Published for SISSA by Springer

Received: *January 18, 2015*
Revised: *February 23, 2015*
Accepted: *March 6, 2015*
Published: *March 25, 2015*

On the $\mathcal{N} = 4$, d = 4 pure spinor measure factor

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Abstract: In this work, we obtain a simple measure factor for the λ and θ zero-mode integrations in the pure-spinor formalism in the context of an $\mathcal{N} = 4$, d = 4 theory. We show that the measure can be defined unambiguously up to BRST-trivial terms and an overall factor, and is much simpler than (although equivalent to) the expression obtained by dimensional reduction from the $\mathcal{N} = 1$, d = 10 measure factor. We also give two explicit examples of how to obtain the dual to a vertex operator using this measure.

Keywords: Superstrings and Heterotic Strings, BRST Symmetry, Extended Supersymmetry

ArXiv ePrint: 1412.5927

doi:10.1007/JHEP03(2015)136

Contents

1 Introduction

The prescription for computing tree-level open superstring scattering amplitudes in a manifestly super-Poincaré covariant manner was given by Berkovits some time ago, in the same paper in which the pure spinor superstring was introduced [1]. In this formalism, the unintegrated vertex operators are in the ghost-number 1 cohomology of the BRST operator

$$Q = \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}z \, \lambda^{\hat{\alpha}} d_{\hat{\alpha}} \tag{1.1}$$

and are functions of the ten-dimensional superspace coordinates x^μ and $\theta^{\hat{\alpha}}$ for $\mu = 0$ to 9 and $\hat{\alpha} = 1$ to 16. More information on notations and conventions can be found in appendix A.

In (1.1), $\lambda^{\hat{\alpha}}$ is a pure-spinor ghost variable, i.e. $\lambda^{\hat{\alpha}}$ satisfies $\lambda\gamma_\mu\lambda = 0$, and $d_{\hat{\alpha}} = p_{\hat{\alpha}} - \frac{2}{\alpha'}\left[\partial x^\mu (\theta\gamma_\mu)_{\hat{\alpha}} + \frac{1}{2}(\theta\gamma^\mu\partial\theta)(\theta\gamma_\mu)_{\hat{\alpha}}\right]$, where $p_{\hat{\alpha}}$ is the conjugate momentum to $\theta^{\hat{\alpha}}$. The OPE's

$$p_{\hat{\alpha}}(z)\,\theta^{\hat{\beta}}(w) \sim \frac{\delta_{\hat{\alpha}}^{\hat{\beta}}}{z-w} \qquad \text{and} \qquad x^\mu(z,\bar{z})\,x^\nu(w,\bar{w}) \sim -\frac{\alpha'}{2}\eta^{\mu\nu}\log|z-w|^2$$

imply

$$d_{\hat{\alpha}}(z)\,\mathcal{F}(x(w),\theta(w)) \sim \frac{1}{z-w} D_{\hat{\alpha}}\mathcal{F}(x(w),\theta(w))\,, \tag{1.2}$$

where $D_{\hat{\alpha}} = \frac{\partial}{\partial\theta^{\hat{\alpha}}} + (\theta\gamma^\mu)_{\hat{\alpha}}\partial_\mu$ and $\mathcal{F}(x,\theta)$ is any superfield. Hence, we write $Q\mathcal{F} = \lambda^{\hat{\alpha}} D_{\hat{\alpha}}\mathcal{F}$.

In addition to knowing the vertex operators and OPE's, one needs to know how to perform the integrations of the λ and θ zero modes. They can be performed by means of the following BRST-invariant measure factor:

$$\left\langle (\lambda\gamma^{\mu}\theta)(\lambda\gamma^{\nu}\theta)(\lambda\gamma^{\rho}\theta)(\theta\gamma_{\mu\nu\rho}\theta) \right\rangle = 1\,. \tag{1.3}$$

More precisely, the integration is done by keeping only the terms proportional to three λ's and five θ's in this combination.

Thus far, we have written everything in an $\mathcal{N} = 1$, d $= 10$ notation. In order to compute superstring scattering amplitudes in an $\mathcal{N} = 4$, d $= 4$ theory — such as the gauge theory describing the effective world-volume degrees of freedom of a D3-brane, for instance — using the pure-spinor formalism, one needs to know how to perform the integrations of the λ and θ zero modes in that case. In other words, one needs to find a BRST-invariant measure factor analogous to (1.3).

At first, it might seem to be just a matter of dimensional reduction. However, although the particular combination of λ's and θ's of (1.3) is special in ten flat dimensions, since it is the unique (up to an overall factor) SO(9, 1) scalar which can be built out of three λ's and five θ's, there is no reason why its dimensional reduction should be preferred over any other BRST-invariant, SO(3, 1)×SU(4) scalar in four dimensions. Therefore, it is important to investigate whether there is any ambiguity in the definition of the $\mathcal{N} = 4$, d $= 4$ measure factor.

In this paper, this issue is studied in detail. In section 2, we write the most general four-dimensional expression with three λ's and five θ's and derive the conditions for it to be BRST invariant. In section 3, we find the independent BRST-trivial combinations of the terms introduced in section 2. In section 4, we present the main results of this paper. We find that the $\mathcal{N} = 4$, d $= 4$ measure factor is unique up to BRST-trivial terms and an overall factor. Moreover, we show that the measure can be written in a much simpler form than the dimensional reduction of (1.3) — the latter has twelve terms, whereas the former has only three. In section 5, we give two examples of the use of this measure factor. Finally, section 6 is devoted to our conclusions.

2 BRST equations

In four-dimensional notation, the most general SO(3, 1)×SU(4)-invariant, real expression one can write with three λ's and five θ's is

$$\begin{aligned}(\lambda^3\theta^5) := {} & c_1\,\varepsilon_{mnj\ell}(\bar\lambda_i\bar\lambda_k)(\lambda^m\theta^n)(\theta^i\theta^j)(\theta^k\theta^\ell)\\ & + c_2\,(\bar\lambda_j\bar\lambda_k)(\bar\lambda_\ell\bar\theta_i)(\theta^i\theta^j)(\theta^k\theta^\ell) + c_3\,\varepsilon_{mnj\ell}(\bar\lambda_i\bar\theta_k)(\lambda^k\theta^\ell)(\lambda^m\theta^n)(\theta^i\theta^j)\\ & + c_4\,(\bar\lambda_\ell\bar\lambda_k)(\lambda^j\theta^\ell)(\theta^i\theta^k)(\bar\theta_i\bar\theta_j) + c_5\,(\bar\lambda_i\bar\theta_k)(\bar\lambda_j\bar\theta_\ell)(\lambda^k\theta^\ell)(\theta^i\theta^j)\\ & + c_6\,\varepsilon_{mn\ell k}(\lambda^i\theta^k)(\lambda^j\theta^\ell)(\lambda^m\theta^n)(\bar\theta_i\bar\theta_j)\\ & + \text{H.c.}\,,\end{aligned} \tag{2.1}$$

where $c_1, \ldots, c_6$ are arbitrary constants and "H.c." means "Hermitian conjugate". We use the standard d=4 two-component spinor notation as described in appendix A.1. In

