T_0(u)K_0^-(u)\hat{T}_0(u), \qquad (3.3)$$

$$\hat{T}_0(u) = R_{01}(u+\theta_1)R_{02}(u+\theta_2)\ldots R_{0N}(u+\theta_N). \qquad (3.4)$$

Then the double-row transfer matrix $t(u)$ of the open spin chain is given by

$$t(u) = tr_0\{K_0^+(u)\mathcal{J}_0(u)\}. \qquad (3.5)$$

From the QYBE and the (dual) RE, one may check that the transfer matrices with different spectral parameters commute with each other: $[t(u),t(v)]=0$. Thus $t(u)$ serves as the generating functional of the conserved quantities, which ensures the integrability of the system.

In this paper, we consider a generic solution $K^-(u)$ to the RE associated with the R-matrix (2.2) [70–74]

$$K^-(u) = \xi + uM, \quad M^2 = 1, \qquad (3.6)$$

where ξ is a boundary parameter and M is an $n\times n$ constant matrix (only depends on boundary parameters). Besides the RE, the K-matrix satisfies the following properties

$$K^-(0) = \xi, \quad K^-(u) = u\,M + \ldots, \quad u\to\infty. \qquad (3.7)$$

Since the second power of M becomes the $n\times n$ identity matrix, the eigenvalues of M must be ±1. Suppose that there are p positive eigenvalues and q negative eigenvalues, then we have $p+q=n$ and $trM=p-q$. At the same time, we introduce the corresponding *dual* K-matrix $K^+(u)$ which is a generic solution of the dual RE (3.2)

$$K^+(u) = \bar{\xi} - \left(u+\frac{n}{2}\eta\right)\bar{M}, \quad \bar{M}^2 = 1, \qquad (3.8)$$

where $\bar{\xi}$ is a boundary parameter and $\bar{M}$ is an $n\times n$ boundary parameter dependent matrix, whose eigenvalues are ±1. Again, we suppose that there are $\bar{p}$ positive eigenvalues and $\bar{q}$ negative eigenvalues, then we have $\bar{p}+\bar{q}=n$ and $tr\bar{M}=\bar{p}-\bar{q}$. Besides the dual RE, the K-matrix also satisfies the following properties

$$K^+\left(-\frac{n}{2}\eta\right) = \bar{\xi}, \quad K^+(u) = -u\,\bar{M} + \ldots, \quad u\to\infty. \qquad (3.9)$$

The Hamiltonian of the open spin chain specified by the K-matrices $K^{\pm}(u)$ (3.6) and (3.8) can be expressed in terms of the transfer matrix (3.5) as

$$\begin{aligned}
H &= \eta\frac{\partial \ln t(u)}{\partial u}|_{u=0,\theta_j=0} \\
&= 2\sum_{j=1}^{N-1}P_{j,j+1} + \eta\frac{tr_0 K_0^{+\prime}(0)}{tr_0 K_0^+(0)} + 2\frac{tr_0 K_0^+(0)P_{0N}}{tr_0K_0^+(0)} + \eta\frac{1}{\xi}K_1^{-\prime}(0). \qquad (3.10)
\end{aligned}$$

3.2 Operator product identities

Similar to the closed spin chain case in the previous section, we apply the fusion technique to study the open spin chain. In this case, we need to use the fusion techniques both for R-matrices [65–69] and for K-matrices [75, 76]. We only consider the antisymmetric fusion procedure which leads to the desired operator identities to determine the spectrum of the transfer matrix $t(u)$ given by (3.5).

Following [75, 76], let us introduce the fused K-matrices and double-row monodromy matrices by the following recursive relations

$$\begin{aligned} K^+_{1,\ldots,m}(u) &= K^+_{\langle 2,\ldots,m\rangle}(u-\eta)R_{1m}(-2u-n\eta+(m-1)\eta)\ldots \\ &\quad \times R_{12}(-2u-n\eta+\eta)K^+_1(u), \end{aligned} \tag{3.11}$$

$$K^+_{\langle 1,\ldots,m\rangle}(u) = P^{(-)}_{1,\ldots,m}K^+_{1,\ldots,m}(u)P^{(-)}_{1,\ldots,m}, \tag{3.12}$$

$$K^-_{1,\ldots,m}(u) = K^-_1(u)R_{21}(2u-\eta)\ldots R_{m1}(2u-(m-1)\eta)K^-_{\langle 2,\ldots,m\rangle}(u-\eta), \tag{3.13}$$

$$K^-_{\langle 1,\ldots,m\rangle}(u) = P^{(-)}_{1,\ldots,m}\,K^-_{1,\ldots,m}(u)\,P^{(-)}_{1,\ldots,m}, \tag{3.14}$$

$$\mathcal{J}_{1,\ldots,m}(u) = \mathcal{J}_1(u)R_{21}(2u-\eta)\ldots R_{m1}(2u-(m-1)\eta)\mathcal{J}_{\langle 2,\ldots,m\rangle}(u-\eta), \tag{3.15}$$

$$\mathcal{J}_{\langle 1,\ldots,m\rangle}(u) = P^{(-)}_{1,\ldots,m}\mathcal{J}_{1,\ldots,m}(u)P^{(-)}_{1,\ldots,m} = T_{\langle 1,\ldots,m\rangle}(u)K^-_{\langle 1,\ldots,m\rangle}(u)\hat{T}_{\langle 1,\ldots,m\rangle}(u), \tag{3.16}$$

where the fused one-row monodromy matrix $T_{\langle 1,\ldots,m\rangle}(u)$ is given by (2.12) and

$$\hat{T}_{\langle 1,\ldots,m\rangle}(u) = P^{(-)}_{1,2,\ldots,m}\,\hat{T}_1(u)\hat{T}_2(u-\eta)\ldots\hat{T}_m(u-(m-1)\eta)\,P^{(-)}_{1,2,\ldots,m}. \tag{3.17}$$

For the open spin chain, the m-th fused transfer matrix $t_m(u)$ constructed by the antisymmetric fusion procedure is given by

$$t_m(u) = tr_{1,\ldots,m}\{K^+_{\langle 1,\ldots,m\rangle}(u)\mathcal{J}_{\langle 1,\ldots,m\rangle}(u)\}, \quad m=1,\ldots,n, \tag{3.18}$$

which includes the fundamental transfer matrix $t(u)$ given by (3.5) as the first one, i.e., $t(u)=t_1(u)$. The relation (3.16) allows us to rewrite the transfer matric $t_m(u)$ in terms of the fused K-matrices and one-row monodromy matrices

$$t_m(u) = tr_{1,\ldots,m}\left\{K^+_{\langle 1,\ldots,m\rangle}(u)\,T_{\langle 1,\ldots,m\rangle}(u)\,K^-_{\langle 1,\ldots,m\rangle}(u)\,\hat{T}_{\langle 1,\ldots,m\rangle}(u)\right\}. \tag{3.19}$$

It follows from the fusion of the R-matrix [65–69] and that of the K-matrices [75, 76] that the fused transfer matrices constitute commutative families, namely,

$$[t_i(u),\, t_j(v)] = 0, \quad i,j=1,\ldots,n. \tag{3.20}$$

Moreover, we remark that $t_n(u)$ is the so-called quantum determinant and that for generic u and $\{\theta_j\}$ it is proportional to the identity operator, namely,

$$\begin{aligned} t_n(u) &= \Delta_q(u)\times \mathrm{id}, \\ \Delta_q(u) &= \Delta_q\{T(u)\}\Delta_q\{\hat{T}(u)\}\Delta_q\{K^+(u)\}\Delta_q\{K^-(u)\} \\ &= \prod_{l=1}^{N}(u-\theta_l+\eta)(u+\theta_l+\eta)\prod_{l=1}^{N}\prod_{k=1}^{n-1}(u-\theta_l-k\eta)(u+\theta_l-k\eta) \end{aligned} \tag{3.21}$$

$$\times \prod_{i=1}^{n-1}\prod_{j=1}^{i}(2u-(i+j)\eta)(-2u+(n-2-i-j)\eta)$$
$$\times(-1)^{q+\bar{q}}\prod_{k=0}^{\bar{q}-1}\Big(-u+\frac{n-2}{2}\eta-\bar{\xi}-k\eta\Big)\prod_{k=0}^{\bar{p}-1}\Big(-u+\frac{n-2}{2}\eta+\bar{\xi}-k\eta\Big)$$
$$\times\prod_{k=0}^{q-1}(u-\xi-k\eta)\prod_{k=0}^{p-1}(u+\xi-k\eta). \tag{3.22}$$

