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Fractional instantons and bions in the $\mathrm{O}(N)$ model with twisted boundary conditions

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Abstract: Recently, multiple fractional instanton configurations with zero instanton charge, called bions, have been revealed to play important roles in quantum field theories on compactified spacetime. In two dimensions, fractional instantons and bions have been extensively studied in the $\mathbb{C}P^{N-1}$ model and the Grassmann sigma model on $\mathbb{R}^1 \times S^1$ with the $\mathbb{Z}_N$ symmetric twisted boundary condition. Fractional instantons in these models are domain walls with a localized U(1) modulus twisted half along their world volume. In this paper, we classify fractional instantons and bions in the $\mathrm{O}(N)$ nonlinear sigma model on $\mathbb{R}^{N-2} \times S^1$ with more general twisted boundary conditions in which arbitrary number of fields change sign. We find that fractional instantons have more general composite structures, that is, a global vortex with an Ising spin (or a half-lump vortex), a half sine-Gordon kink on a domain wall, or a half lump on a "space-filling brane" in the O(3) model ($\mathbb{C}P^1$ model) on $\mathbb{R}^1 \times S^1$, and a global monopole with an Ising spin (or a half-Skyrmion monopole), a half sine-Gordon kink on a global vortex, a half lump on a domain wall, or a half Skyrmion on a "space-filling brane" in the O(4) model (principal chiral model or Skyrme model) on $\mathbb{R}^2 \times S^1$. We also construct bion configurations in these models.

Keywords: Solitons Monopoles and Instantons, Sigma Models

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Contents

1 Introduction

Instantons have been known for long time to play significant roles in non-perturbative dynamics of quantum field theories such as supersymmetric QCD. Recently, multiple fractional instanton configurations with zero instanton charge, called bions, have been revealed to play important roles in quantum field theories on compactified spacetime [1–22]. The prime example which has been studied extensively is QCD with adjoint fermions (adj.) on $\mathbb{R}^3 \times S^1$. Bions can be classified into two classes, magnetic (charged) bions carrying a magnetic charge, and neutral bions carrying no magnetic charge. Magnetic bions are conjectured to lead semiclassical confinement in QCD (adj.) on $\mathbb{R}^3 \times S^1$ [23–33]. On the other hand, neutral bions are identified as the infrared renormalons in field theory [6–15, 34–36], and play an essential role in unambiguous and self-consistent semiclassical definition of quantum field theories in a process known as the resurgence; imaginary ambiguities called renormalon ambiguities arising in non-Borel-summable perturbative series exactly cancel out with those arising in neutral bion's amplitude in the small compactification-scale regime of QCD (adj.) on $\mathbb{R}^3 \times S^1$. It indicates that the full semi-classical expansion, referred as a resurgent expansion [37], that includes both perturbative and non-perturbative sectors, leads to unambiguous and self-consistent definition of field theories. In quantum mechanics this is known as the Bogomol'nyi-Zinn-Justin prescription [38–40].

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On the other hand, two dimensional nonlinear sigma model enjoys a lot of common features with four-dimensional Yang-Mills theory [41] such as asymptotic freedom, dynamical mass generation, and instantons [42–45]. We can further expect a similar correspondence between fractional instantons and bions in nonlinear sigma models on $\mathbb{R}^1 \times S^1$ and those in Yang-Mills theory on $\mathbb{R}^3 \times S^1$. Fractional instantons in the $\mathbb{C}P^{N-1}$ model [46] (see also refs. [47–50]) and the Grassmann sigma model [51, 52] were constructed on $\mathbb{R}^1 \times S^1$ with twisted boundary conditions by using the moduli matrix technique [53–59] (see ref. [60] as a review) and D-brane configurations [51, 52, 61]. Bions and the resurgence have been extensively studied in the $\mathbb{C}P^{N-1}$ model [8–10, 16–18, 21] and the Grassmann sigma model [19] on $\mathbb{R}^1 \times S^1$. In particular in refs. [8, 9], bion configurations in the $\mathbb{C}P^{N-1}$ model were studied based on the dilute instanton description with taking account of interactions between well-separated fractional instantons and anti-instantons, to show explicitly that the imaginary ambiguity in the amplitude of neutral bions has the same magnitude with an opposite sign as the leading ambiguity arising from the non-Borel-summable series expanded around the perturbative vacuum. The ambiguities at higher orders are canceled by amplitudes of bion molecules and the full trans-series expansion around the perturbative and non-perturbative vacua results in unambiguous semiclassical definition of field theories. Furthermore, neutral bion ansatz beyond exact solutions were found in the $\mathbb{C}P^{N-1}$ model [17, 18] and the Grassmann model [19] in terms of the moduli matrix and was found to be consistent with the results from the well-separated instanton gas calculus [8, 9] from all ranges of separations. Bions and resurgence were also studied for principal chiral models [12, 15] and quantum mechanics [11, 13, 14].

In order to understand more precise structures of fractional instantons and bions in generic field theories, it is worth to remind that fractional instantons in the $\mathbb{C}P^{N-1}$ and Grassmann models on $\mathbb{R}^1 \times S^1$ with the $\mathbb{Z}_N$ twisted boundary conditions have a composite soliton structure [46, 51, 52]. When the coordinate x^2 is a compact direction, fractional instantons are domain walls extending to the x^2 direction (perpendicular to the x^1 direction) whose world volume a U(1) modulus is localized on and twisted half along. Fractional instantons can be therefore regarded as half sine-Gordon kinks on a domain wall. Since a domain wall carries unit instanton (lump) charge when the U(1) modulus is twisted once (full sine-Gordon kink) [62–65], the above configuration carries half instanton charge [66, 67]. In this paper, we refer the above domain wall and sine-Gordon kink as a host soliton and daughter soliton, respectively. The simplest among $\mathbb{C}P^{N-1}$ model and the Grassmann model is the $\mathbb{C}P^1$ model, which is equivalent to the O(3) sigma model described by a unit three-vector of scalar fields $\mathbf{n} = \{n_A(x)\}$ $(A = 1, 2, 3)$ with $\mathbf{n}^2 = 1$. The $\mathbb{Z}_2$ symmetric boundary condition reduces to $(n_1, n_2, n_3)(x + R) = (-n_1, -n_2, +n_3)(x)$ in this notation.

In this paper, we classify fractional instantons and bions in the $O(N)$ nonlinear sigma model on $\mathbb{R}^{N-2} \times S^1$ with the twisted boundary conditions in which arbitrary number of fields change signs:

$$(\underbrace{-,-,\cdots}_{s},\underbrace{+,+,\cdots}_{N-s}): \quad (n_1,\cdots,n_N)(x+R) = (-n_1,\cdots,-n_s,+n_{s+1},\cdots,+n_N)(x), \tag{1.1}$$

bulk space	boundary condition	fixed manifold $\mathcal{N}$	host soliton $\pi_n(\mathcal{N})$, codim	modui $\mathcal{M}$ of host soliton	daughter soliton $\pi_m(\mathcal{M})$, codim
$\mathbb{R}^1 \times S^1$	$(-,+,+)$	S^1 $n_2^2+n_3^2=1$	vortex π_1, 2	2 points $n_1=\pm 1$	Ising spin π_0, 0
$\mathbb{R}^1 \times S^1$	$(-,-,+)$	2 points $n_3=\pm 1$	domain wall π_0, 1	S^1 $n_1^2+n_2^2=1$	SG kink π_1, 1
$\mathbb{R}^1 \times S^1$	$(-,-,-)$	non $\{0\}$	space-filling "π_{-1}", 0	S^2 $n_1^2+n_2^2+n_3^2=1$	lump π_2, 2
$\mathbb{R}^2 \times S^1$	$(-,+,+,+)$	S^2 $n_2^2+n_3^2+n_4^2=1$	monopole π_2, 3	2 points $n_1=\pm 1$	Ising spin π_0, 0
$\mathbb{R}^2 \times S^1$	$(-,-,+,+)$	S^1 $n_3^2+n_4^2=1$	vortex π_1, 2	S^1 $n_1^2+n_2^2=1$	SG kink π_1, 1
$\mathbb{R}^2 \times S^1$	$(-,-,-,+)$	2 points $n_4=\pm 1$	domain wall π_0, 1	S^2 $n_1^2+n_2^2+n_3^2=1$	lump π_2, 2
$\mathbb{R}^2 \times S^1$	$(-,-,-,-)$	non $\{0\}$	space-filling "π_{-1}", 0	S^3 $n_1^2+n_2^2+n_3^2+n_4^2=1$	Skyrmion π_3, 3

Table 1. Fractional instantons in the O(3) model on $\mathbb{R}^1 \times S^1$ and the O(4) model on $\mathbb{R}^2 \times S^1$ with twisted boundary conditions. SG denotes sine-Gordon. Host solitons are classified by $\pi_n(\mathcal{N})$, where $\mathcal{N}$ is a fixed manifold. Daughter solitons are classified by $\pi_m(\mathcal{M})$, where M is a moduli space of a host soliton. Daughter solitons are all half quantized carrying a half topological charge. There are the relations among the dimensionality of the homotopy groups, $n+m+1=2$ for the O(3) model and $n+m+1=3$ for the O(4) model. Equivalently, the sum of codimensions of a host soliton and of a daughter soliton is 2 and 3 for the O(3) model on $\mathbb{R}^1 \times S^1$ and the O(4) model on $\mathbb{R}^2 \times S^1$, respectively.

where we have labeled the boundary condition by a set of N signs as $(\underbrace{-,-,\cdots}_{s},\underbrace{+,+,\cdots}_{N-s})$. The O(3) model is equivalent to the $\mathbb{C}P^1$ model for which the $\mathbb{Z}_2$ twisted boundary condition studied before [8–10, 16–18] corresponds to $(-,-,+)$, while the cases of $(-,+,+)$ and $(-,-,-)$ have not been studied before. The O(4) model is equivalent to a principal chiral model with a group SU(2) or a Skyrme model if four derivative (Skyrme) term is added [68, 69], in which fractional instantons or bions were not studied before. We find that general boundary conditions (1.1) induce fractional instantons as various types of composite solitons. Our results are summarized in table 1 and figures 1 and 2. Throughout the paper, red (black) arrows denote fields which are (not) twisted by the twisted boundary condition in eq. (1.1). Depending on the boundary conditions, a fractional instanton in the O(3) model is found to be a global vortex with an Ising spin (or a half-lump vortex) for the boundary condition $(-,+,+)$, a half sine-Gordon kink on a domain wall for $(-,-,+)$, or a half lump on a "space-filling brane" for $(-,-,-)$. The second case was studied before. In the third case we formally consider a space-filling brane for the situation that there is no localized host soliton. A fractional instanton in the O(4) model is found to be a

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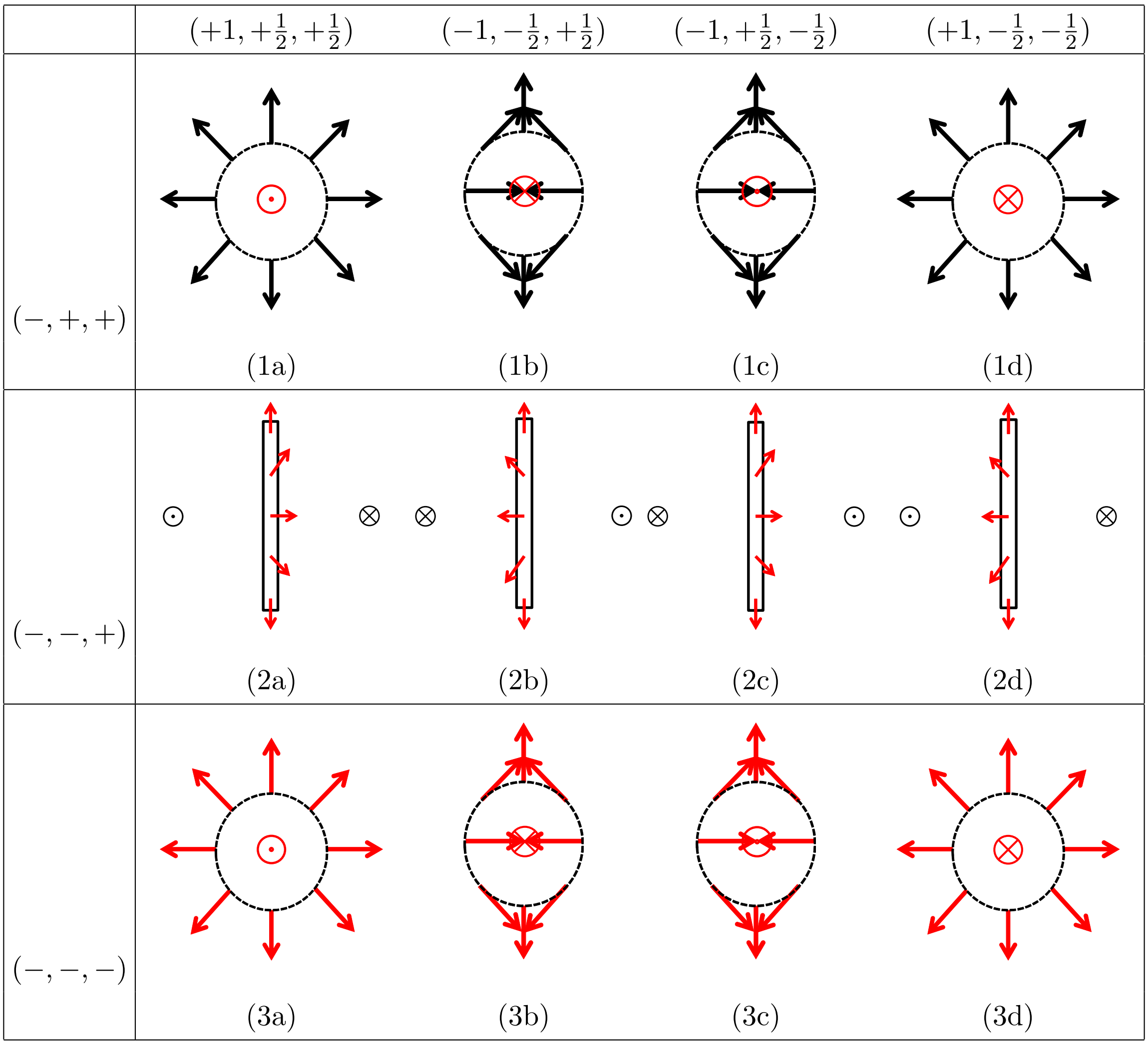

Figure 1. Fractional instantons in the O(3) model with the twisted boundary conditions (1) $(-,+,+)$, (2) $(-,-,+)$ and (3) $(-,-,-)$. Black and red arrows denote the moduli space $\mathcal{N}$ of vacua and the moduli space $\mathcal{M}$ of a host soliton, respectively, as we explain in more detail in later sections. The first lines indicate the topological charges (homotopy groups) characterizing (a host soliton, a daughter soliton, the total instanton charge) are (π_1, π_0, π_2) for (1a)–(1d), (π_0, π_1, π_2) for (2a)–(2d), and (π_{-1}, π_2, π_2) for (3a)–(3d), where π_{-1} is merely formal. For each boundary condition, fractional (anti-)instantons can make following composite structures: (a)+(b) instanton, (c)+(d) anti-instanton, (a)+(c), (b)+(d) bions.

global monopole with an Ising spin (or a half-Skyrmion monopole) for $(-,+,+,+)$, a half sine-Gordon kink on a global vortex for $(-,-,+,+)$, a half lump on a domain wall for $(-,-,-,+)$, or a half Skyrmion on a "space-filling brane" for $(-,-,-,-)$.

