Boundary Weyl anomaly of $\mathcal{N} = (2, 2)$ superconformal models

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ABSTRACT: We calculate the trace and axial anomalies of $\mathcal{N} = (2, 2)$ superconformal theories with exactly marginal deformations, on a surface with boundary. Extending recent work by Gomis et al, we derive the boundary contribution that captures the anomalous scale dependence of the one-point functions of exactly marginal operators. Integration of the bulk super-Weyl anomaly shows that the sphere partition function computes the Kähler potential $K(\lambda, \bar{\lambda})$ on the superconformal manifold. Likewise, our results confirm the conjecture that the partition function on the supersymmetric hemisphere computes the holomorphic central charge, $c^\Omega(\lambda)$, associated with the boundary condition Ω . The boundary entropy, given by a ratio of hemispheres and sphere, is therefore fully determined by anomalies.

KEYWORDS: Anomalies in Field and String Theories, Conformal Field Theory, D-branes

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Dedicated to the memory of Ioannis Bakas.

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1 Introduction and summary

Recently Gomis et al. [1] computed the trace anomaly of two-dimensional $\mathcal{N} = (2, 2)$ superconformal field theories (SCFT) that belong to a continuous family (alias ‘superconformal manifold’) $\mathcal{M}$. The complex moduli $\{\lambda^I\}$ that parametrize $\mathcal{M}$ couple to exactly marginal operators $\{\mathcal{O}_I\}$ whose 2-point functions have a scale anomaly first discussed by Osborn [2]. This anomaly only manifests itself when the λ^I vary in space, but supersymmetry relates it to another term that does not vanish for constant couplings, $\delta \int \sqrt{g} R^{(2)} K(\lambda, \bar{\lambda})$ where K is the Kähler potential on $\mathcal{M}$. Although this term is the variation of a local action, it cannot be supersymmetrized in a local way and is hence an irreducible part of the anomaly. The integrated anomaly thus contains a term $\frac{1}{2} K \chi_M$ with χ_M the Euler characteristic of M .

An important corollary is that the sphere partition function has a universal finite piece that does not depend on the renormalization scheme.¹ Explicitly

$$\mathcal{Z}(S^2) = \left(\frac{r}{r_0}\right)^{c/3} e^{-K(\lambda, \bar{\lambda})}, \quad (1.1)$$

where r is the radius of the sphere, r_0 a short-distance cutoff and c the central charge of the theory. In many cases $\mathcal{Z}(S^2)$ can be computed exactly using localization of the

¹Actually there exists a residual ambiguity [1] that corresponds to Kähler-Weyl transformations of $K(\lambda, \bar{\lambda})$. This will be important in our discussion later on.

path integral [3, 4], which is thus established as a powerful new tool for the calculation of worldsheet-instanton corrections to K , and of the associated Gromov-Witten invariants.

The relation (1.1) was first conjectured by Jockers et al. [5] and established by different methods in [6, 7]. The proof based on the trace anomaly is elegant and more powerful, it can be generalized for example to four dimensions [1]. In this paper we will extend the calculation of the anomaly in another direction, allowing for a surface with boundary. As a corollary we will derive the hemisphere relations

$$\mathcal{Z}_+(D^2) = \left(\frac{r}{r_0}\right)^{c/6} c^\Omega(\lambda), \quad \mathcal{Z}_-(D^2) = \left(\frac{r}{r_0}\right)^{c/6} c^\Omega(\bar{\lambda}), \quad (1.2)$$

where $\pm$ indicate a choice of spin structure, and $c^\Omega(\lambda)$ is the holomorphic central charge of the boundary condition Ω . These relations have been conjectured by Honda and Okuda [8] and by Hori and Romo [9] (see also refs. [10, 11]). Here we will show that they follow from the supersymmetric trace anomaly.

A key ingredient in our calculation is the scale anomaly of the one-point functions $\langle \mathcal{O}_I \rangle_\Omega$, whose supersymmetrization gives a term proportional to the boundary charge c^Ω . Without supersymmetry this term would have been ambiguous, much like the term proportional to K in the bulk. Thanks to supersymmetry it acquires universal meaning. The vacuum degeneracy of the boundary (alias g -function [12]) is $g^\Omega = |c^\Omega| e^{K/2}$. Our result therefore shows that

$$g^\Omega = \sqrt{\mathcal{Z}_+(D^2)\mathcal{Z}_-(D^2)/\mathcal{Z}(S^2)}. \quad (1.3)$$

As a special application, one can use localization to compute Calabi's diastasis function [13].

We now describe these results in more detail. The $\mathcal{N} = (2, 2)$ superconformal theories are defined on a Riemann surface M with boundary. Boundary conditions, Ω , that preserve half of the superconformal symmetry are of two types, A and B. For SCFTs whose target space is a large Calabi-Yau threefold they correspond, respectively, to special-Lagrangian and to holomorphic submanifolds [14] (see also [15, 16]). Mirror symmetry exchanges A with B, so we may restrict to the B-type boundaries only.

The moduli space of $\mathcal{N} = (2, 2)$ SCFTs factorizes locally² into chiral and twisted-chiral deformations, $\mathcal{M} \simeq \mathcal{M}_c \times \mathcal{M}_{tc}$. In the geometric Calabi-Yau limit these factors correspond, respectively, to complex-structure and Kähler moduli. Chiral deformations are generically obstructed at a B-type boundary, i.e. supersymmetry is completely broken [18, 19]. There is no such obstruction for twisted chiral deformations but another, non-generic phenomenon can occur for them: across lines of marginal stability the bulk deformation can induce a boundary renormalization-group flow [20], so that physics at the boundary is discontinuous.³ To avoid these complications we will restrict attention to the (unobstructed) twisted chiral moduli only, and to regions of their moduli space that are free from marginal-stability lines. The λ^I in (1.2) are the twisted-chiral couplings inside such regions.

In calculating the boundary anomaly we rely on different arguments. These include parity invariance, the covariance of the standard trace anomaly, and supersymmetric Ward

²Except when the supersymmetry is enhanced, as discussed in the recent reference [17].

³Such a flow is induced whenever the marginal bulk operator mixes with relevant or marginal operators on the boundary [18]. In string theory these flows are interpreted as D-brane decays.

identities. An important consistency check is that different Kähler-Weyl frames should correspond to different renormalization schemes, i.e. that they should be related by the addition of local gauge invariant counterterms [1]. Another non-trivial check is that the one-point functions of marginal operators, $\langle \mathcal{O}_I \rangle_\Omega$, have the same dependence on K and c^Ω as the Ramond-Ramond charges of Calabi-Yau D-branes [14, 15]. We will see why this is no coincidence.

The paper is organized as follows. In section 2 we review the calculation of ref. [1] paying particular attention to the scale anomaly of two-point functions, and to the role of Kähler-Weyl transformations. Section 3 introduces two simple tricks for restoring supersymmetry in the presence of boundaries: a standard ‘reference’ completion of D-term integrals, and boundary superspace [21, 22]. Section 4 presents the general ansatz for the boundary anomaly that is parity invariant, and consistent with a special supersymmetric Ward identity that we will explain. The ansatz depends on a holomorphic function of the moduli which we identify with the logarithm of the boundary charge c^Ω . This identification is confirmed in section 5 where we extract the moduli-dependence of the one-point functions on the half-plane, and show that they agree with the Ramond-Ramond charges of $|\Omega\rangle\rangle$ derived in [14]. Asking that Kähler-Weyl transformations be cohomologically trivial implies that c^Ω is the section of a holomorphic line bundle. In section 6 we integrate the anomaly for the supersymmetric squashed hemisphere [6] and establish the relations (1.2). The result does not depend on the squashing parameter. In appendix A we summarize conventions.

2 Review of the bulk anomaly

We begin by reviewing the results of Gomis et al. [1]. We consider the $U(1)_V$ supergravity whose gauge field V^μ couples to the vector-like R-symmetry, the one preserved by the B-type boundary conditions. The theory is defined on a surface M (not to be confused with the moduli space $\mathcal{M}$). For details of the $\mathcal{N} = 2$, $d = 2$ supergravity see [23]. In superconformal gauge the graviton multiplet has two real bosonic degrees of freedom, σ and a , which are defined by $g_{\mu\nu} = e^{2\sigma}\eta_{\mu\nu}$ and $V^\mu = \epsilon^{\mu\nu}\partial_\nu a$. They combine to form the lowest component of a twisted chiral field⁴

$$\Sigma(y^\mu) = (\sigma + ia) + \theta^+ \bar{\chi}_+ + \bar{\theta}^- \chi_- + \theta^+ \bar{\theta}^- w, \quad (2.1)$$

where component fields are functions of the coordinates $y^\pm = x^\pm \mp i\theta^\pm \bar{\theta}^\pm$.

This $U(1)_V$ supergravity is coupled to a $\mathcal{N} = (2, 2)$ superconformal field theory which has a set of marginal couplings λ^I that parametrize the superconformal manifold. As explained in the introduction, we restrict ourselves to the twisted chiral deformations. A useful trick [24] is to promote the λ^I to expectation values of the (lowest components of) twisted chiral superfields Λ^I . We will be interested in the anomalous dependence of the partition function $\mathcal{Z}_V(M)$ on the Weyl superfield Σ . The two supersymmetries and the vector-like R-symmetry can be preserved by the regulator, so the anomaly, $iA(\delta\Sigma) := \delta\Sigma \log \mathcal{Z}_V(M)$, must be an invariant combination of the twisted-chiral superfields Σ and Λ^I .

⁴Our superfield conventions are given in appendix A. Note that in ref. [1] twisted chiral fields and twisted chiral couplings are denoted by tildes in order to distinguish them from chiral fields and chiral couplings. Since in this paper we focus on the twisted-chiral sector only, the tildes are dropped.