(2.3) and (3.2) allow us to rewrite the fused K-matrix in another form

$$\begin{aligned}K^+_{\langle 1,\ldots,m\rangle}(u) &= P^{(-)}_{1,\ldots,m}K^+_m(u-(m-1)\eta)R_{m\,m-1}(-2u-n\eta+(2m-3)\eta)\ldots\\ &\quad\times R_{m\,1}(-2u-n\eta+(m-1)\eta)K^+_{\langle 1,\ldots,m-1\rangle}(u)P^{(-)}_{1,\ldots,m}.\end{aligned} \tag{3.23}$$

The above equation, (3.13)–(3.14), (2.5)–(2.7) and the degenerate properties of the R-matrix and the K-matrices:

$$R_{12}(0)=\eta P_{1,2},\quad K^-(0)=\xi,\quad K^+\Big(-\frac{n}{2}\eta\Big)=\bar{\xi}, \tag{3.24}$$

implies that at the following $2m$ special points

$$0,\frac{\eta}{2},\ldots,\frac{m-1}{2}\eta,\ \text{and}\ -\frac{n}{2}\eta+(m-1)\eta,-\frac{n}{2}\eta+(m-1)\eta-\frac{\eta}{2},\ldots,-\frac{n}{2}\eta+\frac{m-1}{2}\eta, \tag{3.25}$$

the transfer matrix $t_m(u)$ may be written in terms of $\{t_l(u)|l=m-1,\ldots,0\}$ (we have used the convention $t_0(u)=\mathrm{id}$), for examples (4.8)–(4.13) for the $su(3)$-case and (B.1)–(B.12) for the $su(4)$-case. The commutativity of the transfer matrices with different spectral parameters implies that they have common eigenstates. Let $|\Psi\rangle$ be a common eigenstate of $\{t_m(u)\}$, which does not depend upon u, with the eigenvalue $\Lambda_m(u)$, i.e.,

$$t_m(u)|\Psi\rangle=\Lambda_m(u)|\Psi\rangle,\quad m=1,\ldots n. \tag{3.26}$$

Now let us evaluate the product of the fundamental transfer matrix and the fused ones at some special points

$$\begin{aligned}t(\pm\theta_j)t_m(\pm\theta_j-\eta) &= tr_{1\ldots m+1}\big\{\mathcal{J}_1^{t_1}(\pm\theta_j)K_1^+(\pm\theta_j)^{t_1}\\ &\quad\times\mathcal{J}_{\langle 2,\ldots,m+1\rangle}(\pm\theta_j-\eta)K^+_{\langle 2,\ldots m+1\rangle}(\pm\theta_j-\eta)\big\}\\ &\overset{(2.6)}{=} \prod_{k=1}^{m}\rho_2^{-1}(\pm2\theta_j-k\eta)\times tr_{1\ldots m+1}\big\{\mathcal{J}_1^{t_1}(\pm\theta_j)K_1^+(\pm\theta_j)^{t_1}\\ &\quad\times R^{t_1}_{12}(\mp2\theta_j+\eta-n\eta)\ldots R^{t_1}_{1\,m+1}(\mp2\theta_j+m\eta-n\eta)\\ &\quad\times R^{t_1}_{1\,m+1}(\pm2\theta_j-m\eta)\ldots R^{t_1}_{12}(\pm2\theta_j-\eta)\\ &\quad\times\,\mathcal{J}_{\langle 2,\ldots,m+1\rangle}(\pm\theta_j-\eta)K^+_{\langle 2,\ldots m+1\rangle}(\pm\theta_j-\eta)\big\}\\ &= \prod_{k=1}^{m}\rho_2^{-1}(\pm2\theta_j-k\eta)\times tr_{1\ldots m+1}\big\{\\ &\quad\times R_{1\,m+1}(\mp2\theta_j+m\eta-n\eta)\ldots R_{12}(\mp2\theta_j+\eta-n\eta)K_1^+(\pm\theta_j)\\ &\quad\times\,\mathcal{J}_{1,\ldots,m+1}(\pm\theta_j)K^+_{\langle 2,\ldots,m+1\rangle}(\pm\theta_j-\eta)\big\}\end{aligned}$$

$$
\begin{aligned}
&= \prod_{k=1}^{m} \rho_2^{-1}(\pm 2\theta_j - k\eta) \times tr_{1\ldots m+1}\Big\{K^{+}_{\langle 2,\ldots,m+1\rangle}(\pm\theta_j - \eta) \\
&\qquad \times R_{1\,m+1}(\mp 2\theta_j + m\eta - n\eta)\ldots R_{12}(\mp 2\theta_j + \eta - n\eta)K^{+}_{1}(\pm\theta_j) \\
&\qquad \times\, \mathcal{J}_{1,\ldots,m+1}(\pm\theta_j)\} \\
&= \prod_{k=1}^{m} \rho_2^{-1}(\pm 2\theta_j - k\eta) \times tr_{1\ldots m+1}\left\{K^{+}_{1,\ldots,m+1}(\pm\theta_j)\mathcal{J}_{1,\ldots,m+1}(\pm\theta_j)\right\} \\
&\overset{(A.2)}{=} \prod_{k=1}^{m} \rho_2^{-1}(\pm 2\theta_j - k\eta) \times tr_{1\ldots m+1}\Big\{K^{+}_{1,\ldots,m+1}(\pm\theta_j)P^{(-)}_{1,\ldots,m+1}\mathcal{J}_{1,\ldots,m+1}(\pm\theta_j)\Big\} \\
&\overset{(A.11)}{=} \prod_{k=1}^{m} \rho_2^{-1}(\pm 2\theta_j - k\eta) \times tr_{1\ldots m+1}\Big\{K^{+}_{\langle 1,\ldots,m+1\rangle}(\pm\theta_j)\mathcal{J}_{1,\ldots,m+1}(\pm\theta_j)P^{(-)}_{1,\ldots,m+1}\Big\} \\
&\overset{(A.10)}{=} \prod_{k=1}^{m} \rho_2^{-1}(\pm 2\theta_j - k\eta) \times tr_{1\ldots m+1}\Big\{K^{+}_{\langle 1,\ldots,m+1\rangle}(\pm\theta_j)\mathcal{J}_{\langle 1,\ldots,m+1\rangle}(\pm\theta_j)\Big\}.
\end{aligned}
$$

According to the definition (3.18), we thus have the following functional relations among the transfer matrices

$$
t(\pm\theta_j)t_m(\pm\theta_j - \eta) = t_{m+1}(\pm\theta_j)\prod_{k=1}^{m} \rho_2^{-1}(\pm 2\theta_j - k\eta), \qquad j = 1,\ldots,N; \quad m = 1,\ldots,n-1. \tag{3.27}
$$

In terms of the corresponding eigenvalues, the above relations become

$$
\Lambda(\pm\theta_j)\Lambda_m(\pm\theta_j - \eta) = \Lambda_{m+1}(\pm\theta_j)\prod_{k=1}^{m} \rho_2^{-1}(\pm 2\theta_j - k\eta), \qquad j = 1,\ldots,N; \quad m = 1,\ldots,n-1. \tag{3.28}
$$

Using the similar method that we have derived the zero points (2.17) of the fused monodromy matrix $T_{\langle 1,\ldots,m\rangle}(u)$, we can figure out the zero points of the fused monodromy matrix $\hat{T}_{\langle 1,\ldots,m\rangle}(u)$ and those of the fused K-matrices $K^{\pm}_{\langle 1,\ldots,m\rangle}(u)$ respectively. Thanks to the alternative expression (3.19) of the fused transfer matrix $t_m(u)$, we know that these zero points all together constitute the zero points of the transfer matrix, which allows us to rewrite the transfer matrix as

$$
\begin{aligned}
t_m(u) = &\prod_{i=1}^{m-1}\prod_{j=1}^{i}(2u - i\eta - j\eta)(-2u + (2m-2-n)\eta - i\eta - j\eta) \\
&\times \prod_{l=1}^{N}\prod_{k=1}^{m-1}(u - \theta_l - k\eta)(u + \theta_l - k\eta)\,\tau_m(u).
\end{aligned} \tag{3.29}
$$

Since the operator $\tau_m(u)$ is proportional to the transfer matrix $t_m(u)$ by c-number coefficient, the corresponding eigenvalue $\bar{\Lambda}_m(u)$ has the following relation with $\Lambda_m(u)$

$$
\begin{aligned}
\Lambda_m(u) = &\prod_{i=1}^{m-1}\prod_{j=1}^{i}(2u - i\eta - j\eta)(-2u + (2m-2-n)\eta - i\eta - j\eta) \\
&\times \prod_{l=1}^{N}\prod_{k=1}^{m-1}(u - \theta_l - k\eta)(u + \theta_l - k\eta)\bar{\Lambda}_m(u).
\end{aligned} \tag{3.30}
$$