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particular, $i, j, \ldots = 1$ to 4 and $\alpha, \dot{\alpha} = 1$ to 2. One can convince oneself these are the only non-zero independent terms which can be constructed, keeping in mind that

$$\lambda^{\alpha i}\bar{\lambda}_i^{\dot{\alpha}} = 0 \qquad \text{and} \qquad (\lambda^i\lambda^j) = \frac{1}{2}\varepsilon^{ijk\ell}(\bar{\lambda}_k\bar{\lambda}_\ell)\,, \tag{2.2}$$

which are the dimensional reduction of $\lambda\gamma^\mu\lambda = 0$. More details on dimensional reduction can be found in appendix A.2. The notation we are going to use throughout the paper is such that

$$(\lambda^3\theta^5) =: \sum_{n=1}^{6} c_n\mathbf{T}_n + \text{H.c.}\,, \tag{2.3}$$

i.e. we define $\mathbf{T}_1, \ldots, \mathbf{T}_6$ to be the independent possible terms as appearing in (2.1). For example, $\mathbf{T}_3 \equiv \varepsilon_{mnj\ell}(\bar{\lambda}_i\bar{\theta}_k)(\lambda^k\theta^\ell)(\lambda^m\theta^n)(\theta^i\theta^j)$. The $\mathbf{T}_n$ and their Hermitian conjugates $\mathbf{T}_n^\dagger$ form a basis for four-dimensional expressions made of three λ's and five θ's. For example, it is not difficult to show the dimensional reduction of (1.3) gives (2.1) with $c_1 = 1$, $c_2 = c_4 = 4$, $c_3 = 3$, $c_5 = 12$ and $c_6 = 2$, up to an overall factor.

Since we are looking for the pure spinor measure, we are interested in expressions which are annihilated by $\lambda^{\hat{\alpha}}D_{\hat{\alpha}} = \lambda^{\alpha p}D_{\alpha p} + \bar{\lambda}_{\dot{\alpha}p}\bar{D}^{\dot{\alpha}p}$. This requirement yields equations for the constants in (2.1). We begin with

$$\Big[\lambda^{\hat{\alpha}}D_{\hat{\alpha}}(\lambda^3\theta^5)\Big]_{\theta^4\bar{\theta}^0} = \lambda^{\alpha p}D_{\alpha p}[c_1\mathbf{T}_1] + \bar{\lambda}_{\dot{\alpha}p}\bar{D}^{\dot{\alpha}p}[c_2\mathbf{T}_2 + c_3\mathbf{T}_3]\,, \tag{2.4}$$

where the subscript $\theta^4\bar{\theta}^0$ means "contributions with four θ's and no $\bar{\theta}$." The explicit calculation gives:

$$\begin{aligned}\lambda^{\alpha p}D_{\alpha p}\mathbf{T}_1 &= \varepsilon_{mnj\ell}(\bar{\lambda}_i\bar{\lambda}_k)\Big[(\lambda^m\lambda^n)(\theta^i\theta^j)(\theta^k\theta^\ell) + (\lambda^m\theta^n)(\theta^i\lambda^j)(\theta^k\theta^\ell) \\ &\qquad\qquad + (\lambda^m\theta^n)(\theta^i\theta^j)(\theta^k\lambda^\ell)\Big] \\ &= 4\,(\bar{\lambda}_i\bar{\lambda}_k)(\bar{\lambda}_j\bar{\lambda}_\ell)(\theta^i\theta^j)(\theta^k\theta^\ell)\,,\end{aligned} \tag{2.5}$$

where we used (2.2) and $\lambda^{\alpha[i|}\lambda^{\beta|j]} = -\frac{1}{2}\varepsilon^{\alpha\beta}(\lambda^i\lambda^j)$. Moreover,

$$\bar{\lambda}_{\dot{\alpha}p}\bar{D}^{\dot{\alpha}p}\mathbf{T}_2 = (\bar{\lambda}_j\bar{\lambda}_k)(\bar{\lambda}_\ell\bar{\lambda}_i)(\theta^i\theta^j)(\theta^k\theta^\ell) \tag{2.6}$$

and $\bar{\lambda}_{\dot{\alpha}p}\bar{D}^{\dot{\alpha}p}\mathbf{T}_3 = 0$. Therefore

$$\boxed{\Big[\lambda^{\hat{\alpha}}D_{\hat{\alpha}}(\lambda^3\theta^5)\Big]_{\theta^4\bar{\theta}^0} = 0 \iff c_2 = 4c_1\,.} \tag{2.7}$$

Proceeding to the next order, we have

$$\Big[\lambda^{\hat{\alpha}}D_{\hat{\alpha}}(\lambda^3\theta^5)\Big]_{\theta^3\bar{\theta}^1} = \lambda^{\alpha p}D_{\alpha p}[c_2\mathbf{T}_2 + c_3\mathbf{T}_3] + \bar{\lambda}_{\dot{\alpha}p}\bar{D}^{\dot{\alpha}p}[c_4\mathbf{T}_4 + c_5\mathbf{T}_5 + c_6\mathbf{T}_6]\,. \tag{2.8}$$

The $\mathbf{T}_2$-contribution is easy to compute. We get

$$\lambda^{\alpha p}D_{\alpha p}\mathbf{T}_2 = -(\bar{\lambda}_j\bar{\lambda}_k)(\bar{\lambda}_\ell\bar{\theta}_i)(\lambda^i\theta^j)(\theta^k\theta^\ell)\,. \tag{2.9}$$

For $\mathbf{T}_3$, we obtain

$$\begin{aligned}\lambda^{\alpha p}D_{\alpha p}\mathbf{T}_3 &= -\varepsilon_{mnj\ell}(\bar\lambda_i\bar\theta_k)\Big[(\lambda^k\lambda^\ell)(\lambda^m\theta^n)(\theta^i\theta^j)-(\lambda^k\theta^\ell)(\lambda^m\lambda^n)(\theta^i\theta^j)\\ &\qquad\qquad -(\lambda^k\theta^\ell)(\lambda^m\theta^n)(\theta^i\lambda^j)\Big]\\ &= 4\,(\bar\lambda_j\bar\lambda_\ell)(\bar\lambda_i\bar\theta_k)(\lambda^k\theta^\ell)(\theta^i\theta^j)\,. \end{aligned}\tag{2.10}$$

The $\mathbf{T}_4$- and $\mathbf{T}_5$-contributions are also simple to calculate:

$$\bar\lambda_{\dot\alpha p}\bar D^{\dot\alpha p}\mathbf{T}_4 = -(\bar\lambda_\ell\bar\lambda_k)(\lambda^j\theta^\ell)(\theta^i\theta^k)(\bar\lambda_i\bar\theta_j)\,,\tag{2.11}$$

$$\bar\lambda_{\dot\alpha p}\bar D^{\dot\alpha p}\mathbf{T}_5 = -(\bar\lambda_i\bar\theta_k)(\bar\lambda_j\bar\lambda_\ell)(\lambda^k\theta^\ell)(\theta^i\theta^j)\,.\tag{2.12}$$

Finally, $\bar\lambda_{\dot\alpha p}\bar D^{\dot\alpha p}\mathbf{T}_6=0$. In all, we get our second equation for the coefficients:

$$\boxed{\Big[\lambda^{\hat\alpha}D_{\hat\alpha}(\lambda^3\theta^5)\Big]_{\theta^3\bar\theta^1}=0 \iff c_2+4c_3=c_4+c_5\,.}\tag{2.13}$$