By using fractional instantons, we can construct neutral bions in the $O(N)$ model. On the other hand, charged bions are not possible in the $O(N)$ model. We note that constituent fractional instantons of bions in a principal chiral model in refs. [12, 15] are not topological because they considered a space $\mathbb{R}^1 \times S^1$, while our case on $\mathbb{R}^2 \times S^1$ is topological. When frac-

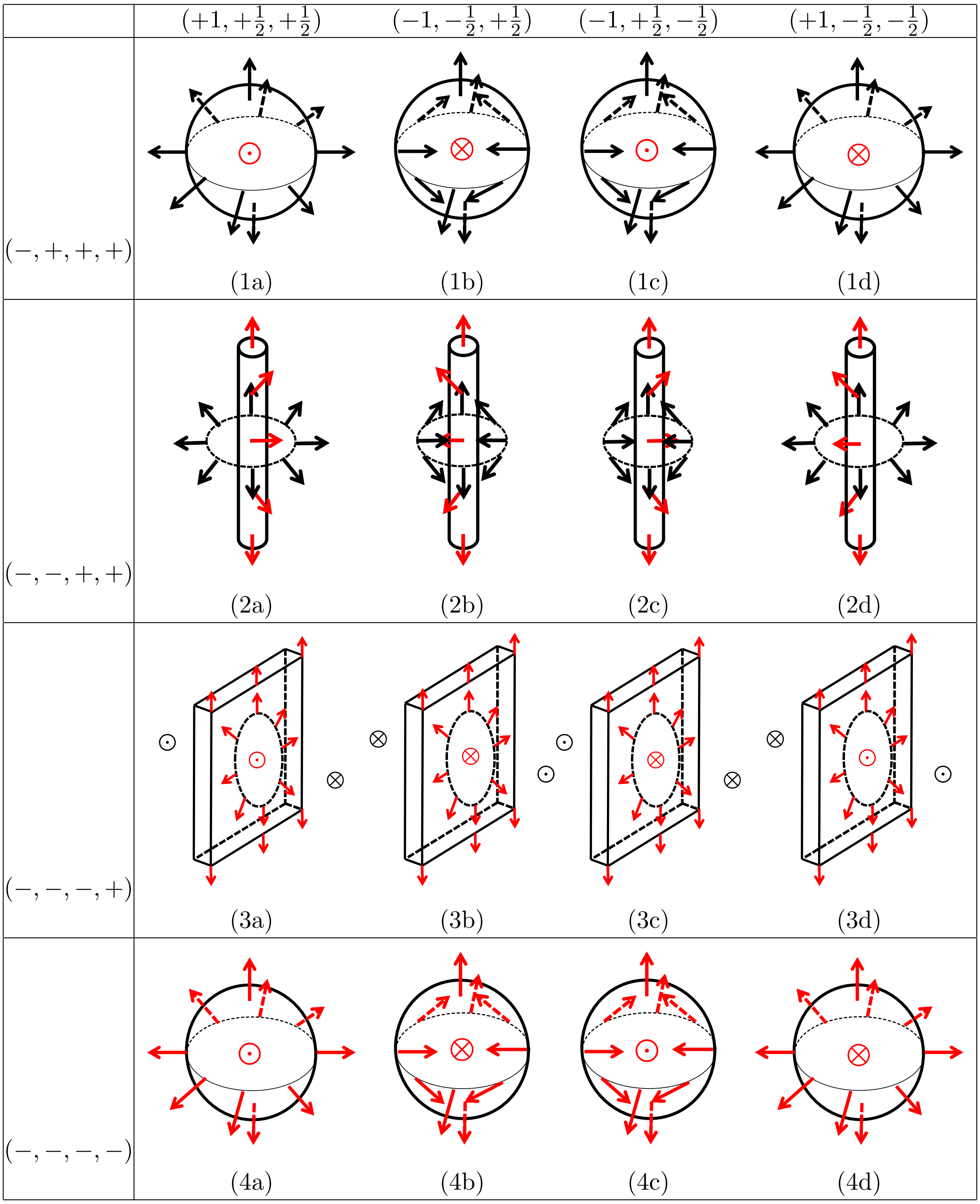

Figure 2. Fractional instantons in the O(4) model with the twisted boundary conditions (1) $(-,+,+,+)$, (2) $(-,-,+,+)$, (3) $(-,-,-,+)$ and (4) $(-,-,-,-)$. The notations of black and red arrows are the same with figure 1. The first lines indicate the topological charges (homotopy groups) characterizing (a host soliton, a daughter soliton, the total instanton charge) are (π_2, π_0, π_3) for (1a)–(1d), (π_1, π_1, π_3) for (2a)–(2d), (π_0, π_2, π_3) for (3a)–(3d), and (π_{-1}, π_3, π_3) for (4a)–(4d), where π_{-1} is merely formal. For each boundary condition, fractional (anti-)instantons can make following composite structures: (a)+(b) instanton, (c)+(d) anti-instanton, (a)+(c), (b)+(d) bions.

tional (anti-)instantons are (anti-) Bogomol'nyi-Prasad-Sommerfield (BPS) [71, 72] or local solitons, the interaction between two of them does not exist or is suppressed exponentially e^{-mr} with the distance r between them, respectively. In either case, the interaction between fractional instanton and anti-instanton is exponentially suppressed, and consequently neutral bions will play a role in resurgence because the energy (action value) of bions is the sum of individual fractional (anti-)instantons when they are well separated. Most of fractional instantons are not BPS except for those of the boundary condition $(-,-,+)$ in the O(3) model, which is the case studied before. We will summarize some modifications which may turn fractional instantons to be local or BPS so that they may play a role in resurgence.

This paper is organized as follows. In section 2, we first give the O(N) model. In section 3, we provide a general framework to construct fractional instantons as composite solitons in the O(N) model with the twisted boundary conditions. In sections 4 and 5, we discuss fractional instantons and bions in the O(3) model on $\mathbb{R}^1 \times S^1$ and the O(4) model on $\mathbb{R}^2 \times S^1$, respectively, with the twisted boundary conditions. Section 6 is devoted to a summary and discussion. We present a list of modification of the models which may make fractional instantons to be local or BPS.

2 O(N) model

We consider an O(N) nonlinear sigma model, whose Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu \mathbf{n}\cdot\partial^\mu \mathbf{n} + \mathcal{L}_{\text{h.d.}} - V(\mathbf{n}), \tag{2.1}$$

with N-component scalar fields $\mathbf{n} = (n_1(x), n_2(x), \cdots, n_N(x))^T$ with a constraint $\mathbf{n}^2 = 1$. We have to consider higher derivative (or the Skyrme) term $\mathcal{L}_{\text{h.d.}}$ to stabilize (fractional) instantons in higher dimensions higher than two or three, or two dimensions with a potential term. In some cases, we also consider a potential term $V(\mathbf{n})$ for the stability of fractional instantons. We compactify the x^{N-1} coordinate to S^1 with a period R.

The target space of the model is $M \simeq S^{N-1}$

$$\pi_{N-1}(S^{N-1}) \simeq \mathbb{Z}, \tag{2.2}$$

which admits topological textures, sine-Gordon kinks ($N = 1$), lumps [42] or baby Skyrmions [73–75] ($N = 2$), Skyrmions [68, 69] ($N = 3$). The topological instanton charges $\pi_2(S^2), \pi_3(S^3)$ can be written as

$$Q_2 = -\frac{1}{8\pi^2}\int d^2x\, \epsilon^{ABC}\epsilon^{ij} n_A \partial_i n_B \partial_j n_C = -\frac{1}{8\pi^2}\int d^2x \epsilon^{ij}\mathbf{n}\cdot\partial_i\mathbf{n}\times\partial_j\mathbf{n}, \tag{2.3}$$

$$Q_3 = -\frac{1}{12\pi^2}\int d^3x\, \epsilon^{ABCD}\epsilon^{ijk} n_A\partial_i n_B \partial_j n_C \partial_k n_D, \tag{2.4}$$

respectively. The charge $\pi_3(S^3)$ is also called the baryon number in the context of the Skyrme model. In general, the instanton charge in $\pi_{N-1}(S^{N-1})$ for the O(N) model is given by (see, e.g., ref. [70])

$$Q_{N-1} = -\frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}\int d^{N-1}x\, \frac{1}{(N-1)!}\epsilon^{i_1\cdots i_{N-1}}\epsilon^{A_1\cdots A_N}\partial_{i_1}n_{A_1}\cdots\partial_{i_{N-1}}n_{A_{N-1}}n_{A_N}. \tag{2.5}$$

The O(3) model is equivalent to the $\mathbb{C}P^1$ model. Let ϕ be a normalized complex two vector ($\phi^\dagger\phi = 1$), and consider the Hopf map from S^3 to S^2 by

$$n_A \equiv \phi^\dagger \sigma_A \phi \tag{2.6}$$

with the Pauli matrices σ_A ($A = 1, 2, 3$). Let us define the stereographic coordinate u of S^2 (projective coordinate of the $\mathbb{C}P^1$) by

$$\phi^T = (1, u)^T / \sqrt{1 + |u|^2}. \tag{2.7}$$

In terms of u, the Lagrangian can be rewritten as

$$\mathcal{L} = 2\frac{|\partial_\mu u|^2}{(1 + |u|^2)^2}. \tag{2.8}$$

In this notation, the topological instanton charge can be rewritten as

$$Q_2 = -\frac{1}{4\pi^2}\int d^2x \frac{i\epsilon^{ij}\partial_i u^* \partial_j u}{(1 + |u|^2)^2}. \tag{2.9}$$

The boundary condition $(-, -, +)$ can be expressed in terms of ϕ and u as

$$(-, -, +): \quad \phi(x + R) = W\phi(x), \quad W \equiv \sigma_3 = \text{diag.}(1, -1) \tag{2.10}$$
$$u(x + R) = -u(x). \tag{2.11}$$

The O(4) model is equivalent to a principal chiral model with a group SU(2) or the Skyrme model if four derivative term is considered. We define an SU(2)-valued field $U(x) \in$ SU(2) in terms of four reals scalar fields $n_A(x)$ ($A = 1, 2, 3, 4$):

$$U = i\sum_{a=1,2,3} n_a\sigma_a + n_4\mathbf{1}_2 \tag{2.12}$$

where σ_a are the Pauli matrices and $\mathbf{n}\cdot\mathbf{n} = 1$ is equivalent to $U^\dagger U = \mathbf{1}_2$. In terms of $U(x)$, the Lagrangian can be rewritten as

$$\mathcal{L} = \text{tr}\,(\partial_\mu U^\dagger \partial^\mu U). \tag{2.13}$$

The symmetry of the Lagrangian is $\tilde{G} = \text{SU}(2)_\text{L}\times\text{SU}(2)_\text{R}$ acting on U as $U \to U' = g_\text{L} U g_\text{R}^\dagger$. This symmetry is spontaneously broken down to $\tilde{H} \simeq \text{SU}(2)_{\text{L+R}}$, which in turn acts as $U \to U' = gUg^\dagger$ so that the target space is $\tilde{G}/\tilde{H} \simeq \text{SU}(2)_{\text{L}-\text{R}} \simeq S^3$. The baryon number (the Skyrme charge) of $Q_3 \in \pi_3(S^3)$ can be rewritten as

$$\begin{aligned} Q_3 &= -\frac{1}{24\pi^2}\int d^3x\ \epsilon^{ijk}\text{tr}\left(U^\dagger\partial_i U U^\dagger \partial_j U U^\dagger \partial_k U\right) \\ &= \frac{1}{24\pi^2}\int d^3x\ \epsilon^{ijk}\text{tr}\left(U^\dagger\partial_i U \partial_j U^\dagger \partial_k U\right). \end{aligned} \tag{2.14}$$

The boundary condition $(-, -, +, +)$ can be expressed in terms of U as

$$(-, -, +, +): \quad U(x + R) = WU(x)W^\dagger, \quad W = \sigma_3 = \text{diag.}(1, -1) \tag{2.15}$$

so that the vacuum is center symmetric.