It follows from the definitions of the fused transfer matrices (3.18) that the eigenvalue $\bar{\Lambda}_m(u)$ of the resulting commutative operator $\tau_m(u)$, as a function of u, is a polynomial of degree $2N+2m$. The functional relations (3.27) give rise to that the eigenvalue $\bar{\Lambda}_m(u)$ of $\tau_m(u)$ satisfies the following relations

$$\bar{\Lambda}(\pm\theta_j)\bar{\Lambda}_m(\pm\theta_j-\eta) = \bar{\Lambda}_{m+1}(\pm\theta_j)\prod_{k=1}^{m}\rho_2^{-1}(\pm 2\theta_j-k\eta)\rho_0(\pm\theta_j), \qquad m=1,\ldots,n-1, \quad j=1,\ldots,N, \tag{3.31}$$

where the function $\rho_0(u)$ is given by

$$\rho_0(u)=\prod_{l=1}^{N}(u-\theta_l-\eta)(u+\theta_l-\eta)\prod_{k=2}^{m+1}(2u-k\eta)(-2u-k\eta+(n-2)\eta).$$

Then $\tau_n(u)$ is proportional to identity operator with a known coefficient $\bar{\Lambda}_n(u)$

$$\begin{aligned}\bar{\Lambda}_n(u)=&\prod_{l=1}^{N}(u-\theta_l+\eta)(u+\theta_l+\eta)\prod_{k=0}^{\bar{q}-1}\Big(-u+\frac{n-2}{2}\eta-\bar{\xi}-k\eta\Big)\\ &\times(-1)^{q+\bar{q}}\prod_{k=0}^{\bar{p}-1}\Big(-u+\frac{n-2}{2}\eta+\bar{\xi}-k\eta\Big)\prod_{k=0}^{q-1}(u-\xi-k\eta)\prod_{k=0}^{p-1}(u+\xi-k\eta).\end{aligned} \tag{3.32}$$

3.3 Asymptotic behaviors of the transfer matrices

The definitions (3.11)–(3.18) of the fused K-matrices, the fused monodromy matrices and the fused transfer matrices and the asymptotic behaviors (3.7) and (3.9) imply that the asymptotic behaviors of the operators $\{\tau_m(u)\}$ given by (3.29) is completely fixed by the eigenvalues of the product matrix $\bar{M}M$ (see (3.36) below). Firstly let us give some properties of the eigenvalues of $\bar{M}M$. Suppose $\{\lambda_l|l=1,\ldots,n\}$ be the eigenvalues. The fact that $M^2=\bar{M}^2=1$ allows one to derive the following relations among the eigenvalues,

$$\sum_{l=1}^{n}\lambda_l^k = tr\{(\bar{M}M)^k\} = tr\{(M\bar{M})^k\} = tr\{(\bar{M}M)^{-k}\} = \sum_{l=1}^{n}\lambda_l^{-k}, \quad \forall k. \tag{3.33}$$

Meanwhile we know that

$$\mathrm{Det}|\bar{M}M| = \lambda_1\ldots\lambda_n = (-1)^{q+\bar{q}}. \tag{3.34}$$

This implies that the eigenvalues of $M\bar{M}$ should take the following form

$$\{\lambda_1,\ldots,\lambda_n\} = \{1,\ldots,1,-1,\ldots,-1,e^{-i\vartheta_1},e^{i\vartheta_1},\ldots,e^{-i\vartheta_r},e^{i\vartheta_r}\}, \tag{3.35}$$

where ϑ_j are some continuous free parameters which are related to boundary interaction terms (e.g., the boundary magnetic fields). The maximum number of the continuous parameters is $n/2$ if n is even and is $(n-1)/2$ if n is odd.

Some remarks are in order. When M and $\bar{M}$ commute with each other and thus can be diagonalized simultaneously by some gauge transformation, the corresponding open spin

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chain can be diagonalized by the algebraic Bethe ansatz method after a global gauge transformation [71]. In case of the boundary parameters (which are related to the matrices M and $\bar{M}$) have some constraints so that a proper "local vacuum state" exists, the generalized algebraic Bethe ansatz method [41, 77, 78] can be used to obtain the Bethe ansatz solutions of the associated open spin chains [79–81]. However, the results in [10–13] strongly suggest that for generic M and $\bar{M}$ such a simple "local vacuum state" do not exist even for the $su(2)$ case.

The asymptotic behaviors (3.7) and (3.9) enable us to derive that the eigenvalue $\bar{\Lambda}_m(u)$ of the operators $\{\tau_m(u)\}$ given by (3.29) have the following asymptotic behaviors

$$\bar{\Lambda}_m(u) = (-1)^m \delta_m \, u^{2N+2m} + \ldots, \quad m = 1, \ldots, n, \quad u \to \infty, \tag{3.36}$$

where

$$\delta_m = \sum_{1 \le i_1 < i_2 \ldots < i_m \le n} \lambda_{i_1} \ldots \lambda_{i_m}, \quad m = 1, \ldots, n.$$

Keeping the fact that $\bar{\Lambda}_n(u)$ has been already fixed (3.32) in the mind, we need to determine the eigenvalues of the other $n-1$ transfer matrices $\{\tau_m(u)|m = 1, \ldots, n-1\}$. It is also known from (3.29) that $\bar{\Lambda}_m(u)$, as a function of u, is a polynomial of degree $2N+2m$. Thanks to the very functional relations (3.31) and the asymptotic behaviors (3.36), one can completely determine the eigenvalues of the transfer matrix and the other higher fused transfer matrices by providing some other values of the eigenvalue functions at $\sum_{m=1}^{n-1} 2m$ special points (3.25) (e.g. see (4.8)–(4.13) for the $su(3)$-case and (B.1)–(B.12) for the $su(4)$-case). The method has been proven in [10–13] to be successful in solving the open spin chains related to $su(2)$ algebra. In the following section, we shall apply the method to solve the open spin chains associated with $su(n)$ algebra.

For this purpose, let us first factorize out the contributions of K-matrices which are relevant to the quantum determinant $\bar{\Lambda}_n(u)$ (3.32) by introducing n functions $\{K^{(l)}(u)|l = 1, \ldots, n\}$ which are polynomials of u with a degree 2. The functions depend only on the boundary parameters ξ and $\bar{\xi}$ and satisfy the following relations

$$\begin{aligned}\prod_{l=1}^{n} K^{(l)}(u-(l-1)\eta) = (-1)^{q+\bar{q}} &\prod_{k=0}^{\bar{q}-1}\left(-u+\frac{n-2}{2}\eta-\bar{\xi}-k\eta\right) \\ &\times \prod_{k=0}^{\bar{p}-1}\left(-u+\frac{n-2}{2}\eta+\bar{\xi}-k\eta\right)\prod_{k=0}^{q-1}(u-\xi-k\eta)\prod_{k=0}^{p-1}(u+\xi-k\eta),\end{aligned} \tag{3.37}$$

$$K^{(l)}(u)K^{(l)}(-u-l\eta) = K^{(l+1)}(u)K^{(l+1)}(-u-l\eta), \quad l = 1, \cdots, n-1. \tag{3.38}$$

From the solution to the above equations, one can construct a nested T-Q ansatz for the eigenvalues $\Lambda_m(u)$. It is remarked that there are some different solutions to the above equations. However, it was shown in [10–13, 58] that for the $su(2)$ open spin chain any choice of the above equation leads to a complete set of solutions of the the corresponding model. It is believed that different choices of the solution might only give rise to different parameterizations of the eigenvalues.

Before closing this section, let us give a summary of the set of properties which characterize the eigenvalue of the transfer matrix $t_m(u)$:

- explicit expression of $t_n(u)$ or the quantum determinant (3.22).
- Analytical property and asymptotical behaviors (3.36) of the transfer matrices.
- Functional relations (3.27) for the fused transfer matrices.
- The values of the transfer matrices at the special points (3.25) (for examples (4.8)–(4.13) for the $su(3)$-case and (B.1)–(B.12) for the $su(4)$-case).

The above condition are believed to determine the eigenvalues of the transfer matrices $t_m(u)$.

4 $su(3)$-invariant spin chain with non-diagonal boundary term

In this section, we use the method outlined in the previous section to give the Bethe ansatz solution of the $su(3)$-invariant spin chain with generic boundary terms. Without loss of generality, we take the corresponding M and $\bar{M}$ with $p = \bar{p} = 1$ and the eigenvalues of $\bar{M}M$ being

$$(\lambda_1, \lambda_2, \lambda_3) = (1, e^{-i\vartheta}, e^{i\vartheta}), \tag{4.1}$$

as an example to demonstrate our method in detail.