For a closed Riemann surface the anomaly has been computed in ref. [1]. It is the sum of two separately-invariant terms,⁵

$$A_{\text{closed}} = \frac{1}{4\pi} \int_M d^2x \int d^4\theta \left[\frac{c}{6} (\delta\Sigma \bar{\Sigma} + \delta\bar{\Sigma} \Sigma) - (\delta\Sigma + \delta\bar{\Sigma}) K(\Lambda, \bar{\Lambda}) \right] := A^{(1)} + A^{(2)}, \quad (2.2)$$

where $A^{(1)}$ is proportional to the central charge c of the SCFT, while $A^{(2)}$ depends on the marginal couplings via the Kähler potential $K(\lambda, \bar{\lambda})$ of the superconformal manifold. The anomaly satisfies the Wess-Zumino consistency condition $\delta_\Sigma A(\delta\Sigma') - \delta_{\Sigma'} A(\delta\Sigma) = 0$, and can thus be integrated with the result

$$\log \mathcal{Z}_V \supset \frac{i}{4\pi} \int_M d^2x \int d^4\theta \left[\frac{c}{6} \Sigma \bar{\Sigma} - (\Sigma + \bar{\Sigma}) K \right]. \quad (2.3)$$

Though local in superconformal gauge, the right hand side cannot be written as the integral of a local *covariant* density — it is cohomologically non-trivial. The term proportional to c is the canonical kinetic action for the $\mathcal{N} = 2$ Liouville superfield Σ , while the less familiar second term is determined by comparison with the two-point function of marginal operators on the plane, as we now explain.

2.1 Anomalous two-point functions and $\mathcal{Z}(S^2)$

The meaning of the anomaly (2.2) is more transparent in component form. We have defined $A^{(1,2)}$ as the D-term superspace integrals,

$$A^{(1)} := \frac{c}{24\pi} \int_M d^2x \int d^4\theta (\delta\Sigma \bar{\Sigma} + \delta\bar{\Sigma} \Sigma) \quad \text{and} \quad A^{(2)} := -\frac{1}{4\pi} \int_M d^2x \int d^4\theta (\delta\Sigma + \delta\bar{\Sigma}) K(\Lambda, \bar{\Lambda}).$$

Expanding $K(\Lambda^I(y^\pm), \bar{\Lambda}^{\bar{I}}(\bar{y}^\pm))$ in Grassman variables, and setting the fermionic and auxiliary components of the coupling superfields to zero gives

$$\begin{aligned} A^{(1)} &= -\frac{c}{12\pi} \int_M d^2x \left[\delta\sigma \square\sigma + \delta a \square a + \frac{1}{2} (\delta w \bar{w} + \delta \bar{w} w) + \partial^\mu b_\mu^{(1)} \right] + \text{fermions}, \\ A^{(2)} &= -\frac{1}{2\pi} \int_M d^2x \left[\delta\sigma (\partial_\mu \lambda^I \partial^\mu \bar{\lambda}^{\bar{J}}) \partial_I \partial_{\bar{J}} K - \frac{1}{2} K \square\delta\sigma - (\partial^\mu \delta a) \mathcal{K}_\mu + \partial^\mu b_\mu^{(2)} \right] \\ \text{where} \quad \mathcal{K}_\mu &:= \frac{i}{2} \left(\partial_I K \partial_\mu \lambda^I - \partial_{\bar{I}} K \partial_\mu \bar{\lambda}^{\bar{I}} \right). \end{aligned} \quad (2.4)$$

The total-derivative terms, which integrate to zero when M is closed, are given by

$$\begin{aligned} b_\mu^{(1)} &= \frac{1}{4} (\partial_\mu \delta\sigma) \sigma - \frac{3}{4} \delta\sigma \partial_\mu \sigma + \frac{1}{4} (\partial_\mu \delta a) a - \frac{3}{4} \delta a \partial_\mu a, \\ b_\mu^{(2)} &= \frac{1}{4} (\partial_\mu \delta\sigma) K - \frac{1}{4} \delta\sigma \partial_\mu K. \end{aligned} \quad (2.5)$$

We keep them because they will be needed for surfaces with boundary.⁶

⁵This expression differs from the one given in [1] by a pure twisted-chiral term and its complex conjugate, $\int_M d^2x \int d^4\theta (\delta\Sigma \Sigma + \delta\bar{\Sigma} \bar{\Sigma})$, which integrate to zero when M is closed. The overall sign corresponds to the case of twisted chiral fields (the ones with tildes) in this reference.

⁶Note that the Wess-Zumino consistency — manifest in the superfield expression (2.2), would be violated if these terms were dropped. For example, the contribution to the pure Weyl anomaly $\int \delta\sigma \square\sigma + \partial^\mu b_\mu^{(1)} = \frac{1}{4} \int [\delta\sigma \square\sigma + \sigma \square\delta\sigma - 2(\partial_\mu \delta\sigma)(\partial^\mu \sigma)]$, is symmetric under $\sigma \leftrightarrow \delta\sigma$, as it should, only if these boundary terms are included.

The first line in (2.4) gives the well-known Weyl anomaly $-\frac{c}{12\pi} \int \delta\sigma \square\sigma = \frac{c}{24\pi} \int \delta\sigma \sqrt{g} R^{(2)}$, and its $\mathcal{N} = 2$ supersymmetric completion. The less familiar second line begins with a term that vanishes when the λ^I are constant. This anomaly, first discussed in ref. [2] by Osborn, captures the logarithmic divergence of the regularized two-point functions of the marginal operators $\mathcal{O}_I$. More explicitly one has

$$\langle \mathcal{O}_I(z) \bar{\mathcal{O}}_{\bar{J}}(w) \rangle = g_{I\bar{J}} \mathcal{R} \frac{1}{|z-w|^4} = g_{I\bar{J}} \frac{1}{2} (\partial\bar{\partial})^2 [\log(|z-w|^2 \mu^2)]^2, \quad (2.6)$$

where $g_{I\bar{J}} = \partial_I \partial_{\bar{J}} K$ is the Zamolodchikov metric [25] at the unperturbed point, $\lambda^I = 0$, of the superconformal manifold, and the symbol $\mathcal{R}$ denotes the differential regularization of the two-point function [2] which should be viewed as a distribution.⁷ In the normalization of [1] one finds for the Euclidean generating function

$$-\mu \frac{\partial}{\partial \mu} \log \mathcal{Z}_V^E \supset -\mu \frac{\partial}{\partial \mu} \int \frac{d^2 z}{\pi} \int \frac{d^2 w}{\pi} \lambda^I(z, \bar{z}) \bar{\lambda}^{\bar{J}}(w, \bar{w}) g_{I\bar{J}} \mathcal{R} \frac{1}{|z-w|^4}. \quad (2.7)$$

Using the identities $\square = 4\partial\bar{\partial}$ and $\partial\bar{\partial} \log|z|^2 = \pi \delta^{(2)}(z)$ it can be checked that this agrees precisely with $A^{(2)}$ for $\delta\sigma = -\delta \log \mu$ constant.

This argument allows us to identify the (a priori arbitrary) function $K(\lambda, \bar{\lambda})$ with the Kähler potential of the superconformal manifold. Now the anomaly $A^{(2)}$ also contains the term $\int \frac{1}{4\pi} K \square \delta\sigma$, which does not vanish when the λ^I are constant. Taken in isolation this term would have been cohomologically trivial since it is the variation of $-\frac{1}{8\pi} \int_M \sqrt{g} R^{(2)} K$, where $R^{(2)}$ is the Ricci scalar of the Riemann surface. Supersymmetry relates it however to the cohomologically non-trivial term $\sim \delta\sigma |\partial\lambda|^2$, so that $\mathcal{N} = 2$ invariant counterterms cannot remove it. This is the key observation in ref. [1]. One important corollary is that the free energy on the round two-sphere gives the Kähler potential of the conformal manifold,

$$\mathcal{Z}_V^E(S^2) = \left(\frac{r}{r_0} \right)^{c/3} e^{-K(\lambda, \bar{\lambda})}, \quad (2.8)$$

where r is the radius of the sphere, r_0 is an ultraviolet cutoff and we used the identity $\int_{S^2} K \square\sigma = -4\pi K$, valid when $K(\lambda, \bar{\lambda})$ is constant on S^2 . This relation was first conjectured by Jockers et al. [5], and proved by a different argument in [6, 7]. The above proof based on the anomaly is powerful and appealing.

The importance of eq. (2.8) stems from the fact that the sphere partition function can be sometimes obtained exactly by localization of the functional integral [3, 4] (as pioneered in [26], see also [27, 28] for recent reviews). This is a powerful new method for computing the quantum Kähler potential on the moduli space of Calabi-Yau threefolds, which does not rely on mirror symmetry.

⁷The reader can easily check that at $z \neq w$ the regularized two-point function equals $|z-w|^{-4}$. By pulling the derivatives in the front one ensures that they can be transferred to test functions of z or w . If these test functions are sufficiently smooth, the singularity at $z = w$ is integrable.

2.2 Kähler-Weyl transformations

An important remark about the above formula concerns the Kähler-Weyl transformations

$$K'(\lambda, \bar{\lambda}) = K(\lambda, \bar{\lambda}) + H(\lambda) + \bar{H}(\bar{\lambda}), \quad (2.9)$$

where H is a holomorphic function of the λ^I . Such transformations do not affect the geometry of the superconformal manifold but they do modify the anomaly (2.2),

$$\Delta_{\text{KW}} A^{(2)} = -\frac{1}{4\pi} \int_M d^2x \int d^4\theta (\delta\Sigma + \delta\bar{\Sigma})H + \text{c.c.} \quad (2.10)$$

This looks paradoxical, at first sight, since physics should be independent of the Kähler-Weyl frame. The puzzle is elegantly resolved [1, 7] by noting that the right-hand side can be equivalently rewritten as a (twisted) F-term,

$$\begin{aligned} \int_M d^2x \int d^4\theta (\delta\Sigma + \delta\bar{\Sigma})H &= \int_M d^2x \int d\theta^+ d\bar{\theta}^- \bar{D}_+ D_- [(\delta\Sigma + \delta\bar{\Sigma})H] + \int_M d^2x (\partial^\mu Y_\mu) \\ &= \int_M d^2x \int d\theta^+ d\bar{\theta}^- (\bar{D}_+ D_- \delta\bar{\Sigma})H + \int_M d^2x (\partial^\mu Y_\mu), \end{aligned} \quad (2.11)$$

where $\partial^\mu Y_\mu$ is a total divergence whose integral over a closed surface M vanishes. We used here the identity $\int d^4\theta X = \int d\theta^+ d\bar{\theta}^- \bar{D}_+ D_- X + \partial^\mu Y_\mu$, valid for any superfield X , as well as the fact that $\bar{D}_+$ and D_- annihilate the twisted chiral superfields $\delta\Sigma$ and $H(\Lambda)$.