3 General framework for fractional instantons in the O(N) model

Here, we provide a general framework to construct fractional instantons in the O(N) model with the boundary condition (1.1). In general, the boundary condition (1.1) defines a fixed manifold

$$\mathcal{N} = \left\{ \sum_{A=s+1}^{N} (n_A)^2 = 1 \right\} \simeq S^{N-s-1} = S^n,$$
$$S^0 \simeq \{n_N = \pm 1\}, \quad n \equiv N - s - 1 \tag{3.1}$$

as the fixed points of the action at the boundary. This is nothing but the moduli space of vacua, since the boundary condition does not induce the gradient energy for the fields n_A $(A = s+1, \cdots, N)$ while it does for that of the rests n_A $(A = 1, \cdots, s)$. From the homotopy group of $\mathcal{N}$,

$$\pi_n(\mathcal{N}) \simeq \mathbb{Z}, \tag{3.2}$$

one finds the existence of a host soliton (defect) in the bulk. Here, we have formally defined π_{-1} for a space-filling brane in the case of $n = -1$ $(s = N)$ for the situation that there is no localized defects.

At the core of the defect, the nonzero fields in the bulk must vanish, and the relation $\sum_{A=s+1}^{N}(n_A)^2 = 0$ holds, which leads

$$\mathcal{M} = \left\{ \sum_{A=1}^{s} (n_A)^2 = 1 \right\} \simeq S^{s-1} = S^m, \quad m \equiv s - 1. \tag{3.3}$$

This is nothing but the moduli localized on the host soliton's world volume (collective coordinates of the host soliton). This has a non-trivial homotopy group

$$\pi_m(\mathcal{M}) \simeq \mathbb{Z}. \tag{3.4}$$

The host soliton has world volume along the compact direction and the rests. Therefore, the moduli $\mathcal{M}$ must be twisted along the world volume in the compact direction with the twisted boundary condition. It inevitably introduces a daughter soliton, which, we find, belongs to a "half" element of the homotopy group in eq. (3.4). In other words, a homotopy group in eq. (3.4) is modified by the boundary condition to take a value in a half integer. While this should be explained by a relative homotopy group more rigorously, we do not do that in this paper. We denote it symbolically by

$$\pi_m^{\text{b.c.}}(\mathcal{M}) \simeq \mathbb{Z} + \frac{1}{2}. \tag{3.5}$$

We thus have a composite soliton. Each composite soliton consists of a daughter soliton, belonging to a half element of the homotopy group $\pi_m^{\text{b.c.}}(\mathcal{M})$ in eq. (3.5) modified by the boundary condition, on a host soliton, belonging to the unit element of the homotopy group $\pi_n(\mathcal{N})$ in eq. (3.2). Consequently, the total homotopy group $\pi_{N-1}(M)$ in eq. (2.2) is a product of the elements in $\pi_n(\mathcal{N})$ in eq. (3.2) and $\pi_m^{\text{b.c.}}(\mathcal{M})$ in eq. (3.5), and so it belongs to a half element of the total homotopy group $\pi_{N-1}(M)$ in eq. (2.2), that is, a half instanton.

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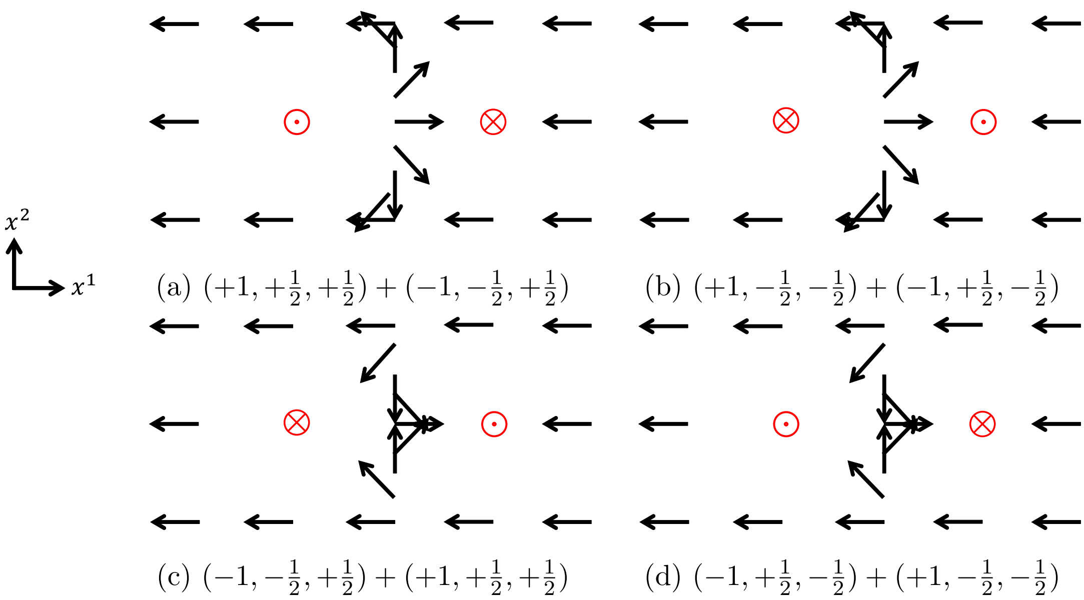

Figure 3. Fractional instantons in the O(3) model with the boundary condition $(-,+,+)$. $\odot$ and $\otimes$ correspond to $n_1 = +1$ and $n_1 = -1$, respectively. Black arrows represent (n_2, n_3) with $n_2^2 + n_3^2 = 1$ $(n_1 = 0)$ parameterizing the moduli space of vacua $\mathcal{N} \simeq S^1$: $\leftarrow$, $\rightarrow$, $\uparrow$, $\downarrow$ correspond to $n_3 = +1$, $n_3 = -1$, $n_2 = +1$, $n_2 = -1$, respectively. We chose the vacuum $n_3 = +1$ at the boundary. Topological charges $(*,*,*)$ denote a host vortex charge π_1, an Ising spin π_0 in its core, and the total instanton charge π_2, respectively. (a) An instanton is split into two fractional instantons $(+1,+\frac{1}{2},+\frac{1}{2})$ and $(-1,-\frac{1}{2},+\frac{1}{2})$ separated into the x^1 direction by a sine-Gordon domain wall. (b) An anti-instanton is split into two fractional anti-instantons $(+1,-\frac{1}{2},-\frac{1}{2})$ and $(-1,+\frac{1}{2},-\frac{1}{2})$ separated into the x^1 direction by a sine-Gordon anti-domain wall. (c) and (d) are isomorphic to (a) and (b), respectively, by a 2π rotation along an axis at the center of the sine-Gordon (anti-)domain wall, which exchanges two fractional instantons.

The sum of codimensions of a host soliton and of a daughter soliton is $N-1$, which is 2 or 3 for the O(3) model on $\mathbb{R}^1 \times S^1$ or the O(4) model on $\mathbb{R}^2 \times S^1$, respectively. Equivalently, there exists a certain relation between the dimensionality of the homotopy groups:

$$n + m + 1 = N - s - 1 + (s-1) + 1 = N - 1 \tag{3.6}$$

which is $n + m + 1 = 2$ for the O(3) model and $n + m + 1 = 3$ for the O(4) model.

In the following sections, we discuss fractional instantons and bions in more detail for each boundary condition in the O(3) and O(4) models.

4 Fractional instantons and bions in the O(3) model

4.1 $(-,+,+)$: global vortex with an Ising spin or half lump-vortex

The fixed manifold is characterized by $n_1 = 0$, equivalently $(n_2)^2 + (n_3)^2 = 1$, which is $\mathcal{N} \simeq S^1$. This is the moduli space of vacua as explained in the last section. It has a nontrivial homotopy $\pi_1(S^1) \simeq \mathbb{Z}$, allowing a global vortex having the winding in $n_2 + in_3$. In the vortex core, the two fields constituting the vortex must vanish $n_2 = n_3 = 0$ and

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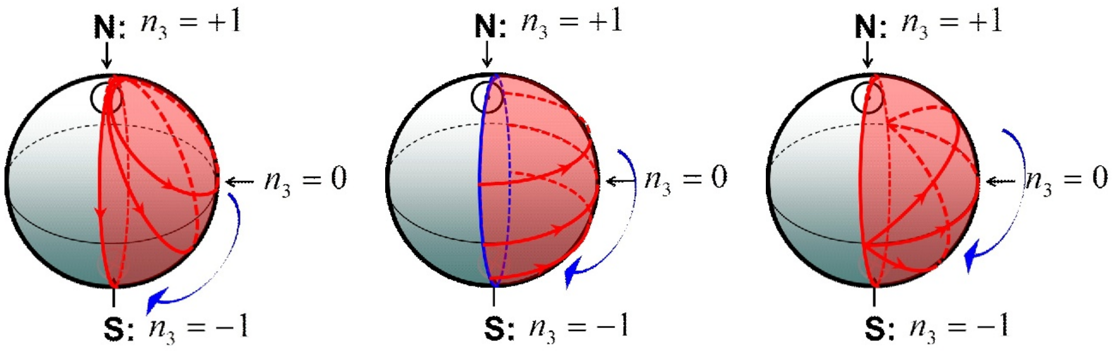
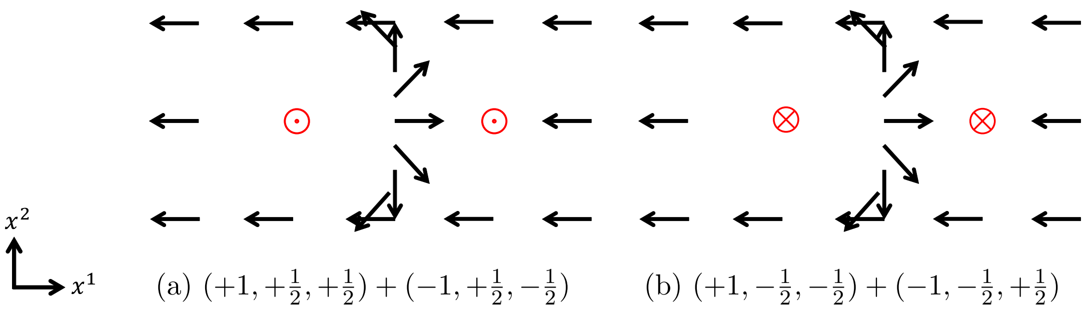
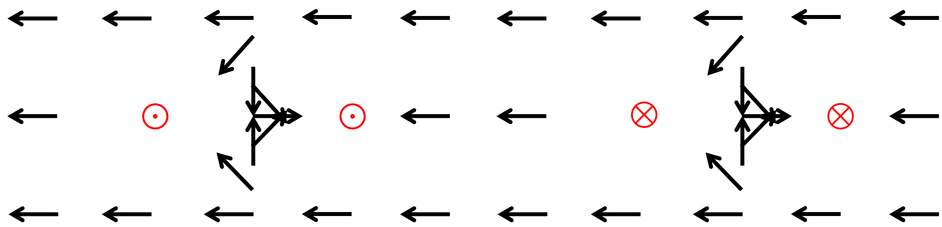


(a) $(-,+,+)$ (b) $(-,-,+)$ (c) $(-,-,-)$

Figure 4. Images of fractional instantons in the target space S^2 of the O(3) model with the boundary conditions (a) $(-,+,+)$, (b) $(-,-,+)$ and (c) $(-,-,-)$. Each path represents an image of $x_1 =$ constant with $x_2 = 0$ to $x_2 = R$, where an arrow indicates a direction. With changing x_1 from $x_1 = -\infty$ to $x_1 = +\infty$, the path moves following the blue arrow to cover a half sphere.



(a) $(+1, +\frac{1}{2}, +\frac{1}{2}) + (-1, +\frac{1}{2}, -\frac{1}{2})$ (b) $(+1, -\frac{1}{2}, -\frac{1}{2}) + (-1, -\frac{1}{2}, +\frac{1}{2})$



(c) $(-1, +\frac{1}{2}, -\frac{1}{2}) + (+1, +\frac{1}{2}, +\frac{1}{2})$ (d) $(-1, -\frac{1}{2}, +\frac{1}{2}) + (+1, -\frac{1}{2}, -\frac{1}{2})$

Figure 5. Bions in the O(3) model with the boundary condition $(-,+,+)$. The notations are the same with figure 3. (c) and (d) are isomorphic to (a) and (b), respectively, by a 2π rotation along an axis at the center of the sine-Gordon (anti-)domain wall.

the rest field n_1 appears taking a value $n_1 = \pm 1$, giving an Ising spin degree of freedom to the vortex. Therefore, the moduli space of the vortex is $\mathcal{M} \simeq \{\pm 1\}$. This is a fractional (anti-)instanton. Depending on the vortex winding and the vortex moduli, there are four possibilities for fractional (anti-)instantons in the boundary condition $(-,+,+)$, as shown in figure 1 (1a)–(1d). A unit (anti-)instanton (lump) can be decomposed into two fractional (anti-)instantons as illustrated in figure 3. Each fractional instanton wraps a half of the target space S^2. For instance, the left half of figure 3(a) wraps a half sphere as in figure 4(a).

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	π_1	π_0	π_2
Figure 1 (1a)	$+1$	$+1/2$	$+1/2$
Figure 1 (1b)	-1	$-1/2$	$+1/2$
Figure 1 (1c)	-1	$+1/2$	$-1/2$
Figure 1 (1d)	$+1$	$-1/2$	$-1/2$

Table 2. Homotopy groups of fractional instantons in the O(3) model with the boundary condition $(-,+,+)$. The columns represent the homotopy groups of a host soliton π_1, a daughter soliton π_0, and the total instanton π_2 from left to right.

If a fractional (anti-)instanton is well separated from the rest and is isolated, it becomes one of figure 1 (1a)–(1d).