Let us introduce 3 functions $\{K^{(l)}|l = 1, 2, 3\}$ as follows

$$K^{(1)}(u) = \left(\bar{\xi} + \frac{1}{2}\eta - u\right)(\xi + u), \tag{4.2}$$

$$K^{(2)}(u) = \left(\bar{\xi} + \frac{3}{2}\eta + u\right)(\xi - u - \eta), \tag{4.3}$$

$$K^{(3)}(u) = \left(\bar{\xi} + \frac{3}{2}\eta + u\right)(\xi - u - \eta), \tag{4.4}$$

which satisfy (3.37) and (3.38) for $n = 3$. From the definitions (3.18) of the fused transfer matrices $t_m(u)$ and the asymptotic behaviors of the K-matrices $K^{\pm}(u)$, we have that the eigenvalues of the transfer matrices have the following asymptotic behaviors

$$\begin{aligned}\bar{\Lambda}(u)|_{u\to\infty} &= -tr(\bar{M}M)u^{2N+2} + \ldots = -\sum_{i=1}^{3}\lambda_i u^{2N+2} + \ldots \\ &= -(1 + 2\cos\vartheta)u^{2N+2} + \ldots,\end{aligned} \tag{4.5}$$

$$\begin{aligned}\bar{\Lambda}_2(u)|_{u\to\infty} &= tr_{12}\left\{P^{(-)}_{1,2}(\bar{M}M)_1(\bar{M}M)_2P^{(-)}_{1,2}\right\}u^{2N+4} + \ldots \\ &= \sum_{1\le i_1<i_2\le 3}\lambda_{i_1}\lambda_{i_2}u^{2N+4} + \ldots \\ &= (2\cos\vartheta + 1)u^{2N+4} + \ldots.\end{aligned} \tag{4.6}$$

The explicit expressions of the K-matrices (3.6) and (3.8) imply the following identities

$$K^-(u)\,K^-(-u)=(\xi^2-u^2),\quad K^+\Big(u+\frac{n}{2}\eta\Big)K^+\Big(-u+\frac{n}{2}\eta\Big)=(\bar{\xi}^2-u^2). \tag{4.7}$$

The above relations and some degenerated properties (3.24) of the R-matrix and the K-matrices allow us to derive that the fused transfer matrices satisfy the following properties at some special points (3.25), namely, 0, $-\frac{3}{2}\eta$; 0, $\frac{\eta}{2}$, $-\frac{\eta}{2}$, $-\eta$ for the $su(3)$-case:

$$t(0) = (-1)^N\xi\prod_{l=1}^N(\theta_l+\eta)(\theta_l-\eta)tr\{K^+(0)\}\ \times\mathrm{id}, \tag{4.8}$$

$$t\Big(-\frac{3}{2}\eta\Big) = (-1)^N\bar{\xi}\prod_{l=1}^N\Big(\theta_l+\frac{3}{2}\eta\Big)\Big(\theta_l-\frac{3}{2}\eta\Big)tr\Big\{K^-\Big(-\frac{3}{2}\eta\Big)\Big\}\ \times\mathrm{id}, \tag{4.9}$$

$$\begin{aligned} t_2\Big(\frac{\eta}{2}\Big) &= tr_{12}\Big\{P^-_{12}K^+_2\Big(-\frac{\eta}{2}\Big)R_{12}(-3\eta)K^+_1\Big(\frac{\eta}{2}\Big)P^-_{12}\Big\}\Big(\frac{\eta^2}{4}-\xi^2\Big)\eta\\ &\quad\times\prod_{l=1}^N\Big(\theta_l+\frac{3}{2}\eta\Big)\Big(\theta_l-\frac{3}{2}\eta\Big)\Big(\theta_l+\frac{\eta}{2}\Big)\Big(\theta_l-\frac{\eta}{2}\Big)\ \times\mathrm{id}, \end{aligned}\tag{4.10}$$

$$\begin{aligned} t_2(-\eta) &= tr_{12}\left\{P^-_{12}K^-_1(-\eta)R_{21}(-3\eta)K^-_2(-2\eta)P^-_{12}\right\}\Big(\frac{\eta^2}{4}-\bar{\xi}^2\Big)\eta\\ &\quad\times\prod_{l=1}^N(\theta_l+\eta)(\theta_l-\eta)(\theta_l+2\eta)(\theta_l-2\eta)\ \times\mathrm{id}, \end{aligned}\tag{4.11}$$

$$t_2(0) = (-1)^N2\xi\eta^2\prod_{l=1}^N(\theta_l+\eta)(\theta_l-\eta)tr\{K^+(0)\}\,t(-\eta), \tag{4.12}$$

$$t_2\Big(-\frac{\eta}{2}\Big) = (-1)^N2\bar{\xi}\eta^2\prod_{l=1}^N\Big(\theta_l+\frac{3}{2}\eta\Big)\Big(\theta_l-\frac{3}{2}\eta\Big)tr\Big\{K^-\Big(-\frac{3}{2}\eta\Big)\Big\}\,t\Big(-\frac{\eta}{2}\Big), \tag{4.13}$$

These relations allow us to derive similar relations of the eigenvalues $\{\bar{\Lambda}_m(u)\}$. Then the resulting relations (total number of the conditions is equal to $2+4=6$), the very relations (3.31) for $n=3$ and the asymptotic behaviors (4.5)–(4.6) allow us to determine the eigenvalues $\bar{\Lambda}_m(u)$ (also $\Lambda_m(u)$ via the relations (3.30)).

Let us define the corresponding $Q^{(r)}(u)$ for the open spin chains

$$Q^{(0)}(u) = \prod_{j=1}^N(u-\theta_j)(u+\theta_j), \tag{4.14}$$

$$Q^{(r)}(u) = \prod_{l=1}^{L_r}(u-\lambda^{(r)}_l)(u+\lambda^{(r)}_l+r\eta),\quad r=1,\ldots,n-1, \tag{4.15}$$

$$Q^{(n)}(u) = 1, \tag{4.16}$$

where $\{L_r|r=1,\ldots,n-1\}$ are some non-negative integers. In the following part of the paper, we adopt the convention

$$a(u)=Q^{(0)}(u+\eta),\quad d(u)=Q^{(0)}(u). \tag{4.17}$$

In order to construct the solution of the open $su(3)$ spin chain, we introduce three functions

$$\tilde{z}_1(u) = z_1(u) + x_1(u), \quad \tilde{z}_2(u) = z_2(u), \quad \tilde{z}_3(u) = z_3(u). \tag{4.18}$$

Here $z_m(u)$ is given by the following relations

$$\begin{aligned} z_m(u) &= \frac{u(u+\frac{3}{2}\eta)}{(u+\frac{(m-1)}{2}\eta)(u+\frac{m}{2}\eta)} K^{(m)}(u)d(u)\frac{Q^{(m-1)}(u+\eta)Q^{(m)}(u-\eta)}{Q^{(m-1)}(u)Q^{(m)}(u)}, \\ & m=1,2,3, \end{aligned} \tag{4.19}$$

with $\{K^{(m)}(u)|m=1,2,3\}$ are given by (4.2)–(4.4) (here we have assumed $Q^{(3)}(u)=1$ since that the $su(3)$-case is considered) and $x_1(u)$ is defined as

$$x_1(u) = u\left(u+\frac{3}{2}\eta\right)a(u)d(u)\frac{F_1(u)}{Q^{(1)}(u)}. \tag{4.20}$$

The nested functional T-Q ansatz is expressed as

$$\Lambda(u) = \sum_{i_1=1}^{3} \tilde{z}_{i_1}(u) = \sum_{i_1=1}^{3} z_{i_1}(u) + u\left(u+\frac{3}{2}\eta\right)a(u)d(u)\frac{F_1(u)}{Q^{(1)}(u)}, \tag{4.21}$$

$$\Lambda_2(u) = \rho_2(2u-\eta)\left\{\sum_{1\le i_1<i_2\le 3} \tilde{z}_{i_1}(u)\tilde{z}_{i_2}(u-\eta) - x_1(u)z_2(u-\eta)\right\}. \tag{4.22}$$