We now analyze the contributions with equal number of θ's and $\bar\theta$'s:

$$\Big[\lambda^{\hat\alpha}D_{\hat\alpha}(\lambda^3\theta^5)\Big]_{\theta^2\bar\theta^2} = \lambda^{\alpha p}D_{\alpha p}[c_4\mathbf{T}_4+c_5\mathbf{T}_5+c_6\mathbf{T}_6]+\bar\lambda_{\dot\alpha p}\bar D^{\dot\alpha p}[\bar c_4\mathbf{T}_4^\dagger+\bar c_5\mathbf{T}_5^\dagger+\bar c_6\mathbf{T}_6^\dagger]\,.\tag{2.14}$$

Again, it is straightforward to compute the contributions from $\mathbf{T}_4$ and $\mathbf{T}_5$:

$$\lambda^{\alpha p}D_{\alpha p}\mathbf{T}_4 = -(\bar\lambda_\ell\bar\lambda_k)(\lambda^j\theta^\ell)(\lambda^i\theta^k)(\bar\theta_i\bar\theta_j)\,,\tag{2.15}$$

$$\lambda^{\alpha p}D_{\alpha p}\mathbf{T}_5 = (\bar\lambda_i\bar\theta_k)(\bar\lambda_j\bar\theta_\ell)(\lambda^k\lambda^\ell)(\theta^i\theta^j)\,.\tag{2.16}$$

The $\mathbf{T}_6$-contribution yields

$$\begin{aligned}\lambda^{\alpha p}D_{\alpha p}\mathbf{T}_6 &= \varepsilon_{mn\ell k}(\bar\theta_i\bar\theta_j)\Big[(\lambda^i\lambda^k)(\lambda^j\theta^\ell)(\lambda^m\theta^n)-(\lambda^i\theta^k)(\lambda^j\lambda^\ell)(\lambda^m\theta^n)\\ &\qquad\qquad +(\lambda^i\theta^k)(\lambda^j\theta^\ell)(\lambda^m\lambda^n)\Big]\\ &= 4\,(\bar\lambda_\ell\bar\lambda_k)(\lambda^i\theta^k)(\lambda^j\theta^\ell)(\bar\theta_i\bar\theta_j)\,. \end{aligned}\tag{2.17}$$

These in turn imply, by Hermitian conjugation,

$$\bar\lambda_{\dot\alpha p}\bar D^{\dot\alpha p}\mathbf{T}_4^\dagger = -(\lambda^\ell\lambda^k)(\bar\lambda_j\bar\theta_\ell)(\bar\lambda_i\bar\theta_k)(\theta^i\theta^j)\,,\tag{2.18}$$

$$\bar\lambda_{\dot\alpha p}\bar D^{\dot\alpha p}\mathbf{T}_5^\dagger = (\lambda^i\theta^k)(\lambda^j\theta^\ell)(\bar\lambda_k\bar\lambda_\ell)(\bar\theta_i\bar\theta_j)\,,\tag{2.19}$$

$$\bar\lambda_{\dot\alpha p}\bar D^{\dot\alpha p}\mathbf{T}_6^\dagger = 4\,(\lambda^\ell\lambda^k)(\bar\lambda_i\bar\theta_k)(\bar\lambda_j\bar\theta_\ell)(\theta^i\theta^j)\,.\tag{2.20}$$

Thus we obtain our last equation:

$$\boxed{\Big[\lambda^{\hat\alpha}D_{\hat\alpha}(\lambda^3\theta^5)\Big]_{\theta^2\bar\theta^2}=0 \iff \bar c_5=c_4+4c_6\,,}\tag{2.21}$$

as well as its complex conjugate.

Note that the vanishing of the orders $\theta^1\bar\theta^3$ and $\theta^0\bar\theta^4$ implies the complex conjugates of (2.7) and (2.13), since they are just the Hermitian conjugates of the orders $\theta^3\bar\theta^1$ and $\theta^4\bar\theta^0$, respectively.

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In summary, we have the following system of equations:

$$\lambda^{\hat{\alpha}} D_{\hat{\alpha}}(\lambda^3\theta^5) = 0 \iff \begin{cases} c_2 = 4c_1 \\ c_2 + 4c_3 = c_4 + c_5 \\ \bar{c}_5 = c_4 + 4c_6 \end{cases}, \tag{2.22}$$

as well as their complex conjugates.

3 BRST-trivial combinations

In the last section, we found the equations which the constants in (2.1) have to satisfy for the expression to be BRST-invariant. Because there are less equations than constants, one might think the $\mathcal{N} = 4$, d = 4 pure spinor measure factor is then not unambiguously defined. Fortunately, that is not the case, and the seemingly independent expressions are actually related by BRST-trivial terms, as we show in the following.

In order to find the independent BRST-trivial combinations of the $\mathbf{T}_n$, i.e. the combinations which equal $\lambda^{\hat{\alpha}} D_{\hat{\alpha}}$ of something, we start by looking for all independent possible terms with two λ's and six θ's. Keeping (2.2) in mind, we find that there are five:

$$\chi_1 := (\lambda^i\theta^j)(\theta^k\theta^\ell)(\bar{\lambda}_k\bar{\theta}_\ell)(\bar{\theta}_i\bar{\theta}_j)\,, \tag{3.1a}$$

$$\chi_2 := \varepsilon_{ijk\ell}(\lambda^i\theta^j)(\lambda^m\theta^k)(\theta^\ell\theta^n)(\bar{\theta}_m\bar{\theta}_n)\,, \tag{3.1b}$$

$$\chi_3 := \varepsilon^{ijk\ell}(\bar{\lambda}_i\bar{\theta}_j)(\bar{\lambda}_m\bar{\theta}_k)(\bar{\theta}_\ell\bar{\theta}_n)(\theta^m\theta^n)\,, \tag{3.1c}$$

$$\chi_4 := \varepsilon_{mnj\ell}(\bar{\lambda}_i\bar{\theta}_k)(\lambda^m\theta^n)(\theta^i\theta^j)(\theta^k\theta^\ell)\,, \tag{3.1d}$$

$$\chi_5 := \varepsilon^{mnj\ell}(\lambda^i\theta^k)(\bar{\lambda}_m\bar{\theta}_n)(\bar{\theta}_i\bar{\theta}_j)(\bar{\theta}_k\bar{\theta}_\ell)\,. \tag{3.1e}$$

Acting with $\lambda^{\hat{\alpha}} D_{\hat{\alpha}} = \lambda^{\alpha p} D_{\alpha p} + \bar{\lambda}_{\dot{\alpha} p}\bar{D}^{\dot{\alpha} p}$ on these terms, we obtain BRST-trivial expressions made of three λ's and five θ's. We begin with the first one:

$$\begin{aligned} \lambda^{\alpha p} D_{\alpha p}\chi_1 &= (\lambda^i\theta^j)(\theta^k\lambda^\ell)(\bar{\lambda}_k\bar{\theta}_\ell)(\bar{\theta}_i\bar{\theta}_j) \\ &= \frac{1}{2}\left[\mathbf{T}_4^\dagger - \mathbf{T}_5^\dagger\right], \end{aligned} \tag{3.2}$$

$$\begin{aligned} \bar{\lambda}_{\dot{\alpha} p}\bar{D}^{\dot{\alpha} p}\chi_1 &= -(\lambda^i\theta^j)(\theta^k\theta^\ell)(\bar{\lambda}_k\bar{\theta}_\ell)(\bar{\theta}_i\bar{\lambda}_j) \\ &= \frac{1}{2}\left[\mathbf{T}_5 - \mathbf{T}_4\right]. \end{aligned} \tag{3.3}$$

Thus we find the first BRST-trivial expression:

$$\boxed{\mathbf{T}_4^\dagger - \mathbf{T}_5^\dagger + \mathbf{T}_5 - \mathbf{T}_4 = \lambda^{\hat{\alpha}} D_{\hat{\alpha}}\left[2\chi_1\right].} \tag{3.4}$$

Of course, we could multiply the expression on the left-hand side of this equation by any constant and it would remain BRST-trivial. The same applies to the other boxed expressions we find in the following.