Now $\bar{D}_+ D_- \bar{\Sigma}$ is a twisted-chiral superfield whose components are the curvature and the field strength of the $U(1)_V$ gauge field,

$$\bar{D}_+ D_- \bar{\Sigma} = -\bar{w} + 4\theta^+ \bar{\theta}^- \partial_+ \partial_- (\sigma - ia) + \dots \quad (2.12)$$

Thus $\int_M d^2x \int d\theta^+ d\bar{\theta}^- \bar{D}_+ D_- \bar{\Sigma} H(\Lambda) + \text{c.c.}$ is a local, supersymmetric, reparametrization-invariant counterterm [7] which can be used to cancel the Kähler-Weyl transformation (2.10). Let us stress this again: the renormalization-scheme ambiguity translates into a freedom of Kähler-Weyl transformations which are thus cohomologically trivial.

It is instructive to also verify this in component form. The Kähler-Weyl transformation changes $\mathcal{K}_\mu$ by the total derivative $\frac{i}{2} \partial_\mu (H - \bar{H})$. The change in the anomaly (2.4) is therefore, after integration by parts, proportional to $(H + \bar{H}) \square \delta\sigma + i(\bar{H} - H) \square \delta a$. Since $\square a = -\epsilon^{\mu\nu} \partial_\mu V_\nu$ and $2\square\sigma = -R^{(2)}\sqrt{g}$, both terms can be cancelled by local gauge- and reparametrization-invariant counterterms, as advertized.

The fact that a Kähler-Weyl transformation amounts to a change of regularization scheme leads to an interesting conjecture [1]. If the superconformal manifold had non-vanishing Kähler class, one could choose space-dependent couplings $\lambda^I(x)$ such that the embedding of $M \subset \mathcal{M}$ is a nontrivial 2-cycle. It would then be impossible to find a regularization scheme, valid everywhere in M , for such sigma models. To avoid this embarrassing situation one must demand that $\mathcal{M}$ have vanishing Kähler class, a non-trivial restriction on superconformal manifolds.

3 Supersymmetry with boundaries

Having summarized the anomaly in the bulk, we turn now our attention to the case when M is an open Riemann surface. The presence of the boundary affects the calculation in different ways. Firstly, the component expressions (2.4) include total-derivative terms which cannot be ignored when M is open. Secondly, these expressions are not invariant under supersymmetry anymore, because the top component of a superfield transforms to a total derivative under a supersymmetry transformation. To cancel this we must add a compensating boundary term, as is well-known from the study of $\mathcal{N} = 2$ sigma-model actions (see for example [29]). Of course, supersymmetry alone does not suffice to fix the boundary anomaly uniquely. There exist several candidates that differ by supersymmetric boundary invariants, and our task will be to find the right one.

We will proceed in two steps. In this section we describe a general algorithm that restores the invariance of a D-term superspace integral $\int_M d^2x \int d^4\theta \mathcal{S}$, for any superfield $\mathcal{S}$, when M has a boundary. All other completions differ from this ‘reference’ one by a superinvariant which can be always written as an integral over boundary superspace [21, 22]. To find the right one we will need extra arguments that are presented in sections 4 and 5.

3.1 Reference completion of bulk D-terms

Let us consider the integral $\int_M d^2x \int d^4\theta \mathcal{S} = \int_M d^2x [\mathcal{S}]_{\text{top}}$, where $\mathcal{S}$ is a real superfield and $[\mathcal{S}]_{\text{top}}$ is its θ^4 component. The B-type supersymmetry transformations, the ones consistent with B-type boundary conditions, are generated by the differential operators in superspace

$$\mathcal{D}_{\text{susy}} = \epsilon (e^{i\beta} Q_+ + e^{-i\beta} Q_-) - \bar{\epsilon} (e^{-i\beta} \bar{Q}_+ + e^{i\beta} \bar{Q}_-), \quad (3.1)$$

where ϵ is a complex Grassmanian parameter, β is an arbitrary phase, and

$$Q_{\pm} = \frac{\partial}{\partial \theta^{\pm}} + i\bar{\theta}^{\pm} \partial_{\pm}, \quad \bar{Q}_{\pm} = -\frac{\partial}{\partial \bar{\theta}^{\pm}} - i\theta^{\pm} \partial_{\pm} \quad (3.2)$$

(see appendix A). Note that the vector and axial R -symmetry charges are $(1, 1)$ for $\bar{Q}_+$, and $(1, -1)$ for $\bar{Q}_-$. If M has a single boundary, one can always reabsorb β by a redefinition of the fermions. We set here $\beta = 0$. We can always restore it later, if needed.

The transformation of $[\mathcal{S}]_{\text{top}}$ is a total derivative,

$$\Delta_{\text{susy}}[\mathcal{S}]_{\text{top}} = \int d^4\theta \mathcal{D}_{\text{susy}} \mathcal{S} = i\epsilon \int d^4\theta (\bar{\theta}^+ \partial_+ \mathcal{S} + \bar{\theta}^- \partial_- \mathcal{S}) + \text{c.c.}, \quad (3.3)$$

which integrates to zero when M is closed, but need not vanish when M is open. Let’s assume that M is the half-space $x^1 \leq 0$. We would like to express the right-hand side of (3.3) as the transformation of a spatial derivative, up to a time derivative,

$$\Delta_{\text{susy}}[\mathcal{S}]_{\text{top}} = -\Delta_{\text{susy}}(\partial_1[\mathcal{S}]_{\text{bnry}}) + \partial_0 Y,$$

so that

$$I_D(\mathcal{S}) := \int d^2x [\mathcal{S}]_{\text{top}} + \int dx^0 [\mathcal{S}]_{\text{bnry}} \quad (3.4)$$

is a supersymmetric invariant. It will serve as our reference completion.

To bring (3.3) to the above form we proceed as follows. Begin with the identity

$$\int d^4\theta Q_+ \mathcal{S} = \int d^4\theta i\bar{\theta}^+ \partial_+ \mathcal{S} = \int d^4\theta i\bar{\theta}^+ \theta^- (Q_- - i\bar{\theta}^- \partial_-) \partial_+ \mathcal{S}, \quad (3.5)$$

where the second equality follows from the rule of integration by parts in Grassman space, $\int d^4\theta \bar{\theta}^+ \theta^- \partial X / \partial \theta^- = \int d^4\theta \bar{\theta}^+ X$ for any X . Adding the vanishing term $\int d^4\theta i\bar{\theta}^+ \theta^- Q_+ \partial_+ \mathcal{S}$, and inserting $\theta^+ Q_+$ in the last term (this just acts as the identity within the superspace integral) allows us to rewrite this equation as

$$\int d^4\theta Q_+ \mathcal{S} = \int d^4\theta i\bar{\theta}^+ \theta^- (Q_+ + Q_-) \partial_+ \mathcal{S} - \int d^4\theta \theta^4 Q_+ \partial_- \partial_+ \mathcal{S},$$

where $\theta^4 = \theta^+ \theta^- \bar{\theta}^- \bar{\theta}^+$. This identity also holds if we exchange $+$ with $-$ indices, so that

$$\begin{aligned} & \int d^4\theta (Q_+ + Q_-) \mathcal{S} \\ &= i \int d^4\theta (\bar{\theta}^+ \theta^- \partial_+ + \bar{\theta}^- \theta^+ \partial_-) (Q_+ + Q_-) \mathcal{S} - \int d^4\theta \theta^4 (Q_+ + Q_-) \partial_- \partial_+ \mathcal{S} \\ &= i \partial_+ [(Q_+ + Q_-) \mathcal{S}]_{\theta^+ \bar{\theta}^-} + i \partial_- [(Q_+ + Q_-) \mathcal{S}]_{\theta^- \bar{\theta}^+} - \partial_- \partial_+ [(Q_+ + Q_-) \mathcal{S}]_{\emptyset}. \end{aligned} \quad (3.6)$$

Here $[X]_{\Theta}$ is the coefficient of the monomial Θ in the expansion of the superfield X , and $[X]_{\emptyset}$ is its bottom component. Multiplying by the anticommuting parameter ϵ , adding the complex conjugate and integrating over M gives

$$\int_M d^2x \int d^4\theta \mathcal{D}_{\text{susy}} \mathcal{S} = \int_{\partial M} dx^0 \left[\frac{i}{2} ([\mathcal{D}_{\text{susy}} \mathcal{S}]_{\theta^+ \bar{\theta}^-} - [\mathcal{D}_{\text{susy}} \mathcal{S}]_{\theta^- \bar{\theta}^+}) + \frac{1}{4} \partial_1 [\mathcal{D}_{\text{susy}} \mathcal{S}]_{\emptyset} \right]. \quad (3.7)$$

Since $[\mathcal{D}_{\text{susy}} X]_{\Theta}$ is the supersymmetry transformation of the superfield component $[X]_{\Theta}$, we have succeeded in what we set out to do, which was to find a boundary term that compensates the supersymmetry transformation of the bulk D-term. From (3.7) one reads

$$[\mathcal{S}]_{\text{bnry}} = -\frac{i}{2} ([\mathcal{S}]_{\theta^+ \bar{\theta}^-} - [\mathcal{S}]_{\theta^- \bar{\theta}^+}) - \frac{1}{4} \partial_1 [\mathcal{S}]_{\emptyset}. \quad (3.8)$$

It is of course also possible to verify the invariance of (3.4) with the above $[\mathcal{S}]_{\text{bnry}}$ by using the transformations of a general superfield given in ref. [23].

The supersymmetric completion (3.4) depends only on the D-term density $\mathcal{S}$. It is not unique, since there exist in general many other combinations of boundary fields that are invariant under the B-type supersymmetry. After completing the bulk Weyl anomaly in this generic ‘reference’ way, we can restrict our search for extra boundary contributions to terms that are separately super-invariant.

3.2 Boundary superspace and invariants

A systematic method to construct boundary superinvariants is by using the formalism of the boundary superspace [21, 22]. The boundary superspace of B-type is defined by the following identifications of coordinates

$$x^+ = x^-, \quad \theta \equiv e^{-i\beta} \theta^+ = e^{i\beta} \theta^-, \quad \bar{\theta} \equiv e^{i\beta} \bar{\theta}^+ = e^{-i\beta} \bar{\theta}^-. \quad (3.9)$$

These identifications are such that the generators (3.1) involve only a derivative in the time direction. We set again $\beta = 0$. The supercovariant boundary derivatives are

$$\bar{D} = \bar{D}_+ + \bar{D}_- = -\frac{\partial}{\partial \bar{\theta}} + i\theta \partial_0, \quad D = D_+ + D_- = \frac{\partial}{\partial \theta} - i\bar{\theta} \partial_0.$$

The restriction of a chiral bulk superfield on a B-type boundary is chiral on the boundary, i.e. it is annihilated by $\bar{D}$, whereas the twisted-chiral bulk superfields do not have any chirality property on the boundary. In particular

$$\Sigma|_{\partial M} = \sigma + ia + \theta \bar{\chi}_+ + \bar{\theta} \chi_- + \theta \bar{\theta} [w - i\partial_1(\sigma + ia)] \quad (3.10)$$

(with all fields functions of x^0) is neither chiral nor antichiral, and the same is true for the coupling superfields $\Lambda^I|_{\partial M}$. The usual D-term and F-term integrals of boundary superfields are invariant under B-type supersymmetry.