We remark that the extra term $x_1(u)$ in (4.18) given by (4.20) does not violate the very functional relation (3.28) with $n=3$ due to the fact $a(\pm\theta_j-\eta)=d(\pm\theta_j)=0$, but it does change the form of the resulting BAEs (see (4.29) below). The function $x_1(u)$ can be determined by regularity of $\Lambda(u)$ and $\Lambda_2(u)$ given by (4.21) and (4.22) and their asymptotic behaviors as follows. The vanishing of the residues of $\Lambda(u)$ at $\lambda_j^{(1)}$ and $-\lambda_j^{(1)}-\eta$ requires

$$F_1(u) = f_1(u)Q^{(2)}(-u-\eta), \tag{4.23}$$

with

$$f_1(u) = f_1(-u-\eta). \tag{4.24}$$

In order not to violate the relations (4.8)–(4.13), let all terms with $x_1(u)$ in $\Lambda_m(u)$ be zero at all the degenerate points considered in (4.8)–(4.13), then the function $f_1(u)$ is given by

$$f_1(u) = c\,u\left(u+\frac{1}{2}\eta\right)^2(u+\eta). \tag{4.25}$$

The asymptotic behaviors of $\Lambda(u)$ and $\Lambda_2(u)$ then fix the constant c,

$$c = 2(\cos\vartheta - 1), \tag{4.26}$$

where ϑ is specified by the eigenvalues of the matrix $\bar{M}M$ (4.1). It is remarked that $F_1(u)$ is a polynomial of degree $2L_1-2N$. Then the above relations lead to the constraint among the non-negative integers L_1 and L_2

$$L_1 = N + L_2 + 2. \tag{4.27}$$

(4.23)–(4.27) and the relations (3.30) between $\bar{\Lambda}_m(u)$ and $\Lambda_m(u)$ lead to that the asymptotic behaviors (4.5)–(4.6) of the eigenvalues $\bar{\Lambda}_m(u)$ are automatically satisfied. Keeping the following identities in mind,

$$a(\pm\theta_j-\eta)=d(\pm\theta_j)=0,\quad j=1,\ldots,N, \tag{4.28}$$

with the help of (3.22), by putting $u=\pm\theta_j,\pm\theta_j-\eta$ in (4.21) and (4.22) we can easily show that the ansatz (4.21)–(4.22) make the very functional relations (3.28) with $n=3$ fulfilled. The regular property of $\Lambda(u)$ and $\Lambda_2(u)$ leads to the associated Bethe ansatz equations,

$$\begin{aligned}1+&\frac{\lambda_l^{(1)}}{\lambda_l^{(1)}+\eta}\frac{K^{(2)}(\lambda_l^{(1)})d(\lambda_l^{(1)})}{K^{(1)}(\lambda_l^{(1)})a(\lambda_l^{(1)})}\frac{Q^{(1)}(\lambda_l^{(1)}+\eta)Q^{(2)}(\lambda_l^{(1)}-\eta)}{Q^{(1)}(\lambda_l^{(1)}-\eta)Q^{(2)}(\lambda_l^{(1)})}\\ &=-c\,\frac{(\lambda_l^{(1)})^2(\lambda_l^{(1)}+\frac{1}{2}\eta)^3(\lambda^{(1)}+\eta)d(\lambda_l^{(1)})Q^{(2)}(\lambda_l^{(1)}-\eta)}{K^{(1)}(\lambda_l^{(1)})Q^{(1)}(\lambda_l^{(1)}-\eta)},\qquad l=1,\ldots,L_1,\end{aligned} \tag{4.29}$$

$$\frac{\lambda_l^{(2)}+\frac{3}{2}\eta}{\lambda_l^{(2)}+\frac{1}{2}\eta}\frac{K^{(2)}(\lambda_l^{(2)})}{K^{(3)}(\lambda_l^{(2)})}\frac{Q^{(1)}(\lambda_l^{(2)}+\eta)Q^{(2)}(\lambda_l^{(2)}-\eta)}{Q^{(1)}(\lambda_l^{(2)})Q^{(2)}(\lambda_l^{(2)}+\eta)}=-1,\qquad l=1,\ldots L_2. \tag{4.30}$$

The eigenvalue of the Hamiltonian (3.10) in the case of $n=3$ is given by

$$E=\sum_{l=1}^{L_1}\frac{2\eta^2}{\lambda_l^{(1)}(\lambda_l^{(1)}+\eta)}+2(N-1)+\eta\frac{\bar{\xi}+\frac{3}{2}\eta-\bar{p}\eta-\xi}{\xi(\bar{\xi}+\frac{3}{2}\eta-\bar{p}\eta)}+\frac{2}{3}, \tag{4.31}$$

where the parameters $\{\lambda_l^{(1)}\}$ are the roots of the BAEs (4.29)–(4.30) in the homogeneous limit $\theta_j=0$.

5 Exact solution of $su(n)$-invariant spin chain with general open boundaries

The analogs of (4.8)–(4.13) for arbitrary n at the special points listed in (3.25) can also be constructed with the properties of (3.24) and (4.7). To show the procedure clearly, we construct those relations for $n=4$ in appendix B. In fact, those relations are ensured for the diagonal case as already demonstrated by the algebraic Bethe ansatz. For the non-diagonal case, since we put $x_i(u)$ to be zero for u at those degenerate points, the relations must also hold no matter how their exact forms are. By following the same procedure as the previous section, we may derive the solutions of the $su(n)$-invariant quantum spin chain with general open boundary conditions. Here we present the final result. The functions $z_m(u)$ now read

$$\begin{aligned}z_m(u)=&\frac{2u(2u+n\eta)}{(2u+(m-1)\eta)(2u+m\eta)}K^{(m)}(u)Q^{(0)}(u)\frac{Q^{(m-1)}(u+\eta)Q^{(m)}(u-\eta)}{Q^{(m-1)}(u)Q^{(m)}(u)},\\ &m=1,\ldots,n,\end{aligned} \tag{5.1}$$

where $\{K^{(l)}(u)|l=1,\ldots,n\}$ satisfy (3.37)–(3.38) and $\{Q^{(m)}(u)|m=0,1,\ldots,n\}$ are given by (4.14)–(4.16). In principle, $K^{(l)}(u)$ could be any decomposition of (3.37). For simplicity, we parameterize them satisfying the following relations

$$K^{(l)}(u) = K^{(l+1)}(-u - l\eta), \quad l = 1, \cdots, n-1. \tag{5.2}$$

Similarly as (4.18), let us introduce the functions $\{\tilde{z}_i(u)|i = 1, \ldots, n\}$ by

$$\tilde{z}_i(u) = z_i(u) + x_i(u), \quad i = 1, \ldots, n, \tag{5.3}$$

where the functions $x_i(u)$ are

$$\begin{cases} x_{2l-1}(u) = u(u + \frac{n}{2}\eta)a(u)d(u)\frac{F_{2l-1}(u)}{Q^{(2l-1)}(u)} \\ x_{2l}(u) = 0 \end{cases}, \tag{5.4}$$

and $l = 1, 2, \ldots, \frac{n}{2}$ if n is even, $l = 1, 2, \ldots, \frac{n-1}{2}$. The functions $\{F_{2l-1}(u)\}$ are given by

$$F_1(u) = f_1(u)Q^{(2)}(-u - \eta), \tag{5.5}$$

$$F_{2l-1}(u) = f_{2l-1}(u)Q^{(2l-2)}(-u - (2l-1)\eta)Q^{(2l)}(-u - (2l-1)\eta)a(-u - (2l-1)\eta), \tag{5.6}$$

where $l = 2, \ldots, \frac{n}{2}$ if n is even and $l = 2, \ldots, \frac{n-1}{2}$ if n is odd, and

$$f_{2l-1}(u) = c_{2l-1}\prod_{k=1}^{n-1}\left(u + \frac{k}{2}\eta\right)\left(u + (2l-1)\eta - \frac{k}{2}\eta\right), \quad l = 1, 2, \cdots. \tag{5.7}$$

The functions $f_{2l-1}(u)$ has the following crossing symmetry relation

$$f_{2l-1}(u) = f_{2l-1}(-u - (2l-1)\eta). \tag{5.8}$$

Here the parameters $\{c_{2l-1}\}$ are determined, with helps of the asymptotic behaviors of the eigenvalues of the transfer matrices, by the following relations

$$\begin{aligned} &\sum_{1\le i_1<i_2<\ldots<i_m\le n}\tilde{c}_{i_1}\tilde{c}_{i_2}\ldots\tilde{c}_{i_m} + \sum_{k=1}^{m_1}\sum_{l=k}^{m_2}\sum_{1\le i_1<i_2<\ldots<i_{2k-2}\le 2l-2}\tilde{c}_{i_1}\tilde{c}_{i_2}\ldots\tilde{c}_{i_{2k-2}}\tilde{c}_{2l-1} \\ &\times\sum_{2l+1\le i_{2k+1}<i_{2k+2}<\ldots<i_m\le n}\tilde{c}_{i_{2k+1}}\tilde{c}_{i_{2k+2}}\ldots\tilde{c}_{i_m} + \sum_{k=2}^{m_3}\sum_{l=k}^{m_4}\sum_{1\le i_1<i_2<\ldots<i_{2k-3}\le 2l-2}\tilde{c}_{i_1} \\ &\times\tilde{c}_{i_2}\ldots\tilde{c}_{i_{2k-3}}\tilde{c}_{2l-1}\sum_{2l+1\le i_{2k}<i_{2k+1}<\ldots<i_m\le n}\tilde{c}_{i_{2k}}\tilde{c}_{i_{2k+1}}\ldots\tilde{c}_{i_m} \\ &= (-1)^m\sum_{1\le i_1<i_2<\ldots<i_m\le n}\lambda_{i_1}\lambda_{i_2}\ldots\lambda_{i_m}, \end{aligned} \tag{5.9}$$