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For the second term in (3.1), we have

$$\begin{aligned}\lambda^{\alpha p} D_{\alpha p} \chi_2 &= \varepsilon_{ijk\ell}(\bar{\theta}_m \bar{\theta}_n)\Big[(\lambda^i\lambda^j)(\lambda^m\theta^k)(\theta^\ell\theta^n) - (\lambda^i\theta^j)(\lambda^m\lambda^k)(\theta^\ell\theta^n) \\ &\qquad + (\lambda^i\theta^j)(\lambda^m\theta^k)(\lambda^\ell\theta^n) - (\lambda^i\theta^j)(\lambda^m\theta^k)(\theta^\ell\lambda^n)\Big] \\ &= 4\mathbf{T}_4 - \mathbf{T}_6\,, \qquad (3.5)\\ \bar{\lambda}_{\dot{\alpha}p}\bar{D}^{\dot{\alpha}p}\chi_2 &= -\varepsilon_{ijk\ell}(\lambda^i\theta^j)(\lambda^m\theta^k)(\theta^\ell\theta^n)(\bar{\theta}_m\bar{\lambda}_n) \\ &= \mathbf{T}_3\,. \qquad (3.6)\end{aligned}$$

Therefore,

$$\boxed{\mathbf{T}_3 + 4\mathbf{T}_4 - \mathbf{T}_6 = \lambda^{\hat{\alpha}} D_{\hat{\alpha}} \chi_2\,.} \qquad (3.7)$$

Since the third term in (3.1) is equal to $\chi_2^\dagger$,

$$\boxed{\mathbf{T}_3^\dagger + 4\mathbf{T}_4^\dagger - \mathbf{T}_6^\dagger = \lambda^{\hat{\alpha}} D_{\hat{\alpha}} \chi_3\,.} \qquad (3.8)$$

For the fourth term,

$$\begin{aligned}\lambda^{\alpha p} D_{\alpha p} \chi_4 &= -\varepsilon_{mnj\ell}(\bar{\lambda}_i \bar{\theta}_k)\Big[(\lambda^m\lambda^n)(\theta^i\theta^j)(\theta^k\theta^\ell) + (\lambda^m\theta^n)(\theta^i\lambda^j)(\theta^k\theta^\ell) \\ &\qquad - (\lambda^m\theta^n)(\theta^i\theta^j)(\lambda^k\theta^\ell) + (\lambda^m\theta^n)(\theta^i\theta^j)(\theta^k\lambda^\ell)\Big] \\ &= 4\mathbf{T}_2 - \mathbf{T}_3\,, \qquad (3.9)\\ \bar{\lambda}_{\dot{\alpha}p}\bar{D}^{\dot{\alpha}p}\chi_4 &= \varepsilon_{mnj\ell}(\bar{\lambda}_i\bar{\lambda}_k)(\lambda^m\theta^n)(\theta^i\theta^j)(\theta^k\theta^\ell) \\ &= \mathbf{T}_1\,. \qquad (3.10)\end{aligned}$$

Therefore,

$$\boxed{\mathbf{T}_1 + 4\mathbf{T}_2 - \mathbf{T}_3 = \lambda^{\hat{\alpha}} D_{\hat{\alpha}} \chi_4\,.} \qquad (3.11)$$

Finally, the last term in (3.1) is equal to $\chi_4^\dagger$, so

$$\boxed{\mathbf{T}_1^\dagger + 4\mathbf{T}_2^\dagger - \mathbf{T}_3^\dagger = \lambda^{\hat{\alpha}} D_{\hat{\alpha}} \chi_5\,.} \qquad (3.12)$$

4 The $\mathcal{N} = 4$, d = 4 measure factor in a simple form

We are now in position to show the $\mathcal{N} = 4$, d = 4 measure factor is unique up to BRST-trivial terms and an overall factor. Consider once again the most general real expression with three λ's and five θ's of (2.1). One has

$$(\lambda^3\theta^5) = c_1\mathbf{T}_1 + c_2\mathbf{T}_2 + c_3\mathbf{T}_3 + c_4\mathbf{T}_4 + c_5\mathbf{T}_5 + c_6\mathbf{T}_6 + \text{H.c.}\,. \qquad (4.1)$$

If this is BRST-invariant, then the constants satisfy the equations (2.22) and their complex conjugates. We are free to add BRST-trivial terms to the above expression. If we add

$$-c_1\left[\mathbf{T}_1 + 4\mathbf{T}_2 - \mathbf{T}_3\right] - \frac{1}{4}(c_4 + c_5)\left[\mathbf{T}_3 + 4\mathbf{T}_4 - \mathbf{T}_6\right] + \text{H.c.}$$

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to $(\lambda^3\theta^5)$, we get[1]

$$(\lambda^3\theta^5) = -c_5\mathbf{T}_4 + c_5\mathbf{T}_5 + \frac{1}{4}(c_5+\bar{c}_5)\mathbf{T}_6 + \text{H.c.}\,, \tag{4.2}$$

where we used (2.22).

Furthermore, if $c_5 = \alpha + \mathrm{i}\beta$, with $\alpha, \beta \in \mathbb{R}$, then we can add the BRST-trivial term

$$-\mathrm{i}\beta\left[\mathbf{T}_5 - \mathbf{T}_5^\dagger - \mathbf{T}_4 + \mathbf{T}_4^\dagger\right]$$

to $(\lambda^3\theta^5)$, thus obtaining

$$\boxed{(\lambda^3\theta^5) = -\alpha\left[\mathbf{T}_4 - \mathbf{T}_5 - \frac{1}{2}\mathbf{T}_6 + \text{H.c.}\right].} \tag{4.3}$$

This shows that the measure is unique up to BRST-trivial terms and an overall factor.