Let us consider as a special case the following composite bulk superfield:

$$\delta \mathcal{S}^H = -\frac{1}{4\pi} (\delta \Sigma + \delta \bar{\Sigma}) [H(\Lambda^I) + \bar{H}(\bar{\Lambda}^{\bar{I}})] \quad (3.11)$$

with H a holomorphic function of the Λ^I . This is the D-term integrand that entered in the Kähler-Weyl transformation of the anomaly, eq. (2.10). Recall that for a closed surface M , $\int_M d^2x \int d^4\theta \mathcal{S}^H$ is a local gauge-invariant counterterm which can be expressed as a twisted F-term in terms of the curvature superfield $\bar{D}_+ D_- \bar{\Sigma}$. The question we would like to ask is whether this local counterterm can be defined when M is open.

The superspace integral of $\delta \mathcal{S}^H$ is given by eqs. (2.4) and (2.5) with K replaced by $H + \bar{H}$. Adding the reference completion (3.4), (3.8) gives after some rearrangements

$$\begin{aligned} \text{I}_D(\mathcal{S}^H) &= \frac{1}{4\pi} \int_M d^2x [(H + \bar{H})\square\sigma - i(H - \bar{H})\square a] + \frac{i}{4\pi} \int_{\partial M} dx^0 (w\bar{H} - \bar{w}H) \\ &+ \frac{1}{4\pi} \int_{\partial M} dx^0 \left[\sigma \partial_1(H + \bar{H}) + i\partial_1 a (H - \bar{H}) + \frac{i}{2}(w + \bar{w})(H - \bar{H}) \right]. \end{aligned} \quad (3.12)$$

The bulk integral is the local counterterm that we encountered already in section 2. The boundary integral, on the other hand, involves the Weyl factor σ with no derivatives, so it cannot be written as the integral of a covariant density. Fortunately the lower line of (3.12) is a boundary superinvariant by itself (this is the reason why we isolated it). It is the boundary D-term $\int dx^0 [\mathcal{B}^H]_{\theta\bar{\theta}}$ with

$$\mathcal{B}^H = \frac{i}{8\pi} (\Sigma + \bar{\Sigma})(H - \bar{H})|_{\partial M}. \quad (3.13)$$

Subtracting this integral from (3.12) gives then a bulk and boundary counterterm,

$$\mathcal{C}^H := \text{I}_D(\mathcal{S}^H) - \int dx^0 [\mathcal{B}^H]_{\theta\bar{\theta}}, \quad (3.14)$$

which is local, supersymmetric and gauge invariant. It can be used to compensate Kähler-Weyl transformations when M is open. As a check, note that we could have written $\mathcal{C}^H$ directly as the more familiar boundary completion of an F-term [15]. Indeed

$$\text{I}_F(\Phi) := \int_M d^2x \int d\theta^+ d\bar{\theta}^- \Phi - i \int_{\partial M} dx^0 [\Phi]_{\emptyset} \quad (3.15)$$

is type-B super-invariant for any twisted chiral field Φ , and $\mathcal{C}^H = \mathbf{I}_F(\Phi) + \text{c.c.}$ for the twisted chiral field $\Phi = -\frac{1}{4\pi} \bar{D}_+ D_- \bar{\Sigma} H$. We leave the proof of these statements as an exercise for the reader.

4 The boundary anomaly

We have now the tools at hand to discuss the supersymmetric boundary completion of the anomaly of [1]. Recall that the bulk anomaly is given by the D-term integral

$$A_{\text{closed}} = \int_M d^2x [\delta\mathcal{S}]_{\text{top}} \quad \text{where} \quad \mathcal{S} = \frac{1}{4\pi} \left[\frac{c}{6} \Sigma \bar{\Sigma} - (\Sigma + \bar{\Sigma}) K \right]. \quad (4.1)$$

As was explained in section 3, the most general boundary completion that is consistent with type-B supersymmetry can be written as

$$A_{\text{open}} = \int_M d^2x [\delta\mathcal{S}]_{\text{top}} + \int_{\partial M} dx^0 ([\delta\mathcal{S}]_{\text{bnry}} + [\delta\mathcal{B}]_{\theta\bar{\theta}}) = \mathbf{I}_D(\delta\mathcal{S}) + \int_{\partial M} dx^0 [\delta\mathcal{B}]_{\theta\bar{\theta}} \quad (4.2)$$

where $\delta\mathcal{B}$ is a boundary superfield made out of $\delta\Sigma|_{\partial M}$, $\Sigma|_{\partial M}$ and $\Lambda^I|_{\partial M}$, and depending in general on the boundary condition Ω . Our task is now to find $\delta\mathcal{B}$.

Since we will need it later, let us record here the reference completion $[\delta\mathcal{S}]_{\text{bnry}}$. The relevant terms in the superfield expansions are

$$\begin{aligned} \Sigma \bar{\Sigma} &= \sigma^2 + a^2 + \theta^+ \bar{\theta}^- w(\sigma - ia) + \theta^- \bar{\theta}^+ \bar{w}(\sigma + ia) + \dots, \\ (\Sigma + \bar{\Sigma})K &= 2\sigma K + \theta^+ \bar{\theta}^- wK + \theta^- \bar{\theta}^+ \bar{w}K + \dots, \end{aligned} \quad (4.3)$$

where the dots stand for other monomials in θ , or terms that involve the fermions and the auxiliary fields in Λ^I which are set to zero. Plugging (4.3) in (3.8) gives

$$[\mathcal{S}]_{\text{bnry}} = -\frac{1}{8\pi} \left[\frac{c}{6} \left(\sigma \partial_1 \sigma + a \partial_1 a + a(w + \bar{w}) + i\sigma(w - \bar{w}) \right) - (\partial_1(\sigma K) + iK(w - \bar{w})) \right]. \quad (4.4)$$

The contribution to the anomaly, $\int_{\partial M} [\delta\mathcal{S}]_{\text{bnry}}$, is the variation of the above expression with respect to the supergravity fields σ, a and w .

4.1 Parity and a general ansatz

An important restriction on $\delta\mathcal{B}$ comes from parity invariance. The parity transformation of B-type acts on superspace coordinates as follows [30]:

$$x^+ \leftrightarrow x^-, \quad e^{-i\beta} \theta^+ \leftrightarrow e^{i\beta} \theta^-, \quad e^{i\beta} \bar{\theta}^+ \leftrightarrow e^{-i\beta} \bar{\theta}^-, \quad \text{and hence} \quad y^\pm \leftrightarrow \bar{y}^\mp.$$

The superspace boundary (3.9) is the invariant locus of this transformation. Parity conjugates the Weyl superfield, $\Sigma \leftrightarrow \bar{\Sigma}$, or explicitly in components

$$\sigma \rightarrow \sigma, \quad a \rightarrow -a, \quad e^{-i\beta} \chi_+ \leftrightarrow e^{i\beta} \chi_-, \quad w \rightarrow \bar{w}. \quad (4.5)$$

That a is indeed a pseudoscalar follows from its definition, $V_\mu = \epsilon_{\mu\nu} \partial^\nu a$. We have introduced in the above transformation the axial phase that entered in the preserved supersymmetry. We will continue to set $\beta = 0$ in what follows.

The $\mathcal{N} = (2, 2)$ SCFT is in general not parity invariant. Since B-type parity conjugates the twisted chiral fields, $\mathcal{O}^I \leftrightarrow \bar{\mathcal{O}}^{\bar{I}}$, the terms coupling to $\text{Im}(\lambda^I)$ are parity odd. The symmetry can be restored if the couplings also transform like twisted chiral fields,⁸

$$\Lambda^I \leftrightarrow \bar{\Lambda}^{\bar{I}}. \quad (4.6)$$

Invariance under (4.6) implies the reality condition $K(\lambda, \bar{\lambda}) = K(\bar{\lambda}, \lambda)$ which follows also from the general form of the instanton expansion of K , see for instance [5].

The Zamolodchikov metric $\partial_I \bar{\partial}_{\bar{J}} K$ is parity invariant, while the Kähler one-form $\mathcal{K}$ is parity-odd. It follows that both the bulk anomaly (2.2)–(2.5), and its reference completion (4.4), are parity-invariant as should have been expected. Note that the integrand of boundary terms must change sign under parity because of the change in the orientation of the surface.⁹ Note also that invariance under parity rules out a bulk anomaly $\sim \int_M \int d^4\theta (\delta\Sigma - \delta\bar{\Sigma}) L(\Lambda, \bar{\Lambda})$ with L a real parity-even function of the couplings, even though this is supersymmetric, cohomologically non-trivial and it obeys the Wess-Zumino consistency condition.

Let us go back now to the boundary anomaly $\int dx^0 [\delta\mathcal{B}]_{\theta\bar{\theta}}$, or better to its integrated form $\int dx^0 [\mathcal{B}]_{\theta\bar{\theta}}$ (locality and Wess-Zumino consistency ensure that this latter form exists). The covariance and scale invariance of the differential anomaly, the fact that the central charge c is constant on $\mathcal{M}$, and parity invariance severely constrain the allowed form of $\mathcal{B}$ which is at most quadratic in the Weyl superfield $\Sigma|_{\partial M}$. The general ansatz reads

$$\mathcal{B} = \frac{i}{8\pi} \left[\# \frac{c}{12} (\Sigma^2 - \bar{\Sigma}^2) + \bar{\Sigma} G^\Omega(\Lambda, \bar{\Lambda}) - \Sigma G^\Omega(\bar{\Lambda}, \Lambda) \right] \Big|_{\partial M} := \mathcal{B}^{(1)} + \mathcal{B}^{(2)}, \quad (4.7)$$

where G^Ω is a function of the couplings (and of the boundary Ω) obeying the reality condition

$$G^\Omega(\bar{\Lambda}, \Lambda) = [G^\Omega(\Lambda, \bar{\Lambda})]^*. \quad (4.8)$$

The meaning of $|_{\partial M}$ is that all bulk superfields must be evaluated at the superspace boundary. As in section 2, we have separated the central-charge anomaly $\mathcal{B}^{(1)}$ from the moduli-dependent anomaly $\mathcal{B}^{(2)}$. We have also extracted in the above expression certain numerical factors by anticipation. They could be reabsorbed in the definition of G^Ω and of the (a priori arbitrary) coefficient denoted $\#$. We will now show that $\# = 1$.