where

$$\tilde{c}_i = \begin{cases} -1 + c_{2l-1}, & i = 2l-1, \\ -1, & i = 2l, \\ -1, & i = n, \end{cases}$$

and

$$(i). \quad m_1 = m_3 = \frac{m}{2}, \quad m_2 = m_4 = \frac{n-m-2k}{2}, \quad \text{if } m \text{ is even and } n \text{ is even;} \tag{5.10}$$

$$(ii).\quad m_1=\frac{m-1}{2},\quad m_2=m_4=\frac{n-m-2k-1}{2},\quad m_3=\frac{m+1}{2},$$
$$\text{if } m \text{ is odd and } n \text{ is even}, \tag{5.11}$$
$$(iii).\quad m_1=m_3=\frac{m}{2},\quad m_2=m_4=\frac{n-m-2k-1}{2},$$
$$\text{if } m \text{ is even and } n \text{ is odd}, \tag{5.12}$$
$$(iv).\quad m_1=\frac{m-1}{2},\quad m_2=\frac{n-m-2k}{2},\quad m_3=\frac{m+1}{2},\quad m_4=\frac{n-m-2k-2}{2},$$
$$\text{if } m \text{ is odd and } n \text{ is odd}. \tag{5.13}$$

The constants $\{\lambda_i|i=1,\ldots,n\}$ are the eigenvalues of the matrix $\bar{M}M$ given by (3.35), while the matrices $\bar{M}$ and M are related to the K-matrices $K^{\pm}(u)$. Then the nested T-Q ansatz of the eigenvalues $\Lambda(u)$ of the transfer matrix $t(u)$ is

$$\Lambda(u)=\sum_{i=1}^{n}\tilde{z}_i(u), \tag{5.14}$$

where the functions $\{\tilde{z}_i(u)\}$ are given by (5.3). The other eigenvalues $\Lambda_m(u)$ of the fused transfer matrix $t_m(u)$ are given by

$$\begin{aligned}\Lambda_m(u)=&\prod_{l=1}^{m-1}\prod_{k=1}^{l}\rho_2(2u-k\eta-l\eta+\eta)\\&\times\Bigg\{\sum_{1\le i_1<i_2<\ldots<i_m\le n}\tilde{z}_{i_1}(u)\tilde{z}_{i_2}(u-\eta)\ldots\tilde{z}_{i_m}(u-(m-1)\eta)\\&-\sum_{k=1}^{m_1}\sum_{l=k}^{m_2}\sum_{1\le i_1<i_2<\ldots<i_{2k-2}\le 2l-2}\tilde{z}_{i_1}(u)\tilde{z}_{i_2}(u-\eta)\ldots\tilde{z}_{i_{2k-2}}(u-(2k-3)\eta)\\&\times f_{2l-1}(u-(2k-2)\eta)\tilde{z}_{2l}(u-(2k-1)\eta)\sum_{2l+1\le i_{2k+1}<i_{2k+2}<\ldots<i_m\le n}\\&\times\tilde{z}_{i_{2k+1}}(u-2k\eta)\tilde{z}_{i_{2k+2}}(u-(2k+1)\eta)\ldots\tilde{z}_{i_m}(u-(m-1)\eta)\\&-\sum_{k=2}^{m_3}\sum_{l=k}^{m_4}\sum_{1\le i_1<i_2<\ldots<i_{2k-3}\le 2l-2}\tilde{z}_{i_1}(u)\tilde{z}_{i_2}(u-\eta)\ldots\tilde{z}_{i_{2k-3}}(u-(2k-4)\eta)\\&\times f_{2l-1}(u-(2k-3)\eta)\tilde{z}_{2l}(u-(2k-2)\eta)\sum_{2l+1\le i_{2k}<i_{2k+1}<\ldots<i_m\le n}\\&\times\tilde{z}_{i_{2k}}(u-(2k-1)\eta)\tilde{z}_{i_{2k+1}}(u-2k\eta)\ldots\tilde{z}_{i_m}(u-(m-1)\eta)\Bigg\},\end{aligned} \tag{5.15}$$

where the m_1, m_2, m_3 and m_4 are the same as those in the equations (5.10)–(5.13). The parameters $\{\lambda_l^{(r)}\}$ satisfy the associated Bethe ansatz equations

$$\begin{aligned}&K^{(1)}(\lambda_j^{(1)})a(\lambda_j^{(1)})Q^{(1)}(\lambda_j^{(1)}-\eta)+\frac{\lambda_j^{(1)}}{\lambda_j^{(1)}+\eta}K^{(2)}(\lambda_j^{(1)})d(\lambda_j^{(1)})Q^{(1)}(\lambda_j^{(1)}+\eta)\frac{Q^{(2)}(\lambda_j^{(1)}-\eta)}{Q^{(2)}(\lambda_j^{(1)})}\\&+\lambda_j^{(1)}\left(\lambda_j^{(1)}+\frac{\eta}{2}\right)a(\lambda_j^{(1)})d(\lambda_j^{(1)})F_1(\lambda_j^{(1)})=0,\quad j=1,\ldots,L_1.\end{aligned} \tag{5.16}$$

$$\frac{2\lambda_k^{(2l)}+(2l+1)\eta}{2\lambda_k^{(2l)}+(2l-1)\eta}\frac{K^{(2l)}(\lambda_k^{(2l)})}{K^{(2l+1)}(\lambda_k^{(2l)})}\frac{Q^{(2l-1)}(\lambda_k^{(2l)}+\eta)Q^{(2l+1)}(\lambda_k^{(2l)})}{Q^{(2l-1)}(\lambda_k^{(2l)})Q^{(2l+1)}(\lambda_k^{(2l)}-\eta)}=-\frac{Q^{(2l)}(\lambda_k^{(2l)}+\eta)}{Q^{(2l)}(\lambda_k^{(2l)}-\eta)},$$
$$k=1,\ldots,L_{2l}, \tag{5.17}$$

$$K^{(2s+1)}(\lambda_j^{(2s+1)})Q^{(2s+1)}(\lambda_j^{(2s+1)}-\eta)+\frac{\lambda_j^{(2s+1)}+s\eta}{\lambda_j^{(2s+1)}+(s+1)\eta}K^{(2s+2)}(\lambda_j^{(2s+1)})$$
$$\times Q^{(2s+1)}(\lambda_j^{(2s+1)}+\eta)\frac{Q^{(2s)}(\lambda_j^{(2s+1)})Q^{(2s+2)}(\lambda_j^{(2s+1)}-\eta)}{Q^{(2s)}(\lambda_j^{(2s+1)}+\eta)Q^{(2s)}(\lambda_j^{(2s+1)})}+(\lambda_j^{(2s+1)}+s\eta)$$
$$\times\left(\lambda_j^{(2s+1)}+\frac{2s+1}{2}\eta\right)a(\lambda_j^{(2s+1)})\frac{Q^{(2s)}(\lambda_j^{(2s+1)})}{Q^{(2s)}(\lambda_j^{(2s+1)}+\eta)}F_{2s+1}(\lambda_j^{(2s+1)})=0, \tag{5.18}$$
$$j=1,\ldots,L_{2s+1},$$

where $l=s=1,\ldots,\frac{n}{2}-1$ if n is even, $l=1,\ldots,\frac{n-1}{2}$ and $s=1,\ldots,\frac{n-1}{2}-1$ if n is odd.

The rule for constructing $\{x_i(u)|i=1,\ldots,n\}$ is the following: (1)they must vanish for u at all special points appeared in the operator identities (3.28) and in (3.25) and therefore their existence does not affect the operator identities and the analogs of (4.8)–(4.13); (2)the denominators must not generate new poles and therefore should be $Q_i(u)$ or part of them; (3)the functions $F_i(u)$ must satisfy the corresponding crossing symmetry properties to keep the self-consistency of the BAEs. The explicit expressions (5.4)–(5.7) of the functions $\{x_i(u)|i=1,\ldots,n\}$ and the nested T-Q ansatz (5.14)–(5.15) of the eigenvalues $\Lambda_i(u)$ imply that the nested T-Q ansatz does satisfy the very function relations (3.28) due to the fact $a(\pm\theta_j-\eta)=d(\pm\theta_j)=0$. Since that these functions $\{x_i(u)|i=1,\ldots,n\}$ also vanish at the degenerated points (3.25), hence they do not violate the relations (for examples (4.8)–(4.13) for the $su(3)$-case and (B.1)–(B.12) for the $su(4)$-case) of the transfer matrices at these degenerated points. Moreover, the special choice of these functions (5.4)–(5.7) also assures that the regularity of the all eigenvalues $\{\Lambda_i(u)|i=1,\ldots,n\}$ of the transfer matrices can be guaranteed consistently by the resulting BAEs (5.16)–(5.18). The asymptotic behaviors (3.36) of the eigenvalues of the transfer matrices lead to the equations (5.9)–(5.13), which fixes the constants $\{c_i\}$ in (5.7).