In order to write down explicit configurations, it is useful to define a complex coordinate by $z \equiv x_1 + ix_2$. Then, asymptotic forms near fractional instantons located at $z = 0$ can be given by

$$(1\mathrm{a}): \quad u \sim z, \quad (1\mathrm{b}): \quad u \sim 1/z, \tag{4.1}$$

$$(1\mathrm{c}): \quad u \sim \bar{z}, \quad (1\mathrm{d}): \quad u \sim 1/\bar{z}. \tag{4.2}$$

This expression is also good for large compactification radius R. The topological charges of fractional (anti-)instantons with the boundary condition $(-,+,+)$ are summarized in table 2. Here, we have defined the value of π_0 for the Ising spin to be $\pm 1/2$ to be consistent with the other boundary conditions discussed below.

Let us discuss the interaction between fractional instantons. When constituent fractional instantons are well separated at distance r in a large compactification radius R, the interaction between them is $E_{\mathrm{int}} \sim \pm \log r$ (the force is $F \sim \pm 1/r$) because they are global vortices. Here, the interaction is repulsive for a pair of (anti-)vortices, and attractive for a pair of a vortex and an anti-vortex. Therefore, it is attractive for a pair of fractional (anti-)instantons constituting an (anti-)instanton. On the other hand, when the compactification radius R is as the same as the size of fractional instantons as in figure 5, a sine-Gordon (anti-)kink connects a fractional instanton and anti-instanton so that they are confined by a linear interaction energy $E_{\mathrm{int}} \sim r$ with distance r and the force between them is constant.

Next let us discuss bion configurations. Configurations near a bion can be written as

$$(1\mathrm{b})+(1\mathrm{d}): \quad u \sim \frac{\alpha_1}{z - z_1} + \frac{\bar{\alpha}_1}{\bar{z} - \bar{z}_1} + \beta_1 \tag{4.3}$$

$$(1\mathrm{a})+(1\mathrm{c}): \quad u \sim \frac{(z - z_1)(\bar{z} - \bar{z}_2)}{\alpha z + \beta \bar{z} + \gamma}. \tag{4.4}$$

This is good for large compactification radius R. Bions for small compactification radius R are schematically drawn in figure 5. Each of figure 5 shows a sine-Gordon (anti-)kink connecting two fractional (anti-)instantons for small compactification radius R, and consequently they are confined by a linear potential $E_{\mathrm{int}} \sim r$ with distance r for large separation.

Before going to the other boundary conditions, let us make a comment on fractional instantons in related models. There exist topologically the same fractional instantons

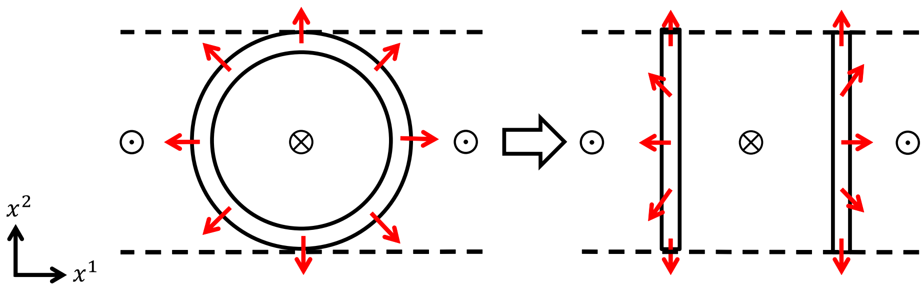


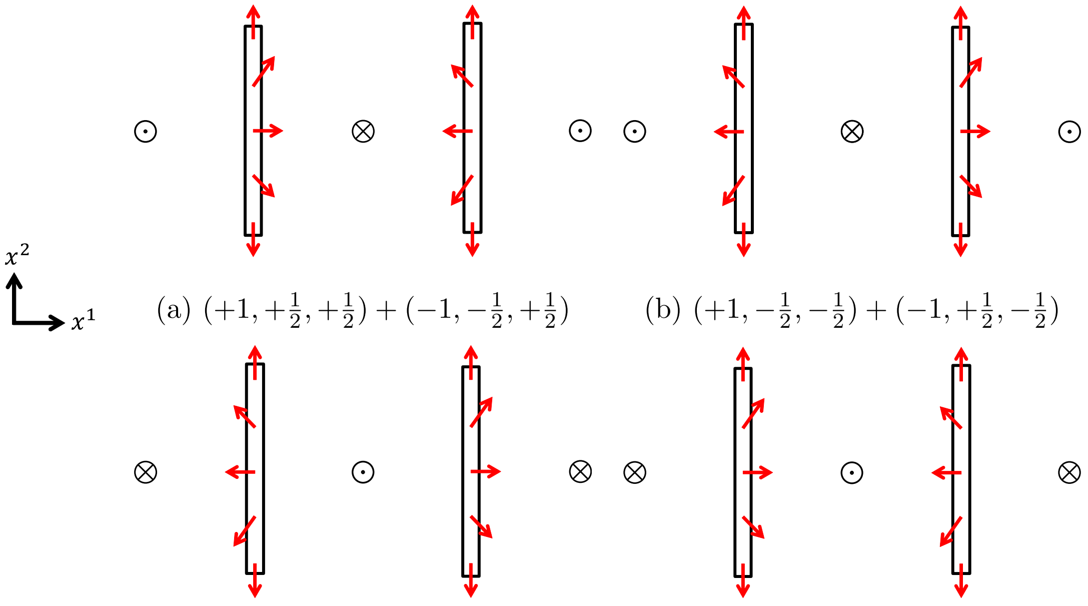
Figure 6. Twisted domain wall ring decaying into two fractional instantons in the O(3) model with the boundary condition $(-,-,+)$. $\odot$ and $\otimes$ correspond to $n_3 = +1$ and $n_3 = -1$, respectively, representing vacua. Red arrows represent (n_1, n_2) with $n_1^2 + n_2^2 = 1$ $(n_3 = 0)$ parameterizing the moduli space of a domain wall $\mathcal{M} \simeq S^1$: $\leftarrow$, $\rightarrow$, $\uparrow$, $\downarrow$ correspond to $n_1 = -1$, $n_1 = +1$, $n_2 = +1$, $n_2 = -1$, respectively. The dotted lines denote the boundary at $x^2 = 0$ and $x^2 = R$. When the size of a domain wall ring is of that of the compact direction, it decays thorough a reconnection into two fractional instantons, which are domain walls with half twisted U(1) moduli.

on $\mathbb{R}^2$ *without* twisted boundary condition. One is baby Skyrmions [73–75] in an O(3) sigma model with a potential term $V = m^2 n_1^2$ and a four derivative (baby Skyrme) term [76–78]. The other is a vortex in a U(1) gauged O(3) sigma model with a potential term $V = m^2 n_1^2$, in which the U(1) acting on $n_2 + in_3$ is gauged [79–83]. Vortices in this case are local, that is, of Abrikosov-Nielsen-Olesen (ANO) type [84, 85]. In both cases, the potential term plays an alternative role of the twisted boundary condition. Interactions between fractional instantons are rather different from our case of the twisted boundary condition. In the former, the interaction between fractional instantons constituting an instanton is attractive at large distance and repulsive at short distance, resulting in a stable molecule [76–78]. In the latter, the interaction between them is exponentially suppressed which is either repulsive or attractive for type-II or type-I superconductor, and non-interactive for the critical limit, which is BPS [82].

4.2 $(-,-,+)$: a half sine-Gordon kink inside a domain wall

This is only the case studied in the literature. This case is equivalent to the $\mathbb{C}P^1$ model with $\mathbb{Z}_2$ symmetric boundary condition, allowing fractional instantons [46, 51, 52, 60] and bions [8–10, 16–18].

Instantons in the O(3) model can be represented as a domain wall ring along which a U(1) modulus is twisted [62, 64, 86–88]. When the size of the domain wall ring is that of the compactification radius R, the top and bottom of the domain wall ring touch each other through the compact direction x^2 with the twisted boundary condition. Then, a reconnection of two parts of the ring occurs, and it can be split into two domain wall lines separated into the x^1 direction, as shown in figure 6. The U(1) modulus is twisted half along the domain lines extending to the x^2 direction, resulting in fractional (anti-)instantons. We have two pairs for instanton and anti-instanton respectively as seen in figure 7. We thus find four kinds of fractional (anti-)instantons shown in figure 1 (2a)–(2d). Each fractional



(a) $(+1,+\frac{1}{2},+\frac{1}{2})+(-1,-\frac{1}{2},+\frac{1}{2})$ (b) $(+1,-\frac{1}{2},-\frac{1}{2})+(-1,+\frac{1}{2},-\frac{1}{2})$

(c) $(-1,-\frac{1}{2},+\frac{1}{2})+(+1,+\frac{1}{2},+\frac{1}{2})$ (d) $(-1,+\frac{1}{2},-\frac{1}{2})+(+1,-\frac{1}{2},-\frac{1}{2})$

Figure 7. Fractional instantons in the O(3) model with the boundary condition $(-,-,+)$. The notations are the same with figure 6. Topological charges $(*,*,*)$ denote a host domain wall charge π_0, a sine-Gordon kink charge π_1 on it, and the total instanton charge π_2, respectively. (a) An instanton is split into two fractional instantons $(+1,+\frac{1}{2},+\frac{1}{2})$ and $(-1,-\frac{1}{2},+\frac{1}{2})$ separated by the vacuum $\otimes$. (b) An anti-instanton is split into two fractional anti-instantons $(+1,-\frac{1}{2},-\frac{1}{2})$ and $(-1,+\frac{1}{2},-\frac{1}{2})$ separated by the vacuum $\otimes$. (c) and (d) are obtained from (a) and (b), respectively, by exchanging the positions of the fractional instanton and anti-instanton.

instanton wraps a half sphere of the target space S^2. For instance, the left half of figure 7(a) wraps a half sphere as in figure 4(b).

Explicit configurations of isolated fractional (anti-)instantons can be given as

$$(2\text{a}): \quad u = e^{\pi z}, \quad (2\text{b}): \quad u = e^{-\pi z}, \tag{4.5}$$

$$(2\text{c}): \quad u = e^{\pi \bar{z}}, \quad (2\text{d}): \quad u = e^{-\pi \bar{z}}. \tag{4.6}$$

The topological charges of fractional (anti-)instantons with the boundary condition $(-,-,+)$ are summarized in table 3.

Bions with the boundary condition $(-,-,+)$ are shown in figure 8. Explicit bion ansatz can be constructed as

$$(2\text{d}) + (2\text{a}): \quad u = e^{-\pi(\bar{z}-\bar{z}_1)} + e^{\pi(z-z_2)}, \tag{4.7}$$

$$(2\text{b}) + (2\text{c}): \quad u = e^{-\pi(z-z_1)} + e^{\pi(\bar{z}-\bar{z}_2)}. \tag{4.8}$$

These ansatz are different from ref. [17, 18], but their asymptotic behaviors are the same. The interactions between fractional (anti-)instantons are exponentially suppressed so that the total action becomes a sum of each action when they are well separated.

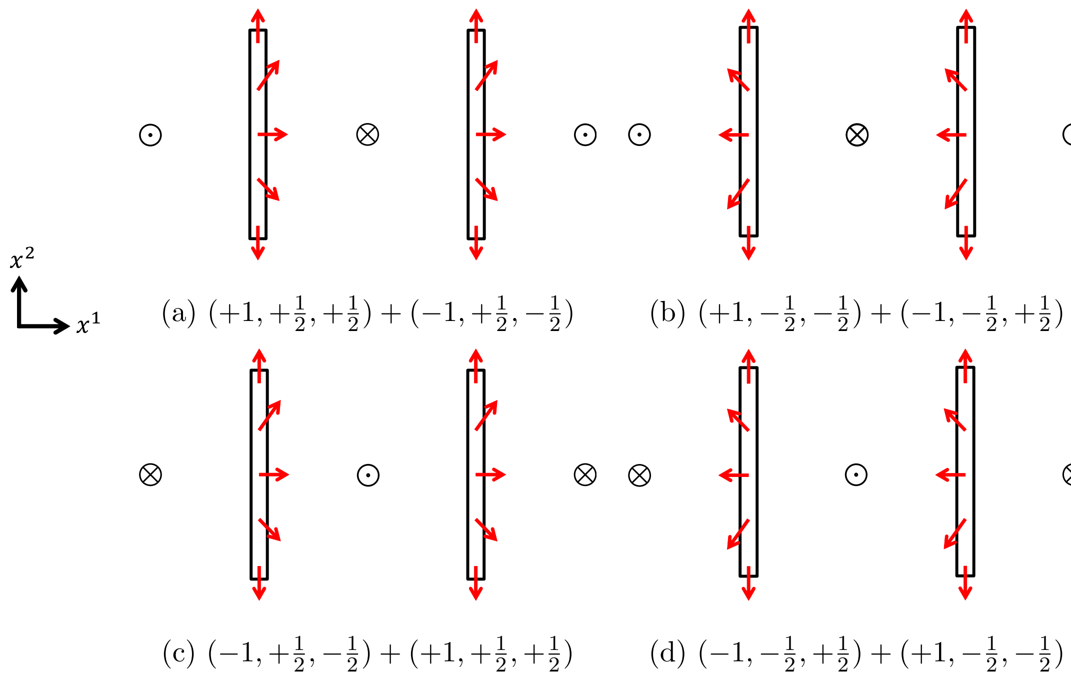


Figure 8. Bions in the O(3) model with the boundary condition $(-,-,+)$. The notations are the same with figure 6. (c) and (d) are obtained from (a) and (b), respectively, by exchanging the positions of the fractional instanton and anti-instanton.

	π_0	π_1	π_2
Figure 1 (2a)	$+1$	$+1/2$	$+1/2$
Figure 1 (2b)	-1	$-1/2$	$+1/2$
Figure 1 (2c)	-1	$+1/2$	$-1/2$
Figure 1 (2d)	$+1$	$-1/2$	$-1/2$

Table 3. Homotopy groups of fractional instantons in the O(3) model with the boundary condition $(-,-,+)$. The columns represent the homotopy groups of a host soliton π_0, a daughter soliton π_1, and the total instanton π_2 from left to right.