In order to fix this coefficient we consider pure Weyl transformations for which $\delta a = \delta w = \delta\chi_\pm = 0$. Using the top component of the boundary superfield

$$[\delta\mathcal{B}^{(1)}]_{\theta\bar{\theta}} = \# \frac{c}{24\pi} \delta \left[\sigma \partial_1 \sigma - a \partial_1 a + \frac{i}{2} \sigma (w - \bar{w}) - \frac{1}{2} a (w + \bar{w}) - \frac{i}{2} (\bar{\chi}_+ \chi_- - \bar{\chi}_- \chi_+) \right],$$

together with eqs. (2.4), (2.5) and (4.4), leads to the following central-charge anomaly on a surface M with boundary:

$$-i\delta_\sigma \log \mathcal{Z}_V \supset -\frac{c}{12\pi} \int d^2x \delta\sigma \square\sigma + \frac{c}{24\pi} \int dx^0 [(1+\#)\delta\sigma \partial_1 \sigma - (1-\#)(\sigma \partial_1 \delta\sigma - \delta\sigma \text{Im} w)].$$

⁸Frequently in the literature the parity-odd terms couple to the *real* parts of the complex moduli. This convention and ours differ by the redefinition $\lambda^I \rightarrow i\lambda^I$.

⁹For example $\int d^2x \partial^\mu \partial_\mu \phi = \int dx^0 \partial_1 \phi$ is invariant if ϕ is a scalar, in which case $\partial_1 \phi \rightarrow -\partial_1 \phi$ under parity.

This should be compared to the standard Weyl anomaly of a CFT on an open Riemann surface. Covariance and Wess-Zumino consistency fix completely the boundary anomaly, in this case (see chapter 3 of Polchinski's book [31]) with the result

$$A_{\text{book}} = \frac{c}{24\pi} \left[\int_M \delta\sigma \sqrt{g} R + 2 \int_{\partial M} \delta\sigma k \right] = -\frac{c}{12\pi} \left[\int d^2x \delta\sigma \square\sigma - \int dx^0 \delta\sigma \partial_1\sigma \right]. \quad (4.9)$$

Here k is the extrinsic curvature of the boundary, equal to the outward-normal derivative of the Weyl factor.¹⁰ Matching our result to (4.9) shows that $\# = 1$. Notice that the dangerous term $\sigma \partial_1 \delta\sigma$, which would have given a non-covariant anomaly, drops out for $\# = 1$, as does the term depending on the auxiliary field w .

4.2 The holomorphic boundary charge

We turn next to the moduli-dependent term $\mathcal{B}^{(2)}$ which features the new function G^Ω . We will show that this must be of the form

$$G^\Omega(\lambda, \bar{\lambda}) = K(\lambda, \bar{\lambda}) + 2h^\Omega(\lambda), \quad (4.10)$$

where h^Ω is a holomorphic function of λ^I . Parity invariance and reality of the anomaly impose the condition $(h^\Omega(\bar{\lambda}))^* = h^\Omega(\lambda)$, which means that this holomorphic function admits expansions with real coefficients. Later, we will identify $\exp(h^\Omega)$ with the central charge of the boundary state.

The starting point for proving (4.10) is the linearized coupling of the SCFT to the $U(1)_V$ supergravity fields [1, 23]. Following appendix C of ref. [1], we assume that in superconformal gauge this coupling can be written as a twisted F-term,

$$\int \delta\mathcal{L}_{\text{sugra}} = \int_M \int d\theta^+ d\bar{\theta}^- (\delta\Sigma T) - i \int_{\partial M} [\delta\Sigma T]_\emptyset + \text{c.c.}, \quad (4.11)$$

where T is a twisted-chiral field related to the $\mathcal{R}$ -supermultiplet of the energy-momentum tensor, i.e. the supermultiplet whose lowest component is the vector-like R-symmetry current. The boundary term in the above expression is the one that preserves the supersymmetry of twisted F-terms, eq. (3.15). If we set $\delta\bar{\Sigma} = 0$, this coupling is $\bar{Q}_B$ -exact, with $\bar{Q}_B = \bar{Q}_+ + \bar{Q}_-$. This fact follows from the simple identity

$$F(x) = \{\bar{Q}_B, [Q_+, \phi(x)]\} + i\partial_1\phi(x)$$

which is valid for any twisted chiral field with components $(\phi, \psi_-, \bar{\psi}_+, F)$. The same argument actually shows that the marginal deformations of the SCFT

$$\int \delta\mathcal{L}_{\text{SCFT}} = \frac{1}{\pi} \int_M \int d\theta^+ d\bar{\theta}^- (\Lambda^I \Phi_I) - \frac{i}{\pi} \int_{\partial M} [\Lambda^I \Phi_I]_\emptyset + \text{c.c.},$$

¹⁰In Minkowski signature and in conformal gauge the general expression for the extrinsic curvature is $k = t^\mu n_\nu \partial_\mu t^\nu - t^\nu t_\nu n^\mu \partial_\mu \sigma$, where t^ν and n^μ are the unit tangent and outward normal vectors. Here we work with a timelike boundary so $t^\nu t_\nu = -1$. For the extension of the standard Weyl anomaly to higher dimensional manifolds with boundary see ref. [32].

would be $\bar{Q}_B$ -exact if $\bar{\Lambda}^I = 0$. Here Φ_I is the twisted-chiral superfield whose top component is the marginal operator $\mathcal{O}_I$. Since $\bar{Q}_B^2 = 0$ and $\bar{Q}_B$ annihilates the boundary state, we conclude that for $\delta\bar{\Sigma} = \bar{\Lambda}^I = 0$ the following Ward identity holds

$$\left\langle \int \delta\mathcal{L}_{\text{sugra}} \int \delta\mathcal{L}_{\text{SCFT}} \right\rangle = 0. \quad (4.12)$$

Put differently, there cannot exist terms in the effective action proportional to $\delta\Sigma\Lambda^I$ or to $\delta\bar{\Sigma}\bar{\Lambda}^I$. Since all we use is type-B supersymmetry which is preserved by the regulator, this identity is exact even after including contact terms.

Let us first check that it is verified by the bulk anomaly (2.4). The relevant terms in $A^{(2)}$ have derivatives acting only on the holomorphic or antiholomorphic couplings, but not both (the term $\sim \partial\lambda\partial\bar{\lambda}$ gives an anomaly of a mixed correlator to which the above argument does not apply). After integration by parts of $\int K\Box\delta\sigma$ one finds

$$A^{(2)} \supset -\frac{1}{4\pi} \int_M d^2x \left[\partial_\mu(\delta\sigma - i\delta a) \partial_I K \partial^\mu \lambda^I + \partial_\mu(\delta\sigma + i\delta a) \partial_{\bar{I}} K \partial^\mu \bar{\lambda}^{\bar{I}} \right].$$

This vanishes if $(\delta\sigma - i\delta a) = \bar{\lambda}^{\bar{I}} = 0$, as advertized.

Consider next the boundary anomaly. Its general form, eq. (4.2), includes three different contributions: (i) the total derivatives in (2.4) and (2.5) to which we should add the term from integrating by parts $\int K\Box\delta\sigma$, as just discussed; (ii) the reference boundary completion (4.4); and (iii) the boundary superinvariant $\int_{\partial M} [\mathcal{B}]_{\theta\bar{\theta}}$ with $\mathcal{B}$ given by eq. (4.7). We focus on the moduli-dependent anomaly. Collecting everything gives

$$\begin{aligned} -i\delta \log \mathcal{Z}_V \supset & \frac{1}{4\pi} \int dx^0 \left[\partial_1(\delta\sigma K) + \frac{i}{2}(\delta w - \delta\bar{w})K \right. \\ & \left. + \frac{i}{2} \left((\delta\sigma - i\delta a)i(\bar{\partial}_{\bar{I}} G^\Omega \partial_1 \bar{\lambda}^{\bar{I}} - \partial_I G^\Omega \partial_1 \lambda^I) + (\delta\bar{w} + i\partial_1(\delta\sigma - i\delta a))G^\Omega - \text{c.c.} \right) \right], \end{aligned} \quad (4.13)$$

where the top line is the sum of contributions (i) and (ii), and the lower line is the contribution (iii). This latter is the integral of

$$[\mathcal{B}^{(2)}]_{\theta\bar{\theta}} = \frac{i}{8\pi} \left[(\sigma - ia)i(\bar{\partial}_{\bar{I}} G^\Omega \partial_1 \bar{\lambda}^{\bar{I}} - \partial_I G^\Omega \partial_1 \lambda^I) + (\bar{w} + i\partial_1(\sigma - ia))G^\Omega \right] + \text{c.c.} \quad (4.14)$$

as follows from the boundary restrictions of Σ , eq. (3.10), and of G^Ω ,

$$G^\Omega(\Lambda, \bar{\Lambda}) \Big|_{\partial M} = G^\Omega + i\theta\bar{\theta} (\bar{\partial}_{\bar{I}} G^\Omega \partial_1 \bar{\lambda}^{\bar{I}} - \partial_I G^\Omega \partial_1 \lambda^I).$$

The expression (4.13) does not, in general, vanish when $\delta\bar{\Sigma} = \bar{\lambda}^{\bar{I}} = 0$. One notes however that for $G^\Omega = K$ the sum of the top and bottom lines collapses to

$$\frac{1}{4\pi} \int dx^0 (\delta\sigma - i\delta a) \partial_I K \partial_1 \lambda^I + \text{c.c.},$$

which does have the desired property. Furthermore, the lower line would vanish separately if and only if G^Ω were a holomorphic function of the couplings.¹¹ This establishes the general form of G^Ω , eq. (4.10).

¹¹Recall that λ^I are deformation parameters, and we can set $G^\Omega(0,0) = 0$.