Some remarks about the existence of solutions of (5.9) are in order. Due to the construction rule (5.15) for $\Lambda_m(u)$, the asymptotic behaviors of $\Lambda_m(u)$ and $\Lambda_{n-m}(u)$ give the same condition in (5.9), namely, the total number of the independent equations in (5.9) is $n/2$ if n is even and is $(n-1)/2$ if n is odd. Thus, the equations (5.9)–(5.13) will fix the constants $\{c_i\}$ in (5.7). We have checked that for $n=3$ there exists an unique solution to (5.9)–(5.13) (e.g. (4.26)) and that for $n>4$ there are more solutions (actually there are two solutions for $n=4$) to (5.9)–(5.13). For an example, the independent equations for $n=4$ and $p=\bar{p}=2$ are

$$\begin{cases}-4+c_1+c_3=-2\cos\vartheta_1-2\cos\vartheta_2\\ 4-2c_1-2c_3+c_1c_3=4\cos\vartheta_1\cos\vartheta_2\end{cases}. \tag{5.19}$$

However, the different solutions only give different parameterizations of $\Lambda_i(u)$ in the T-Q-type. It has been shown in [10–13] that even for the open chains related to the $su(2)$ case there indeed exist different T-Q ansatz for the eigenvalue of the transfer matrix. The numerical check [10–13, 58] for the small sites of lattice shows that any of them gives the complete set of eigenvalues.

The eigenvalue of the Hamiltonian (3.10) is

$$E = \sum_{l=1}^{L_1} \frac{2\eta^2}{\lambda_l^{(1)}(\lambda_l^{(1)}+\eta)} + 2(N-1) + \eta \frac{[K^{(1)}(u)]'}{K^{(1)}(u)}\,|_{u\to 0} + \frac{2}{n}, \tag{5.20}$$

where the parameters $\{\lambda_l^{(1)}\}$ are the roots of the BAEs (5.16)–(5.18) with $\theta_j = 0$.

When two K-matrices $K^+(u)$ and $K^-(u)$ are both diagonal matrices, or they can be diagonalized simultaneously by some gauge transformation, all the parameters c_{2l-1} vanish, leading to $F_{2l-1}(u) = 0$. The eigenvalue $\Lambda(u)$ of the transfer matrix $t(u)$ and the BAEs recover those obtained by the other Bethe ansatz methods [77, 78, 82–88].

6 Conclusions

In this paper, we propose the nested off-diagonal Bethe ansatz method for solving the multi-component integrable models with generic integrable boundaries, a generalization of the method proposed in [10–13] (related to $su(2)$ algebra) for integrable models associated with higher rank algebras. In the method some functional relations (for the $su(n)$ case such as (2.16) for the closed chain or (3.27) for the open chain) among the antisymmetric fused transfer matrices play a very important role. Taking the $su(n)$-invariant spin chain model with both periodic and non-diagonal boundaries as examples, we elucidate how the method works for constructing the Bethe ansatz solutions of the model. For the $su(n)$-invariant closed chain, we re-derive the results obtained previously by other methods [62–64], but with a simplified process. For the open boundary case specified by the most general K-matrices (3.6) and (3.8), the very functional relations (3.27) are derived only via some properties of the R-matrix and K-matrices. Based on these relations, the asymptotic behaviors (3.36) and the values of the eigenvalue functions (for examples, (4.8)–(4.13) for the $su(3)$-case and (B.1)–(B.12) for the $su(4)$-case) at $\sum_{m=1}^{n-1} 2m$ special points (3.25), we obtain the eigenvalues of the transfer matrix. When the K-matrices are both diagonal ones, our results can be reduced to those obtained by the conventional Bethe ansatz methods. Therefore, our method provides an unified procedure for approaching the integrable models both with and without U(1) symmetry. We remark that this method might also be applied to other quantum integrable models defined in other algebras.

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A Proofs of the operator identities

In this appendix, we give the detailed proof of the following identities which are crucial to obtain the functional relations (2.16) and (3.27):

$$T_1(\theta_j)T_{\langle 2,3,\ldots,m\rangle}(\theta_j-\eta) = P^{(-)}_{1,2,\ldots,m}T_1(\theta_j)T_2(\theta_j-\eta)\ldots T_m(\theta_j-(m-1)\eta)\,P^{(-)}_{2,\ldots,m}, \tag{A.1}$$

$$\mathcal{J}_{1,\ldots,m}(\pm\theta_j) = P^{(-)}_{1,2,\ldots,m}\mathcal{J}_{1,\ldots,m}(\pm\theta_j). \tag{A.2}$$

The exchange relation (2.10) of the one row monodromy matrix $T(u)$ implies

$$R_{\bar{2}\bar{1}}(-\eta)T_{\bar{2}}(u-\eta)T_{\bar{1}}(u) = T_{\bar{1}}(u)T_{\bar{2}}(u-\eta)R_{\bar{2}\bar{1}}(-\eta).$$

The above relation and the fusion condition (2.7) allow one to derive the following identity

$$P^{(-)}_{\bar{1},\bar{2}}\,T_{\bar{1}}(u)T_{\bar{2}}(u-\eta)R_{\bar{1}\bar{2}}(-\eta) = T_{\bar{1}}(u)T_{\bar{2}}(u-\eta)R_{\bar{1}\bar{2}}(-\eta). \tag{A.3}$$

Let us evaluate the product of the operators $T_{\bar{1}}(\theta_j)$ and $T_{\bar{2}}(\theta_j-\eta)$

$$\begin{aligned}
T_{\bar{1}}(\theta_j)T_{\bar{2}}(\theta_j-\eta) = &R_{\bar{1}N}(\theta_j-\theta_N)\ldots R_{\bar{1}\,j+1}(\theta_j-\theta_{j+1})R_{\bar{1}j}(0)R_{\bar{1}\,j-1}(\theta_j-\theta_{j-1})\ldots R_{\bar{1}\,1}(\theta_j-\theta_1)\\
&\times R_{\bar{2}N}(\theta_j-\theta_N-\eta)\ldots R_{\bar{2}\,j+1}(\theta_j-\theta_{j+1}-\eta)R_{\bar{2}j}(-\eta)\\
&\times R_{\bar{2}\,j-1}(\theta_j-\theta_{j-1}-\eta)\ldots R_{\bar{2}\,1}(\theta_j-\theta_1-\eta)\\
=&R_{j\,j-1}(\theta_j-\theta_{j-1})\ldots R_{j\,1}(\theta_j-\theta_1)\\
&\times R_{\bar{1}N}(\theta_j-\theta_N)\ldots R_{\bar{1}j+1}(\theta_j-\theta_{j+1})\\
&\times R_{\bar{2}N}(\theta_j-\theta_N-\eta)\ldots R_{\bar{2}j+1}(\theta_j-\theta_{j+1}-\eta)R_{\bar{2}\bar{1}}(-\eta)\\
&\times R_{\bar{1}j}(0)R_{\bar{2}\,j-1}(\theta_j-\theta_{j-1}-\eta)\ldots R_{\bar{2}\,1}(\theta_j-\theta_1-\eta)\\
\overset{(A.3)}{=}&R_{j\,j-1}(\theta_j-\theta_{j-1})\ldots R_{j\,1}(\theta_j-\theta_1)\,P^{(-)}_{\bar{1},\bar{2}}\\
&\times R_{\bar{1}N}(\theta_j-\theta_N)\ldots R_{\bar{1}j+1}(\theta_j-\theta_{j+1})\\
&\times R_{\bar{2}N}(\theta_j-\theta_N-\eta)\ldots R_{\bar{2}j+1}(\theta_j-\theta_{j+1}-\eta)R_{\bar{2}\bar{1}}(-\eta)\\
&\times R_{\bar{1}j}(0)R_{\bar{2}\,j-1}(\theta_j-\theta_{j-1}-\eta)\ldots R_{\bar{2}\,1}(\theta_j-\theta_1-\eta)\\
=&P^{(-)}_{\bar{1},\bar{2}}\,T_{\bar{1}}(\theta_j)T_{\bar{2}}(\theta_j-\eta),
\end{aligned}$$

namely, we have

$$T_1(\theta_j)T_2(\theta_j-\eta) = P^{(-)}_{1,2}\,T_1(\theta_j)T_2(\theta_j-\eta), \quad j=1,\ldots,N. \tag{A.4}$$