Before going to the next case, let us give a brief comment on a relation to dimensional reduction in this case. In the zero radius limit of the compact direction, the theory is dimensionally reduced. By assuming the dependence of the fields on the compact direction x^2 as

$$(n_1, n_2) = \left(\hat{n}_1(x^1)\cos\frac{\pi}{R}x^2, \hat{n}_2(x^1)\sin\frac{\pi}{R}x^2\right) \tag{4.9}$$

in the presence of the twisted boundary condition, we see that a potential term is effectively induced from the gradient term of the fields:

$$V = \int_0^R dx^2\left[(\partial_2 n_1)^2 + (\partial_2 n_2)^2\right] = m^2(\hat{n}_1^2+\hat{n}_2^2) = m^2(1-\hat{n}_3^2), \quad m^2 \equiv \frac{\pi^2}{4R}. \tag{4.10}$$

This is known as the Scherk-Schwarz dimensional reduction in the context of supersymmetric theories in which the induced mass is called a twisted mass. The dimensionally

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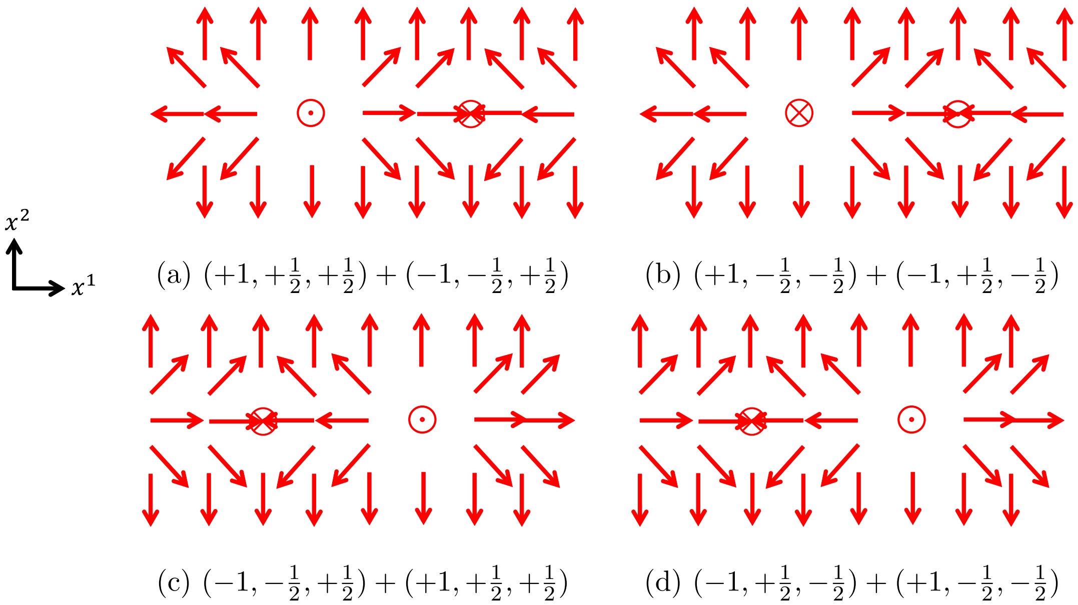


(a) $(+1,+\frac{1}{2},+\frac{1}{2})+(-1,-\frac{1}{2},+\frac{1}{2})$ (b) $(+1,-\frac{1}{2},-\frac{1}{2})+(-1,+\frac{1}{2},-\frac{1}{2})$

(c) $(-1,-\frac{1}{2},+\frac{1}{2})+(+1,+\frac{1}{2},+\frac{1}{2})$ (d) $(-1,+\frac{1}{2},-\frac{1}{2})+(+1,-\frac{1}{2},-\frac{1}{2})$

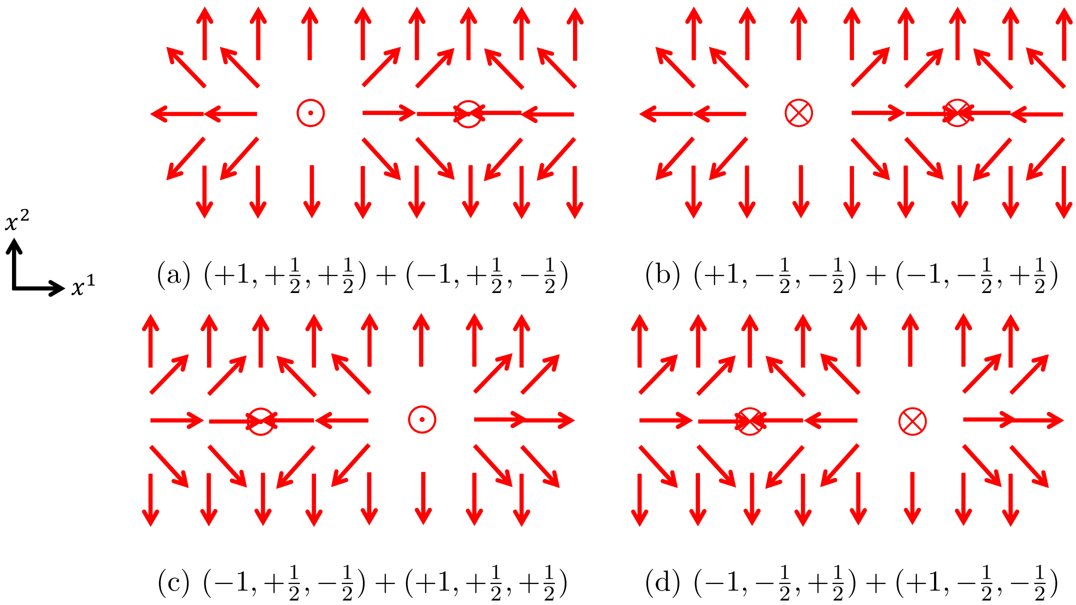
Figure 9. Fractional instantons in the O(3) model with the boundary condition $(-,-,-)$. $\odot$ and $\otimes$ correspond to $n_3 = +1$ and $n_3 = -1$, respectively, and $\leftarrow$, $\rightarrow$, $\uparrow$, $\downarrow$ correspond to $n_1 = -1$, $n_1 = +1$, $n_2 = +1$, $n_2 = -1$, respectively. Topological charges $(*,*,*)$ denote a host space-filling soliton charge π_{-1} which is merely formal, a lump charge π_2 on it, and the total instanton charge π_2, respectively. (a) An instanton is split into two fractional instantons $(+1,+\frac{1}{2},+\frac{1}{2})+(-1,-\frac{1}{2},+\frac{1}{2})$ and $(+1,-\frac{1}{2},-\frac{1}{2})+(-1,+\frac{1}{2},-\frac{1}{2})$ separated by a half sine-Gordon domain wall with opposite orientation with the boundary at $x^1 = \pm\infty$. (b) An anti-instanton is split into two fractional anti-instantons $(-1,-\frac{1}{2},+\frac{1}{2}) + (+1,+\frac{1}{2},+\frac{1}{2})$ and $(-1,+\frac{1}{2},-\frac{1}{2}) + (+1,-\frac{1}{2},-\frac{1}{2})$ separated by a half sine-Gordon domain wall with opposite orientation with the boundary $x^1 = \pm\infty$. (c) and (d) are isomorphic to (a) and (b), respectively, by a 2π rotation along an axis at the center.

reduced $\mathbb{C}P^1$ model is often called as the massive $\mathbb{C}P^1$ model in the context of supersymmetry. Lumps (instantons) are reduced to (a pair of) domain walls in the massive $\mathbb{C}P^1$ model [89–93]. This case was generalized to the $\mathbb{C}P^{N-1}$ and Grassmann sigma models, for which domain walls [53, 54, 61], instantons (lumps) [94, 95], fractional instantons [51, 52], bions [19] were studied.

4.3 $(-,-,-)$

There are no fixed points for the boundary condition $(-,-,-)$ unlike the above cases. This implies that there is no vacuum. In fact, even in the least energy configuration, the fields must be twisted because of the boundary condition, and there exist gradient energy. We do not have localized solitons wrapping around a fixed manifold. We interpret this situation that there is a space-filling soliton of codimension zero to be consistent with the other cases. Then, we interpret the original target space S^2 as moduli of the space-filling soliton.

An (anti-)instanton is separated into two fractional (anti-)instantons with the boundary condition $(-,-,-)$ as shown in figure 9. The ansatz for an isolated fractional



(a) $(+1,+\frac{1}{2},+\frac{1}{2})+(-1,+\frac{1}{2},-\frac{1}{2})$ (b) $(+1,-\frac{1}{2},-\frac{1}{2})+(-1,-\frac{1}{2},+\frac{1}{2})$

(c) $(-1,+\frac{1}{2},-\frac{1}{2})+(+1,+\frac{1}{2},+\frac{1}{2})$ (d) $(-1,-\frac{1}{2},+\frac{1}{2})+(+1,-\frac{1}{2},-\frac{1}{2})$

Figure 10. Bions in the O(3) model with the boundary condition $(-,+,+)$. The notations are the same with figure 9. (c) and (d) are isomorphic to (a) and (b), respectively, by a 2π rotation along an axis at the center of the domain wall.

	π_{-1}	π_2	π_2
Figure 1 (3a)	$+1$	$+1/2$	$+1/2$
Figure 1 (3b)	-1	$-1/2$	$+1/2$
Figure 1 (3c)	-1	$+1/2$	$-1/2$
Figure 1 (3d)	$+1$	$-1/2$	$-1/2$

Table 4. Homotopy groups of fractional instantons in the O(3) model with the boundary condition $(-,-,-)$. The columns represent the homotopy groups of a host soliton π_{-1}, a daughter soliton π_2, and the total instanton π_2 from left to right. π_{-1} is merely formal.

(anti-)instanton can be given as

$$\begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos f(x^1) & \mp\sin f(x^1) \\ 0 & \pm\sin f(x^1) & \cos f(x^1) \end{pmatrix} \begin{pmatrix} -\cos\frac{\pi}{R}x^2 \\ \sin\frac{\pi}{R}x^2 \\ 0 \end{pmatrix} = \begin{pmatrix} -\cos f(x^1) \\ \cos f(x^1)\sin\frac{\pi}{R}x^2 \\ \pm\sin f(x^1)\sin\frac{\pi}{R}x^2 \end{pmatrix}, \tag{4.11}$$

$$f(x^1=-\infty)=0, \quad f(x^1=+\infty)=\pi. \tag{4.12}$$

Each fractional (anti-)instanton wraps a half sphere of the target space S^2. For instance, the left half of figure 9(a) wraps a half sphere as in figure 4(c). The topological charges of fractional (anti-)instantons are summarized in table 4. Here, we formally use π_{-1} for space-filling solitons of codimension zero, to be consistent with the other cases.

It may be worth to mention that the existence of a daughter soliton is not required from the boundary condition, because the x^1-dependent rotation in eq. (4.11) is not necessary and a configuration $(n_1, n_2, n_3) = (-\cos\frac{\pi}{R}x_2, \sin\frac{\pi}{R}x_2, 0)$ is in fact a minimum energy state. This is in contrast to the other boundary conditions in which the existence of a daughter soliton is required in the presence of a host soliton.

In this case, the boundary condition is not enough to stabilize fractional instantons, unlike the other two boundary conditions. One needs to add a potential term

$$V = m^2 n_3^2 \tag{4.13}$$

to the original Lagrangian for the stability of half instantons. For this particular potential term, the function f in eq. (4.11) is a sine-Gordon kink, $f = \arctan\exp(mx^1)$.

For one instanton, one chooses the boundary condition of f as $f = 0$ at $x^1 \to -\infty$ and $f = 2\pi$ at $x^1 \to +\infty$, instead of eq. (4.12). The interaction energy between two fractional instantons would be suppressed exponentially because the energy density between the two fractional instantons is the same with that outside them, and there is no confining force between them. The detail form depends on the choice of a potential term to stabilize half instantons. For the potential in eq. (4.13), the interaction is that of two sine-Gordon kinks, which is repulsive.

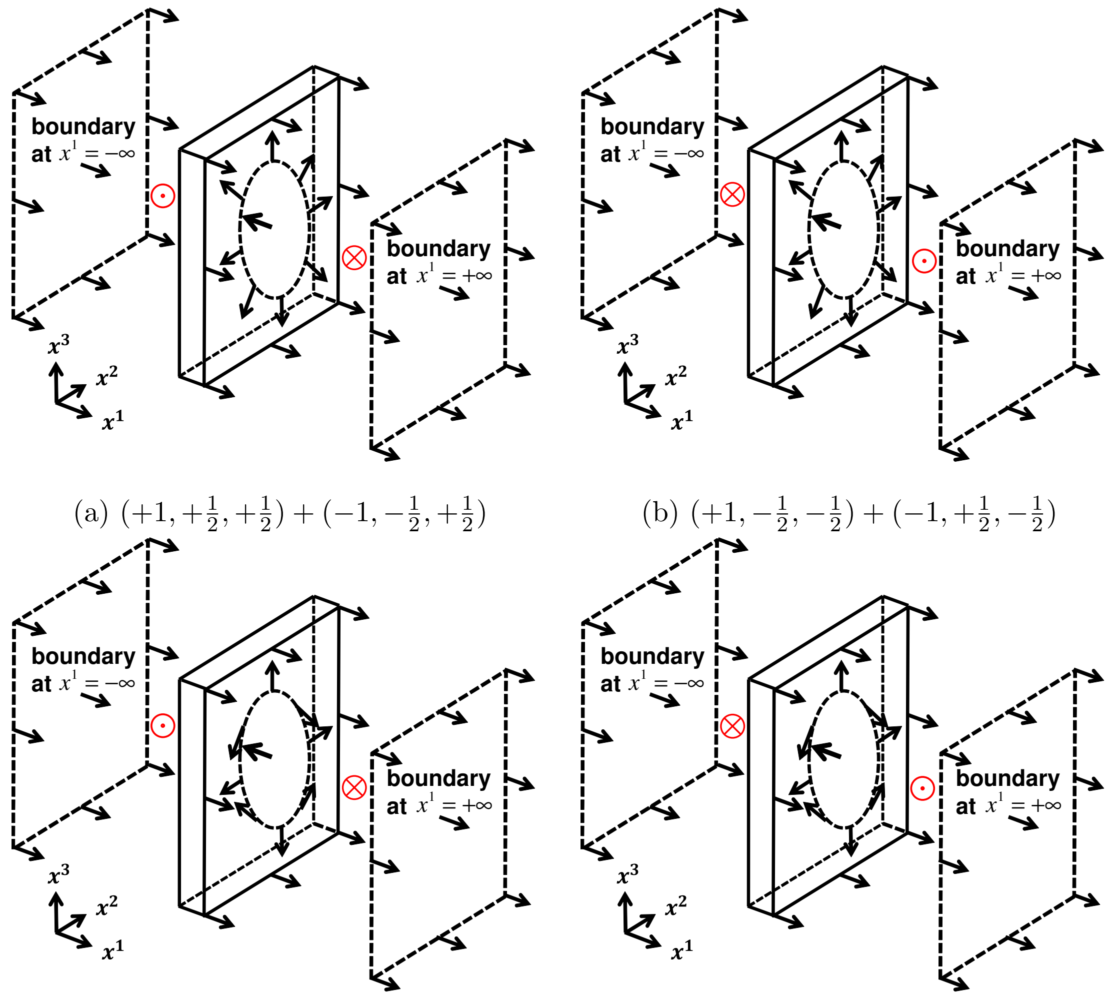
Bions with the boundary condition $(-,-,-)$ are shown in figure 10. The function f in eq. (4.11) behaves as as $f = 0$ at $x^1 - \infty$, $f \sim \pi$ in some intermediate region and back to $f = 0$ at $x^1 \to +\infty$. The interaction energy between two fractional instantons constituting a bion would be suppressed exponentially because of the same reason with the above while the detailed form depends on the choice of the potential term. For the potential term in eq. (4.13), the interaction is that between a sine-Gordon kink and an anti-kink.