The measure can be even further simplified, provided that we relax its manifest reality condition. By adding the BRST-trivial terms

$$\alpha\left[\mathbf{T}_4^\dagger - \mathbf{T}_5^\dagger + \mathbf{T}_5 - \mathbf{T}_4\right] + \frac{1}{2}\alpha\left[\mathbf{T}_3 + 4\mathbf{T}_4 - \mathbf{T}_6\right], \tag{4.4}$$

we arrive at

$$(\lambda^3\theta^5) = \frac{1}{2}\alpha\left[\mathbf{T}_3 + 4\mathbf{T}_5 + \mathbf{T}_6^\dagger\right], \tag{4.5}$$

or, more explicitly,

$$\boxed{\begin{aligned}\Big\langle \varepsilon_{mnj\ell}(\bar{\lambda}_i\bar{\theta}_k)(\lambda^k\theta^\ell)(\lambda^m\theta^n)(\theta^i\theta^j) + 4\,(\bar{\lambda}_i\bar{\theta}_k)(\bar{\lambda}_j\bar{\theta}_\ell)(\lambda^k\theta^\ell)(\theta^i\theta^j)&\\ + \varepsilon^{mn\ell k}(\bar{\lambda}_i\bar{\theta}_k)(\bar{\lambda}_j\bar{\theta}_\ell)(\bar{\lambda}_m\bar{\theta}_n)(\theta^i\theta^j)\Big\rangle = \frac{1}{240}\,,&\end{aligned}} \tag{4.6}$$

where we recovered the explicit form of each term and used the normalization of (1.3).

This simple $\mathcal{N}=4$, $\mathrm{d}=4$ pure spinor measure factor is the main result of this work. Note that, while the dimensional reduction of (1.3) yields twelve independent terms, this expression has only three. In the next section, we give two explicit examples of how to obtain the dual to a vertex operator using this measure factor.

5 Examples

5.1 Gluino

Consider the following vertex operator from [2]:

$$\widetilde{V}_{\text{gluino}} = \frac{1}{4}(\lambda\gamma^\mu\theta)(\lambda\gamma^\nu\theta)(\theta\gamma_{\mu\nu}\xi^*)\,, \tag{5.1}$$

[1] Note the equal sign here means "equal up to BRST-trivial terms."

where $\xi^*_{\hat{\alpha}}$ is the zero-momentum gluino antifield.[2] It is easy to show this operator is annihilated by Q. The dimensional reduction yields

$$\begin{aligned}
\widetilde{V}_{\text{gluino}} = {} & 2\,(\bar{\lambda}_j\bar{\lambda}_k)(\theta^j\theta^\ell)(\theta^k\xi^*_\ell) + 3\,(\bar{\lambda}_i\bar{\theta}_j)(\theta^i\theta^k)(\bar{\lambda}_k\bar{\xi}^{*j}) - 2\,(\bar{\lambda}_j\bar{\theta}_i)(\theta^i\theta^k)(\bar{\lambda}_k\bar{\xi}^{*j}) \\
& - 6\,(\bar{\lambda}_i\bar{\theta}_j)(\lambda^j\theta^k)(\theta^i\xi^*_k) - \varepsilon_{ijk\ell}(\lambda^i\theta^j)(\lambda^n\theta^\ell)(\theta^k\xi^*_n) + 4\,(\bar{\lambda}_j\bar{\theta}_i)(\lambda^k\theta^i)(\theta^j\xi^*_k) \\
& - 2\,(\bar{\lambda}_i\bar{\theta}_j)(\lambda^k\theta^i)(\theta^j\xi^*_k) + (\bar{\lambda}_i\bar{\lambda}_j)(\theta^k\theta^i)(\bar{\theta}_k\bar{\xi}^{*j}) - 3\,\varepsilon_{ijk\ell}(\lambda^m\theta^i)(\lambda^k\theta^\ell)(\bar{\theta}_m\bar{\xi}^{*j}) \\
& + 2\,\varepsilon_{ijk\ell}(\lambda^i\theta^j)(\theta^\ell\theta^m)(\bar{\lambda}_m\bar{\xi}^{*k}) \\
& + \text{H.c.}\,.
\end{aligned} \tag{5.2}$$

We can simplify this expression by adding BRST-trivial terms. For example,

$$\begin{aligned}
Q\left[\varepsilon_{ijk\ell}(\lambda^i\theta^j)(\theta^n\theta^\ell)(\theta^k\xi^*_n)\right] &= -4\,(\bar{\lambda}_\ell\bar{\lambda}_k)(\theta^n\theta^\ell)(\theta^k\xi^*_n) - \varepsilon_{ijk\ell}\,(\lambda^i\theta^j)(\lambda^n\theta^\ell)(\theta^k\xi^*_n) \\
&= -4\mathbf{t}_1 - \mathbf{t}_5\,,
\end{aligned} \tag{5.3a}$$

where $\mathbf{t}_N$ refers to the N-th term in $\widetilde{V}_{\text{gluino}}$ as appearing in (5.2), without the numerical factor (e.g. $\mathbf{t}_1 \equiv (\bar{\lambda}_j\bar{\lambda}_k)(\theta^j\theta^\ell)(\theta^k\xi^*_\ell)$). So the combination $4\mathbf{t}_1 + \mathbf{t}_5$ is BRST-trivial. Of course, this implies $4\mathbf{t}_1^\dagger + \mathbf{t}_5^\dagger$ is also BRST-trivial.

One can also show

$$Q\left[(\bar{\lambda}_i\bar{\theta}_j)(\theta^j\theta^k)(\theta^i\xi^*_k)\right] = -\,\mathbf{t}_1 - \mathbf{t}_4 - \mathbf{t}_6\,, \tag{5.3b}$$

$$Q\left[(\bar{\theta}_i\bar{\theta}_j)(\lambda^j\theta^k)(\theta^i\xi^*_k)\right] = \mathbf{t}_4 + \mathbf{t}_2^\dagger - \mathbf{t}_8^\dagger\,, \tag{5.3c}$$

$$Q\left[(\bar{\theta}_i\bar{\theta}_j)(\lambda^k\theta^i)(\theta^j\xi^*_k)\right] = \mathbf{t}_6 + \mathbf{t}_7 + \mathbf{t}_3^\dagger + \mathbf{t}_8^\dagger\,, \tag{5.3d}$$

$$Q\left[\varepsilon_{ijk\ell}(\lambda^i\theta^j)(\theta^\ell\theta^m)(\bar{\theta}_m\bar{\xi}^{*k})\right] = -\,3\mathbf{t}_8 - \mathbf{t}_9 - \mathbf{t}_{10}\,, \tag{5.3e}$$

as well as their Hermitian conjugates. Then we can add the BRST-trivial amount

$$-\,2\mathbf{t}_1 - 3\mathbf{t}_2 + 2\mathbf{t}_3 - 9\mathbf{t}_4 + \mathbf{t}_5 - 4\mathbf{t}_6 + 2\mathbf{t}_7 - \mathbf{t}_8 - 2\mathbf{t}_9 - 2\mathbf{t}_{10} + \text{H.c.} \tag{5.4}$$

to $\widetilde{V}_{\text{gluino}}$ to obtain a BRST-equivalent vertex operator given by

$$\widetilde{V}'_{\text{gluino}} = -15\,(\bar{\lambda}_i\bar{\theta}_j)(\lambda^j\theta^k)(\theta^i\xi^*_k) - 5\,\varepsilon_{ijk\ell}(\lambda^m\theta^i)(\lambda^k\theta^\ell)(\bar{\theta}_m\bar{\xi}^{*j}) + \text{H.c.}\,. \tag{5.5}$$

Finally, multiplying by an overall factor and dropping the prime, we arrive at the simplest form

$$\boxed{\widetilde{V}_{\text{gluino}} = 3\,(\bar{\lambda}_i\bar{\theta}_j)(\lambda^j\theta^k)(\theta^i\xi^*_k) + \varepsilon_{ijk\ell}(\lambda^m\theta^i)(\lambda^k\theta^\ell)(\bar{\theta}_m\bar{\xi}^{*j}) + \text{H.c.}\,.} \tag{5.6}$$

Now we may look for the dual to this vertex operator, meaning the BRST-closed expression with one λ and two θ's whose product with $\widetilde{V}_{\text{gluino}}$ gives something proportional

[2] We choose to work at zero-momentum for the sake of simplicity, and also because only in this computation is the explicit form of the measure factor needed. Momentum corrections at higher θ-orders can in principle be obtained from the ten-dimensional expressions by carrying out the same steps as in this section (dimensionally reducing, simplifying by adding BRST-trivial terms and then looking for the dual to the vertex operator), but beyond zero-momentum one just has to impose that the terms with more than five θ's in the amplitude cancel.