Similarly, we have

$$\hat{T}_1(-\theta_j)\hat{T}_2(-\theta_j-\eta) = P^{(-)}_{1,2}\,\hat{T}_1(-\theta_j)\hat{T}_2(-\theta_j-\eta), \quad j=1,\ldots,N. \tag{A.5}$$

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Due to the fact that $R_{12}(-\eta)$ is proportional to the antisymmetric projector (2.7), the relation (A.3) also implies

$$T_{\langle 1,2\rangle}(u) = P^{(-)}_{1,2} T_1(u) T_2(u-\eta) P^{(-)}_{1,2} = T_1(u) T_2(u-\eta) P^{(-)}_{1,2}. \tag{A.6}$$

Using similar method to derive the above relation and following the procedure [65–69], we can derive the following relations

$$\begin{aligned} T_{\langle 1,2,\ldots,m\rangle}(u) &= P^{(-)}_{1,2,\ldots,m} T_1(u) T_2(u-\eta) \ldots T_m(u-(m-1)\eta) P^{(-)}_{1,2,\ldots,m} \\ &= T_1(u) T_2(u-\eta) \ldots T_m(u-(m-1)\eta) P^{(-)}_{1,2,\ldots,m}. \end{aligned} \tag{A.7}$$

Combining the above relation with (A.4), we can show that

$$P_{l,l+1}\, T_1(\theta_j) T_{\langle 2,3,\ldots,m\rangle}(\theta_j-\eta) = -T_1(\theta_j) T_{\langle 2,3,\ldots,m\rangle}(\theta_j-\eta), \quad l=1,\ldots,m-1. \tag{A.8}$$

Then we can conclude that $T_1(\theta_j)T_{\langle 2,3,\ldots,m\rangle}(\theta_j-\eta)$ satisfy the relation (A.1).

With the similar method used to prove (A.4) and the reflection equation (3.1), we can obtain the following relations:

$$\begin{aligned} \mathcal{J}_1(\pm\theta_j) R_{21}(\pm 2\theta_j - \eta) \mathcal{J}_2(\pm\theta_j - \eta) &= P^{(-)}_{1,2} \mathcal{J}_1(\pm\theta_j) R_{21}(\pm 2\theta_j - \eta) \mathcal{J}_2(\pm\theta_j - \eta), \\ & \qquad j = 1,\ldots,N. \end{aligned} \tag{A.9}$$

$$\mathcal{J}_{\langle 1,\ldots,m\rangle}(u) = \mathcal{J}_{1,\ldots,m}(u)\, P^{(-)}_{1,\ldots,m},\ m=1,\ldots n, \tag{A.10}$$

$$K^{+}_{\langle 1,\ldots,m\rangle}(u) = K^{+}_{1,\ldots,m}(u)\, P^{(-)}_{1,\ldots,m},\ m=1,\ldots n. \tag{A.11}$$

Using the relations (A.9) and (A.10), we can derive that

$$P_{l,l+1}\, \mathcal{J}_{1,\ldots,m}(\pm\theta_j) = -\mathcal{J}_{1,\ldots,m}(\pm\theta_j), \quad l=1,\ldots,m-1. \tag{A.12}$$

(A.2) is a consequence of the above relations. Hence we complete the proof of (A.2).

B Higher rank analogs of (4.8)–(4.13)

The explicit expressions of the K-matrices (3.6) and (3.8) imply some identities (4.7) among them. These identities and some degenerated properties (3.24) of the R-matrix and the K-matrices allow one to derive that the fused transfer matrices satisfy certain relations at some special points (3.25). For the $su(3)$-case they are given by (4.8)–(4.13). Here we present their analogs for the $su(4)$-case:

$$t(0) = (-1)^N \xi \prod_{l=1}^{N} (\theta_l+\eta)(\theta_l-\eta) tr\{K^+(0)\} \times \mathrm{id}, \tag{B.1}$$

$$t(-2\eta) = (-1)^N \bar{\xi} \prod_{l=1}^{N} \left(\theta_l + \frac{3}{2}\eta\right)(\theta_l - 2\eta) tr\{K^-(-2\eta)\} \times \mathrm{id}, \tag{B.2}$$

$$t_2(0) = 3(-1)^N \xi \eta^2 \prod_{l=1}^{N} (\theta_l+\eta)(\theta_l-\eta) tr\{K^+(0)\}\, t_1(-\eta), \tag{B.3}$$

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$$
\begin{aligned}
t_2\left(\frac{\eta}{2}\right) &= tr_{12}\left\{K^+_{\langle 12\rangle}\left(\frac{\eta}{2}\right)\right\}\eta\left(\frac{\eta^2}{4}-\xi^2\right) \\
&\quad\times\prod_{l=1}^{N}\left(\theta_l-\frac{\eta}{2}\right)\left(\theta_l+\frac{\eta}{2}\right)\left(\theta_l-\frac{3}{2}\eta\right)\left(\theta_l+\frac{3}{2}\eta\right)\times\mathrm{id}, \qquad (B.4)\\
t_2\left(-\frac{3}{2}\eta\right) &= \eta\left(\frac{\eta^2}{4}-\bar{\xi}^2\right)\prod_{l=1}^{N}\left(\theta_l-\frac{5}{2}\eta\right)\left(\theta_l+\frac{5}{2}\eta\right)\left(\theta_l-\frac{3}{2}\eta\right)\left(\theta_l+\frac{3}{2}\eta\right)\\
&\quad\times tr_{12}\left\{K^-_{\langle 12\rangle}\left(-\frac{3}{2}\eta\right)\right\}\times\mathrm{id}, \qquad (B.5)\\
t_2(-\eta) &= 3(-1)^N\bar{\xi}\eta^2\prod_{l=1}^{N}(\theta_l+2\eta)(\theta_l-2\eta)tr\{K^-(2\eta)\}\,t_1(-\eta), \qquad (B.6)
\end{aligned}
$$

and

$$
\begin{aligned}
t_3(0) &= 12(-1)^N\xi\eta^4\prod_{l=1}^{N}(\theta_l+\eta)(\theta_l-\eta)tr\{K^+(0)\}\,t_2(-\eta), \qquad (B.7)\\
t_3(0) &= 12(-1)^N\bar{\xi}\eta^4\prod_{l=1}^{N}(\theta_l+2\eta)(\theta_l-2\eta)tr\{K^-(-2\eta)\}\,t_2(0), \qquad (B.8)\\
t_3\left(\frac{\eta}{2}\right) &= 12tr_{12}\left\{K^+_{\langle 12\rangle}\left(\frac{\eta}{2}\right)\right\}\eta^5\left(\frac{\eta^2}{4}-\xi^2\right)t_1\left(-\frac{3}{2}\eta\right)\\
&\quad\times\prod_{l=1}^{N}\left(\theta_l-\frac{\eta}{2}\right)\left(\theta_l+\frac{\eta}{2}\right)\left(\theta_l-\frac{3}{2}\eta\right)\left(\theta_l+\frac{3}{2}\eta\right), \qquad (B.9)\\
t_3\left(-\frac{\eta}{2}\right) &= 12\eta^5\left(\frac{\eta^2}{4}-\bar{\xi}^2\right)tr_{23}\left\{K^-_{\langle 23\rangle}\left(-\frac{3}{2}\eta\right)\right\}t_1\left(-\frac{\eta}{2}\right)\\
&\quad\times\prod_{l=1}^{N}\left(\theta_l-\frac{5}{2}\eta\right)\left(\theta_l+\frac{5}{2}\eta\right)\left(\theta_l-\frac{3}{2}\eta\right)\left(\theta_l+\frac{3}{2}\eta\right), \qquad (B.10)\\
\frac{\partial}{\partial u}t_3(u)\,|_{u=\eta} &= 4\xi\eta^2(\xi^2-\eta^2)(-1)^N tr_{123}\{K^+_{\langle 123\rangle}(\eta)\}\\
&\quad\times\prod_{l=1}^{N}\theta_l^2(\theta_l-\eta)(\theta_l+\eta)(\theta_l-2\eta)(\theta_l+2\eta)\times\mathrm{id}, \qquad (B.11)\\
\frac{\partial}{\partial u}t_3(u)\,|_{u=-\eta} &= 4\bar{\xi}\eta^2(\eta^2-\bar{\xi}^2)(-1)^N tr_{123}\{K^-_{\langle 123\rangle}(-\eta)\}\\
&\quad\times\prod_{l=1}^{N}(\theta_l-\eta)(\theta_l+\eta)(\theta_l-2\eta)(\theta_l+2\eta)(\theta_l-3\eta)(\theta_l+3\eta)\times\mathrm{id}. \qquad (B.12)
\end{aligned}
$$

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