We should also examine how the anomaly changes under Kähler-Weyl transformations. These latter act as follows on the bulk superfield $\mathcal{S}$ and the boundary superfield $\mathcal{B}$ that enter in the expression (4.2):

$$\mathcal{S} \rightarrow \mathcal{S} + \mathcal{S}^H, \quad \text{and} \quad \mathcal{B} \rightarrow \mathcal{B} + \frac{i}{8\pi}(\bar{\Sigma} - \Sigma)(H + \bar{H})\Big|_{\partial M} + \frac{i}{4\pi}(\bar{\Sigma} \Delta h^\Omega - \Sigma \Delta \bar{h}^\Omega)\Big|_{\partial M}$$

where Δh^Ω denotes the transformation of h^Ω . Now use $I_D(\delta\mathcal{S} + \delta\mathcal{S}^H) = I_D(\delta\mathcal{S}) + I_D(\delta\mathcal{S}^H)$, and the local counterterm, $\mathcal{C}^H$, that compensates Kähler-Weyl transformations, and which we computed in eqs. (3.14), (3.13). Putting these two facts together implies that the anomaly transforms (in the sense of cohomology, i.e. up to local counterterms) as

$$\Delta_{\text{KW}} A_{\text{open}} \simeq \frac{i}{4\pi} \int dx^0 [\delta\bar{\Sigma}(H + \Delta h^\Omega) - \delta\Sigma(\bar{H} + \Delta \bar{h}^\Omega)]_{\theta\bar{\theta}}.$$

Invariance then implies that under Kähler-Weyl transformations

$$h^\Omega \rightarrow h^\Omega - H, \tag{4.15}$$

which means that e^{h^Ω} is a section of a holomorphic line bundle.

5 Anomalous one-point functions

Taking stock of the analysis of the previous section, we can now write the complete moduli-dependent part of the super-Weyl anomaly,

$$\begin{aligned} A_{\text{open}} \supset \frac{1}{4\pi} \int d^2x \left[K \square \delta\sigma - 2\delta\sigma \partial_\mu \lambda^I \partial^\mu \bar{\lambda}^{\bar{J}} g_{I\bar{J}} + 2(\partial_\mu \delta a) \mathcal{K}^\mu \right] + \frac{1}{4\pi} \int dx^0 \left[-K \partial_1 \delta\sigma \right. \\ \left. + (\delta\sigma - i\delta a) \partial_I (K + h^\Omega) \partial_1 \lambda^I + i(\delta\bar{w} + i\partial_1(\delta\sigma - i\delta a)) h^\Omega + \text{c.c.} \right]. \end{aligned} \tag{5.1}$$

The terms that survive when $\delta\sigma = -\delta \log \mu$ is constant, capture scale anomalies of correlation functions in the SCFT. The bulk term multiplying $\partial_\mu \lambda^I \partial^\mu \bar{\lambda}^{\bar{J}}$ corresponds to the anomalous two-point functions discussed in section 2. Likewise, the boundary term proportional to $\partial_1 \lambda^I$ corresponds to anomalous one-point functions of marginal operators on the half-plane. In this section we will compute these one-point functions, and compare them to known results about Ramond-Ramond charges of D-branes [14]. Matching the two will lead to the identification of $\exp(h^\Omega)$ as the boundary charge.

Translation invariance and the scaling dimension $\Delta = 2$ determine the one-point functions up to unknown coefficients,

$$\langle \mathcal{O}_I(x) \rangle_\Omega = d_I^\Omega \mathcal{R} \frac{1}{|x_1|^2} = d_I^\Omega \partial_1^2 [\Theta(-x^1) \log |x^1 \mu|]. \tag{5.2}$$

We have here introduced a differential regularization similar to that of the two-point functions, eq. (2.6). The step function is $\Theta = 1$ in the half-space $x^1 < 0$, and $\Theta = 0$ outside. Indeed, for any twice differentiable test function the integral of the right-hand-side is finite

at $x^1 = 0$. With the help of the identity $\partial_1 \Theta(-x^1) = -\delta(x^1)$ one derives the following scale dependence of the partition function

$$-i\mu \frac{d}{d\mu} \log \mathcal{Z}_V \supset -\mu \frac{d}{d\mu} \int \frac{d^2x}{\pi} \lambda^I d_I^\Omega \mathcal{R} \frac{1}{|x^1|^2} = -\frac{1}{\pi} \int dx^0 d_I^\Omega \partial_1 \lambda^I. \quad (5.3)$$

It matches precisely with (5.1), for $\delta\sigma = -\delta \log \mu$, provided that

$$d_I^\Omega = \frac{1}{4} \partial_I (K + h^\Omega). \quad (5.4)$$

Thus the one-point functions of marginal operators must be completely determined by K and h^Ω . Notice that the d_I^Ω are Kähler-Weyl invariant, as expected for the data of a SCFT.

These relations are reminiscent of those obeyed by the RR charges of D-branes in Calabi-Yau compactifications [14]. We will see that this is no coincidence.

One more remark is in order here. Contrary to what happened for the two-point functions, which only had a scale anomaly, the one-point functions also suffer from a global axial anomaly captured by the contributions that do not vanish for constant δa . This anomaly reflects a contact term in the two-point functions $\langle (\partial_\mu j_A^\mu) \mathcal{O}_I \rangle_\Omega$ where j_A^μ is the axial R-symmetry current. One could compute this contact term directly along the lines of refs. [33, 34]. Here we have derived it from the supersymmetric Ward identity (4.12), which related it to the scale anomaly as we have explained.

5.1 Ramond-Ramond charges

The RR charges of a D-brane are the overlaps of the corresponding boundary state $|\Omega\rangle\rangle$ with the supersymmetric vacua of the worldsheet theory. These latter are of two kinds: (i) the canonical ground state $|0\rangle_{\text{RR}}$ obtained from the Neveu-Schwarz vacuum by spectral flow, and (ii) the states $|I\rangle_{\text{RR}}$ obtained by spectral flow from the Neveu-Schwarz states $\phi^I(0)|0\rangle_{\text{NS}}$, where ϕ^I are the lowest components of twisted-chiral superfields whose highest components are the marginal operators $\mathcal{O}^I$. The geometry of this ‘vacuum bundle’, i.e. how the collection of these vacuum states varies as a function of the moduli, has been described in the classical work of Cecotti and Vafa [35, 36].

The boundary state is a formal sum of Ishibashi states, one for each representation of the $\mathcal{N} = (2, 2)$ superconformal algebra. The coefficients in this sum are determined by the inner product of $|\Omega\rangle\rangle$ with the highest-weight states in each Ishibashi state. Of particular interest is the projection Π_{vac} of $|\Omega\rangle\rangle$ on the supersymmetric ground states¹²

$$\Pi_{\text{vac}} |\Omega\rangle\rangle := c^\Omega |0\rangle_{\text{RR}} + \sum_I c_I^\Omega |I\rangle_{\text{RR}}.$$

The key observation of Ooguri, Oz and Yin [14] (see also [15]) is that this is a flat section of an improved connection $\nabla = D - C$ on the (twisted-chiral) moduli space, where $(C_I)^J_K$ are the structure constants of the twisted-chiral ring, and the overall normalization is chosen

¹²A B-type brane has no component along ground states obtained by spectral flow from chiral (c, c) fields. So ground states here refers implicitly to those obtained by flowing from the twisted chiral (a, c) fields only.

so that $|0\rangle_{\text{RR}}$ has holomorphic dependence on the moduli. In practice, for our purposes here, these statements imply

$$\bar{\partial}_I c^\Omega = 0, \quad \partial_I c^\Omega + (\partial_I K) c^\Omega - c_I^\Omega = 0,$$

from which one finds easily

$$\frac{c_I^\Omega}{c^\Omega} = \partial_I (K + \log c^\Omega). \quad (5.5)$$

We will now argue that $c_I^\Omega/c^\Omega = 4d_I^\Omega$, so that comparing (5.4) with (5.5) identifies h^Ω with the logarithm of the canonical RR charge of the Ω brane,

$$h^\Omega(\lambda) = \log c^\Omega(\lambda). \quad (5.6)$$

To relate the one-point function coefficients with the RR charges of the boundary D-brane we perform the conformal map from the half-plane to the semi-infinite cylinder, This reads

$$y = \frac{z+1}{z-1} \implies \frac{\partial y}{\partial z} = -\frac{2}{(z-1)^2},$$

where z parametrizes the half-plane ($\text{Re} z \leq 0$ with $z = x^1 - ix^2$) and y parametrizes the cylinder ($\log y = \tau + i\varphi$ with $\tau \leq 0$). For any conformal scalar primary field Ψ_Δ with scaling dimension Δ one has

$$\langle \Psi_\Delta(z=-1) \rangle_{\text{half-plane}} = \lim_{\tau \rightarrow -\infty} \frac{1}{2^\Delta} e^{-\Delta\tau} \langle \Psi_\Delta(y) \rangle_{\text{half-cylinder}} = \frac{1}{2^\Delta} \frac{\langle \langle \Omega | \Psi_\Delta \rangle_{\text{NS}} \rangle}{\langle \langle \Omega | 0 \rangle_{\text{NS}} \rangle},$$

where $|\Psi_\Delta\rangle$ is the state created by acting with Ψ_Δ on the NS vacuum, and the overall normalization was fixed so that insertion of the identity operator gives 1. Pick $\Psi_\Delta = \mathcal{O}_I$, so that $\Delta = 2$, and insert in the inner products on the right-hand side the spectral flow operator $e^{i\hat{\xi}}$. This latter maps NS states to RR states,¹³ while its action on the boundary state is a pure phase, $e^{i\hat{\xi}}|\Omega\rangle\rangle = e^{i\xi^\Omega}|\Omega\rangle\rangle$. It follows easily that

$$d_I^\Omega = \frac{c_I^\Omega}{4c^\Omega} \quad (5.7)$$

which is the sought-for relationship.

Equations (5.4) and (5.5) have been obtained from different routes, so the fact that they match is a confirmation of our result for the super-Weyl anomaly.

6 Hemisphere partition functions

Up to now we have considered the generating functional of correlation functions expanded around the flat-metric background, i.e. for $\delta\Sigma$ very small. In this section we will integrate

¹³Usually the starting point of the spectral flow is the state created by the lowest component of the marginal superfield, but this is charged so its OPE with $e^{i\hat{\xi}}$ is singular. Since we are interested here in normalizations, it is preferable to start with the top component which is neutral and has therefore a non-singular OPE. As the end states of the flow lie in the same Ishibashi block, they have the same coupling to the boundary state.

the anomaly and calculate $\mathcal{Z}(D^2)$ for supersymmetric backgrounds with the topology of the disk. We would like, in particular, to prove the conjecture of refs. [8, 9] that the round-hemisphere partition function computes the holomorphic boundary charge.