5 Fractional instantons and bions in the O(4) model

5.1 $(-,+,+,+)$: global monopole with an Ising spin or half Skyrmion-monopole

The fixed manifold is characterized by $n_1 = 0$, equivalently $(n_2)^2 + (n_3)^2 + (n_4)^2 = 1$, which is the moduli space of vacua $\mathcal{N} \simeq S^2$. Therefore, it has a nontrivial homotopy $\pi_2(S^2) \simeq \mathbb{Z}$, allowing a global monopole. In the monopole core $n_2 = n_3 = n_4 = 0$, the field n_1 appears taking a value $n_1 = \pm 1$ in the center, giving an Ising spin degree of freedom to the monopole, that is, the moduli space of the monopole is $\mathcal{M} \simeq \{\pm 1\}$. This is a fractional (anti-)instanton with the boundary condition $(-,+,+,+)$ as drawn in figure 2 (1a)–(1d). Apparently, each fractional instanton wraps a half of the target space S^3. A unit (anti-)instanton can be separated into two fractional (anti-)instantons as shown in figure 11. Again, each fractional instanton wraps a half of the target space S^3. If one is well separated from the rests, it becomes one of figure 2 (1a)–(1d). The topological charges of fractional (anti-)instantons with the boundary condition $(-,+,+,+)$ are summarized in table 5. Here, we have defined the value of π_0 for the Ising spin to be $\pm 1/2$ to be consistent with the other boundary conditions discussed below.

We need higher derivative (Skyrme) term for the stability of fractional instantons (Skyrmions) [96], as is so for usual Skyrmions.



(a) $(+1, +\frac{1}{2}, +\frac{1}{2}) + (-1, -\frac{1}{2}, +\frac{1}{2})$ (b) $(+1, -\frac{1}{2}, -\frac{1}{2}) + (-1, +\frac{1}{2}, -\frac{1}{2})$

(c) $(-1, -\frac{1}{2}, +\frac{1}{2}) + (+1, +\frac{1}{2}, +\frac{1}{2})$ (d) $(-1, +\frac{1}{2}, -\frac{1}{2}) + (+1, -\frac{1}{2}, -\frac{1}{2})$

Figure 11. Fractional instantons in the O(4) model with the boundary condition $(-, +, +, +)$. $\odot$ and $\otimes$ correspond to $n_1 = +1$ and $n_1 = -1$, respectively, representing the moduli space (an Ising spin) $\mathcal{M}$ of a monopole Black arrows represent (n_2, n_3, n_4) with $n_2^2 + n_3^2 + n_3^2 = 1$ $(n_1 = 0)$ parameterizing the moduli space of vacua $\mathcal{N} \simeq S^2$. Brackets $(*, *, *)$ denote topological charges for a host monopole characterized by π_2, that for an Ising spin characterized by π_0, and that for an instanton characterized by π_3. (a) An instanton is split into two fractional instantons $(+1, +\frac{1}{2}, +\frac{1}{2})$ and $(-1, -\frac{1}{2}, +\frac{1}{2})$ separated by a lump. (b) An anti-instanton is split into two fractional anti-instantons $(+1, -\frac{1}{2}, -\frac{1}{2})$ and $(-1, +\frac{1}{2}, -\frac{1}{2})$ separated by an anti-lump. (c) and (d) are isomorphic to (a) and (b), respectively, by a 2π rotation along the x^1 axis.

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	π_2	π_0	π_3
Figure 2 (1a)	$+1$	$+1/2$	$+1/2$
Figure 2 (1b)	-1	$-1/2$	$+1/2$
Figure 2 (1c)	-1	$+1/2$	$-1/2$
Figure 2 (1d)	$+1$	$-1/2$	$-1/2$

Table 5. Homotopy groups of fractional instantons in the O(4) model with the boundary condition $(-,+,+,+)$. The columns represent the homotopy groups of a host soliton π_2, a daughter soliton π_0, and the total instanton π_3 from left to right.

When the compactification radius R is large, the interaction between two well-separated fractional instantons is the same with that of global monopoles at large distance. For a small compactification radius R of the order of fractional instanton size, the interaction between two well-separated fractional instantons at distance r is $E_{\rm int} \sim r$ because of a lump string connecting them.

Bions with the boundary condition $(-,+,+,+)$ are schematically drawn in figure 12. While each domain separated by a lump an instanton charge, the total instanton charges are canceled out. Again, the two well separated fractional instantons at distance r are confined by a linear potential $E_{\rm int} \sim r$ for a compactification radius R of the order of fractional instanton size.

If one gauges the SO(3) symmetry acting on (n_2, n_3, n_4), a half-Skyrmion monopole becomes local, that is, of 't Hooft-Polyakov type [97, 98] having finite energy. This is in fact the case of the SO(3) gauged model with a potential term $V = m^2 n_1^2$ defined on $\mathbb{R}^3$ *without* twisted boundary condition [99–103].

5.2 $(-,-,+,+)$: a half sine-Gordon kink inside a vortex

The twisted boundary condition $(-,-,+,+)$ is equivalent to eq. (2.15) in terms of the SU(2)-valued field $U(x)$. Here we use the original notation of the four real scalar fields $n_A(x)$. The fixed manifold is characterized by $n_1 = n_2 = 0$, equivalently $(n_3)^2 + (n_4)^2 = 1$, which is the moduli space of vacua $\mathcal{N} \simeq S^1$. It has a nontrivial homotopy $\pi_1(S^1) \simeq \mathbb{Z}$, allowing a global vortex having a winding in $n_3 + in_4$. In the vortex core, the winding field must vanish $n_3 = n_4 = 0$, and the other fields n_1 and n_2 appear with a constraint $(n_1)^2 + (n_2)^2 = 1$, giving a modulus $\mathcal{M} \simeq$ U(1) to the vortex. For a fractional instanton, this U(1) modulus is twisted half along the vortex string extending to the compactified direction, as described below.

An instanton (Skyrmion) can be represented by (a global analog of) a vorton [104–108], that is, a vortex ring along which a U(1) modulus is twisted. This fact was first found in the context of Bose-Einstein condensates (BEC) [109–115] (see also [116]), and stable solutions in a Skyrme model were also constructed in refs. [117–119]. Configurations of Skyrmions as vortons are shown in figure 13. The decomposition of an instanton into fractional instantons can be understood as higher dimensional analog of a domain wall ring in the O(3) model with the twisted boundary condition $(-,-,+)$. When the size of

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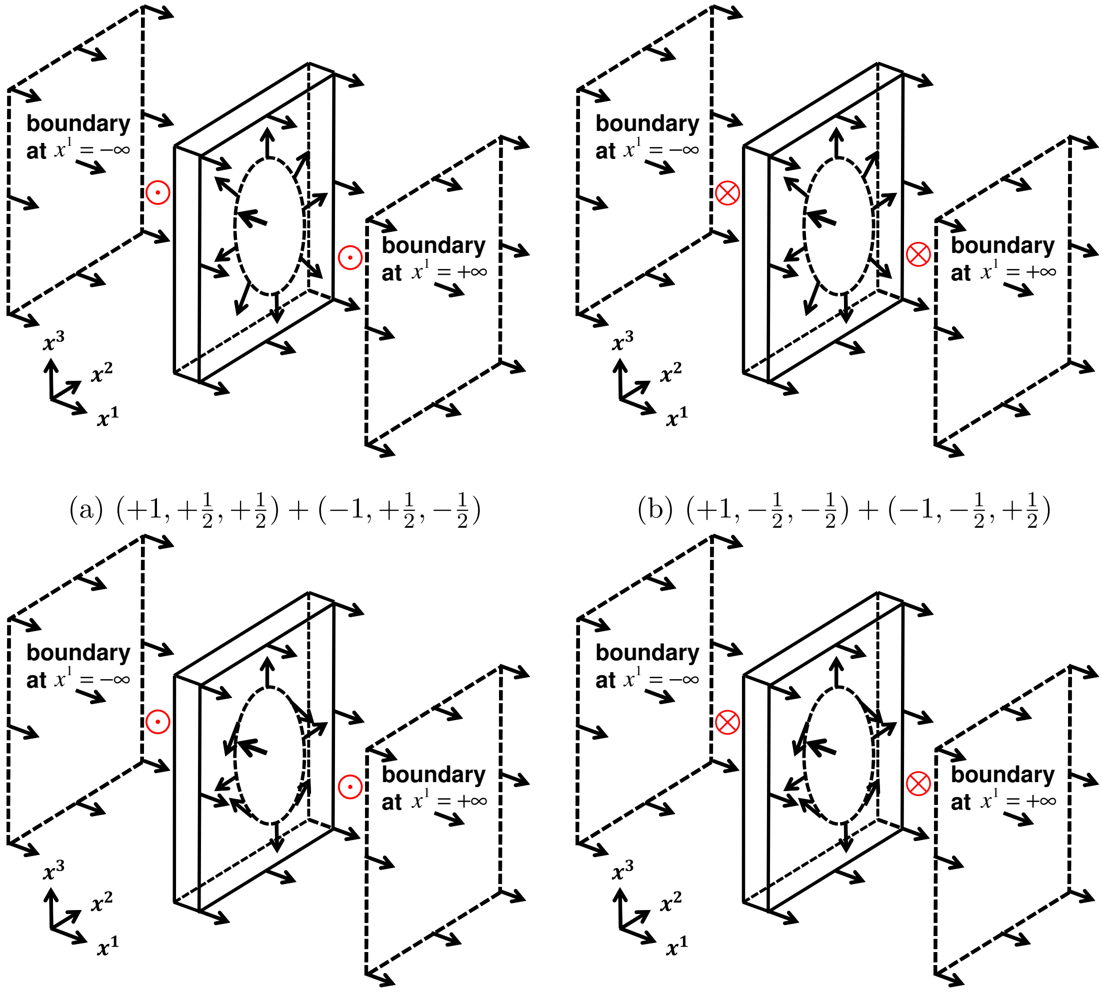


(a) $(+1, +\frac{1}{2}, +\frac{1}{2}) + (-1, +\frac{1}{2}, -\frac{1}{2})$ (b) $(+1, -\frac{1}{2}, -\frac{1}{2}) + (-1, -\frac{1}{2}, +\frac{1}{2})$

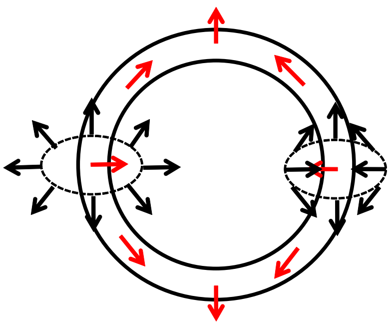
(c) $(-1, +\frac{1}{2}, -\frac{1}{2}) + (+1, +\frac{1}{2}, +\frac{1}{2})$ (d) $(-1, -\frac{1}{2}, +\frac{1}{2}) + (+1, -\frac{1}{2}, -\frac{1}{2})$

Figure 12. Bions in the O(4) model with the boundary condition $(-, +, +, +)$. The notations are the same with figure 11. (c) and (d) are isomorphic to (a) and (b), respectively, by a 2π rotation along an axis parallel to the x^1 axis at the center of the domain wall.

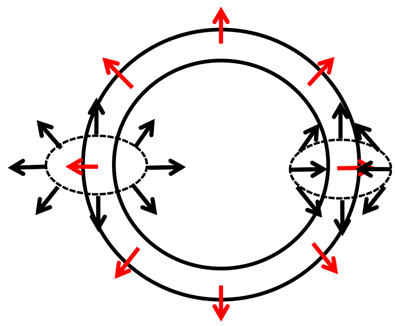
a vortex ring is the same with that of the compactification scale R, the top and bottom parts of the vortex ring touch each other through the compact x^3 direction with the twisted boundary condition. Then, a reconnection of two fractions of the ring can occur (see [120] for a reconnection of strings with moduli), the ring can be split into two vortex strings stretched along the compact direction, and subsequently they are separated into the x^1-x^2 plane as shown in figure 14. The U(1) modulus is twisted half along each string, resulting in a fractional (anti-)instanton. These twisted vortices in the Skyrme model were numerically constructed in ref. [121]. By considering all possibilities of twisted vortex rings, we find four kinds of fractional (anti-)instantons, as summarized in figure 2 (2a)–(2d).