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to the measure (4.6). The dual is certainly going to contain the gluino field $\xi^{\hat{\alpha}}$, and the substitution

$$\xi^*_{\hat{\beta}}\xi^{\hat{\alpha}} \longrightarrow \delta^{\hat{\alpha}}_{\hat{\beta}} \tag{5.7}$$

can be used to determine it.

Comparing (5.6) with (4.6), we expect the dual to contain a term with $\varepsilon_{ijk\ell}$. It is easy to see there is only one such term:

$$V^{[1]}_{\text{gluino}} = \kappa_1\, \varepsilon_{ijk\ell}(\lambda^i\theta^j)(\theta^k\xi^\ell)\,, \tag{5.8}$$

where κ_1 is a constant to be determined. Then, using (5.7), we obtain

$$\widetilde{V}_{\text{gluino}}\, V^{[1]}_{\text{gluino}} = \kappa_1\, [3\mathbf{T}_3 + 3\mathbf{T}_5]\,. \tag{5.9}$$

To obtain a term proportional to $\mathbf{T}_6^\dagger$, we need a term with $\bar{\lambda}$. There are not many, and it is not difficult to show the following one works:

$$V^{[2]}_{\text{gluino}} = \kappa_2\, (\bar{\lambda}_i\bar{\theta}_j)(\theta^i\xi^j)\,, \tag{5.10}$$

for a constant κ_2 to be determined shortly. Making use of (5.7) once again, we get

$$\widetilde{V}_{\text{gluino}}\, V^{[2]}_{\text{gluino}} = -\kappa_2\, \left[3\mathbf{T}_5 + \mathbf{T}_6^\dagger\right]. \tag{5.11}$$

Hence, if $\kappa_1 = \frac{1}{3}$ and $\kappa_2 = -1$, we have

$$\widetilde{V}_{\text{gluino}}\, V_{\text{gluino}} := \widetilde{V}_{\text{gluino}}(V^{[1]}_{\text{gluino}} + V^{[2]}_{\text{gluino}}) = \mathbf{T}_3 + 4\mathbf{T}_5 + \mathbf{T}_6^\dagger\,, \tag{5.12}$$

which means the dual to $\widetilde{V}_{\text{gluino}}$ is given by

$$\boxed{V_{\text{gluino}} = \frac{1}{3}\varepsilon_{ijk\ell}(\lambda^i\theta^j)(\theta^k\xi^\ell) - (\bar{\lambda}_i\bar{\theta}_j)(\theta^i\xi^j)\,.} \tag{5.13}$$

One can show this expression is BRST-invariant, as it should be.

5.2 Gluon

We now derive, for the gluon, expressions analogous to those obtained for the gluino in the previous subsection. The vertex operator for the zero-momentum gluon antifiled a^*_b ($b = 0$ to 3) can also be found in [2], and is given by

$$\widetilde{V}_{\text{gluon}} = \frac{1}{4}(\lambda\gamma^\mu\theta)(\lambda\gamma^\nu\theta)(\theta\gamma_{\mu\nu}{}^b\theta)a^*_b\,. \tag{5.14}$$

Again, it is easy to show this operator is annihilated by Q. The dimensional reduction can be easily obtained from the gluino case, by noting that $\xi^* \longmapsto (\gamma^b\theta)a^*_b \implies \widetilde{V}_{\text{gluino}} \longmapsto \widetilde{V}_{\text{gluon}}$. One gets

$$\begin{aligned}
\widetilde{V}_{\text{gluon}} = {} & 3\,(\bar{\lambda}_j\bar{\lambda}_k)(\theta^j\theta^\ell)(\theta^k a^*\bar{\theta}_\ell) - 3\,(\bar{\lambda}_i\bar{\theta}_j)(\theta^i\theta^k)(\theta^j a^*\bar{\lambda}_k) + 2\,(\bar{\lambda}_j\bar{\theta}_i)(\theta^i\theta^k)(\theta^j a^*\bar{\lambda}_k) \\
& - 8\,(\bar{\lambda}_i\bar{\theta}_j)(\lambda^j\theta^k)(\theta^i a^*\bar{\theta}_k) - 4\,\varepsilon_{ijk\ell}(\lambda^i\theta^j)(\lambda^n\theta^\ell)(\theta^k a^*\bar{\theta}_n) + 4\,(\bar{\lambda}_j\bar{\theta}_i)(\lambda^k\theta^i)(\theta^j a^*\bar{\theta}_k) \\
& - 2\,\varepsilon_{ijk\ell}(\lambda^i\theta^j)(\theta^\ell\theta^m)(\theta^k a^*\bar{\lambda}_m) \\
& + \text{H.c.}\,,
\end{aligned} \tag{5.15}$$

where $(\theta^i a^*\bar{\lambda}_j) = \theta^{\alpha i}a^*_{\alpha\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}}_j$, with $a^*_{\alpha\dot{\alpha}} = (\sigma^b)_{\alpha\dot{\alpha}}a^*_b$.

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To simplify this expression, one can add combinations of the following BRST-trivial terms:

$$Q\left[\varepsilon_{ijk\ell}(\lambda^i\theta^j)(\theta^n\theta^\ell)(\theta^k a^*\bar{\theta}_n)\right] = -4\mathbf{u}_1 - \mathbf{u}_5 + \mathbf{u}_7\,, \tag{5.16a}$$

$$Q\left[(\bar{\lambda}_i\bar{\theta}_j)(\theta^j\theta^k)(\theta^i a^*\bar{\theta}_k)\right] = -\mathbf{u}_1 + \mathbf{u}_3 - \mathbf{u}_4 - \mathbf{u}_6\,, \tag{5.16b}$$

$$Q\left[(\theta^j\theta^k)(\bar{\theta}_k\bar{\theta}_j)(\theta^i a^*\bar{\lambda}_i)\right] = -2\,(\mathbf{u}_2 + \mathbf{u}_3 + \mathbf{u}_4 + \mathbf{u}_6)\,, \tag{5.16c}$$

as well as their Hermitian conjugates, where $\mathbf{u}_N$ refers to the N-th term in $\widetilde{V}_{\text{gluon}}$ as appearing in (5.15), without the numerical factor (e.g. $\mathbf{u}_1 \equiv (\bar{\lambda}_j\bar{\lambda}_k)(\theta^j\theta^\ell)(\theta^k a^*\bar{\theta}_\ell)$). Then one can show the following vertex operator is equivalent to $\widetilde{V}_{\text{gluon}}$:

$$\widetilde{V}'_{\text{gluon}} = -6\mathbf{u}_5 + 12\mathbf{u}_6 + \text{H.c.}\,. \tag{5.17}$$