We begin by restricting the anomaly to the case of constant λ^I . Dropping all derivative terms in eq. (5.1) (as well as the moduli-independent terms) we find

$$A_{\text{open}} \supset \delta \left\{ -\frac{1}{4\pi} \int d^2x \left[\square(\sigma - ia) h^\Omega + \square(\sigma + ia) \bar{h}^\Omega \right] + \frac{i}{4\pi} \int dx^0 \left[\bar{w} h^\Omega - w \bar{h}^\Omega \right] \right\}. \quad (6.1)$$

As was the case for closed M , here also the constant- λ anomaly is the variation of a local covariant action. It would have been cohomologically trivial in a bosonic theory, but acquires universal meaning thanks to $\mathcal{N} = (2, 2)$ supersymmetry.

The expression inside the curly brackets is the integrated anomaly, $-i \log \mathcal{Z}$. In writing it we have converted the boundary terms to bulk integrals of total derivatives. This does not change the anomaly, but it ensures that $-i \log \mathcal{Z}(M) \rightarrow 0$ when M is a vanishingly-small disk. Consider for instance terms in (5.1) that are proportional to the Kähler potential. If we convert the boundary term to a total-derivative the two such terms cancel each other. Otherwise $K(\int d^2x \square \sigma - \int dx^0 \partial_1 \sigma) = -2\pi K \chi_M$ with χ_M the Euler characteristic of the surface. Neither choice contributes to $\delta \log \mathcal{Z}$, but only the first one guarantees that excising a tiny bit of surface from M (e.g. as part of the regularization) does not change the free energy by a finite amount. Furthermore, by converting the Wilson line to integrated flux we make it insensitive to gauge-choice singularities. The contribution of auxiliary fields was left as a boundary integral with the implicit understanding that w is non-singular, i.e. that $\oint_{\mathcal{C}} w \rightarrow 0$ for any shrinking cycle $\mathcal{C}$ in the interior of M .

6.1 Killing spinor equations

In the integrated anomaly (6.1) the dependence on the Kähler potential dropped out, so $\mathcal{Z}(M)$ depends only on the boundary charge. To compute $\mathcal{Z}(M)$ we need, in addition to the metric and gauge field, also the auxiliary fields $w, \bar{w}$. For supersymmetric backgrounds these follow from the Killing-spinor equations of the $\mathcal{N} = 2$ supergravity, whose covariant form can be found in ref. [23]. We take here the simpler route [1] of working directly in superconformal gauge, where these equations reduce to the condition that the twisted-chiral-field background $\exp(\Sigma_{\text{backgr}})$ be annihilated by global superconformal transformations.

The standard supersymmetry transformations of a twisted chiral field with components $(\phi, \psi_-, \bar{\psi}_+, F)$, and of the conjugate twisted antichiral field, read

$$\begin{aligned} \delta_{\text{susy}} \phi &= \bar{\epsilon}_+ \psi_- - \epsilon_- \bar{\psi}_+, & \delta_{\text{susy}} \bar{\phi} &= \bar{\epsilon}_- \psi_+ - \epsilon_+ \bar{\psi}_-, \\ \delta_{\text{susy}} \psi_- &= -2i\epsilon_+ \partial_- \phi + \epsilon_- F, & \delta_{\text{susy}} \bar{\psi}_- &= 2i\bar{\epsilon}_+ \partial_- \bar{\phi} + \bar{\epsilon}_- \bar{F}, \\ \delta_{\text{susy}} \bar{\psi}_+ &= 2i\bar{\epsilon}_- \partial_+ \phi + \bar{\epsilon}_+ F, & \delta_{\text{susy}} \psi_+ &= -2i\epsilon_- \partial_+ \bar{\phi} + \epsilon_+ \bar{F}, \\ \delta_{\text{susy}} F &= -2i\epsilon_+ \partial_- \bar{\psi}_+ - 2i\bar{\epsilon}_- \partial_+ \psi_-, & \delta_{\text{susy}} \bar{F} &= -2i\bar{\epsilon}_+ \partial_- \psi_+ - 2i\epsilon_- \partial_+ \bar{\psi}_-. \end{aligned}$$

Assume that ϕ is a conformal primary field with conformal dimensions (Δ_+, Δ_-) , so that the fermions ψ_+ and ψ_- have dimensions $(\Delta_+ + \frac{1}{2}, \Delta_-)$ and $(\Delta_+, \Delta_- + \frac{1}{2})$ while the auxiliary

field F has dimensions $(\Delta_+ + \frac{1}{2}, \Delta_- + \frac{1}{2})$. We may render the above equations covariant under arbitrary conformal transformations by letting the parameters ϵ_- and ϵ_+ transform as conformal tensors of dimensions $(-\frac{1}{2}, 0)$ and $(0, -\frac{1}{2})$, and by modifying appropriately the derivatives,

$$\begin{aligned}\delta_{\text{susy}}\phi &= \bar{\epsilon}_+\psi_- - \epsilon_-\bar{\psi}_+, & \delta_{\text{susy}}F &= -2i\epsilon_+D_-\bar{\psi}_+ - 2i\bar{\epsilon}_-D_+\psi_-, \\ \delta_{\text{susy}}\psi_- &= -2i\epsilon_+D_-\phi + \epsilon_-F, & \delta_{\text{susy}}\bar{\psi}_+ &= 2i\bar{\epsilon}_-D_+\phi + \bar{\epsilon}_+F,\end{aligned}\quad (6.2)$$

where for a conformal tensor X with dimensions (Δ_+, Δ_-) the covariant derivatives are

$$\epsilon_+D_-X := \epsilon_+\partial_-X + 2\Delta_-(\partial_-\epsilon_+)X, \quad \bar{\epsilon}_-D_+X := \bar{\epsilon}_-\partial_+X + 2\Delta_+(\partial_+\bar{\epsilon}_-)X. \quad (6.3)$$

Similar formulae apply to the twisted anti-chiral field. The modified transformations reduce to the standard ones for constant $\epsilon_\pm$, $\bar{\epsilon}_\pm$, and behave homogeneously under changes of the conformal frame provided that ϵ_+ , $\bar{\epsilon}_+$ are functions of x^- , and ϵ_- , $\bar{\epsilon}_-$ functions of x^+ . Changes of frame make these functions arbitrary.

Following [1] we can express the Killing spinor equations as the conditions that e^Σ be left invariant by a globally-defined superconformal transformation. The exponential of the Weyl superfield is a twisted chiral field with dimensions $(\frac{1}{2}, \frac{1}{2})$.¹⁴ Its bosonic components are $\phi = \exp(\sigma + ia)$, $F = \exp(\sigma + ia)w$, $\bar{\phi} = \exp(\sigma - ia)$, $\bar{F} = \exp(\sigma - ia)\bar{w}$, while the fermionic background is set to zero. Invariance under (6.2) then implies

$$\begin{aligned}\epsilon_-w &= 2i\epsilon_+\partial_-(\sigma + ia + \log \epsilon_+), & \bar{\epsilon}_+w &= -2i\bar{\epsilon}_-\partial_+(\sigma + ia + \log \bar{\epsilon}_-), \\ \epsilon_+\bar{w} &= 2i\epsilon_-\partial_+(\sigma - ia + \log \epsilon_-), & \bar{\epsilon}_-\bar{w} &= -2i\bar{\epsilon}_+\partial_-(\sigma - ia + \log \bar{\epsilon}_+).\end{aligned}\quad (6.4)$$

We are interested in backgrounds with Euclidean signature, so we perform the Wick rotation $x^0 \rightarrow -ix^2$ which maps (x^+, x^-) to $(\bar{z}, -z)$ with $z := x^1 + ix^2$. We also write the Wick rotated ϵ 's (which we continue to label with subscripts ' $\pm$ ') as constant anticommuting parameters multiplying holomorphic or antiholomorphic Killing spinors

$$\epsilon_+ = \epsilon \zeta^-(z), \quad \epsilon_- = -\epsilon \zeta^+(\bar{z}), \quad \bar{\epsilon}_+ = \bar{\epsilon} \bar{\zeta}^-(z), \quad \bar{\epsilon}_- = -\bar{\epsilon} \bar{\zeta}^+(\bar{z}). \quad (6.5)$$

Depending on the unbroken supersymmetries some of these parameters could be set to zero. We are interested in supersymmetries compatible with the B-type boundary conditions, for which all the ζ 's are non-vanishing. With the above conventions the auxiliary fields read

$$\begin{aligned}w &= 2i\frac{\zeta^-}{\zeta^+}\partial_z(\sigma + ia + \log \zeta^-) = 2i\frac{\bar{\zeta}^+}{\bar{\zeta}^-}\partial_{\bar{z}}(\sigma + ia + \log \bar{\zeta}^+), \\ \bar{w} &= -2i\frac{\zeta^+}{\zeta^-}\partial_{\bar{z}}(\sigma - ia + \log \zeta^+) = -2i\frac{\bar{\zeta}^-}{\bar{\zeta}^+}\partial_z(\sigma - ia + \log \bar{\zeta}^-).\end{aligned}\quad (6.6)$$

These are the relations derived in appendix D of ref. [1].

¹⁴This field transforms homogeneously under the reparametrizations and $U(1)_V$ gauge transformations that preserve the superconformal gauge: $x^{\pm'} = f^{\pm}(x^{\pm})$ and $V' = V + dg$ with $g = g^+(x^+) + g^-(x^-)$. A simple calculation gives $\exp(\sigma' + ia') = (df^+/dx^+)^{-1/2} (df^-/dx^-)^{-1/2} \exp(ig^+ - ig^-) \exp(\sigma + ia)$. Note that a $U(1)_V$ gauge transformation with angle $(g^+ + g^-)$ is equivalent to a $U(1)_A$ gauge transformation with angle $(g^+ - g^-)$. Thus the lowest component of $\exp(\Sigma)$ behaves under this restricted class of transformations as a $(1/2, 1/2)$ conformal tensor with unit axial-R charge.