The ansatz for fractional (anti-)instanton configurations with the boundary condition $(-, -, +, +)$ is given as

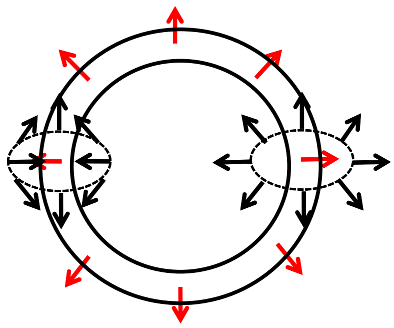
$$n_3 + i n_4 = \sin g(r) e^{i\theta}, \qquad n_1 + i n_2 = \cos g(r) e^{i\zeta(z)}, \tag{5.1}$$



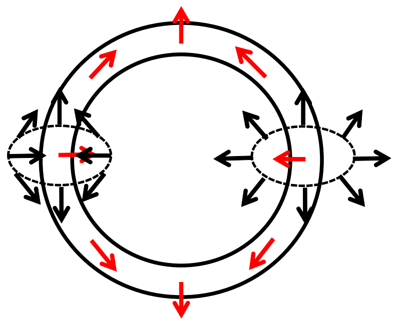
(a) $(+1, +\frac{1}{2}, +\frac{1}{2}) + (-1, -\frac{1}{2}, +\frac{1}{2})$



(b) $(+1, -\frac{1}{2}, -\frac{1}{2}) + (-1, +\frac{1}{2}, -\frac{1}{2})$



(c) $(-1, -\frac{1}{2}, +\frac{1}{2}) + (+1, +\frac{1}{2}, +\frac{1}{2})$



(d) $(-1, +\frac{1}{2}, -\frac{1}{2}) + (+1, -\frac{1}{2}, -\frac{1}{2})$

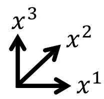


Figure 13. Fractional instantons from instantons in the O(4) model with the boundary condition $(-,-,+,+)$. Black arrows represent (n_1, n_2) with $n_1^2 + n_2^2 = 1$ $(n_3 = n_4 = 0)$ parameterizing the moduli space of vacua $\mathcal{N} \simeq S^1$, while red arrows represent (n_3, n_4) with $n_3^2 + n_4^2 = 1$ $(n_1 = n_2 = 0)$ parameterizing the moduli space of a vortex $\mathcal{M} \simeq S^1$. An instanton can be represented as a vorton, that is, a vortex ring along which the U(1) modulus is twisted once. Brackets $(*, *, *)$ denote topological charges for a host global vortex characterized by π_1, that for a sine-Gordon kink characterized by π_1, and that for an instanton characterized by π_3. (a) An instanton (vorton) can be split into two fractional instantons $(+1, +\frac{1}{2}, +\frac{1}{2})$ and $(-1, -\frac{1}{2}, +\frac{1}{2})$. (b) An anti-instanton (anti-vorton) can be split into two fractional anti-instantons $(+1, -\frac{1}{2}, -\frac{1}{2})$ and $(-1, +\frac{1}{2}, -\frac{1}{2})$. (c) and (d) are isomorphic to (a) and (b), respectively, by a 2π rotation along an axis parallel to the x^1 axis.

$$g(0) = 0, \qquad\qquad g(\infty) = \pm\frac{\pi}{2}, \tag{5.2}$$

$$\zeta(z = R) = \zeta(z = 0) \pm \pi, \tag{5.3}$$

where (r, θ, z) are cylindrical coordinates. The topological instanton charge (Skyrmion charge or baryon number) can be calculated as

$$Q_3 = \frac{1}{16\pi^2} \int d^3x \, \frac{1}{r} \sin(g) g_r \zeta_z = \frac{1}{2\pi} [\zeta]_{z=0}^{z=R} = \pm\frac{1}{2}. \tag{5.4}$$

General formula of the instanton (Skyrme) charge for a vortex string with the winding

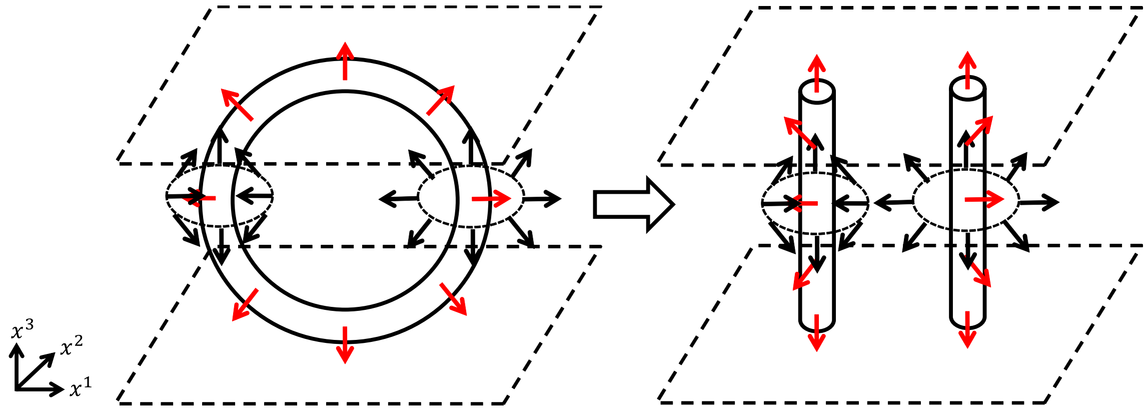


Figure 14. A twisted vortex ring of the size of the compact direction decays into two fractional instantons through a reconnection in the O(4) model with the boundary condition $(-,-,+,+)$. The notations are the same with figure 13. The dotted planes denote the boundary at $x^3 = 0$ and $x^3 = R$. When the top and bottom of the ring touch each other through the compact direction x^3 with the twisted boundary condition, a reconnection of the two parts of the string occurs and the ring is split into two fractional (anti-)instantons, vortices with the half twisted U(1) moduli.

	π_1	π_1	π_3
Figure 2 (2a)	$+1$	$+1/2$	$+1/2$
Figure 2 (2b)	-1	$-1/2$	$+1/2$
Figure 2 (2c)	-1	$+1/2$	$-1/2$
Figure 2 (2d)	$+1$	$-1/2$	$-1/2$

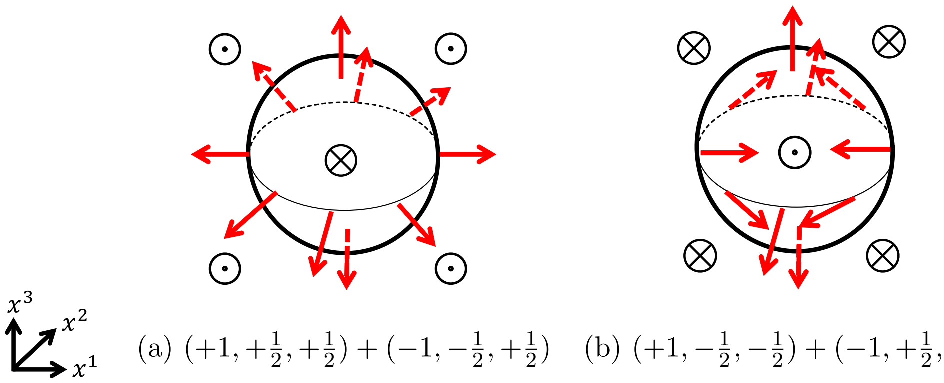
Table 6. Homotopy groups of fractional instantons in the O(4) model with the boundary condition $(-,-,+,+)$. The columns represent the homotopy groups of a host soliton π_1, a daughter soliton π_1, and the total instanton π_3 from left to right.

number Q, along which the U(1) modulus is twisted P times, was calculated to be PQ in ref. [122] in the context of Hopfions and in ref. [117] for Skyrmions. The topological charges of fractional (anti-)instantons with the boundary condition $(-,-,+,+)$ are summarized in table 6.

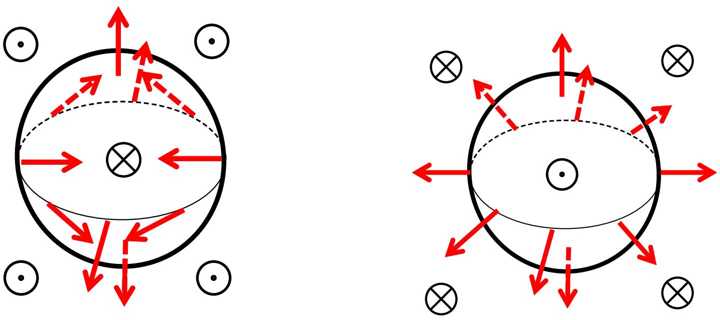
Interestingly, we do not need higher derivative (Skyrme) term even though fractional instantons are Skyrmions. Indeed, stable configurations of (half) Skyrmions inside a vortex string was constructed without the Skyrme term in ref. [117] on $\mathbb{R}^3$ without twisted boundary condition.

Fractional instantons with the boundary condition $(-,-,+,+)$ are global vortices in the x^1-x^2 plane so that the interaction between them is $E_{\rm int} \sim \pm \log r$ with distance r for large separation (the force is $F \sim \pm 1/r$), where positive sign is for a pair of (anti-)vortices and negative sign is for a pair of a vortex and anti-vortex.

Bions can be constructed by combining configurations in (2a) and (2c) in figure 2, or (2b) and (2d) in figure 2. In the both cases, instanton charges are canceled out. The interaction between fractional instantons constituting a bion is $E_{\rm int} \sim -\log r$ with distance r for large separation and $F \sim -1/r$, because they are a pair of a global vortex and global anti-vortex.



(a) $(+1,+\frac{1}{2},+\frac{1}{2})+(-1,-\frac{1}{2},+\frac{1}{2})$ (b) $(+1,-\frac{1}{2},-\frac{1}{2})+(-1,+\frac{1}{2},-\frac{1}{2})$



(c) $(-1,-\frac{1}{2},+\frac{1}{2})+(+1,+\frac{1}{2},+\frac{1}{2})$ (d) $(-1,+\frac{1}{2},-\frac{1}{2})+(+1,-\frac{1}{2},-\frac{1}{2})$

Figure 15. Fractional instantons from instantons in the O(4) model with the boundary condition $(-,-,-,+)$. $\otimes$ and $\odot$ denote $n_4=+1$ and $n_4=-1$, representing the vacua $\mathcal{N}=\{\pm 1\}$. Red arrows represent (n_1,n_2,n_3) parameterizing the moduli space of a domain wall $\mathcal{M}\simeq S^2$. An instanton can be represented as a twisted domain wall, that is, a spherical domain wall around which S^2 moduli are wound once. Brackets $(*,*,*)$ denote topological charges for a host domain wall characterized by π_0, that for a lump characterized by π_2, and that for an instanton characterized by π_3. (a) An instanton can be split into two fractional instantons $(+1,+\frac{1}{2},+\frac{1}{2})$ and $(-1,-\frac{1}{2},+\frac{1}{2})$. (b) An anti-instanton can be split into two fractional anti-instantons $(+1,-\frac{1}{2},-\frac{1}{2})$ and $(-1,+\frac{1}{2},-\frac{1}{2})$. (c) and (d) are isomorphic to (a) and (b), respectively, by a 2π rotation along an axis parallel to the x^1 axis.

As in eq. (4.10) for the O(3) model with the boundary condition $(-,-,+)$, the Scherk-Schwarz dimension reduction to two dimensions induces a potential term

$$V=m^2(\hat{n}_1^2+\hat{n}_2^2)=m^2(1-\hat{n}_3^2-\hat{n}_4^2). \tag{5.5}$$

If one gauges the U(1) symmetry acting on n_3+in_4, vortices become local vortices of the ANO type, having finite energy.

5.3 $(-,-,-,+)$: a half lump inside a domain wall

The fixed manifold is characterized by $n_1=n_2=n_3=0$ which is two discrete points characterized by $n_4=\pm 1$. The moduli space of vacua is $\mathcal{N}\simeq\{\pm 1\}$. It has a nontrivial homotopy $\pi_0(\pm 1)\simeq\mathbb{Z}_2$, allowing a domain wall. In the domain wall core, the field making a domain wall vanishes $n_4=0$ and the other fields n_1,n_2 and n_3 appear with a constraint $(n_1)^2+(n_2)^2+(n_3)^2=1$, giving the moduli $\mathcal{M}\simeq S^2$ to a domain wall. For a fractional

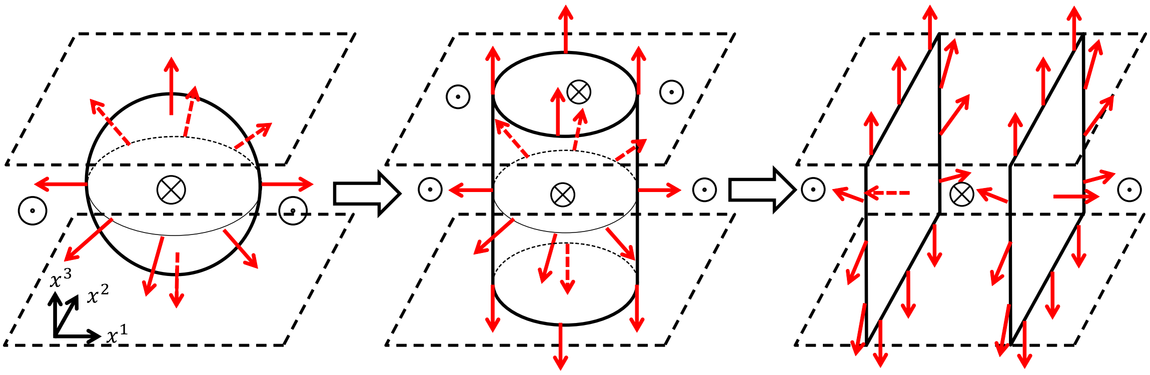


Figure 16. Deformation of a twisted spherical domain wall into two fractional instantons in the O(4) model with the boundary condition $(-,+,+,+)$. The notations are the same with figure 15. The dotted planes denote the boundary at $x^3 = 0$ and $x^3 = R$. A twisted spherical domain wall turns to a twisted domain wall tube when the top and bottom of the sphere touch each other through the compact direction x^3 with the twisted boundary condition. If it is further stretched into the infinities in the x^2 direction, it can be deformed to two fractional instantons.

instanton, these S^2 moduli are wound half in the wall world volume with the compact direction, as described below.