Finally, multiplying by an overall factor and dropping the prime, we arrive at the simplest form

$$\boxed{\widetilde{V}_{\text{gluon}} = \varepsilon_{ijk\ell}(\lambda^i\theta^j)(\lambda^n\theta^\ell)(\theta^k a^*\bar{\theta}_n) - 2\,(\bar{\lambda}_j\bar{\theta}_i)(\lambda^k\theta^i)(\theta^j a^*\bar{\theta}_k) + \text{H.c.}\,.} \tag{5.18}$$

The dual to this vertex operator is a BRST-closed expression with one $\bar{\lambda}$ and one θ whose product with $\widetilde{V}_{\text{gluon}}$ gives something proportional to the measure (4.6). This expression should also contain the gluon field a_b. It is easy to see there is only one possibility:

$$\boxed{V_{\text{gluon}} = (\theta^i a\,\bar{\lambda}_i)\,.} \tag{5.19}$$

Then, using $a_b^* a_c \longrightarrow \eta_{bc}$, it is not difficult to show $\widetilde{V}_{\text{gluon}}V_{\text{gluon}}$ is indeed proportional to (4.6).

6 Conclusion

In this work, we have obtained a simple measure factor for the λ and θ zero-mode integrations in the pure-spinor formalism in the context of an $\mathcal{N} = 4$, d $= 4$ theory. We have shown that the measure can be defined unambiguously up to BRST-trivial terms and an overall factor, and is much simpler than (although equivalent to) the expression obtained by dimensional reduction from the $\mathcal{N} = 1$, d $= 10$ measure factor. We have also given two explicit examples of how to obtain the dual to a vertex operator using this measure.

We expect these results to be useful for the computation of disk scattering amplitudes of states propagating in the world-volume of a D3-brane, as well as open-closed superstring amplitudes of states close to the AdS_5 boundary [6]. It would also be interesting to make contact with recent results obtained for ten-dimensional superstring amplitudes, such as those in [7–9].

Acknowledgments

I would like to thank Thiago Fleury, Renann Jusinskas and my advisor Nathan Berkovits for useful discussions and suggestions. This work was supported by FAPESP grants 2010/19596-2 and 11/11973-4.

A Notation and conventions

A.1 Two-component spinor notation

The four-dimensional Lorentz group SO(3, 1) is locally isomorphic to $\mathrm{SL}(2,\mathbb{C})$, which has two distinct fundamental representations. One of them is described by a pair of complex numbers [3]

$$\psi_\alpha = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} , \tag{A.1}$$

with transformation law

$$\psi'_\alpha = \Lambda_\alpha^{\ \beta}\psi_\beta \,, \qquad \Lambda \in \mathrm{SL}(2,\mathbb{C}) \,, \tag{A.2}$$

and is called $\left(\frac{1}{2},0\right)$ or left-handed chiral representation.

The other fundamental representation, called $\left(0,\frac{1}{2}\right)$ or right-handed chiral, is obtained by complex conjugation:

$$\bar\psi'_{\dot\alpha} = \bar\Lambda_{\dot\alpha}^{\ \dot\beta}\bar\psi_{\dot\beta} \,, \qquad \bar\Lambda_{\dot\alpha}^{\ \dot\beta} = \overline{(\Lambda_\alpha^{\ \beta})} \,. \tag{A.3}$$

The dot over the indices indicates the representation to which we refer.

The indices with and without dot are raised and lowered in the following way:

$$\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta \,, \qquad \bar\chi^{\dot\alpha} = \varepsilon^{\dot\alpha\dot\beta}\bar\chi_{\dot\beta} \,; \tag{A.4a}$$

$$\psi_\alpha = \varepsilon_{\alpha\beta}\psi^\beta \,, \qquad \bar\chi_{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar\chi^{\dot\beta} \,, \tag{A.4b}$$

where ε is antisymmetric and has the properties

$$\varepsilon^{12} = \varepsilon^{\dot 1\dot 2} = -\varepsilon_{12} = -\varepsilon_{\dot 1\dot 2} = 1 \quad \Longrightarrow \quad \varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha^\gamma \,, \qquad \varepsilon_{\dot\alpha\dot\beta}\varepsilon^{\dot\beta\dot\gamma} = \delta_{\dot\alpha}^{\dot\gamma} \,. \tag{A.5}$$

For spinorial derivatives, raising or lowering the indices involve an extra sign. For example, $D_i^\alpha = -\varepsilon^{\alpha\beta}D_{\beta i}$.

The convention for contraction of spinorial indices is

$$\psi^\alpha\lambda_\alpha =: (\psi\lambda) \,, \qquad \bar\chi_{\dot\alpha}\bar\xi^{\dot\alpha} =: (\bar\chi\bar\xi) \,. \tag{A.6}$$

In $\mathrm{SL}(2,\mathbb{C})$ notation, a four-component Dirac spinor is represented by a pair of chiral spinors:

$$\Psi_{\mathrm{D}} = \begin{pmatrix} \psi_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix} . \tag{A.7}$$

For a Majorana spinor, $\bar\chi_{\dot\alpha} = \overline{(\psi_\alpha)}$. The Dirac matrices are

$$\Sigma^a = \begin{pmatrix} 0 & (\sigma^a)_{\alpha\dot\alpha} \\ (\tilde\sigma^a)^{\dot\alpha\alpha} & 0 \end{pmatrix} , \tag{A.8}$$

where the matrices σ^a $(a = 0, \ldots, 3)$ are defined as

$$(\sigma^a)_{\alpha\dot\alpha} = (-\mathbb{I}_2, \vec\sigma)_{\alpha\dot\alpha} \,, \qquad (\tilde\sigma^a)^{\dot\alpha\alpha} = \varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}(\sigma^a)_{\beta\dot\beta} = (-\mathbb{I}_2, -\vec\sigma)^{\dot\alpha\alpha} \,, \tag{A.9}$$

with $\mathbb{I}_2$ the 2×2 identity matrix and $\vec{\sigma}$ the Pauli matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.10}$$

and have the following properties:

$$\begin{aligned} (\sigma^a)_{\alpha\dot\alpha}(\tilde\sigma_a)^{\dot\beta\beta} &= -2\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta}, & (\sigma_a)_{\alpha\dot\alpha}(\tilde\sigma^b)^{\dot\alpha\alpha} &= -2\delta_a^b, \\ \sigma^a\tilde\sigma^b &= -\eta^{ab} + \sigma^{ab}, & \tilde\sigma^a\sigma^b &= -\eta^{ab} + \tilde\sigma^{ab}, \\ \sigma^{ab} &= -\sigma^{ba}, & \tilde\sigma^{ab} &= -\tilde\sigma^{ba}, \qquad (\sigma^{ab})_\alpha{}^\alpha = (\tilde\sigma^{ab})^{\dot\alpha}{}_{\dot\alpha} = 0, \end{aligned} \tag{A.11}$$

with $\eta^{ab} = \mathrm{diag}(-1,1,1,1)$. These properties imply $\{\Sigma^a,\Sigma^b\} = -2\eta^{ab}\,\mathbb{I}_4$.