6.2 The (squashed) hemisphere

To find non-trivial solutions of these equations one must allow for non-hermitean backgrounds in which the metric factor σ is real, but a is allowed to be complex and $(\bar{w})^* \neq w$. Consider a surface with disk topology parametrized by $\{z \in \mathbb{C}; |z| \leq 1\}$. We are interested in solutions that obey the B-type boundary conditions at $|z| = 1$:

$$\bar{z}^{-\frac{1}{2}} \zeta^+ = e^{-2i\beta} z^{-\frac{1}{2}} \zeta^-, \quad \bar{z}^{-\frac{1}{2}} \bar{\zeta}^+ = e^{2i\beta} z^{-\frac{1}{2}} \bar{\zeta}^-. \quad (6.7)$$

Here β is the axial phase introduced in section 3, and we have transformed to the cylindrical coordinate $\log z = \tau + i\varphi$ using the fact that ϵ_+ and ϵ_- are conformal tensors with weight $(0, -\frac{1}{2})$ and $(-\frac{1}{2}, 0)$. These boundary conditions admit two inequivalent solutions for given β . If $\beta = 0$ they read

$$\begin{aligned} (+): \quad & \zeta^- = 1, \quad \zeta^+ = \bar{z}, \quad \bar{\zeta}^- = z, \quad \bar{\zeta}^+ = 1, \\ (-): \quad & \zeta^- = z, \quad \zeta^+ = 1, \quad \bar{\zeta}^- = 1, \quad \bar{\zeta}^+ = \bar{z}. \end{aligned} \quad (6.8)$$

These two choices, related by CPT, correspond to the two choices of spin structure in the conformal Killing spinor equation on the hemisphere [9].¹⁵ Inserting these expressions in (6.6) shows that both σ and a must be independent of the phase of z . Indeed, the solutions of the Killing spinor equations with the boundary conditions (6.7) leave unbroken a $SU(1|1)$ subgroup of the global superconformal group $OSp(2|2, \mathbb{C})$. The Killing isometry corresponds to the $U(1)$ factor of this unbroken symmetry.

A particular supersymmetric background is the perfectly round hemisphere with vanishing $U(1)_V$ gauge-field:

$$\sigma = -\log(1 + z\bar{z}) + \text{constant}, \quad a = 0.$$

For this background one finds the auxiliary fields

$$(+): \quad w = \bar{w} = -\frac{2i}{1 + z\bar{z}}, \quad (-): \quad w = \bar{w} = \frac{2i}{1 + z\bar{z}},$$

which are smooth in the interior and take constant values on the boundary, $w = \bar{w} = \mp i$. Inserting the above expressions in (6.1), and recalling that $h^\Omega = \log c^\Omega$, leads finally to the hemisphere partition functions

$$Z_+(D^2, \Omega) = \mathcal{Z}_0 c^\Omega(\lambda), \quad Z_-(D^2, \Omega) = \mathcal{Z}_0 c^\Omega(\bar{\lambda}). \quad (6.9)$$

This establishes the conjecture of refs. [8, 9]. The moduli-independent factor $\mathcal{Z}_0$,¹⁶ is a priori scheme-dependent, and hence uninteresting. It can be determined in accordance with the 2-sphere as in the eqs. (1.2) of the introduction.

¹⁵In [9] the minus spin structure is defined with phase $\beta = \pi/2$, which arises naturally from CPT. Since for a single boundary the gluing phase is irrelevant, we here set it to zero.

¹⁶In principle, the free energy could contain non-local terms that are Weyl-invariant and a -independent and hence make no contribution to the anomaly. We are assuming that if such terms exist they are λ -independent.

It is in fact straightforward to extend the calculation to more general metric and gauge-field backgrounds that respect the symmetry under phase rotations of z . An example is the squashed-hemisphere background of [6, 8]. Let $\delta\sigma(z\bar{z})$ and $a(z\bar{z})$ be the deformations of the round hemisphere background. Inserting eqs. (6.6) and (6.7) in the integrated anomaly (the expression in curly brackets in (6.1)) shows that all dependence on $\delta\sigma$ and a drops out as long as the $z \rightarrow 0$ region is smooth, i.e. free from conical and Dirac-string singularities. For all such backgrounds the results (6.9) continue to hold.

We conclude with some remarks. First, it follows from the Kähler-Weyl transformation (4.15) that the hemisphere partition functions are sections of (anti)holomorphic line bundles. The unambiguous quantities are the partition-function ratios $Z_{\pm}(D^2, \Omega_1)/Z_{\pm}(D^2, \Omega_2)$ for pairs of different boundary conditions, as well as the g -function (1.3) which is the physical degeneracy of the boundary. It is interesting that for $\mathcal{N} = 2$ boundaries the g -function can be determined entirely by anomalies.

A second remark concerns the superconformal interfaces that transport the SCFT along its moduli space $\mathcal{M}$. These can be mapped to boundaries by folding the surface along the interface and complex conjugating the folded theory [37]. The central charge and entropy of such interfaces depends on the analytic extension of the Kähler potential, [13]

$$c^{\Omega} = e^{-\frac{1}{2}K(\lambda_1, \bar{\lambda}_2)}, \quad 2\log g^{\Omega} = K(\lambda_1, \bar{\lambda}_1) + K(\lambda_2, \bar{\lambda}_2) - K(\lambda_1, \bar{\lambda}_2) - K(\lambda_2, \bar{\lambda}_1).$$

This extension can be therefore computed by localization of the hemisphere partition function.

Finally, it should be possible to extend the analysis of [1] to four-dimensional manifolds with boundary. A localization calculation of an $\mathcal{N} = 2$ supersymmetric gauge theory on the four-dimensional hemisphere has been performed recently in [38]. Another interesting question concerns the dependence of c^{Ω} on the boundary (or open-string) moduli. Since this is part of the definition of Ω it should be also accessible by localization techniques.

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A Notation and conventions

We follow the conventions in chapter 12 of [22]. Superfields are functions on $\mathcal{N} = (2, 2)$ superspace with coordinates $(x^{\pm}, \theta^{\pm}, \bar{\theta}^{\pm})$, where $x^{\pm} = x^0 \pm x^1$. The flat Minkowski metric is $\eta_{00} = -\eta_{11} = -1$, so that $\square = -4\partial_+\partial_-$. The Wick rotation sets $x^0 = -ix^2$. Complex conjugation flips the order of the fermionic coordinates and acts on them as $(\theta^{\pm})^* = \bar{\theta}^{\pm}$. The Grassmann integration measure is $d^4\theta := d\theta^+d\theta^-d\bar{\theta}^-d\bar{\theta}^+$.

A general supersymmetry transformation reads

$$\Delta_{susy} = \epsilon_+ Q_- - \epsilon_- Q_+ - \bar{\epsilon}_+ \bar{Q}_- + \bar{\epsilon}_- \bar{Q}_+ \quad (\text{A.1})$$

where the ϵ are anticommuting parameters and

$$Q_\pm = \frac{\partial}{\partial \theta^\pm} + i\bar{\theta}^\pm \partial_\pm, \quad \bar{Q}_\pm = -\frac{\partial}{\partial \bar{\theta}^\pm} - i\theta^\pm \partial_\pm. \quad (\text{A.2})$$

Another useful set of differential operators is

$$D_\pm = \frac{\partial}{\partial \theta^\pm} - i\bar{\theta}^\pm \partial_\pm, \quad \bar{D}_\pm = -\frac{\partial}{\partial \bar{\theta}^\pm} + i\theta^\pm \partial_\pm.$$

Twisted chiral fields obey the relations

$$\bar{D}_+ \Phi = D_- \Phi = 0,$$

and have the following expansion in components

$$\Phi = \phi(y^\pm) + \theta^+ \bar{\psi}_+(y^\pm) + \bar{\theta}^- \psi_-(y^\pm) + \theta^+ \bar{\theta}^- F(y^\pm), \quad (\text{A.3})$$

where $y^\pm = x^\pm \mp i\theta^\pm \bar{\theta}^\pm$. The operators $D_\pm$ have trivial cohomology, meaning that $D_+ \mathcal{F} = 0$ implies that $\mathcal{F} = D_+ \mathcal{G}$ for some superfield $\mathcal{G}$.

The supersymmetry variation of a twisted chiral field in terms of components reads

$$\begin{aligned} \delta_{\text{susy}} \phi &= \bar{\epsilon}_+ \psi_- - \epsilon_- \bar{\psi}_+, & \delta_{\text{susy}} \bar{\phi} &= \bar{\epsilon}_- \psi_+ - \epsilon_+ \bar{\psi}_-, \\ \delta_{\text{susy}} \psi_- &= -2i\epsilon_+ \partial_- \phi + \epsilon_- F, & \delta_{\text{susy}} \bar{\psi}_- &= 2i\bar{\epsilon}_+ \partial_- \bar{\phi} + \bar{\epsilon}_- \bar{F}, \\ \delta_{\text{susy}} \bar{\psi}_+ &= 2i\bar{\epsilon}_- \partial_+ \phi + \bar{\epsilon}_+ F, & \delta_{\text{susy}} \psi_+ &= -2i\epsilon_- \partial_+ \bar{\phi} + \epsilon_+ \bar{F}, \\ \delta_{\text{susy}} F &= -2i\epsilon_+ \partial_- \bar{\psi}_+ - 2i\bar{\epsilon}_- \partial_+ \psi_-, & \delta_{\text{susy}} \bar{F} &= -2i\bar{\epsilon}_+ \partial_- \psi_+ - 2i\epsilon_- \partial_+ \bar{\psi}_-. \end{aligned}$$

On a B-type boundary we have the following identifications of coordinates,

$$x^+ = x^-, \quad \theta := e^{-i\beta} \theta^+ = e^{i\beta} \theta^-, \quad \bar{\theta} := e^{i\beta} \bar{\theta}^+ = e^{-i\beta} \bar{\theta}^-. \quad (\text{A.4})$$

The unbroken supersymmetries are generated by (A.1) with $\epsilon := e^{-i\beta} \epsilon_+ = -e^{i\beta} \epsilon_-$ and $\bar{\epsilon} := e^{i\beta} \bar{\epsilon}_+ = -e^{-i\beta} \bar{\epsilon}_-$. Unless indicated otherwise, we set the phase $\beta = 0$.

The Euler density is $\sqrt{g}R = -2\Box\sigma$ and the Euler characteristic reads

$$\chi_M = \frac{1}{4\pi} \left[\int_M \sqrt{g}R + 2 \int_{\partial M} k \right] = -\frac{1}{2\pi} \left[\int_M \Box\sigma - \int_{\partial M} \partial_\perp \sigma \right] = 2 - 2h_M - b_M \quad (\text{A.5})$$

where h_M is the number of handles and b_M the number of boundaries of the surface M . The normal derivative of the Weyl factor is in the outward direction.

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