An instanton (Skyrmion) can be represented as a twisted spherical domain wall, that is, a spherical domain wall around which S^2 moduli are wound [123]. Configurations of Skyrmions as twisted spherical domain walls are shown in figure 15. Deformation of a twisted spherical domain wall into two fractional instantons can be explained as follows. First, when the size of a sphere is the same with that of the compactification radius R, the top and bottom of the sphere touch and join each other through the compact direction x^3 with the twisted boundary condition, and the sphere turns to a twisted domain wall tube as in left to middle in figure 16. Second, if the tube is further stretched to the infinities in the x^2 direction, it can be deformed to two surfaces (domain walls), separated into the x^1 direction as middle to right in figure 16, where the domain wall world volume extend to the x^2 and x^3 coordinates. The S^2 moduli are twisted half inside the domain wall world volumes, giving rise to fractional (anti-)instantons. While the first process can occur energetically, the second process cannot occur energetically because it needs infinite world volumes unless the x^2 direction is compactified. Still the final configurations themselves are possible. We thus find four possibilities of fractional (anti-)instantons as shown in figure 2 (3a)–(3d).

The ansatz for fractional (anti-)instanton with the boundary condition $(-,-,-,+)$ is given as

$$\mathbf{n} = (b_1(x^1,x^2)\sin f(x^3), b_2(x^1,x^2)\sin f(x^3), b_3(x^1,x^2)\sin f(x^3), \cos f(x^3)), \quad (5.6)$$

$$f(0) = 0, \qquad f(\infty) = \pi. \quad (5.7)$$

The fields $\mathbf{b} = (b_1, b_2, b_3)(x^1, x^2)$ are induced on a domain wall, satisfying the same boundary condition with $(-,-,-)$ of the O(3) model. Then we can consider a half lump given in eq. (4.11). The topological instanton charge (Skyrmion charge or baryon number) can

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	π_0	π_2	π_3
Figure 2 (3a)	$+1$	$+1/2$	$+1/2$
Figure 2 (3b)	-1	$-1/2$	$+1/2$
Figure 2 (3c)	-1	$+1/2$	$-1/2$
Figure 2 (3d)	$+1$	$-1/2$	$-1/2$

Table 7. Homotopy groups of fractional instantons in the O(4) model with the boundary condition $(-,-,-,+)$. The columns represent the homotopy groups of a host soliton π_0, a daughter soliton π_2, and the total instanton π_3 from left to right.

be calculated as [126]

$$Q_3 = \frac{1}{\pi}\int d^3x\ \mathcal{Q} f_x = \int d^2x \mathcal{Q} \equiv Q_2, \tag{5.8}$$

$$\mathcal{Q} = \frac{1}{8\pi}\epsilon^{ij}\mathbf{b}\cdot\partial_i\mathbf{b}\times\partial_j\mathbf{b}. \tag{5.9}$$

The topological charges of fractional (anti-)instantons with the boundary condition $(-,-,-,+)$ are summarized in table 7.

Interestingly, we do not need higher derivative (Skyrme) term even though fractional instantons are Skyrmions, as in the case of the boundary condition $(-,-,+,+)$. Indeed, stable configurations of a unit (not half) Skyrmion inside a domain wall was constructed in $\mathbb{R}^3$ without twisted boundary condition [124–126]. We expect that the same holds for half instantons (Skyrmions).

Fractional instantons with the boundary condition $(-,-,-,+)$ are domain walls perpendicular to the x^1 coordinate so that the interaction between them is $E_{\rm int} \sim -e^{-mr}$ with distance r for large separation. The energy of domain walls are linearly divergent in the x^2 direction.

Bions can be constructed by combining configurations in (3a) and (3c) in figure 2, or (3b) and (3d) in figure 2, where the instanton charge is canceled out. The interaction between fractional instantons constituting a bion is attractive and exponentially suppressed $E_{\rm int} \sim -e^{-mr}$.

5.4 $(-,-,-,-)$

There are no fixed points for the boundary condition $(-,-,-,-)$ as the case with the boundary condition $(-,-,-)$ in the O(3) model. We do not have localized solitons wrapping around a fixed manifold. Again, we regard that there is a space-filling soliton (brane) with the moduli $\mathcal{M} \simeq S^3$.

One (anti-)instanton is separated into two fractional (anti-)instantons with the boundary condition $(-,-,-,-)$, as summarized in figure 2 (4a)–(4b). Each fractional instanton wraps a half sphere of the target space S^3.

We need higher derivative (Skyrme) term for the stability of fractional instantons (Skyrmions).

	π_{-1}	π_3	π_3
Figure 2 (4a)	$+1$	$+1/2$	$+1/2$
Figure 2 (4b)	-1	$-1/2$	$+1/2$
Figure 2 (4c)	-1	$+1/2$	$-1/2$
Figure 2 (4d)	$+1$	$-1/2$	$-1/2$

Table 8. Homotopy groups of fractional instantons in the O(4) model with the boundary condition $(-,-,-,-)$. The columns represent the homotopy groups of a host soliton π_{-1}, a daughter soliton π_3, and the total instanton π_3 from left to right. Here, π_{-1} is merely formal.

The Scherk-Schwarz dimension reduction can be discussed as in eq. (4.10) for $(-,-,+,+)$. However, by assuming the dependence of the fields on the compact direction x^2 as

$$(n_1,n_2,n_3,n_4)=\left(\hat{n}_1(x^1)\cos\frac{\pi}{R}x^2,\hat{n}_2(x^1)\sin\frac{\pi}{R}x^2,\hat{n}_3(x^1)\cos\frac{\pi}{R}x^2,\hat{n}_4(x^1)\sin\frac{\pi}{R}x^2\right) \tag{5.10}$$

in the presence of the twisted boundary condition $(-,-,-,-)$, we see that it does not give a nontrivial potential:

$$V=m^2(\hat{n}_1^2+\hat{n}_2^2+\hat{n}_3^2+\hat{n}_4^2)=m^2, \tag{5.11}$$

with m in eq. (4.10).

6 Summary and discussion

We have found that a fractional instanton in the O(3) model is a global vortex with an Ising spin for $(-,+,+)$, a half sine-Gordon kink on a domain wall for $(-,-,+)$, or a half lump on a "space-filling brane" for $(-,-,-)$, and that a fractional instanton in the O(4) model is a global monopole with an Ising spin for $(-,+,+,+)$, a half sine-Gordon kink on a global vortex for $(-,-,+,+)$, a half lump on a domain wall for $(-,-,-,+)$, or a half Skyrmion on a "space-filling brane" for $(-,-,-,-)$. As from general argument in section 3, the above classification holds for the O(N) model with arbitrary N. We have also constructed neutral bions the O(3) and O(4) models but have found that charged bions are not possible. We have seen that when the number of minus signs in the boundary condition is even, a small compactification limit gives the Scherk-Schwarz dimensional reduction which induces a potential term as in eq. (4.10).

If the interaction energy of two fractional instantons is exponentially suppressed $E_{\rm int}\sim e^{-mr}$ when they are well separated at distance r, the total energy of well separated fractional instantons is just of the sum of those of individual fractional instantons. In this case, they would play a role in resurgence of quantum field theory. This is indeed the case of the O(3) model with the boundary condition $(-,-,+)$ [8, 9, 17, 18] in which fractional (anti-)instantons are (anti-)BPS so that there exists no interaction between fractional BPS instantons, or between fractional anti-BPS instantons, and exponentially suppressed interaction $E_{\rm int}\sim e^{-mr}$ between a BPS and an anti-BPS fractional instantons.

Fractional instantons are not local or BPS in the other cases discussed in this paper as they are. However, with suitable modifications, some of them may become BPS or local as summarized as follows:

1. The O(3) model with $(-,+,+)$. If the U(1) symmetry acting on $n_2 + in_3$ is gauged, half lump-vortices become local vortices having finite energy, in which case the interaction between them would be exponentially suppressed. This is because the U(1) gauged O(3) model with a potential $V = m^2 n_1^2$ on $\mathbb{R}^2$ allows local vortices [79–83]. If we further choose the gauge coupling to be $e^2 = m^2$, fractional (anti-)instantons become (anti-)BPS and the theory can be made supersymmetric [82]. In our case, the twisted boundary condition would play a role of the potential, so the gauge coupling should be correlated to the compactification radius for vortices to be BPS.

2. The O(3) model with only four derivative term. An O(3) model consists of only four derivative (Skyrme) term and a suitable potential term is known as a BPS baby Skyrme model, admitting BPS instantons (lumps, baby Skyrmions) on $\mathbb{R}^2$ [127, 128]. This model can be made supersymmetric [129–132]. A generalization of the model to $\mathbb{R}^1 \times S^1$ with twisted boundary condition is expected to admit BPS fractional instantons.

3. The O(4) model with $(-,+,+,+)$. If one gauges the SO(3) symmetry action on (n_2, n_3, n_4), a half-Skyrmion monopole becomes local, that is, of 't Hooft-Polyakov type having finite energy, in which case the interaction between them is exponentially suppressed. This is because an SO(3) gauged Skyrme model with a potential $V = m^2 n_1^2$ on $\mathbb{R}^3$ allows a local 't Hooft-Polyakov type monopole with finite energy [99–103]. A BPS limit is not known in this case. Again in our case, the twisted boundary condition would play a role of the potential.

4. The O(4) model with $(-,-,+,+)$. If one gauges the U(1) symmetry acting on $n_3 + in_4$, vortices as half instantons become local vortices having finite energy, in which case the interaction between them would be exponentially suppressed.

5. The O(4) model on $S^2 \times S^1$. If we consider a geometry $S^2 \times S^1$ instead of $\mathbb{R}^2 \times S^1$, instantons (Skyrmions) are BPS for untwisted boundary condition [133]. An extension to a twisted boundary condition should be possible, in which case fractional (anti-)instantons may be also (anti-)BPS.

6. The O(4) model with only a six derivative term. If we consider Lagrangian containing only a six derivative term, which is baryon charge density squared, and a suitable potential term, instantons (Skyrmions) are BPS, which is indeed the case of $\mathbb{R}^3$ [134, 135]. It may be generalized to the case of $\mathbb{R}^2 \times S^1$ with a twisted boundary condition, in which case fractional (anti-)instantons may be also (anti-)BPS.

In these cases, fractional instantons will play a role in resurgence, which is indeed the case of the O(3) model with the boundary condition $(-,-,+)$ [8, 9, 17, 18] as denoted above.

When we compactify more than one directions, we can consider more general twisted boundary conditions. For instance, we may consider the O(3) model on $\mathbb{R}^n \times (S^1)^2$ with a twisted boundary condition $(-,-,+)$ for one direction and $(+,-,-)$ for the other direction. A complete classification of these more general cases remain as an interesting problem.

A lattice of half Skyrmion appear in finite baryon density [136–138]. There may be certain relation with our half Skyrmions in the presence of a compact direction with twisted boundary conditions.

Hopfions are knot like solitons supported by the Hopf charge $\pi_3(S^2) \simeq \mathbb{Z}$ in the O(3) model with four derivative (Faddeev-Skyrme) term [139–143]. Since Hopfions on $\mathbb{R}^3$ are closed lump strings along which U(1) moduli are twisted (see, e.g. ref. [144, 145]), those on $\mathbb{R}^2 \times S^1$ with an untwisted boundary condition can be twisted closed lump strings wrapping around S^1 [122, 146–149]. If we impose twisted boundary conditions, we will be able to obtain a fractional Hopfion as a half-twisted lump string wrapping around S^1.

By applying our method to non-Abelian gauge theories, classification of fractional Yang-Mills instantons may be possible, which would be important toward the resurgence of gauge theories. To this end, realizations of Yang-Mills instantons as various composite solitons summarized in ref. [150] will be useful, as has been demonstrated for Skyrmions in this paper. Yang-Mills instantons are Skyrmions inside a domain wall [151], lumps inside a vortex [46, 56, 152–154], or sine-Gordon kinks on a monopole string [63]. Investigating boundary conditions realizing these would be an important first step toward the resurgence of gauge theories.

Finally, let us make a comment on duality. As seen in this paper, a $\mathbb{C}P^1$ instanton with the boundary condition $(-,-,+)$ is decomposed into a set of two fractional instantons which are half twisted domain walls, as seen in figure 6, and one of them becomes a domain wall in a small compactification radius limit in which the other is removed to infinity [46]. The same relation holds between a Yang-Mills instanton and a BPS monopole, which can be also understood as a T-duality acting on D-branes in type-II string theory [155]. In ref. [46], $\mathbb{C}P^{N-1}$ fractional instantons were realized as fractional Yang-Mills instantons trapped inside a vortex in a $U(N)$ gauge theory, which explains a relation between the above mentioned two T-dualities. Here, in this paper, we have added one more example, that is, a T-duality between a Skyrmion and a vortex. In the O(4) model with the boundary condition $(-,-,+,+)$, equivalently eq. (2.15), a Skyrmion is decomposed into a set of two fractional instantons which are half twisted vortex strings as seen in figure 14. One of them becomes a vortex in a small compactification radius limit, in which the other is removed to infinity. We think that a further T-duality maps this configuration to a domain wall through a domain wall Skyrmion.

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