A.2 Dimensional reduction

Since in the text we write expressions both in ten- and four-dimensional notation, it is important to clarify our notation and conventions. Breaking the SO(9,1) Lorentz symmetry to $\mathrm{SO}(3,1)\times\mathrm{SO}(6)\simeq\mathrm{SO}(3,1)\times\mathrm{SU}(4)$, an SO(9,1) vector v^μ ($\mu=0,\dots,9$) decomposes as

$$v^\mu \longmapsto (v^a, v^{[ij]}), \tag{A.12}$$

where v^a ($a=0,\dots,3$) transforms under the representation **4** of SO(3,1) and $v^{[ij]} = -v^{[ji]}$ ($i,j=1,\dots,4$) transforms under the **6** of SU(4). The relation between the **6** of SU(4) and the **6** of SO(6) is given by the SO(6) Pauli matrices $(\rho_I)^{ij} = -(\rho_I)^{ji}$ ($I=1,\dots,6$) in the following way:

$$v^{[ij]} = \frac{1}{2\mathrm{i}}(\rho_I)^{ij}v^{I+3}. \tag{A.13}$$

These matrices have the properties [4]

$$\begin{aligned} (\rho^I)^{ij}(\rho^J)_{jk} + (\rho^J)^{ij}(\rho^I)_{jk} &= 2\eta^{IJ}\delta_k^i, \\ (\rho^I)_{ij} &= \frac{1}{2}\varepsilon_{ijk\ell}(\rho^I)^{k\ell}, \\ (\rho^I)_{ij}(\rho_I)_{k\ell} &= -2\varepsilon_{ijk\ell}, \end{aligned} \tag{A.14}$$

where $\eta^{IJ} = \mathrm{diag}(1,1,1,1,1,1)$ and $\varepsilon_{ijk\ell}$ is the SU(4)-invariant, totally antisymmetric tensor such that $\varepsilon_{1234}=1$. Analogously, one can define the tensor $\varepsilon^{ijk\ell}$ such that $\varepsilon^{1234}=1$. These satisfy the relation

$$\varepsilon_{ijk\ell}\varepsilon^{k\ell mn} = 4\delta_{[i}^m\delta_{j]}^n. \tag{A.15}$$

A left-handed Majorana-Weyl spinor $\xi^{\hat\alpha}$ ($\hat\alpha = 1,\dots,16$) transforming under the **16** of SO(9,1) decomposes as

$$\xi^{\hat\alpha} \longmapsto (\xi^{\alpha i}, \bar\xi_j^{\dot\alpha}), \tag{A.16}$$

where we use the standard two-component notation for chiral spinors ($\alpha=1,2\,;\dot\alpha=\dot1,\dot2$) and $\xi^{\alpha i}$ (resp. $\bar\xi_j^{\dot\alpha}$) transforms under the representation **4** (resp. $\bar{\mathbf{4}}$) of SU(4). Analogous conventions apply to right-handed Majorana-Weyl spinors of SO(9,1).

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We also need to know how to translate the SO(9,1) Pauli matrices $(\gamma^\mu)_{\hat{\alpha}\hat{\beta}}$ and $(\gamma^\mu)^{\hat{\alpha}\hat{\beta}}$ to the language of $\mathrm{SO}(3,1)\times\mathrm{SU}(4)$. Based on [5], we propose the following *ansatz* for the non-vanishing components:

$$\begin{aligned}(\gamma^a)_{(\alpha i)\binom{\dot\alpha}{j}} &= \delta_i^j(\sigma^a)_{\alpha\dot\alpha} = (\gamma^a)_{\binom{\dot\alpha}{j}(\alpha i)}\\ (\gamma^{[k\ell]})_{(\alpha i)(\beta j)} &= 2\varepsilon_{\alpha\beta}\delta^k_{[i}\delta^\ell_{j]}\\ (\gamma^{[k\ell]})_{\binom{\dot\alpha}{i}\binom{\dot\beta}{j}} &= \varepsilon_{\dot\alpha\dot\beta}\varepsilon^{ijk\ell}\end{aligned} \tag{A.17}$$

for $(\gamma^\mu)_{\hat{\alpha}\hat{\beta}}$ and

$$\begin{aligned}(\gamma^a)^{(\alpha i)\binom{\dot\alpha}{j}} &= \delta^i_j(\tilde\sigma^a)^{\dot\alpha\alpha} = (\gamma^a)^{\binom{\dot\alpha}{j}(\alpha i)}\\ (\gamma^{[k\ell]})^{(\alpha i)(\beta j)} &= \varepsilon^{\alpha\beta}\varepsilon^{ijk\ell}\\ (\gamma^{[k\ell]})^{\binom{\dot\alpha}{i}\binom{\dot\beta}{j}} &= 2\varepsilon^{\dot\alpha\dot\beta}\delta^k_{[i}\delta^\ell_{j]}\end{aligned} \tag{A.18}$$

for $(\gamma^\mu)^{\hat{\alpha}\hat{\beta}}$. It is straightforward to show that the above matrices satisfy the usual relation

$$(\gamma^\mu)_{\hat\alpha\hat\beta}(\gamma^\nu)^{\hat\beta\hat\gamma} + (\gamma^\nu)_{\hat\alpha\hat\beta}(\gamma^\mu)^{\hat\beta\hat\gamma} = -2\eta^{\mu\nu}\delta^{\hat\gamma}_{\hat\alpha}\,, \tag{A.19}$$

with $\eta^{[ij][k\ell]} := \frac{1}{2}\varepsilon^{ijk\ell}$.

As an example, we show how to obtain the dimensional reduction of the pure spinor constraints $\lambda\gamma^\mu\lambda = 0$ using (A.17). For $\lambda\gamma^a\lambda = 0$, we have

$$\lambda^{\hat\alpha}(\gamma^a)_{\hat\alpha\hat\beta}\lambda^{\hat\beta} = 0 \iff \lambda^{\alpha i}(\gamma^a)_{(\alpha i)\binom{\dot\alpha}{j}}\bar\lambda^{\dot\alpha}_j + \bar\lambda^{\dot\alpha}_j(\gamma^a)_{\binom{\dot\alpha}{j}(\alpha i)}\lambda^{\alpha i} = 2\lambda^{\alpha i}(\sigma^a)_{\alpha\dot\alpha}\bar\lambda^{\dot\alpha}_i = 0\,,$$

whence

$$\lambda^{\alpha i}\bar\lambda^{\dot\alpha}_i = 0\,. \tag{A.20}$$

For $\lambda\gamma^{[ij]}\lambda = 0$, we have

$$\lambda^{\hat\alpha}(\gamma^{[ij]})_{\hat\alpha\hat\beta}\lambda^{\hat\beta} = 0 \iff \lambda^{\alpha k}(\gamma^{[ij]})_{(\alpha k)(\beta\ell)}\lambda^{\beta\ell} + \bar\lambda^{\dot\alpha}_k(\gamma^{[ij]})_{\binom{\dot\alpha}{k}\binom{\dot\beta}{\ell}}\bar\lambda^{\dot\beta}_\ell = 2(\lambda^i\lambda^j) - \varepsilon^{ijk\ell}(\bar\lambda_k\bar\lambda_\ell) = 0\,,$$

whence

$$(\lambda^i\lambda^j) = \frac{1}{2}\varepsilon^{ijk\ell}(\bar\lambda_k\bar\lambda_\ell)\,. \tag{A.21}$$

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